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Analysis and Construction of Logical Systems: A Category-Theoretic Approach

PhD dissertation

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aware of legal responsibility I hereby declare that I have written this dissertation myself and all the contents of the dissertation have been obtained by legal means.

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Abstract

The aim of this dissertation is to develop categorical foundations for studying lambda calculi and their logics formed into logical systems. We show how internal models for polymorphic lambda calculi arise in any 2-category with a notion of discreteness. We generalise to a 2-categorical setting the famous theorem of Peter Freyd saying that there are no sufficiently (co)complete non-posetal categories. As a simple corollary, we obtain a variant of Freyd's theorem for categories internal to any tensored category. We introduce the concept of an associated category, and relying on it, provide a representation theorem relating our internal models with well-studied fibrational models for polymorphism. Finally, we define Yoneda triangles as relativisations of internal adjunctions, and use them to characterise universes that admit a notion of convolution. We show that such universes induce semantics for lambda calculi. We prove that a construction analogical to enriched Day convolution works for categories internal to a locally cartesian closed category with finite colimits.

Contents

Introduction

There are mainly two distinct approaches to logic. One is the "model-theoretic" approach. This approach deals with a set of formulae *Sen* and a class of models *Mod* together with a binary relation $\models \subseteq \mathit{Mod} \times \mathit{Sen}$ saying which formulae are true in which models.

Example 0.1 (Model-theoretic propositional logic). *Let Σ_{Prop} be the propositional signature — that is, the signature consisting of two nullary symbols $\top, \bot$ and three binary symbols $\wedge, \vee, \Rightarrow$. A propositional syntax Prop is the free algebra over Σ_{Prop} on a countable set of generators Var (the set of variables). Denote by Bool the class of pairs $\langle B, \nu\colon \mathit{Var} \to |B|\rangle$, where B is a Boolean algebra over Σ_{Prop}, $|B|$ is its carrier, and $\nu\colon \mathit{Var} \to |B|$ is the valuation of variables Var in the carrier of B. By freeness of Prop, every valuation $\nu \in |B|^{\mathit{Var}}$ uniquely extends to the homomorphism $\nu^{\#} \in B^{\mathit{Prop}}$. We define the model-theoretic classical propositional logic as the relation $\models_B \subseteq \mathit{Bool} \times |\mathit{Prop}|$*

$$\langle B, \nu\colon \mathit{Var} \to |B|\rangle \models_B \phi \;\; \textit{iff} \;\; \nu^{\#}(\phi) = \top_B$$

where $\top_B$ is the interpretation of $\top$ in algebra B.

As usual, we say that a formula ϕ is valid if it is satisfied in every model, that is, for every model B, and every valuation $\nu \in |B|^{\mathit{Var}}$ the following holds:

$$\langle B, \nu\rangle \models_B \phi$$

If we replace Boolean algebras with Heyting algebras in the above definition, we obtain the model-theoretic intuitionistic propositional logic:

$$\models_H \subseteq \mathit{Heyting} \times |\mathit{Prop}|$$

Recalling that a 2-valued Boolean algebra $2 = \{0 \leq 1\}$ is complete for the classical propositional logic[1] we may reach the following compact characterisation.

Example 0.2 (Model-theoretic 2-valued logic). *With the notation of Example 0.1 we define the model-theoretic* 2*-valued propositional logic to be the relation* $\models_2 \subseteq |2|^{Var} \times |Prop|$

$$\nu \models_2 \phi \ \textit{ iff } \ \nu^{\#}(\phi) = \top_2$$

By completeness of 2*-valued models, validity of formulas wrt. model-theoretic propositional logic from Example 0.1 and wrt. the above logic coincide.*

In the above example logical connectives were defined internally to the logic — i.e. were defined inductively over the syntax to imitate operations from a Boolean algebra. Another way is to define logical connectives externally to the logic. This idea may be found in the theory of specifications, where it is used to give an abstract characterisation of logical connectives, or to enrich logical systems with some "missing" connectives (Example 4.1.41 of [ST12]). For example we can extend the set $|Prop|$ by formulae of the form $\urcorner\phi$ and put:

$$\nu \models \urcorner\phi \ \textit{ iff } \ \nu \not\models \phi$$

Connectives defined in such a way do not depend on the structure of the Boolean algebra 2, but on the connectives from the "external" logic — i.e. the meta-logic that defines the relation $\models$. We shall call these connectives "extensionally defined" as they rely on logical values of $\models$.

Another approach to logic is the "proof-theoretic" approach. This approach deals with deductive systems, or categories. Following Lambek and Scott [LS86] we define a graph G to be a quadruple $\langle Obj; Arr; src, trg \colon Arr \to Obj\rangle$, and shall call elements of Obj objects (or formulae), elements of Arr arrows (morphisms, proofs, or deductions) and write $f \colon A \to B \in G$ for $f \in Arr \wedge src(f) = A \wedge trg(f) = B$. Then a deductive system is a graph in which with every object A there is associated an arrow id_A, and with every pair of compatible arrows $f \colon A \to B$, $g \colon B \to C$ there is associated an arrow $g \circ f \colon A \to C$ — that is, a deductive system is a collection of proofs (deductions) between formulae together with at least one axiom $\dfrac{}{A \xrightarrow{id_A} A}$ (identity),

[1] And the Heyting algebra consisting of the open subset of the real line is complete for intuinitionistic logic.

and at least one rule $\dfrac{A \xrightarrow{f} B \quad B \xrightarrow{g} C}{A \xrightarrow{g \circ f} C}$ (cut). A category is a deductive system satisfying the obvious coherence conditions — for every compatible arrows: $A \xrightarrow{f} B$, $B \xrightarrow{g} C$, $C \xrightarrow{h} D$ the following holds: $h \circ (g \circ f) = (h \circ g) \circ f$ and $f \circ id_A = f = id_B \circ f$.

Example 0.3 (Proof-theoretic propositional logic)**.** *The proof-theoretic classical propositional logic is the smallest (posetal*[2]*) category having as objects elements from the carrier of the syntactic algebra Prop (recall Example 0.1) and arrows generated by the following rules:*

$$\frac{}{\phi \longrightarrow \phi}\ \textit{(identity)} \qquad \frac{\phi \longrightarrow \psi \quad \psi \longrightarrow \gamma}{\phi \longrightarrow \gamma}\ \textit{(cut)}$$

$$\frac{}{\phi \longrightarrow \top}\ \textit{(true)} \qquad \frac{}{\bot \longrightarrow \phi}\ \textit{(false)}$$

$$\frac{\phi \longrightarrow \psi \quad \phi \longrightarrow \gamma}{\phi \longrightarrow \psi \wedge \gamma}\ \textit{(}\wedge\textit{-intro)} \qquad \frac{\phi \longrightarrow \psi \wedge \gamma}{\phi \longrightarrow \psi}\ \textit{(}\wedge\textit{-left-elim)}$$

$$\frac{\phi \longrightarrow \psi \wedge \gamma}{\phi \longrightarrow \gamma}\ \textit{(}\wedge\textit{-right-elim)} \qquad \frac{\phi \longrightarrow \psi \quad \gamma \longrightarrow \psi}{\phi \vee \gamma \longrightarrow \psi}\ \textit{(}\vee\textit{-intro)}$$

$$\frac{\phi \vee \gamma \longrightarrow \psi}{\phi \longrightarrow \psi}\ \textit{(}\vee\textit{-left-elim)} \qquad \frac{\phi \vee \gamma \longrightarrow \psi}{\gamma \longrightarrow \psi}\ \textit{(}\vee\textit{-right-elim)}$$

$$\frac{\phi \wedge \psi \longrightarrow \gamma}{\psi \longrightarrow \phi \Rightarrow \gamma}\ \textit{(}\Rightarrow\textit{-intro)} \qquad \frac{\psi \longrightarrow \phi \Rightarrow \gamma}{\phi \wedge \psi \longrightarrow \gamma}\ \textit{(}\Rightarrow\textit{-elim)}$$

$$\frac{}{(\phi \Rightarrow \bot) \Rightarrow \bot \longrightarrow \phi}\ \textit{(double negation)}$$

We say that a formula ϕ is valid if there is a morphism $\top \to \phi$.

Observe that while the connectives in the above example were defined by the generating rules of deductions, we could equivalently characterise the connectives as categorical products, coproducts and exponents (i.e. in a posetal category the above rules coincide with the rules defining products, coproducts

[2]A category is posetal if there is at most one morphism between any two objects.

and exponents), making them "external" to the logic. Furthermore, we shall call these connectives "intensional" as they rely on categorical properties.

The above requires more elaboration. A standard way to define categorical products, coproducts and exponents is via the Yoneda lemma:

$$\frac{X \mapsto \{*\}}{\overline{\hom(X,1)}}\ (1\text{-rule}) \qquad \frac{X \mapsto \{*\}}{\overline{\hom(0,X)}}\ (0\text{-rule})$$

$$\frac{\hom(X,Y) \times \hom(X,Z)}{\overline{\hom(X, Y \times Z)}}\ (\times\text{-rule}) \qquad \frac{\hom(Y,X) \times \hom(Z,X)}{\overline{\hom(Y \sqcup Z, X)}}\ (\sqcup\text{-rule})$$

$$\frac{\hom(X \times Y, Z)}{\overline{\hom(X, Z^Y)}}\ ((-)^{(=)}\text{-rule})$$

where the double vertical bars should be read as the existences of natural isomorphisms between **Set**-valued functors. Now, if a category is posetal, then the "naturality condition" is always satisfied. Moreover, for sets A, B with cardinality less-or-equal than one, the following holds: A is isomorphic to B iff there are functions $A \to B$ and $B \to A$. Therefore, in posetal categories, the above rules for connectives can be split into pairs of rules (introduction and elimination), and writing $X \to Y$ for $\hom(X,Y)$, we can obtain rules from Example 0.3.

The two approaches described in the above often come together — if a logic is defined to be a deductive system, a key problem is to find a desirable class of models over which the logic is sound (and, ideally, complete); conversely — if a logic is defined to be a satisfaction relation, a key problem is to find a sound (and, ideally, complete) deductive system for the logic. By linking these two approaches we obtain the concept of a logical system.

Example 0.4 (Logical consequence)**.** *Let* $\models \subseteq Mod \times Sen$ *be a model-theoretic logic. The logical consequence relation* $\models_{Sen} \subseteq Sen \times Sen$ *is defined as follows:*

$$\phi \models_{Sen} \psi \ \textit{ iff } \ \forall_{M \in Mod}\ M \models \phi \ \textit{ implies } \ M \models \psi$$

This relation induces the structure of a category on Sen, which is compatible with satisfaction in the sense:

$$M \models \phi,\ \ \phi \models_{Sen} \psi \ \Rightarrow\ M \models \psi$$

Proof-theoretic propositional logic (Example 0.3) is the category induced by logical consequence from model-theoretic propositional logic (Example 0.2).

Generally, if $\models\ \subseteq Mod \times |\mathbf{Sen}|$ is a model-theoretic logic, and **Sen** is a posetal category build upon $|\mathbf{Sen}|$, we shall say that proof-theoretic logic **Sen** is sound over model-theoretic logic $\models$ if the compatibility condition:

$$M \models \phi,\ \ \phi \to \psi \ \Rightarrow\ M \models \psi$$

holds — i.e. if $M \models \phi$ and there exists a morphism $\phi \to \psi$, then $M \models \psi$. Because we deal exclusively with sound systems, we define a logical system to be a relation between a collection *Mod* and the collection of objects of a posetal category **Sen** satisfying compatibility condition.

The aim of this dissertation is to develop purely categorical foundations for studying lambda calculi and their logics formed into logical systems. In Chapter 1 we provide a general 2-categorical setting for intensional calculi and study its properties. This extends and gives a new perspective on the fibrational models for polymorphic lambda calculi — we believe that our 2-categorical models are more natural and easier to understand. We state and prove a suitable version of Peter Freyd's incompleteness theorem. In Chapter 2 we provide a general 2-categorical setting for extensional calculi and show how intensional and extensional calculi can be related in logical systems. The chapter focuses on transporting the notion of Day convolution to a 2-categorical framework. We define the concept of Yoneda (bi)triangle, and show how objects in a Yoneda bitriangle get extensional semantics "for free". This includes the usual semantics for propositional calculi, Kripke semantics for intuitionistic calculi and ternary frame semantics for substructural calculi including Lambek's lambda calculi, relevance and linear logics. We show how in this setting one may use a model-theoretic logic to induce a structure of proof-theoretic logic. Appendix A recalls some basic categorical concepts and notions.

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thank an anonymous reviewer of the paper corresponding to Chapter 1 of this dissertation for suggesting a characterisation of the left adjoint to the representation functor (Theorem 1.76).

Notational conventions

Throughout the dissertation we shall use the following notational conventions. Categories whose names consist of a single character are denoted by wide capital letters $\mathbb{A}, \mathbb{B}, \mathbb{C}$; categories whose names consist of multiple characters are usually denoted by bold letters starting from a capital letter $\mathbf{Set}, \mathbf{Top}, \mathbf{Cat}$. An exception from this rule is the (2-)category **cat** of small categories. Likewise, we write $\mathbf{cat}(\mathbb{C})$ for the (2-)category of *small* categories *internal* to a finitely complete category $\mathbb{C}$. By **Cat** we mean the (2-)category of *locally small* categories, and by $\mathbf{Cat}(\mathbb{C})$ we mean the (2-)category of categories enriched over a monoidal category $\mathbb{C}$. Notice, that we do not have a general notation for *locally small internal* categories, nor for *small enriched* categories — the reason is that the natural notion for internal category is a small internal category, whereas the natural notion for enriched category is a locally small enriched category, and thus the other notions are rarely used. Hence, if not stated otherwise, by "internal category" we will mean "small internal category", and by "enriched category" we will mean "locally small enriched category". We shall use the term "posetal category" for a degenerate category that has at most one morphism between any two objects. Every "posetal category" can be thought of as a pre-ordered set in a natural way (in fact, since pre-ordered sets and partially ordered sets are categorically equivalent, if it will not lead to confusion, we shall not distinguish between these two concepts and call them both: partially ordered sets, or "posets").

Usually, objects in a category are denoted by capital letters from initial and final segments of Latin alphabet A, B, C, X, Y, Z, functors by the other Latin capital letters F, G, H, and usual morphisms by non-capital letters f, g, h. The fact that a morphism f has domain A and codomain B is denoted by $f\colon A \to B$ or in a more compact form by $A \xrightarrow{f} B$. We indicate the fact that a morphism $f\colon A \to B$ plays the role of "relation" by drawing a vertical bar $f\colon A \nrightarrow B$. The external object of morphisms from A to B is denoted by $\hom(A, B)$, with an optional subscript denoting the ambient category. B^A denotes the internal object of morphisms — i.e. the (linear) exponential object from A to B. Note that the objects $\hom(A, B)$ and B^A

coincide in categories enriched over themselves (for more details see Appendix A.3 and Definition A.33 there). Usually, the notion B^A will be restricted to cartesian exponents, and non-cartesian (linear) exponents from A to B will be denoted by $A \multimap B$. Small letters from Greek alphabet α, β, γ are generally used for 2-morphisms and, particularly, for natural transformations. The A-th component of a natural transformation $\alpha\colon F \to G$ is written as $\alpha_A\colon F(A) \to G(A)$, or if it does not lead to confusion, as $\alpha\colon F(A) \to G(A)$ — without any subscript. The identity morphism on an object A is denoted by id_A or by the object A itself. We use the circle symbol $\circ$ to denote the usual categorical composition. Thus, if $f\colon A \to B$ and $g\colon B \to C$ are morphisms, then $g \circ f\colon A \to C$ is the composition of f with g. Furthermore, if it will not lead to confusion, we sometimes omit the composition symbol $\circ$ writing $gf\colon A \to C$ for the composition of f with g. In case of (possibly weak) 2-categories, the other composition (i.e. the internal composition) is denoted by the solid disc $\bullet$. In expressions involving both compositions, it is assumed that the usual categorical composition $\circ$ has higher priority than the internal composition $\bullet$. Therefore expressions like:

$$g \circ \beta \bullet \alpha \circ f$$

are always parsed as:

$$(g \circ \beta) \bullet (\alpha \circ f)$$

We shall use the term "weak 2-category" to mean a 2-category-like structure, where the composition "$\circ$" does not have to satisfy associativity and unity rules on-the-nose, but only up to canonical 2-morphisms [Lei04]. Throughout the dissertation we shall only use general arguments about weak 2-categories that are true in all reasonable models of weak 2-categories. If one is not comfortable with such a notion, then one may read "weak 2-category" as "bicategory", and "weak 2-functor" as homomorphism of bicategories in the sense of Jean Benabou [Bén67].

If not stated otherwise, all diagrams are commutative.

Chapter 1

Internal calculi

The well-known Lambek-Curry-Howard isomorphism [LS86] in its simplest form establishes a link between cartesian closed categories, simply typed lambda calculi and propositional intuitionistic logics:

Category	λ-calculus	Logic
1	$\{\bullet\}$	$\top$
$A \times B$	$A \times B$	$A \wedge B$
B^A	$A \to B$	$A \Rightarrow B$
0	$\emptyset$	$\bot$
$A \sqcup B$	$A \sqcup B$	$A \vee B$

To a *two*-category theorist, a category is just an object in a very well-behaved 2-category **Cat** of (locally small) categories. A natural question then is to ask what properties a 2-category has to posses to allow establishing the above connection inside the 2-category; and more importantly — what can be gained by such considerations?

An open and still very active area of research in category theory is to give a reasonable characterisation of a 2-category that allows describing categorical constructions inside the 2-category. Some constructions like adjunctions, Kan extensions/liftings and fibrations/opfibrations [Joh93] are easily definable in any 2-category. Others like pointwise Kan extensions/liftings require existence of particular finite limits. Some others like internal limits/colimits are much harder and require additional conditions or structures on the 2-category [Woo82] [Woo85] [SW78] [Web07]. In the next chapter we propose our approach to *internal* 2-categorical constructions through the concept of a

Yoneda (bi)triangle. In this chapter we shall investigate internal 2-categorical constructions through *discreteness*. The following definitions are standard.

Definition 1.1 (Adjunction). *A morphism $f\colon A \to B$ is left adjoint to a morphism $g\colon B \to A$ (equivalently, g is right adjoint to f) in a 2-category $\mathbb{W}$, if there exists a 2-morphism $\eta\colon id_A \to g \circ f$, called the unit of the adjunction, and a 2-morphism $\epsilon\colon f \circ g \to id_B$, called the counit of the adjunction that satisfy the triangle equalities:*

$$\begin{aligned}(\eta \circ id_g) \bullet (id_g \circ \epsilon) &= id_g \\ (id_f \circ \eta) \bullet (\epsilon \circ id_f) &= id_f\end{aligned}$$

In such a case we write $f \dashv g$.

Example 1.2 (Adjunction between categories). *A functor $F\colon \mathbb{A} \to \mathbb{B}$ is left adjoint to a functor $G\colon \mathbb{B} \to \mathbb{A}$ in the 2-category* **Cat** *of locally small categories, functors and natural transformations, iff F is left adjoint to G in the usual sense — iff there are bijections natural in $A \in \mathbb{A}$ and $B \in \mathbb{B}$:*

$$\hom_{\mathbb{B}}(F(A), B) \approx \hom_{\mathbb{A}}(A, G(B))$$

Example 1.3 (Adjunction between 2-categories). *A 2-functor $F\colon \mathbb{A} \to \mathbb{B}$ is (strictly) left adjoint to a 2-functor $G\colon \mathbb{B} \to \mathbb{A}$ in the 2-category* **2Cat** *of locally small 2-categories, 2-functors and 2-natural transformations, iff there are natural in $A \in \mathbb{A}$ and $B \in \mathbb{B}$ isomorphisms of categories:*

$$\hom_{\mathbb{B}}(F(A), B) \approx \hom_{\mathbb{A}}(A, G(B))$$

There is an obvious underlying functor U from 2-category of (locally small) 2-categories **2Cat** to 2-category of (locally small) categories **Cat**, which forgets 2-morphisms. If $\mathbb{W}, \mathbb{V}$ are 2-categories, then we call a functor $F\colon U(\mathbb{W}) \to U(\mathbb{V})$ between their underlying categories, a 1-functor from $\mathbb{W}$ to $\mathbb{V}$. Likewise, an adjunction between underlying categories is called a 1-adjunction.

Definition 1.4 (Discreteness). *Let $\mathbb{W} \underset{F}{\overset{U}{\rightleftarrows}} \mathbb{C}$ be an adjunction between categories $\mathbb{C}$ and $\mathbb{W}$ with F left adjoint to U. This adjunction gives a notion of discreteness on category $\mathbb{W}$ if the unit of the adjunction is an isomorphism.*

Because the unit of an adjunction $F \dashv U$ is an isomorphism if and only if the left adjoint F is fully faithful, we may identify $\mathbb{C}$ with the full image of F and write $Disc_F(\mathbb{W})$ for it, dropping the subscript if F is known from the context. The right adjoint to the inclusion will be usually denoted by $|{-}|$, so that for an object $A \in \mathbb{C}$ we have $U(F(A)) = |A|$, and the coreflection $|A| \to A$ (the counit of the adjunction) will be denoted by ϵ. Examples of discreteness abound: Example 1.9, Example 1.5, Example 1.6.

Example 1.5 (Discrete graph)**.** *Let* **Graph** *be the category of undirected graphs and graph homomorphisms. Its full subcategory* $Disc(\mathbf{Graph})$ *consisting of graphs without edges gives a notion of discreteness on* **Graph***, with a discretisation functor* $|{-}|\colon \mathbf{Graph} \to Disc(\mathbf{Graph})$ *discarding all edges from a graph. Clearly, there is a natural isomorphism* $\hom(D, G) \approx \hom(D, |G|)$*, where* D *is a discrete graph.*

Example 1.6 (Discrete topological space)**.** *Let* **Top** *be the category of topological spaces and continuous functions. Its full subcategory* $Disc(\mathbf{Top})$ *consisting of topological spaces for which every set is open, gives a notion of discreteness on* **Top***, with a discretisation functor* $|{-}|\colon \mathbf{Top} \to Disc(\mathbf{Top})$ *"upgrading" a topology on a space to the finest topology (i.e. every set is open) on the space — every function from a discrete space* D *to any space* W *is automatically continuous, since inverse image of any set is open in* D*; therefore, we have a natural isomorphism* $\hom(D, W) \approx \hom(D, |W|)$*.*

Definition 1.4 is rather general and we may find many unintended examples of discreteness. One such example is presented in Section 1.3 as Example 1.67. Here is another one.

Example 1.7 (Unintended discreteness of monoids)**.** *Let* **Mon** *be the category of monoids and monoid homomorphisms, and* **Grp** *be its full subcategory of groups. This subcategory gives an "unintended notion of discreteness" on* **Mon***. The right adjoint* $U\colon \mathbf{Mon} \to \mathbf{Grp}$ *to the inclusion maps monoids to their groups of units. Indeed, a homomorphism* $h\colon G \to M$ *from a group* G *to a monoid* M*, has to assign to each element* $x \in G$ *a unital element* $h(x) \in M$ *— this is because by the property of a homomorphism, we have a chain of equalities:*

$$\epsilon_M = h(\epsilon_G) = h(x \bullet_G x^{-1}) = h(x) \bullet_M h(x^{-1})$$

and thus $h(x)$ *is unital. Therefore, homomorphisms* $h\colon G \to M$ *are tantamount to homomorphisms* $h\colon G \to U(M)$*, where* $U(M)$ *is the group of units of monoid* M*.*

In fact, if $\mathbb{C}$ gives a notion of discreteness on $\mathbb{W}$, then any coreflective subcategory of $\mathbb{C}$ does.

A special care has to be taken if $\mathbb{W}$ is a 2-category and $\mathbb{C}$ is a 1-category — here a notion of discreteness is induced by a 1-adjunction $F \dashv U$ between underlying 1-categories with F a 2-fully faithful functor; that is, there can be no non-trivial 2-morphisms in the full subcategory on the image of F.

Definition 1.8 (Discreteness of a 2-category)**.** *Let* $\mathbb{W} \underset{F}{\overset{U}{\rightleftarrows}} \mathbb{C}$ *be a 1-adjunction between a 1-category* $\mathbb{C}$ *and a 2-category* $\mathbb{W}$*, where* F *is a 2-fully faithful functor which is left 1-adjoint to a 1-functor* U*. This adjunction gives a notion of discreteness on category* $\mathbb{W}$ *if the unit of the adjunction is an isomorphism.*

Example 1.9 (Discrete category)**.** *Let* **cat** *be the 2-category of small categories, functors and natural transformations. The category* **Set** *of sets and functions is its full subcategory inducing a notion of discreteness on* **cat***. The discretisation functor* $|-|\colon \mathbf{cat} \to \mathbf{Set}$ *sends a category to its underlying set of objects. The natural isomorphism:*

$$\hom_{\mathbf{cat}}(X, \mathbb{C}) \approx \hom_{\mathbf{Set}}(X, |\mathbb{C}|)$$

follows directly from the definition of a functor. The situation generalises to any 2-category $\mathbf{cat}(\mathbb{C})$ *of categories internal to a finitely complete category* $\mathbb{C}$*. Moreover, this situation also generalizes to any 2-category* $\mathbf{Cat}(V)$ *of categories enriched in a cartesian closed category* $\mathbb{V}$ *with an initial object.*

Although **cat** is a 2-category, we could not demand the inclusion $\mathbf{Set} \to \mathbf{cat}$ to have right 2-adjoint — clearly because there are no non-trivial 2-morphisms in a 1-category. In this example we could also characterise discrete categories X as precisely these categories that satisfy the property: for every category $\mathbb{C}$ and every parallel functors $F, G\colon \mathbb{C} \to X$ there are no non-trivial (i.e. other than identities) natural transformation $F \to G$. This suggests a very important generic notion of discreteness, which we shall call "the canonical notion of discreteness".

Definition 1.10 (Canonical discreteness)**.** *Let* $\mathbb{W}$ *be a 2-category. Let us write Disc*$(\mathbb{W})$ *for the full subcategory of* $\mathbb{W}$ *consisting of these objects* X*, for which the category* $\hom(C, X)$ *is discrete in the sense of Example 1.9 for every object* $C \in \mathbb{W}$*. We shall say that* $\mathbb{W}$ *has the canonical notion of discreteness if the inclusion Disc*$(\mathbb{W}) \to \mathbb{W}$ *has right 1-adjoint.*

Not every 2-category has the canonical notion of discreteness: consider the full 2-subcategory of **cat** consisting of all small categories excluding *infinite* discrete categories. Clearly, the inclusion from the category $\mathbf{Set}_{\aleph_0}$ of finite sets and functions does not have a right adjoint.

Note 1.11. *One may notice that the concept of discreteness as introduced in Definition 1.8 is not 2-categorical. A possible solution to this "issue" is to replace the concept of "discreteness" with the concept of "discreteness up to equivalence" (a category which is equivalent to a discrete one is called "elementary"; therefore we could call the concept of "discreteness up to equivalence" — "elementariness"). Nonetheless, for the purpose of this dissertation it is sufficient and far easier to work with the strict notion of discreteness. For the same reason, whenever it does not exclude interesting examples, we prefer to define our concepts for strict 2-categories (as opposed to weak 2-categories). In a few examples, this leads to a minor inconvenience: sometimes we have to replace a (weak) 2-category with a (weakly) 2-equivalent (strict) 2-category with "good" notion of strict discreteness (for example: we prefer to work with strict indexed categories instead of fibrations).*

Throughout this chapter the concept of discreteness serves threefold purpose: in the next section it allows us to capture a good notion of internal cartesian closedness and a good notion of internal products, whereas in the third section it allows us to introduce the concept of an associated category.

1.1 Internal lambda calculi

Let us recall that in any cartesian category $\mathbb{W}$ (i.e. category with finite products) every object $A \in \mathbb{W}$ carries a unique comonoid structure $1 \xleftarrow{!} A \xrightarrow{\Delta} A \times A$, where $\Delta = \langle id, id \rangle$ is the diagonal morphism. In case $\mathbb{W} = \mathbf{Cat}$, we obtain the usual notion of terminal (initial) object and binary products (coproducts) in $\mathbb{A} \in \mathbf{Cat}$ by taking right (resp. left) adjoint to the comonoid structure on $\mathbb{A}$. It seems reasonable then, to internalise the notion of cartesian structure inside any cartesian 2-category $\mathbb{W}$ in the following way.

Definition 1.12 (Internally (co)cartesian connectives)**.** *Let us assume that a 2-category $\mathbb{W}$ has finite products. An object $A \in \mathbb{W}$ has an* internal *terminal value $\{\bullet\}_A$ (initial value $\{\}_A$) if the unique morphism $A \xrightarrow{!} 1$ has right adjoint $1 \xrightarrow{\{\bullet\}_A} A$ (resp. left adjoint $1 \xrightarrow{\{\}_A} A$), and it has* internal *products*

$\times_A$ *(coproducts* $\sqcup_A$*) if the diagonal* $A \stackrel{\Delta_A}{\to} A \times A$ *has right adjoint* $A \times A \stackrel{\times_A}{\to} A$ *(resp. left adjoint* $A \times A \stackrel{\sqcup_A}{\to} A$ *).*

Example 1.13 (Internal connectives in fibration)**.** *Let* $Fib(\mathbb{B})$ *be the 2-category of fibrations, fibred functors and fibred natural transformation over category* $\mathbb{B}$ *(see Definition A.20 from the Appendix A.2). A fibration* $p\colon \mathbb{E} \to \mathbb{B}$ *has internal terminal object as an object of* $Fib(\mathbb{B})$ *in the sense of Definition 1.12 iff each of its fibres has a terminal object that is preserved by reindexing morphisms. Similarly,* $p\colon \mathbb{E} \to \mathbb{B}$ *has internal binary products/initial object/binary coproducts iff each of its fibre has binary products/initial object/binary coproducts stable under reindexing morphisms.*

Yoneda lemma for 2-categories[1] implies that for any (locally small) 2-category $\mathbb{W}$ the assignment:

$$A \in \mathbb{W} \mapsto \hom_{\mathbb{W}}(-, A) \in \mathbf{Cat}^{\mathbb{W}^{op}}$$

extends to a fully faithful 2-embedding:

$$y\colon \mathbb{W} \to \mathbf{Cat}^{\mathbb{W}^{op}}$$

called "2-Yoneda functor". Therefore, a morphism f is adjoint to g in $\mathbb{W}$ iff the transformation $\hom(-, f)$ is adjoint to the transformation $\hom(-, g)$ in $\mathbf{Cat}^{\mathbb{W}^{op}}$. Because 2-Yoneda functor also preserves finite products, it is possible to coherently give an external characterisation of *internal* connectives in $\mathbb{W}$, even in case $\mathbb{W}$ does not have all finite products. Generally, we shall say that an object $A \in \mathbb{W}$ has a *virtual* property, if its representable 2-functor $\hom(-, A)\colon \mathbb{W}^{op} \to \mathbf{Cat}$ has that property as an object in $\mathbf{Cat}^{\mathbb{W}^{op}}$. Thus, an object $A \in \mathbb{W}$ has a virtual internal terminal value (initial value, products, coproducts) if $\hom(-, A)\colon \mathbb{W}^{op} \to \mathbf{Cat}$ has internal terminal value (resp. false value, products, coproducts) as an object in $\mathbf{Cat}^{\mathbb{W}^{op}}$. The essence of virtual values is that although sometimes we may not have an access to the defining morphisms, there is always a natural assignment of parametrised values via universal properties. Recalling from [BW85] (Chapter I, Sections 4 and 5) the notion of generalised elements, let us write $\tau_X, \sigma_X \in A$ for morphisms $X \longrightarrow A$, and then, given $s\colon A \to B$, $s(\tau_X) \in B$ for $s \circ \tau_X$. If an object $A \in \mathbb{W}$ has an internal virtual terminal value, then for every object $X \in \mathbb{W}$ there is a natural way to form a constant element

[1] Yoneda lemma for **Cat**-enriched categories, see Appendix A.3.

$1_X \in A$ sending everything from X to the virtual terminal value of A — it is given by the functor $(\{\bullet\}_{\hom(-,A)})_X \colon 1 \to \hom(X, A)$ applied to the single object of the terminal category 1. Similarly, given two generalised elements $\tau_X, \sigma_X \in A$ there is a canonical generalised element $\tau_X \times \sigma_X \in A$, provided A has virtual internal products — it is given by the product functor $(\times_{\hom(-,A)})_X \colon \hom(X, A) \times \hom(X, A) \to \hom(X, A)$ applied to τ_X and σ_X.

The definition of *internal* cartesian closedness is less obvious. One may pursue an approach of Mark Weber [Web07] (Definition 8.1) and say that an object A of a 2-category with finite products is internally cartesian closed if for every global element $1 \xrightarrow{x} A$ the morphism $A \xrightarrow{id_A \times x} A \times A \xrightarrow{\times_A} A$ has a right adjoint. Unfortunately, this definition is inadequate in various contexts — including fibred and internal categories. For this reason we shall call the above concept "naive cartesian closedness". Let us first recall that the concept of being cartesian closed is not stable under exponentiation.

Example 1.14 (Stability of cartesian closedness under exponentiation)**.** *The concept of cartesian closedness is not stable under exponentiation (this is Exercise 5 in Section 6 Chapter 4 of [ML78]). If $\mathbb{A}$ is a cartesian closed category and $\mathbb{X}$ a category then, by Yoneda-like argument, $\mathbb{A}^{\mathbb{X}}$ is cartesian closed iff the canonical limits defining exponents exist. In particular, if a cartesian closed category $\mathbb{A}$ is complete and $\mathbb{X}$ is small then $\mathbb{A}^{\mathbb{X}}$ is cartesian closed.*

Therefore, to show an example of a cartesian closed category $\mathbb{A}$ and a category $\mathbb{X}$ such that $\mathbb{A}^{\mathbb{X}}$ is not cartesian closed, one has to choose for $\mathbb{A}$ a category that misses some classes of limits. Let **FinSet** *be the category of finite sets and functions. By Yoneda lemma* **FinSet** *is a free cocompletion of the terminal category* 1 *under finite colimits. Thus, one should expect that* **FinSet** *misses many infinite limits, and for some infinite categories $\mathbb{X}$ the functor category $\mathbf{FinSet}^{\mathbb{X}}$ will not be cartesian closed. Let $\mathbb{X} = P(\mathcal{N})$ be the category of subsets of natural numbers ordered by inclusion. If $F, G \colon P(\mathcal{N}) \to \mathbf{FinSet}$ are two functors, then by Yoneda-like argument, their exponent G^F, if exists, can be evaluated at the empty set as: $G^F(\{\}) = \hom_{\mathbf{FinSet}^{\mathbb{X}}}(F, G)$. Consider $F = G$ that maps the empty set to the empty set, singletons to the truth-set $\Omega = \{0, 1\}$, and the other sets to the singleton. Then $|\hom_{\mathbf{FinSet}^{\mathbb{X}}}(F, F)| = \Omega^{\Omega \times \aleph_0} = \mathfrak{c}$ is not finite, hence the exponent F^F does not exist.*

One may obtain a similar example by considering the category $\mathbf{FinSet}^{\mathcal{N}_\perp}$, where $\mathcal{N}_\perp$ is the poset $\mathcal{N} \cup \{\perp\}$, with $\perp < n$ for all $n \in \mathcal{N}$ and no other *(non-trivial) relations*[2].

Example 1.15 (Failure of naive cartesian closedness). *A split indexed category is cartesian closed iff its every fibre is cartesian closed and reindexing morphisms preserve cartesian closed structure. Let $\mathbb{A}$ be a cartesian closed category for which there exists a category $\mathbb{X}$ such that $\mathbb{A}^{\mathbb{X}}$ is not cartesian closed (Example 1.14). The 2-Yoneda functor gives an indexed category:*

$$\hom(-, \mathbb{A}) \colon \mathbf{Cat}^{op} \to \mathbf{Cat}$$

which (again by Yoneda lemma as explained in the above) is naively cartesian closed as an object in $\mathbf{Cat}^{\mathbf{Cat}^{op}}$. However, it is not a cartesian closed indexed category — the fibre $\hom(\mathbb{X}, \mathbb{A}) = \mathbb{A}^{\mathbb{X}}$ over $\mathbb{X}$ is not cartesian closed. The problem with the naive definition is that choosing an element $x_1 \colon 1 \to \hom(1, A)$ by naturality of x, chooses constant morphisms in every fibre. Therefore, naive cartesian closedness expresses existence of exponents of "constant objects".

We shall generalise the idea of cartesian closedness provided by Bart Jacobs[3] in Definition 3.9 in [Jac93] for fibrations and adopt it to arbitrary cartesian 2-categories with a notion of discreteness[4].

Definition 1.16 (Internally cartesian closed connectives). *Let $\mathbb{W}$ be a cartesian 2-category with a notion of discreteness. An object $A \in \mathbb{W}$ is* internally *cartesian closed if it has internal products and the morphism:*

$$A \times |A| \xrightarrow{\langle \times_A \circ (id \times \epsilon_A), \pi_{|A|} \rangle} A \times |A|$$

has a right adjoint, where ϵ_A is the counit of the adjunction that gives the notion of discreteness on $\mathbb{W}$.

The idea is that while the definition proposed by Mark Weber constructs internal exponents at "each internal object" separately, our definition constructs internal exponents at "all internal objects" simultaneously. Putting

[2] This has been suggested by the reviewer of the dissertation J. van Oosten.

[3] We would get a proper generalisation if we substituted the notion of discreteness with the notion of "grupoidalness". Nonetheless, for the purpose of this dissertation it suffices to work with much simpler, yet not 2-categorical, concept of discreteness.

[4] There is also a general notion of an internally closed object within $\star$-autonomous 2-categories (Definition 10 in [DS97]), however it cannot be generalised to our setting because cartesian 2-$\star$-autonomous categories are necessarily posetal.

it another way — constructively, there is a difference between "each" (separately) and "all" (simultaneously) and the idea of being "internally closed" is "having all internal exponents".

