University of Warsaw

Faculty of Philosophy and Sociology

Marek Czarnecki

Foundations of Mathematics without Actual Infinity

Podstawy matematyki bez aktualnej nieskończoności

Ph.D. Thesis

Thesis Advisor:
Prof. Marcin Mostowski
Thesis Assistant Advisor:
Ph.D. Konrad Zdanowski

Warsaw
May 24, 2014

Abstract

Contemporary mathematics significantly uses notions which belong to *ideal mathematics* (in Hilbert's sense) – which is expressed in language which essentially uses actual infinity. However, we do not have a meaningful notion of truth for such languages. We can only reduce the notion of truth to finitistic mathematics via axiomatic theories. Nevertheless, justification of truth of axioms themselves exceeds the capabilities of the theory based on these axioms.

On the other hand, we can easily decide the truth or falsity of a statement in finite structures. The aim of this dissertation is to identify the fragment of mathematics, which is of the finitistic character. The fragment of mathematics which can be described without actual infinity. This is the part of mathematics which can be described in finite models and for which the truth of its statements can be verified within finite models. We call this fragment of mathematics with a term introduced by Knuth – the *concrete mathematics*. This part of mathematics is of computational character and it is *closer to our empirical base*, which makes it more difficult.

We consider concrete foundations of mathematics, in particular the concrete model theory and semantics without actual infinity. We base on the notion of FM–representability, introduced by Mostowski, as an explication of expressibility without actual infinity. By the Mostowski's FM–representability theorem, FM–representable notions are exactly those, which are recursive with the halting problem as an oracle.

We show how to express basic concepts of model theory in the language without actual infinity. We investigate feasibility of the classical model–theoretic constructions in the concrete model theory. We present the Concrete Completeness Theorem and the Low Completeness Theorem; the Concrete Omitting Types Theorem; and Preservation Theorems. We identify the constructions which are not admissible in the concrete model theory by showing stages of these constructions which are not allowed in the concrete framework. We show which arguments from the axiomatic model theory fail in the concrete model theory.

Moreover, we investigate how to approximate truth for finite models. In particular we study the properties of approximate FM–truth definitions which are expressible in modal logic. We introduce modal logic SL, axioms of which mimic the properties of a specific approximate FM–truth definition. We show that SL is the modal logic of any approximate FM–truth definition. This is done by proving a theorem analogous to Solovay's completeness theorem for modal logic GL.

Streszczenie

Współczesna matematyka w znaczącej mierze posługuje się pojęciami, które należą do *matematyki idealnej* (w sensie Hilberta) – wyrażona jest w języku istotnie wykorzystującym aktualną nieskończoność. Dla tego typu języków nie posiadamy sensownego kryterium prawdziwości. Jesteśmy w stanie jedynie redukować je do matematyki skończonościowej poprzez teorie aksjomatyczne. Niemniej uzasadnianie prawdziwości samych aksjomatów znajduje się poza zasięgiem teorii na nich opartej.

Z drugiej strony w strukturach skończonych jesteśmy w stanie w prosty sposób rozstrzygać prawdziwość i fałszywość twierdzeń. Celem niniejszej rozprawy jest identyfikacja fragmentu matematyki, który ma skończonościowy charakter. Fragmentu matematyki, do którego opisu nie jest niezbędna aktualna nieskończoność, a wystarczy jedynie nieskończoność potencjalna. Jest to ta część matematyki, której pojęcia można wyrazić w modelach skończonych oraz prawdziwość twierdzeń której można w nich zweryfikować. Tę część matematyki, za Knuthem, nazywamy matematyką konkretną. Ma ona obliczeniowy, kombinatoryczny charakter i jest *bliższa naszemu doświadczeniu* niż matematyka idealna, a co za tym idzie jest trudniejsza.

Rozważamy konkretne podstawy matematyki, w szczególności konkretną teorię modeli oraz semantykę bez aktualnej nieskończoności. Opieramy się na wprowadzonym przez Mostowskiego pojęciu FM–reprezentowalności, jako eksplikacji wyrażalności bez aktualnej nieskończoności oraz twierdzeniu o FM–reprezentowalności identyfikującym FM–reprezentowalne pojęcia z tymi, które są obliczalne z problemem stopu jako wyrocznią.

Pokazujemy w jaki sposób można zinterpretować podstawowe pojęcia teorii modeli w języku bez aktualnej nieskończoności. Następnie badamy klasyczne konstrukcje teoriomodelowe pod kątem ich wykonalności w obszarze matematyki konkretnej. Prezentujemy twierdzenie o konkretnej pełności oraz twierdzenie o łatwej pełności, twierdzenie o omijaniu typów oraz twierdzenia o zachowaniu. Przedstawiamy konstrukcje, które są niewykonalne dla modeli konkretnych, identyfikując etapy konstrukcji teoriomodelowych, które nie są wykonalne w teorii modeli konkretnych. Identyfikujemy argumenty z aksjomatycznej teorii mnogości, które nie są dopuszczalne w konkretnej teorii modeli.

Ponadto, badamy możliwość przybliżania prawdy arytmetycznej w modelach skończonych. W szczególności rozważamy te własności przybliżonych predykatów prawdy dla modeli skończonych, które wyrażalne są w logice modalnej. Wprowadzamy logikę modalną SL, której aksjomaty odzwierciedlają własności przybliżonych predykatów prawdy. Pokazujemy, że logika SL jest logiką przybliżonych predykatów prawdy – dowodzimy twierdzenia analogicznego do twierdzenia o pełności dla logiki GL udowodnionego przez Solovaya.

Acknowledgements

This dissertation could not have been completed without substantial support I have received during my PhD studies.

First of all, I thank my thesis supervisor, Marcin Mostowski. He introduced me to mathematical logic and thanks to him I got attracted to the problems concerning potential and actual infinity. He also suggested to me pursuing my research in foundations of mathematics without actual infinity.

I express my gratitude to thesis assistant supervisor, Konrad Zdanowski. I could not count the number of suggestions he gave me to improve this thesis. Moreover, one of the chapters of this dissertation is based on our collaborative research.

I thank Leszek Aleksander Kołodziejczyk for his support during my first steps in logic. Leszek and Konrad are still my role models of hardcore logicians who know everything in their fields of interests. Here is the place to express my gratitude to both of them.

I also thank Damian Niwiński, Paweł Urzyczyn and Mikołaj Bojańczyk who showed me the beauty of computability theory. I also express my gratitude to Nina Giersimczuk, Jakub Szymanik and Rafał Urbaniak for showing me how philosophy and logic can be combined in results that contribute to both of these fields.

I thank the other people whom I have had the pleasure to collaborate with and who obviously influenced my scientific development: Krzysztof Kapulkin, Michał Tomasz Godziszewski and Dariusz Kalociński. Moreover, Michał helped with improving the language of this dissertation and his inquisitiveness resulted in lots of clarifications.

Finally, I thank my parents for all their support and my wife Joanna, who valiantly could stand the growing heaps of printouts piled up around my desk.

Contents

[...] οὐκ ἀφαιρεῖται δ᾽ ὁ λόγος οὐδὲ τοὺς μαθηματικοὺς τὴν θεωρίαν, ἀναιρῶν οὕτως εἶναι ἄπειρον ὥστε ἐνεργείᾳ εἶναι ἐπὶ τὴν αὔξησιν ἀδιεξίτητον· οὐδὲ γὰρ νῦν δέονται τοῦ ἀπείρου (οὐ γὰρ χρῶνται), ἀλλὰ μόνον εἶναι ὅσην ἂν βούλωνται πεπερασμένην· τῷ δὲ μεγίστῳ μεγέθει τὸν αὐτὸν ἔστι τετμῆσθαι λόγον ὁπηλικονοῦν μέγεθος ἕτερον. ὥστε πρὸς μὲν τὸ δεῖξαι ἐκείνοις οὐδὲν διοίσει τὸ [δ᾽] εἶναι ἐν τοῖς οὖσιν μεγέθεσιν.

Aristotle "Physics", Book 3, part 7, circa 350BC

Ατήματα.
[...]
β΄. Καὶ πεπερασμένην εὐθεῖαν κατὰ τὸ συνεχὲς ἐπ᾽ εὐθείας ἐκβαλεῖν.

Euclid "Elements", Book 1, circa 300BC

[...] protestire ich zuvörderst gegen den Gebrauch einer unendlichen Grösse als einer *Vollendeten*, welcher in der Mathematik niemals erlaubt ist. Das Unendliche ist nur eine Façon de parler, indem man eigentlich von Grenzen spricht, denen gewisse Verhältnisse so nahe kommen als man will, während andern ohne Einschränkung zu wachsen verstattet ist.

Carl Friedrich Gauss in a letter to Heinrich Christian Schumacher, 12 July 1831[1]

[1]For English translations see [Aria], [Hea56], [Fra76]. They are also present in this dissertation on pages 6–7.

Preface

An Outline and Results

In Chapter 1 we discus the motivations for the research presented in this dissertation.[2]

In Chapter 2 we present basic definitions and a historical outline of the logical framework we work in. We recall some results from computability theory and the structure of Turing degrees. We also present the basics of Mostowski's approach to the study of potentially infinite worlds – in particular we discuss FM–domains, FM–representability theorem and the FM–version of the undefinability of truth theorem. The latter states that there are no FM–truth definitions for *sufficiently strong* FM–domains with *sl*–semantics e.g. for FM($\mathcal{N}$).

In Chapter 3 we study FM–truth–like predicates for FM($\mathcal{N}$). We construct the predicate $\mathrm{Tr}_{\mathrm{sl}}$ – an approximate FM–truth predicate – which resembles some properties of the full FM–truth predicate. Further, we consider properties of approximate FM–truth predicates which are expressible in modal logic. We introduce the modal logic SL as an extension of the modal logic K by axioms $\Box(\neg\varphi) \equiv (\neg\Box\varphi)$. We also introduce the modal logic L_{Tr} as the logic containing modal formulae whose every translation into the arithmetical language is true in FM($\mathcal{N}$). Therefore L_{Tr} may be considered as the modal logic of $\mathrm{Tr}_{\mathrm{sl}}$. Using fixpoint extensions SL^* and ${\mathrm{L}_{\mathrm{Tr}}}^*$ of logics SL and L_{Tr}, we show that these logics are equivalent. It demonstrates that SL is the modal logic of the approximate FM–truth predicate $\mathrm{Tr}_{\mathrm{sl}}$. It follows that $\mathrm{Tr}_{\mathrm{sl}}$ has all the properties of the FM–truth predicates which are expressible in modal logic. Chapter 3 is entirely based on the joint work with Zdanowski ([CZ10]).

The main part of the dissertation is Chapter 4 where we study what amount of the model theory can be performed without the use of actual infinity. We introduce the notion of a concrete structure and of a concrete model. A concrete structure may be seen as an FM–representation of a model understood in the standard sense. We require both the structure and the satisfaction relation of a concrete model to be FM–representable. Therefore,

[2] The research program was first formulated by Mostowski in [Mos11].

concrete structures are possible ontologies and concrete models are possible semantics representable with the use of potential infinity only. We transfer basic model–theoretic concepts such as submodels and elementary submodels, chains of models and various morphisms between models to the concrete models framework. We point out difficulties that arise in the context of concrete models. We prove Kleene–style (see [Kle52]) and low completeness theorems and the omitting types theorem in the concrete models context.

We are particularly concerned with model–theoretic constructions and their feasibility. We show how concrete chains and concrete towers of concrete models can be constructed. We show that, although it is not the case for concrete chains, concrete elementary chains can be summed to obtain a concrete model. We show that the construction of Σ_n chains of models by Chang and Keisler fails in the concrete context. We prove the preservation results for consistent recursive theories: T is preserved under unions of chains if and only if T has a Π_2 axiomatisation and T is preserved under homomorphisms if and only if T has a positive axiomatisation. We show that, for concrete models, Robinson's construction fails to prove Craig's Interpolation Lemma and Robinson's Joint Consistency Theorem. The problem stems from the fact that glueing concrete models is not feasible.

Quotations

In this section we present important quotations which we refer to in the next chapter.

Quotation 1 ([Aria])

> *Hence this infinite is potential, never actual: the number of parts that can be taken always surpasses any assigned number. But this number is not separable from the process of bisection, and its infinity is not a permanent actuality but consists in a process of coming to be, like time and the number of time. [...] Our account does not rob the mathematicians of their science, by disproving the actual existence of the infinite in the direction of increase [...]. In point of fact they do not need the infinite and do not use it. They postulate only that the finite straight line may be produced as far as they wish. It is possible to have divided in the same ratio as the largest quantity another magnitude of any size you like. Hence, for the purposes of proof, it will make no difference to them to have such an infinite instead, while its existence will be in the sphere of real magnitudes.*
>
> ***Aristotle "Physics", Book 3, part 7. Translated by R. P. Hardie and R. K. Gaye.***

Quotation 2 ([Hea56])

> *Postulate 2.*
>
> *[...]*
>
> *To produce a finite straight line continuously in a straight line.*
>
> ***Euclid "Elements", Book 1. Translated by Sir Thomas L. Heath***

Quotation 3 ([Fra76])

> *I protest [...] against the use of infinite magnitude as if it were something finished; this use is not admissible in mathematics. The infinite is only a façon de parler: one has in mind limits approached by certain ratios as closely as desirable while other ratios may increase indefinitely.*
>
> ***Carl F. Gauss in a letter to Heinrich C. Schumacher, 12 July 1831. Translated by Abraham A. Fraenkel.***

Quotation 4 ([Hil02])

> *The form of logical inference in which this conception finds its expression – namely, those that we employ when, for example, we deal with* all *real numbers having a certain property or assert that* there exist *real numbers having a certain property – are called upon quite without restriction and are used again and again by Weierstrass precisely when he is establishing the foundations of analysis.*
>
> *Thus, the infinite, in a disguised form, was able to worm its way back into Weierstrass' theory and escape the sharp edge of his critique; therefore it is the* problem of the infinite *in the sense just indicated that still needs to be conclusively solved. And just as the infinite, in the sense of the infinitely small and the infinitely large, could, in the case of the limiting processes of the infinitesimal calculus, be shown to be a mere way of speaking, so we must recognize that the infinite in the sense of the infinite totality (wherever we still come upon it in the modes of inference) is something merely apparent. And just as operations with the infinitely small were replaced by processes in the finite that have quite the same results and lead to quite the same elegant formal relations, so the models of inference employing the infinite must be replaced generally by finite processes that have precisely the same results, that is, that permit us to carry out proofs along the same lines and to use the same methods of obtaining formulas and theorems.*

> *That, then, is the purpose of my theory. [...]*
>
> *But in clarifying the notion of the infinite we must still take into consideration a more general aspect of the question. If we pay close attention, we find that the literature of mathematics is replete with absurdities and inanities, which can usually be blamed on the infinite.*
>
> ***David Hilbert "On the infinite" 1925. Translated by Stefan Bauer-Mengelberg.***

Quotation 5 ([Eps79])

> *Another reason is because A is computable relative to $\mathbf{0}'$ if and only if it is the constructive limit of constructive procedures. The study of $\mathcal{D}(\leqslant \mathbf{0}')$ is the study of what we can computably approximate. The tension between the finite and the infinite is the vibrating center of the subject. In a full approximation construction we give a uniform computable procedure – a uniform finitistic process – which we claim satisfies some property in the limit. These are truly constructions, not constructions "relative to." These constructions have the same flavor as arguments in other areas of finite mathematics, such as number theory or graph theory.*
>
> ***Richard L. Epstein "Degrees of Unsolvability: Structure and Theory" 1979.***

Chapter 1

Motivations

1.1 Potential and Actual Infinity

The distinction between potential and actual infinity was introduced by Aristotle in [Aria] (see Quotation 1). Potential infinity may be identified with a process with arbitrarily many steps, such that only finitely many new objects are considered at each of these steps. On the other hand, actual infinity is a completed one, e.g. the collection of all objects appearing in an infinite process. We may consider, for instance, natural numbers as a completed – actually infinite – collection $\{0, 1, 2, \dots\}$ or as a potentially infinite process staring with $\{0\}$, and at each step adding the successor of the greatest number we had after the previous step. At every stage we have a finite set $\{0, 1, \dots, n\}$, but we can always *add* a larger number if we need. Aristotle treats only potential infinity as *existing*. Moreover, he claims that mathematicians do not need actual infinity. Euclid in his Elements, for instance, considers a very different concept of a line than contemporary mathematicians do – Euclid's lines are potentially infinite – they can be arbitrarily extended, yet they are always finite.