Example 1.17 (Internally cartesian closed indexed category)**.** *A split indexed category* $\Phi\colon \mathbb{C}^{op} \to \mathbf{Cat}$ *is discrete in the sense of Definition 1.10 iff it is discrete in the usual sense — i.e. each of its fibres is a discrete category. Therefore* Φ *is a cartesian closed indexed category iff it is internally cartesian closed in the sense of Definition 1.16.*

We would like to extend the calculus of parametrised elements to internally closed connectives, but Example 1.14 shows that it is impossible in the full generality — if $\mathbb{A}, \mathbb{X} \in \mathbf{Cat}$ are such that $\mathbb{A}$ is cartesian closed and $\mathbb{A}^{\mathbb{X}}$ is *not* cartesian closed, then there is no way to form an exponent $\tau_{\mathbb{X}}^{\sigma_{\mathbb{X}}}$ for every pair of parametrised elements $\tau_{\mathbb{X}}, \sigma_{\mathbb{X}} \in \mathbb{A}$. However, this is possible if $\mathbb{X}$ is discrete. We shall postpone the proof of the following theorem until Section 1.3 (Theorem 1.70).

Theorem 1.18 (Parametrised simply typed lambda calculus)**.** *Let* $\mathbb{W}$ *be a cartesian 2-category with a notion of discreteness, and assume that an object* $A \in \mathbb{W}$ *is internally cartesian closed. Then for every discrete object* $X \in \mathit{Disc}(\mathbb{W})$ *the category* $\hom_{\mathbb{W}}(X, A)$ *is cartesian closed. Moreover, if* A *is internally cocartesian (i.e has an internal initial value and internal binary coproducts), then* $\hom_{\mathbb{W}}(X, A)$ *is cocartesian.*

Therefore, an internally cartesian closed and cocartesian object $A \in \mathbb{W}$ for every discrete object $X \in \mathit{Disc}(\mathbb{W})$ gives an interpretation of the following system of rules:

$$\frac{}{\tau_X \xrightarrow{id_{\tau_X}} \tau_X}\ (\mathrm{id}) \qquad \frac{\tau_X \xrightarrow{f} \sigma_X \quad \sigma_X \xrightarrow{g} \rho_X}{\tau_X \xrightarrow{g \circ f} \rho_X}\ (\mathrm{com})$$

$$\frac{}{\tau_X \xrightarrow{!} 1}\ (1\text{-int}) \qquad \frac{}{0 \xrightarrow{*} \tau_X}\ (0\text{-int})$$

$$\frac{\rho_X \xrightarrow{f} \tau_X \quad \rho_X \xrightarrow{g} \sigma_X}{\rho_X \xrightarrow{\langle f, g\rangle} \tau_X \times \sigma_X}\ (\times\text{-int}) \qquad \frac{\rho_X \xrightarrow{f} \tau_X \times \sigma_X}{\rho_X \xrightarrow{\pi_{\tau_X} \circ f} \tau_X \quad \rho_X \xrightarrow{\pi_{\sigma_X} \circ f} \sigma_X}\ (\times\text{-eli})$$

$$\frac{\tau_X \xrightarrow{f} \rho_X \quad \sigma_X \xrightarrow{g} \rho_X}{\tau_X \sqcup \sigma_X \xrightarrow{[f, g]} \rho_X}\ (\sqcup\text{-int}) \qquad \frac{\tau_X \sqcup \sigma_X \xrightarrow{f} \rho_X}{\tau_X \xrightarrow{f \circ \iota_{\tau_X}} \rho_X \quad \sigma_X \xrightarrow{f \circ \iota_{\sigma_X}} \rho_X}\ (\sqcup\text{-eli})$$

$$\frac{\tau_X \times \sigma_X \xrightarrow{f} \rho_X}{\tau_X \xrightarrow{\lambda y : \sigma_X . f(-, y)} {\rho_X}^{\sigma_X}}\ (\lambda\text{-int}) \qquad \frac{\tau_X \xrightarrow{f} {\rho_X}^{\sigma_X}}{\tau_X \times \sigma_X \xrightarrow{f(-)(=)} \rho_X}\ (\lambda\text{-eli})$$

which by the Lambek-Curry-Howard isomorphism [LS86] gives rise to a simply typed lambda calculus. Moreover, such systems parametrized by $X \in \mathit{Disc}(\mathbb{W})$ give interpretation of polymorphism (Example 1.33).

More generally, given any morphism $r \colon A \times A \to A$, we shall say that object A is *internally* left (resp. right) r-closed if the morphism: $A \times |A| \xrightarrow{\langle r \circ (id \times \epsilon_A), \pi_{|A|}\rangle} A \times |A|$ (resp. $|A| \times A \xrightarrow{\langle r \circ (\epsilon_A \times id), \pi_{|A|}\rangle} A \times |A|$) has a right adjoint. Following the terminology of Bourbaki we shall call an object A together with a morphism $r \colon A \times A \to A$ a "magma", and internally r-left and r-right closed object a "(bi)closed magma".

Example 1.19 (Monoidal closed structure)**.** *A monoidal structure $\langle I, \otimes\rangle$ on a category $\mathbb{C}$ is left (resp. right) closed in the usual sense if it is internally left (resp. right) $\otimes$-closed.*

Example 1.20 (Lambek category)**.** *Let us recall that a Lambek category is a category $\mathbb{C}$ together with a functor $R \colon \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ such that for every object $A \in \mathbb{C}$ both $R(A, -)$ and $R(-, A)$ have right adjoints. A Lambek category is precisely a category which is an internally left and right R-closed magma.*

We can go a bit further and define r-closedness in a general monoidal 2-category.

Definition 1.21 (Internally closed connectives within a monoidal 2-category)**.** *Let $\mathbb{W}$ be a monoidal 2-category*[5]*, $\mathbb{C}$ a cartesian 1-category and $F\colon \mathbb{C} \to \mathbb{W}$ an op-lax monoidal embedding (see [GPS95]) making $\mathbb{C}$ a 1-coreflective subcategory of $\mathbb{W}$ by a 1-functor $|-|\colon \mathbb{W} \to \mathbb{C}$.*

An object $A \in \mathbb{W}$ together with a morphism $r\colon A \otimes A \to A$ is internally *left r-closed if:*

$$A \otimes F(|A|) \xrightarrow{id \otimes \theta \circ F(\Delta_{|A|})} A \otimes F(|A|) \otimes F(|A|) \xrightarrow{id \otimes \epsilon \otimes id} A \otimes A \otimes F(|A|) \xrightarrow{r \otimes id} A \otimes F(|A|)$$

has a right adjoint, where $\epsilon\colon F(|A|) \to A$ is the counit of adjunction $F \vdash |-|$, $\theta\colon F(|A| \times |A|) \to F(|A|) \otimes F(|A|)$ is the structure morphism from the definition of op-lax monoidal functor, and the natural isomorphisms expressing associativity of the tensor product $\otimes$ have been omitted for clarity. Similarly, object A is right r-closed if the morphism:

$$F(|A|) \otimes A \xrightarrow{\theta \circ F(\Delta_{|A|}) \otimes id} F(|A|) \otimes F(|A|) \otimes A \xrightarrow{id \otimes \epsilon \otimes id} F(|A|) \otimes A \otimes A \xrightarrow{id \otimes r} F(|A|) \otimes A$$

has right adjoint.

Example 1.22 (Internal closedness of enriched categories)**.** *Let $\langle \mathbb{C}, I, \otimes \rangle$ be a symmetric monoidal closed category. There is an underlying monoidal 2-functor $U\colon \mathbf{Cat}(\mathbb{C}) \to \mathbf{Cat}$ that assigns to a $\mathbb{C}$-enriched category $\mathbb{X}$ an ordinary category $\hom_{\mathbf{Cat}(\mathbb{C})}(\mathbb{I}, \mathbb{X})$, where $\mathbb{I}$ is the unit $\mathbb{C}$-enriched category. If $\mathbb{C}$ has small coproducts, then the underlying functor has a monoidal left adjoint $L\colon \mathbf{Cat} \to \mathbf{Cat}(\mathbb{C})$, which takes an ordinary category $\mathbb{A}$ and yields a $\mathbb{C}$-enriched category consisting of the same objects as $\mathbb{C}$ and morphisms from an object A to an object B defined in the following way:*

$$\hom_{L(\mathbb{A})}(A, B) = \coprod_{\hom_{\mathbb{A}}(A,B)} I$$

We shall write $|-|\colon \mathbf{Cat}(\mathbb{C}) \to \mathbf{Set}$ for the composition of U with the usual discretisation functor $\mathbf{Cat} \to \mathbf{Set}$ for ordinary categories, and we shall write $F\colon \mathbf{Set} \to \mathbf{Cat}(\mathbb{C})$ for its left adjoint — i.e. the composition of $L\colon \mathbf{Cat} \to \mathbf{Cat}(\mathbb{C})$ with the embedding of sets into ordinary categories. Observe, that to define the above adjunction $F \dashv |-|$ we do not need to assume the existence of

[5]For simplicity we shall assume that the 2-category is strict. Note 1.11 points how to generalise such notions to weak 2-categories.

all small coproducts, but only one coproduct: the nullary coproduct (i.e. the initial object 0*).*

We say that a $\mathbb{C}$*-enriched category* $\mathbb{A}$ *is enriched-discrete iff:*

$$\mathrm{hom}_{\mathbb{A}}(X,Y) \approx \begin{cases} 0 & \text{for } X \neq Y \\ I & \text{for } X = Y \end{cases}$$

with trivial units and compositions. Observe that enriched-discreteness does not imply discreteness of the underlying category (take for $\mathbb{V}$ *the category of abelian groups* **Ab** *or the category* **Vect** *of vector spaces with the usual tensor product). It implies, however, if the enrichment is defined in a cartesian closed structure. On the other hand, enriched-discreteness behaves similarly to the usual discreteness when it comes to parametrisation — if* $S, T \colon \mathbb{X} \to \mathbb{A}$ *are two parallel enriched functors from an enriched-discrete category* $\mathbb{X}$*, then every transformation* $\tau \colon S \to T$ *is automatically a natural transformation. To see this, consider the diagram of enriched naturality [Kel82]:*

$$\begin{tikzcd}
& I \otimes \mathrm{hom}_{\mathbb{X}}(X,Y) \arrow[r, "\tau_X \otimes S"] & \mathrm{hom}_{\mathbb{A}}(S(Y),T(Y)) \otimes \mathrm{hom}_{\mathbb{A}}(S(X),S(Y)) \arrow[dr, "\mu"] & \\
\mathrm{hom}_{\mathbb{X}}(X,Y) \arrow[ur, "l"] \arrow[dr, "r"'] & & & \mathrm{hom}_{\mathbb{X}}(S(X),T(Y)) \\
& \mathrm{hom}_{\mathbb{X}}(X,Y) \otimes I \arrow[r, "T \otimes \tau_Y"] & \mathrm{hom}_{\mathbb{X}}(T(X),T(Y)) \otimes \mathrm{hom}_{\mathbb{A}}(S(X),T(X)) \arrow[ur, "\mu"'] &
\end{tikzcd}$$

where l, r *are coherence morphisms from the definition of a monoidal category. By definition of enriched-discreteness, if* $X \neq Y$ *then* $\mathrm{hom}_{\mathbb{X}}(X,Y)$ *is initial, thus equalises every pair of morphisms; and if* $X = Y$ *then* $\mathrm{hom}_{\mathbb{X}}(X,Y)$ *is the unit of the tensor, thus the above diagram reduces to the coherence laws from a monoidal category* $\mathbb{V}$ *and of enriched functors* S, T*. An analogical argument shows that if tensor* $\otimes$ *preserves the initial object* 0 *then for every enriched functors* $S, T \colon \mathbb{K} \otimes \mathbb{X} \to \mathbb{A}$ *every transformation* $S \to T$ *that is natural in* $\mathbb{K}$ *is automatically natural in* $\mathbb{K} \otimes \mathbb{X}$*, provided* $\mathbb{X}$ *is discrete.*

Let us now consider internal closedness of a $\mathbb{C}$*-enriched category* $\mathbb{A}$ *together with a* $\mathbb{C}$*-enriched functor* $R \colon \mathbb{A} \otimes \mathbb{A} \to \mathbb{A}$*. The functor* $A \otimes F(|A|) \to A \otimes F(|A|)$ *from the definition of left* R*-closedness is given by:*

$$\langle A, X \rangle \mapsto \langle A, X, X \rangle \mapsto \langle R(A,X), X \rangle$$

By the definition of the tensor product for enriched categories:

$$\mathrm{hom}_{\mathbb{A} \otimes F(|\mathbb{A}|)}(\langle R(A,X), X \rangle, \langle B, Y \rangle) = \mathrm{hom}_{\mathbb{A}}(R(A,X), B) \otimes \mathrm{hom}_{F(|\mathbb{A}|)}(X,Y)$$

Let us assume that for every $X \in F(|\mathbb{A}|)$ the functor $R(-,X)$ has right adjoint $X \multimap (-)$. We claim that $\langle B, Y\rangle \mapsto \langle Y \multimap B, Y\rangle$ is right adjoint to $\langle A, X\rangle \mapsto \langle R(A,X), X\rangle$. The proof is as follows. If we use the definition of the tensor product of categories:

$$\hom_{\mathbb{A}\otimes F(|\mathbb{A}|)}(\langle A, X\rangle, \langle Y \multimap B, Y\rangle) = \hom_{\mathbb{A}}(A, Y \multimap B) \otimes \hom_{F(|\mathbb{A}|)}(X, Y)$$

then it remains to show:

$$\hom_{\mathbb{A}}(R(A,X), B) \otimes \hom_{F(|\mathbb{A}|)}(X,Y) \approx \hom_{\mathbb{A}}(A, Y \multimap B) \otimes \hom_{F(|\mathbb{A}|)}(X,Y)$$

Because $F(|\mathbb{A}|)$ is enriched-discrete we can argue by cases. If $X \neq Y$, then by discreteness $\hom(X,Y) = 0$*, and by preservation of initial object by the tensor:*

$$\hom_{\mathbb{A}}(R(A,X), B) \otimes 0 \approx 0 \approx \hom_{\mathbb{A}}(A, Y \multimap B) \otimes 0$$

On the other hand, for $X = Y$ the situation reduces to the adjunction between $R(-,X)$ and $X \multimap (-)$. Hence, if $\mathbb{A}$ is left R-closed in the usual sense, it is left R-closed in the sense of Definition 1.21. To see that the converse is true as well, it suffices put $Y = X$ in the above formula. A symmetric argument shows that $\mathbb{A}$ is right R-closed iff for every $X \in F(|\mathbb{A}|)$ the functor $R(X,-)$ has right adjoint.

In case $\mathbb{W}$ is a cartesian 2-category and $r\colon A \times A \to A$ is the diagonal morphism, by universal properties of products, Definition 1.16 coincides with Definition 1.21.

Example 1.23 (Topological spaces)**.** *Although category of topological spaces is not cartesian closed, very many interesting topological spaces are exponentiable. In fact for a topological space A there exists right adjoint to $- \times A\colon$ **Top** $\to$ **Top** if and only if A is a core-compact space [Isb86], which means that the underlying locale of its open sets is continuous. One then may think that a restriction to the subcategory of topological spaces consisting of core-compact spaces could work. However, this again is not the case, because an exponent of two core-compact spaces need not be core-compact*[6]*. This example shows that sometimes we need even more general notion of internal closedness of one object with respect to another object. Formally, we shall say that given any morphisms $j\colon B \to A$ and $r\colon A \times A \to A$, an object*

[6] An example of a subcategory of topological spaces that is cartesian closed is the category of compactly generated topological spaces [Ste67] [ELS04].

A is internally left (resp. right) r-closed with respect to "the inclusion" j if: $A \times |B| \xrightarrow{\langle r \circ (id \times j \circ \epsilon_B), \pi_{|B|} \rangle} A \times |B|$ *(resp.* $A \times |B| \xrightarrow{\langle r \circ (j \circ \epsilon_B \times id), \pi_{|B|} \rangle} A \times |B|$*) has a right adjoint. According to this definition* **Top** *is cartesian closed with respect to the subcategory of core-compact spaces.*

The next example shows how the notion of internal closedness can vary with the change of the notion of discreteness.

Example 1.24 (Cartesian closed functor)**.** *A functor* $F \colon \mathbb{A} \to \mathbb{B}$ *is said to be cartesian closed if categories* $\mathbb{A}$ *and* $\mathbb{B}$ *are cartesian closed and* F *preserves exponents. This terminology can be justified by the observation that functors thought of as* $\mathbf{Set}^{\{0 \leq 1\}}$*-internal categories are internally cartesian closed (as internal categories) if both their domains and codomains are cartesian closed and they preserve the cartesian closed structures.*

In other words, being a cartesian closed functor means being internally cartesian closed in the 2-category $\mathbf{cat}^{\{0 \leq 1\}}$ *with the canonical notion of discreteness.*

On the other hand, category **Set** *provides a different notion of discreteness on* $\mathbf{cat}^{\{0 \leq 1\}}$*. The embedding* $J \colon \mathbf{Set} \to \mathbf{cat}^{\{0 \leq 1\}}$ *sends a set* X *to the identity* id_X *on the discrete category* X*, and the coreflector* $R \colon \mathbf{cat}^{\{0 \leq 1\}} \to \mathbf{Set}$ *assigns to a functor* $F \colon \mathbb{A} \to \mathbb{B}$ *the set of objects* $|\mathbb{A}|$ *of its domain* $\mathbb{A}$ *— i.e. there is a chain of natural equivalences:*

$$\hom_{\mathbf{Set}}(X, |dom(F)|) \approx \hom_{\mathbf{cat}}(X, dom(F)) \approx \hom_{\mathbf{cat}^{\{0 \leq 1\}}}(id_X, F)$$

According to this notion of discreteness, a functor $F \colon \mathbb{A} \to \mathbb{B}$ *is internally cartesian closed iff its domain* $\mathbb{A}$ *is cartesian closed and* F *maps exponents from* $\mathbb{A}$ *to exponents in* $\mathbb{B}$ *(which implies that* $\mathbb{B}$ *is exponentiable with objects of the form* $F(X)$ *for* $X \in \mathbb{A}$*, but it does not imply that* $\mathbb{B}$ *is exponentiable with every object from* $\mathbb{B}$*). Indeed, by Definition 1.16,* $F \colon \mathbb{A} \to \mathbb{B}$ *is internally cartesian closed if the product with the second coordinate:*

$$F \times id_{|dom(F)|} \to F \times id_{|dom(F)|}$$

has right adjoint. This product morphism consists of a pair of product functors like on the diagram:

$$\begin{tikzcd}
\mathbb{A} \times |\mathbb{A}| \arrow[r, "\langle \times_{\mathbb{A}} \circ (id \times \epsilon_{\mathbb{A}}), \pi_{|\mathbb{A}|} \rangle"] \arrow[d, "F \times id_{|\mathbb{A}|}"'] & \mathbb{A} \times |\mathbb{A}| \arrow[d, "F \times id_{|\mathbb{A}|}"] \\
\mathbb{B} \times |\mathbb{A}| \arrow[r, "\langle \times_{\mathbb{B}} \circ (id \times F \circ \epsilon_{\mathbb{A}}), \pi_{|\mathbb{A}|} \rangle"] & \mathbb{B} \times |\mathbb{A}|
\end{tikzcd}$$

and by Beck-Chevalley, the product morphism has right adjoint if both $\mathbb{A} \times |\mathbb{A}| \to \mathbb{A} \times |\mathbb{A}|$ *and* $\mathbb{B} \times |\mathbb{A}| \to \mathbb{B} \times |\mathbb{A}|$ *have right adjoints compatible with* F*.*

One may wonder if Definition 1.16 can have even more exotic examples than Example 1.24. Nonetheless, the main results of Section 1.3 — i.e. Theorem 1.65 and Theorem 1.76 together with Theorem 1.70 — show that under some mild conditions, we can treat objects from an arbitrary 2-category with a notion of discreteness as internal categories and, moreover, in such a way that these objects have internal connectives iff the associated internal categories have corresponding connectives in the usual sense. Therefore, one should be sceptical about existence of such exotic examples.

Before we extend our lambda calculi by a notion of polymorphism, we have to recall the notion of a comma object.

Definition 1.25 (Comma object)**.** *Let* $f\colon A \to C$ *and* $g\colon B \to C$ *be morphisms in a 2-category* $\mathbb{W}$*. A comma object from* f *to* g *is an object* $f{\downarrow}g$ *together with projection morphisms* $\pi_f\colon f{\downarrow}g \to A$*,* $\pi_g\colon f{\downarrow}g \to B$ *and a 2-morphism* $\pi\colon f \circ \pi_f \to g \circ \pi_g$ *satisfying universal properties:*

- *for any object* X *together with morphisms* $a\colon X \to A$*,* $b\colon X \to B$ *and a 2-morphism* $\sigma\colon f \circ a \to g \circ b$ *there is a unique morphism* $h\colon X \to f{\downarrow}g$ *such that:*

$$\begin{aligned} a &= \pi_f \circ h \\ b &= \pi_g \circ h \\ \sigma &= \pi \circ h \end{aligned}$$

diagrammatically:

$$\begin{tikzcd}
X \arrow[dr, dashed, "h"', "\Nearrow\sigma"] \arrow[drr, bend left, "b"] \arrow[ddr, bend right, "a"'] & & \\
& f\downarrow g \arrow[r, "\pi_g"] \arrow[d, "\pi_f"'] \arrow[dr, phantom, "\Nearrow\pi"] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}$$

- *for any object X, any pair of parallel morphisms $h, k\colon X \to f{\downarrow}g$ and 2-morphisms $\alpha\colon \pi_f \circ h \to \pi_f \circ k$, $\beta\colon \pi_g \circ h \to \pi_g \circ k$ satisfying: $g \circ \beta \bullet \pi \circ h = \pi \circ k \bullet f \circ \alpha$, there is a unique 2-morphism $\lambda\colon h \to k$ such that:*

$$\begin{aligned} \alpha &= \pi_f \circ \lambda \\ \beta &= \pi_g \circ \lambda \end{aligned}$$

Example 1.26 (Comma category)**.** *A comma object in 2-category* **Cat** *from a functor $F\colon \mathbb{A} \to \mathbb{C}$ to a functor $G\colon \mathbb{B} \to \mathbb{C}$ is the usual comma category $F{\downarrow}G$ defined as follows:*

- *objects consist of triples $\langle A \in \mathbb{A}, f\colon F(A) \to G(B), B \in \mathbb{B}\rangle$*
- *a morphism from an object $\langle A \in \mathbb{A}, f\colon F(A) \to G(B), B \in \mathbb{B}\rangle$ to an object $\langle A' \in \mathbb{A}, f'\colon F(A') \to G(B'), B' \in \mathbb{B}\rangle$ consists of a pair of morphisms $\langle h\colon A \to A', k\colon B \to B'\rangle$ such that: $f' \circ F(h) = G(k) \circ f$*

with the obvious projections.

In Section 1.3 we elaborate more on the highly related notion of *inserter* — if a 2-category is sufficiently complete, then a comma object $f{\downarrow}g$ can be obtained as the inserter on the product of domains of f and g (Corollary 1.50 Section 1.3). Conversely, an inserter of two parallel morphisms can be obtained as an equaliser of their comma object.

Definition 1.27 (Parametrised (co)products)**.** *Let $\mathbb{W}$ be a 2-category. Consider an object $A \in \mathbb{W}$, and a morphism $s\colon X \to Y \in \mathbb{W}$. A parametrised element $\tau_X \in A$ has a (co)product along s if the right (resp. left) Kan extension $\prod_s \tau_X$ (resp. $\coprod_s \tau_X$) of τ_X along s exists. That is, there is a morphism $\prod_s \tau_X\colon Y \to A$ (resp. $\coprod_s \tau_X$) and natural in $h\colon Y \to A$ bijections* $\hom(h, \prod_s \tau_X) \approx \hom(h \circ s, \tau_X)$ *(resp.* $\hom(\coprod_s \tau_X, h) \approx \hom(\tau_X, h \circ s)$ *).*

Moreover, we call the (co)product stable if the Kan extension is pointwise, meaning that the Kan extension is stable under comma objects. That is, for any diagram with a comma object square:

$$\begin{tikzcd}
i{\downarrow}s \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, "\Nearrow \pi"] & X \arrow[r, "\tau_X"] \arrow[d, "s"] \arrow[dr, phantom, "\Nearrow \epsilon"] & A \\
I \arrow[r, "i"'] & Y \arrow[ur, "\prod_s \tau_X"'] &
\end{tikzcd}$$

the composition:

$$\epsilon \circ \pi_1 \bullet (\prod_s \tau_X) \circ \pi$$

exhibits $(\prod_s \tau_X) \circ i$ *as the right Kan extension of* $\tau_X \circ \pi_1$ *along* π_2*; and dually, for any diagram with a comma object square:*

$$\begin{tikzcd}
s{\downarrow}i \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, "\Swarrow\pi"] & X \arrow[r, "\tau_X"] \arrow[d, "s"] \arrow[dr, phantom, "\Swarrow\eta"] & A \\
I \arrow[r, "i"'] & Y \arrow[ur, "\coprod_s \tau_X"'] &
\end{tikzcd}$$

the composition:

$$(\coprod_s \tau_X) \circ \pi \bullet \eta \circ \pi_1$$

exhibits $(\coprod_s \tau_X) \circ i$ *as the left Kan extension of* $\tau_X \circ \pi_1$ *along* π_2*.*

Example 1.28 (Internal (co)products)**.** *Let* $\mathbb{W}$ *be a finitely complete 2-category with coproducts and* $A \in \mathbb{W}$ *an object with internal (co)products. Then for every object* $X \in \mathbb{W}$ *and every pair of parametrised elements* $\tau_X, \sigma_X \in A$ *the parametrised stable (co)product of cotuple* $[\tau_X, \sigma_X]$ *along the codiagonal* $\nabla \colon X \sqcup X \to X$ *exists*

$$\begin{tikzcd}
X \sqcup X \arrow[r, "{[\tau_X,\sigma_X]}"] \arrow[d, "\nabla"'] \arrow[dr, phantom, "\Nearrow\ (\Swarrow)"] & A \\
X \arrow[ur, "\prod_\nabla[\tau_X,\sigma_X]"', "(\coprod_\nabla[\tau_X,\sigma_X])"' near end] &
\end{tikzcd}$$

and is equal to the internal (co)product $\tau_X \times_A \sigma_X$ *(resp.* $\tau_X \sqcup_A \sigma_X$*). Indeed, by definition of Kan extensions we are looking for adjoint to:*

$$\hom(X, A) \xrightarrow{(-)\circ\nabla} \hom(X \sqcup X, A) \approx \hom(X, A) \times \hom(X, A)$$

However by the universal property of an adjunction this morphism is isomorphic to the diagonal functor:

$$\hom(X, A) \xrightarrow{\Delta} \hom(X, A) \times \hom(X, A)$$

which by the usual 2-Yoneda argument has right (resp. left) adjoint since $A \xrightarrow{\Delta} A \times A$ *does.*

Let us elaborate on the stability condition. Given a diagram like in Definition 1.27, we extend it by taking generalised elements $i_I \in Y, j_I \in X$ together with a generalised arrow $i_I \xrightarrow{k} s(j_I)$, and form a comma object:

$$
\begin{tikzcd}
I \arrow[dr, dashed] \arrow[drr, bend left, "j_I"] \arrow[ddr, bend right, "id"'] \arrow[dr, phantom, shift left=3ex, "\Nearrow k"] & & & \\
& i_I{\downarrow}s \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, "\Nearrow \pi"] & X \arrow[r, "\tau_X"] \arrow[d, "s"] \arrow[dr, phantom, "\Nearrow \epsilon"] & A \\
& I \arrow[r, "i_I"'] & Y \arrow[ur, "\prod_s \tau_X"'] &
\end{tikzcd}
$$

Intuitively, the stability condition tells us that we may define the product $\prod_s \tau_X$, which is a Y-indexed family, on each index $i_I \in Y$ separately by multiplying over generalised arrows $i_I \longrightarrow s(j_I)$, that is:

$$\{\prod_s \tau_X\}_{i_I} = \prod_{i_I \to s(j_I)} \{\tau_X\}_{j_I}$$

In case Y is canonically discrete, every line shrinks to a point, the comma object turns into pullback, and the above formula simplifies to:

$$\{\prod_s \tau_X\}_{i_I} = \prod_{i_I = s(j_I)} \{\tau_X\}_{j_I}$$

In the rest of the chapter we shall mostly restrict to (co)products parametrised by discrete objects (restricting also the stability condition in Definition 1.27 to the subcategory of discrete objects), and call the (co)products polymorphic objects. Such polymorphism induces two additional rules for products:

$$
\dfrac{\left\{\sigma_{s(j)} \xrightarrow{f_j} \tau_j\right\}_{j \in Y}}{\left\{\sigma_i \xrightarrow{\langle f_j \rangle_{i=s(j)}} \prod_{i=s(j)} \tau_j\right\}_{i \in X}}
\qquad
\dfrac{\left\{\sigma_i \xrightarrow{f_i} \prod_{i=s(j)} \tau_j\right\}_{i \in X}}{\left\{\sigma_{s(j)} \xrightarrow{\pi_j \circ f_{s(j)}} \tau_j\right\}_{j \in Y}}
$$

($\prod$-int) ($\prod$-eli)

and dual for coproducts. It is easiest to grasp the rules by the following example.

Example 1.29 (Polymorphism in **Cat**)**.** *Let* **Cat** *be the 2-category of locally small categories. Consider two sets X, Y interpreted as categories in* **Cat***. A functor $F\colon X \times Y \to \mathbb{C}$ may be thought of as an X, Y-indexed family $\{\tau_{i,j}\}_{i \in X, j \in Y}$ of objects $\tau_{i,j} \in \mathbb{C}$, where $\tau_{i,j} = F(i,j)$. If $\mathbb{C}$ has Y-indexed products (in the usual sense), then with every such family, we may associate an X-indexed family $\{\prod_{j \in Y} \tau_{i,j}\}_{i \in X}$. Furthermore, this family satisfies the following universal property: for every X-indexed collection $\{\sigma_i\}_{i \in X}$ from $\mathbb{C}$ and every X, Y-indexed collection $\{f_{i,j}\colon \sigma_i \to \tau_{i,j}\}_{i \in X, j \in Y}$ of morphism from $\mathbb{C}$ there exists a unique collection of X-indexed morphisms $\{h_i\colon \sigma_i \to \prod_{j \in Y} \tau_{i,j}\}_{i \in X}$ from $\mathbb{C}$ such that $\pi^i_j \circ h_i = f_{i,j}$, where $\pi^i_j\colon \prod_{j \in Y} \tau_{i,j} \to \tau_{i,j}$ is the j-th projection of i-th element of the family. When X is the singleton, the above reduces to "internalisation" of an external (that is set-indexed) collection of objects (types) $\{\tau_j\}_{j \in Y}$ into a single product object (type) $\prod_{j \in Y} \tau_j$.*

In the above case, the product is taken along the cartesian projection $\pi_X\colon X \times Y \to X$. More generally, we may form a product along any function $s\colon Z \to X$ — it assigns to a Z-indexed collection $\{\tau_j\}_{j \in Z}$ of objects $\tau_j \in \mathbb{C}$ an X-indexed collection $\{\prod_{i = s(j)} \tau_j\}_{i \in X}$.

The example shows that polymorphism in **Cat** is really an "ad hoc polymorphism". This is because every discrete category $\mathbb{X}$ is isomorphic to the coproduct over terminal category $\coprod_{|\mathbb{X}|} 1$, and every morphism between discrete categories is induced by a function between indexes of the coproducts. Generally, we shall call such polymorphism "ad hoc" to stress the fact, that we are able to freely choose every element of the collection by choosing a generalised element on each of its components. It is better perhaps to think of $\coprod_\lambda \mathbb{A}$ as tensor of $\mathbb{A}$ with a discrete category λ. Here, we shall recall the notion of tensor in an arbitrary 2-category.

Definition 1.30 ((co)Tensor)**.** *Let $\mathbb{W}$ be a 2-category, A an object in $\mathbb{W}$, and λ an ordinary small category. The tensor of A with λ exists, and is denoted by $\lambda \otimes A$, if there exists a 2-natural isomorphism of 2-functors:*

$$\hom_{\mathbf{Cat}}(\lambda, \hom_{\mathbb{W}}(A, -)) \approx \hom_{\mathbb{W}}(\lambda \otimes A, -)$$

Dually, the cotensor of A with λ exists, and is denoted by $\lambda \pitchfork A$, if there exists a 2-natural isomorphism of 2-functors:

$$\hom_{\mathbf{Cat}}(\lambda, \hom_{\mathbb{W}}(-, A)) \approx \hom_{\mathbb{W}}(-, \lambda \pitchfork A)$$

If λ is a set thought of as a discrete category, then the notion of tensor with λ coincides with the coproduct over λ — clearly by the definition of a coproduct $\hom(\coprod_\lambda A, -) \approx \hom(\lambda, \hom(A, -))$ therefore $\lambda \otimes A \approx \coprod_\lambda A$. The usual codiagonal morphism $\nabla\colon \coprod_\lambda A \to A$ is the projection morphism $\pi\colon \lambda \otimes A \to A$ obtained via the transposition of the functor $\lambda \to \hom(A, A)$ sending everything from λ to the identity on A. There is also a diagonal functor $\Delta\colon \lambda \to \hom(A, \lambda \otimes A)$ given by the transposition of the identity functor $id_{\lambda\otimes A}\colon \lambda \otimes A \to \lambda \otimes A$. Then every function between indexes $s\colon \lambda' \to \lambda$ induces a reindexing morphism $s \otimes A\colon \lambda' \otimes A \to \lambda \otimes A$, which is the transposition of $\Delta \circ s\colon \lambda' \to \hom(A, \lambda \otimes A)$. An ad hoc polymorphism is a polymorphism along such reindexing morphisms.

Definition 1.31 (Ad hoc polymorphism)**.** *Let $A, X \in \mathbb{W}$ be two objects in a 2-category, and assume that the tensors $\lambda \otimes X$ and $\lambda' \otimes X$ with sets λ and λ' exist. An ad hoc $\lambda' \otimes X$-parametrised family $\tau\colon \lambda' \otimes X \to A$ has an ad hoc (co)product along a function $s\colon \lambda' \to \lambda$ if the parametrised (co)product of τ along the reindexing morphism $s \otimes X\colon \lambda' \otimes X \to \lambda \otimes X$ exists. In case the (co)product is taken over cartesian projection $\lambda \times \lambda' \to \lambda$ we write $\prod_{i\in\lambda'} \tau_i$ (resp. $\coprod_{i\in\lambda'} \tau_i$) for the ad hoc (co)product and call it "simple (co)product".*

The next example shows that in other 2-categories, other variants of polymorphisms are possible.

Example 1.32 (Polymorphism in $\mathbf{cat}(\omega\mathbf{Set})$)**.** *Let $\omega\mathbf{Set}$ be the category whose objects are sets X of pairs $\langle x, n\rangle$, where n is a natural number, and whose morphisms $f\colon X \to Y$ are functions $f\colon \pi_1[X] \to \pi_1[Y]$ such that there exists a partial recursive function e with the property: if $\langle x, n\rangle \in X$ then $\langle f(x), e(n)\rangle \in Y$. One may think of ω-sets as of sets enhanced by "proofs" of the fact that elements belong to the set. Then a function between ω-sets has to computably translate the proofs. In the above notation $\pi_1[-]$ is really a functor $\omega\mathbf{Set} \to \mathbf{Set}$ forgetting the proofs. Furthermore, it has right adjoint $F\colon \mathbf{Set} \to \omega\mathbf{Set}$ assigning to a set X the ω-set $\{\langle x, n\rangle : x \in X, n \in N\}$, which means "everything is a proof that an element belongs to the set for those elements that belong to the set", and making $\mathbf{Set}$ a reflective subcategory of $\omega\mathbf{Set}$. The category of ω-sets has finite limits, therefore we may define the 2-category $\mathbf{cat}(\omega\mathbf{Set})$ of categories internal to $\omega\mathbf{Set}$. We start with a definition of an ordinary category $\mathbf{PER}$ — its objects are partial equivalence relations on the set of natural numbers, and its morphisms $f\colon A \to B$ from a PER A to a PER B are functions $f\colon N/A \to N/B$ between quotients of* *the relations, for which there exist partial recursive functions e on natural numbers satisfying $f([a]_A) = [e(a)]_B$. One may think of category* **PER** *as realisation of Reynold's system R [Rey83] [AP90]. A PER A corresponds to a "type". Two elements a, a' are "the same" from the perspective of type A if aAa', and an element a belongs to type A if A recognises it, that is, if aAa. A function from a type A to a type B is thus a function between elements that maps "the same" elements to "the same" elements. We shall see that* **PER** *has also a natural ω-set structure. First, let us observe that* **PER** *is cartesian closed — a product of two PER's A and B is given by:*

$$x(A \times B)y \Leftrightarrow \pi_1(x)A\pi_1(y) \wedge \pi_2(x)B\pi_2(y)$$

where $\pi_1, \pi_2 \colon (N \times N \approx N) \to N$ are some chosen partial recursive projections, and the exponent is given by:

$$eB^A r \Leftrightarrow \forall_{a,a'} aAa' \Rightarrow e(a)Br(a')$$

under some chosen partial recursive enumeration of partial recursive functions. Therefore, **PER** *may be thought of as a category enriched over itself. Then, observe that* **PER** *is a reflective subcategory of ω***Set** *— the embedding* **PER** $\to \omega$**Set** *sends a PER A to the ω-set of quotients:*

$$\{\langle [n]_A, n\rangle \colon nAn\}$$

and its left adjoint identifies elements along their proofs — it sends an ω-set X to the relation $\widehat{X}$:

$$n\widehat{X}m \Leftrightarrow \exists_{\langle x,n\rangle, \langle x',m\rangle \in X}\ x \cong x'$$

where two elements belong to the same equivalence class of equivalence relation $\cong$ if they share a common proof: that is, $\cong$ is generated by $x \cong x'$, such that $\langle x, e\rangle \in X$ and $\langle x', e\rangle \in X$ for some e. Observer that $\widehat{X}$ is really a partial equivalence relation: symmetry is obvious; for transitivity, assume that both $n\hat{X}m$ and $m\hat{X}k$ hold and let us show that $n\hat{X}k$ holds as well. By the definition of $\hat{X}$, we know that there exist pairs:

$$\langle x, n\rangle, \langle x', m\rangle, \langle y, m\rangle, \langle y', k\rangle$$

such that:

$$x \cong x' \wedge y \cong y'$$

Because x' and y share common m in their pairs, by the definition of $\cong$, we have $x' \cong y$. Thus, by the fact that $\cong$ is an equivalence relation: $x \cong y'$. Therefore, $\langle x, n\rangle$ together with $\langle y', k\rangle$ form a proof that $n\hat{X}k$.

The above means that **PER** *may be thought of as a category enriched over* ω**Set**. *Finally, observe that we may glue* hom-ω*-sets of such enriched category into a single ω-set $\mathbf{PER}_1 = \{\langle\langle A, B, [n]_{B^A}\rangle, n\rangle \colon A, B$ are PER's and $nB^An\}$, making* **PER** *an* ω**Set***-internal category. Now, if X is an ordinary set, then ω-functors (i.e. ω***Set***-internal functors) $\tau_X, \sigma_X \colon X \to \mathbf{PER}$ are ordinary families of PERs. However, an ω-natural transformation (i.e. ω***Set***-internal natural transformation) $\alpha \colon \tau_X \to \sigma_X$ has to satisfy a uniformity condition:*

$$\bigcap_{x\in X} \alpha(x) \neq \emptyset$$

This means that $\alpha \colon \tau_X \to \sigma_X$ is determined by a single partial recursive function $e \colon N \to N$ such that for all $x \in X$ we have $a\tau_X(x)a' \Rightarrow e(a)\sigma_X(x)e(a')$. Therefore, the parametrised product of σ_X is given by $\bigcap_{x\in X} \sigma_X(x)$:

$$\begin{tikzcd} \tau \arrow[d, dashed, "\exists![e]"'] \arrow[dr, "{[e]}"] & \\ \bigcap_{x\in X} \sigma_X(x) \arrow[r, "\pi_x"] & \sigma_X(x) \end{tikzcd}$$

The projections $\bigcap_{x\in X} \sigma_X(x) \xrightarrow{\pi_x} \sigma_X(x)$ are induced by the identity function. For every constant ω-functor $\tau \colon X \to \mathbf{PER}$, an ω-natural transformation $\tau \to \sigma_X$ is induced by e satisfying $\forall_{x\in X} a\tau a' \to e(a)\sigma_X(x)e(a')$. The last condition is equivalent to $a\tau a' \to e(a)(\bigcap_{x\in X} \sigma_X(x))e(a')$. Therefore, every ω-natural transformation $\tau \to \sigma_X$ uniquely determines a morphism $\tau \longrightarrow \bigcap_{x\in X} \sigma_X(x)$. One may find that such products resemble usual rules for intersection types in lambda calculi:

$$\frac{\tau \xrightarrow{f} \sigma_X}{\tau \xrightarrow{f} \bigcap_{x\in X} \sigma_X(x)} \ (\textstyle\bigcap\text{-}int) \qquad \frac{\tau \xrightarrow{f} \bigcap_{x\in X} \sigma_X(x)}{\tau \xrightarrow{f} \sigma_X} \ (\textstyle\bigcap\text{-}eli)$$

By similar considerations, we get that the parametrised coproduct of σ_X is $\bigcup_{x\in X} \sigma_X(x)$. An extension of Example 1.28 shows that internal (finite) (co)products may be obtained by using tensors $X \otimes 1$ in parametrisation instead of X. There is also an intermediate construction between X and $X \otimes 1$ that yields uniform quantifiers. We may reach this construction by parameterising a category via the internal natural number object $N_\omega = \{\langle n, n\rangle \colon n \in N\}$

*in ω***Set***. An N_ω-parametrised collection of objects from* **PER** *is any countable collection $\sigma(n)_{n\in N}$ of PER's. A product $\prod_{n\in N}\sigma_X(n)$, which in this context may be denoted by $\forall_{n\in N}\sigma_X(n)$, consists of partial recursive functions e which applied to the n-th index return an element of $\sigma_X(n)$, that is: $e(\forall_{n\in N}\sigma_X(n))r \Leftrightarrow \forall_{n\in N}\ e(n)\sigma_X(n)r(n)$. It should be noted that the last construction reduces to the usual dependent product in the ordinary category* **PER** *since the internal natural number object in* **PER** *is the same as the internal number object in ω***Set***.*

Example 1.33 (Second-order Lambda Caluclus)**.** *Let us briefly recall the system of second-order polymorphic lambda calculus. In second-order polymorphic lambda calculus there are terms, types and a kind Ω. Every term M has a type σ, which is denoted by $M:\sigma$, and every type σ has kind Ω, which is denoted by $\sigma:\Omega$. We will call a finite sequence $\sigma_1:\Omega,\sigma_2:\Omega,\cdots,\sigma_n:\Omega$, where σ_i are types, a kind environment. A well-formed type-sequent is of the form:*

$$\sigma_1:\Omega,\sigma_2:\Omega,\cdots,\sigma_n:\Omega\vdash\tau:\Omega$$

There are eight basic rules for type construction:

Type Constructors

$$\frac{}{\Xi, x\colon\Omega\ \vdash x\colon\Omega}\qquad\qquad \frac{\Xi\vdash\sigma\colon\Omega\quad \Xi\vdash\tau\colon\Omega}{\Xi\vdash\sigma\Rightarrow\tau\colon\Omega}$$

$$\frac{}{\vdash\{*\}\colon\Omega}\qquad\qquad \frac{}{\vdash\emptyset\colon\Omega}$$

$$\frac{\Xi\vdash\sigma\colon\Omega\quad \Xi\vdash\tau\colon\Omega}{\Xi\vdash\sigma\times\tau\colon\Omega}\qquad\qquad \frac{\Xi\vdash\sigma\colon\Omega\quad \Xi\vdash\tau\colon\Omega}{\Xi\vdash\sigma+\tau\colon\Omega}$$

$$\frac{\Xi, x\colon\Omega\vdash\sigma_x:\Omega}{\Xi\vdash\prod_{x:\Omega}\sigma_x:\Omega}\qquad\qquad \frac{\Xi, x\colon\Omega\vdash\sigma_x:\Omega}{\Xi\vdash\sum_{x:\Omega}\sigma_x:\Omega}$$

Similarly, we will call a finite sequence $M_1:\sigma_1,M_2:\sigma_2,\cdots,M_n:\sigma_n$, where M_i are terms and σ_i are types, a type environment. A well-formed sequent is of the form:

$$\sigma_1:\Omega,\sigma_2:\Omega,\cdots,\sigma_n:\Omega;M_1:\tau_1,M_2:\tau_2,\cdots,M_k:\tau_k\vdash N:\rho$$

where semicolon ; separates kind environment from type environment.

The standard rules for terms include the identity rule:

$$\frac{}{\Xi;\Gamma, x\colon \sigma \vdash x\colon \sigma}$$

and introduction and elimination rules for connectives:

$$\frac{}{\vdash *\colon \{*\}}\ \textit{(singleton)} \qquad \frac{}{\Xi;\Gamma, x\colon \emptyset \vdash \bot\colon \sigma}\ \textit{(empty type)}$$

$$\frac{\Xi;\Gamma \vdash M : \sigma \quad \Xi;\Gamma \vdash N : \tau}{\Xi;\Gamma \vdash \langle M, N\rangle\colon \sigma \times \tau}\ \textit{(}\times\textit{-int)} \qquad \frac{\Xi;\Gamma \vdash M\colon \sigma \times \tau}{\Xi;\Gamma \vdash \pi_\sigma M : \sigma \quad \Xi;\Gamma \vdash \pi_\tau M : \tau}\ \textit{(}\times\textit{-eli)}$$

$$\frac{\Xi;\Gamma \vdash M\colon \sigma}{\Xi;\Gamma \vdash \iota_\sigma M\colon \sigma + \tau}\ \textit{(+-int)} \qquad \frac{\Xi;\Gamma \vdash N\colon \tau}{\Xi;\Gamma \vdash \iota_\tau N\colon \sigma + \tau}\ \textit{(+-int)}$$

$$\frac{\Xi;\Gamma \vdash M\colon \sigma + \tau \quad \Xi;\Gamma, x\colon \sigma \vdash R\colon \gamma \quad \Xi;\Gamma, y\colon \tau \vdash S\colon \gamma}{\Xi;\Gamma \vdash (\mathit{continue}\ M\ \ \mathit{as}\ x\ \mathit{in}\ R\ \ \mathit{or}\ \mathit{as}\ y\ \mathit{in}\ S)\colon \gamma}\ \textit{(+-eli)}$$

$$\frac{\Xi;\Gamma, x\colon \sigma \vdash M\colon \tau}{\Xi;\Gamma \vdash \lambda x : \sigma.M : \sigma \Rightarrow \tau}\ \textit{(}\lambda\textit{-int)} \qquad \frac{\Xi;\Gamma \vdash M\colon \sigma \Rightarrow \tau \quad \Xi;\Gamma \vdash N\colon \sigma}{\Xi;\Gamma \vdash MN\colon \tau}\ \textit{(}\lambda\textit{-eli)}$$

$$\frac{\Xi, x\colon \Omega;\Gamma \vdash M\colon \sigma_x}{\Xi;\Gamma \vdash \lambda x : \Omega.M : \prod_{x:\Omega}\sigma_x}\ \textit{(}\textstyle\prod\textit{-int)} \qquad \frac{\Xi;\Gamma \vdash M\colon \prod_{x:\Omega}\sigma_x \quad \Xi \vdash \tau\colon \Omega}{\Xi;\Gamma \vdash M\tau\colon \sigma_\tau}\ \textit{(}\textstyle\prod\textit{-eli)}$$

$$\frac{\Xi \vdash \sigma\colon \Omega \quad \Xi;\Gamma \vdash M\colon \tau_\sigma}{\Xi;\Gamma \vdash \langle M, \sigma\rangle\colon \sum_{x:\Omega}\tau_x}\ \textit{(}\textstyle\sum\textit{-int)} \qquad \frac{\Xi \vdash \rho : \Omega \quad \Xi, x\colon \Omega;\Gamma, y : \sigma_x \vdash M : \rho}{\Xi;\Gamma, z\colon \sum_{x:\Omega}\sigma_x \vdash \mathit{let}\ \langle y, x\rangle = z\ \mathit{in}\ M\colon \rho}\ \textit{(}\textstyle\sum\textit{-eli)}$$

Notice that in $\prod$-int rule environment Γ may not depend on x, and similarly in $\sum$-eli rule environment Γ and type ρ may not depend on x. Moreover, the usual conversions of lambda calculus apply:

$$\begin{array}{lll} (\lambda x : \tau.M)N \equiv M[N/x] & (\lambda x : \Omega.M)\tau \equiv M[\tau/x] & \textit{(}\beta\textit{-conversion)} \\ \lambda x : \tau.(Mx) \equiv M & \lambda x : \Omega.(Mx) \equiv M & \textit{(}\eta\textit{-conversion)} \end{array}$$

together with similar conversions for other connectives. There are very many systems of the second-order lambda calculus. Its inventor, Jean-Yves Girard, prefers to present a minimal set of rules (i.e. rules for λ and $\prod$) and then use the following encoding for the rest [Gir86] [Gir11]:

- $\sigma \times \tau = \prod_{x\colon \Omega}(\tau \Rightarrow \sigma \Rightarrow x) \Rightarrow x$
- $\sigma \sqcup \tau = \prod_{x\colon \Omega}(\tau \Rightarrow x) \Rightarrow (\sigma \Rightarrow x) \Rightarrow x$
- $0 = \prod_{x\colon \Omega} x$
- $\sum_{x\colon \Omega} \tau = \prod_{y\colon \Omega}(\prod_{x\colon \Omega} \tau \Rightarrow y) \Rightarrow y$

(Girard omits the encoding of the singleton, but from the text one may infer that if included, the encoding would probably have been: $\{*\} = \prod_{x\colon \Omega} x \Rightarrow x$.)

Because we are more concerned in giving explicit interpretations for all rules with explicit environments, we have presented the system in its full form.

Let A be an internally cartesian closed object with internal (co)products in a cartesian 2-category $\mathbb{W}$ with a notion of discreteness. Furthermore, assume that A has stable parametrised (co)products along projections $|A|^n \times |A| \to |A|^n$, for every natural number n. We will give semantics for second-order lambda calculus in object A. The interpretation of kind Ω is $|A|$. Interpretations of type sequents are given by morphisms $|A|^n \to |A|$, where n is a natural number. Notice that by coreflexivity of discrete objects, these morphisms correspond to morphisms $|A|^n \to A$, which will give denotations for types. The meaning of rules for type constructions is obvious. For example, the identity rule:

$$\frac{}{\Xi, x\colon \Omega; \vdash x\colon \Omega}$$

where the type-environment Ξ consists of n type variables, introduces the projection morphism $|A|^n \times |A| \xrightarrow{\pi_{n+1}} |A|$, the rule of function-space formation:

$$\frac{\Xi \vdash \sigma\colon \Omega \quad \Xi \vdash \tau\colon \Omega}{\Xi \vdash \sigma \Rightarrow \tau\colon \Omega}$$

introduces the internal exponentiation: given $|A|^n \xrightarrow{\sigma} A$ and $|A|^n \xrightarrow{\tau} A$, by internal cartesian closedness, there is a morphism $|A|^n \xrightarrow{\tau^\sigma} A$, and the rule for products:

$$\frac{\Xi, x\colon \Omega \vdash \sigma_x : \Omega}{\Xi \vdash \prod_{x:\Omega} \sigma_x : \Omega}$$

introduces the parametric product along projection $|A|^n \times |A| \longrightarrow |A|^n$. *Semantics of sequents:*

$$\sigma_1 : \Omega, \sigma_2 : \Omega, \cdots, \sigma_n : \Omega; M_1 : \tau_1, M_2 : \tau_2, \cdots, M_k : \tau_k \vdash N : \rho$$

are given by 2-morphisms between parallel 1-morphisms $|A|^n \to A$*: i.e. if* $\tau_1, \tau_2, \cdots, \tau_k, \rho$ *are morphisms* $|A|^n \to A$*, then semantics for a sequent of the above form is given by a 2-morphism* $\tau_1 \times \tau_2 \times \cdots \times \tau_k \to \rho$.

- *(singleton) the element* $*$ *of singleton* $\{*\}$ *is the identity 2-morphism on the internal terminal value*

- *(empty type) the element* $\perp$ *of* $\sigma \colon |A|^n \to A$ *is the unique 2-morphism from* $\Gamma \times 0$ *to* σ*, where* 0 *is the internal initial value*

- *(*$\times$*-int/eli) elements are given by the usual categorical rules for products*

- *(+-int) elements are given by the usual categorical rules for coproducts*

- *(+-eli) elements are given by continuation-like semantics; given four 1-morphisms:* $\Gamma, \sigma, \tau, \rho \colon |A|^n \to A$ *and 2-morphisms* $M \colon \Gamma \to \sigma \sqcup \tau$, $R \colon \Gamma \times \sigma \to \rho$ *and* $S \colon \Gamma \times \tau \to \rho$*, we construct a 2-morphism:*

  $$(\textit{continue } M \ \textit{ as } x \textit{ in } R \ \textit{ or as } y \textit{ in } S) \colon \Gamma \to \rho$$

  as follows: first, by universal properties of internal coproducts and exponents, we obtain a 2-morphism $R \sqcup S \colon \Gamma \times (\sigma \sqcup \tau) \to \rho$*, and then we precompose it with morphism* $\langle id, M \rangle \colon \Gamma \to \Gamma \times (\sigma \sqcup \tau)$

- *(*λ*-int/eli) elements are given by the usual categorical rules for exponents*

- *(*$\prod$*-int) let* $\sigma \colon |A|^n \times |A| \to A$ *be the interpretation of type* σ, $\Gamma \colon |A|^n \to A$ *be the interpretation of environment* Γ*, and* $M \colon \Gamma \circ \pi \to \sigma$ *the interpretation of term* M*, where* $\pi \colon |A|^n \times |A| \to |A|^n$ *is the cartesian projection; we obtain the interpretation of* $\lambda x : \Omega . M$ *as transposition of* M *provided by right Kan extension of* σ *along* π

- *(*$\prod$*-eli) let* $\prod_{x \colon |A|} \sigma_x \colon |A|^n \to A$ *be the interpretation of type* $\prod\limits_{x \colon \Omega} \sigma_x$, $\Gamma \colon |A|^n \to A$ *be the interpretation of environment* Γ, $\tau \colon |A|^n \to A$ *be the interpretation of type* τ*, and* $M \colon \Gamma \to \sigma$ *be the interpretation of* *term M; by transposition of M, there is a 2-morphism $M^*\colon \Gamma \circ \pi \to \sigma_x$; we obtain the interpretation of $M\tau$ as the precomposition of 2-morphism M^* with 1-morphism $\langle id, \tau\rangle\colon |A|^n \to |A|^n \times |A|$*

- *($\sum$-int) let $\tau_x\colon |A|^n \times |A| \to A$ be the interpretation of type τ_x, $\sigma\colon |A|^n \to A$ be the interpretation of type σ, $\Gamma\colon |A|^n \to A$ be the interpretation of environment Γ, and $M\colon \Gamma \to \tau_x \circ \langle id, \sigma\rangle$ the interpretation of term M; we obtain the interpretation of term $\langle M, \sigma\rangle$ as precomposition of term M with $\eta \circ \langle id, \sigma\rangle$, where η is the 2-morphism that defines the left Kan extension of τ_x along cartesian projection $\pi\colon |A|^n \times |A| \to |A|^n$*

- *($\sum$-eli) let $\rho\colon |A|^n \to A$ be the interpretation of type ρ, $\sigma\colon |A|^n \times |A| \to A$ be the interpretation of type σ_x, $\Gamma\colon |A|^n \to A$ be the interpretation of environment Γ, and $M\colon (\Gamma \circ \pi) \times \sigma_x \to \rho$ the interpretation of term M, where $\pi\colon |A|^n \times |A| \to |A|^n$ is the cartesian projection; we obtain the interpretation of let $\langle y, x\rangle = z in\ M$ as transposition of M provided by left Kan extension of $(\Gamma \circ \pi) \times \sigma_x$ along π; notice that the types agree, since $\coprod_\pi((\Gamma \circ \pi) \times \sigma_x) \approx \Gamma \times \coprod_\pi \sigma_x$ in an internally cartesian closed object*

In case $\mathbb{W}$ is the category of internal categories with the canonical notion of discreteness, the above semantics reduces to the usual interpretation of connectives in internal PL-categories [See87].

If $\tau\colon X \to C$ is an X-parametrised element of C, then one may try to compute the parametrised product of τ along *itself*:

$$\begin{tikzcd}
X \arrow[r, "\tau"] \arrow[d, "\tau"'] & C \\
C \arrow[ur, "\prod_\tau \tau"'] \arrow[u, phantom, ""] &
\arrow[from=1-1, to=2-1, phantom, shift left=6, "\Nearrow"]
\end{tikzcd}$$

Definition 1.34 (Density (co)product)**.** *A density (co)product $T_\tau\colon C \to C$ (resp. $D_\tau\colon C \to C$) of a parametrised element $\tau\colon X \to C$ is the (co)product of $\tau\colon X \to C$ along itself.*

Example 1.35 (Logical consequence)**.** *Let* $\mathbf{Cat}(2)$ *be the 2-category of categories enriched in a 2-valued Boolean algebra $2 = \{0 \to 1\}$. A 2-enriched category is tantamount to a partially ordered set (poset), and a 2-enriched functor is essentially a monotonic function between posets [Kel82]. Let us consider a relation:*

$$\models\ \subseteq Mod \times Sen$$

thought of as a satisfaction relation between a set of models Mod and a set of sentences Sen. By transposition, relation $\models$ *yields the "theory" function* $th\colon Mod \to 2^{Sen}$, *where* 2^{Sen} *is the poset of functions* $Sen \to 2$, *or equivalently the poset of subsets of Sen .*

Since "power" posets 2^{Sen} *are internally complete in the 2-category* $\mathbf{Cat}(2)$ *[Kel82], the stable density product of* $th\colon Mod \to 2^{Sen}$ *exists:*

$$\begin{tikzcd} Mod \arrow[r, "th"] \arrow[d, "th"'] \arrow[dr, phantom, "\Nearrow"] & 2^{Sen} \\ 2^{Sen} \arrow[ur, "T_{th}"'] & \end{tikzcd}$$

and is given by the 2-enriched end [Str74]:

$$T_{th}(\Gamma)(\psi) = \int_{M \in Mod} th(M)(\psi)^{\hom(\Gamma, th(M)(-))}$$

where $\psi \in Sen$ *is a sentence, and* $\Gamma \in 2^{Sen}$ *is a set of sentences. We are interested in values of* T_{th} *on representable functors (i.e. single sentences)* $\hom_{Sen}(-,\phi)$*:*

$$\begin{aligned} T_{th}(\hom_{Sen}(-,\phi))(\psi) &= \int_{M \in Mod} th(M)(\psi)^{\hom(\hom_{Sen}(-,\phi), th(M)(-))} \\ &\approx \int_{M \in Mod} th(M)(\psi)^{th(M)(\phi)} \end{aligned}$$

where the isomorphism follows from the Yoneda reduction. Observe that the exponent $th(M)(\psi)^{th(M)(\phi)}$ *in a 2-enriched world may be expressed by the implication "*$th(M)(\phi) \Rightarrow th(M)(\psi)$*", or just "*$M \models \phi \Rightarrow M \models \psi$*", where every component of the implication is interpreted as a logical value in the 2-valued Boolean algebra. Furthermore, ends turn into universal quantifiers, when we move to 2-enriched world. So, the end* $\int_{M \in Mod} th(M)(\psi)^{th(M)(\phi)}$ *is equivalent to the meta formula "*$\forall_{M \in Mod}\,(M \models \phi \Rightarrow M \models \psi)$*", which is just the definition of logical consequence:*

$$\phi \models_{Sen} \psi \quad \textit{iff} \quad \forall_{M \in Mod}\,(M \models \phi \Rightarrow M \models \psi)$$

The general case, where Γ *is not necessarily representable, is similar:*

$$T_{th}(\Gamma)(\psi) \quad \textit{iff} \quad \forall_{M \in Mod}\,((\forall_{\phi \in \Gamma} M \models \phi) \Rightarrow M \models \psi)$$

Therefore, the density product of a satisfaction relation reassembles the semantic consequence relation.

A density product morphism $T_\tau = \prod_\tau \tau$, if it exists, is always a part of a monad structure. The unit $\eta \colon id_C \to T_\tau$ is the unique 2-morphism to the product induced by the identity $id_\tau \colon \tau \to \tau$; similarly the multiplication $\mu \colon T_\tau \circ T_\tau \to T_\tau$ is given as the unique 2-morphism to the product induced by $\epsilon \bullet T_\tau \circ \epsilon$, where $\epsilon \colon T_\tau \circ \tau \to \tau$ is the product's 2-morphism. By duality, a coproduct morphism $D_\tau = \coprod_\tau \tau$, provided it exists, is always a part of a comonad structure. In case of functors between ordinary categories the density coproduct is known as density comonad, and density product is sometimes called a "codensity monad". The terminology comes from the fact that a functor $F \colon \mathbb{A} \to \mathbb{B}$ between categories $\mathbb{A}$ and $\mathbb{B}$ is dense iff the identity on $\mathbb{B}$ is the parametrised coproduct of F with itself. In a sense the density comonad on a functor exhibits the "defect" of the functor to be dense.

There are variety of examples of density comonad and codensity monad that may be found in ordinary mathematics. Here, generalising the idea from [Lei13], we use codensity monad to define internal ultraproducts.

Example 1.36 (Internal ultraproducts)**.** *First let us recall the concept of a finitely presentable object (Chapter 5.1, Volume 2 of [Bor94]). Let $\mathbb{C}$ be a locally small category with filtered colimits. An object $A \in \mathbb{C}$ is finitely presentable if:*

$$\hom_{\mathbb{C}}(A, -) \colon \mathbb{C} \to \mathbf{Set}$$

preserves filtered colimits. For example, a set $A \in \mathbf{Set}$ is finitely presentable iff it is finite in the usual sense[7]*. More generally, an algebra is finitely presentable if it can be defined on a finite set of generators and satisfying a finite set of relations (Chapter 3.8, Volume 2 of [Bor94]).*

If we denote by $Fin(\mathbb{C})$ the full subcategory of $\mathbb{C}$ that consists of finitely presentable objects, then we say that $\mathbb{C}$ has ultrafilters if there is a density product $T_J \colon \mathbb{C} \to \mathbb{C}$ of the inclusion $J \colon Fin(\mathbb{C}) \to \mathbb{C}$. In this case we call T_J an ultrafilter monad. It is well-known (Section 8 of [Lei13]) that the functor $T_J \colon \mathbf{Set} \to \mathbf{Set}$ defined:

$$\begin{aligned} T_J(X) &= \{F \colon F \text{ is an ultrafilter on } X\} \\ T_J(A \xrightarrow{f} B) &= F \in T(A) \mapsto \{B_0 \subseteq B \colon f^{-1}[B_0] \in F\} \end{aligned}$$

is a part of the codensity monad on the inclusion of finite sets $\mathbf{FinSet} \approx Fin(\mathbf{Set})$ to sets $\mathbf{Set}$ with units $\eta_A \colon A \to T_J(A)$:

$$\eta_A(a) = \{A_0 \subseteq A \colon a \in A_0\}$$

[7]Here we assume that the Axiom of Choice holds in the metatheory.

and multiplications $\mu_A \colon T_J(T_J(A)) \to T_J(A)$*:*

$$\mu_A(FF) = \{A_0 \subseteq A \colon \{F \in T_J(A) \colon A_0 \in F\} \in FF\}$$

Now, let $\mathbb{W}$ *be a 2-category with a notion of discreteness. Furthermore, assume that the category of discrete objects* $Disc(\mathbb{W})$ *has ultrafilters* $T_J \colon Disc(\mathbb{W}) \to Disc(\mathbb{W})$*. Let* A *be an object of* $\mathbb{W}$*. There is a diagram (where* $(-)^*$ *sends a category to its opposite):*

$$\begin{tikzcd}
Disc(\mathbb{W})^{op} \arrow[r, "\hom_{\mathbb{W}}(-{,}A)^*"] & \mathbf{Cat} \\
Fin(Disc(\mathbb{W}))^{op} \arrow[u, "J"] \arrow[ur, "\hom_{\mathbb{W}}(J(-){,}A)^*"'] &
\end{tikzcd}$$

which under Grothendieck construction induces a functor:

$$\int \hom_{\mathbb{W}}(J(-), A)^* \xrightarrow{\overline{J}} \int \hom_{\mathbb{W}}(-, A)^*$$

Explicitly, functor $\overline{J}$ *is given as the following pullback:*

$$\begin{tikzcd}
\int \hom_{\mathbb{W}}(J(-), A)^* \arrow[r, "\overline{J}"] \arrow[d, "\pi_{\hom_{\mathbb{W}}(J(-),A)^*}"'] & \int \hom_{\mathbb{W}}(-, A)^* \arrow[d, "\pi_{\hom_{\mathbb{W}}(-,A)^*}"'] \\
Fin(Disc(\mathbb{W})) \arrow[r, "J"] & Disc(\mathbb{W})
\end{tikzcd}$$

The codensity monad $T_{\overline{J}} \colon \int \hom_{\mathbb{W}}(-, A)^* \to \int \hom_{\mathbb{W}}(-, A)^*$ *on* $\overline{J}$*, provided it exists, is the ultraproduct functor. It maps elements* $\tau \colon X \to A$ *parametrised by a discrete object* X *to elements* $T_{\overline{J}}(\tau) \colon T_J(X) \to A$ *parametrised by ultrafilters* $T_J(X)$ *on* X*. It follows from [Lei13] that in case* $\mathbb{W} = \mathbf{Cat}$ *and* $A = \mathbf{Set}$ *the functor* $T_{\overline{J}}$ *is given by:*

$$T_{\overline{J}}(\tau) = (\prod_F \tau)_{F \in T_J(X)}$$

where $\prod_F \tau$ *is the usual ultraproduct of family of sets* τ *over ultrafilter* F *— i.e.* $\prod_F \tau$ *is the colimit of the product functor:*

$$\prod_{x \in (-)} \tau(x) \colon F^{op} \to \mathbf{Set}$$

where F^{op} is viewed as the category consisting of sets from ultrafilter F ordered by reverse-inclusion. It is best to understand this construction in case $\delta_{\{0,1\}} \colon \mathcal{N} \to \mathbf{Set}$ is the constant collection assigning to every natural number from $\mathcal{N}$ two-element set $\{0,1\}$ (i.e. $\delta_{\{0,1\}}(n) = \{0,1\}$), and F is the Frechet filter[8] *of cofinite subsets of natural numbers $\mathcal{N}$ (i.e. subsets whose complements are finite), then the ultraproduct $\prod\limits_F \delta_{\{0,1\}}$ consists of the set of all (countable) cofinite sequences over $\{0,1\}$ (i.e. it is the quotient of the set $\{0,1\}^{\mathcal{N}}$ of all sequences over $\{0,1\}$, by the equivalence relation: $(a_i)_{i\in\mathcal{N}} \sim (a_i)_{i\in\mathcal{N}}$ iff $\{i \in \mathcal{N} \colon a_i = b_i\}$ is cofinite).*

More generally, $T_{\overline{J}}$ agrees with the usual notion of ultraproduct for $A = Alg(\Sigma)$, where $Alg(\Sigma)$ is the category of algebras and algebra homomorphisms over a signature Σ.

1.2 Internal incompleteness theorem

The classical result of Freyd shows that categories that are both small and complete are preorders. Let us recall the argument. If $\mathbb{C}$ is a small category, then there exists a set of all morphisms of $\mathbb{C}$ with cardinality λ. Let us assume that there is a pair of distinct parallel morphisms $f, g \colon A \to B$ in $\mathbb{C}$. We may form a product of λ-copies of B, provided $\mathbb{C}$ is sufficiently complete:

$$\begin{tikzcd} A \arrow[d, dashed, "\exists! h"'] \arrow[dr, "f{,}g"] & \\ \prod_\lambda B \arrow[r, "\pi_{j\in\lambda}"] & B \end{tikzcd}$$

Now, for each index $j \in \lambda$ we may *freely* choose either a morphism f or g to make a cone over B's. There are $\{f,g\}^\lambda$ of such cones. Because, by the property of product $\prod_\lambda B$, each cone uniquely determines a morphism $h \colon A \to \prod_\lambda B$, the cardinality of the set $\hom(A, \prod_\lambda B)$ is at least $\{f,g\}^\lambda$. This contradicts our claim that the set of all morphism has cardinality λ, since in ZFC there could be no injection $2^\lambda \to \lambda$.

The result relies on two fundamental properties of standard set theory. One is non-uniformity of set-indexed collections; or arbitrary richness of set-

[8] Technically, Frechet filters are generally not ultrafilters, but the construction is essentially the same. Notice, that it is difficult to give an *explicit* example of an ultrafilter, for which the construction does not trivialise (because non-principal ultrafilters are non-constructive).

indexed collections — for any cardinal λ and any set K, we may make a free/independent/non-uniform choice of one of the elements of K for each index $j \in \lambda$. Another is the property of being 2-valued. We say that a set theory is 2-valued if the set $2 = 1 \sqcup 1$ forms the subset classifier. By the classical diagonal argument one may show that in any topos with a subobject classifier Ω there could be no injection $\Omega^A \to A$. Therefore, the contradiction in the Freyd's argument follows from the fact that the subobject classifier in ZFC has only two elements.

One may wonder if the above properties are crucial to the result of Freyd. And the answer is — yes, but in two different ways. In late 80's Martin Hyland showed that there exists a small (weakly) complete non-posetal category internal to the effective topos [Pit87] [Hyl88]. The key argument in his work is that the cones in the mentioned category have to satisfy a suitable smoothness condition (recall Example 1.32) — there is no way to form an arbitrary collection $\{f, g\}^\lambda$ as in the above proof. On the other hand, the result of Freyd carries to any cocomplete topos as we show below in Theorem 1.41 (also see: Corollary 1.45), in particular, to any Grothendieck topos — no matter how "big", or "complicated" the subobject classifier in the topos is. In a sense, the second property is used on a higher meta-level than the first one[9], and we shall not investigate it in this dissertation.

Now, we try to reproduce the result of Freyd in any sufficiently cocomplete 2-category.

Lemma 1.37. *Let $\mathbb{W}$ be a 2-category. Consider a pair of parallel morphisms $a, b \colon X \to C$, and a pair of distinct parallel 2-morphisms $f, g \colon a \to b$ in $\mathbb{W}$. Let us assume that for a set λ the 2-coproduct $\coprod_\lambda X$ exists, and that there is a right Kan extension $Ran_\nabla(b \circ \nabla)$ of $b \circ \nabla \colon \coprod_\lambda X \to C$ along $\nabla \colon \coprod_\lambda X \to X$, where ∇ is the coproduct codiagonal. Then the set $\hom(a, Ran_\nabla(b \circ \nabla))$ has cardinality at least 2^λ.*

[9]The second property refers to the ambient category of the 2-category of internal categories. It is worth pointing out that contrary to some common beliefs the above argument is purely constructive — even though it may not imply that the set $\hom(A, B)$ has cardinality less than 2.

Proof. Consider a diagram that satisfies the hypothesis of the lemma:

$$\begin{tikzcd}
\coprod_\lambda X \arrow[r, "\nabla"] \arrow[d, "\nabla"'] & X \arrow[l, bend right, "\iota_{i\in\lambda}"'] \arrow[r, bend left, "a"] \arrow[r, bend right, "b"'] \arrow[r, phantom, "f\Downarrow\ \Downarrow g"] & C \\
X \arrow[urr, bend right, "Ran_\nabla(b\circ\nabla)"'] \arrow[ur, phantom, "\Nwarrow\epsilon"] & &
\end{tikzcd}$$

where $\iota_{i\in\lambda}$ are coproduct's injections. We form two cocones — one by constantly choosing a, and another by constantly choosing b for each index $i \in \lambda$. By the universal property of coproduct $\coprod_\lambda X$ these cocones induce unique morphisms $a \circ \nabla \colon \coprod_\lambda X \to C$ and $b \circ \nabla \colon \coprod_\lambda X \to C$, respectively. We may form a transformation of cones by independently choosing either a 2-morphism $f \colon a \to b$ or $g \colon a \to b$ for each index $i \in \lambda$. There are $\{f, g\}^\lambda$ of such transformations, and by the universal property of 2-coproduct, each transformation uniquely determines a 2-morphism $a \circ \nabla \to b \circ \nabla$. Therefore, $\hom(a \circ \nabla, b \circ \nabla)$ has cardinality at least 2^λ. The definition of the right Kan extension $Ran_\nabla(b \circ \nabla)$ says that there is a natural isomorphism:

$$\hom(a, Ran_\nabla(b \circ \nabla)) \approx \hom(a \circ \nabla, b \circ \nabla)$$

thus, by the above, $\hom(a, Ran_\nabla(b \circ \nabla))$ has cardinality at least 2^λ, which completes the proof. $\square$

There is an obvious generalisation of the above lemma, which may be obtained by replacing cardinal λ with arbitrary category, and coproduct $\coprod_\lambda X$ with tensor $\lambda \otimes X$. Indeed, by the definition of tensor $\hom(a \circ \pi, b \circ \pi) \approx \hom(\Delta(a), \Delta(b))$, where $\Delta(a), \Delta(b) \colon \lambda \to \hom(X, C)$ are constant functors assigning everything to a and b respectively, and π plays the role of the co-diagonal ∇. Therefore $\hom(a, Ran_\pi(b \circ \pi)) \approx \hom(\Delta(a), \Delta(b))$. Choosing discrete λ puts no constraints on transformations $\Delta(a) \to \Delta(b)$ and leads to the conclusion $\hom(\Delta(a), \Delta(b)) \approx \hom(a, b)^\lambda$

Before we state the 2-categorical incompleteness theorem, let us write explicitly definition of a representable poset and of a 2-generating family.

Definition 1.38 (Representable poset)**.** *An object A from a 2-category $\mathbb{W}$ is representably posetal if for every object $X \in \mathbb{W}$, the category* $\hom(X, A)$ *is a poset.*

Definition 1.39 (2-generating family)**.** *A class of objects G from a 2-category $\mathbb{W}$ is called a 2-generating family if for every pair of 2-morphisms α, β between parallel 1-morphisms from an object $A \in \mathbb{W}$ to an object $B \in \mathbb{W}$ the following holds: if for every 2-morphism τ between parallel one morphisms from an object $X \in G$ to object A the equality of compositions holds $\alpha \circ \tau = \beta \circ \tau$, then $\alpha = \beta$.*

We shall also recall the notion of density (see Definition A.36 in Appendix A.3) in the context of 2-categories.

Definition 1.40 (Density)**.** *A 2-functor $F \colon \mathbb{C} \to \mathbb{D}$ is dense if the 2-functor $A \mapsto \hom_{\mathbb{D}}(F(-), A)$ is fully faithful.*

Theorem 1.41 (Incompleteness theorem)**.** *Let $\mathbb{W}$ be a locally small 2-category and $G \subset \mathbb{W}$ a 2-generating family. Furthermore, assume that objects from G have tensors with sets. If an object $C \in \mathbb{W}$ has all ad hoc simple products parametrised by G, then C is representably posetal.*

Proof. Let X be an object in $G \subset \mathbb{W}$. Let us assume that there exists a pair of distinct 2-morphisms $f, g \colon a \to b \in \hom(X, C)$, and choose a cardinal λ equal to the cardinality of the underlying set of morphisms of $\hom(X, C)$. By Lemma 1.37, $\hom(a, \prod_{i \in \lambda} b)$ has cardinality at least 2^λ, which leads to the contradiction $2^\lambda \leq \lambda$ in ZFC. Therefore, $\hom(X, C)$ is a poset on each $X \in G$, thus by the property of a 2-generating family, C is representably posetal. $\square$

There is also a version of the incompleteness theorem directly using adjunctions to codiagonals (recall Example 1.28).