With the development of mathematical methods, especially in analysis and geometry, actual infinity started to *leak* into mathematics. This, however, resulted with objections of such great mathematicians as Gauss ([GSP60]) or Kronecker ([Kro87]). In 1831 in a letter to Schumacher Gauss protests against the use of actually infinite concepts in mathematics (see Quotation 3). It seems that Gauss shared Aristotle's view on actual infinity. While the infinite naturally appears in mathematics, it is admissible only as a way of speaking about the limits of finitistic, potentially infinite processes and should never be treated as something completed.

Nevertheless, infinitary methods in mathematics were more and more common. This is due to their effectiveness in solving mathematical problems. Actual infinity simplifies mathematical reasonings, even in such areas as combinatorics. However, there is no clear criterion of truth for actually infinite models. On the other hand, we easily decide the truth, in finite models.

1.2 Hilbert's Programme

Aristotle was one of the main proponents of the axiomatic method ([Arib]). In the late 19th century we learned how to use this method to manipulate such finite objects as formulae and sequences of formulae. It was Frege ([Fre79]) who showed how the axiomatic method can be reduced to combinatorics on finite objects. This way we obtain an axiomatic presentation of the world of actual infinities. Moreover, this presentation is reduced to manipulations of finite combinatorial objects. The question whether a statement is true has been reduced to the question whether it has a proof in a certain theory. Unfortunately, Frege's theory was inconsistent.

David Hilbert founded a research project concerning the foundations of mathematics and the scientific methods. He desired to implement it towards grounding mathematical knowledge. This research project is known as the Hilbert's programme. The paper in which his ideas are most explicitly presented is probably [Hil26] (see also [Hil02]). Hilbert clearly identifies the main source of problems in the foundations of mathematics with the use of the notion of infinity (see Quotation 4). In fact, he indicates that the problem stems from the use of infinity as a given actual totality. He aims at eliminating actual infinity from the logical inference and replacing it with potentially infinite limiting processes.

The aim of Hilbert's programme was to give a reliable mathematical basis to entire mathematics including parts employing notions involving actual infinity. Hilbert's idea was to develop a *finitistic* axiomatic theory, whose axioms would be undoubtedly true and capable of proving the consistency of the *ideal* mathematics as well as to decide every statement of it.

In his famous paper [Göd31] Kurt Gödel proves that no consistent axiomatic theory containing sufficiently many truths about natural numbers, can be complete. It follows that Hilbert's style justification of mathematics of actual infinity is not sufficient. For each current axiomatisation we can find a relatively simple (Π_1) independent sentence. This means that Hilbert's programme is not feasible. However, we still have no better idea than to use axiomatisations as foundational tools, following Hilbert. Therefore, in modern mathematics, axiomatic representations of theories involving actual infinity are pretty standard.

Following Hilbert's historical programme, the term *finitistic mathematics* is used sometimes as equivalent to being interpretable in primitive recursive arithmetic (see [Sim10], [Tai81])[1]. Therefore, we rather use the term *concrete*

[1] This approach is particularly preferred in *reverse mathematics* research area. It is devoted to describing logical relationships between classical mathematical theorems. Taking

mathematics. We intend to study concrete mathematics and particularly the concrete foundations of mathematics.

1.3 Concrete Mathematics

An important philosophical question remains: what amount of mathematics is independent of the problems caused by the use of actual infinity? This is the part of mathematics in which the truth can be decided in finite models. This is the topic of the concrete mathematics. We do not want to give any sort of new foundations of mathematics neither to reform the mathematics. We want to investigate the part of mathematics which requires no axiomatic presentation.

The choice of the term *concrete* is not accidental. It refers to *concrete mathematics* as introduced in [GKP94] by Knuth et al. for describing mathematics based on computations, algorithms and constructions. The concrete mathematics is to be *an antidote* to the abstract mathematics which is performed without any references to concrete problems and which focusses on mere generalisations. On the other hand, concrete mathematicians do not automatically rule out the abstract mathematics from their scope of interests. As discussed later in this introductory section, the computer scientific, concrete, way of thinking about mathematics and its foundations turns out to be exactly the one needed to capture the part of mathematics of potential infinity only i.e. without actual infinity involved.

In [Myc81] Mycielski presents his approach to analysis without actual infinity. He considers arbitrarily large initial segments of natural numbers. It is well known that rational numbers can be represented by natural numbers via a suitable pairing function. Thus, greater initial segments of ω contain more and more rational numbers. Real numbers are represented by rational numbers with sufficiently large denominators. Mycielski reconstructs in this framework such basic notions of analysis as the notion of a limit, continuity, derivative and integral. This seems to be quite a surprising result, since analysis seems to be a branch of mathematics inseparably associated with uncountable sets like real and complex numbers. Mycielski shows that basic notions of analysis can be approximated with the use of finite sets only. Therefore, even analysis can be developed without referring to actually infinite sets. Let us observe that the same idea is used in computer representations of real numbers.

In [Mos01] and [Mos03] Mostowski considers another approach to foundations of mathematics without actual infinity. His approach essentially conforms with that by Mycielski [Myc81].[2] Nevertheless Mostowski's idea is

---

a weaker basic theory results in obtaining more subtle description.

[2] Mycielski in his later papers (see e.g. [Myc86]) investigates locally finite theories – theories with the property that each finite subtheory has finite models. This approach

restricted to finite models only. The basic notions of Mostowski's approach are FM–domains, families of initial segments of arithmetical models, and *sl*–semantics, satisfiability in sufficiently large models.

FM–domains may be understood as models of potentially infinite worlds. Mostowski's aim was to transfer the Tarski's method of truth definitions of separating classes of formulae to the realm of finite models. One of the questions that naturally arose during the investigations was the problem of representing basic syntactic and semantic concepts with respect to the *sl*–semantics in FM–domains. An arithmetical formula φ FM–represents a relation R if for every tuple $\overline{a}$ the relation $R(\overline{a})$ holds if and only if $\varphi(\overline{a})$ is true in all sufficiently large finite models and $\neg R(\overline{a})$ holds if and only if $\neg\varphi(\overline{a})$ is true in all sufficiently large finite models. The notion of FM–representability can be explained in the following way. A relation R is FM–representable if it can be "decided" within a potentially infinite process, but still in finite models.

One of the most important results by Mostowski is the FM–representability theorem. It states that the class of FM–representable relations is equal to Δ_2^0 (for a complete survey see [MZ05b] and [Mos08]). Identifying the class of FM–representable relations with Δ_2^0 relations gives an additional insight into notions meaningful without actual infinity. By the Limit Lemma (see [Sho59]) the FM–representable notions are exactly those which are computable in the limit i.e. those which are of degree $\leqslant \mathbf{0}'$. Moreover, by the investigations on the theory of algorithmic learning due to Gold ([Gol65]) and Putnam ([Put65]) those are exactly algorithmically learnable relations. Therefore, researchers working in various fields of logic and interested in not necessarily finite but computable processes discovered Δ_2^0 as the limit for the notions they investigated.

Yet another example comes from Epstein's survey on degrees of unsolvability [Eps79]. He justifies why we should be particularly interested in degrees of unsolvability $\leqslant \mathbf{0}'$ (see Quotation 5). Epstein clearly points out that concepts $\leqslant \mathbf{0}'$ are those for which the infinite can be replaced by a finitistic – potentially infinite – processes which are constructive in the strict sense.

1.4 Concrete and Axiomatic Model Theory

In the very beginning, model theory started with a concrete approach. Kurt Gödel in [Göd30] proves the first version of completeness theorem for first order logic, by a concrete construction. Having a consistent first order sentence φ he constructs a model $\mathcal{M}$ such that $\mathcal{M} \models \varphi$ as the limit of a chain of finite models

$$\mathcal{M}_1 \subseteq \mathcal{M}_2 \subseteq \mathcal{M}_3 \subseteq \ldots$$

have less in common with Mostowski's than the approach from [Myc81].

Contemporary model theory, which started with Tarski's paper [Tar56], went into another direction. Model–theoretic constructions are performed within axiomatic set theory. In the classical monograph by Chang and Keisler [CK73] model theory is developed inside a fixed axiomatic set theory – Bernays–Morse set theory. In this way we can consider models and perform model–theoretic constructions within the *Cantor's paradise.*[3] However, *there is no such thing as a free lunch.* In this way, in model theory, we prove theorems of the form: if there exists an inaccessible cardinal, then there is a model of Zermelo–Fraenkel set theory; and: if generalised continuum hypothesis holds, then every model has a saturated elementary extension ([Hod93]).

On the other hand statements based on our computational experience are meaningful independently of decidability in any axiomatic framework. We still believe that open combinatorial problems like: the twin primes conjecture, "P=NP?", "is integer factorisation in P?" and many others are meaningful independently of any axiomatic framework. These problems belong to concrete mathematics. We search for true mathematical theorems, not only those that are provable in some axiomatic theories.

In [HB39] Hilbert and Bernays gave an arithmetised proof of the completeness theorem for first order logic. This led Kleene (see [Kle52]) to show that the model obtained by the Gödel's construction of a model of an irrefutable sentence is Δ^0_2. Therefore, the obtained model is concrete.

We call the approach to model theory presented by Chang and Keisler in [CK73] the *axiomatic model theory.* Another approach is suggested by Gödel's construction and Kleene's result. We call it the *concrete model theory.* It is developed in this dissertation.

In this dissertation we focus on the feasibility of model–theoretic constructions in concrete model theory. We consider classical model–theoretic constructions searching for their concrete content. We show difficulties of the concrete model theory and in particular, we explain which classical model–theoretic constructions fail and why they fail.

1.5 Recursive and Constructive Model Theories

The constructive and computational approach to model theory was first motivated by intuitionism. Such researchers as Kleene ([Kle52]) and Markov ([Mar54]) considered Brouwer's notion of a construction too vague to be used in rigorous mathematical discourse. They preferred to use the notion of recursivity instead of the notion of construction.

This approach leaked into various fields of mathematics, including model theory. In the 1960's, independently, two schools of mathematicians, working on different sides of the Iron Curtain, studied the topic. In America and Australia computable model theory was developed by such investigators as

[3] See [Hil02].

(among others) Nerode, Millar and Harizanov (see [Har98] for a survey). In Soviet Russia constructible model theory was developed in parallel by (among others) Nurtazin, Goncharov, Ershov and Peretyat'kin (see [Ers73]).

Some of the results obtained by both of the schools of mathematics are relevant to our investigations. Especially, by relativisation, many results concerning (purely) recursive model theory can be interpreted as the results of concrete model theory. However, the structure of degrees of unsolvability $\leqslant \mathbf{0}'$ is very rich. Therefore, in concrete model theory, there is a lot of constructions which are not just relativisations of those performed on recursive models.

Moreover, the motivations of researchers from both schools were not the same as of Mostowski who launched the project of recognizing concrete fragment of mathematics. They did not intend to identify the safe, finitistic content of model theory. They rather studied relative computational complexity of various constructions, not paying attention to the distinction between concrete and ideal mathematics.

1.6 Semantics without Actual Infinity

During his investigations on mathematics without actual infinity Mostowski observed that semantic paradoxes do not require actual infinity. In fact, a lot of them can be reproduced in finite models. As a consequence a very natural question arose: how can we represent truth in finite models? In [Mos01] and [Mos03] Mostowski investigates this topic. His research led him to the finite models version of the undefinability of truth theorem.

Therefore, there is no full (first order) notion of truth in finite models. However, questions remains: which features of truth can be represented in finite models? and is there a way to approximate the full notion of truth in finite models?

In [CZ10] Zdanowski and the author consider predicates which have certain properties of the truth predicate – the approximate truth definitions for finite models (approximate FM–truth definitions). We investigate properties of these predicates, which can be expressed in modal logic. We consider modal logic SL, which mimics the properties of approximate FM–truth definitions. As the result we obtain a Solovay–style completeness theorem ([Sol76]) which shows that SL is the modal logic of approximate FM–truth definitions.

Chapter 2

Preliminaries

In this chapter we introduce basic notions used in the dissertation and sketch historical and methodological background. We briefly present early developments of the twentieth century's mathematical logic. We present the basics of the work of Gödel: the completeness theorem and arithmetisation of syntax; the work of Turing and his descendants: Turing machines and degrees of unsolvability; and Tarski's research: the approach on the mathematisation of the concept of truth. Further, we present more recent results by Mostowski, concerning representing concepts in potentially infinite worlds. Finally, we sketch the basics of modal logic, which are used in Chapter 3.

2.1 Basic Notions

In this dissertation we work with first order logic (in Chapter 4) and modal logic (in Chapter 3). We consider arbitrary relational vocabularies with finitely many predicates built from the following predicates:

$$\mathrm{P}^0_0, \mathrm{P}^0_1, \ldots, \mathrm{P}^1_0, \mathrm{P}^1_1, \ldots, \mathrm{P}^2_0, \mathrm{P}^2_1, \ldots,$$

where the upper index indicates the arity of a predicate and the lower index is the position of a predicate in sequence of predicates of given arity. We write $ar(P)$ for the arity of a predicate P.

We additionally augment the language with an infinite set of constants $\mathrm{Consts} = \{\mathfrak{c}_0, \mathfrak{c}_1, \ldots\}$.

We therefore consider vocabularies of the form $\sigma = (P_1, \ldots, P_k, ar, C)$, where $C \subseteq \mathrm{Consts}$, $P_1, \ldots, P_k$ are predicates and $ar : \{P_1, \ldots, P_k\} \to \omega$ is the arity function. Since there are no function symbols, the only terms are variables $\mathrm{Var} = \{v_0, v_1, \ldots\}$ and constants Consts. Then, the set Term_σ of terms of a vocabulary $\sigma = (P_1, \ldots, P_k, ar, C)$ is defined as $\mathrm{Var} \cup C$. The set of atomic formulae in a vocabulary σ (atomic σ–formulae), in symbols FormAt_σ is defined as follows:

- If $t, s \in \mathrm{Term}_\sigma$, then $t = s$ is as an atomic σ–formula,

- If P is a predicate in σ such that $ar(P) = n$ and $t_1 \ldots, t_n \in \mathrm{Term}_\sigma$, then $P(t_1, \ldots, t_n)$ is an atomic σ–formula.