Corollary 1.42 (Special incompleteness theorem)**.** *Let $A \in \mathbb{W}$. If for every set X the constant product $\prod_X A$ exists, and the diagonal $\Delta \colon A \to \prod_X A$ has right adjoint, then A is representably posetal.*

Example 1.43 (Freyd's theorem)**.** *The classical Freyd's theorem is obtained from Theorem 1.41 by taking $\mathbb{W} =$ **cat**, and recalling that the terminal category 1 is a 2-generator in **cat**. Alternatively, one may use the special incompleteness theorem in the following way: in **cat** cotensors $X \pitchfork \mathbb{A} = \mathbb{A}^X = \prod_X \mathbb{A}$ exist for any small category $\mathbb{A}$ and every set X; Corollary 1.42 says that if for every X there is a right adjoint to the diagonal $\Delta \colon \mathbb{A} \to \mathbb{A}^X$ then $\mathbb{A}$ is posetal.*

We shall observe in the next section that for a 2-category of internal categories, the above notion of being representably posetal coincides with the usual notion of an internal poset (Corollary 1.55), and ad hoc products parametrised by discrete objects correspond to the internal products in the usual sense (Corollary 1.72).

Definition 1.44 (Internal poset)**.** *Let $\mathbb{C}$ be a category with finite limits. A $\mathbb{C}$-internal poset A is a $\mathbb{C}$-internal category for which the domain and codomain morphisms $dom, cod \colon A_1 \to A_0$ are jointly mono, meaning that the morphism $\langle dom, cod \rangle \colon A_1 \to A_0 \times A_0$ is mono.*

Therefore, we may write the following corollary.

Corollary 1.45. *Let $\mathbf{cat}(\mathbb{C})$ be the 2-category of categories internal to a finitely complete locally small category $\mathbb{C}$ that has tensors with sets. If a $\mathbb{C}$-internal category $C \in \mathbf{cat}(\mathbb{C})$ has simple ad hoc polymorphism then it is an internal poset.*

Proof. The category $\mathbb{C}$ is a 2-dense subcategory of $\mathbf{cat}(\mathbb{C})$ spanned on discrete objects (i.e. the inclusion functor is dense), therefore the class of discrete objects is a 2-generating family. $\square$

A direct consequence of Corollary 1.45 is that there are no small complete non-posetal categories internal to a Grothendieck topos.

We can also get instantly from Theorem 1.41 the incompleteness theorem for enriched categories.

Corollary 1.46. *Let $\mathbb{V}$ be a monoidal category. If a small $\mathbb{V}$-enriched category is complete, then it is representably posetal.*

Proof. The 2-category of small $\mathbb{V}$-enriched categories has small coproducts inherited from **Set**. $\square$

Example 1.47 (ω**Set** and Hyland's effective topos)**.** *The incompleteness theorem does not work in $\mathbf{cat}(\omega\mathbf{Set})$ nor in the categories internal to Hyland's effective topos, because these categories do not have "sufficiently big" coproducts (see for example [vO08] Section 3.2.2). Let us show that ω**Set** does not have even countable coproducts on the terminal object. To obtain a contradiction, assume that a coproduct $\coprod_{n \in N} 1$ exists. Consider the natural number object in ω-sets $N_\omega = \{\langle n, n\rangle \colon n \in N\}$. Every ω-function $k \colon 1 \to N_\omega$*

is uniquely determined by a natural number $k \in N$, and by the universal property of coproduct $\coprod_{n \in N} 1$, every family $n \mapsto k_n$ indexed by natural numbers $n \in N$ uniquely determines an ω-function $h\colon \coprod_{n \in N} 1 \to N_\omega$ with $h(n) = k_n$. Because proofs in N_ω are disjoint, h is determined by a partial recursive function. This leads to a contradiction since not every function $N \to N$ is partial recursive.

In fact, category **PER** *internal to* ω**Set** *is internally complete and cocomplete; particularly, for each morphism $f\colon X \to Y$ every functor:*

$$\mathbf{PER}^f \colon \mathbf{PER}^Y \to \mathbf{PER}^X$$

has both left and right adjoint as sketched in Example 1.32.

Intuitively, Theorem 1.41 may be read as an instance of the well-known mathematical dichotomy that says that either a system or its meta-system has to be incomplete. However, one should be a bit suspicious about the fact that the reasoning in Theorem 1.41, which is about general (possibly constructive) systems, relies on the classical "cardinality argument". To understand better this phenomenon, one should recall that the completeness of a category (or a higher category) depends on the meta-foundations — a category can be complete from the perspective of some foundations, but may miss (co)limits from the perspective of other foundations. For example, a finitely complete and cocomplete locally cartesian closed category is always (co)complete from its own perspective, but may still miss infinite (co)limits (e.g. the category **FinSet** of finite sets from the perspective of the finite world is complete and cocomplete).

Corollary 1.45 says that the underlying 2-category of a (large) $\mathbb{C}$-internal 2-category cannot be sufficiently cocomplete if there are non-posetal internally (co)complete $\mathbb{C}$-internal categories. This does not imply that the $\mathbb{C}$-internal 2-category itself cannot be cocomplete. The situation should not be strange — after all, such phenomenon also occurs in lower dimensions: ω**Set**-internal category **PER** from Example 1.32 is complete and cocomplete as an ω**Set**-internal category, but its underlying classical category is not (because it is non-posetal).

A natural question is then: can there be an internally complete and cocomplete (large) $\mathbb{C}$-internal 2-category, whose $\mathbb{C}$ internal categories may be non-posetal and (co)complete? This question has the affirmative answer. One may understand the (large) $\mathbb{C}$-internal 2-category of $\mathbb{C}$-internal categories as the (weak) functor $cat(\mathbb{C})\colon \mathbb{C}^{op} \to \mathbf{2Cat}$ that assigns to an object $X \in \mathbb{C}$ a 2-category $\mathbf{cat}(\mathbb{C}/X)$ and to a morphism $f\colon X \to Y$ in $\mathbb{C}$

the categorfication $\mathbf{cat}(f^*)\colon \mathbf{cat}(\mathbb{C}/Y) \to \mathbf{cat}(\mathbb{C}/X)$ of the pullback functor $f^*\colon \mathbb{C}/Y \to \mathbb{C}/X$. This definition is sound because the construction **cat** from categories with pullbacks to internal categories is 2-functorial[10]. The classical 2-category $\mathbf{cat}(\mathbb{C})$ may then be seen as the fibre of $cat(\mathbb{C})$ at the global object — i.e. $\mathbf{cat}(\mathbb{C}) = cat(\mathbb{C})(1)$. Now, assume that $\mathbb{C}$ is finitely complete and cocomplete locally cartesian closed category. Its not hard to see that each fibre of $cat(\mathbb{C})$ is finitely weighted complete and cocomplete and, moreover, the reindexing functors have both left and right adjoints that satisfy Beck-Chevalley condition (since $\mathbb{C}$ is locally cartesian closed, every $f^*\colon \mathbb{C}/Y \to \mathbb{C}/X$ has both left and right adjoints satisfying Beck-Chevalley; and since these conditions are equationally defined, they are preserved by 2-functor **cat**). Therefore, **PER** is an example of a non-posetal complete and cocomplete internal category living in a complete and cocomplete (large) internal 2-category.

1.3 The associated category

This section is intended to provide a framework that allows us to better understand 2-categorical models for lambda calculi, and under some conditions embed them into a very well-behaved 2-category — the 2-category of internal categories. We start with an explicit description of a category associated to an object from a 2-category with a notion of *canonical* discreteness, and then move to a more abstract framework.

In the reminder of the section, we shall use extensively the notion of "inserter", which we recall below.

Definition 1.48 (Inserter)**.** *Let $f, g\colon A \to B$ be two parallel morphisms in a 2-category $\mathbb{W}$. The inserter of f and g is an object I in $\mathbb{W}$ together with a morphism $i\colon I \to A$ and a 2-morphism $\pi\colon f \circ i \to g \circ i$ that is universal in the following sense:*

$$\begin{tikzcd} X \arrow[d, dashed, "h_x"'] \arrow[dr, "x"] & & \\ I \arrow[r, "i"] & A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & B \end{tikzcd} \qquad \begin{array}{c} f \circ i \overset{\pi}{\Longrightarrow} g \circ i \\ f \circ x \overset{\alpha}{\Longrightarrow} g \circ x \end{array}$$

[10]See Section 1.3 for more details.

for every diagram $\langle x\colon X \to A, \alpha\colon f \circ x \to g \circ x\rangle$ there exists a unique morphism $h_x\colon X \to I$ such that $i \circ h_x = x$ and $\pi \circ h_x = \alpha$; and for every diagram $\langle x'\colon X \to A, \alpha'\colon f \circ x' \to g \circ x'\rangle$ and a 2-morphism $\gamma\colon x \to x'$ that is a morphism of diagrams, i.e. $(g \circ \gamma) \bullet \alpha = \alpha' \bullet (f \circ \gamma)$, there exists a unique 2-morphism $h_\gamma\colon h_x \to h_{x'}$ such that $i \circ h_\gamma = \gamma$.

Let us rewrite the definition of an inserter in more explicit terms. A functor $F\colon \{ \bullet \rightrightarrows \bullet \} \to \mathbb{W}$ corresponds to a diagram $\{ A \underset{g}{\overset{f}{\rightrightarrows}} B \}$ in $\mathbb{W}$. A natural transformation in $\hom(W(-), \hom(X, F(-)))$ chooses a morphism $x\colon X \to A$ together with a 2-morphism $\alpha\colon f \circ x \to g \circ x$ like on the picture:

$$
\begin{tikzcd}
X \arrow[r, "x"] & A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & B
\end{tikzcd}
\qquad f \circ x \overset{\alpha}{\Longrightarrow} g \circ x
$$

We call a pair $\langle x\colon X \to A, \alpha\colon f \circ x \to g \circ x\rangle$ an "inserter cone over X". A morphism between parallel natural transformations $W(-) \to \hom_{\mathbb{W}}(X, F(-))$ is a *modification*. If $\langle x\colon X \to A, \alpha\colon f \circ x \to g \circ x\rangle$ and $\langle x'\colon X \to A, \alpha'\colon f \circ x' \to g \circ x'\rangle$ are two inserter cones over X induced by natural transformations $W(-) \to \hom_{\mathbb{W}}(X, F(-))$, then a modification between the natural transformations corresponds to a single 2-morphism $\gamma\colon x \to x'$ in $\mathbb{W}$ such that $(g \circ \gamma) \bullet \alpha = \alpha' \bullet (f \circ \gamma)$. Therefore, we may write *Inserter*$(X; f, g)$ for the category of inserter cones over X of the shape of F, which is isomorphic to $\hom(W(-), \hom_{\mathbb{W}}(X, F(-)))$. Then, the assignment $X \mapsto$ *Inserter*$(X; f, g)$ extends by composition to a functor:

$$\mathit{Inserter}(-; f, g)\colon \mathbb{W}^{op} \to \mathbf{Cat}$$

The inserter I of f, g is a 2-representation

$$\hom_{\mathbb{W}}(-, I)\colon \mathbb{W}^{op} \to \mathbf{Cat}$$

of *Inserter*$(-; f, g)$. That is, the inserter is an object I together with a morphism $i\colon I \to A$ and a 2-morphism $\pi\colon f \circ i \to g \circ i$ that is universal in the following sense:

$$
\begin{tikzcd}
 & X \arrow[dl, dashed, "h_x"'] \arrow[dr, "x"] & & \\
I \arrow[rr, "i"] & & A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & B
\end{tikzcd}
\qquad
\begin{array}{c}
f \circ i \overset{\pi}{\Longrightarrow} g \circ i \\
f \circ x \overset{\alpha}{\Longrightarrow} g \circ x
\end{array}
$$

for every diagram $\langle x\colon X \to A, \alpha\colon f \circ x \to g \circ x\rangle$ there exists a unique morphism $h_x\colon X \to I$ such that $i \circ h_x = x$ and $\pi \circ h_x = \alpha$; and for every diagram $\langle x'\colon X \to A, \alpha'\colon f \circ x' \to g \circ x'\rangle$ and a 2-morphism $\gamma\colon x \to x'$ that is a morphism of diagrams, i.e. $(g \circ \gamma) \bullet \alpha = \alpha' \bullet (f \circ \gamma)$, there exists a unique 2-morphism $h_\gamma\colon h_x \to h_{x'}$ such that $i \circ h_\gamma = \gamma$.

By the above characterisation, we instantly get the following corollary.

Corollary 1.49. *In any 2-category a morphism $i\colon I \to A$ of an inserter $\langle I, i\colon I \to A\rangle$ is discrete — i.e. it is representably faithful and conservative, which means that for every object X the functor* $\hom(X, i)$ *is faithful and conservative.*

As mentioned in the previous section, in the presence of finite conical limits, comma objects are constructible from inserters and vice versa.

Corollary 1.50 (Comma objects from inserters and products)**.** *Let $f\colon A \to C$ and $g\colon B \to C$ be morphisms in a 2-category $\mathbb{W}$ with products and inserters. Then the comma object $f{\downarrow}g$ exists and can be constructed as the inserter of $f \circ \pi_A$ with $g \circ \pi_B$, where $\pi_A\colon A \times B \to A$, $\pi_B\colon A \times B \to B$ are the product's projections:*

$$\begin{tikzcd}
f{\downarrow}g \arrow[dr, "i"] \arrow[drr, bend left, "\pi_g=\pi_B\circ i"] \arrow[ddr, bend right, "\pi_f=\pi_A\circ i"'] & & \\
& A \times B \arrow[r, "\pi_B"] \arrow[d, "\pi_A"'] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}$$

Proof. We have to check that the universal properties of Definition 1.25 are satisfied.

First, let us assume, that there is an object X together with morphisms $a\colon X \to A$, $b\colon X \to B$ and a 2-morphism $\sigma\colon f \circ a \to g \circ b$. We construct morphism $h\colon X \to f{\downarrow}g$ as the universal morphism to the inserter corresponding to a 2-morphism $\sigma\colon f \circ \pi_A \circ \langle a, b\rangle \to g \circ \pi_B \circ \langle a, b\rangle$. The equations are trivially satisfied.

Similarly, let us assume that $h, k\colon X \to f{\downarrow}g$ are two parallel morphism and $\alpha\colon \pi_f \circ h \to \pi_f \circ k$, $\beta\colon \pi_g \circ h \to \pi_g \circ k$ are 2-morphisms satisfying: $g \circ$

$\beta \bullet \pi \circ h = \pi \circ k \bullet f \circ \alpha$. We construct 2-morphism $\lambda\colon h \to k$ as the universal 2-morphism to the inserter corresponding to a 2-morphism $\langle \alpha, \beta \rangle$. Also, the equations are trivially satisfied by the universal property of the inserter. $\square$

In fact, comma objects, inserters, tensors and conical limits are all special cases of **Cat**-weighted limits.

Example 1.51 (Tensor as a finite weighted limit)**.** *Let* $\mathbb{W}$ *be a 2-category, and* λ *a category. The cotensor* $\lambda \pitchfork A$ *in the sense of Definition 1.30 is the limit of the functor* $\ulcorner A \urcorner\colon 1 \to \mathbb{W}$*, which chooses object* A*, weighted by the functor* $\ulcorner \lambda \urcorner\colon 1 \to \mathbf{Cat}$*, which chooses category* λ*. Indeed, by the definition of the weighted limit, we have:*

$$\hom_{\mathbb{W}}(X, \mathit{lim}_{\ulcorner \lambda \urcorner}(\ulcorner A \urcorner)) \approx \hom_{\mathbf{Cat}}(\lambda, \hom_{\mathbb{W}}(X, A))$$

Therefore, $\mathit{lim}_{\ulcorner \lambda \urcorner}(\ulcorner A \urcorner) \approx \lambda \pitchfork A$*.*

Example 1.52 (Conical limit as a finite weighted limit)**.** *The classical limit (i.e. a conical limit) of a 2-functor* $F\colon \mathbb{D} \to \mathbb{W}$ *is the same as limit of* F *weighted by the constant functor* $\Delta(1)\colon \mathbb{D} \to \mathbf{Cat}$*. To see this, let us unwind the definition of a weighted limit:*

$$\frac{\frac{\frac{\hom_{\mathbf{Cat}^{\mathbb{D}}}(\Delta(1)(-), \hom_{\mathbb{W}}(X, F(-)))}{\hom_{\mathbf{Cat}}(1, \mathit{lim}\ \hom_{\mathbb{W}}(X, F(-)))}}{\hom_{\mathbf{Cat}}(1, \hom_{\mathbb{W}}(X, \mathit{lim}\ F))}}{\hom_{\mathbb{W}}(X, \mathit{lim}\ F)}$$

where the first isomorphism follows from the usual definition of classical limit in **Cat***, the second follows from Yoneda lemma for 2-categories, and the third from the cartesian closed structure of* **Cat***.*

If $\mathit{Disc}(\mathbb{W})$ gives the canonical notion of discreteness on a finitely (weighted) complete[11] 2-category is $\mathbb{W}$, then with every object $A \in \mathbb{W}$ we may associate a $\mathit{Disc}(\mathbb{W})$-internal category $\mathbb{A}$. Given $A \in \mathbb{W}$ we define the "object of objects" $\mathbb{A}_0$ as $|A|$. Then we shall define the "object of morphisms" $\mathbb{A}_1$ as the

[11] A finitely weighted complete 2-category is a 2-category where every functor from a finite 2-category weighted by finite categories has a limit in the sense of **Cat**-enriched categories (see Definition A.37 in Appendix A.3).

inserter of the following diagram (notice that $|A \times A| \approx |A| \times |A|$ since $|-|$ is right adjoint):

$$\begin{tikzcd} \mathbb{A}_1 \arrow[r, "\langle dom, cod \rangle"] & {|A \times A|} \arrow[r, bend left, "{\pi_0}_{|A|}"] \arrow[r, bend right, "{\pi_1}_{|A|}"'] & {|A|} \arrow[r, "\epsilon_A"] & A \end{tikzcd}$$

together with the "choosing" 2-morphism: $\alpha\colon \epsilon_A \circ dom \to \epsilon_A \circ cod$. We have to show that $\mathbb{A}_1$ is discrete. However this is a straightforward consequence of Corollary 1.49.

Corollary 1.53. *An inserter $\langle I, i\colon I \to A\rangle$ on a discrete object A is a discrete object.*

The internal identity $\eta_{\mathbb{A}}\colon \mathbb{A}_0 \to \mathbb{A}_1$ is given as the unique morphism to the inserter induced by the identity 2-morphism on:

$$\begin{tikzcd} & \mathbb{A}_0 \arrow[dl, dashed, "\eta_{\mathbb{A}}"'] \arrow[d, "|\Delta_A|"] & & \\ \mathbb{A}_1 \arrow[r, "\langle dom, cod \rangle"] & {|A \times A|} \arrow[r, bend left, "{\pi_0}_{|A|}"] \arrow[r, bend right, "{\pi_1}_{|A|}"'] & {|A|} \arrow[r, "\epsilon_A"] & A \end{tikzcd}$$

To define the internal composition, let us first form the pullback:

$$\begin{tikzcd} \mathbb{A}_2 \arrow[r, "p_1"] \arrow[d, "p_2"'] & \mathbb{A}_1 \arrow[d, "dom"] \\ \mathbb{A}_1 \arrow[r, "cod"'] & \mathbb{A}_0 \end{tikzcd}$$

and take the composition $\mu_{\mathbb{A}}\colon \mathbb{A}_2 \to \mathbb{A}_1$ to be the unique morphism to the inserter induced by the 2-morphism:

$$\alpha p_2 \bullet \alpha p_1\colon \epsilon_A \circ dom \circ p_1 \to \epsilon_A \circ cod \circ p_2$$

of the diagram:

$$\begin{tikzcd} & \mathbb{A}_2 \arrow[dl, dashed, "\mu_{\mathbb{A}}"'] \arrow[d, "\langle dom \circ p_1, cod \circ p_2 \rangle"] & & \\ \mathbb{A}_1 \arrow[r, "\langle dom, cod \rangle"] & {|A \times A|} \arrow[r, bend left, "{\pi_0}_{|A|}"] \arrow[r, bend right, "{\pi_1}_{|A|}"'] & {|A|} \arrow[r, "\epsilon_A"] & A \end{tikzcd}$$

Definition 1.54 (Canonically associated category)**.** *Let $\mathbb{W}$ be a finitely complete 2-category with a canonical notion of discreteness. With the notation as above, we define an associated category to an object $A \in \mathbb{W}$ to be the $Disc(\mathbb{W})$-internal category $\mathbb{A} = \langle \mathbb{A}_0, \mathbb{A}_1,\ \mathbb{A}_1 \overset{dom}{\underset{cod}{\rightrightarrows}} \mathbb{A}_0\ , \mathbb{A}_0 \xrightarrow{\eta_{\mathbb{A}}} \mathbb{A}_1, \mathbb{A}_2 \xrightarrow{\mu_{\mathbb{A}}} \mathbb{A}_1 \rangle$.*

Similarly, every morphism $f\colon A \to B$ induces an internal functor $F\colon \mathbb{A} \to \mathbb{B}$, and every 2-morphism $\tau\colon f \to g$ induces an internal natural transformation $\tau\colon F \to G$ between internal functors induced by f and g. This gives a 2-functor $E\colon \mathbb{W} \to \mathbf{cat}(Disc(\mathbb{W}))$. We shall see that $Disc(\mathbb{W})$ is a 2-dense subcategory of $\mathbb{W}$ iff $E\colon \mathbb{W} \to \mathbf{cat}(Disc(\mathbb{W}))$ is a fully faithful embedding. One then instantly gets the following: the 2-functor $E\colon \mathbf{cat}(\mathbb{C}) \to \mathbf{cat}(Disc(\mathbf{cat}(\mathbb{C})))$ is a 2-equivalence of 2-categories for any category $\mathbb{C}$ with pullbacks.

Corollary 1.55. *A category $\mathbb{A}$ internal to a finitely complete category $\mathbb{C}$ is representably posetal iff it is an internal poset.*

Proof. Let X be an object in $\mathbb{C}$. We shall think of X as a discrete $\mathbb{C}$-internal category. An internal functor $f\colon X \to \mathbb{A}$ is tantamount to a single morphism $f\colon X \to \mathbb{A}_0$ in $\mathbb{C}$. An internal natural transformation between such functors $f, g\colon X \to \mathbb{A}$ consists of a morphism $\tau\colon X \to \mathbb{A}_1$ satisfying $\langle dom \circ \tau, cod \circ \tau \rangle = \langle f, g \rangle$. Therefore $\langle dom, cod \rangle$ is mono precisely when over any $\langle f, g \rangle$ there is at most one internal natural transformation. On the other hand if $\langle dom, cod \rangle$ is mono, the condition $\langle dom \circ \tau, cod \circ \tau \rangle = \langle f_0, g_0 \rangle$ ensures that $\hom(\mathbb{X}, \mathbb{A})$ is posetal for any $\mathbb{C}$-internal category $\mathbb{X}$. □

There is also a construction in the other direction $I\colon \mathbf{cat}(Disc(\mathbb{W})) \to \mathbb{W}$, provided $\mathbb{W}$ has enough (weighted) colimits. But first, let us recall the definition of a family fibration from Chapter 7.3 of [Jac99].

Definition 1.56 (Externalisation of a category)**.** *For every category $\mathbb{A}$ internal to a finitely complete category $\mathbb{C}$ one may construct a split indexed category (the externalisation of a category): $fam(\mathbb{A})\colon \mathbb{C}^{op} \to \mathbf{Cat}$ as follows:*

- *$fam(\mathbb{A})(X)$ is the category whose objects are morphisms $X \to \mathbb{A}_0$ in $\mathbb{C}$, whose morphisms from an object $x\colon X \to \mathbb{A}_0$ to an object $y\colon X \to \mathbb{A}_0$ are morphisms $f\colon X \to \mathbb{A}_1$ in $\mathbb{C}$ such that $\langle dom, cod \rangle \circ f = \langle x, y \rangle$ and with the identities and compositions inherited from $\mathbb{A}$*

- *for a morphism $f\colon X \to Y$ the functor $fam(\mathbb{A})(f) = (-) \circ f$ is the post-composition with f.*

Abusing notation a bit, we shall also write $fam(\mathbb{C})\colon \mathbb{C}^{op} \to \mathbf{Cat}$ *for the split indexed category associated to the fundamental indexing over* $\mathbb{C}$ *via the fibred Yoneda lemma (Appendix A.2 Theorem A.23). In more detail, it is defined as:*

$$fam(\mathbb{C})(X) = psfn(\hom_{\mathbb{C}}(-, X), \mathbb{C}^*/(-))$$

where $\mathbb{C}^*/(f\colon X \to Y)\colon \mathbb{C}/Y \to \mathbb{C}/X$ *is the pullback functor along* $f\colon X \to Y$ *(i.e.* $\mathbb{C}^*/(-)$ *is a component-wise right-adjoint to the usual slice functor), and psfn is the "*hom*" of pseudofunctors as defined in Appendix A.2.*

Let $F\colon \mathit{Disc}(\mathbb{W}) \to \mathbb{W}$ be the inclusion from the category of discrete objects. Consider a $\mathit{Disc}(\mathbb{W})$-internal category $\mathbb{A}$ together with its externalisation $fam(\mathbb{A})\colon \mathit{Disc}(\mathbb{W})^{op} \to \mathbf{Cat}$. The corresponding object $I(\mathbb{A}) \in \mathbb{W}$, if it exists, is the colimit of F weighted by $fam(\mathbb{A})$. Therefore, if $\mathbb{W}$ has enough (weighted) colimits then there exists a 2-functor $I\colon \mathbf{cat}(\mathit{Disc}(\mathbb{W})) \to \mathbb{W}$, which is left adjoint to $E\colon \mathbb{W} \to \mathbf{cat}(\mathit{Disc}(\mathbb{W}))$.

Instead of directly proving the above facts, we generalise the construction of an associated category to any notion of discreteness and prove more general theorems. Let us first generalise the construction of family fibration from Definition 1.56. Because, by Theorem A.22 from the Appendix, fibrations are equivalent to indexed categories, we use these concepts interchangeably.

Definition 1.57 (Generalised family fibration)**.** *Let* $F\colon \mathbb{C} \to \mathbb{W}$ *be a functor from a 1-category to a 2-category. Every object* $A \in \mathbb{W}$ *induces a split indexed category:* $\hom(F(-), A)\colon \mathbb{C}^{op} \to \mathbf{Cat}$*, which we shall call "family fibration" and denote by* $fam_F(A)$*.*

Example 1.58 (Canonical family fibration)**.** *Let* $\mathbb{A}$ *be a* $\mathbb{C}$*-internal category. Its externalisation* $fam(\mathbb{A})\colon \mathbb{C}^{op} \to \mathbf{Cat}$ *coincides with the family fibration in the sense of Definition 1.56:* $fam_F(\mathbb{A})\colon \mathit{Disc}(\mathbf{cat}(\mathbb{C}))^{op} \to \mathbf{Cat}$ *where* $\mathit{Disc}(\mathbf{cat}(\mathbb{C})) \approx \mathbb{C}$ *and* $F\colon \mathbb{C} \to \mathbf{Cat}(\mathbb{C})$ *gives the canonical notion of discreteness. More generally, if* $\mathbb{A}$ *is a category relative to a monoidal fibration [GR76] [Prz07] [Shu13], then its externalisation as defined in Chapter 1.5 of [Prz07] also coincides with the family fibration.*

The assignment $A \mapsto fam_F(A)$ extends to a 2-functor: $fam_F\colon \mathbb{W} \to \mathbf{Cat}^{\mathbb{C}^{op}}$ which will be called "the family functor". We shall recall the definitions of a generic object, locally small, and small indexed category [Bor94] [Joh02] [Jac99] [Pho92].

Definition 1.59 (Generic object)**.** *A split indexed category* $\Theta\colon \mathbb{C}^{op} \to \mathbf{Cat}$ *has a generic object* $\Omega \in \mathbb{C}$ *if its underlying discrete indexed category:*

$$\mathbb{C}^{op} \xrightarrow{\Theta} \mathbf{Cat} \xrightarrow{|-|} \mathbf{Set}$$

is represented by:

$$\hom_{\mathbb{C}}(-, \Omega)$$

Definition 1.60 (Locall smallness)**.** *A split indexed category* $\Theta\colon \mathbb{C}^{op} \to \mathbf{Cat}$ *is locally small if for every object* $I \in \mathbb{C}$ *and every pair of objects* $X, Y \in \Theta(I)$ *there exists an object* $\underline{\hom(X,Y)} \in \mathbb{C}$ *together with a morphism* $p\colon \underline{\hom(X,Y)} \to I$*, and a vertical morphism* $\chi\colon \Theta(p)(X) \to \Theta(p)(Y)$ *over* $\underline{\hom(X,Y)}$ *such that for any morphism* $q\colon J \to I \in \mathbb{C}$ *and any vertical morphism* $\beta\colon \Theta(q)(X) \to \Theta(q)(Y)$ *over* J *there exists a unique morphism* $h\colon J \to \underline{\hom(X,Y)}$ *such that* $p \circ h = q$ *and* $\Theta(h)(\chi) = \beta$*.*

Definition 1.61 (Smallness)**.** *A split indexed category is small if it has a generic object and is locally small.*

It is well-known that (split) small categories indexed over a category $\mathbb{C}$ with finite limits are equivalent to $\mathbb{C}$-internal categories (Proposition 7.3.8 in [Jac99]). We show that if $\mathbb{C}$ is a coreflective subcategory of a finitely complete 2-category $\mathbb{W}$, then $\mathbb{C}$-indexed family fibrations of $\mathbb{W}$ are small, thus have associated $\mathbb{C}$-internal categories.

Theorem 1.62. *For every* $A \in \mathbb{W}$ *the indexed category* $fam_F(A)$ *has a generic object iff* $F\colon \mathbb{C} \to \mathbb{W}$ *has a (1-)right adjoint* $U\colon \mathbb{W} \to \mathbb{C}$*.*

Proof. The theorem is almost tautological. The definition of an adjunction says that for every A there is a natural isomorphism: $\hom(F(-), A) \approx \hom(-, U(A))$ between **Set**-valued functors, but this is exactly the definition of a generic object $\Omega = U(A)$. $\square$

Theorem 1.63. *If* $\mathbb{W}$ *has (weighted) finite limits and an adjunction* $\mathbb{W} \underset{F}{\overset{U}{\rightleftarrows}} \mathbb{C}$ *makes* $\mathbb{C}$ *a coreflective subcategory of* $\mathbb{W}$*, then for every object* $A \in \mathbb{W}$ *family fibration* $fam_F(A)$ *is locally small.*

Proof. Let I be an object in $\mathbb{C}$, and $x, y\colon F(I) \to A$ two parallel morphisms. Let us write $\pi\colon \underline{\hom(x,y)} \to F(I)$ for the inserter of x, y, and $\alpha\colon x \circ \pi \to y \circ \pi$

for the inserter's 2-morphism. We shall show that such data mapped by the functor U make $fam_F(A)$ a locally small fibration. Formally, let $p = \eta_I^{-1} \circ U(\pi)$, and $\chi = \alpha \circ \epsilon_{\underline{\hom(x,y)}}$. Observe that χ is really a 2-morphism $x \circ F(p) \to y \circ F(p)$:

$$\begin{tikzcd}
\underline{\hom(x,y)} \arrow[r, "\pi"] & F(I) & \\
FU(\underline{\hom(x,y)}) \arrow[u, "\epsilon_{\underline{\hom(x,y)}}"'] \arrow[r, "FU(\pi)"] & FUF(I) \arrow[u, "\epsilon_{F(I)}"'] \arrow[r, "F(\eta_I^{-1})"] & F(I) \arrow[ul, "id"']
\end{tikzcd}$$

The square commutes by naturality of the counit ϵ, and commutativity of the triangle on the right side follows from triangle equality of the adjunction. We have to show that for any $q\colon J \to I$ and any 2-morphism $\beta\colon x \circ F(q) \to y \circ F(q)$ there exists a unique morphism $h\colon J \to U(\underline{\hom(x,y)})$ such that $p \circ h = q$ and $\chi \circ F(h) = \beta$. By the definition of inserter $\underline{\hom(x,y)}$, we get a morphism $\widehat{h}\colon F(J) \to \underline{\hom(x,y)}$ like on the diagram

$$\begin{tikzcd}
F(J) \arrow[d, dashed, "\widehat{h}"'] \arrow[dr, "F(q)"] & & \\
\underline{\hom(x,y)} \arrow[r, "\pi"] & F(I) \arrow[r, bend left, "x"] \arrow[r, bend right, "y"'] & A
\end{tikzcd}$$

$$x \circ \pi \overset{\alpha}{\Longrightarrow} y \circ \pi$$

$$x \circ F(q) \overset{\beta}{\Longrightarrow} y \circ F(q)$$

which via transposition gives a morphism $h\colon J \to U(\underline{\hom(x,y)})$. The above conditions follows directly from the coreflectivity of $\mathbb{C}$ and the definition of the inserter. We have $p \circ h = \eta_I^{-1} \circ U(\pi) \circ U(\widehat{h}) \circ \eta_J = \eta_I^{-1} \circ UF(q) \circ \eta_J = q$, and $\chi \circ F(h) = \alpha \circ \epsilon_{\underline{\hom(x,y)}} \circ F(h) = \alpha \circ \widehat{h} = \beta$. For the uniqueness, let us assume that $h\colon J \to U(\underline{\hom(x,y)})$ is such that $p \circ h = q$ and $\chi \circ F(h) = \beta$. Since h and $\widehat{h}$ uniquely determines each other, it suffices to show the following $\pi \circ \widehat{h} = \pi \circ \epsilon_{\underline{\hom(x,y)}} \circ F(h) = F(p) \circ F(h) = F(q)$, and $\alpha \circ \widehat{h} = \alpha \circ \epsilon_{\underline{\hom(x,y)}} \circ F(h) = \chi \circ F(h) = \beta$. $\square$

Corollary 1.64. *If $\mathbb{W}$ has finite (weighted) limits and the adjunction* $\mathbb{W} \underset{F}{\overset{U}{\rightleftarrows}} \mathbb{C}$ *makes $\mathbb{C}$ a coreflective subcategory of $\mathbb{W}$, then every indexed category $fam_F(A)$ is small.*

Theorem 1.65 (Representation theorem)**.** *Let $\mathbb{W}$ be a finitely (weighted) complete 2-category, and assume that there is an adjunction* $\mathbb{W} \underset{F}{\overset{U}{\rightleftarrows}} \mathbb{C}$ *making $\mathbb{C}$ a coreflective subcategory of $\mathbb{W}$. With every object $A \in \mathbb{W}$ we may associate, in a canonical way, a $\mathbb{C}$-internal category. Moreover, this assignment makes $\mathbb{W}$ a full (necessarily dense) 2-subcategory of* $\mathbf{cat}(\mathbb{C})$ *iff $\mathbb{C}$ is a dense subcategory of $\mathbb{W}$.*

Proof. Density of $\mathbb{C}$ in $\mathbb{W}$ by definition is equivalent to the condition that the 2-functor $fam_F \colon \mathbb{W} \to \mathbf{Cat}^{\mathbb{C}^{op}}$ is fully faithful. It is then also essentially injective on objects. Therefore, by Corollary 1.64, $\mathbb{W}$ is equivalent to a full subcategory of $\mathbb{C}$-internal categories. □

Example 1.66 (**Cat** with canonical discreteness)**.** *The canonical externalisation of a category $\mathbb{C}$ gives the usual family fibration $fam(\mathbb{C}) \colon \mathbf{Set}^{op} \to \mathbf{Cat}$. This fibration is small precisely when category $\mathbb{C}$ is small. The category associated to $\mathbb{C}$ is (equivalent to) the same category.*

Example 1.67 (**Cat** with 0)**.** *The subcategory of* **Cat** *consisting of a single empty category* 0 *gives a non-dense notion of discreteness on* **Cat***. Since $\mathbb{C}^0 \approx 1$ for any category $\mathbb{C}$, there is only one associated category to every object in* **Cat***.*

Example 1.68 (**Cat** with 1)**.** *The subcategory of* **Cat** *consisting of a terminal category* 1 *does not give a notion of discreteness on* **Cat***, simply because the terminal category functor $1 \to \mathbf{Cat}$ does not have right adjoint. However, the terminal category is a 2-generator in* **Cat***. The family fibration does not loose any information about objects in* **Cat***, but every non-trivial fibration is not small, therefore does not have the associated category.*

We shall write $E \colon \mathbb{W} \to \mathbf{cat}(Disc(\mathbb{W}))$ for the functor from Theorem 1.65 representing an object from $\mathbb{W}$ as an internal category.

Lemma 1.69. *Let $\mathbb{W}$ be a 2-category with a notion of discreteness. The functor $E \colon \mathbb{W} \to \mathbf{cat}(Disc(\mathbb{W}))$ preserves limits and discrete objects.*

Proof. It preserves limits by 2-Yoneda lemma, and discrete objects by the definition of discreteness. □

Theorem 1.70. *Let $\mathbb{W}$ be a finitely (weighted) complete 2-category with a notion of discreteness. If an object $A \in \mathbb{W}$ has internal connectives (internal terminal/initial value, internal (closed) products, coproducts) then its* *associated category $E(A)$ has corresponding connectives in the usual sense. Moreover, if discrete objects are dense, then the converse holds as well.*

Proof. One direction follows from Lemma 1.69 and the fact that 2-functors preserve adjunctions. The other direction follows from the same facts plus Theorem 1.65 saying that $\mathbb{W}$ is a full subcategory of $\mathbf{cat}(Disc(\mathbb{W}))$ provided $Disc(\mathbb{W})$ is dense. $\square$

Corollary 1.71. *Theorem 1.18 from Section 1.1 holds.*

The notion of an associated category allows us to better understand the Beck-Chevalley condition for fibred (co)products [Jac99]. Let us recall that a fibration represented as an indexed category $\Phi\colon \mathbb{C}^{op} \to \mathbf{Cat}$ over a finitely complete category $\mathbb{C}$ has (co)products if for each morphism $s\colon X \to Y$ in $\mathbb{C}$ the functor $\Phi(s)$ has right $\prod_s$ (resp. left $\coprod_s$) adjoint. Furthermore, the (co)products satisfy the Beck-Chevalley condition if for every pullback:

$$\begin{tikzcd} P \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & X \arrow[d, "s"] \\ I \arrow[r, "i"'] & Y \end{tikzcd}$$

the canonical natural transformation $\Phi(i) \circ \prod_s \to \prod_{\pi_2} \circ \pi_1$ (resp. $\coprod_{\pi_2} \circ \pi_1 \to \Phi(i) \circ \coprod_s$) is an isomorphism.