The set of first order formulae of a vocabulary σ (σ–formulae), in symbols Form_σ, is defined inductively as follows:

- If $\varphi \in \mathrm{FormAt}_\sigma$, then φ is a formula,
- If φ is a formula, then $\neg\varphi$ is a formula,
- If φ, ψ are formulae, then $(\varphi \wedge \psi)$ is a formula,
- If φ is formula and x is a variable, then $(\exists x\, \varphi)$ is a formula.

Other connectives are considered as abbreviations: $(\varphi \Rightarrow \psi)$ stands for $\neg(\varphi \wedge \neg\psi)$, $(\varphi \vee \psi)$ stands for $\neg(\neg\varphi \wedge \neg\psi)$ and finally $(\forall x\, \varphi)$ stands for $\neg(\exists x\, \neg\varphi)$. We also skip unnecessary parentheses. The set Lit_σ of the literals in a vocabulary σ is defined as the set of atomic σ–formulae and their negations.

Since we restrict our considerations to first order logic we refer to first order formulae as to formulae. The notion of a free variable is defined inductively in a standard way. For terms $\mathrm{FV}(x) = \{x\}$ and $\mathrm{FV}(c) = \varnothing$, where $x \in \mathrm{Var}$ and $c \in \mathrm{Consts}$. For formulae $\mathrm{FV}(t_1 = t_2) = \mathrm{FV}(t_1) \cup \mathrm{FV}(t_2)$, $\mathrm{FV}(P(t_1, \ldots, t_{ar(P)})) = \bigcup_{i=1,\ldots,ar(P)} \mathrm{FV}(t_i)$, $\mathrm{FV}(\neg\varphi) = \mathrm{FV}(\varphi)$, $\mathrm{FV}(\varphi \wedge \psi) = \mathrm{FV}(\varphi) \cup \mathrm{FV}(\psi)$ and finally $\mathrm{FV}(\exists x\, \varphi) = \mathrm{FV}(\varphi) - \{x\}$. We say that a term t is closed if it has no free variables and that a formula φ is a sentence if it has no free variables i.e. $\mathrm{FV}(t) = \varnothing$ and $\mathrm{FV}(\varphi) = \varnothing$. We denote the set of sentences of a vocabulary σ by Sent_σ and we refer to its elements as σ–sentences.

A theory is an arbitrary set of sentences. A sentence φ is a consequence of a theory T, in symbols $T \vdash \varphi$, if there is a finite sequence of sentences $\varphi_0, \varphi_1, \ldots, \varphi_n$ such that $\varphi = \varphi_n$ and for every $i = 0, \ldots, n$ it is either the case that φ_i is a logical axiom, $\varphi_i \in T$ or there are $j, k < i$ such that $\varphi_k = (\varphi_j \Rightarrow \varphi_i)$. In the definition above we use arbitrary fixed complete axiomatisation of first order logic i.e. a set of axioms which enables to prove every logical tautology (see e.g. [AZ11]). The set of all consequences of a theory T is denoted by $\mathrm{Cn}(T)$. Note that we do not require theories to be closed under consequence.

We use the symbol $\perp$ to abbreviate a contradictory sentence e.g. $\exists x\, x \neq x$. A theory T is consistent if $\perp \notin \mathrm{Cn}(T)$. A theory T is complete in a vocabulary σ if for every σ–sentence φ it is either the case that $\varphi \in T$ or $\neg\varphi \in T$. A theory T has the witness property for σ in some set of constants C if for every σ–formula $\varphi(x)$ such that $\mathrm{FV}(\varphi) = \{x\}$, there is $c \in C$ such that $(\exists x\, \varphi(x) \Rightarrow \varphi(c)) \in T$.[1] Theories which are consistent, complete in σ and which have witness property for σ in some set of constants C play an

[1]Formulae of the form: $\exists x\, \varphi(x) \Rightarrow \varphi(c)$ are called Henkin's axioms.

important role in our work. If a theory T has this properties we say that T is $\mathrm{CCW}(\sigma, C)$.

For $n \in \omega$, we use a shorthand $\exists^{=n}$ for quantifiers "there exist exactly n elements" and $\exists^{>n}$ for the quantifiers "there exist more than n elements". Similarly, we use also $\exists^{<n}$, $\exists^{\leqslant n}$ and $\exists^{\geqslant n}$ in obvious meanings.

The quantifier rank of a formula φ, $\mathrm{rk}(\varphi)$, is defined in a usual way, i.e. $\mathrm{rk}(\varphi) = 0$ if φ is an atomic formula, $\mathrm{rk}(\neg\varphi) = \mathrm{rk}(\varphi)$, $\mathrm{rk}(\varphi \wedge \psi) = \max\{\mathrm{rk}(\varphi), \mathrm{rk}(\psi)\}$, and $\mathrm{rk}(\exists x\varphi) = 1 + \mathrm{rk}(\varphi)$.

2.2 Early Foundations

2.2.1 Arithmetics and Arithmetisation

Since we work with relational vocabularies, we consider a relational version of the standard model of arithmetic: $\mathcal{N} = (\omega, R_+, R_\times, \leqslant)$. Here R_+ is the ternary relation of addition, $R_\times$ is the ternary relation of multiplication and $\leqslant$ is – the usual – binary ordering relation. In Chapter 3 we consider the following expansion of the standard model of arithmetic: $(\mathcal{N}, n)_{n\in\omega}$ instead of $\mathcal{N}$. This allows to have terms naming every natural number without introducing any function symbols to the vocabulary. By $\underline{n}$ we denote the n–th numeral.

An arithmetical formula φ is bounded or Δ_0 if all quantifiers occurring in φ are of the form $(Qx \leqslant t)$, where $Q \in \{\exists, \forall\}$ and t is a term in which x does not occur. By Σ_n we denote the set of formulae which begin with a block of existential quantifiers and have $n-1$ alternations of blocks quantifiers followed by a bounded formula. Similarly, φ is Π_n if it begins with a block of universal quantifiers and has $n-1$ alternations of blocks of quantifiers followed by a bounded formula. Let us observe that Σ_0 as well as Π_0 formulae are exactly bounded formulae.[2]

An arithmetical relation R is Σ^0_n (Π^0_n) if it is defined (in $\mathcal{N}$) by a Σ_n (Π_n) formula. A relation is Δ^0_n if it is both Σ^0_n and Π^0_n. By Tarski's undefinability of truth theorem, which is discussed later, it is known that these classes of formulae form a strict hierarchy of definable arithmetical relations – so called arithmetical hierarchy. The arithmetical hierarchy is presented in the following diagram.

[2]We use the notation Σ_n and Π_n not only for arithmetical formulae, but also for formulae of arbitrary vocabulary. In such contexts the base case are quantifier free formulae. The inductive case remains the same.

$$
\begin{tikzcd}
& \Pi^0_1 \arrow[dr, phantom, sloped, "\subsetneq"] & & \Pi^0_2 \arrow[dr, phantom, sloped, "\subsetneq"] & & \Pi^0_3 & \\
\Delta^0_1 \arrow[ur, phantom, sloped, "\subsetneq"] \arrow[dr, phantom, sloped, "\subsetneq"] & & \Delta^0_2 \arrow[ur, phantom, sloped, "\subsetneq"] \arrow[dr, phantom, sloped, "\subsetneq"] & & \Delta^0_3 \arrow[ur, phantom, sloped, "\subsetneq"] \arrow[dr, phantom, sloped, "\subsetneq"] & & \cdots \\
& \Sigma^0_1 \arrow[ur, phantom, sloped, "\subsetneq"] & & \Sigma^0_2 \arrow[ur, phantom, sloped, "\subsetneq"] & & \Sigma^0_3 &
\end{tikzcd}
$$

Given a little bit of arithmetic we can encode syntax via standard Gödel numbering. First, we need to set codes of basic (primitive) elements of the language. We want to use only finitely many primitive symbols. Therefore, all infinite sets such as variables, constants and predicates are encoded by (binary) sequences of primitive symbols. To encode variables we introduce two primitive symbols 0_v and 1_v. For $i \in \omega$, the variable v_i is written as a sequence of 0_vs and 1_vs corresponding to the binary presentation of i. Similarly, for constants, we introduce two additional symbols 0_c and 1_c. For predicates we need to express both the arity and the *position* in the enumeration of all predicates with given arity. Therefore, we introduce 0_{ar}, 1_{ar}, 0_P and 1_P. Therefore, variable v_3 is written as 1_v1_v, constant $\mathfrak{c}_4$ is written as $1_c0_c0_c$ and predicate P^1_2 is written as $1_{ar}1_P0_P$.

The length of the variable v_i for $i > 0$ is equal to $\lceil \log_2(i+1) \rceil$. Similarly, the length of the constant $\mathfrak{c}_i$ for $i > 0$ is equal to $\lceil \log_2(i+1) \rceil$. For $i, j > 0$, the length of the predicate P^i_j is equal to $\lceil \log_2(i+1) \rceil + \lceil \log_2(j+1) \rceil$. For a formula φ by $|\varphi|$ we denote the length of φ.

The other primitive symbols that we need to encode the syntax of first order logic are: the existential quantifier $\exists$, propositional connectives $\neg$ and $\wedge$, the identity symbol $=$, parentheses: (and) and a comma. We put: $\ulcorner 0_v \urcorner = 1$, $\ulcorner 1_v \urcorner = 2$, $\ulcorner 0_c \urcorner = 3$, $\ulcorner 1_c \urcorner = 4$, $\ulcorner 0_{ar} \urcorner = 5$, $\ulcorner 1_{ar} \urcorner = 6$, $\ulcorner 0_P \urcorner = 7$, $\ulcorner 1_P \urcorner = 8$, $\ulcorner \exists \urcorner = 9$, $\ulcorner \neg \urcorner = 10$, $\ulcorner \wedge \urcorner = 11$, $\ulcorner = \urcorner = 12$, $\ulcorner (\urcorner = 13$, $\ulcorner) \urcorner = 14$ and finally $\ulcorner , \urcorner = 15$.

Let $p_0, p_1, \ldots$ be an increasing enumeration of prime numbers. We can extend the function $\ulcorner - \urcorner$ to the function over arbitrary strings $A^{<\omega}$ over an alphabet A containing all primitive symbols in the standard way:

$$\ulcorner - \urcorner \colon A^{<\omega} \to \omega,$$

by putting for $\alpha = a_0 \ldots a_n \in A^{<\omega}$:

$$\ulcorner \alpha \urcorner = \ulcorner a_0 \ldots a_n \urcorner = p_0^{1+a_0} \cdot \cdots \cdot p_n^{1+a_n}.$$

The function $\ulcorner - \urcorner$ enables to encode and decode effectively every string of primitive symbols. Therefore, with $\ulcorner - \urcorner$ we can encode every formula and every sequence of formulae as a natural number. This encoding is proper i.e. injective (by the fundamental theorem of arithmetic), recursive and with recursive decoding. Moreover, we also use $\ulcorner - \urcorner$ to encode finite sequences of natural numbers and – in general – any finite object.

In Chapter 4 we mention axiomatic theories such as Peano Arithmetic PA and Zermelo–Fraenkel set theory ZF. We take arbitrary axiomatisations of these theories. In the case of PA we need to translate the axioms into the relational vocabulary.

2.2.2 Model Theory – Mathematisation of Truth

In [Tar33] Tarski considers the notion of truth for formalised languages. His considerations are the milestone for the whole branch of mathematics called model theory, which emerged in his and his student's later papers ([Tar56], [TV58]). The ideas introduced by Tarski seem to be very simple. Yet, in the times of his investigations on the notion of truth, semantics – in general – were considered metaphysical and therefore not admissible in scientific discourse.

Tarski introduced the notion of a model and gave the rigorous definition of the satisfaction relation. It is his investigations that allow to deal with semantics in a strict mathematical manner.

We present the notion of truth definition in the context of arithmetics. We say that τ is a truth definition for arithmetical vocabulary if for every arithmetical sentence ψ it holds that:

$$\mathcal{N} \models \psi \equiv \tau(\ulcorner \psi \urcorner).$$

Tarski proved in [Tar33] the famous undefinability of truth theorem which states that there is no arithmetical truth definition for arithmetical sentences.

Theorem 2.2.1 (Tarski's Undefinability of Truth Theorem ([Tar33])) *There is no first order arithmetical formula τ such that for every first order arithmetical sentence ψ:*

$$\mathcal{N} \models \psi \equiv \tau(\ulcorner \psi \urcorner).$$

The undefinability of truth theorem can be stated in various forms for any sufficiently expressible language. Tarski invented the method of truth definitions which enables separate languages with respect to their expressive power. It follows from his consideration that the arithmetical hierarchy is strict and that for any vocabulary there is a truth definition for n–th order logic sentences in the $(n+1)$–th order logic.

Some semantical remarks and considerations were present in mathematics even before Tarski. The aim of the Hilbert's programme, for instance, was to find a *finitistic* axiomatic theory which is complete and capable of deciding the truth of every mathematical sentence. Such theories, often called systems, were the first attempts on grasping mathematical truth – by provability within an appropriate system. A notion similar to the notion of a model was also used before Tarski. Those were – so called – domains of individuals. However, there was no notion of valuation in a domain of individuals nor the notion of satisfaction, but still the notion of truth of a sentence in a given domain of individuals was used as an intuitive notion.

In his doctoral dissertation from 1929 Gödel considers and proves the completeness theorem for first order logic. It was first stated by Russell and Whitehead in Principia Mathematica ([WR27]). However, since the theorem is semantical in its nature, Russell and Whitehead were not very concerned with it, since it exceeded the provability within the system they introduced (see [Göd02a]).

Gödel's completeness theorem was presented in The Königsberg Congress in 1930. However – again – since the theorem has a semantical nature, his presentation did not raise great interest. A sentence is called refutable if $\vdash \neg\psi$ i.e. if the negation of ψ is provable from the axioms of first order logic. A sentence ψ is satisfiable if there is a *domain of individuals* in which ψ is *true*. The completeness theorem is stated as follows.

Theorem 2.2.2 (Gödel's Completeness Theorem ([Göd30])) *Let ψ be a first order sentence. Then one of the following holds:*

- *ψ is refutable,*
- *ψ is satisfiable.*

The original proof of Gödel's completeness theorem (see [Göd30]) is especially interesting for us, because the key argument used in it requires no actual infinity. In the case of ψ not being refutable Gödel constructs a countable domain of individuals in which ψ is true. The construction goes by a sequence of finite models forming a chain (in the modern terminology) such that its union satisfies the irrefutable sentence ψ. Gödel's proof of the completeness theorem is one of the motivations of this dissertation. It shows that no actual infinity is involved in the notion of completeness. However, the tools provided by Tarski enabled one to study model theory within set theory using the whole variety of infinitary methods. The modern model theory is developed in a suitable axiomatic set theory, which is explicitly stated in textbooks on the subject e.g. [CK73].