Corollary 1.72. *Let $\mathbb{W}$ be a 2-category with a notion of discreteness. An object $A \in \mathbb{W}$ has polymorphic (co)products iff its family fibration has (co)products along all morphisms. Moreover if $\mathbb{W}$ has finite limits, then these (co)products are stable iff in the family fibration (co)products satisfy the Beck-Chevalley condition.*

The above corollary together with Theorem 1.70 imply that our 2-categorical models for polymorphism externalise to fibrational models [Jac99] [See87]. However, if discrete objects are not dense, we may not rely on the external fibrational semantics. To see this, consider a monoidal-enriched category — its externalisation gives the usual family fibration on the *underlying* category; therefore fibrational semantics discard enrichment and collapse to semantics in an ordinary category.

We can define very many categorical concepts relatively to $\mathbb{W}$ by means of the same concepts internal to the associated categories. Here is one more example.

Example 1.73 (Associated internal two-sided fibrations)**.** *Let us recall from [BCSW83] that if $\mathbb{M}$ is a (possibly weak) 2-category, then a pair of monads $S\colon A \to A$, $T\colon B \to B$ internal to $\mathbb{M}$ induces an ordinary monad $T \circ (-) \circ S$ on the hom category* $\hom_{\mathbb{M}}(A, B)$ *via composition. An S-T-module is an object in the Eilenberg-Moore resolution for monad $T \circ (-) \circ S$ (i.e. it is a $T \circ (-) \circ S$-algebra). Moreover, if $\mathbb{M}$ has local coequalisers, then modules can be composed in the natural way: i.e. if $\alpha\colon S \to T$ is an S-T-module and $\beta\colon T \to R$ is an T-R-module, then the coequaliser of $\eta_T \circ \alpha \circ id_S$ and $id_S \circ \beta \circ \eta_R$ acquires a structure of an S-R-module. These compositions are then associative up to canonical isomorphisms and organize modules into a (weak) 2-category $Mod(\mathbb{M})$.*

Let $\mathbb{C}$ be a category with pullbacks. Then, there exists a weak 2-category $Span(\mathbb{C})$ of spans in $\mathbb{C}$ with the same objects as in $\mathbb{C}$, with 1-morphisms from A to B consisting of pairs of arrows $A \leftarrow X \to B$ in $\mathbb{C}$ with (weak) compositions given by pullbacks, and with 2-morphisms $A \xleftarrow{\gamma_0} X \xrightarrow{\gamma_1} B \Rightarrow A \xleftarrow{\gamma'_0} Y \xrightarrow{\gamma'_1} B$ consisting of a single morphism $f\colon X \to Y$ from $\mathbb{C}$ such that $\langle \gamma'_0 \circ f, \gamma'_1 \circ f\rangle = \langle \gamma_0, \gamma_1\rangle$. A $\mathbb{C}$-internal category is a monad internal to $Span(\mathbb{C})$. An A-B-module between internal categories A, B gives the usual notion of internal profunctor from category A to category B. In more detail, a $\mathbb{C}$-internal profunctor from a $\mathbb{C}$-internal category A to a $\mathbb{C}$-internal category B is given by a span $A \xleftarrow{s} R \xrightarrow{t} B$ together with a morphism of spans $B \circ R \circ A \xrightarrow{\phi} R$ in $Span(\mathbb{C})$ (all squares on the diagram are pullbacks):

$$\begin{tikzcd}
 & & & B \circ R \circ A \arrow[dl] \arrow[dr] \arrow[dd, dashed, "\phi"'] & & & \\
 & & R \circ A \arrow[dl] \arrow[dr] & & B \circ R \arrow[dl] \arrow[dr] & & \\
 & A_1 \arrow[dl, "\delta_0"'] \arrow[dr, "\delta_1"] & & R \arrow[dl, "s"'] \arrow[dr, "t"] & & B_1 \arrow[dl, "\delta_0"'] \arrow[dr, "\delta_1"] & \\
A_0 & & A_0 & & B_0 & & B_0
\end{tikzcd}$$

which is compatible with identities and compositions in A and B.

A two-sided internal fibration from $X \in \mathbb{W}$ to $Y \in \mathbb{W}$ in the sense of Street [Str80] may be defined as a profunctor from the category associated to X to the category associated to Y, under the identity coreflection on $\mathbb{W}$.

Lemma 1.74. *Let $\mathbb{W}$ be a 2-category, and assume that there is an adjunction*

$\mathbb{W} \underset{F}{\overset{U}{\rightleftarrows}} \mathbb{C}$ *making 1-category* $\mathbb{C}$ *a coreflective subcategory of* $\mathbb{W}$*. The 2-functor* $\mathit{fam}_F(-)\colon \mathbb{W} \to \mathbf{Cat}^{\mathbb{C}^{op}}$ *has left adjoint* $L\colon \mathbf{Cat}^{\mathbb{C}^{op}} \to \mathbb{W}$ *expressed as the coend:*

$$L(H) = \int^{X\in\mathbb{C}} H(X) \times F(X)$$

provided $\mathbb{W}$ *is sufficiently cocomplete. Moreover, if* $\mathbb{W}$ *is finitely complete, the above formula induces adjunction* $\mathbb{W} \leftrightarrows \mathbf{cat}(\mathbb{C})$*.*

Proof. Let $H\colon \mathbb{C}^{op} \to \mathbf{Cat}$ be an indexed category, and A an object in $\mathbb{W}$. There are natural 2-isomorphisms:

$$\begin{array}{c}
\hom_{\mathbb{W}}(\int^{X\in\mathbb{C}} H(X) \times F(X), A) \\ \hline\hline
\int_{X\in\mathbb{C}} \hom_{\mathbb{W}}(H(X) \times F(X), A) \\ \hline\hline
\int_{X\in\mathbb{C}} \hom_{\mathbf{Cat}}(H(X), \hom_{\mathbb{W}}(F(X), A)) \\ \hline\hline
\hom_{\mathbf{Cat}}(H, \hom(F(-), A)) \\ \hline\hline
\hom_{\mathbf{Cat}}(H, \mathit{fam}_F(A))
\end{array}$$

where the first isomorphism exists because hom-functors turn colimits into limits, the second is the definition of the tensor with a category, the third is the definition of the object of natural transformation, and the last one is the definition of the family variation. By Theorem 1.65 the above restricts to the adjunction $\mathbb{W} \leftrightarrows \mathbf{cat}(\mathbb{C})$. □

Generally $\mathit{fam}_F(-)\colon \mathbb{W} \to \mathbf{Cat}^{\mathbb{C}^{op}}$ hardly has a left adjoint, because there are no reasons to expect that such big coends exist for all $H\colon \mathbb{C}^{op} \to \mathbf{Cat}$. One may expect, however, a more reasonable characterisation of the left adjoint to the representation functor $E\colon \mathbb{W} \to \mathbf{cat}(\mathbb{C})$.

Consider a functor $K\colon \mathbb{A} \to \mathbb{X}$ between ordinary categories, and let us try to describe it through mappings to category $\mathbb{X}$ from the discrete data associated with $\mathbb{A}$. Functor K induces the mapping of objects $K_0\colon \mathbb{A}_0 \longrightarrow \mathbb{X}_0 \longrightarrow \mathbb{X}$, and the mapping of morphisms K_1. The later can be described as the natural transformation $K_1\colon K_0 \circ \mathit{dom} \to K_0 \circ \mathit{cod}$ like on the following diagram:

$$\begin{tikzcd}
\mathbb{A}_0 \arrow[r, bend left, "dom"{name=U}] \arrow[r, bend right, "cod"'{name=D}] \arrow[from=U, to=D, phantom, "K_1\Downarrow"] & \mathbb{A}_1 \arrow[r, "K_0"] & \mathbb{X}
\end{tikzcd}$$

Not every diagram of the above form induces a functor $\mathbb{A} \to \mathbb{X}$, because a functor has to preserve identities and compositions. To express these extra conditions, one may first extend the diagram by additional functions:

$$\begin{tikzcd}
\mathbb{A}_2 \arrow[r, bend left, "p_2"] \arrow[r, "\mu"] \arrow[r, bend right, "p_1"'] & \mathbb{A}_1 \arrow[r, bend left, "dom"] \arrow[r, bend right, "cod"'] & \mathbb{A}_0 \arrow[l, "\eta"'] \arrow[r, "F_0"] & \mathbb{X}
\end{tikzcd}$$

and impose some constraints. This idea leads to the concept of codescent objects [Str10] [Lac02].

Example 1.75 (Codescent object)**.** *Let Δ be the skeletal full subcategory of* **Cat** *consisting of finite ordinal numbers. We denote by $k = \{0 \leq 1 \leq \ldots \leq k-1\}$ the k-th ordinal number [ML78]. The (2-)truncated simplicial category E is the (non-full) subcategory of Δ generated by the following diagram:*

$$\begin{tikzcd}
1 \arrow[r, bend left, "\delta^1_0"] \arrow[r, bend right, "\delta^1_1"'] & 2 \arrow[l, "!"'] \arrow[r, bend left, "\delta^2_0"] \arrow[r, "\delta^2_1"] \arrow[r, bend right, "\delta^2_2"'] & 3
\end{tikzcd}$$

where $\sigma^k_j \colon k \to k+1$ for $0 \leq j \leq k$ is the unique injective function whose image omits j, and $! \colon 2 \to 1$ is the terminal function. Equivalently, one may describe E as the free category generated by a graph like the above and subject to the following equalities:

- $\delta^2_0 \circ \delta^1_0 = \delta^2_1 \circ \delta^1_0$
- $\delta^2_0 \circ \delta^1_1 = \delta^2_2 \circ \delta^1_0$
- $\delta^2_1 \circ \delta^1_1 = \delta^2_2 \circ \delta^1_1$
- $! \circ \delta^1_0 = id_1 = ! \circ \delta^1_1$

A truncated cosimplicial diagram in a 2-category $\mathbb{W}$ is a functor $F \colon E^{op} \to \mathbb{W}$. A (strict) codescent object of a truncated cosimplicial diagram $F \colon E^{op} \to \mathbb{W}$ is an object $C \in \mathbb{W}$ together with a 2-natural isomorphism:

$$\hom_{\mathbb{W}}(C, -_1) \approx \hom_{\mathbf{Cat}^E}(G(-_2), \hom_{\mathbb{W}}(F(-_2), -_1))$$

*where $G \colon E \to$ **Cat** is the natural embedding $E \to \Delta \to$ **Cat**.*

The above definition says that morphisms from codescent object C to an object X are in an isomorphic correspondence with morphisms of diagrams $G(-) \to \hom_{\mathbb{W}}(F(-), X)$. Generally, internal categories are the most natural examples of codescent objects — a $\mathbb{C}$-internal category $\mathbb{A}$ is the codescent object of its own truncated cosimplicial diagram [Str10] [Str04]:

$$\begin{tikzcd} \mathbb{A}_2 \arrow[r, bend left, "p_2"] \arrow[r, "\mu" description] \arrow[r, bend right, "p_1"'] & \mathbb{A}_1 \arrow[r, bend left, "dom"] \arrow[r, bend right, "cod"'] & \mathbb{A}_0 \arrow[l, "\eta" description] \arrow[r, dashed, "c"] & \mathbb{A} \end{tikzcd}$$

where $\mathbb{A}_0, \mathbb{A}_1, \mathbb{A}_2$ are the objects of: objects, morphisms, and composable pairs of morphisms respectively, dom, cod, p_1, p_2 are projections, μ is the internal composition and η assigns internal identities to objects, as defined earlier in this section.

In fact, codescent objects are colimits weighted by the weight (finite) $E \to \mathbf{Cat}$. Observe that for an object $K \in \mathbb{C}$ thought of as a discrete internal category, $L(K) = \int^{X \in \mathbb{C}} \hom_{\mathbf{cat}(\mathbb{C})}(X, K) \times F(X) = \int^{X \in \mathbb{C}} \hom_{\mathbb{C}}(X, K) \times F(X) = F(K)$, where the first equality is the definition of L, the second follows from discreteness of K and the third is the Yoneda reduction. Because weighted colimits are preserved by left adjoint functors, $L(\mathbb{A})$ has to be the codescent object of the diagram:

$$\begin{tikzcd} F(\mathbb{A}_2) \arrow[r, bend left, "F(p_2)"] \arrow[r, "F(\mu)" description] \arrow[r, bend right, "F(p_1)"'] & F(\mathbb{A}_1) \arrow[r, bend left, "F(dom)"] \arrow[r, bend right, "F(cod)"'] & F(\mathbb{A}_0) \arrow[l, "F(\eta)" description] \arrow[r, dashed, "c"] & L(\mathbb{A}) \end{tikzcd}$$

Therefore if $\mathbb{W}$ has enough *finite* weighted colimits then the representation functor has left adjoint.

Theorem 1.76 (Reflective representation)**.** *Let $\mathbb{W}$ be a 2-category, and assume that there is an adjunction $\mathbb{W} \underset{F}{\overset{U}{\rightleftarrows}} \mathbb{C}$ making 1-category $\mathbb{C}$ a coreflective subcategory of $\mathbb{W}$. The representation 2-functor $E \colon \mathbb{W} \to \mathbf{cat}(\mathbb{C})$ has the left adjoint iff $\mathbb{W}$ has codescent objects of truncated cosimplicial diagrams formed in $\mathbb{C}$.*

Corollary 1.77. *Let $\mathbb{W}$ be a finitely complete 2-category with a dense notion of discreteness. Furthermore, assume that $\mathbb{W}$ has codescent objects of discrete truncated cosimplicial diagrams. Then the representation 2-functor $E \colon \mathbb{W} \to \mathbf{cat}(Disc(\mathbb{W}))$ makes $\mathbb{W}$ a dense reflective 2-subcategory of $\mathbf{cat}(Disc(\mathbb{W}))$.*

Chapter 2

Free semantics

A well-known Mac Lane's slogan says "adjunctions arise everywhere". One may find adjunctions in variety of concepts from theoretical computer science: definitions of Galois correspondence between syntax and semantics together with Dedekind-MacNeille completion as the fixed-point of the adjunction, power objects, function spaces (in the form of lambda abstraction), logical connectives and quantifications, to classical mathematics: free structures, definitions of tensor products, distributions, and many more. In some cases, however, the definition of an adjunction is too restrictive. We have already observed in Example 1.23 that a category may not be closed in itself, but in a larger embedding category. Actually, the situation from the example was quite simple, because objects from the subcategory were exponentiable with respect to *all* objects in the embedding category. Here is a less trivial example.

Example 2.1 (Partial recursive functions)**.** *Let us consider a category consisting of two objects — the set of natural numbers* $\mathbb{N}$*, together with partial recursive functions, and a singleton* 1*, with all singleton maps* $1 \to \mathbb{N}$*. This category is not cartesian — a pairing cannot be partial recursive — was it, one could decide the halting problem*[1]*. In more detail, let us assume that the product* $\mathbb{N} \times \mathbb{N}$ *exists with projections* $\pi_1, \pi_2 \colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$*. Consider a partial recursive function* f *and the constant function* 0 *that assigns every natural*

[1] If the pairing is effective then it has to be ambiguous.

number to zero:

$$\begin{tikzcd} & \mathbb{N} \arrow[dl, "f"'] \arrow[d, dashed, "\langle f,0\rangle"] \arrow[dr, "0"] & \\ \mathbb{N} & \mathbb{N}\times\mathbb{N} \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & \mathbb{N} \end{tikzcd}$$

Because 0 *is a total function,* $\langle f, 0\rangle$ *has to be total as well. Consider natural numbers* $n, m \notin dom(f)$ *that do not belong to the domain of* f*. I claim that:*

$$\langle f, 0\rangle(n) = \langle f, 0\rangle(m)$$

because, otherwise, one could swap these values obtaining another pairing function, which would contradict its uniqueness. Since being partial recursive is invariant under incrementation, without loss of generality we may assume that f *is not surjective. Let* $k \notin dom(f)$ *be any natural number that does not belong to the domain of* f*. There is a recursive function (where "halt" does not belong to the image of* f*):*

$$t_f(x) = \begin{cases} halt & \text{if } \langle f, 0\rangle(x) = \langle f, 0\rangle(k) \\ f(x) & \text{otherwise} \end{cases}$$

Observe that $f(x) = t_f(x)$ *whenever* $x \in dom(f)$*. This leads to the contradiction, since not every partial recursive function* f *can be extended to a total function.*

On the other hand, this category is "cartesian" in the category of Π^0_2 *functions in the sense of arithmetic hierarchy. Similarly, the category of* Π^0_2 *functions is not cartesian in itself. In fact, a simple diagonalisation argument shows that none of* $\Sigma^m_n, \Pi^m_n, \Delta^m_n$ *is cartesian.*

Such situations frequently occur when a construction over an object is of a poorer quality than the original object. Here is our driving example

Example 2.2 (Russell paradox)**.** *In a ZFC set theory*[2] *there can be no set* $\mathcal{U}$ *universal for all sets — i.e. there is no set* $\mathcal{U}$ *such that every set is isomorphic to exactly one element of* $\mathcal{U}$*. However, there exists a (necessarily proper*[3]*) family of sets* $\mathcal{U}_0 \subseteq \mathcal{U}_1 \subseteq \mathcal{U}_2 \subseteq \cdots$ *that is collectively universal, which means that for every set* A *there exists* $\mathcal{U}_k$ *and exactly one* $X \in \mathcal{U}_k$ *with* $A \approx X$*. An example of such a family is the ascending chain of all cardinal numbers.*

[2] The result generalises to any higher-order type theory [BF97].

[3] Otherwise, by the axiom of union we could form $A_0 = \bigcup_k \bigcup \mathcal{U}_k$, and $A = P(A_0)$ would not be classified by any $\mathcal{U}_k$.

Intuitively, we should think of universes as 2-dimensional analogue of the internal truth-values object Ω in a topos — just like Ω classifies *internal logic* of a category, a universal object tries to classify the *external logic*. The attempt to classify the full external logic is, however, futile, as stated in the above example. Therefore, we have to focus on a classification of some parts of the external logic.

In this chapter we set forth categorical foundations for "2-powers", which shall generalise partial classifiers of external logics for objects in a 2-category. We show how internal logical systems in any 2-category with 2-powers carry free semantics on objects. We propose a notion of a Yoneda (bi)triangle and use it to generalise to the 2-categorical setting the famous construction of convolution introduced by Brian Day in his PhD dissertation [Day70] for enriched categories. As a complementary result we prove the convolution theorem for internal categories.

2.1 Categorical 2-powers

To better understand our definition of "2-powers", let us first recall how one may define ordinary powers. With every (well-powered) regular category[4] $\mathbb{C}$ there is associated a 2-poset (i.e. poset-enriched category[5]) of its internal relations $\mathbf{Rel}(\mathbb{C})$ together with a bijective-on-objects functor $J\colon \mathbb{C} \to \mathbf{Rel}(\mathbb{C})$. Furthermore, the right adjoint of J, if it exists, $P\colon \mathbb{C} \to \mathbf{Rel}(\mathbb{C})$ induces the natural isomorphism [Fe90]:

$$\frac{\hom_{\mathbf{Rel}(\mathbb{C})}(J(A), B)}{\hom_{\mathbb{C}}(A, P(B))}$$

If additionally $\mathbb{C}$ has a terminal object 1 then, recalling the definition of an internal relation (Example 2.3.8 in [Jac99]), one gets:

$$\begin{array}{c} sub(\mathbb{C})(A) \\ \hline\hline \hom_{\mathbf{Rel}(\mathbb{C})}(J(A), 1) \\ \hline\hline \hom_{\mathbb{C}}(A, P(1)) \end{array}$$

[4]A category is called regular (Chapter 2, Volume 2 of [Bor94], Chapter A1.3 of [Joh02], Chapter 4, Section 4 of [Jac99], (Chapter 1.5 of [Fe90]) if it has finite limits, regular epimorphisms are stable under pullbacks, and every morphism factors as a regular epimorphism followed by a monomorphism.

[5]Poset-enriched category, also called 2-poset, is a category enriched in the category of posets.

which makes $\mathbb{C}$ a topos with power functor P and the subobject classifier $\Omega = P(1)$ (see: Chapter 5 of [Jac99]). All of the above may be abstractly characterised by starting with a regular fibration $p\colon \mathbb{E} \to \mathbb{C}$ on a finitely complete category $\mathbb{C}$, then constructing the category of p-internal relations $\mathbf{Rel}(p)$ and a bijective-on-objects functor $J\colon \mathbb{C} \to \mathbf{Rel}(p)$ (see: Chapter 4 of [Jac99]). We shall recover the classical situation by taking for p the usual subobject fibration. Now we would like to argue that the right notion of the category of relations over $\mathbb{C}$ is encapsulated by *any* bijective-on-objects functor $J\colon \mathbb{C} \to \mathbb{D}$, where $\mathbb{D}$ is a 2-poset. Here are some intuitions. Let us recall [JH06] that any such bijective-on-objects functor corresponds to a poset-enriched module monad ($\mathbf{Pos}$ is the category of partial ordered sets and pointwise-ordered monotonic functions):

$$\hom_{\mathbb{D}}(J(-_1), J(-_2))\colon \mathbb{C}^{op} \times \mathbb{C} \to \mathbf{Pos}$$

with unit:

$$\eta\colon\ \hom_{\mathbb{C}}(-_1, -_2) \to \hom_{\mathbb{D}}(J(-_1), J(-_2))$$

given by the action of J, and multiplication:

$$\mu\colon \int^{B\in\mathbb{C}} \hom_{\mathbb{D}}(J(-_1), J(B)) \times \hom_{\mathbb{D}}(J(B), J(-_2)) \to \hom_{\mathbb{D}}(J(-_1), J(-_2))$$

induced by the composition from $\mathbb{D}$. This monad gives a "fibred span" (i.e. a span where one leg is a fibration and the other is an opfibration) $\mathbb{C} \overset{\pi_1}{\leftarrow} \int \hom_{\mathbb{D}}(J(-_1), J(-_2)) \overset{\pi_2}{\to} \mathbb{C}$ with a monoidal action induced by a generalised Grothendieck construction — the total category is defined as the following coend:

$$\int \hom_{\mathbb{D}}(J(-_1), J(-_2)) = \int^{X,Y\in\mathbb{C}} \mathbb{C}/X \times \hom_{\mathbb{D}}(J(X), J(Y)) \times Y/\mathbb{C}$$

where:

$$\mathbb{C}/(-)\colon \mathbb{C} \to \mathbf{Cat}$$
$$(-)/\mathbb{C}\colon \mathbb{C}^{op} \to \mathbf{Cat}$$

are the slice an coslice functors, and π_1, π_2 are the obvious projections. If we assume that $\mathbb{C}$ has a terminal object 1, then:

$$\hom_{\mathbb{D}}(J(1), J(-))\colon \mathbb{C} \to \mathbf{Pos}$$

by Grothendieck construction corresponds to an opfibration:

$$\pi_{\hom_{\mathbb{D}}(J(1),J(-))} \colon \int \hom_{\mathbb{D}}(J(1), J(-)) \to \mathbb{C}$$

and:

$$\hom_{\mathbb{D}}(J(-), J(1)) \colon \mathbb{C}^{op} \to \mathbf{Pos}$$

corresponds to a fibration:

$$\pi_{\hom_{\mathbb{D}}(J(-),J(1))} \colon \int \hom_{\mathbb{D}}(J(-), J(1)) \to \mathbb{C}$$

In case $\mathbb{D} = \mathbf{Rel}(p)$ these two functors are "essentially the same" and encode the fibration $p \colon \mathbb{E} \to \mathbb{C}$; one may check that our fibred span arises by pulling back $p \colon \mathbb{E} \to \mathbb{C}$ along the Cartesian product functor $\times \colon \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ to obtain a bifibration $rel(p) \colon \mathbf{Rel}(p) \to \mathbb{C} \times \mathbb{C}$ and postcomposing it with two projections. For this reason, the functor $J \colon \mathbb{C} \to \mathbb{D}$ does not lose any information about the regular logic associated to $\mathbb{C}$. Second, we do believe that a more natural setting for relations is a fibred span than a bifibration — this allows us to distinguish between relations $A \nrightarrow B$ from relations $B \nrightarrow A$ and generalise the construction to higher categories. For example, as suggested by Jean Benabou, the role of relations between categories should be played by profunctors. For any complete and cocomplete symmetric monoidal closed category $\mathbb{C}$, we may define a 2-profunctor:

$$\hom_{\mathbf{Dist}(\mathbb{C})} \colon \mathbf{Cat}(\mathbb{C})^{op} \times \mathbf{Cat}(\mathbb{C}) \to \mathbf{Cat}$$

sending $\mathbb{C}$-enriched categories A, B to the category of $\mathbb{C}$-enriched profunctors $A \nrightarrow B$ and $\mathbb{C}$-enriched natural transformations. Because:

$$\hom_{\mathbf{Dist}(\mathbb{C})}(A, B) \not\approx \hom_{\mathbf{Dist}(\mathbb{C})}(B, A)$$

the 2-profunctor $\hom_{\mathbf{Dist}(\mathbb{C})}$ is not induced by any (co)indexed 2-category.

Example 2.3 (Allegory)**.** *Another way to look at these concepts is through the notion of an* allegory *[Fe90]. An allegory is a pair* $\langle \mathbb{A}, (-)^\star \colon \mathbb{A} \to \mathbb{A}^{op} \rangle$, *where* $\mathbb{A}$ *is a poset-enriched category (2-poset),* $(-)^\star \colon \mathbb{A} \to \mathbb{A}^{op}$ *is an identity-on-objects duality involution, and:*

- *for each* $A, B \in \mathbb{A}$ *the poset* $\hom(A, B)$ *has binary conjunctions*

- *for each triple of morphisms* $\begin{tikzcd} A \arrow[r, "f"'] \arrow[rr, bend left, "h"] & B \arrow[r, "g"'] & C \end{tikzcd}$ *the following holds:*

$$(g \circ f) \wedge h \leq g \circ (g^\star \circ h \wedge f)$$

Every allegory $\mathbb{A}$ induces a bijective on objects embedding $J \colon \mathbb{C} \to \mathbb{A}$ by forming a subcategory $\mathbb{C}$ consisting of morphisms that have right adjoints. Moreover, if $\mathbb{A}$ is a tabular allegory[6]*, then $\mathbf{Rel}(\mathbb{C}) \approx \mathbb{A}$ and $\mathbb{C}$ is (locally) regular (2.148 in [Fe90]; moreover, the converse is true by 2.132 in the same book).*

As mentioned earlier, a (locally small) category with finite limits has power objects iff it is regular and the induced functor $J \colon \mathbb{C} \to \mathbf{Rel}(\mathbb{C})$ has right adjoint. It is natural then to provide the following generalisation of a power functor. If $J \colon \mathbb{C} \to \mathbb{D}$ is a bijective on objects functor, then we say that $P(B) \in \mathbb{C}$ is a J-power of $B \in \mathbb{D}$ if there is a representation:

$$\hom_{\mathbb{D}}(J(-), B) \approx \hom_{\mathbb{C}}(-, P(B))$$

If a representation $P(B)$ exists for every $B \in \mathbb{D}$, i.e. J has the right adjoint $P \colon \mathbb{D} \to \mathbb{C}$, we say that $\mathbb{C}$ has J-powers.

Example 2.4 (Topos)**.** *Let $\mathbb{C}$ be a (locally small) regular category, and $J \colon \mathbb{C} \to \mathbf{Rel}(\mathbb{C})$ its inclusion functor into the category of relations. $\mathbb{C}$ is a topos iff it has J-powers.*

Example 2.5 (Quasitopos)**.** *Let $\mathbb{C}$ be a finitely complete and cocomplete locally cartesian closed category, such that its fibration of regular subobjects*[7] *is regular*[8]*, and $J \colon \mathbb{C} \to \mathbf{RegRel}(\mathbb{C})$ its inclusion functor into the category of regular relations. $\mathbb{C}$ is a quasitopos iff it has J-powers.*

Example 2.6 (Regular fibration)**.** *More generally, let $p \colon \mathbb{E} \to \mathbb{C}$ be a regular fibration on a finitely complete category $\mathbb{C}$ (Definition 4.2.1 in [Jac99]). If $J \colon \mathbb{C} \to \mathbf{Rel}(p)$ has a right adjoint, then $p \colon \mathbb{E} \to \mathbb{C}$ has a generic object. The converse is true provided that $\mathbb{C}$ is cartesian closed.*

---

[6] An allegory is tabular if every morphism h admits a decomposition $h = g^\star \circ f$ such that $f^\star \circ f \wedge g^\star \circ g = id$.

[7] A regular subobject of A is a (equivalence class of) regular monomorphism with codomain A. A regular monomorphism is a monomorphism that arises as an equaliser.

[8] See Chapter 4, Section 4 of [Jac99]

Still, as exposed in the introduction to this chapter, such definition is too strong to embrace many interesting examples. Here is another one. Let **cat** be the 2-category of small categories, and **dist** the 2-category of profunctors, with the usual bijective on objects embedding $J\colon \mathbf{cat} \to \mathbf{dist}$ defined on functors:

$$J(F) = \hom_{\mathbf{dist}}(-_1, F(-_2))$$

Then **cat** does not have J-powers due to the size issues — profunctors $\mathbb{A} \nrightarrow \mathbb{B}$ correspond to functors $\mathbb{A} \to \mathbf{Set}^{\mathbb{B}^{op}}$, but the category $\mathbf{Set}^{\mathbb{B}^{op}}$ usually is not small, nor even equivalent to a small one. Unfortunately, these size issues are fundamental — there is no sensible restriction on the sizes of objects and morphism to make **cat** admit J-powers. However, *some* of the profunctors *are* classified in such a way. These observations lead us to the concept of a Yoneda triangle.

Definition 2.7 (Yoneda triangle)**.** *Let $\mathbb{W}$ be a 2-category. A Yoneda triangle in $\mathbb{W}$, written $\eta\colon y \triangleright \langle f, g\rangle$, consists of three morphisms $y\colon A \to \overline{A}$, $f\colon A \to B$ and $g\colon B \to \overline{A}$ together with a 2-morphism $\eta\colon y \to g \circ f$ which exhibits g as a pointwise left Kan extension of y along f, and exhibits f as an absolute left Kan lifting*[9] *of y along g:*

$$\begin{tikzcd}
A \arrow[r, "y"] \arrow[d, "f=Lift_g(y)"'] \arrow[dr, phantom, "\swArrow\eta"] & \overline{A} \\
B \arrow[ur, "g=Lan_f(y)"'] &
\end{tikzcd}$$

The absoluteness of a Kan lifting means that the lifting is preserved by any morphism $k\colon K \to A$ — i.e. the 2-morphism $\eta \circ k$ exhibits $Lift_g(y) \circ k$ as $Lift_g(y \circ k)$.

The idea of a Yoneda triangle is that, we have a morphism $y\colon A \to \overline{A}$ which plays the role of a "defective identity" and for a given morphism $f\colon A \to B$ we try to characterise its right adjoint up to the "defective identity" y.

[9] The concept of a Kan lifting is the *opposite* of the concept of Kan extension — i.e. a Kan lifting in $\mathbb{W}$ is a Kan extension in $\mathbb{W}^{op}$.

Example 2.8 (Adjunction as Yoneda triangle)**.** *A 1-morphism $f\colon A \to B$ in a 2-category $\mathbb{W}$ has a right adjoint $g\colon B \to A$ with unit $\eta\colon id \to g \circ f$ precisely when $\eta\colon id \triangleright \langle f, g\rangle$ is a Yoneda triangle:*

$$\begin{tikzcd}
B \arrow[d, "g"'] \arrow[dr, "g"] \arrow[dd, bend right=60, "id"'] & \\
A \arrow[r, "id"] \arrow[d, "f=Lift_g(id)"'] \arrow[l, phantom, "\swArrow\epsilon"] \arrow[dr, phantom, "\swArrow\eta"] & A \\
B \arrow[ur, "g=Lan_f(id)"'] &
\end{tikzcd}$$

Since $f = Lift_g(id)$ is an absolute lifting, $f \circ g$ is a lifting of g through g with $\eta \circ g\colon g \to g \circ f \circ g$. By the universal property of the lifting, there is a unique 2-morphism $\epsilon\colon f \circ g \to id$ such that $g \circ \epsilon \bullet \eta \circ g = id$, which may be defined as the counit of the adjunction. We have to show that also the other triangle equality holds. Let us first postcompose the equation $g \circ \epsilon \bullet \eta \circ g = id$ with f to obtain $g \circ \epsilon \circ f \bullet \eta \circ g \circ f = id$. Then postcompose it with η to get $g \circ \epsilon \circ f \bullet \eta \circ g \circ f \bullet \eta = \eta$. But this equation, under bijection provided by $Lift_g(id)$, corresponds to the equation $\epsilon \circ f \bullet f \circ \eta = id$, which is the required triangle equality.

On the other hand, let us assume that f is left adjoint to g with unit $\eta\colon id \to g \circ f$ and counit $\epsilon\colon f \circ g \to id$. We shall see that for every $k\colon C \to A$ the composite $\eta \circ k$ exhibits $f \circ k$ as the left Kan lifting of k along g:

$$\begin{tikzcd}
C \arrow[r, "k"] \arrow[d, "f\circ k"] \arrow[d, bend right=60, "h"', phantom, "\swArrow\alpha"] \arrow[dr, phantom, "\swArrow\eta\circ k"] \arrow[d, bend right=60, "h"'] & A \\
B \arrow[ur, "g"'] &
\end{tikzcd}$$

We have to show that the assignments:

$$f \circ k \xrightarrow{\alpha} h \mapsto k \xrightarrow{g\circ\alpha\bullet\eta\circ k} g \circ h$$

and:

$$k \xrightarrow{\beta} g \circ h \mapsto f \circ k \xrightarrow{\epsilon\circ h\bullet f\circ\beta} h$$

are inverse of each other. Let us check the first composition:

$$
\begin{aligned}
\epsilon \circ h \bullet f \circ (g \circ \alpha \bullet \eta \circ k) &= \epsilon \circ h \bullet f \circ g \circ \alpha \bullet f \circ \eta \circ k \\
&= \alpha \bullet \epsilon \circ f \circ k \bullet f \circ \eta \circ k \\
&= \alpha
\end{aligned}
$$

where the first and second equalities follow from the interchange law of a 2-category, and the last one follows by the triangle equation. Similarly we may check the second composition:

$$
\begin{aligned}
g \circ (\epsilon \circ h \bullet f \circ \beta) \bullet \eta \circ k &= g \circ \epsilon \circ h \bullet g \circ f \circ \beta \bullet \eta \circ k \\
&= g \circ \epsilon \circ h \bullet \eta \circ g \circ h \bullet \beta \\
&= \beta
\end{aligned}
$$

The fact that g is a pointwise left extension of id along f follows from a more general observation that a left Kan extension along a left adjoint always exists and is pointwise (Proposition 20 in [Str74]). However, it is illustrative to see how the bijections defining Kan extensions are constructed in our particular case. Let us extend the diagram of adjunction $f \dashv g$ by generalised elements $a_X \in A$, $b_X \in B$:

$$
\begin{tikzcd}
X \arrow[dr, dashed, "\exists! h"] \arrow[drr, bend left, "g(b_X)"] \arrow[ddr, bend right, "id"'] \arrow[drr, phantom, "\Swarrow\epsilon_b"] & & & \\
& f{\downarrow}b \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, "\Swarrow\pi"] & A \arrow[r, "id"] \arrow[d, "f"] \arrow[dr, phantom, "\Swarrow\eta"] & A \\
& X \arrow[r, "b_X"] \arrow[urr, bend right=60, "a_X"'] & B \arrow[ur, "g"'] \arrow[r, phantom, "\Swarrow\beta"] &
\end{tikzcd}
$$

where $h\colon X \to f{\downarrow}b$ is the unique morphism to the comma object induced by the counit $\epsilon_b\colon f(g(b_X)) \to b_X$. Then, one part of the bijective correspondence is given by assigning to $\beta\colon g(b_X) \to a_X$ an arrow $\beta \circ \pi_2 \bullet g \circ \pi \bullet \eta \circ \pi_1 \colon \pi_1 \to a_X \circ \pi_2$, and the other is given by composition with h.

Generally, a Yoneda-like triangle $\eta\colon y \triangleright \langle f, g\rangle$ where g is not assumed to be the left Kan extension of y along f is called an adjunction relative to y [Ulm68]. Note however, that in such a case g need not be uniquely determined by f.

The term "Yoneda triangle" has been chosen to suggest a very important source of examples — namely, Yoneda structures as defined in [Web07] (Definition 3.1 there). This source is explained in Example 2.11 and continued in the discussion following the example.

To establish a more direct connection between Yoneda triangles and Yoneda structures, let us introduce one more concept.

Example 2.9 (Strong internal density)**.** *Let $\mathbb{W}$ be a finitely complete 2-category. Definition 1.34 from Chapter 1 internalises the concept of density (co)product and, in particular, the concept of a dense morphism. That is, a morphism $y\colon A \to \overline{A}$ from $\mathbb{W}$ is dense if the identity on $\overline{A}$ is a pointwise left Kan extension of y along itself. As mentioned there, this definition agrees with the usual definition of density of a functor between ordinary categories, but in general 2-categories, may be too weak.*

One may use Yoneda triangles to give a stronger definition. Recall from Example 2.14 that if a functor $Y\colon \mathbb{A} \to \overline{\mathbb{A}}$ is dense, then any functor $F\colon \mathbb{A} \to \mathbb{B}$ that is an absolute left Kan lifting of $Y\colon \mathbb{A} \to \overline{\mathbb{A}}$ along $G\colon \mathbb{B} \to \overline{\mathbb{A}}$ makes automatically G a pointwise left Kan extension of Y along F. By internalising the above property, we get the following definition of "strong internal density". A morphism $y\colon A \to \overline{A}$ from $\mathbb{W}$ is strongly dense if every triangle like on the below diagram:

$$\begin{tikzcd} A \arrow[r, "y"] \arrow[d, "f"'] \arrow[dr, phantom, "\swarrow\eta"] & \overline{A} \\ B \arrow[ur, "g"'] & \end{tikzcd}$$

where f is a y-relative right adjoint to g (i.e. f is an absolute left Kan lifting of y along g) forms a Yoneda triangle. Because any morphism $y\colon A \to \overline{A}$ is always an absolute left Kan lifting of itself along the identity on $\overline{A}$, the condition of "strong internal density" implies the ordinary condition of "internal density".

A similar idea lead M. Weber in [Web07] to the notion of "good Yoneda structure". A good Yoneda structure in a 2-category $\mathbb{W}$ consists of:

- a right ideal of morphisms in $\mathbb{W}$, which are called "admissible morphisms"

- for each admissible identity $id\colon A \to A$ in $\mathbb{W}$, a morphism $y\colon A \to \overline{A}$, which is thought of as internal Yoneda embedding of an object A into "object of internal presheaves" $\overline{A}$

These data have to satisfy some properties to make it possible to work with "internal Yoneda embeddings" like with ordinary Yoneda embeddings for categories. It turned out, that for many purposes it suffices to ensure that "object of internal presheaves" are sufficiently (co)complete and that internal Yoneda embeddings satisfy the "strong density condition" as defined in Example 2.9, but restricted to "admissible morphisms" only.

Just like in the previous chapter we provided an elementary description of pointwise Kan extensions, we shall now give a similar characterisation of absolute Kan liftings. Let us extend the diagram of a Yoneda triangle $\eta\colon y \triangleright \langle f, g\rangle$, by taking generalised elements $a_X \in A, b_X \in B$ and a generalised arrow $f(a_X) \xrightarrow{k} b_X$:

$$
\begin{tikzcd}
& X \arrow[d, "a_X"] \arrow[dr, "y(a_X)"] \arrow[dd, bend right=60, "b_X"'] & \\
{} \arrow[r, phantom, "\swarrow k"] & A \arrow[r, "y"] \arrow[d, "f"] \arrow[dr, phantom, "\swarrow \eta"] & \overline{A} \\
& B \arrow[ur, "g"'] &
\end{tikzcd}
$$

The absoluteness of a left Kan lifting says that there is a bijective correspondence:

$$
\begin{array}{ccc}
f(a_X) & \overset{k}{\to} & b_X \\
\hline\hline
y(a_X) & \overset{\eta_a \circ k}{\to} & g(b_X)
\end{array}
$$

which clearly resembles the usual hom-definition of adjunction on generalised elements. Moreover, using the formula for pointwise left Kan extension, we may write:

$$
g(b_X) = \coprod_{f(a_X) \to b_X} y(a_X)
$$

which in case of **Cat** may be interpreted as the colimit of y taken over comma category $f{\downarrow}\Delta_{b_X}$. In particular, we have the following characterisation of Yoneda triangles in **Cat**.

Example 2.10 (Yoneda triangles in **Cat**)**.** *If we take $\mathbb{W}$ to be the 2-category* **Cat** *of locally small categories, functors and natural transformations, then the condition that G is a pointwise left Kan extension of $Y\colon \mathbb{A} \to \overline{\mathbb{A}}$ along $F\colon \mathbb{A} \to \mathbb{B}$ reduces to:*

$$G(-) = \int^{A\in\mathbb{A}} \hom_{\mathbb{B}}(F(A), -) \times Y(A)$$

In case category $\overline{\mathbb{A}}$ is not tensored over **Set** *the above coend has to be interpreted as the colimit of Y weighted by:*

$$\hom_{\mathbb{B}}(F(-_1), -_2)$$

The condition that F is an absolute left Kan lifting of Y along G reduces to:

$$\hom_{\mathbb{B}}(F(-_1), -_2) \approx \hom_{\overline{\mathbb{A}}}(Y(-_1), G(-_2))$$

Furthermore, if Y is dense, then G is automatically a pointwise Kan extension in a canonical way — from density we have:

$$G(-) \approx \int^{A\in\mathbb{A}} \hom(Y(A), G(-)) \times Y(A)$$

and using the formula for an absolute lifting:

$$G(-) \approx \int^{A\in\mathbb{A}} \hom(F(A), -) \times Y(A)$$

This example needs more elaboration. In the literature, there exist two essentially different notions of pointwise Kan extensions. The older, provided by Eduardo Dubuc [Dub70] for enriched categories, defines pointwise Kan extensions as appropriate enriched (co)ends:

$$\begin{aligned} Ran_F(Y) &= \int_{A\in\mathbb{A}} Y(A)^{\hom_{\mathbb{B}}(-,F(C))} \\ Lan_F(Y) &= \int^{A\in\mathbb{A}} \hom_{\mathbb{B}}(F(C), -) \otimes Y(A) \end{aligned}$$

The newer, provided by Ross Street [Str74], works in the general context of (sufficiently complete) 2-categories, and has been used in the dissertation up to this point. As mentioned in [Str74] [Kel82] these definitions agree for categories enriched in **Set**, and for categories enriched in the 2-valued Boolean algebra 2, but Street's definition is stronger than Dubuc's one for general enriched categories (it is strictly stronger for categories enriched in abelian groups **Ab**, and for categories enriched in **Cat**). Steve Lack [Lac10] blamed for this mismatch the definition of a category of $\mathbb{C}$-enriched categories, which "can't see" the extra structure of a $\mathbb{C}$-enriched category on functor categories $\hom(A, B)$. Although it is certainly true that the category $\mathbf{Cat}(\mathbb{C})$ of $\mathbb{C}$-enriched categories is more than a 2-category — after all, it is a $\mathbf{Cat}(\mathbb{C})$-enriched category with an underlying 2-category — the reasoning is not correct. Technically, the reasoning cannot be right, because treating a 2-category as a $\mathbf{Cat}(\mathbb{C})$-enriched category and carrying to this setting Street's definition of pointwise Kan extension may only strengthen the concept of a Kan extension, which is, actually, in its ordinary 2-categorical form, stronger than Dubuc's one. More importantly, also philosophically the reasoning cannot be right — the enrichment of $\mathbf{Cat}(\mathbb{C})$ in $\mathbf{Cat}(\mathbb{C})$ is a self-enrichment, which means that it is completely recoverable from its underlying 2-category; the idea behind Street's pointwise Kan extensions was to define the Kan extension at "every generalised 2-point" just to evade defining it on "enriched objects".

Example 2.11 (Yoneda triangle along Yoneda embedding)**.** *For any functor $F\colon \mathbb{A} \to \mathbb{B}$ between locally small categories, there is a Yoneda triangle:*

$$\begin{tikzcd}
\mathbb{A} \arrow[r, "y_{\mathbb{A}}"] \arrow[d, "F"'] \arrow[dr, phantom, "\swArrow\eta"] & \mathbf{Set}^{\mathbb{A}^{op}} \\
\mathbb{B} \arrow[ur, "\hom_{\mathbb{B}}(F(-_2),-_1)"'] &
\end{tikzcd}$$

which resembles the fact that every functor always has a "distributional" right adjoint[10]*. The same is true for internal categories and for categories enriched in a cocomplete symmetric monoidal closed category, and generally (almost by definition) for any 2-category equipped with a Yoneda structure in the sense of [SW78].*

The essence of the example is that because the Yoneda functor $y_{\mathbb{B}}\colon \mathbb{B} \to \mathbf{Set}^{\mathbb{B}^{op}}$ is a full and faithful embedding, functors $F\colon \mathbb{A} \to \mathbb{B}$ may be thought as of

[10]Every functor has a right adjoint in the weak 2-category of profunctors.

profunctors

$$y_{\mathbb{B}} \circ F = \hom_{\mathbb{B}}(-_2, F(-_1))$$

Every profunctor arisen in this way has a right adjoint profunctor $\hom_{\mathbb{B}}(F(-_2), -_1)$ in the (weak) 2-category of profunctors. The profunctor $\hom_{\mathbb{B}}(F(-_2), -_1)$ actually has type $\mathbb{B} \to \mathbf{Set}^{\mathbb{A}^{op}}$, which is the only reason that may prevent F from having the ordinary (functorial) right adjoint $G \colon \mathbb{B} \to \mathbb{A}$. Formally, we say that F has a right adjoint, if there exists G such that:

$$y_{\mathbb{A}} \circ G \approx \hom_{\mathbb{B}}(F(-_2), -_1)$$

which means:

$$\hom_{\mathbb{A}}(-_2, G(-_1)) \approx \hom_{\mathbb{B}}(F(-_2), -_1)$$

Of course, a Yoneda 2-triangle is a Yoneda triangle in the (2-)category of 2-categories, 2-functors, and 2-natural transformations. However, in the light of our elaboration on "pointwiseness", we shall weaken the definition of pointwise Kan extension to the one suitable for enriched categories — as it is much easier and convenient to work with.

Definition 2.12 (Yoneda bitriangle)**.** *A Yoneda bitriangle $\eta \colon Y \triangleright \langle F, G\rangle$ consists of pseudofunctors $Y \colon \mathbb{A} \to \overline{\mathbb{A}}$, $F \colon \mathbb{A} \to \mathbb{B}$, $G \colon \mathbb{B} \to \overline{\mathbb{A}}$ between (weak) 2-categories $\mathbb{A}, \overline{\mathbb{A}}, \mathbb{B}$ and a pseudonatural transformation $\eta \colon Y \to G \circ F$ that induces natural equivalences between functors:*

$$\hom_{\overline{\mathbb{A}}}(Y(-_2), G(-_1)) \approx \hom_{\mathbb{B}}(F(-_2), -_1)$$

and between functors:

$$\hom_{\overline{\mathbb{A}}}(G(-_1), -_2) \approx \hom(\hom_{\mathbb{B}}(F(-_3), -_1), \hom_{\overline{\mathbb{A}}}(Y(-_3), -_2))$$

The last equivalence should be informally understood as the following "equivalence"[11]:

$$G(-) \approx \int^{A \in \mathbb{A}} \hom_{\mathbb{B}}(F(A), -) \times Y(A)$$

Observe that even in case $Y \colon \mathbb{A} \to \overline{\mathbb{A}}$ is only weakly 2-dense, i.e. there is a canonical equivalence:

$$\hom(\hom_{\overline{\mathbb{A}}}(Y(-_1), -_2), \hom_{\overline{\mathbb{A}}}(Y(-_1), -_3)) \approx \hom_{\overline{\mathbb{A}}}(-_2, -_3)$$

[11]It may be expressed as such an equivalence in case objects on the right hand side are well-defined.

if G satisfies the first condition, then it automatically satisfies the second as well:

$$\begin{aligned}
& \hom(\hom_{\mathbb{B}}(F(-_3), -_1), \hom_{\overline{\mathbb{A}}}(Y(-_3), -_2)) \\
\approx\ & \hom(\hom_{\overline{\mathbb{A}}}(Y(-_3), G(-_1)), \hom_{\overline{\mathbb{A}}}(Y(-_3), -_2)) \\
\approx\ & \hom_{\overline{\mathbb{A}}}(G(-_1), -_2)
\end{aligned}$$

We shall be mostly interested in Yoneda bitriangles arisen from proarrow equipment [Woo82] [Woo85], which we recall below. Let $J\colon \mathbb{A} \to \mathbb{B}$ be a (weak) 2-functor from a (strict) 2-category $\mathbb{A}$ to a (weak) 2-category $\mathbb{B}$. We say that J equips $\mathbb{A}$ with proarrows if the following holds:

- J is bijective on objects
- J is locally fully faithful, which means that for every pair of objects $X, Y \in \mathbb{A}$ the induced functor $\hom_{\mathbb{A}}(X, Y) \to \hom_{\mathbb{B}}(J(X), J(Y))$ is fully faithful
- for every 1-morphism $f\colon X \to Y$ in $\mathbb{A}$ the corresponding morphism $J(f)\colon J(X) \to J(Y)$ in $\mathbb{B}$ has a right adjoint

A proarrow equipment resembles the concept of an allegory in a 2-dimensional context.

Definition 2.13 (2-powers)**.** *Let $J\colon \mathbb{A} \to \mathbb{B}$ be an equipment of $\mathbb{A}$ with proarrows, and $Y\colon \mathbb{A} \to \overline{\mathbb{A}}$ a 2-functor making $\mathbb{A}$ a full 2-subcategory of $\overline{\mathbb{A}}$. Then $\mathbb{A}$ has J-relative 2-powers if J and Y can be completed to a Yoneda bitriangle $\eta\colon Y \triangleright \langle J, P\rangle$ with $P\colon \mathbb{B} \to \overline{\mathbb{A}}$ and $\eta\colon Y \to P \circ J$.*

Example 2.14 (Categorical 2-powers)**.** *The archetypical situation is when we take $\eta\colon Y\colon \mathbf{cat} \to \mathbf{Cat} \triangleright \langle J\colon \mathbf{cat} \to \mathbf{dist}, P\colon \mathbf{dist} \to \mathbf{Cat}\rangle$, where* **cat** *is the 2-category of small categories,* **Cat** *is the 2-category of locally small categories, and* **dist** *is the (weak) 2-category of profunctors between small categories. Then $J\colon \mathbf{cat} \to \mathbf{dist}$, $Y\colon \mathbf{cat} \to \mathbf{Cat}$ are the usual embeddings, $P\colon \mathbf{dist} \to \mathbf{Cat}$ is the covariant 2-power pseudofunctor $\mathbf{Set}^{(-)^{op}}$ defined on profunctors via left Kan extensions, and $\eta_{\mathbb{A}}\colon \mathbb{A} \to \mathbf{Set}^{\mathbb{A}^{op}}$ is the Yoneda embedding of a small category $\mathbb{A}$. There are isomorphisms of categories:*

$$\hom_{\mathbf{dist}}(\mathbb{A}, \mathbb{B}) \approx \hom_{\mathbf{Cat}}(\mathbb{A}, \mathbf{Set}^{\mathbb{B}^{op}})$$

where $\mathbb{A}$ and $\mathbb{B}$ are small. Therefore, to show that P is a (weak) pointwise left Kan extension it suffices to show that Y is 2-dense. However, Y is obviously 2-dense, because the terminal category 1 is a 2-dense subcategory of **Cat** *and Y is fully faithful.*

Here is a similar result for internal categories.

Theorem 2.15 ($\mathbb{C}$-internal 2-powers)**.** *Let $\mathbb{C}$ be a finitely cocomplete locally cartesian closed category. There is a Yoneda bitriangle:*

$$\eta\colon \mathit{fam}\colon \mathbf{cat}(\mathbb{C}) \to \mathbf{Cat}^{\mathbb{C}^{op}} \triangleright \langle J\colon \mathbf{cat}(\mathbb{C}) \to \mathbf{dist}(\mathbb{C}), P\colon \mathbf{dist}(\mathbb{C}) \to \mathbf{Cat}^{\mathbb{C}^{op}}\rangle$$

where $\mathbf{cat}(\mathbb{C})$ is the 2-category of $\mathbb{C}$-internal categories, $\mathbf{dist}(\mathbb{C})$ is the (weak) 2-category of $\mathbb{C}$-internal profunctors with J the usual embedding, and:

$$\mathit{fam}\colon \mathbf{cat}(\mathbb{C}) \to \mathbf{Cat}^{\mathbb{C}^{op}}$$

is the canonical family functor (the externalisation functor). Pseudofunctor:

$$P\colon \mathbf{dist}(\mathbb{C}) \to \mathbf{Cat}^{\mathbb{C}^{op}}$$

is given by:

$$\begin{array}{rcl} P(A) & = & \mathit{fam}(\mathbb{C})^{\mathit{fam}(A)^{op}} \\ P(A \overset{F}{\nrightarrow} B) & = & \mathit{Lan}_{y_A}(F) \end{array}$$

where $\mathit{fam}(\mathbb{C})$ is a split indexed category corresponding to the fundamental (i.e. codomain) fibration (Chapter 1, Definition 1.56), and:

$$y_A\colon \mathit{fam}(A) \to \mathit{fam}(\mathbb{C})^{\mathit{fam}(A)^{op}}$$

is the usual internal Yoneda embedding defined as the cartesian transposition of:

$$\underline{\hom(-_2,-_1)}\colon \mathit{fam}(A) \times \mathit{fam}(A)^{op} \to \mathit{fam}(\mathbb{C})$$

Proof. By universal properties of Kan extensions, P is a pseudofunctor $\mathbf{dist}(\mathbb{C}) \longrightarrow \mathbf{Cat}^{\mathbb{C}^{op}}$. There is an equivalence of categories[12] (Section 4 of [BCSW83], Section 3 of [Str05]):

$$\hom_{\mathbf{dist}(\mathbb{C})}(A, B) \approx \hom_{\mathbf{Cat}^{\mathbb{C}^{op}}}(\mathit{fam}(A), \mathit{fam}(\mathbb{C})^{\mathit{fam}(B)^{op}})$$

[12]In fact this equivalence is almost a definition of the category $\mathbf{dist}(\mathbb{C})$.

To show that P is a (pointwise) left Kan extension it suffices to show that *fam* is 2-dense. However, *fam* on discrete internal categories is clearly 2-dense by (weak) 2-Yoneda lemma, and discrete internal categories form a full 2-subcategory of all categories. Therefore *fam* is 2-dense. □

It requires much more work to obtain analogical result for enriched categories. The difficulty is of the same kind as we encountered earlier — discrete objects in the category of enriched categories are generally not dense (more — they rarely constitute a generating family) and there is no canonical candidate for any subcategory giving a dense notion of discreteness. First, let us observe that every enriched category is a canonical limit over its full subcategories consisting of at most three objects.

Lemma 2.16 (On a 2-dense subcategory of $\mathbf{Cat}(\mathbb{C})$)**.** *Let $\langle I, \otimes, \mathbb{C}\rangle$ be a complete and cocomplete symmetric monoidal closed category. The category of small $\mathbb{C}$-enriched categories is a 2-dense subcategory of all $\mathbb{C}$-enriched categories.*

Proof. We have to show that the following categories of natural transformations are isomorphic in a canonical way for all $\mathbb{C}$-enriched categories $A, B \in \mathbf{Cat}(\mathbb{C})$:

$$\begin{aligned}\hom_{\mathbf{Cat}^{\mathbf{Cat}(\mathbb{C})^{op}}}(\hom_{\mathbf{Cat}(\mathbb{C})}(-,A), \hom_{\mathbf{Cat}(\mathbb{C})}(-,B)) \approx \\ \hom_{\mathbf{Cat}^{\mathbf{Cat}_S(\mathbb{C})^{op}}}(\hom_{\mathbf{Cat}(\mathbb{C})}(Y(-),A), \hom_{\mathbf{Cat}(\mathbb{C})}(Y(-),B))\end{aligned}$$

where $Y \colon \mathbf{Cat}_S(\mathbb{C}) \to \mathbf{Cat}(\mathbb{C})$ is the embedding of small categories $\mathbf{Cat}_S(\mathbb{C})$ into all (locally small) categories $\mathbf{Cat}(\mathbb{C})$. To simplify the proof, let us observe that it suffices to show that the underlying sets of the above natural transformation objects are naturally bijective (i.e. that $\mathbf{Cat}_S(\mathbb{C})$ is 1-dense in $\mathbf{Cat}(\mathbb{C})$). Because $\mathbf{Cat}(\mathbb{C})$ is cotensored over small categories, we have natural in $X \in \mathbf{cat}$ bijections:

$$\begin{aligned}&\hom_{\mathbf{Cat}^{\mathbf{Cat}(\mathbb{C})^{op}}}(\hom_{\mathbf{Cat}(\mathbb{C})}(-,A), \hom_{\mathbf{Cat}(\mathbb{C})}(-,X \pitchfork B)) \approx \\ &\hom_{\mathbf{Cat}^{\mathbf{Cat}(\mathbb{C})^{op}}}(\hom_{\mathbf{Cat}(\mathbb{C})}(-,A), \hom_{\mathbf{Cat}(\mathbb{C})}(-,B)^X) \approx \\ &\qquad \hom_{\mathbf{Cat}}(X, \hom_{\mathbf{Cat}^{\mathbf{Cat}(\mathbb{C})^{op}}}(\hom_{\mathbf{Cat}(\mathbb{C})}(-,A), \hom_{\mathbf{Cat}(\mathbb{C})}(-,B)))\end{aligned}$$

and similarly — because $\mathbf{Cat}_S(\mathbb{C})$ is cotensored over small categories:

$$\hom_{\mathbf{Cat}^{\mathbf{Cat}_S(\mathbb{C})^{op}}}(\hom_{\mathbf{Cat}(\mathbb{C})}(Y(-),A),\hom_{\mathbf{Cat}(\mathbb{C})}(Y(-),X \pitchfork B)) \approx$$
$$\hom_{\mathbf{Cat}^{\mathbf{Cat}_S(\mathbb{C})^{op}}}(\hom_{\mathbf{Cat}(\mathbb{C})}(Y(-),A),\hom_{\mathbf{Cat}(\mathbb{C})}(Y(-),B)^X) \approx$$
$$\hom_{\mathbf{Cat}}(X,\hom_{\mathbf{Cat}^{\mathbf{Cat}_S(\mathbb{C})^{op}}}(\hom_{\mathbf{Cat}(\mathbb{C})}(Y(-),A),\hom_{\mathbf{Cat}(\mathbb{C})}(Y(-),B)))$$

By the usual argument, categories:

$$\hom_{\mathbf{Cat}^{\mathbf{Cat}(\mathbb{C})^{op}}}(\hom_{\mathbf{Cat}(\mathbb{C})}(-,A),\hom_{\mathbf{Cat}(\mathbb{C})}(-,B))$$

and:

$$\hom_{\mathbf{Cat}^{\mathbf{Cat}_S(\mathbb{C})^{op}}}(\hom_{\mathbf{Cat}(\mathbb{C})}(Y(-),A),\hom_{\mathbf{Cat}(\mathbb{C})}(Y(-),B))$$

are isomorphic iff the sets:

$$\hom_{\mathbf{SET}}(X,\hom_{\mathbf{Cat}^{\mathbf{Cat}(\mathbb{C})^{op}}}(\hom_{\mathbf{Cat}(\mathbb{C})}(-,A),\hom_{\mathbf{Cat}(\mathbb{C})}(-,B)))$$

and:

$$\hom_{\mathbf{SET}}(X,\hom_{\mathbf{Cat}^{\mathbf{Cat}_S(\mathbb{C})^{op}}}(\hom_{\mathbf{Cat}(\mathbb{C})}(Y(-),A),\hom_{\mathbf{Cat}(\mathbb{C})}(Y(-),B)))$$

are naturally bijective in $X \in \mathbf{cat}$. Therefore, if the canonical function:

$$\begin{array}{c}\hom_{\mathbf{Cat}^{\mathbf{Cat}(\mathbb{C})^{op}}}(\hom_{\mathbf{Cat}(\mathbb{C})}(-,A),\hom_{\mathbf{Cat}(\mathbb{C})}(-,X \pitchfork B)) \\ \downarrow \\ \hom_{\mathbf{Cat}^{\mathbf{Cat}_S(\mathbb{C})^{op}}}(\hom_{\mathbf{Cat}(\mathbb{C})}(Y(-),A),\hom_{\mathbf{Cat}(\mathbb{C})}(Y(-),X \pitchfork B))\end{array}$$

is a bijection, then the canonical functor:

$$\begin{array}{c}\hom_{\mathbf{Cat}^{\mathbf{Cat}(\mathbb{C})^{op}}}(\hom_{\mathbf{Cat}(\mathbb{C})}(-,A),\hom_{\mathbf{Cat}(\mathbb{C})}(-,B)) \approx \\ \downarrow \\ \hom_{\mathbf{Cat}^{\mathbf{Cat}_S(\mathbb{C})^{op}}}(\hom_{\mathbf{Cat}(\mathbb{C})}(Y(-),A),\hom_{\mathbf{Cat}(\mathbb{C})}(Y(-),B))\end{array}$$

is an isomorphism.

Denote by $\mathbf{Cat}_3(\mathbb{C})$ the full 1-subcategory of $\mathbf{Cat}(\mathbb{C})$ consisting of categories having at most three objects, and by $K\colon \mathbf{Cat}_3(\mathbb{C}) \to \mathbf{Cat}(\mathbb{C})$ its embedding. We show that $\mathbf{Cat}_3(\mathbb{C})$ is a 1-dense subcategory of $\mathbf{Cat}(\mathbb{C})$, which by fully-faithfulness of Y implies that $\mathbf{Cat}_S(\mathbb{C})$ is 1-dense subcategory of $\mathbf{Cat}(\mathbb{C})$, and by the above that it is 2-dense.

The direction showing that the canonical morphism is injective is easy — if $\alpha\colon \hom_{\mathbf{Cat}(\mathbb{C})}(-,A) \to \hom_{\mathbf{Cat}(\mathbb{C})}(-,B)$ is a natural transformation, then its restriction $\widetilde{\alpha}\colon \hom_{\mathbf{Cat}(\mathbb{C})}(K(-),A) \to \hom_{\mathbf{Cat}(\mathbb{C})}(K(-),B)$ to a subcategory is natural as well, and since $\mathbf{Cat}_3(\mathbb{C})$ is clearly a generating subcategory, then this assignment is injective. So let us focus on the other direction.

Observe that every $\mathbb{C}$-enriched category A may be canonically represented as a colimit over at most three-object categories:

- for every triple of objects $X, Y, Z \in A$, let $A_{X,Y,Z}$ be the full subcategory of A on this triple with injection $j^A_{X,Y,Z}\colon A_{X,Y,Z} \to A$; similarly define $j^{X,Y,Z}_{X,Y}\colon A_{X,Y} \to A_{X,Y,Z}$ for the full subcategory $A_{X,Y}$ of $A_{X,Y,Z}$ on every pair $X, Y \in A_{X,Y,Z}$, and $j^{X,Y}_X\colon A_X \to A_{X,Y}$ for the full one-object subcategory on every object $X \in A_{X,Y}$

- for diagram D_A consisting of all such defined injections $j^{X,Y,Z}_{X,Y}\colon A_{X,Y} \to A_{X,Y,Z}$, $j^{X,Y}_X\colon A_X \to A_{X,Y}$, the category A together with $j^A_{X,Y,Z}\colon A_{X,Y,Z} \to A$ is the colimit of D_A — if B is another category with cocone $\tau^A_{X,Y,Z}\colon A_{X,Y,Z} \to B$ then the unique functor $H\colon A \to B$ is given on objects by $H(X) = (\tau^A_{X,Y,Z} \circ j^{X,Y,Z}_{X,Y} \circ j^{X,Y}_X)(X)$, and similarly on morphisms; the compositions are preserved by H, because they are preserved pairwise by each $\tau^A_{X,Y,Z}$, and preservation of identities is obvious.

Let $\widetilde{\alpha}\colon \hom_{\mathbf{Cat}(\mathbb{C})}(K(-), A) \to \hom_{\mathbf{Cat}(\mathbb{C})}(K(-), B)$ be a natural transformation. By naturality, the diagram D_A is mapped by $\widetilde{\alpha}$ to a cocone under B. By universal property of colimits, this cocone induces a morphism $c\colon A \to B$, which by Yoneda lemma is tantamount to the natural transformation:

$$\hom_{\mathbf{Cat}(\mathbb{C})}(-, c)\colon \hom_{\mathbf{Cat}(\mathbb{C})}(-, A) \to \hom_{\mathbf{Cat}(\mathbb{C})}(-, B)$$

We have to show that $\hom_{\mathbf{Cat}(\mathbb{C})}(-, c)$ on $\mathbf{Cat}_3(\mathbb{C})$ is equal to $\widetilde{\alpha}$, that is: for any at most three-element category M and a functor $f\colon M \to A$ the composite $c \circ f$ is equal to $\widetilde{\alpha}(f)$. But this is easy. Let us assume that M has exactly three objects X, Y, Z then $f\colon M \to A$ factors as $g\colon M \to A_{f(X),f(Y),f(Z)}$ through injection $j^A_{f(X),f(Y),f(Z)}\colon A_{f(X),f(Y),f(Z)} \to A$. By naturality of $\widetilde{\alpha}$ we have: $\widetilde{\alpha}(j^A_{f(X),f(Y),f(Z)}) \circ g = \widetilde{\alpha}(j^A_{f(X),f(Y),f(Z)}) \circ g) = \widetilde{\alpha}(f)$ and by the definition: $c \circ j^A_{f(X),f(Y),f(Z)} = \widetilde{\alpha}(j^A_{f(X),f(Y),f(Z)})$. Therefore $c \circ f = \widetilde{\alpha}(f)$. A similar argument exhibits equality between components of natural transformations on less than three object categories. $\square$

Theorem 2.17 ($\mathbb{C}$-enriched 2-powers)**.** *Let $\langle I, \otimes, \mathbb{C}\rangle$ be a complete and cocomplete symmetric monoidal closed category. There is a Yoneda bitriangle:*

$$\eta\colon Y\colon \mathbf{Cat}_S(\mathbb{C}) \to \mathbf{Cat}(\mathbb{C}) \triangleright \langle J\colon \mathbf{Cat}_S(\mathbb{C}) \to \mathbf{Dist}_S(\mathbb{C}), P\colon \mathbf{Dist}_S(\mathbb{C}) \to \mathbf{Cat}(\mathbb{C})\rangle$$

where $\mathbf{Cat}_S(\mathbb{C})$ is the 2-category of small $\mathbb{C}$-enriched categories, $\mathbf{Cat}(\mathbb{C})$ is the 2-category of all (i.e. locally small) $\mathbb{C}$-enriched categories, $\mathbf{Dist}_S(\mathbb{C})$ is *the (weak) 2-category of* $\mathbb{C}$*-enriched profunctors between small categories, and* J, Y *are the canonical embeddings. Pseudofunctor:*

$$P\colon \mathbf{Dist}_S(\mathbb{C}) \to \mathbf{Cat}(\mathbb{C})$$

is given by:

$$\begin{array}{rcl} P(A) & = & \mathbb{C}^{A^{op}} \\ P(A \overset{F}{\nrightarrow} B) & = & Lan_{y_A}(F) \end{array}$$

where $y_A\colon A \to \mathbb{C}^{A^{op}}$ *is the enriched Yoneda functor.*

Proof. By definition of $\mathbf{Dist}_S(\mathbb{C})$ there is an equivalence of categories:

$$\hom_{\mathbf{Dist}_S(\mathbb{C})}(\mathbb{A}, \mathbb{B}) \approx \hom_{\mathbf{Cat}(\mathbb{C})}(\mathbb{A}, \mathbb{C}^{\mathbb{B}^{op}})$$

By Lemma 2.16 category $\mathbf{Cat}_S(\mathbb{C})$ is a 2-dense subcategory of $\mathbf{Cat}(\mathbb{C})$; therefore P is a pointwise left Kan extension of Y along G. $\square$

It should be noted that the proarrow equipments in the above examples are canonically determined by the 2-categories of internal and enriched categories respectively — in fact the categories of profunctors are equivalent to the (weak) 2-categories of codiscrete cofibred spans (Theorem 14 in [BCSW83]) in these categories. One can seek a characterisation of a 2-topos along this line, but we leave it for a careful reader, as it is mostly irrelevant for our considerations.

2.2 Power semantics

If $\models\, \subseteq M \times S$ is a binary relation between two sets: M, which is thought of as a set of models, and S, which is thought of as a set of syntactic elements (sentences), then we have "for free" Boolean semantics for propositional connectives formed over set S:

$$\begin{array}{lll} M \models \top & \text{iff} & \text{true} \\ M \models \bot & \text{iff} & \text{false} \\ M \models x \wedge y & \text{iff} & M \models x \quad \text{and} \quad M \models y \\ M \models x \vee y & \text{iff} & M \models x \quad \text{and} \quad M \models y \\ M \models x \Rightarrow y & \text{iff} & M \models x \quad \text{implies} \quad M \models y \end{array}$$

More generally, in any topos with a subobject classifier Ω, a relation $\models\, \subseteq M \times S$ corresponds to a morphism $\nu\colon S \to \Omega^M$ (recall Section 2.1). Since for every object M the power object Ω^M inherits an internal Heyting algebra structure from Ω, we may give the valuation semantics for propositional connectives in S via the composition:

$$\begin{array}{lcl} \nu(\top) & \approx & \top \\ \nu(\bot) & \approx & \bot \\ \nu(x \wedge y) & \approx & \nu(x) \wedge \nu(y) \\ \nu(x \vee y) & \approx & \nu(x) \vee \nu(y) \\ \nu(x \Rightarrow y) & \approx & \nu(x) \Rightarrow \nu(y) \end{array}$$

where $x, y \in S$ are generalised elements. The above should be read as follows — given any generalised elements $X \xrightarrow{x,y} S$ there is a diagram:

$$\begin{tikzcd} X \arrow[r, bend left, "x"] \arrow[r, bend right, "y"'] & S \arrow[r, "\nu"] & \Omega^M \end{tikzcd}$$

then the semantics of meta-formula "$x \wedge y$" is:

$$\wedge \circ (\nu \circ x \times \nu \circ y)$$

where $\Omega^M \times \Omega^M \xrightarrow{\wedge} \Omega^M$ is the internal conjunction morphism in internal Heyting algebra Ω^M; similarly for the other connectives.

Example 2.18 (Free propositional semantics)**.** *Let us start with a set Var and the equality relation* $= \subseteq Var \times Var$*. Since every set is isomorphic to a coproduct on singletons, all generalised elements of a set are recoverable from its global elements. Therefore, we may restrict our semantics to global elements only. For every pair of elements* $x, y \in Var$ *the free semantics for the meta-conjunction* $x \wedge y$ *is* $\nu(x) \wedge \nu(y) = v \mapsto (x = v) \wedge (y = v)$*, and similarly for other connectives. Observe that this gives semantics for a pair* $x, y \in Var$ *interpreted as conjunction* $x \wedge y$*, without saying what exactly* $x \wedge y$ *is. If one is not comfortable with such semantics then one may "materialize" elements by forming an initial algebra like in Example 0.1. Formally, for a given set Var let us define an endofunctor on* **Set***:*

$$F(X) = (X \times X) \sqcup (X \times X) \sqcup (X \times X) \sqcup 1 \sqcup 1$$

and $Prop_{Var}$ as the initial algebra for $F(X) \sqcup Var$ [JR97]. Now, the free semantics of $= \subseteq Var \times Var$ may be extended to the semantics for $Prop_{Var}$ via the unique morphism from the initial algebra to the algebra:

$$(2^{Var} \times 2^{Var}) \sqcup (2^{Var} \times 2^{Var}) \sqcup (2^{Var} \times 2^{Var}) \sqcup 1 \sqcup 1 \sqcup Var \xrightarrow{[\wedge,\vee,\Rightarrow,\top,\bot,=]} 2^{Var}$$

Much more is true. Not only does the power object Ω^M have all propositional connectives, in a sense, which we make precise in this section, Ω^M has all possible connectives.

Example 2.19 (Relational semantics in **Set**)**.** *Let $r \subseteq M \times M \times M$ be a ternary relation on a set M. Then there is a corresponding binary operation $\otimes_r$ on Ω^M defined as follows:*

$$f \otimes_r g = \lambda x \mapsto \underset{a,b\in M}{\exists} f(a) \wedge g(b) \wedge r(x,a,b)$$

Moreover, $\otimes_r$ has "exponentiations" on each of its coordinates. They are given by the following formulae:

$$\begin{aligned} f \overset{L}{\multimap}_r g &= \lambda a \mapsto \underset{x,b\in M}{\forall} f(b) \wedge r(x,a,b) \Rightarrow g(x) \\ f \overset{R}{\multimap}_r g &= \lambda b \mapsto \underset{x,a\in M}{\forall} f(a) \wedge r(x,a,b) \Rightarrow g(x) \end{aligned}$$

We get the usual propositional connectives by considering relations associated to the unique comonoid structure $\langle !\colon M \to 1, \Delta\colon M \to M \times M\rangle$ in a cartesian closed category **Set** *— for $\phi, \psi\colon M \to 2$:*

$$\begin{aligned} (\phi \wedge \psi)(x) \quad &\textit{iff} \quad \underset{a,b\in M}{\exists} \phi(a) \wedge \psi(b) \wedge \langle x,x\rangle = \langle a,b\rangle \\ &\textit{iff} \quad \phi(x) \wedge \psi(x) \end{aligned}$$

and:

$$\begin{aligned} (\phi \overset{L}{\Rightarrow} \psi)(a) \quad &\textit{iff} \quad \underset{x,b\in M}{\forall} \phi(b) \wedge \langle x,x\rangle = \langle a,b\rangle \Rightarrow \psi(x) \\ &\textit{iff} \quad \phi(a) \Rightarrow \psi(a) \\ &\textit{iff} \quad (\phi \overset{R}{\Rightarrow} \psi)(a) \end{aligned}$$

One may recognise in the above example the concept of ternary frame semantics for substructural logics [Doš92]. The crucial point, however, is that such defined semantics have 2-dimensional analogues. The next example was the subject of Brian Day's thesis [Day70].

Example 2.20 (Day convolution)**.** *Let $\langle \mathbb{C}, \otimes, I\rangle$ be a complete and cocomplete symmetric monoidal closed category. Suppose $M \colon \mathbb{A} \otimes \mathbb{A} \nrightarrow \mathbb{A}$ is a $\mathbb{C}$-enriched profunctor. The convolution of M is a functor $\otimes_M \colon \mathbb{C}^{\mathbb{A}^{op}} \otimes \mathbb{C}^{\mathbb{A}^{op}} \to \mathbb{C}^{\mathbb{A}^{op}}$ defined by the coeand:*

$$(F \otimes_M G)(-) = \int^{B,C\in\mathbb{A}} F(B) \otimes G(C) \otimes M(-, B, C)$$

If $\langle \mathbb{A}, M \colon \mathbb{A} \otimes \mathbb{A} \nrightarrow \mathbb{A}, J \colon \mathbb{A}^{op} \to \mathbb{C}\rangle$ is a $\mathbb{C}$-promonoidal category (i.e. a weak monoid in a (weak) 2-category of $\mathbb{C}$-enriched profunctors (Chapter 2 of [Day70])). The induced by convolution operation on $\mathbb{C}^{\mathbb{A}^{op}}$ yields a monoidal structure $\langle \mathbb{C}^{\mathbb{A}^{op}}, \otimes_M, J\rangle$ on $\mathbb{C}^{\mathbb{A}^{op}}$. First observe that J is the right unit of $\otimes_M$:

$$\begin{aligned}
(F \otimes_M J)(-) &= \int^{B,C\in\mathbb{A}} F(B) \otimes J(C) \otimes M(-, B, C) \\
&\approx \int^{B\in\mathbb{A}} F(B) \otimes \hom_{\mathbb{A}}(-, B) \\
&\approx F(-)
\end{aligned}$$

where $\int^{C\in\mathbb{A}} J(C) \otimes M(-, B, C) \approx \hom_{\mathbb{A}}(-, B)$ because J is the promonoidal right unit of M. Similarly, J is the left unit of $\otimes_M$. If the promonoidal structure on $\mathbb{A}$ is induced by a monoidal structure — i.e. if:

$$M(-, B, C) = \hom_{\mathbb{A}}(-, B \otimes_M C)$$

then this structure is preserved by the Yoneda embedding — there is a natural isomorphism:

$$\hom_{\mathbb{A}}(-, X) \otimes_M \hom_{\mathbb{A}}(-, Y) \approx M(-, X, Y)$$

Indeed, by definition

$$\hom_{\mathbb{A}}(-, X) \otimes_M \hom_{\mathbb{A}}(-, Y) = \int^{B,C\in\mathbb{A}} \hom_{\mathbb{A}}(B, X) \otimes \hom_{\mathbb{A}}(C, Y) \otimes M(-, B, C)$$

which via Yoneda reduction is isomorphic to $M(-, X, Y)$.