2.3 Computability Theory

2.3.1 Turing Machines

In 1936 Turing published his famous paper [Tur36] on the notion of computability. It was historically the first so comprehensive and philosophically grounded elaboration of the subject. Turing introduces his *automatic machines* which are commonly known today as Turing machines. Turing machines have a certain place in majority of the modern academic courses in computability theory. These machines may look quite strange on a first sight and it is mostly hard to understand the underlying algorithm that is encoded in a transition function, but Turing put a lot of effort in justifying the final appearance of his model of computation. This clearly is an abstraction from observations of how we perform computations based on a given procedure and also from investigations which procedures are admissible.

Intuitively, a person performing a computation – a computer – needs only a sheet of paper, a pencil and a rubber. The computer starts with a problem written down on a sheet of paper and uses a pencil and a rubber to write down new symbols on it or erase symbols from it to get the answer to the problem. Symbols that appear during the computation have to be distinguishable from one another thus they have to be no smaller than of certain size – therefore we may assume that the sheet is squared and each square can contain exactly one symbol out of a finite number of symbols or remain blank. The computer during the computation is in one of the finite number of mental states that reflect the sub–task of the computation she currently preforms. Turing pointed out also that the person performing the computation has only finite range of sight – thus we may assume without loss of generality that at one time she only reads a symbol from exactly one square and can change it according to its current mental state. From this intuition Turing abstracted the notion of *automatic machine* i.e. the notion of a Turing machine.

A Turing machine consists of (potentially) infinite tape which is divided into cells and a head that moves through the tape reading and writing on cells. The tape is an abstraction of the sheet of paper and the head is an abstraction of a pencil. The mental states of the computer become the states of the machine in Turing's abstraction.

The formal definition of the Turing machine is as follows. A Turing machine is a tuple $M = (Q, \Gamma, BLANK, \Sigma, \delta, q_0, F)$, where Q is a finite, non–empty set of states, Γ is a finite, non–empty set of tape symbols, $BLANK \in \Gamma$ is a special blank symbol, $\Sigma \subseteq \Gamma - \{BLANK\}$ is the alphabet of the input data, $\delta\colon (Q - F) \times \Gamma \to Q \times \Gamma \times \{L, 0, R\}$ encodes an algorithm that the machine performs, $q_0 \in Q$ is the initial state at which the computation starts and $F \subseteq Q$ is the set of final states at which the computation stops.

Note that every Turing machine is a finite object – it consists of finite sets – and therefore it can be effectively encoded by a single natural number e.g. its Gödel number. This encoding can be made such that from a given code one can algorithmically extract the the full description of the Turing machine.

By Φ_i we denote the i–th Turing machine with respect to the size of its encoding. For $i, n \in \omega$ we write $\Phi_i(n)\downarrow$ if the i–th Turing machine eventually halts given n as an input and $\Phi_i(n)\uparrow$ otherwise. Turing machines can give outputs of computations for instance as numbers encoded by binary sequences that remain on the tape after the machine stops. This enables Turing machines to compute functions or to decide sets i.e. compute their characteristic functions. For $i, n, k \in \omega$ we write $\Phi_i(n)\downarrow = k$ when the output of the i–th Turing machine is k given input n. A set A is said to be partially decided (partially computed) by a Turing machine Φ_i if $A = \{n \in \omega \colon \Phi_i(n)\downarrow = 1\}$. A set A is decided (computed) by a Turing machine Φ_i if it is partially decided by Φ_i and $\omega - A = \{n \in \omega \colon \Phi_i(n)\downarrow = 0\}$. A set is partially decidable (partially recursive, recursively enumerable) if there is a Turing machine which partially decides it. A set is decidable (recursive, computable) if there is a Turing machine which decides it.

Since Turing machines can be encoded as natural numbers, we can give as an input for a Turing machine U a pair of a code of another Turing machine M and an input to it n and make U simulate the computation of M on the input n. Such U is called a universal Turing machine. The existence of universal Turing machines lead to the famous undecidability of the halting problem – there is no algorithm (Turing machine) which takes on the input pairs $(\ulcorner M \urcorner, n)$, where M is a Turing machine M and decides whether M halts on the input n. Therefore the set $K = \{(i, j) \colon \Phi_i(j)\downarrow\}$ – the halting problem – is undecidable. Similarly undecidable is the set $S = \{i \colon \Phi_i(i)\downarrow\}$. Both sets K and S are computationally equivalent (see the following section for precise statement of this equivalence). Therefore we refer to both of them as to the halting problem. The statement of the undecidability of hating problem is the following.

Theorem 2.3.1 (Undecidability of Halting Problem (see [Sho59])) *The halting problem is undecidable.*

Apart from Turing machines there are several different models of computations that played an important role in the theory of computability. One of the most significant are lambda calculus, RAM machines, while–programs, partially recursive functions and Σ_1 formulae. These were all shown to be equivalent to Turing machines i.e. functions (partially) computable in each of these models are exactly the same as functions (partially) computable by Turing machines. Most of the modern programming languages resemble while–programs. This is because while–programs are easier to understand and analyse by humans than other computation models. We therefore, for the sake of clarity, write while–program–like algorithms in this dissertation, rather than use any other model of computations.

We finish this section with two important in computability theory theorems, which characterise recursive sets.

Theorem 2.3.2 (Post's Theorem (see [Sho59])) *Let $A \subseteq \omega$. The following are equivalent:*

- *A is recursive,*
- *A and $\omega - A$ are recursively enumerable.*

Theorem 2.3.3 ((See [Sho59])) *Let $A \subseteq \omega$. The following are equivalent:*

- *A is recursive,*
- *there is an increasing enumeration $a_0, a_1, \dots$ of elements of A and an algorithm computing $i \mapsto a_i$.*

An increasing enumeration of a recursive set A as in the previous theorem we call an effective presentation of A.

2.3.2 Degrees of Unsolvability

In this section we show how to extend the model of computation based on Turing machines with some additional capabilities. We introduce relativised computations i.e. we allow Turing machines to use as a black–box, an oracle, some set for which the machine can ask for the membership in this set. The computation with such an oracle is called a computation relative to the oracle. Such computations were first mentioned by Turing in [Tur39] and further developed by Turing, Post and Kleene ([Pos44], [KP54]).

Oracle Turing machines have an additional oracle tape and oracle head used only for querying the oracle. Oracle Turing machines have also additional states to control the head of the oracle and additional states ASK, YES, NO such that if an oracle Turing machine is in state ASK the next state becomes YES if the number encoded by symbols on the oracle tape is in the oracle and NO otherwise. In both cases the oracle tape is cleared after the answer is given. In while–program–like algorithms we present in this dissertation relativised computations are introduced by adding an additional expression of the form $i \in \mathcal{O}$, where $i \in \omega$ to the syntax. The meaning of $i \in \mathcal{O}$ is obviously "i belongs to the oracle".

Recall that every Turing machine, therefore every algorithm, can by encoded as a natural number. This encoding is effective in sense that given a Turing machine M one can recursively calculate its code $\ulcorner M \urcorner$ and given a natural number i one can recursively decode it to obtain full description of a Turing machine M such that $i = \ulcorner M \urcorner$. Oracle Turing machines are not very different structurally from the ordinary Turing machines – they are only enriched with additional instructions for asking the oracle, thus they can be encoded similarly to the ordinary Turing machines. For $i \in \omega$ and $f \in 2^\omega$, by Φ_i^f we denote the oracle Turing machine with code i and oracle f.[3] Similarly to the regular Turing machines we write $\Phi_i^f(j)\downarrow$, if the i–th oracle Turing

[3]Here we identify sets with their characteristic functions, which are more convenient for presentation of the content of this section.

machine with oracle f halts on input j and $\Phi_i^f(j)\uparrow$ otherwise. We also write $\Phi_i^f(j)\downarrow = k$ to indicate that the i–th oracle Turing machine with oracle f halts on input j and outputs k and if $k \in \{0,1\}$ we can understand the machine to compute a characteristic function i.e. to decide the membership relation of some set.

The set of natural numbers $\{j \in \omega \colon \Phi_i^f(j)\downarrow = 1\}$ is said to be partially computed in f (partially decided in f) by the i–th Turing machine with oracle f. If it holds that for every $j \in \omega$, $\Phi_i^f(j)\downarrow$ and outputs 0 or 1 then the i–th oracle Turing machine with oracle f is said to compute (or decide) the set $\{j \in \omega \colon \Phi_i^f(j)\downarrow = 1\}$ in f. A set is partially decidable in f (partially computable in f, recursively enumerable in f), if there is an oracle Turing machine which partially computes it, with f as an oracle.

Let $f, g \in 2^\omega$ be sets of natural numbers. We say that f is recursive in g (computable in g, decidable in g), in symbols $f \leqslant_T g$, if there is an oracle Turing machine which, with g as an oracle, computes f.

Let $\mathcal{K} \subseteq \mathcal{P}(\omega)$ be a family of sets. We say that g is $\mathcal{K}$–hard if for every $f \in \mathcal{K}$ it holds that $f \leqslant_T g$. We say that g is $\mathcal{K}$–complete if $g \in \mathcal{K}$ and g is $\mathcal{K}$–hard.

Note that $\leqslant_T$ is a pre–order on subsets of ω. We say that f and g are Turing equivalent, in symbols $f \equiv_T g$, if both $f \leqslant_T g$ and $g \leqslant_T f$ hold. Turing equivalence is an equivalence relation on subsets of ω. Equivalence classes of relation $\equiv_T$ are called Turing degrees or degrees of unsolvability and for $f \in 2^\omega$ the degree of f is denoted by $\deg(f)$. The pre-order $\leqslant_T$ on subsets of ω induces an order $\leqslant$ on Turing degrees, which is defined by $\deg(f) \leqslant \deg(g)$ if and only if $f \leqslant_T g$. The set of all Turing degrees is denoted by $\mathcal{D}$. Usually Turing degrees are denoted by lowercase boldface letters $\boldsymbol{a}, \boldsymbol{b}, \ldots$.

We focus on the properties of $\mathcal{D}$. First let us note that there is a least element with respect to $\leqslant$, denoted by $\mathbf{0}$, in $\mathcal{D}$ – it is the degree of all recursive sets. For every two sets $f, g \in 2^\omega$ we define $f \oplus g$ as their recursive sum i.e. for every $i \in \omega$, $(f \oplus g)(2i) = 1$ if and only if $f(i) = 1$ and $(f \oplus g)(2i+1) = 1$ if and only if $g(i) = 1$. The operation $\oplus$ induces a (recursive) join operation on Turing degrees which is denoted by $\cup$ and is defined in the following way: $\deg(f) \cup \deg(g) = \deg(f \oplus g)$. Taking recursive sum of two oracles as an oracle may be seen as enabling Turing machines to use both oracles during their computations. The degree $\deg(f \oplus g)$ is the least upper–bound of $\deg(f)$ and $\deg(g)$ in $(\mathcal{D}, \leqslant)$, hence $(\mathcal{D}, \leqslant)$ is an upper semi–lattice. There is one more very important operation on sets of natural numbers which transfers to Turing degrees – this operation is connected with halting problem i.e. $f^* = \{i \in \omega \colon \Phi_i^f(i)\downarrow\}$. f^* is called the halting problem for f. By f' we denote $\deg(f^*)$ and call it the (Turing) jump of $\deg(f)$. The argument analogous to the one used to show that the halting problem for ordinary Turing machines is not recursive can be generalised to work for oracle Turing machines too – showing that for every $f \in 2^\omega$ the set f^* is computably harder then f i.e. $\deg(f) < \deg(f^*) = \deg(f)'$. Another important property of jump operator is its monotonicity i.e. for each $f, g \in 2^\omega$ if $\deg(f) \leqslant \deg(g)$, then $\deg(f)' \leqslant \deg(g)'$.

The structure $(\mathcal{D}, \leqslant, \cup, ')$ of Turing degrees is known to be extremely complicated. It is known (see [Sim77]) that it is bi–interpretable with the second order theory of the standard model of arithmetic $\mathcal{N}$. We are interested in a small part $\mathcal{D}(\leqslant \mathbf{0}')$ of $\mathcal{D}$ – only those Turing degrees which are $\leqslant \mathbf{0}'$. Among these degrees, there is a particularly interesting class of Turing degrees – the low degrees i.e. those $\boldsymbol{a}$ for which it holds that $\boldsymbol{a}' \leqslant \mathbf{0}'$. A set is said to be low if it belongs to a low degree.

When we consider a specific set or relation that is computed by a Turing machine with a specific oracle we need to encode the oracle, as obviously different oracles lead to different performance of a Turing machine. Since we are especially interested in $\leqslant \mathbf{0}'$ concepts we need only to be able to effectively encode Turing machines with recursively enumerable oracles. This can be done easily by representing Turing machines with recursively enumerable oracles as pairs of codes of Turing machines $(\ulcorner M \urcorner, \ulcorner O \urcorner)$, where M computes the algorithm (with additional oracle instructions) and O is a Turing machine partially computing the oracle.

We say that $T \subseteq 2^{<\omega}$ is a binary tree if for all $\sigma, \tau \in 2^{<\omega}$ if $\tau \subseteq \sigma$ and $\sigma \in T$, then $\tau \in T$, where $\tau \subseteq \sigma$ means that there is $\tau' \in 2^{<\omega}$ such that $\tau\tau' = \sigma$ i.e. τ is an initial segment of the sequence σ. Thus we identify binary trees with sets of their branches encoded by binary sequences where empty sequence stands for the root of the tree and for every finite sequence σ, $\sigma 0$ is its left and $\sigma 1$ its right child. Since we only consider binary trees we refer to them as trees. We say that a tree T has a certain computational property if relation $\sigma \in T$ has this property e.g. T is recursive if relation $\sigma \in T$ is recursive and T is low if $\sigma \in T$ is low. Here we need, of course, to code nodes of the tree in some suitable way and we use Gödel numbering for this purpose i.e. $\ulcorner \sigma \urcorner = p_0^{1+\sigma(0)} p_1^{1+\sigma(1)} \dots p_{\mathrm{lh}(\sigma)-1}^{1+\sigma(\mathrm{lh}(\sigma)-1)}$. For $f \in 2^\omega$ and $i \in \omega$ by $f{\upharpoonright}i$ we denote a finite sequence σ of length i such that for all $n < i$, $f(n) = \sigma(n)$ holds.

With every tree T we associate a class of subsets of ω which are coded by its infinite branches i.e. $[T] = \{f \in 2^\omega \colon \forall x \in \omega\, f{\upharpoonright}x \in T\}$.

Definition 2.3.4 (Π^0_1–class) *A class $\mathcal{C} \subseteq 2^\omega$ is called a Π^0_1 class if there is a recursive set A such that $\mathcal{C} = \{f \in 2^\omega \colon \forall i \in \omega\, A(f{\upharpoonright}i)\}$.*

We are ready to state the Low Basis Theorem.

Theorem 2.3.5 (Low Basis Theorem ([JS72])) *Every non-empty Π^0_1 class contains a low member.*

The original proof of the theorem can be found in [JS72]. However, it uses a topological argument and gives a rather vague insight on how the low member really looks like. We present our own very different – algorithmic – proof of the Low Basis Theorem by showing the explicit $\mathbf{0}'$ oracle algorithm for computing a low member of a Π^0_1 class. We base our algorithm on the original proof from [JS72] but we construct the low member rather inductively than by taking an element of the intersection of an infinite sequence of recursive trees. The original statement of the Low Basis Theorem is too weak to our purposes but, nevertheless, the proof itself can be strengthened easily to a form that we can use in our further investigations.

It is also more convenient to work with trees rather than classes. In what follows we work with trees understood both as sets of their finite branches $\sigma \in 2^{<\omega}$ and as sets of codes of those branches. To avoid ambiguity we write $T(i) = 1$ for *the branch with Gödel number equal to* i *is in* T and $\sigma \in T$ for the standard meaning – the branch σ is in T.

We introduce one more abbreviation for $i, j \in \omega$ and $g \in 2^\omega$: $\Phi_i^{\sigma \oplus g}(j)\downarrow$ – it means that the i–th oracle Turing machine halts on input j after at most $\mathrm{lh}(\sigma)$ steps using a partial oracle $\sigma \oplus g$. This means that for $k \geqslant \mathrm{lh}(\sigma)$, whenever the machine asks: "$2k \in \sigma \oplus g$?"[4], it loops. Note that for $f, g \in 2^\omega$, $\Phi_i^{f \oplus g}(j)\downarrow$ if and only if there is $n \in \omega$ such that $\Phi_i^{f\upharpoonright n \oplus g}(j)\downarrow$ and dually $\Phi_i^{f \oplus g}(j)\uparrow$ if and only if for each $n \in \omega$ it holds that $\Phi_i^{f\upharpoonright n \oplus g}(j)\uparrow$.

Our aim is to prove the Low Basis Theorem in the following form: Let $\mathcal{T}$ be an infinite low tree. Then there is a low $f \in 2^\omega$ such that $f \in [T]$ and $f \oplus \mathcal{T}$ is low.

Let $\mathcal{T}$ be an infinite low tree. We present the general construction and proceed with a series of lemmata concerning its properties. Then, we prove the Low Basis Theorem.

First let us define a sequence of trees recursive in $\mathcal{T}$:

$$U_n = \{\sigma \in 2^{<\omega} \colon \Phi_n^{\sigma \oplus \mathcal{T}}(n)\uparrow\}.$$

Now let us inductively define a descending sequence of trees as follows.

$$T_0 = \mathcal{T},$$

$$T_{n+1} = \begin{cases} T_n & \text{if } T_n \cap U_n \text{ is finite} \\ T_n \cap U_n & \text{else} \end{cases}.$$

Lemma 2.3.6 *For every* $n \in \omega$ *the tree* T_n *is infinite and recursive in* $\mathcal{T}$*, thus low.*

Proof: The proof goes by induction. For $n = 0$ we have $T_0 = \mathcal{T}$, thus T_0 is obviously infinite and recursive in $\mathcal{T}$. Suppose for the inductive hypothesis that for $n \in \omega$, T_n is recursive in $\mathcal{T}$ and infinite. If $T_n \cap U_n$ is finite,

[4] Which is equivalent to "$k \in \sigma$?$''$.

then $T_{n+1} = T_n$ and the thesis holds by the induction hypothesis. Otherwise $T_n \cap U_n$ is infinite and $T_{n+1} = T_n \cap U_n$ which is infinite by the assumption and since both T_n and U_n are recursive in $\mathcal{T}$, also T_{n+1} is recursive in $\mathcal{T}$. $\square$

For each $i \in \omega$ we consider the following algorithm M_i with oracle g. It halts if and only if $g(i) = 1$.

Algorithm 1 Procedure M_i

```
if g(i) = 1 then
  return true
else
  while true do
  end while
end if
```

Each of the algorithms M_i ignores its input and halts if and only if i is in the oracle. It is obvious that the mapping $i \mapsto \ulcorner M_i \urcorner$ is recursive.

Let us define the set f that is further shown to be low and such that $f \in [\mathcal{T}]$ and $\deg(f \oplus \mathcal{T})' \leqslant \mathbf{0}'$. For every $i \in \omega$ we put $f(i) = 1$ if and only if $T_{\ulcorner M_{2i} \urcorner + 1} \cap U_{\ulcorner M_{2i} \urcorner + 1}$ if finite. Algorithm 2 computes f using an oracle of degree $\mathbf{0}'$. Note that in our algorithms we use a certain convention from the objective programming paradigm: we use objects, such as for instance recursive in $\mathcal{T}$ trees, in our programs. This, however, is not a problem since every low tree has the degree $\leqslant \mathbf{0}'$. Therefore, it can be represented as a finite object – as a pair of natural numbers, as we explained earlier in this section. We also use the procedure IsFinite which takes a low tree as an input and returns 1 if the given tree is finite and 0 otherwise. This procedure can be performed with an oracle of degree $\mathbf{0}'$ for every low tree, since finiteness of a low tree T can be defined as follows:

$$\exists n \, \forall \sigma \, (\mathrm{lh}(\sigma) = n \Rightarrow \sigma \notin T).$$

This is existential quantification over a recursive in T, thus low, set. Observe also that the intersection of two recursive in $\mathcal{T}$ trees is also a recursive in $\mathcal{T}$ tree.

Algorithm 2 Low Basis Algorithm computing low infinite branch of $\mathcal{T}$

Input: $i \in \omega$
Output: $f(i)$

```
Array⟨recursive in 𝒯 tree⟩ T
Array⟨recursive in 𝒯 tree⟩ U
T[0] ← 𝒯
a ← ⌜M_{2i}⌝ + 1
for j ← 0 to a do
  U[j] ← {σ ∈ 2^{<ω} : Φ_j^{σ⊕𝒯}(j) ↑}
  if IsFinite(T[j] ∩ U[j]) then
    if j = a then
      return 1
    end if
    T[j+1] ← T[j]
  else
    if j = a then
      return 0
    end if
    T[j+1] ← T[j] ∩ U[j]
  end if
  j ← j + 1
end for
```

We are going to show a series of lemmata in our way to prove the Low Basis Theorem.

Lemma 2.3.7

$$\forall i \in \omega \, \forall g \in 2^{\omega} \, \Phi^{g}_{\ulcorner M_i \urcorner}(\ulcorner M_i \urcorner)\downarrow \;\equiv\; g(i) = 1.$$

Proof: Fix $i \in \omega$ and $g \in 2^{\omega}$. The following are equivalent:

- $\Phi^{g}_{\ulcorner M_i \urcorner}(\ulcorner M_i \urcorner)\downarrow$,
- the algorithm M_i halts,
- $g(i) = 1$.

□

Lemma 2.3.8

$$\forall i \in \omega \, ((T_i \cap U_i \text{ is finite} \Rightarrow \forall g \in [T_{i+1}] \, \Phi_i^{g\oplus\mathcal{T}}(i)\downarrow) \wedge$$
$$(T_i \cap U_i \text{ is infinite} \Rightarrow \forall g \in [T_{i+1}] \, \Phi_i^{g\oplus\mathcal{T}}(i)\uparrow)).$$

Proof: Fix $i \in \omega$.

Suppose that $T_i \cap U_i$ is finite – then $T_{i+1} = T_i$. Fix $g \in [T_i]$. It is not the case that $g \in [U_i]$ as otherwise $T_i \cap U_i$ would be infinite. Therefore, by the definition of U_i it holds that there is $n \in \omega$ such that $g{\restriction}n \notin U_i$ which means that $\Phi_i^{g\restriction n \oplus \mathcal{T}}(i){\downarrow}$ and therefore also $\Phi_i^{g \oplus \mathcal{T}}(i){\downarrow}$.

Suppose now that $T_i \cap U_i$ is infinite. Then $T_{i+1} = T_i \cap U_i$. Let $g \in [T_{i+1}]$. Then for every $n \in \omega$ it holds that $g{\restriction}n \in T_i \cap U_i$. Therefore, by the definition of U_i it holds that $\forall n \in \omega\, \Phi_i^{g\restriction n \oplus \mathcal{T}}(i){\uparrow}$, hence $\Phi_i^{g \oplus \mathcal{T}}(i){\uparrow}$. □

Lemma 2.3.9 *For every $i \in \omega$ the following are equivalent:*

1. $T_{\ulcorner M_i \urcorner + 1} \cap U_{\ulcorner M_i \urcorner + 1}$ *is finite,*
2. $\forall g \in [T_{\ulcorner M_i \urcorner + 1}]\, \Phi_{\ulcorner M_i \urcorner}^{g \oplus \mathcal{T}}(\ulcorner M_i \urcorner){\downarrow}$,
3. $\forall g \in [T_{\ulcorner M_i \urcorner + 1}]\, (g \oplus \mathcal{T})(i) = 1$.

Proof: Fix $i \in \omega$. The equivalence ($1 \Leftrightarrow 2$) follows from 2.3.8 and the equivalence ($2 \Leftrightarrow 3$) follows directly from 2.3.7. □

Lemma 2.3.10

$$\forall i \in \omega\, \forall g \in [T_{\ulcorner M_{2i} \urcorner + 1}]\ \ g{\restriction}(i+1) = f{\restriction}(i+1).$$

Proof: The proof goes by induction on i.

For the base step suppose that $i = 0$ and fix $g \in [T_{\ulcorner M_0 \urcorner + 1}]$. If $T_{\ulcorner M_0 \urcorner + 1} \cap U_{\ulcorner M_0 \urcorner + 1}$ is finite then by Lemma 2.3.9 it holds that $(g \oplus \mathcal{T})(0) = 1$ i.e. $g(0) = 1$. By the definition of f it also holds that $f(0) = 1$. If $T_{\ulcorner M_0 \urcorner + 1} \cap U_{\ulcorner M_0 \urcorner + 1}$ is infinite, then by Lemma 2.3.9 it holds that $(g \oplus \mathcal{T})(0) = 0$ i.e. $g(0) = 0$. Then, by the definition $f(0) = 0$. This ends the base step.

Now suppose for the induction hypothesis that for $i \in \omega$ it holds that $\forall g \in [T_{\ulcorner M_{2i} \urcorner + 1}]\, g{\restriction}(i+1) = f{\restriction}(i+1)$. Fix $g \in [T_{\ulcorner M_{2(i+1)} \urcorner + 1}]$. Then $g \in [T_{\ulcorner M_{2i} \urcorner + 1}]$ and therefore by the induction hypothesis $g{\restriction}(i+1) = f{\restriction}(i+1)$. It remains to show that $f(i+1) = g(i+1)$. The argument is analogous to the one in the base step of induction. If $T_{\ulcorner M_{2(i+1)} \urcorner + 1} \cap U_{\ulcorner M_{2(i+1)} \urcorner + 1}$ is finite, then by Lemma 2.3.9 it holds that $(g \oplus \mathcal{T})(2(i+1)) = 1$ i.e. $g(i+1) = 1$ and by the definition of f it also holds that that $f(i+1) = 1$. If $T_{\ulcorner M_{2(i+1)} \urcorner + 1} \cap U_{\ulcorner M_{2(i+1)} \urcorner + 1}$ is infinite, then by Lemma 2.3.9 it holds that $(g \oplus \mathcal{T})(2(i+1)) = 0$, i.e. $g(i+1) = 0$ and by the definition $f(i+1) = 0$. This ends the proof. □

Lemma 2.3.11 *For every $k \in \omega$ it holds that $f \in [T_k]$.*

Proof: It is sufficient to show that:

$$\forall k, n \in \omega\, f{\upharpoonright}(n+1) \in T_k.$$

Fix $k, n \in \omega$. First, let us recall that by Lemma 2.3.6 each T_j is infinite, which is equivalent to the fact that $[T_j]$ is not empty. Therefore $[T_{\ulcorner M_{2n} \urcorner+1}]$ is not empty and by Lemma 2.3.10 for every $g \in [T_{\ulcorner M_{2n} \urcorner+1}]$ it holds that $f{\upharpoonright}(n+1) = g{\upharpoonright}(n+1)$. Now let $l = \max\{\ulcorner M_{2n} \urcorner + 1, k\}$ – then certainly $[T_l] \subseteq [T_k]$ and $[T_l] \subseteq [T_{\ulcorner M_{2n} \urcorner+1}]$. Hence, for every $g \in [T_l]$ we have $f{\upharpoonright}(n+1) = g{\upharpoonright}(n+1)$. Let $g \in [T_l]$, then $g{\upharpoonright}(n+1) \in T_l$ and therefore $f{\upharpoonright}(n+1) \in T_l \subseteq T_k$. This ends the proof. $\square$

Lemma 2.3.12 *For every $i \in \omega$ the following are equivalent:*

1. *$T_i \cap U_i$ is finite,*
2. *$\forall g \in [T_{i+1}]\, \Phi_i^{g \oplus \mathcal{T}}(i){\downarrow}$,*
3. *$\Phi_i^{f \oplus \mathcal{T}}(i){\downarrow}$.*

Proof: Fix $i \in \omega$. The equivalence ($1 \Leftrightarrow 2$) follows directly from lemma 2.3.8. The equivalence ($2 \Leftrightarrow 3$) follows from lemmata 2.3.8 and 2.3.11. $\square$

By Lemma 2.3.12, we can decide $\Phi_i^{f \oplus \mathcal{T}}(i){\downarrow}$ by deciding whether $T_i \cap U_i$ is finite. Therefore, Algorithm 3 with an oracle of degree $\mathbf{0}'$ computes the halting problem for $f \oplus \mathcal{T}$. It instantly follows that both $f \oplus \mathcal{T}$ and f are low.

Algorithm 3 Algorithm computing the halting problem for $f \oplus \mathcal{T}$

Input: $i \in \omega$
Output: truth value of $\Phi_i^{f\oplus\mathcal{T}}(i)\downarrow$

```
Array⟨recursive in 𝒯 tree⟩ T
Array⟨recursive in 𝒯 tree⟩ U
T[0] ← 𝒯
for j ← 0 to i do
  U[j] ← {σ : Φ_j^{σ⊕𝒯}(j) ↑}
  if IsFinite(T[j] ∩ U[j]) then
    if j = i then
      return true
    end if
    T[j + 1] ← T[j]
  else
    if j = i then
      return false
    end if
    T[j + 1] ← T[j] ∩ U[j]
  end if
  j ← j + 1
end for
```

We therefore have the following theorem.

Theorem 2.3.13 (Low Basis Theorem) *Let $\mathcal{T}$ be an infinite low tree. Then there is a low $f \in [\mathcal{T}]$ such that $f \oplus \mathcal{T}$ is low.*

Proof: Let $\mathcal{T}$ be an infinite low tree. We perform the above construction of descending sequence of trees for $\mathcal{T}$. Lemma 2.3.11 shows that $f \in [\mathcal{T}]$. Since the procedure IsFinite can be performed with an oracle of degree $\mathbf{0}'$, Algorithm 3 shows that $f \oplus \mathcal{T}$ is low, thus f is also low. $\square$

Note that the construction of a low branch of a low tree presented here is uniform and can be made effective. We show how to effectively get the low branch of a low infinite tree, but first we need to make some observations.

In this dissertation we construct mostly algorithms with K – the halting problem – as an oracle. This is because there is a natural way of presentation for such algorithms (see Theorem 2.4.4). Let $f \in 2^\omega$ and suppose we have an algorithm M which decides f in K. Then $\Phi_i^f(i) = \Phi_{g(i)}^K(i)$,[5] for some recursive function g which depends only on M. The function g takes a code of an oracle Turing machine as an input and outputs the code of the oracle machine where every query to the oracle $j \in \mathcal{O}$ is replaced by $M(j) = 1$.