Brian Day showed more — every monoidal structure induced via convolution is a *(bi)closed* monoidal structure. The left linear exponent is defined by the end:

$$(F \multimap^L_M G)(-) = \int_{A,C\in\mathbb{A}} G(A)^{F(C)\otimes M(A,-,C)}$$

and the right linear exponent by the end:

$$(F \multimap^R_M G)(-) = \int_{A,B \in \mathbb{A}} G(A)^{F(B) \otimes M(A,B,-)}$$

We have to show that:

$$\hom(H, F \multimap^L_M G) \approx \hom(H \otimes_M F, G)$$

Unwinding the right hand side, we get:

$$\begin{aligned}
\hom(H \otimes_M F, G) &= \hom(\int^{B,C \in \mathbb{A}} H(B) \otimes F(C) \otimes M(-, B, C), G) \\
&\approx \int_{B,C \in \mathbb{A}} \hom(H(B) \otimes F(C) \otimes M(-, B, C), G) \\
&\approx \int_{A,B,C \in \mathbb{A}} G(A)^{H(B) \otimes F(C) \otimes M(A,B,C)} \\
&\approx \int_{A,B,C \in \mathbb{A}} (G(A)^{F(C) \otimes M(A,B,C)})^{H(B)} \\
&\approx \int_{A,C \in \mathbb{A}} \hom(H, G(A)^{F(C) \otimes M(A,-,C)}) \\
&\approx \hom(H, \int_{A,C \in \mathbb{A}} G(A)^{F(C) \otimes M(A,-,C)}) \\
&\approx \hom(H, F \multimap^L_M G)
\end{aligned}$$

and similarly for the other variable.

We show that a similar phenomenon occurs for internal categories. In [DS97] Brian Day and Ross Street defined a notion of convolution within a monoidal (weak) 2-category (Proposition 4). For a reason that shall become clear in a moment, we are willing to call it "virtual convolution". Here are the definitions (essentially taken form [GPS95] and [DS97]).

Definition 2.21 (Monoidal (weak) 2-category)**.** *A monoidal bicategory consists of a bicategory* $\mathbb{W}$*, a (weak) 2-functor* $\otimes \colon \mathbb{W} \times \mathbb{W} \to \mathbb{W}$*, an object* $I \in \mathbb{W}$ *and natural equivalences* α, l, r*:*

$$\alpha \colon \otimes \circ (\otimes \times id_{\mathbb{W}}) \to \otimes \circ (id_{\mathbb{W}} \times \otimes)$$

$$l \colon \otimes \circ (I \times id_{\mathbb{W}}) \to id_{\mathbb{W}}$$

$$r\colon id_{\mathbb{W}} \to \otimes \circ (id_{\mathbb{W}} \times I)$$

and isomorphic modifications:

$$(\otimes\circ(id\times\alpha))\bullet(\alpha\circ(id\times\otimes\times id))\bullet(\otimes\circ(\alpha\times id)) \approx (\alpha\circ(id\times id\times\otimes))\bullet(\alpha\circ(\otimes\times id\times id))$$

$$(\otimes \circ (id \times l)) \bullet (\alpha \circ (id \times I \times id)) \bullet (\otimes \circ (r \times id)) \approx id$$

$$(l \circ \otimes) \bullet (\alpha \circ (I \times id \times id)) \approx \otimes \circ (l \times id)$$

$$(\alpha \circ (id \times id \times I)) \bullet (r \circ \otimes) \approx \otimes \circ (id \times r)$$

subject to the usual coherence laws as in [GPS95].

Definition 2.22 (Weak monoid)**.** *Let $\mathbb{W}$ be a monoidal (weak) 2-category. A weak monoid (or a pseudomonoid) in $\mathbb{W}$ consists of the following data:*

- *an object B in $\mathbb{W}$*
- *a 1-morphism $B \otimes B \xrightarrow{\mu} B$ in $\mathbb{W}$*
- *a 1-morphism $I \xrightarrow{\eta} B$ in $\mathbb{W}$*
- *a 2-morphism $\mu \circ (\mu \otimes id_B) \xrightarrow{\alpha} \mu \circ (id_B \otimes \mu)$ in $\mathbb{W}$*
- *a 2-morphism $\mu \circ (\eta \otimes id_B) \xrightarrow{l} id_B$ in $\mathbb{W}$*
- *a 2-morphism $\mu \circ (id_B \otimes \eta) \xrightarrow{r} id_B$ in $\mathbb{W}$*

subject to the usual coherence laws as in [DS97]. A weak symmetric monoid is additionally equipped with a 2-morphism $\rho\colon B \otimes B \to B \otimes B$ satisfying the usual laws for symmetry [DS97]. A weak (symmetric) comonoid in $\mathbb{W}$ consists of the same data with the directions of 1-morphisms reversed. To simplify notation, we shall sometimes write a weak (co)monoid as a triple:

$$\langle B, \mu\colon B \otimes B \to B, \eta\colon I \to B\rangle$$

leaving coherence 2-morphisms implicit.

Let

$$\langle A, \delta\colon A \to A \otimes A, \epsilon\colon A \to I\rangle$$

be a weak comonoid, and:

$$\langle B, \mu\colon B \otimes B \to B, \eta\colon I \to B \rangle$$

be a weak monoid in a monoidal (weak) 2-category with tensor $\otimes$ and unit I, then:

$$\langle \hom(A, B), \star, i \rangle$$

is a monoidal category by:

$$\begin{aligned} f \star g &= \mu \circ (f \otimes g) \circ \delta \\ i &= \eta \circ \epsilon \end{aligned}$$

So the "virtual convolution" structure exists "virtually" — on hom-categories. If a monoidal 2-category admits all right Kan liftings, then the induced monoidal category $\langle \hom(I, B), \star, i \rangle$ for trivial comonoid on I is monoidal (bi)closed by:

$$f \overset{L}{\multimap} h = \mathit{Rift}_{\mu \circ (f \otimes id)}(h)$$

$$f \overset{R}{\multimap} h = \mathit{Rift}_{\mu \circ (id \otimes f)}(h)$$

where $\mathit{Rift}_{\mu \circ (f \otimes id)}(h)$ is the right Kan lifting of h along $\mu \circ (f \otimes id)$ and $\mathit{Rift}_{\mu \circ (id \otimes f)}(h)$ is the right Kan lifting of h along $\mu \circ (id \otimes f)$.

Taking for the monoidal 2-category the category of profunctors, we obtain the well-known formula for convolution. However, in the general setting, such induced structure is far weaker than one would wish to have — for example in the category of profunctors enriched over a monoidal category $\mathbb{C}$ the induced convolution instead of giving a monoidal structure on the category of enriched presheaves:

$$\mathbb{C}^{B^{op}}$$

merely gives a monoidal structure on the underlying (**Set**-enriched) category[13]:

$$\hom(I, \mathbb{C}^{B^{op}})$$

The solution is to find a way to "materialize" the "virtual convolution". Here is a materialization for internal categories.

[13] There is a work-around for this issue in the context of enriched categories, as suggested in the [DS97], but the general weakness of "virtual convolution" is obvious.

Theorem 2.23 (Internal convolution)**.** *Let $\mathbb{C}$ be a locally cartesian closed category with finite colimits. For every $\mathbb{C}$-internal profunctor $\mu\colon A \times A \nrightarrow A$ there is a canonical (bi)closed magma (biclosed functor) on $\mathit{fam}(\mathbb{C})^{\mathit{fam}(A)^{op}}$. Furthermore, if $\mu\colon A \times A \nrightarrow A$ together with $\eta\colon 1 \nrightarrow A$ is a weak (symmetric) monoid, then the induced magma is weak (symmetric) monoidal.*

Intuitively, the term "canonical" in the above theorem means that the induced structure on $\mathit{fam}(\mathbb{C})^{\mathit{fam}(A)^{op}}$ is obtained by the usual Yoneda-like construction. A detailed discussion is included at the end of this chapter in subsection Another Approach. We shall also present some ideas here. As defined in Theorem 2.15, for every locally cartesian closed category with finite colimits, there is a pseudofunctor:

$$P\colon \mathbf{dist}(\mathbb{C}) \to \mathbf{Cat}^{\mathbb{C}^{op}}$$

which "embeds" $\mathbb{C}$-internal profunctors into large $\mathbb{C}$-internal categories. Therefore, for any $\mathbb{C}$-internal profunctor:

$$\mu\colon A \times A \nrightarrow A$$

there is a large $\mathbb{C}$-internal functor:

$$P(\mu)\colon P(A \times A) \to P(A)$$

The canonical structure on $P(A) = \mathit{fam}(\mathbb{C})^{\mathit{fam}(A)^{op}}$ can be obtained as the composition of the above large internal functor $P(\mu)$ with mediating map $P(A) \times P(A) \to P(A \times A)$. This mediating map can be obtained fibrewise by Yoneda-like argument — i.e.: recalling that $P(A)(X) = \hom_{\mathbf{dist}(\mathbb{C})}(X, A)$, by Yoneda for the diagonal internal functor $\Delta_X\colon X \times X \to X$ thought of as profunctor, there is an ordinary functor:

$$\hom_{\mathbf{dist}(\mathbb{C})}(X \times X, A \times A) \xrightarrow{\hom_{\mathbf{dist}(\mathbb{C})}(\Delta_X, id_{A \times A})} \to \hom_{\mathbf{dist}(\mathbb{C})}(X, A \times A)$$

which composed with the tensor product $\times_{\mathbf{dist}(\mathbb{C})}$ on $\mathbf{dist}(\mathbb{C})$:

$$\hom_{\mathbf{dist}(\mathbb{C})}(X, A) \times \hom_{\mathbf{dist}(\mathbb{C})}(X, A) \xrightarrow{\times_{\mathbf{dist}(\mathbb{C})}} \hom_{\mathbf{dist}(\mathbb{C})}(X \times X, A \times A)$$

yields the desired mediating map.

Proof of Theorem 2.23. We shall present a proof for a promonoidal structure on A. The case of (bi)closed magma is analogical.

Since $\mathbb{C}$ is locally cartesian closed, every existing colimit in $\mathbb{C}$ is stable under pullbacks. In particular, coequalisers are stable under pullbacks, and we may form the (weak) 2-category of $\mathbb{C}$-internal profunctors with compositions defined in the usual tensor-like manner (Example 1.73 from Section 1.3, Section 3 of [BCSW83], Section 3 of [Str05]). Moreover, local cartesian closedness allows us to "transpose" compositions (where coequalisers turn into equalisers, and pullbacks turn into local exponents) which makes the category of profunctors admit all right Kan liftings[14]. We have to show that given a promonoidal structure

$$\langle A, \mu \colon A \times A \nrightarrow A, \eta \colon 1 \nrightarrow A \rangle$$

there is a corresponding monoidal (bi)closed structure on:

$$\mathit{fam}(\mathbb{C})^{\mathit{fam}(A)^{op}}$$

i.e. each fibre of $\mathit{fam}(\mathbb{C})^{\mathit{fam}(A)^{op}}$ is a monoidal closed category and reindexing functors preserve these monoidal structures. For $K \in \mathbb{C}$ interpreted as a discrete $\mathbb{C}$-internal category, there are isomorphisms:

$$\begin{array}{rcl} \mathit{fam}(\mathbb{C})^{\mathit{fam}(A)^{op}}(K) & \approx & \hom(\hom(-, K), \mathit{fam}(\mathbb{C})^{\mathit{fam}(A)^{op}}) \\ & \approx & \hom(1, \mathit{fam}(\mathbb{C})^{\hom(-,K) \times \mathit{fam}(A)^{op}}) \\ & = & \hom_{\mathbf{dist}(\mathbb{C})}(1, K \times A) \end{array}$$

where the first isomorphism is the fibred Yoneda lemma, and the second is induced by cartesian closedness of $\mathbf{Cat}^{\mathbb{C}^{op}}$ and the fact that $K = K^{op}$ for discrete internal category K.

Since K has a trivial promonoidal structure:

$$\langle K, K \times K \overset{\Delta^*}{\nrightarrow} K, 1 \overset{!^*}{\nrightarrow} K \rangle$$

we obtain a "product" promonoidal structure on $K \times A$:

$$\begin{array}{rcl} K \times A \times K \times A & \overset{\Delta^* \times \mu}{\nrightarrow} & K \times A \\ 1 & \overset{\langle !^*, \eta \rangle}{\nrightarrow} & K \times A \end{array}$$

[14] See [BCSW83] (Section 4)

Explaining the above notion in more details — observe that because $\mathbb{C}$ is cartesian, every object $K \in \mathbb{C}$ carries a unique comonoid structure:

$$\begin{array}{ccc} K & \overset{\Delta}{\to} & K \times K \\ K & \overset{!}{\to} & 1 \end{array}$$

which has a promonoidal right adjoint structure $\langle \Delta^*, !^* \rangle$ in the category of internal profunctors. The product of the above two promonoidal structures is given by the usual cartesian product of internal categories (note, it is not a product in the category of internal profunctors) followed by the internal product functor $\mathit{fam}(\mathbb{C}) \times \mathit{fam}(\mathbb{C}) \overset{\mathit{prod}}{\to} \mathit{fam}(\mathbb{C})$.

Then, by "virtual convolution" (Proposition 4 and Proposition 6 in [DS97]) there is a monoidal (bi)closed structure on $\hom_{\mathbf{dist}(\mathbb{C})}(1, K \times A) \approx \hom_{\mathbf{dist}(\mathbb{C})}(K, A)$. Therefore each fibre $\mathit{fam}(\mathbb{C})^{\mathit{fam}(A)^{op}}(K)$ is a monoidal (bi)closed category. The preservation of monoidal structures by cartesian functors is almost tautological — we have to show that the following diagram commutes:

$$\begin{tikzcd}
K \arrow[r, "\Delta_K"] & K \times K \arrow[r, "f \times g"] & A \times A \arrow[r, "\mu"] & A \\
L \arrow[u, "h"] \arrow[r, "\Delta_L"] & L \times L \arrow[u, "h \times h"] \arrow[r, "f \times g"] & A \times A \arrow[u, "id_A \times id_A"] \arrow[r, "\mu"] & A \arrow[u, "id_A"]
\end{tikzcd}$$

The left square commutes because every morphism $h \colon L \to K$ is a homomorphism of unique comonoidal structures, and two other squares are identity squares. Furthermore, closedness of convoluted structures is preserved, since in a locally cartesian closed category, pullback functors preserve local exponents. □

Let us work out the concept of internal Day convolution in case $\mathbb{C} = \mathbf{Set}$, and see that it agrees with the usual formula for convolution.

Example 2.24 (**Set**-internal convolution)**.** *The split family fibration (or more accurately, the indexed functor corresponding to the family fibration) for a (locally) small category A:*

$$\mathit{fam}(A) \colon \mathbf{Set}^{op} \to \mathbf{Cat}$$

is defined as follows:

$$\begin{array}{rcl} \mathit{fam}(A)(K \in \mathbf{Set}) & = & A^K \\ \mathit{fam}(A)(K \overset{f}{\to} L) & = & A^L \overset{(-)\circ f}{\to} A^K \end{array}$$

where K, L are sets and $K \xrightarrow{f} L$ is a function between sets. One may think of category A^K as of the category of K-indexed tuples of objects and morphisms from A. Given any monoidal structure on a small category

$$\langle A, \otimes \colon A \times A \to A, I \colon 1 \to A \rangle$$

the usual notion of convolution induces a monoidal structure on $\mathbf{Set}^{A^{op}}$:

$$\langle F, G \rangle \mapsto F \otimes G = \int^{B,C \in A} F(B) \times G(C) \times \hom(-, B \otimes C)$$

The split fibration:

$$\mathit{fam}(\mathbf{Set})^{\mathit{fam}(A)^{op}} \colon \mathbf{Set}^{op} \to \mathbf{Cat}$$

may be characterised as follows:

$$\begin{aligned} \mathit{fam}(\mathbf{Set})^{\mathit{fam}(A)^{op}}(K \in \mathbf{Set}) &= \mathbf{Set}^{A^{op} \times K} \\ \mathit{fam}(\mathbf{Set})^{\mathit{fam}(A)^{op}}(K \xrightarrow{f} L) &= \mathbf{Set}^{A^{op} \times L} \xrightarrow{(-)\circ(id \times f)} \mathbf{Set}^{A^{op} \times K} \end{aligned}$$

Since $\mathbf{Set}^{A^{op} \times K} \approx (\mathbf{Set}^{A^{op}})^K$ we may think of $\mathbf{Set}^{A^{op} \times K}$ as of K-indexed tuples of functors $A^{op} \to \mathbf{Set}$. In fact:

$$\mathit{fam}(\mathbf{Set})^{\mathit{fam}(A)^{op}} \approx \mathit{fam}(\mathbf{Set}^{A^{op}})$$

It is natural then to extend the monoidal structure induced on $\mathbf{Set}^{A^{op}}$ point-wise to $(\mathbf{Set}^{A^{op}})^K$:

$$(F \otimes G)(k) = \int^{B,C \in A} F(k)(B) \times G(k)(C) \times \hom(-, B \otimes C)$$

where $k \in K$. On the other hand, using the internal formula for convolution, we get (up to a permutation of arguments):

$$\frac{\dfrac{\int^{B,C \in A, \beta, \gamma \in K} F(B, \beta) \times G(C, \gamma) \times \hom(\Delta(k), \langle \beta, \gamma \rangle) \times \hom(-, B \otimes C)}{\int^{B,C \in A, \beta, \gamma \in K} F(B, \beta) \times G(C, \gamma) \times \hom(k, \beta) \times \hom(k, \gamma) \times \hom(-, B \otimes C)}}{\int^{B,C \in A} F(B, k) \times G(C, k) \times \hom(-, B \otimes C)}$$

where the first equivalence is the definition of a diagonal Δ, and the second one is by "Yoneda reduction" applied twice.

Note that the local cartesian closedness of the ambient category $\mathbb{C}$ was crucial for the proof. There is always the trivial (cartesian) monoidal structure on the terminal category 1 internal to $\mathbb{C}$, but if $\mathbb{C}$ is not locally cartesian closed than its fundamental fibration $\mathit{fam}(\mathbb{C}) \approx \mathit{fam}(\mathbb{C})^1$ is not a cartesian closed fibration.

There are various possibilities to define universes that induce free semantics. Here is the weakest one.

Definition 2.25 (Power semantics universe)**.** *Let $\eta\colon Y \triangleright \langle F, G\rangle$ be a Yoneda bitriangle $Y\colon \mathbb{A} \to \overline{\mathbb{A}}$, $F\colon \mathbb{A} \to \mathbb{B}$, $G\colon \mathbb{B} \to \overline{\mathbb{A}}$, where $\mathbb{A}$ is a monoidal 2-category that admits a notion of discreteness, $\overline{\mathbb{A}}$ is a monoidal 2-category with finite coproducts that admits a notion of discreteness with op-lax monoidal free functors (recall Definition 1.21), and $\mathbb{B}$ is a monoidal 2-category. Furthermore assume, that Y preserves discrete objects, Y and F preserve magma structures and G maps magmas from $\mathbb{B}$ to internally (bi)closed magmas in $\overline{\mathbb{A}}$. We call the triangle $\eta\colon Y \triangleright \langle F, G\rangle$ a power semantics universe for magmas if for every $V \in \mathit{Disc}(\mathbb{A})$ the functor:*

$$F_V(X) \mapsto Y(V) \sqcup (X \otimes X) \sqcup (|X| \otimes X) \sqcup (|X| \otimes X)$$

has an initial algebra Lambek_V.

If $\eta\colon Y \triangleright \langle F, G\rangle$ is a power semantics universe, then for every magma $R\colon M \otimes M \to M$ and every $\models\colon V \to M$ in $\mathbb{B}$, the free semantics of V by R is defined to be the the unique "semantic" morphism $\mathit{Lambek}_V \to P(M)$ from the initial algebra Lambek_V to the algebra:

$$Y(V) \sqcup (P(M) \otimes P(M)) \sqcup (|P(M)| \otimes P(M)) \sqcup (|P(M)| \otimes P(M)) \xrightarrow{\models, \otimes_R, \overset{L}{\multimap}_R, \overset{R}{\multimap}_R} P(M)$$

where $\otimes_R, \overset{L}{\multimap}_R, \overset{R}{\multimap}_R$ is the internally (bi)closed magma on $P(M)$ induced by R. Similarly, we may introduce power semantics universe for (weak) monoids. Moreover, in many cases (universes induced by categories enriched over a complete and cocomplete symmetric monoidal category, and in universes induced by categories internal to a finitely cocomplete locally cartesian closed category) power objects $P(M)$ are internally cocomplete, thus, in particular, have internal coproducts. This observation makes it possible to extend the above semantics by propositional disjunctions and "false" value.

Example 2.26 (Kripke semantics). *A Kripke structure is a triple $\langle S, \leq \subset S \times S, \Vdash \subseteq S \times |Prop_V|\rangle$, where $\leq$ is a partial order on S, $Prop_V$ is the propositional syntax on a set of variables V, and $\Vdash$ is a "forcing" relation satisfying:*

- *(compatibility on variables) if $A \in V$ and $p, q \in S$ such that $p \leq q$ then $p \Vdash A \Rightarrow q \Vdash A$*
- *(extensional true) $p \Vdash \top$ always holds*
- *(extensional false) $p \Vdash \bot$ never holds*
- *(extensional and) $p \Vdash \phi \wedge \psi$ iff $p \Vdash \phi$ and $p \Vdash \psi$*
- *(extensional or) $p \Vdash \phi \vee \psi$ iff $p \Vdash \phi$ or $p \Vdash \psi$*
- *(extensional implication) $p \Vdash \phi \Rightarrow \psi$ iff for all $q \in S$ such that $p \leq q$ we have: $q \Vdash \phi$ implies $q \Vdash \psi$*

The compatibility condition on variables implies compatibility condition on all formulae, so every Kripke structure gives rise to logical system $\Vdash\colon \langle S, \geq\rangle^{op} \times Prop_V \to 2$, where $\langle S, \geq\rangle^{op} = \langle S, \leq\rangle$ is a posetal category, and $Prop_V$ is the category induced by the logical consequence of $\Vdash$.

Kripke structures may be rediscovered as power semantics for trivial comonoidal structure in the power semantics universe of 2-enriched categories. A poset $\leq \subseteq S \times S$ is exactly a 2-enriched category S. Moreover, S^{op} has the trivial comonoidal structure $\Delta\colon S^{op} \to S^{op} \times S^{op}$, which induces a promonoidal structure $\Delta^*\colon S^{op} \times S^{op} \nrightarrow S^{op}$.

Given a "forcing" relation on variables $\Vdash_V \subseteq S \times V$ that satisfies compatibility condition (i.e. is a 2-enriched profunctor $\Vdash_V\colon V \nrightarrow S^{op}$), there is the semantics homomorphism $Lambek_V \to 2^S$ induced by initiality of $Lambek_V$ and algebraic structure $\langle \Vdash_V, \times, \overset{L}{\Rightarrow}, \overset{R}{\Rightarrow}\rangle$, where $\times = 2^{\Delta^*}$ is the usual cartesian product. Observe that since Δ^* is symmetric, both exponents $\overset{L}{\Rightarrow}$ and $\overset{R}{\Rightarrow}$ are essentially the same, and we may drop one of them from our signature. Furthermore, because Δ^* has also a unit, and 2-enriched presheaves are cocomplete, one may extend the signature functor by additional operations representing true/false objects and disjunctions:

$$L_V(X) \mapsto Y(V) \sqcup 1 \sqcup 1 \sqcup (X \times X) \sqcup (|X| \times X)$$

The initial algebra for L_V is the discrete propositional category $|Prop_V|$ and the Kripke semantics $\Vdash \subseteq S \times |Prop_V|$ is obtained as the transposition of the unique homomorphism to the algebra $\langle \Vdash_V, 1, 0, \times, \sqcup, \Rightarrow \rangle$.

Let us elaborate a bit more on the connectives. The explanation is similar to Example 1.35. The induced convolution structure on the category of 2-enriched (co)presheaves 2^S is cartesian. Products and coproducts may be computed pointwise: i.e. if $\phi, \psi \colon S \to 2$ are two such (co)presheaves, then:

- $(\phi \wedge \psi)(p) = \phi(p) \wedge \psi(p)$
- $(\phi \vee \psi)(p) = \phi(p) \vee \psi(p)$

and exponents, by Yoneda lemma for 2-enriched categories, are given by 2-enriched end:

$$(\phi \Rightarrow \psi)(p) = \int_{q \in S} p \leq q \wedge \phi(q) \Rightarrow \psi(q)$$

which may be rewritten as a first-order formula:

$$(\phi \Rightarrow \psi)(p) = \forall_{q \in S, p \leq q}\ \phi(q) \Rightarrow \psi(q)$$

Writing $p \Vdash \phi$ for $\phi(p)$ we obtain the usual rules of Kripke structures by induction over syntax of $Prop_V$.

In Example 1.35 from the previous chapter we have seen that the satisfaction relation $\models \subseteq Mod \times Sen$ induces semantics consequence relation $\models_{Sen} \subseteq Sen \times Sen$ via the density product. We have also seen, that density products are always equipped with a monad structure. In fact $\models_{Sen} \subseteq Sen \times Sen$ thought of as a 2-enriched profunctor acquires the monad structure from the density product. Because the 2-category of 2-enriched profunctors is sufficiently cocomplete, this monad has a resolution as a Kleisli object Sen_K. In more details, a Kleisli object in the 2-category of 2-enriched profunctors may be described by generalised Grothendieck construction [Bén00] — objects in Sen_K are the same as in Sen, whereas morphisms in Sen_K are defined by:

$$\hom_{Sen_K}(\phi, \psi) = \phi \models_{Sen} \psi$$

Identities and compositions are induced by monad's unit and multiplication respectively. Then by definition of density product the relation $\models \subseteq Mod \times Sen$ extends to the relation:

$$\models \subseteq Mod \times Sen_K$$

In an essentially the same manner one may extend the forcing relation $\Vdash \subseteq S \times |Prop_V|$ of the above Example 2.26 to the relation:

$$\Vdash \subseteq S \times Prop_V$$

where $Prop_V$ is the Kleisli resolution for the density product on the forcing relation.

The next example generalises semantics in Kripke structures.

Example 2.27 (Ternary frame)**.** *A ternary frame [Doš92] is a pair $\langle X, R\rangle$, where X is a set, and $R\colon X \times X \times X \to 2$ is a ternary relation on X. Ternary frames were proposed as generalisations of Kripke structures to model substructural logics. Let Σ_{Lambek} be the signature consisting of three binary symbols $\otimes$, $\overset{L}{\multimap}$ and $\overset{R}{\multimap}$. The semantics for Lambek syntax in ternary frame $\langle X, R\rangle$ is a relation $\Vdash \subseteq X \times Lambek_{Var}$ satisfying:*

- $x \Vdash \phi \otimes \psi$ *iff* $\underset{y,z \in X}{\exists}\; y \Vdash \phi \wedge z \Vdash \psi \wedge R(x, y, z)$

- $y \Vdash \phi \overset{L}{\multimap} \psi$ *iff* $\underset{x,z \in X}{\forall}\; z \Vdash \phi \wedge R(x, y, z) \Rightarrow x \Vdash \psi$

- $z \Vdash \phi \overset{R}{\multimap} \psi$ *iff* $\underset{x,y \in X}{\forall}\; y \Vdash \phi \wedge R(x, y, z) \Rightarrow x \Vdash \psi$

Connectives are defined according to the nonassociative Lambek calculus induced on 2^X via the convolution of R.

Another approach

There is another, more abstract, road to Day convolution for internal categories. Recall from Chapter 1 (Definition 1.57) that if $F\colon \mathbb{C} \to \mathbb{W}$ is a functor from a 1-category $\mathbb{C}$ to a 2-category $\mathbb{W}$, then the F-externalisation $fam_F(A)$ of an object $A \in \mathbb{W}$ is defined to be the functor:

$$\hom_{\mathbb{W}}(F(-), A)\colon \mathbb{C}^{op} \to \mathbf{Cat}$$

For example, in Theorem 2.15, $fam(A)$ is an F-externalisation of an object (i.e. internal category) $A \in \mathbf{cat}(\mathbb{C})$. However, the 2-power $P(A)$ may be itself defined as an "externalisation" — namely, the $J \circ F$ "externalisation" of A.

By fibred Yoneda lemma $P(A) \approx fam_{J \circ F}(A)$ iff there is a natural in $X \in \mathbb{C}$ isomorphism:

$$\hom_{\mathbf{cat}^{\mathbb{C}^{op}}}(\hom_{\mathbb{C}}(-, X), P(A)) \approx \hom_{\mathbf{cat}^{\mathbb{C}^{op}}}(\hom_{\mathbb{C}}(-, X), fam_{J \circ F}(A))$$

The left hand side by definition is isomorphic to $\hom_{\mathbf{dist}(\mathbb{C})}(J(F(X)), A)$, and by Fibred Yoneda lemma, the right hand side is isomorphic to $fam_{J \circ F}(A)(X)$, which by definition equals $\hom_{\mathbf{dist}(\mathbb{C})}(J(F(X)), A)$.

Therefore, the Yoneda bitriangle for internal powers may be redrawn as follows:

$$\begin{tikzcd}
\mathbf{cat}^{\mathbb{C}^{op}} & \mathbf{cat}^{\mathbf{cat}(\mathbb{C})^{op}} \arrow[l, "(-)\circ F^{op}"'] & \mathbf{cat}^{\mathbf{dist}(\mathbb{C})^{op}} \arrow[l, "(-)\circ J^{op}"'] \\
\mathbb{C} \arrow[u, dashed, "y_{\mathbb{C}}"] \arrow[r, "F"] \arrow[ur, phantom, "="] & \mathbf{cat}(\mathbb{C}) \arrow[u, "y_{\mathbf{cat}(\mathbb{C})}"] \arrow[r, "J"] \arrow[ur, phantom, "\Rightarrow J_1"] & \mathbf{dist}(\mathbb{C}) \arrow[u, "y_{\mathbf{dist}(\mathbb{C})}"']
\end{tikzcd}$$

where $F \colon \mathbb{C} \to \mathbf{cat}(\mathbb{C})$ is a strong (cartesian) monoidal functor, $J \colon \mathbf{cat}(\mathbb{C}) \to \mathbf{dist}(\mathbb{C})$ is strong monoidal by the definition of tensor product on $\mathbf{dist}(\mathbb{C})$. We would like to argue that both $F^{op} \circ (-)$ and $J^{op} \circ (-)$ are lax monoidal, because F and J are strong monoidal, and that the natural transformation:

$$J_1 \colon \hom_{\mathbf{dist}(\mathbb{C})}(J(-_1), J(-_2)) \to \hom_{\mathbf{cat}(\mathbb{C})}(-_1, -_2)$$

induced by the "arrows-part" of monoidal functor $J \colon \mathbf{cat}(\mathbb{C}) \to \mathbf{dist}(\mathbb{C})$ is monoidal. Then we could provide the following definitions.

Definition 2.28 (Monoidal Yoneda (bi)triangle)**.** *A Yoneda (bi)triangle $\eta \colon y \triangleright \langle f, g \rangle$ is monoidal if f and g are lax monoidal morphisms between (weak) monoidal objects, and η is a monoidal 2-morphism.*

Definition 2.29 (Power universe)**.** *Let $\eta \colon Y \triangleright \langle F, G \rangle$ be a Yoneda bitriangle $Y \colon \mathbb{A} \to \overline{\mathbb{A}}$, $F \colon \mathbb{A} \to \mathbb{B}$, $G \colon \mathbb{B} \to \overline{\mathbb{A}}$, where $\mathbb{A}$ admits a notion of discreteness, and $\overline{\mathbb{A}}$ has finite coproducts and admits a notion of discreteness with op-lax monoidal free functors. We call bitriangle $\eta \colon Y \triangleright \langle F, G \rangle$ the power universe, if magmas mapped by G are (bi)closed, Y preserves discrete objects, and for every $V \in Disc(\mathbb{A})$ the functor:*

$$F_V(X) \mapsto Y(V) \sqcup (X \otimes X) \sqcup (|X| \otimes X) \sqcup (|X| \otimes X)$$

has an initial algebra $Lambek_V$.

The reason that we did not follow this approach is that we missed some crucial notions and theorems for monoidal bicategories to justify the above observations. For example, we did not succeed in finding an abstract argument that shows that the pseudofunctor $J^{op} \circ (-)$ is really lax monoidal. We leave this for further work.

Chapter 3

Conclusions and further work

In the dissertation we showed that a natural categorical framework for lambda calculi is encapsulated by a 2-category with a notion of discreteness. Then we provided a natural power semantic for internal calculi defined in this framework.

Our first contribution is in extending the definition of fibred/internal connectives (Definition 1.12 and Definition 1.16) together with its generalisation Definition 1.21, polymorphism (Definition 1.27, Definition 1.31 and Definition 1.34) and internal ultraproducts (Example 1.36) to an arbitrary 2-category with a notion of discreteness, and in showing that a naive approach to internally closed connectives like in [Web07] does not work properly. To justify that our proposed definitions give an appropriate extension, we provided a concept of an "associated category" (Definition 1.54 and Definition 1.57). This leads to our second contribution — we showed that with every finitely complete 2-category $\mathbb{W}$ that admits a notion of discreteness, one may associate a 2-functor realising $\mathbb{W}$ in a 2-category $\mathbf{cat}(Disc(\mathbb{W}))$ of categories internal to the discrete objects of $\mathbb{W}$, in such a way that internal connectives and polymorphic objects are preserved (Corollary 1.64). This realisation gives an equivalence of 2-categories if and only if discrete objects are dense (Theorem 1.65), and has a left adjoint iff $\mathbb{W}$ has codescent objects of discrete truncated cosimplicial diagrams (Theorem 1.76 and Corollary 1.77). This sheds new light on the nature of fibred (co)products and their stability condition (i.e. the Beck-Chevalley condition). Moreover, because in the world of enriched categories discrete objects generally are not dense, we have to use our 2-categorical definitions, since the usual fibrational definitions lose information about categories. For the third contribution, we generalised the classical result of Freyd saying that the (co)completeness of a non-posetal category has to be at a lower level on the set-theoretic hierarchy than the category itself, which, intuitively, is just another incarnation of Russel's paradox, Cantor's diagonal argument, Goedel's incompleteness theorem, or the result of Reynolds about non-existence of non-posetal set-theoretic models for parametric polymorphism [Rey84]. We showed that if a 2-category is sufficiently rich, then its objects cannot have all internal products (therefore cannot be internally complete), unless are posetal (Theorem 1.41). Using the concept of an associated category, we obtained the Freyd's theorem for categories internal to any *tensored* category (Corollary 1.45). The fourth contribution is in providing a notion of a generalised adjunction, which we call a Yoneda triangle (Definition 2.7 and Definition 2.12), and showing that many natural concepts in category theory may be characterised as (higher) Yoneda triangles (Theorem 2.15 and Theorem 2.17). There comes our last contribution — we showed that the natural setting for convolution is a Yoneda (bi)triangle, and proved Day convolution theorem for internal categories (Theorem 2.23). Such Yoneda (bi)triangles admitting convolutions provide a semantics universe (Definition 2.25), where objects get their semantics (almost) for free — this includes the usual semantics for propositional calculi, Kripke semantics for intuitionistic calculi and ternary frame semantics for substructural calculi including Lambek's lambda calculi, relevance and linear logics.

We are not fully satisfied with our Definition 2.25. In fact we think that a more appropriate setting would be to consider monoidal Yoneda (bi)triangles — i.e. Yoneda (bi)triangles consisting of lax monoidal morphism and lax monoidal 2-morphisms and give an abstract characterisation of the concept of convolution along this line (see Section 2.2). We leave this for further work.

Another interesting problem is to extend the convolution theorem from monoidal enriched categories and internal categories to more general categories enriched over a bicategory [BCSW83].

Appendix A

Basic concepts and definitions

A.1 Internal categories

A comprehensive treatment of internal categories may be found in [Bor94], [Jac99] and [Ehr63]. Additional information may be found in [Bet97], [Joh77] and [Str10].