[5] This is the equality in the strong sense: both computations halt on the same inputs and, if they halt, the outputs are equal.

Observe that the halting problem naturally corresponds to quantification. Let R be a binary relation. Then $R^* = \{i \in \omega \colon \Phi_i^R(i)\downarrow\}$ is the halting problem for R. It is easy to see that R^* can be defined by a Σ_1–formula in a vocabulary extended by a predicate interpreted as R. On the other hand, a relation defined by a formula $\exists k\, R(k, n)$ is recursive in R^*. Consider the following algorithm.

Algorithm 4 Algorithm partially deciding $\exists k\, R(k, n)$ using R as an oracle

```
Input: n
  k = 0
  while true do
    if R(k, n) then
      return true
    end if
    k = k + 1
  end while
```

Algorithm 4 halts if and only if $\exists k\, R(k, n)$. Therefore to decide $\exists k\, R(k, n)$ we can use R^* as an oracle. It suffices to compute the code m of Algorithm 4 and check if $\Phi_m^R(n)\downarrow$. This is easily computable in R^*. For the universal quantification the reasoning is analogous.

Note also that from a code of an algorithm which computes f^* in K we can recursively compute a code of an algorithm computing f in K.

In the light of the above remarks we have the following theorem.

Theorem 2.3.14 (Constructivity of the Low Basis Theorem) *There is a recursive procedure $s \mapsto (a, b, c, d)$, such that if:*

- Φ_s^K *computes the halting problem $\mathcal{T}^*$ for a low infinite tree $\mathcal{T}$,*

then:

- Φ_a^K *computes f, where f is the low infinite branch of $\mathcal{T}$ obtained by means of the Low Basis Theorem (2.3.13),*
- Φ_b^K *computes f^*,*
- Φ_c^K *computes $f \oplus \mathcal{T}$,*
- Φ_d^K *computes $(f \oplus \mathcal{T})^*$.*

Proof: Let $\mathcal{T}$ be a low infinite tree. Let s be such that Φ_s^K computes the halting problem $\mathcal{T}^*$ for $\mathcal{T}$.

Observe that from the code of an algorithm which computes $(f \oplus \mathcal{T})^*$ in K we can recursively compute codes of algorithms which compute f, f^* and $f \oplus \mathcal{T}$ in K. Therefore, it is sufficient to construct from s, an algorithm which computes $(f \oplus \mathcal{T})^*$ in K. Similarly the code t of the algorithm such that Φ_t^K computes $\mathcal{T}$ can be recursively produced from s.

Our recursive procedure outputs the code of Algorithm 3, where we use Φ_t^K to compute the code of $j \mapsto \{\sigma\colon \Phi_j^{\sigma \oplus \mathcal{T}}(j)\uparrow\}$ and Φ_s^K to compute procedure IsFinite. □

By Theorem 2.3.14 the Low Basis Theorem is constructive in a sense that it translates algorithms which, in K, compute halting problems for infinite low trees to algorithms computing, in K, their low branches and halting problems for these branches. This is shown to be essential in various model–theoretic constructions we perform in Chapter 4. Its power is that it allows us to iterate potentially infinitely many times certain reasonings on low objects (trees, theories, models etc.) and still get low objects as an output. This helps us build various chains and towers of models.

It is also worth noting that there is no recursive, nor even recursive in K, procedure computing the code of an algorithm which decides in K the halting problem for a low set A, from the code of an algorithm which decides A in K ([Mon14]). Therefore, we need an explicitly given algorithm which decides, in K, the halting problem for a low infinite tree T, to effectively produce the low branch of T – by means of Theorem 2.3.14.

We end this section with a relativised version of Post's theorem.

Theorem 2.3.15 (Relativised Post's Theorem (see [Sho59]))
Let $A, B \subseteq \omega$. The following are equivalent:

- *A is recursive in B,*
- *A and $\omega - A$ are recursively enumerable in B.*

In the following section we show why relations computable in K are essential in this dissertation. It is also explained why we consider algorithms relative to K instead of relative to any other oracle.

2.4 Finite Models and Potentially Infinite Domains

In this section we present basic definitions and theorems introduced by Mostowski (mostly in [Mos01] and [Mos03]) in his account of potentially infinite domains.

2.4.1 Representing Concepts in a Language without Actual Infinity

Consider an arithmetical model $\mathcal{A} = (\omega, \{R_i\}_{i \in I}, \{a_j\}_{j \in J})$. We consider the following finite approximations of $\mathcal{A}$ – initial segments of $\mathcal{A}$:

$$\mathcal{A}_n = (\{0, \dots, n\}, \{R_i^n\}_{i \in I}, \{a_j^n\}_{j \in J}, n).$$

Each model $\mathcal{A}_n$ is a finite model in a vocabulary of $\mathcal{A}$ extended by a constant MAX which is interpreted as the greatest element n of the model $\mathcal{A}_n$. For each $i \in I$, the relation R_i^n is a restriction of R_i to the set $\{0, \dots, n\}$. For each $j \in J$, $a_j^n = a_j$, if $a_j \leqslant n$ or $a_j^n = n$ otherwise.

The family of all initial segments of $\mathcal{A}$ is called an FM–domain of $\mathcal{A}$ and is denoted by FM($\mathcal{A}$). Therefore:

$$\mathrm{FM}(\mathcal{A}) = \{\mathcal{A}_n \colon n \in \omega\}.$$

FM($\mathcal{A}$) may be considered as a model of a potentially infinite world – the family of finite approximations of the model $\mathcal{A}$. For each $n \in \omega$ we can take $m > n$ and a model $\mathcal{A}_m$, if $\mathcal{A}_n$ is too small for our purposes. However, every element of FM($\mathcal{A}$) is always a finite model.

We follow Mostowski's approach and define the *sl*–semantics for FM–domains.

Definition 2.4.1 (*sl*–semantics)
Let φ be a formula and let $a_1, \dots, a_r \in \omega$.

$$\mathrm{FM}(\mathcal{A}) \models_{sl} \varphi[a_1, \dots, a_r] \quad \textit{if and only if} \quad \exists k \, \forall n > k \, \mathcal{A}_n \models \varphi[a_1^n, \dots, a_r^n],$$

where for $i = 1, \dots, r$, $a_i^n = \min\{a_i, n\}$.

sl–semantics express the asymptotic behaviour of formulae in FM–domains. A valuation $\overline{a}$ *sl*–satisfies a formula φ if $\overline{a}$ satisfies φ in all sufficiently large initial segments of $\mathcal{A}$.

We are particularly interested in properties of the standard FM–domain FM($\mathcal{N}$), which is an FM–domain of the standard model of arithmetic $\mathcal{N} = (\omega, R_+, R_\times, \leqslant)$.[6]

One of the main directions of Mostowski's investigations was to identify the notions that are meaningful without actual infinity – those which can be represented in FM–domains.

Definition 2.4.2 (FM–representability)
Let $R \subseteq \omega^r$. R is FM–represented by a formula $\varphi(x_1, \dots, x_r)$ if for every $a_1, \dots, a_r \in \omega$:

1. $(a_1, \dots, a_r) \in R$ *if and only if* $\mathrm{FM}(\mathcal{N}) \models_{sl} \varphi[a_1, \dots, a_r]$,
2. $(a_1, \dots, a_r) \notin R$ *if and only if* $\mathrm{FM}(\mathcal{N}) \models_{sl} \neg\varphi[a_1, \dots, a_r]$.

R is FM–representable if there exists a formula φ FM–representing R.

[6] For investigations on FM–domains in weaker arithmetic languages see: multiplication [KZ05], divisibility [MW04], coprimality [MZ05a].

Therefore, a relation R is FM–representable if there is a formula φ such that for every instance $\overline{a}$ the the truth with respect to *sl*–semantics of φ under valuation $\overline{a}$ fixes and coincides with $R(\overline{a})$. This means that R is FM–representable if every instance of is resolvable within some finite model. This justifies why FM–representability may serve as an explication of expressibility without actual infinity.

The following theorem is due to Mostowski and is essential for our further considerations.

Theorem 2.4.3 (FM–representability Theorem ([Mos01])) *Let $R \subseteq \omega^r$, then the following are equivalent:*

- *R is FM–representable,*
- *R is Δ^0_2,*
- *R is recursive with a recursively enumerable oracle,*
- *$R \leqslant \mathbf{0}'$.*

Therefore, FM–representable notions – those which are meaningful without actual infinity – are exactly those which are $\leqslant \mathbf{0}'$, or by Shoenfield's (see [Sho59]) Limit Lemma recursive in the limit. Recall Epstein's quote from [Eps79] presented in Chapter 1 in which he describes sets $\leqslant \mathbf{0}'$ as those which are *truly constructive* and as recognisable as *limits of uniform finitistic processes*. This serves as another justification for taking FM–representability as an explication of meaningfulness without actual infinity.

For a formula $\varphi(x_1, \ldots, x_r)$ by $\varphi^{\mathrm{FM}(\mathcal{N})}$ we denote the set

$$\{(a_1, \ldots, a_r) \in \omega^r : \mathrm{FM}(\mathcal{N}) \models_{sl} \varphi[a_1, \ldots, a_r]\}.$$

Note that functions which are recursive in the halting problem can be defined by formulae such that in each finite model $\mathcal{N}_n$ the uniqueness holds and with respect to *sl*–semantics the existence holds. Whenever we consider a formula which FM–represents a function we mean a formula with those properties.

From the proof of the FM–representability theorem the following theorem can be easily deduced.

Theorem 2.4.4 *Let R be an FM–representable relation.*

1. *There is an algorithm which, given on an input the Gödel number of a formula φ which FM–represents R, outputs the Gödel number of an algorithm which decides R using the halting problem as an oracle.*
2. *There is an algorithm which, given on an input the Gödel number of an algorithm which decides R using the halting problem as an oracle, outputs a formula φ, which FM–represents R.*

Proof: Let R be an FM–representable relation with arity equal k.

For the proof of point 1 suppose that φ FM–represents R. The algorithm deciding R with K as an oracle is the following.

Algorithm 5 Algorithm deciding R

Input: $n_1, \ldots, n_k$
Output: truth value of $R(n_1, \ldots, n_k)$
  $i = 0$
  while true do
    if $\forall k \geqslant i\, \mathcal{N}_k \models \varphi(n_1, \ldots, n_k)$ **then**
      return true
    end if
    if $\forall k \geqslant i\, \mathcal{N}_k \models \neg\varphi(n_1, \ldots, n_k)$ **then**
      return false
    end if
    $i = i + 1$
  end while

In Algorithm 5 we use a procedure which computes in K the truth value of $\forall k \geqslant i\, \mathcal{N}_k \models \varphi(n_1, \ldots, n_k)$. Since φ FM–represents R the algorithm always halts and decides R with K as an oracle.

Note that Algorithm 5 depends on φ. The recursive procedure we search for takes φ as an input and outputs the code of Algorithm 5 for φ.

For the proof of point 2 we need to show how to define a formula φ which FM–represents R given the code of an algorithm M which decides R with K as an oracle. Let ψ be a Σ_1–formula FM–representing K. It is sufficient to take the formula expressing "there exists an accepting oracle computation of M on input $n_1, \ldots, n_k$, using ψ queries instead of oracle queries". Such formula can be effectively computed from the code of M. Moreover it can be expressed in a Σ_2 form and it FM–represents R (see [Mos01]).

Note that similarly we could effectively produce a Π_2 formula expressing "every oracle computation of M on input $n_1, \ldots, n_k$, using ψ queries instead of oracle queries, is accepting". It is easy to show to that such a formula also FM–represents R. $\square$

By Theorem 2.4.4 there is an effective way of passing from formulae FM–representing relations to algorithms which decide these relations in K and the other way round. Thus, we further freely pass from such formulae to suitable algorithms and back. Further, in most contexts, we use the term concrete: concrete sets, concrete relations, concrete functions; whenever we consider FM–representable or recursive in K sets, relations and functions.

2.4.2 FM–Truth Definitions

One of the motivations of Mostowski's investigations on FM–domains and *sl*–semantics was to transfer Tarski's method of truth definitions for separating languages with respect to their expressive power to the domain of finite models. Tarski's original method of truth definitions works only for infinite models. Mostowski introduced the following notion of an FM–truth definition.

Definition 2.4.5 (FM–truth Definition) *A formula $\varphi(x)$ is an* FM*–truth definition if for every sentence ψ it holds that:*

$$\mathrm{FM}(\mathcal{N}) \models_{sl} \psi \equiv \varphi(\ulcorner \psi \urcorner).$$

By the FM–representability theorem it is easy to prove the following FM–version of the diagonal lemma.

Theorem 2.4.6 (FM–version of the Diagonal Lemma ([Mos01])) *For every formula $\varphi(x)$ there is a sentence ψ such that:*

$$\mathrm{FM}(\mathcal{N}) \models_{sl} \psi \equiv \varphi(\ulcorner \psi \urcorner).$$

As a consequence we get the FM–version of the undefinability of truth theorem.

Theorem 2.4.7 (FM–version of the Undefinability of Truth Theorem ([Mos01]))
There is no first order FM*–truth definition for first order sentences.*

The method of FM–truth definitions is a useful tool for separating languages with respect to their expressive power in finite models. Therefore, it enables one to separate logics which occur in descriptive complexity theory (see e.g. [Koł04]).

There is also a positive result about FM–truth definitions similar to Tarski's result on the existence of truth definition in higher order logics.

Theorem 2.4.8 (Finite Models Expressibility Hierarchy ([Mos01]))
There is an $(i+2)$–th order FM*–truth definition for i–th order sentences.*

Note that in order to define FM–truth we need to go two levels up, not just one as in the case of truth definitions in infinite models. This is due to the lack of paring function in finite models. We use additional level to cover arbitrary arities of predicates. It remains an open problem whether there is a $(i+1)$–th order FM–truth definition for i–th order logic.

2.5 Modal Logic Basics

In this section we present the basics of modal logic which we use in Chapter 3.

Formulae of modal logics are generated by the following grammar:

$$\varphi \longmapsto \bot \,|\, p \,|\, c \,|\, \varphi \wedge \varphi \,|\, \neg\varphi \,|\, \Box\varphi,$$

where p is an element of the set PROP of propositional variables and c is an element of the set CONSTS of propositional constants. We assume PROP $\cap$ CONSTS $= \varnothing$. We introduce the following abbreviations: $\varphi \vee \psi = \neg(\neg\varphi \wedge \neg\psi)$, $\varphi \Rightarrow \psi = \neg(\varphi \wedge \neg\psi)$ and $\Diamond\varphi = \neg\Box\neg\varphi$.

Similarly to the quantifier rank in first order logic we define the modal depth of a modal formula as follows: $\mathrm{md}(p) = 0$ for every propositional variable p, $\mathrm{md}(c) = 0$ for every propositional constant, $\mathrm{md}(\neg\varphi) = \mathrm{md}(\varphi)$, $\mathrm{md}(\varphi \wedge \psi) = \max\{\mathrm{md}(\varphi), \mathrm{md}(\psi)\}$ and $\mathrm{md}(\Box\varphi) = 1 + \mathrm{md}(\varphi)$.

A Kripke frame is a pair (W, R), where $W \neq \emptyset$ is a set of the – so called – possible worlds and $R \subseteq W^2$ is an accessibility relation. For a fixed Kripke frame $F = (W, R)$ a valuation in F is a function $V\colon (W \times (\mathrm{PROP} \cup \mathrm{CONSTS})) \to \{0, 1\}$. We call a triple (W, R, V) a Kripke model when (W, R) is a Kripke frame and V is a valuation on it. If it is clear from the context that we refer to a Kripke model we will call it a model for short. The semantics for modal logics is defined inductively on the construction of a formula. For a Kripke model $M = (W, R, V)$ and a formula φ let $[\![\varphi]\!]_M$ be a set of worlds at which φ is true in M. Then,

- $[\![\bot]\!]_M = \varnothing$,
- $[\![p]\!]_M = \{w \in W\colon V(w, p) = 1\}$,
- $[\![c]\!]_M = \{w \in W\colon V(w, c) = 1\}$,
- $[\![\varphi \wedge \psi]\!]_M = [\![\varphi]\!]_M \cap [\![\psi]\!]_M$,
- $[\![\neg\varphi]\!]_M = W - [\![\varphi]\!]_M$,
- $[\![\Box\varphi]\!]_M = \{w \in W\colon \forall v \in W\, (wRv \Rightarrow v \in [\![\varphi]\!]_M)\}$.

We say that a formula φ is true at a world $w \in W$ of a model $M = (W, R, V)$ when $w \in [\![\varphi]\!]_M$. We denote this fact also by $M, w \models \varphi$ or, if we consider a Kripke frame $F = (W, R)$ with a valuation V, we denote it by $F, w \models \varphi[V]$. We write $F \models \varphi[V]$ if for each $w \in W$, $F, w \models \varphi[V]$ holds.

We present below the most basic modal logic K, which is a base of all modal logics considered here, in a sense that every modal logic is an extension of K by some additional axioms.

Definition 2.5.1 (K) *The modal logic* K *is an extension of classical propositional logic by axioms:*

$$\Box(\varphi \Rightarrow \psi) \Rightarrow (\Box\varphi \Rightarrow \Box\psi),$$

where φ *and* ψ *are arbitrary modal formulae and by the necessitation rule, i.e., if we proved* φ *we can write in a proof also* $\Box\varphi$*, for any formula* φ*.*

Any modal logic which contains K and is closed on modus ponens, necessitation and substitution is called normal. In this work we consider only normal logics with one natural proviso. We allow substitution only for propositional variables of the basic language, not for propositional constants.

Definition 2.5.2 (Extension of a Modal Logic)
Let $\mathrm{L}_1, \mathrm{L}_2$ *and* L *be modal logics. We say that* L_2 *extends* L_1 *in the language of* L *and write* $\mathrm{L}_1 \leqslant_{\mathrm{L}} \mathrm{L}_2$ *if for every* L*–formula* φ *if* $\mathrm{L}_1 \vdash \varphi$*, then* $\mathrm{L}_2 \vdash \varphi$*.*

Chapter 3

Approximating Truth in Finite Models

We already mentioned the Mostowski's FM–version of undefinability of truth theorem which states that there is no FM–truth definition for the standard FM–domain $\mathrm{FM}(\mathcal{N})$. In this chapter we investigate how one can approximate FM–truth definitions i.e. define arithmetical predicates that have certain (as many as possible) properties of an FM–truth definition, but do not fall under the assumptions of FM–version of undefinability of truth theorem.

Since there is no FM–truth definition i.e. there is no formula $\psi(x)$ such that for every arithmetical sentence φ it holds that

$$\mathrm{FM}(\mathcal{N}) \models_{sl} \psi(\ulcorner\varphi\urcorner) \equiv \varphi,$$

we need to weaken the requirements on such ψ.

It is known that the property of a sentence φ of being true in almost all finite models ($\mathrm{FM}(\mathcal{N}) \models_{sl} \varphi$), is Σ^0_2–complete in the arithmetical hierarchy (see [MZ05b]). The upper bound can be clearly seen just from the arithmetical definition of the property:

$$\exists k \forall n \geqslant k\, \mathcal{N}_n \models \varphi.$$

On the other hand, if for a given formula $\varphi(x)$ we consider a set X_φ defined as

$$X_\varphi = \{n \in \omega \colon \mathrm{FM}(\mathcal{N}) \models_{sl} \varphi(n)\},$$

then for first order arithmetical formulae φ we get exactly the sets in Σ^0_2 (see [MZ05b]). It follows that there is an arithmetical formula ψ such that for each arithmetical sentence φ we have the following equivalence:

$$\mathrm{FM}(\mathcal{N}) \models_{sl} \varphi \text{ if and only if } \mathrm{FM}(\mathcal{N}) \models_{sl} \psi(\ulcorner\varphi\urcorner).$$

We construct the predicate $\mathrm{Tr_{sl}}$ which has the above–mentioned property and, moreover, commutes with propositional connectives under *sl*–semantics.

We extract the basic properties of $\mathrm{Tr}_{\mathrm{sl}}$ defining what we call an approximate FM–truth definition. We cannot expect it to commute with quantifiers as this would give a usual FM–truth definition which is impossible.

Next we study those properties of approximate FM–truth definitions which are expressible in modal logic. We define L_{Tr} as the set of modal formulae φ such that for any translation[1] tr of modal formulae to arithmetical sentences, the arithmetical sentence φ^{tr} is true in almost all finite models from $\mathrm{FM}(\mathcal{N})$. In the main theorem of this chapter we characterise L_{Tr} as an extension of the basic modal logic K by axioms:

$$\Box(\neg\varphi) \equiv \neg\Box\varphi,$$

for each formula φ. We call this extension SL.

Thus, the modal properties of an approximate FM–truth definition may be contrasted with that of provability predicate which corresponds to the, so called, Gödel–Löb modal logic GL – an extension of K by a scheme $\Box(\Box\varphi \Rightarrow \varphi) \Rightarrow \Box\varphi$ corresponding to the Löb's theorem. (see e.g. [Boo93]). Indeed, L_{Tr} and GL are incomparable. This fact may be somehow expected since our approximate FM–truth predicate captures certain semantics while GL is a logic which captures the properties of provability – a very different concept.

The proof of the main result of this chapter shows also that, unlike in the case of GL, we cannot consistently extend SL by any axiom scheme without contradicting some of the properties of approximate FM–truth definitions. This means that the modal logic SL is the strongest modal logic of approximate FM–truth definitions and that approximate FM–truth definitions have all the properties of FM–truth definitions expressible in the modal logic.

The method of proving that SL and L_{Tr} are equivalent is by extending both of them by a fixpoint construction obtaining logics SL^* and ${\mathrm{L}_{\mathrm{Tr}}}^*$. Then, we prove that ${\mathrm{L}_{\mathrm{Tr}}}^*$ is conservative over SL in the vocabulary of SL using SL^* as an auxiliary logic.

The result presented in this chapter can be seen as a contribution to the study of what fragments of finite models semantics can be expressed within finite models and what are modal logic properties of these fragments of finite models semantics.

3.1 A Truth Definition for Almost All Finite Models

In this chapter we work with an extended relational arithmetical vocabulary. This means that we consider the expansion $(\mathcal{N}, n)_{n\in\omega}$ of the standard model of arithmetic. Recall that by $\underline{n}$ we denote the constant naming n in $(\mathcal{N}, n)_{n\in\omega}$.

[1] Conditions imposed on translations are specified later.

Moreover considering FM–domains we have an additional constant MAX which names the largest element of every finite model of the FM–domain. We refer to the vocabulary of $\mathrm{FM}((\mathcal{N}, n)_{n\in\omega})$ as the extended arithmetical vocabulary with MAX. Further we denote $(\mathcal{N}, n)_{n\in\omega}$ by $\mathcal{N}$, remembering that we have constants naming every natural number.

We proceed to the construction of the approximate FM–truth predicate $\mathrm{Tr}_{\mathrm{sl}}$.

It is folklore that there is no Δ_0 truth definition for Δ_0 formulae. Since both relations Name and Subst are Δ_0 and Δ_0 is closed under negation it falls under the assumptions of Tarski's undefinability of truth theorem. It is also known that there exists a Σ_1 truth definition for Δ_0 sentences. In fact there is also a Σ_1 satisfaction definition – $\mathrm{Sat}_{\Delta_0}(x, v)$ for Δ_0 formulae. It holds that $\mathcal{N} \models \mathrm{Sat}_{\Delta_0}(\ulcorner\psi\urcorner, \ulcorner\overline{a}\urcorner)$ if and only if ψ is a Δ_0 formula, $\overline{a}$ is a valuation on free variables of ψ and it holds that $\mathcal{N} \models \psi[\overline{a}]$. Further, if a formula ψ is known from the context, we refer to the valuations on free variables of ψ as to valuations.

The formula $\mathrm{Sat}_{\Delta_0}(x, v)$ is defined as follows. Let ψ be a given Δ_0 formula and let $\overline{a}$ be a valuation on free variables of ψ. We consider a quantifier free formula $\widetilde{\psi}$ obtained from ψ by replacing bounded quantifiers $Qx \leqslant t\varphi(x)$ by finite conjunctions (in the case of universal quantifiers) and disjunctions (in the case of existential quantifiers) of $\varphi(\underline{0}), \ldots, \varphi(\underline{\mathrm{Val}(t, \overline{a})})$. The formula $\mathrm{Sat}_{\Delta_0}(\ulcorner\psi\urcorner, \ulcorner\overline{a}\urcorner)$ expresses the existence of such a labelling of the syntactic tree of $\widetilde{\psi}$ with 0s and 1s which labels leaves of that tree with 1s if and only if atomic formulae on leaves are satisfied and with 0s otherwise. Further the labelling is propagating up to the root of the syntactic tree preserving the truth value, so that finally the formula is true if and only if the label of the root is 1.

Therefore Sat_{Δ_0} is a Σ_1 formula i.e. it is of the form $\exists y\, \mathrm{Sat}^0_{\Delta_0}(x, v, y)$, where $\mathrm{Sat}^0_{\Delta_0} \in \Delta_0$. Despite the fact that we cannot get rid of this leading existential quantifier in $\mathrm{Sat}_{\Delta_0}(x, v)$ we know how to estimate a value of a witness for it (see [HP93]). For the sake of syntax encoding we suppose that for each $n > 0$ it holds that $|\underline{n}| = \lceil \log_2(n + 1)\rceil$, similarly as for variables we have $|v_n| = \lceil \log_2(n + 1)\rceil$ (for $n = 0$ the lengths are equal 1). Since the estimation of the witness for Sat_{Δ_0} from [HP93] is for the standard arithmetical vocabulary and we work with an extended relational arithmetical vocabulary we present the estimation that fits our purposes.

Our aim is to compute the upper bound for the code of a labelled syntactic tree for $\widetilde{\psi}$ of a Δ_0 formula ψ on a given valuation $\overline{a}$. Such a tree can be expressed as a sequence of pairs (φ, V), where φ is a subformula of $\widetilde{\psi}$ and V encodes the truth value i.e. it is either $\underline{0}$ or $\underline{1}$. The number of pairs in the computation of truth value of a formula ψ under a valuation $\overline{a}$ is equal to the number of subformulae of $\widetilde{\psi}$. Therefore, we need to estimate the number of subformulae of a $\widetilde{\psi}$.

Fix a valuation $\overline{a}$. First by the induction on the complexity of a formula

φ we estimate the number of subformulae of $\widetilde{\varphi}$. By $\mathrm{SubForm}(\varphi, \overline{a})$ we denote the number of subformulae of φ under the valuation $\overline{a}$. If φ is atomic, then $\mathrm{SubForm}(\widetilde{\varphi}, \overline{a}) = 1$. For conjunction, if $\varphi = \varphi_1 \wedge \varphi_2$, then it holds that

$$\mathrm{SubForm}(\widetilde{\varphi}, \overline{a}) = \mathrm{SubForm}(\widetilde{\varphi_1}, \overline{a}) + \mathrm{SubForm}(\widetilde{\varphi_2}, \overline{a}) + 1.$$

For the case of negation, if $\varphi = \neg\psi$, then it holds that

$$\mathrm{SubForm}(\widetilde{\varphi}, \overline{a}) = \mathrm{SubForm}(\widetilde{\psi}, \overline{a}) + 1.$$

Finally, for the case of quantification, if $Q \in \{\forall, \exists\}$ and $\varphi = (Qx \leqslant t\ \psi(x))$, then it holds that

$$\mathrm{SubForm}(\widetilde{\varphi}, \overline{a}) = \mathrm{SubForm}(\widetilde{\psi}, \overline{a}) \cdot (\mathrm{Val}(t, \overline{a}) + 1) + 1.$$

It is easy to show by the induction on the complexity of formulae that:

$$\mathrm{SubForm}(\widetilde{\varphi}, \overline{a}) \leqslant (\mathrm{Val}(t, \overline{a}) + 1)^{|\varphi|}.$$

It is also easy to see that for a term t it holds that

$$\mathrm{Val}(t, \overline{a}) \leqslant (\max\{\overline{a}\} + 2)^{|t|}.$$

Therefore the length of the sequence witnessing that a Δ_0 formula φ is either satisfied by $\overline{a}$ or not can be estimated by

$$((\max\{\overline{a}\} + 2)^{|\varphi|} + 1)^{|\varphi|} \leqslant (\max\{\overline{a}\} + 2)^{2 \cdot |\varphi|^2}.$$

Each element of this sequence is a pair consisting of a subformula of φ, $\underline{0}$ or $\underline{1}$ and two commas. Therefore the entire sequence has no greater length than

$$(|\varphi| + 3) \cdot (\max\{\overline{a}\} + 2)^{2 \cdot |\varphi|^2} \leqslant (\max\{\overline{a}\} + 2)^{3 \cdot |\varphi|^2}.$$

Recall that the Gödel number of a sequence $s = s_0, \ldots, s_{k-1}$ is equal $p_0^{1+s_0} \cdot \ldots \cdot p_{k-1}^{1+s_{k-1}}$. Let c be the greatest code of a primitive symbol in the Gödel numbering. Then since for each $i \in \omega$ it holds that $p_{i+1} \leqslant 2p_i$ it follows that

$$\ulcorner s \urcorner \leqslant \prod_{i=0}^{k-1} p_i^{1+c} = \left(\prod_{i=0}^{k-1} p_i\right)^{1+c} \leqslant \left(2^{\sum_{i=0}^{k-1} 1+i}\right)^{1+c} \leqslant 2^{(1+c) \cdot k^2}.$$

The length k of the sequence was estimated by $(\max\{\overline{a}\}+2)^{3 \cdot |\varphi|^2}$ therefore the Gödel number of the sequence computing truth or falsity of a Δ_0 formula φ under valuation $\overline{a}$ is estimated by

$$2^{(1+c) \cdot \left((\max\{\overline{a}\}+2)^{3 \cdot |\varphi|^2}\right)^2}$$

and for Gödel numberings with only reasonably many primitive symbols[2] this can be estimated by

$$2^{(\max\{\overline{a}\}+2)^{9\cdot|\varphi|^2}}.$$

Let $h(\ulcorner\varphi\urcorner, y)$ be a function defined as

$$h(\ulcorner\varphi\urcorner, y) = \max\{2^{(y+2)^{9\cdot|\varphi|^2}}\} \cup \{\ulcorner\overline{a}\urcorner\colon \overline{a} \text{ is a valuation on } \varphi \text{ with values } \leqslant y\}^3.$$

Let $\varphi(\overline{x})$ be a Δ_0 formula and $\overline{a}$ be a valuation. Then the following are equivalent:

- $\mathcal{N} \models \mathrm{Sat}_{\Delta_0}(\ulcorner\varphi\urcorner, \ulcorner\overline{a}\urcorner)$,
- $\mathcal{N} \models \exists y\, \mathrm{Sat}^0_{\Delta_0}(\ulcorner\varphi\urcorner, \ulcorner\overline{a}\urcorner, y)$,
- $\mathcal{N} \models \exists y \leqslant h(\ulcorner\varphi\urcorner, \max\{\overline{a}\})\, \mathrm{Sat}^0_{\Delta_0}(\ulcorner\varphi\urcorner, \ulcorner\overline{a}\urcorner, y)$.