Definition A.1 (Internal category)**.** *Let $\mathbb{B}$ be a category with pullbacks. A $\mathbb{B}$-internal category consists of:*

- *an object $C_0 \in \mathbb{B}$,*
- *an object $C_1 \in \mathbb{B}$ together with indexing morphisms $dom, cod\colon C_1 \to C_0$*
- *a morphism $\mu\colon C_2 \to C_1$, where C_2 is a pullback of dom with cod and the following diagrams are commutative:*

$$
\begin{tikzcd}
C_2 \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & C_1 \arrow[d, "dom"] \\
C_1 \arrow[r, "cod"'] & C_0
\end{tikzcd}
\qquad
\begin{tikzcd}
C_1 \arrow[d, "cod"'] & C_2 \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "\mu"] & C_1 \arrow[d, "dom"] \\
C_0 & C_1 \arrow[l, "cod"] \arrow[r, "dom"'] & C_0
\end{tikzcd}
$$

- *a morphism $\eta\colon C_0 \to C_1$ satisfying: $dom \circ \eta = cod \circ \eta = id_{C_0}$*

subject to the laws expressed by commutativity of the following diagrams:

- *multiplication*

$$\begin{tikzcd} C_3 \arrow[r, "id\times\mu"] \arrow[d, "\mu\times id"'] & C_2 \arrow[d, "\mu"] \\ C_2 \arrow[r, "\mu"'] & C_1 \end{tikzcd}$$

where C_3 *is the pullback:*

$$\begin{tikzcd} C_3 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C_1 \arrow[d, "dom"] \\ C_2 \arrow[r, "cod\circ\pi_1"'] & C_0 \end{tikzcd}$$

- *unit*

$$\begin{tikzcd} C_0 \times_{C_0} C_1 \arrow[r, "\eta\times id"] \arrow[dr, "\pi_2"'] & C_2 \arrow[d, "\mu"] & C_1 \times_{C_0} C_0 \arrow[l, "id\times\eta"'] \arrow[dl, "\pi_1"] \\ & C_1 & \end{tikzcd}$$

Definition A.2 (Internal functor)**.** *Let*

$$\begin{aligned} \mathbb{A} &= \langle \mathbb{A}_0, \mathbb{A}_1,\ \mathbb{A}_1 \underset{cod_{\mathbb{A}}}{\overset{dom_{\mathbb{A}}}{\rightrightarrows}} \mathbb{A}_0\ , \mathbb{A}_0 \xrightarrow{\eta_{\mathbb{A}}} \mathbb{A}_1, \mathbb{A}_2 \xrightarrow{\mu_{\mathbb{A}}} \mathbb{A}_1 \rangle \\ \mathbb{B} &= \langle \mathbb{B}_0, \mathbb{B}_1,\ \mathbb{B}_1 \underset{cod_{\mathbb{B}}}{\overset{dom_{\mathbb{B}}}{\rightrightarrows}} \mathbb{B}_0\ , \mathbb{B}_0 \xrightarrow{\eta_{\mathbb{B}}} \mathbb{B}_1, \mathbb{B}_2 \xrightarrow{\mu_{\mathbb{B}}} \mathbb{B}_1 \rangle \end{aligned}$$

be two $\mathbb{C}$*-internal categories. A* $\mathbb{C}$*-internal functor* $\mathbb{A} \xrightarrow{F} \mathbb{B}$ *consists of a pair of morphisms* $\langle F_0 \colon \mathbb{A}_0 \to \mathbb{B}_0, F_1 \colon \mathbb{A}_1 \to \mathbb{B}_1 \rangle$ *in* $\mathbb{C}$ *subject to the following laws:*

- *preservation of domains and codomains:* $F_0 \circ dom_{\mathbb{A}} = dom_{\mathbb{B}} \circ F_1$ *and* $F_0 \circ cod_{\mathbb{A}} = cod_{\mathbb{B}} \circ F_1$
- *preservation of unit:* $F_1 \circ \eta_{\mathbb{A}} = \eta_{\mathbb{B}} \circ F_0$

- *preservation of multiplication:* $F_1 \circ \mu_{\mathbb{A}} = \mu_{\mathbb{B}} \circ (F_1 \times F_1)$

Definition A.3 (Internal natural transformation)**.** *Let* $F, G \colon \mathbb{A} \to \mathbb{B}$ *be two parallel* $\mathbb{C}$*-internal functors. A* $\mathbb{C}$*-internal natural transformation* $F \xrightarrow{\alpha} G$ *consists of a single morphism* $\alpha \colon \mathbb{A}_0 \to \mathbb{B}_1$ *in* $\mathbb{C}$ *subject to the following laws:*

- $dom_{\mathbb{B}} \circ \alpha = F_0$ *and* $cod_{\mathbb{B}} \circ \alpha = G_0$
- $\mu_{\mathbb{B}} \circ \langle \alpha \circ dom_{\mathbb{A}}, G_1 \rangle = \mu_{\mathbb{B}} \circ \langle F_1, \alpha \circ cod_{\mathbb{A}} \rangle$

Corollary A.4. $\mathbb{C}$*-internal categories, functors and natural transformations form a 2-category* $\mathbf{cat}(\mathbb{C})$

One definition of a $\mathbb{C}$*-internal profunctor (distributor) appears in Example 1.73 of Chapter 1 Section 1.3. Here, assuming that* $\mathbb{C}$ *is a locally cartesian closed category with finite colimits, we present two alternative definitions.*

Definition A.5 (Internal profunctor (external definition))**.** *A* $\mathbb{C}$*-internal distributor* Ψ *from a* $\mathbb{C}$*-internal category* $\mathbb{X}$ *to a* $\mathbb{C}$*-internal category* $\mathbb{Y}$*, denoted by* $\Psi \colon \mathbb{X} \nrightarrow \mathbb{Y}$*, is a fibred functor* $fam(\mathbb{Y})^{op} \times fam(\mathbb{X}) \to fam(\mathbb{C})$*, where:*

$$fam \colon \mathbf{cat}(\mathbb{B}) \to \mathbf{Cat}^{\mathbb{B}^{op}}$$

is the usual externalisation functor defined in Chapter 1 Section 1.3 Definition 1.56. The composition of $\mathbb{C}$*-internal distributors is given by left Kan extensions along internal Yoneda embeddings.*

Definition A.6 (Internal profunctor (strict definition))**.** *A* $\mathbb{C}$*-internal distributor from a* $\mathbb{C}$*-internal category* $\mathbb{X}$ *to a* $\mathbb{C}$*-internal category* $\mathbb{Y}$ *is a fibred functor* $fam(\mathbb{C})^{\mathbb{X}^{op}} \to fam(\mathbb{C})^{\mathbb{Y}^{op}}$ *that has a right adjoint in the 2-category of of fibrations over* $\mathbb{C}$*.*

All (weak) 2-categories of profunctors induced by definitions given in Example 1.73, Definition A.5 and Definition A.6 are (weakly) equivalent.

A.2 Fibrations

The concept of fibration is a categorical abstraction for "coherent collection of objects". A good introduction to fibrations may be found in [Pho92, Jac99, Str99, Bor94, Joh02]. A general 2-categorical treatment of fibrations was investigated by Ross Street in [Str80, Str87].

Definition A.7 (Vertical morphism)**.** *Let $p\colon \mathbb{C} \to \mathbb{B}$ be a functor. A morphism $v\colon X \to Y$ is p-vertical (or just vertical, if p is known from the context) if:*

$$p(v) = id$$

Definition A.8 (Fibre)**.** *Let $p\colon \mathbb{C} \to \mathbb{B}$ be a functor and J an object in $\mathbb{B}$. A fibre over J is a full subcategory $\mathbb{C}_J$ of $\mathbb{C}$ consisting of all vertical morphism over J.*

Definition A.9 (Cartesian morphism)**.** *Let $p\colon \mathbb{C} \to \mathbb{B}$ be a functor. A morphism $f\colon Y \to Z$ in $\mathbb{C}$ is cartesian, if for every morphism $g\colon X \to Z$ in $\mathbb{C}$ and every decomposition $p(g) = p(f) \circ k$ there is exactly one morphism h such that $p(h) = k$ and $g = f \circ h$:*

$$\begin{tikzcd}
 & X \arrow[dr, dashed, "h"'] \arrow[drr, bend left, "g"] & & \\
\mathbb{C} \arrow[dd, "p"'] & & Y \arrow[r, "f"'] & Z \\
 & p(X) \arrow[dr, "k"'] \arrow[drr, bend left, "p(g)"] & & \\
\mathbb{B} & & p(Y) \arrow[r, "p(f)"'] & p(Z)
\end{tikzcd}$$

Definition A.10 (Cartesian lifting)**.** *Let $p\colon \mathbb{C} \to \mathbb{B}$ be a functor. For an object $Z_K \in \mathbb{C}$ and a morphism $j\colon J \to p(Z_K)$ in $\mathbb{B}$, we shall say that a morphism lift $j\colon Z_J \to Z_K$ is a cartesian (or final) lifting of j if it is cartesian and it is over j — i.e. lift j is cartesian and $p($lift $j) = j$:*

$$\begin{tikzcd}
\mathbb{C} \arrow[d, "p"'] & Z_J \arrow[r, dashed, "\text{lift } j"] & Z_K \\
\mathbb{B} & J \arrow[r, "j"] & p(Z_K)
\end{tikzcd}$$

Definition A.11 (Fibration)**.** *A functor $p\colon \mathbb{C} \to \mathbb{B}$ is a fibration if for every object $Z_K \in \mathbb{C}$ and every morphism $j\colon J \to p(Z_K)$ there is a cartesian lifting of j. In such a case we call $\mathbb{C}$ the total category of fibration p and $\mathbb{B}$ the base category of fibration p.*

Definition A.12 (Opfibration)**.** *A functor* $p\colon \mathbb{C} \to \mathbb{B}$ *is an opfibration if* $p^{op}\colon \mathbb{C}^{op} \to \mathbb{B}^{op}$ *is a fibration.*

Definition A.13 (Cloven fibration)**.** *A fibration* $p\colon \mathbb{C} \to \mathbb{B}$ *is cloven if it is equipped with an assignment of a cartesian lifting lift* $j\colon j^*(Z_J) \to Z_J$ *to every pair of object* $Z_J \in \mathbb{C}$ *and morphism* $j\colon I \to J$ *in* $\mathbb{B}$*.*

Theorem A.14 (Reindexing functor)**.** *Let* $p\colon \mathbb{C} \to \mathbb{B}$ *be a cloven fibration and* $j\colon I \to J$ *a chosen morphism in the base category. The assignment* $Z_J \mapsto j^*(Z_J)$ *canonically extends to a functor between fibres* $j^*\colon \mathbb{C}_J \to \mathbb{C}_I$*, which shall be called "the reindexing functor through (or along) j". Moreover, for any* $j\colon I \to J$*,* $k\colon J \to K$ *and* $l\colon K \to L$ *there exist natural isomorphisms:*

- $\eta^J\colon id_{\mathbb{C}_J} \to (id_J)^*$
- $\lambda^{j,k}\colon j^* \circ k^* \to (k \circ j)^*$

satisfying monoidal-like laws:

- *unit:*
$$\lambda^{id_I,j} \bullet (\eta^I \circ j^*) = id_{j^*} = \lambda^{j,id_J} \bullet (j^* \circ \eta^J)$$
diagrammatically:
$$\begin{tikzcd}
(id_I)^* \circ j^* \arrow[r, "\lambda^{id_I,j}"] & j^* \arrow[d, "id"] & j^* \circ (id_J)^* \arrow[l, "\lambda^{j,id_J}"'] \\
 & j^* \arrow[ul, "\eta^I \circ j^*"] \arrow[u] \arrow[ur, "j^* \circ \eta^J"'] &
\end{tikzcd}$$

- *multiplication:*
$$\lambda^{k \circ j,l} \bullet (\lambda^{j,k} \circ l^*) = \lambda^{j,l \circ k} \bullet (j^* \circ \lambda^{k,l})$$
diagrammatically:
$$\begin{tikzcd}
j^* \circ k^* \circ l^* \arrow[r, "j^* \circ \lambda^{k,l}"] \arrow[d, "\lambda^{j,k} \circ l^*"'] & j^* \circ (l \circ k)^* \arrow[d, "\lambda^{j,l \circ k}"] \\
(k \circ j)^* \circ l^* \arrow[r, "\lambda^{k \circ j,l}"'] & (l \circ k \circ j)^*
\end{tikzcd}$$

The above laws say that operation $(-)^*$ extends to a pesudofunctor $(-)^*\colon \mathbb{B}^{op} \to \mathbf{Cat}$.

Definition A.15 (Split fibration)**.** *A cloven fibration $p\colon \mathbb{C} \to \mathbb{B}$ is split if the natural isomorphisms given by Theorem A.14 are identities.*

Definition A.16 (Fibred functor)**.** *Let $p\colon \mathbb{C} \to \mathbb{B}$ and $q\colon \mathbb{D} \to \mathbb{E}$ be fibrations. A fibred functor $p \longrightarrow q$ is a pair of functors $\langle F\colon \mathbb{C} \to \mathbb{D}, L\colon \mathbb{B} \to \mathbb{E}\rangle$ such that $q \circ F = L \circ p$:*

$$\begin{tikzcd} \mathbb{C} \arrow[r, "F"] \arrow[d, "p"'] & \mathbb{D} \arrow[d, "q"] \\ \mathbb{B} \arrow[r, "L"] & \mathbb{E} \end{tikzcd}$$

and F preserves cartesian morphisms. Moreover, if p and q are split fibrations, then we say that a fibred functor is split if it preserves chosen *cartesian morphisms.*

Definition A.17 (Fibred natural transformation)**.** *Let $p\colon \mathbb{C} \to \mathbb{B}$, $q\colon \mathbb{D} \to \mathbb{E}$ be fibrations and $\langle F\colon \mathbb{C} \to \mathbb{D}, L\colon \mathbb{B} \to \mathbb{E}\rangle$, $\langle G\colon \mathbb{C} \to \mathbb{D}, K\colon \mathbb{B} \to \mathbb{E}\rangle$ be a pair of parallel fibred functors. A fibred natural transformation $\langle F, L\rangle \longrightarrow \langle G, K\rangle$ consists of a pair of natural transformations $\langle \tau\colon F \to G, \sigma\colon L \to K\rangle$ such that for any $C \in \mathbb{C}$ we have that τ_C is cartesian in q over $\sigma_{p(C)}$:*

$$\begin{tikzcd}[row sep=huge] \mathbb{C} \arrow[rr, bend left, "F"] \arrow[rr, bend right, "G"'] \arrow[rr, phantom, "\tau\Downarrow"] \arrow[d, "p"'] & & \mathbb{D} \arrow[d, "q"] \\ \mathbb{B} \arrow[rr, bend left, "L"] \arrow[rr, bend right, "K"'] \arrow[rr, phantom, "\sigma\Downarrow"] & & \mathbb{E} \end{tikzcd}$$

Corollary A.18 (**Fib**)**.** *Fibrations, fibred functors and fibred natural transformations form a 2-category* **Fib**.

Corollary A.19 (**Split**)**.** *Split fibrations, split fibred functors and fibred natural transformations form a 2-category* **Split**.

In the dissertation we will mostly work with fibrations over a chosen base category.

Definition A.20 ($\mathbf{Fib}(\mathbb{B})$)**.** *Let $\mathbb{B}$ be a category. Fibrations over $\mathbb{B}$ together with fibred functors $\langle F, id_{\mathbb{B}}\rangle$ and natural transformations form a 2-subcategory* $\mathbf{Fib}(\mathbb{B})$ *of* $\mathbf{Fib}$*.*

Definition A.21 ($\mathbf{Split}(\mathbb{B})$)**.** *Let $\mathbb{B}$ be a category. Split fibrations over $\mathbb{B}$ together with split fibred functors $\langle F, id_{\mathbb{B}}\rangle$ and natural transformations form a 2-subcategory* $\mathbf{Split}(\mathbb{B})$ *of* $\mathbf{Split}$*.*

Under the axiom of choice in the meta set theory, the 2-category $\mathbf{Fib}(\mathbb{B})$ is 2-equivalent to the 2-category of cloven fibrations. Throughout the dissertation, if not stated otherwise, we shall assume that every fibration is equipped with cleavage.

Theorem A.22 (Fibrations as indexed categories)**.** $\mathbf{Fib}(\mathbb{B})$ *is 2-equivalent to the 2-category $psfn(\mathbb{B}^{op}, \mathbf{Cat})$ of pseudofunctors $\mathbb{B}^{op} \to \mathbf{Cat}$ (called "indexed categories"), pseudonatural transformations and modifications (Appendix A.3 Definition A.39), and* $\mathbf{Split}(\mathbb{B})$ *is 2-equivalent to the 2-category* $\hom(\mathbb{B}^{op}, \mathbf{Cat})$ *of functors $\mathbb{B}^{op} \to \mathbf{Cat}$ (called "split indexed categories"), natural transformations and modifications. These equivalences are given by the Grothendieck construction and its inverse (Theorem A.14). If $\Theta \colon \mathbb{B}^{op} \to \mathbf{Cat}$ is a psuedofunctor then the Grothendieck construction over it will be denoted by $\int \Theta \overset{\pi_\Theta}{\to} \mathbb{B}$. The category $\int \Theta$ may be defined as the coend:*

$$\int^{B \in \mathbb{B}} (\mathbb{B}/B) \times \Theta(B)$$

where $\mathbb{B}/(-) \colon \mathbb{B} \to \mathbf{Cat}$ is the usual slice functor.

The following theorem is a special case of a Yoneda lemma for weak 2-categories.

Theorem A.23 (Fibred Yoneda lemma)**.** *For any pseudofunctor $\Phi \colon \mathbb{B}^{op} \to \mathbf{Cat}$ there is an equivalence:*

$$psfn(\hom_{\mathbb{B}}(-, X), \Phi) \approx \Phi(X)$$

which is natural in $X \in \mathbb{B}$. Moreover the assignment:

$$\Phi \colon \mathbb{B}^{op} \to \mathbf{Cat} \mapsto (\lambda X \in \mathbb{B} \mapsto psfn(\hom_{\mathbb{B}}(-, X), \Phi))$$

induces a weak 2-equivalence between the 2-category $psfn(\mathbb{B}^{op}, \mathbf{Cat})$ of pseudofunctors, pseudonatural transformations and modifications and the 2-category $\mathbf{Cat}^{\mathbb{B}^{op}}$ of functors, natural transformations and modifications.

There is also a connection between fibrations and internal categories.

Theorem A.24 (Externalisation)**.** *Let $\mathbb{B}$ be a category with pullbacks. The 2-category $\mathbf{cat}(\mathbb{B})$ of $\mathbb{B}$-internal categories is a full 2-subcategory of $\mathbf{Cat}^{\mathbb{B}^{op}}$. The inclusion is given by the externalisation functor:*

$$fam\colon \mathbf{cat}(\mathbb{B}) \to \mathbf{Cat}^{\mathbb{B}^{op}}$$

Fibrations which are equivalent to externalised $\mathbb{B}$-internal categories are called small.

The term *small* from the above definition is justified by the observation:

$$\mathbf{cat}^{\mathbb{B}^{op}} \approx \mathbf{cat}(\mathbf{Set}^{\mathbb{B}^{op}})$$

since $\mathbf{Set}^{\mathbb{B}^{op}}$ may be seen as an enlargement of universe $\mathbb{B}$.

The following diagram summarizes above observations:

$$\begin{tikzcd}
 & \mathbf{Split}(\mathbb{B}) \arrow[r, leftrightarrow, dashed, "fib\,Yoneda"] \arrow[d, leftrightarrow, "Grothendieck"] & \mathbf{Clov}(\mathbb{B}) \arrow[r, leftrightarrow, "AC"] \arrow[d, leftrightarrow, "Grothendieck"] & \mathbf{Fib}(\mathbb{B}) \\
\mathbf{cat}(\mathbf{Set}^{\mathbb{B}^{op}}) \arrow[r, leftrightarrow] & \mathbf{cat}^{\mathbb{B}^{op}} \arrow[r, leftrightarrow, dashed, "fib\,Yoneda"] & psfn(\mathbb{B}^{op}, \mathbf{cat}) & \\
 & \mathbf{cat}(\mathbb{B}) \arrow[u, hook, "fam"'] \arrow[ul, hook, "Yoneda"] & &
\end{tikzcd}$$

where $\mathbf{Clov}(\mathbb{B})$ is the 2-category of cloven fibrations over $\mathbb{B}$.

A.3 Enriched categories

Most of the definitions and theorems presented in this section are from Max Kelly's monograph [Kel82].

Definition A.25 (Monoidal category)**.** *A monoidal category consists of a category $\mathbb{V}$, a functor $\otimes\colon \mathbb{V} \times \mathbb{V} \to \mathbb{V}$ and an object $I \in \mathbb{V}$, together with natural isomorphism:*

- $\alpha_{A,B,C}\colon (A \otimes B) \otimes C \to A \otimes (B \otimes C)$

- $l_A \colon I \otimes A \to A$
- $r_A \colon A \otimes I \to A$

satisfying the following laws:

- *associativity:*

$$(\alpha_{A,B,C} \otimes id_D) \circ \alpha_{A,B\otimes C,D} \circ (id_A \otimes \alpha_{B,C,D}) = \alpha_{A,B,C\otimes D} \circ \alpha_{A\otimes B,C,D}$$

diagrammatically:

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[r, "\alpha"] \arrow[d, "\alpha\otimes id"] & (A \otimes B) \otimes (C \otimes D) \arrow[r, "\alpha"] & A \otimes (B \otimes (C \otimes D)) \\
(A \otimes (B \otimes C)) \otimes D \arrow[rr, "\alpha"] & & A \otimes ((B \otimes C) \otimes D) \arrow[u, "id\otimes\alpha"']
\end{tikzcd}$$

- *unit:*

$$(id_A \otimes l_B) \circ \alpha_{A,I,B} = r_A \otimes id_B$$

diagrammatically:

$$\begin{tikzcd}
(A \otimes I) \otimes C \arrow[rr, "\alpha"] \arrow[dr, "r\otimes id"'] & & A \otimes (I \otimes C) \arrow[dl, "id\otimes l"] \\
& A \otimes C &
\end{tikzcd}$$

We say that a monoidal category is left (resp. right) closed if for every object $A \in \mathbb{V}$ the functor $A \otimes (-) \colon \mathbb{V} \to \mathbb{V}$ (resp. $(-) \otimes A \colon \mathbb{V} \to \mathbb{V}$) has right adjoint. A monoidal category which is both left and right closed is called a (bi-)closed monoidal category.

A monoidal category is symmetric if it is equipped with a family of natural in $A, B \in \mathbb{V}$ isomorphisms $\sigma_{A,B} \colon A \otimes B \to B \otimes A$ satisfying:

- $l_A \circ \sigma_{A,I} = r_A$
- $\sigma_{B,A} \circ \sigma_{A,B} = id_{A\otimes B}$
- $(id_B \otimes \sigma_{A,C}) \circ \alpha_{B,A,C} \circ (\sigma_{A,B} \otimes id_C) = \alpha_{B,C,A} \circ \sigma_{A,B\otimes C} \circ \alpha_{A,B,C}$

Definition A.26 (Monoidal functor)**.** *Let $\langle \mathbb{C}, \otimes^{\mathbb{C}}, I^{\mathbb{C}} \rangle$ and $\langle \mathbb{D}, \otimes^{\mathbb{D}}, I^{\mathbb{D}} \rangle$ be two monoidal categories. A lax monoidal functor $F \colon \mathbb{C} \to \mathbb{D}$ is a triple $\langle \dot{F}, \theta, \xi \rangle$, where:*

- $\dot{F}$ *is a functor* $\mathbb{C} \longrightarrow \mathbb{D}$,
- θ *is a natural transformation* $\dot{F}(-) \otimes^{\mathbb{D}} \dot{F}(-) \longrightarrow \dot{F}(- \otimes^{\mathbb{C}} -)$,
- ξ *is a morphism* $I^{\mathbb{D}} \longrightarrow \dot{F}(I^{\mathbb{C}})$

satisfy the following laws (for simplicity we shall write F *for* $\dot{F}$*)*

- *associativity*

$$\begin{tikzcd}
(F(A) \otimes^{\mathbb{D}} F(B)) \otimes^{\mathbb{D}} F(C) \arrow[r, "\alpha^{\mathbb{D}}"] \arrow[d, "\theta \otimes id"] & F(A) \otimes^{\mathbb{D}} (F(B) \otimes^{\mathbb{D}} F(C)) \arrow[d, "id \otimes \theta"] \\
F(A \otimes^{\mathbb{C}} B) \otimes^{\mathbb{D}} F(C) \arrow[d, "\theta"] & F(A) \otimes^{\mathbb{D}} F(B \otimes^{\mathbb{C}} C) \arrow[d, "\theta"] \\
F((A \otimes^{\mathbb{C}} B) \otimes^{\mathbb{C}} C) \arrow[r, "F(\alpha^{\mathbb{C}})"] & F(A \otimes^{\mathbb{C}} (B \otimes^{\mathbb{C}} C))
\end{tikzcd}$$

- *right unit*

$$\begin{tikzcd}
F(A) \otimes^{\mathbb{D}} I^{\mathbb{D}} \arrow[r, "id \otimes \xi^{\mathbb{D}}"] \arrow[d, "r^{\mathbb{D}}"] & F(A) \otimes^{\mathbb{D}} F(I^{\mathbb{C}}) \arrow[d, "\theta"] \\
F(A) & F(A \otimes^{\mathbb{C}} I^{\mathbb{C}}) \arrow[l, "F(r^{\mathbb{C}})"]
\end{tikzcd}$$

- *left unit*

$$\begin{tikzcd}
I^{\mathbb{D}} \otimes^{\mathbb{D}} F(A) \arrow[r, "\xi^{\mathbb{D}} \otimes id"] \arrow[d, "l^{\mathbb{D}}"] & F(I^{\mathbb{C}}) \otimes^{\mathbb{D}} F(A) \arrow[d, "\theta"] \\
F(A) & F(I^{\mathbb{C}} \otimes^{\mathbb{C}} A) \arrow[l, "F(l^{\mathbb{C}})"]
\end{tikzcd}$$

We say that a functor F *is op-lax monoidal if* F^{op} *is lax monoidal.*

Definition A.27 (Strong functor)**.** *A lax monoidal functor* $F\colon \mathbb{C} \to \mathbb{D}$ *given by* $\langle \dot{F}, \theta, \xi \rangle$ *is strong if* θ *and* ξ *are isomorphisms.*

Definition A.28 (Strict monoidal functor)**.** *A lax monoidal functor* $F\colon \mathbb{C} \to \mathbb{D}$ *given by* $\langle \dot{F}, \theta, \xi \rangle$ *is strict if* θ *and* ξ *are* identities*.*

Definition A.29 (Monoidal natural transformation)**.** *Let $\langle F, \theta, \xi \rangle$, $\langle G, \vartheta, \varsigma \rangle$ be two parallel lax monoidal functors between monoidal categories $\langle \mathbb{C}, \otimes^{\mathbb{C}}, I^{\mathbb{C}} \rangle$ and $\langle \mathbb{D}, \otimes^{\mathbb{D}}, I^{\mathbb{D}} \rangle$. A monoidal natural transformation τ from F to G is a natural transformation $\tau \colon F \to G$ satisfying:*

- *preservation of multiplications*

$$\begin{tikzcd} F(A) \otimes^{\mathbb{D}} F(B) \arrow[r, "\tau_A \otimes^{\mathbb{D}} \tau_B"] \arrow[d, "\theta"'] & G(A) \otimes^{\mathbb{D}} G(B) \arrow[d, "\vartheta"] \\ F(A \otimes^{\mathbb{C}} B) \arrow[r, "\tau_{A \otimes^{\mathbb{C}} B}"'] & G(A \otimes^{\mathbb{C}} B)' \end{tikzcd}$$

- *preservation of units*

$$\begin{tikzcd} & I^{\mathbb{D}} \arrow[dl, "\xi"'] \arrow[dr, "\varsigma"] & \\ F(I^{\mathbb{C}}) \arrow[rr, "\tau_{I^{\mathbb{C}}}"'] & & G(I^{\mathbb{C}}) \end{tikzcd}$$

We shall restrict our attention to enrichment over cosmoi $\langle \mathbb{V}, \otimes, I \rangle$ — that is — complete and cocomplete symmetric monoidal closed categories.

Definition A.30 (Enriched category)**.** *A $\mathbb{V}$-enriched category $\mathbb{C}$ consists of:*

- *collection $\mathbb{C}_0$*
- *collection $\mathbb{C}(x, y)_{x,y \colon \mathbb{C}_0}$ of objects from $\mathbb{V}$*
- *collection $(\mu_{x,y,z} \colon \mathbb{C}(y, z) \otimes \mathbb{C}(x, y) \longrightarrow \mathbb{C}(x, z))_{x,y,z \colon \mathbb{C}_0}$ of morphisms in $\mathbb{V}$*
- *collection $(\eta_x \colon I \longrightarrow \mathbb{C}(x, x))_{x \colon \mathbb{C}_0}$ of morphisms in $\mathbb{V}$*

subject to the following coherence conditions:

- *associativity of multiplication:*

$$\mu_{x,y,v} \circ (\mu_{y,z,v} \otimes id_{\mathbb{C}(x,y)}) = \mu_{x,z,v} \circ (id_{\mathbb{C}(z,v)} \otimes \mu_{x,y,z}) \circ \alpha_{\mathbb{C}(z,v),\mathbb{C}(y,z),\mathbb{C}(x,y)}$$

diagrammatically:

$$\begin{tikzcd}
(\mathbb{C}(z,v)\otimes\mathbb{C}(y,z))\otimes\mathbb{C}(x,y) \arrow[rr, "\alpha"] \arrow[d, "\mu\otimes id"'] & & \mathbb{C}(z,v)\otimes(\mathbb{C}(y,z)\otimes\mathbb{C}(x,y)) \arrow[d, "id\otimes\mu"] \\
\mathbb{C}(y,v)\otimes\mathbb{C}(x,y) \arrow[r, "\mu"'] & \mathbb{C}(x,y) & \mathbb{C}(z,v)\otimes\mathbb{C}(x,z) \arrow[l, "\mu"]
\end{tikzcd}$$

- *identity:*

$$\begin{aligned}
l_{\mathbb{C}(x,y)} &= \mu_{x,y,y}\circ(\eta_y\otimes id_{\mathbb{C}(x,y)}) \\
r_{\mathbb{C}(x,y)} &= \mu_{x,x,y}\circ(id_{\mathbb{C}(x,y)}\otimes\eta_x)
\end{aligned}$$

diagrammatically:

$$\begin{tikzcd}
\mathbb{C}(y,y)\otimes\mathbb{C}(x,y) \arrow[r, "\mu"] & \mathbb{C}(x,y) & \mathbb{C}(x,y)\otimes\mathbb{C}(x,x) \arrow[l, "\mu"'] \\
I\otimes\mathbb{C}(x,y) \arrow[u, "\eta_y\otimes id"] \arrow[ur, "l"'] & & \mathbb{C}(x,y)\otimes I \arrow[u, "id\otimes\eta_x"'] \arrow[ul, "r"]
\end{tikzcd}$$

According to the notational conventions, elements form $\mathbb{C}_0$ will be usually denoted by capital letters $X, Y, Z, \ldots$ and objects $\mathbb{C}(X,Y)$ by $\hom_{\mathbb{C}}(X,Y)$.

Definition A.31 (Enriched functor)**.** *A $\mathbb{V}$-enriched functor from a $\mathbb{V}$-enriched category $\mathbb{C}$ to a $\mathbb{V}$-enriched category $\mathbb{D}$ denoted by $F\colon\mathbb{C}\to\mathbb{D}$ consists of a function $F_0\colon\mathbb{C}_0\to\mathbb{D}_0$ and a collection of morphisms:*

$$(F_1)_{X,Y}\colon\ \hom_{\mathbb{C}}(X,Y)\to\hom_{\mathbb{D}}(F_0(X),F_0(Y))$$

subject to the laws:

- *preservation of multiplication:* $F_1\circ\mu^C=\mu^D\circ(F_1\otimes F_1)$
- *preservation of identities:* $F_1\circ\eta^C=\eta^D$

Definition A.32 (Enriched natural transformation)**.** *A $\mathbb{V}$-enriched natural transformation from a $\mathbb{V}$-enriched functor F to a $\mathbb{V}$-enriched functor G between $\mathbb{V}$-enriched categories $\mathbb{C}$ and $\mathbb{D}$ consists of a collection of morphisms $(\tau_X\colon I\to\hom_{\mathbb{D}}(F_0(X),G_0(X)))_{X\colon\mathbb{C}_0}$ satisfying the $\mathbb{V}$-naturality condition: $\mu^D\circ(\tau_Y\otimes F_1)\circ l=\mu^D\circ(G_1\otimes\tau_X)\circ r$, where l and r are left and right units from the definition of the monoidal category.*

Definition A.33 (Self-enrichment)**.** *A monoidal closed category $\mathbb{V}$ can be naturally seen as a $\mathbb{V}$-enriched category (i.e. it is a self-enriched category). This $\mathbb{V}$-enriched structure is given as follows:*

- *objects of $\mathbb{V}$-enriched category $\mathbb{V}$ are the same as the objects in ordinary category $\mathbb{V}$*
- *morphisms $\hom_{\mathbb{V}}(X, Y)$ from X to Y of $\mathbb{V}$-enriched category $\mathbb{V}$ are the linear exponents $X \multimap Y$ in $\mathbb{V}$, i.e. $\hom_{\mathbb{V}}(X, Y) = X \multimap Y$*
- *identities and compositions are inherited from $\mathbb{V}$*

Definition A.34 (Enriched distributor)**.** *A $\mathbb{V}$-enriched distributor Ψ from a $\mathbb{V}$-enriched category $\mathbb{X}$ to a $\mathbb{V}$-enriched category $\mathbb{Y}$, denoted by $\Psi\colon \mathbb{X} \nrightarrow \mathbb{Y}$, is a $\mathbb{V}$-enriched functor $\mathbb{Y}^{op} \otimes \mathbb{X} \to \mathbb{V}$, where $\mathbb{V}$ is viewed as a category enriched over itself, and $(-)^{op}$ is the obvious duality involution.*

Theorem A.35 (Enriched Yoneda lemma)**.** *For every (small) $\mathbb{V}$-enriched category $\mathbb{C}$, there is a fully faithful $\mathbb{V}$-enriched functor:*

$$y_{\mathbb{C}}\colon \mathbb{C} \to \mathbb{V}^{\mathbb{C}^{op}}$$

given by:

$$y_{\mathbb{C}}(A) = \hom_{\mathbb{C}}(-, A)$$

Definition A.36 (Enriched density)**.** *A $\mathbb{V}$-enriched functor $F\colon \mathbb{C} \to \mathbb{D}$ is ($\mathbb{V}$-enriched) dense if the functor:*

$$\hom(F(-_2), -_1)\colon \mathbb{D} \to \mathbb{V}^{\mathbb{C}^{op}}$$

is fully faithful.

The classical concept of (conical) limit is insufficient in general theory of enriched categories. To get an appropriate notion of completeness for enriched categories, we have to study so called "weighted limits".

Definition A.37 (Weighted limit)**.** *A $\mathbb{V}$-enriched functor $F\colon \mathbb{C} \to \mathbb{D}$ has a limit weighted by a $\mathbb{V}$-enriched functor (copresheaf) $G\colon \mathbb{C} \to \mathbb{V}$ if there exists an object $lim_G(F)$ and a natural isomorphism:*

$$\hom_{\mathbb{D}}(-_1, lim_G(F)) \approx \hom_{\mathbb{V}^{\mathbb{C}}}(G(-_2), \hom_{\mathbb{D}}(-_1, F(-_2)))$$

In such a case we call $\lim_G(F)$ *the limit of* F *weighted by* G. *A* $\mathbb{V}$*-enriched category* $\mathbb{D}$ *is weighted complete (or just complete) if for every small* $\mathbb{V}$*-enriched category* $\mathbb{C}$ *and every functor* $F\colon \mathbb{C} \to \mathbb{D}$ *there exists a limit weighted by any functor* $G\colon \mathbb{C} \to \mathbb{V}$.

The concept of weighted colimit is the categorical dual to the concept of weighted limit — i.e. a weighted limit in $\mathbb{W}^{op}$ *is called a weighted colimit in* $\mathbb{W}$.

Definition A.38 (2-category)**.** *A 2-category is a category enriched in* $\langle \mathbf{Cat}, \times, 1\rangle$. *We write* **2Cat** *for the 2-category of 2-categories, 2-functors (i.e.* **Cat***-enriched functors) and 2-natural transformations (i.e.* **Cat***-enriched natural transformations).*

Because 2-category **2Cat** is cartesian closed as a 2-category, it is furthermore enriched over itself, and therefore forms a 3-category (i.e. a category enriched in a **2Cat**). Three-morphisms in **2Cat** are called modifications.

Definition A.39 (Modification)**.** *Let* $\mathbb{A}, \mathbb{B}$ *be 2-categories,* $F, G\colon \mathbb{A} \to \mathbb{B}$ *parallel 2-functors from* $\mathbb{A}$ *to* $\mathbb{B}$*, and* $\tau, \sigma\colon F \to G$ *parallel 2-natural transformations from* F *to* G. *A modification* $\xi\colon \tau \to \sigma$ *is a collection of 2-morphisms* $\xi_A\colon \tau_A \to \sigma_A$ *indexed by objects* $A \in \mathbb{A}$ *satisfying for every* $A, B \in \mathbb{A}$*, every parallel 1-morphisms* $f, g\colon A \to B$ *and every 2-morphism* $\alpha\colon f \to g$ *the following naturality condition:*

$$\xi_B \circ F(\alpha) = G(\alpha) \circ \xi_A$$

A.4 Cantor's diagonal argument

The classical Cantor's diagonal argument is purely constructive and as such carries to any higher-order type theory, and more generally, to any elementary topos[1]. Let $\mathbb{C}$ be an elementary topos with a subobject classifier Ω, and let us assume that there is an injection $j\colon \Omega^A \to A \in \mathbb{C}$. We may form a paradoxical subset of Ω^A:

$$P = \{x \in A\colon \forall_{y \in \Omega^A}(x \in y) \to (x \neq j(y))\}$$

[1]Notice however, that it heavily relies on impredicativity of the topos, so it does not literally carry over to predicative type theories.

Let us consider $p = j(P)$. If $p \in P$ then according to the definition of P:

$$(p \in y) \to (p \neq j(y)) \qquad (*)$$

for all $y \in \Omega^A$, so particularly for $y = P$, we have:

$$(p \in P) \to (p \neq j(P))$$

and by using (again) the assumption $p \in P$, we can derive $p \neq j(P)$. The last formula, by the definition of p is equivalent to "false" $\perp$. Therefore, we have constructed a method of turning a statement $p \in P$ into absurd, that is $(p \in P) \to \perp$. On the other hand, we may show that the formula $(*)$ holds for every y as follows. By the definition of p, it is equivalent to:

$$(p \in y) \to (j(P) \neq j(y))$$

and by the definition of the implication, to:

$$(j(P) = j(y)) \to ((p \in y) \to \perp) \qquad (**)$$

Now, we may observe that formula:

$$(P = y \wedge p \in y) \to \perp$$

holds because $(p \in P) \to \perp$ as has been shown in the first part of the proof. Therefore $(**)$ holds as the composition of the above formula with our extra assumption saying that j is injective:

$$(j(P) = j(y)) \to (P = y)$$

Finally, by comprehension, $p \in P$. So:

$$(p \in P) \wedge ((p \in P) \to \perp)$$

thus:

$$\perp$$

which means that truth $\top$ is equivalent to absurd $\perp$.

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