Our aim is to use the formula $\mathrm{Sat}^0_{\Delta_0}$ to capture certain properties of the satisfaction relation in finite models. For a formula in the vocabulary with MAX we define its restriction in the following way. Let ψ be a formula in the extended arithmetical vocabulary with MAX and let $\overline{a}$ be a valuation on ψ. By $\psi^{\leqslant y}_{\overline{a}}$ we denote the formula obtained by:

- replacing every occurrence of MAX in ψ by $\underline{\mathrm{Val}(y, \overline{a})}$,
- bounding every quantifier in ψ by $\underline{\mathrm{Val}(y, \overline{a})}$,
- adding to ψ a conjunct $\bigwedge_{v_i \in \mathrm{FV}(\psi)} v_i \leqslant \underline{\mathrm{Val}(y, \overline{a})}$.

Therefore we have the following. Let ψ be a formula in the extended arithmetical vocabulary with MAX, let $k \in \omega$ and let $\overline{a}$ a valuation on ψ with values $\leqslant k$. Then $\psi^{\leqslant k}_{\overline{a}}$ is a Δ_0 formula in a vocabulary without MAX such that for $n \geqslant k$ it holds that

$$\mathcal{N}_k \models \psi[\overline{a}] \text{ if and only if } \mathcal{N}_n \models \psi^{\leqslant k}_{\overline{a}}[\overline{a}].$$

Now let us introduce a Δ_0 *truth definition* $\alpha(x, v, k, z)$ with two additional parameters k and z. Here $x = \ulcorner\psi\urcorner$ for some formula in a vocabulary with MAX, v is a code of a valuation, k is a bound for quantifiers in ψ and values of v as in $\psi^{\leqslant k}_{v}$ and, finally, z is a bound for existential quantifier in $\mathrm{Sat}_{\Delta_0}(x, v)$. Thus we define:

$$\alpha(x, v, k, z) = \exists y \leqslant z\, \mathrm{Sat}^0_{\Delta_0}(x^{\leqslant k}_{v}, v, y).$$

[2]In fact even for Gödel numberings with 2^{20} primitive symbols.

[3]We want to ensure that both the code of the tree witnessing the truth of φ under a valuation with the maximal value not exceeding y, and the Gödel numbers of all valuations on φ with values not greater than y, are less than $h(\ulcorner\varphi\urcorner, y)$.

Let

$$f(x,k) = \max\{h(\ulcorner \psi_{\overline{a}}^{\leqslant l} \urcorner, l) \colon \ulcorner \psi \urcorner \leqslant x, l \leqslant k \text{ and}$$
$$\overline{a} \text{ is a valuation on } \psi \text{ with values } \leqslant l\}.$$

The function f defined this way is monotone in both arguments and the following are equivalent:

1. $\mathcal{N}_k \models \varphi[\overline{a}]$,
2. $\mathcal{N}_{f(\ulcorner \varphi \urcorner, k)} \models \alpha(\ulcorner \varphi \urcorner, \ulcorner \overline{a} \urcorner, k, \mathrm{MAX})$,
3. $\forall n \geqslant f(\ulcorner \varphi \urcorner, k)\ \mathcal{N}_n \models \alpha(\ulcorner \varphi \urcorner, \ulcorner \overline{a} \urcorner, k, \mathrm{MAX})$.

Part 3 in the equivalence above is essential for our purpose as we investigate asymptotic properties of formulae.

Now, we take $F(x) = f(x,x)$ and define the relation

$$k = F^{-1}(x) \equiv_{df} x \in [F(k), F(k+1)).$$

The notation is slightly abused since it may happen that x is not a value of F. Nevertheless, it is justified by a close correspondence between the relation $k = F^{-1}(x)$ and the inverse image of F.

Observe that

$$\forall k \in \omega\, \exists x\, k = F^{-1}(x)$$

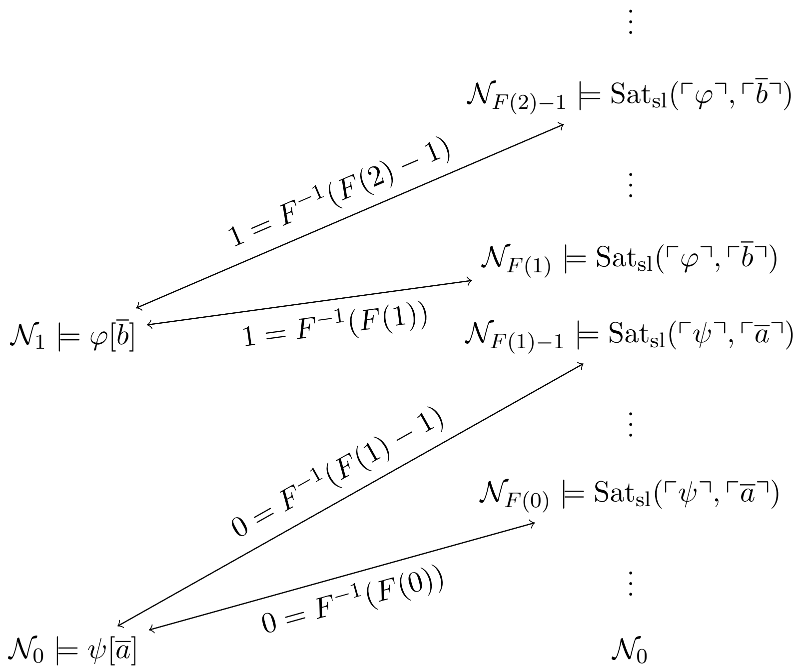
and

$$\forall x \geqslant F(0)\, \exists^{=1} k\, x \in [F(k), F(k+1)).$$

The formula $\alpha(x, v, k, z)$ is written in a Δ_0 form. Similarly, the relation $z = F^{-1}(x)$ is Δ_0–definable. It follows that there is a one arithmetical formula which defines in a given finite model $\mathcal{N}_m$ the restriction of the relation $z = F^{-1}(x)$ to the universe of $\mathcal{N}_m$. Finally, the formula $\mathrm{Sat}_{\mathrm{sl}}(x,v)$ approximating FM–satisfaction is defined as:

$$\mathrm{Sat}_{\mathrm{sl}}(x,v) \equiv_{df} \exists k\, (k = F^{-1}(\mathrm{MAX}) \wedge \alpha(x, v, k, \mathrm{MAX})).$$

By our discussion the above formula is Δ_0. Intuitively, the formula $\mathrm{Sat}_{\mathrm{sl}}(x,v)$ states that a formula with Gödel number x is satisfied by a valuation with code v in a finite model $\mathcal{N}_k$ such that $k = F^{-1}(\mathrm{MAX})$. This is shown by the following picture.



Definition 3.1.1 *We say that an arithmetical formula $\tau(x)$ is an approximate* FM*–satisfaction definition if for all quantifier free formulae $\psi(x_1, \ldots, x_k)$ and for every valuation $\overline{a} = a_1, \ldots, a_k$ it holds that*

1. $\mathrm{FM}(\mathcal{N}) \models_{sl} (\psi(\underline{a_1}/x_1, \ldots, \underline{a_k}/x_k) \equiv \tau(\ulcorner\psi\urcorner, \ulcorner a_1, \ldots, a_k\urcorner))$

and for all formulae φ, ψ it holds that

2. $\mathrm{FM}(\mathcal{N}) \models_{sl} \varphi[\overline{a}]$ *if and only if* $\mathrm{FM}(\mathcal{N}) \models_{sl} \tau(\ulcorner\varphi\urcorner, \ulcorner\overline{a}\urcorner)$,

3. $\mathrm{FM}(\mathcal{N}) \models_{sl} \tau(\ulcorner\neg\varphi\urcorner, \ulcorner\overline{a}\urcorner) \equiv \neg\tau(\ulcorner\varphi\urcorner, \ulcorner\overline{a}\urcorner)$,

4. $\mathrm{FM}(\mathcal{N}) \models_{sl} \tau(\ulcorner\varphi \wedge \psi\urcorner, \ulcorner\overline{a}\urcorner) \equiv (\tau(\ulcorner\varphi\urcorner, \ulcorner\overline{a}\urcorner) \wedge \tau(\ulcorner\psi\urcorner, \ulcorner\overline{a}\urcorner))$.

We do not expect an approximate FM–satisfaction definition to commute with quantifiers, because this would give us a regular FM–satisfaction definition which does not exist. However, we could write an FM–satisfaction definition $\psi(x, v)$ satisfying one of the implications, either

$$\mathrm{FM}(\mathcal{N}) \models_{sl} \psi(\ulcorner\forall x\varphi\urcorner, \ulcorner\overline{a}\urcorner) \Rightarrow \forall n\ \psi(\ulcorner\varphi(\underline{n})\urcorner, \ulcorner\overline{a}\urcorner)$$

or

$$\mathrm{FM}(\mathcal{N}) \models_{sl} \forall n\ \psi(\ulcorner\varphi(\underline{n})\urcorner, \ulcorner\overline{a}\urcorner) \Rightarrow \psi(\ulcorner\forall x\varphi\urcorner).$$

The following theorem justifies why we say that $\mathrm{Sat}_{\mathrm{sl}}(x, v)$ approximates FM–satisfaction.

Theorem 3.1.2 $\mathrm{Sat}_{\mathrm{sl}}$ *is an approximate* FM*–satisfaction definition.*

Proof: We prove consecutive points of Definition 3.1.1.

1. Let $\overline{a} = a_1, \ldots, a_k \in \omega$. First, let us observe that for every closed quantifier–free formula $\psi(\underline{a_1}, \ldots, \underline{a_k})$ we can eliminate MAX with respect to *sl*–semantics. The closed atomic formulae of the form MAX $=$ MAX, MAX $\leqslant$ MAX, $R_+(\mathrm{MAX}, \underline{0}, \mathrm{MAX})$, $R_+(\underline{0}, \mathrm{MAX}, \mathrm{MAX})$, $R_\times(\underline{0}, \mathrm{MAX}, \underline{0})$, $R_\times(\mathrm{MAX}, \underline{0}, \underline{0})$, $R_\times(\underline{1}, \mathrm{MAX}, \mathrm{MAX})$, $R_\times(\mathrm{MAX}, \underline{1}, \mathrm{MAX})$ and $\underline{n} \leqslant \mathrm{MAX}$ for arbitrary $n \in \omega$ are *sl*–true. The rest of the closed atomic formulae with MAX are *sl*–false. Therefore, in every closed quantifier free formula we can eliminate every occurrence of MAX. Thus we can assume that $\psi(\underline{a_1}, \ldots, \underline{a_k})$ is a closed quantifier free formula without occurrences of MAX. Therefore it follows that either:

   - for each $m \geqslant \max\{a_1, \ldots, a_k\}$ it holds that $\mathcal{N}_m \models \psi(\underline{a_1}, \ldots, \underline{a_k})$
   - for each $m \geqslant \max\{a_1, \ldots, a_k\}$ it holds that $\mathcal{N}_m \not\models \psi(\underline{a_1}, \ldots, \underline{a_k})$.

   Let $M = \max\{\ulcorner\psi(x_1, \ldots, x_k)\urcorner, a_1, \ldots, 1_k\}$. In the first case, for each $m \geqslant F(M)$ it holds that:

   - $\mathcal{N}_m \models \mathrm{Sat}_{\mathrm{sl}}(\ulcorner\psi(x_1, \ldots, x_k)\urcorner, \ulcorner\overline{a}\urcorner)$,
   - $\mathcal{N}_m \models \psi(\underline{a_1}, \ldots, \underline{a_k}) \equiv \mathrm{Sat}_{\mathrm{sl}}(\ulcorner\psi(x_1, \ldots, x_k)\urcorner, \ulcorner\overline{a}\urcorner)$.

   In the other case, for $m \geqslant F(M)$ it holds that:

   - $\mathcal{N}_m \models \neg\mathrm{Sat}_{\mathrm{sl}}(\ulcorner\psi(x_1, \ldots, x_k)\urcorner, \ulcorner\overline{a}\urcorner)$,
   - $\mathcal{N}_m \models \psi(\underline{a_1}, \ldots, \underline{a_k}) \equiv \mathrm{Sat}_{\mathrm{sl}}(\ulcorner\psi(x_1, \ldots, x_k)\urcorner, \ulcorner\overline{a}\urcorner)$.

   Thus $\mathrm{FM}(\mathcal{N}) \models_{sl} \psi(\underline{a_1}, \ldots, \underline{a_k}) \equiv \mathrm{Sat}_{\mathrm{sl}}(\ulcorner\psi(x_1, \ldots, x_k)\urcorner, \ulcorner\overline{a}\urcorner)$.

2. Fix a formula φ. Let $k \geqslant \ulcorner\varphi\urcorner$ and let $n \in [F(k), F(k+1))$. Then for every valuation $\overline{a}$ in $\mathcal{N}_k$ the following are equivalent:

   - $\mathcal{N}_k \models \varphi[\overline{a}]$,
   - $\mathcal{N}_n \models \mathrm{Sat}_{\mathrm{sl}}(\ulcorner\varphi\urcorner, \ulcorner\overline{a}\urcorner)$.

   Therefore for every valuation $\overline{a}$ in $\mathcal{N}$ the following are equivalent:

   - $\mathrm{FM}(\mathcal{N}) \models_{sl} \varphi[\overline{a}]$,
   - $\mathrm{FM}(\mathcal{N}) \models_{sl} \mathrm{Sat}_{\mathrm{sl}}(\ulcorner\varphi\urcorner, \ulcorner\overline{a}\urcorner)$.

3. Fix a formula φ. Let $k \geqslant \ulcorner\neg\varphi\urcorner > \ulcorner\varphi\urcorner$ and let $n \in [F(k), F(k+1))$. Then for every valuation $\overline{a}$ in $\mathcal{N}_k$ the following are equivalent:

   - $\mathcal{N}_n \models \mathrm{Sat}_{\mathrm{sl}}(\ulcorner\neg\varphi\urcorner, \ulcorner\overline{a}\urcorner)$,
   - $\mathcal{N}_k \models \neg\varphi[\overline{a}]$,

- $\mathcal{N}_k \not\models \varphi[\overline{a}]$,
- $\mathcal{N}_n \not\models \mathrm{Sat}_{\mathrm{sl}}(\ulcorner\varphi\urcorner, \ulcorner\overline{a}\urcorner)$,
- $\mathcal{N}_n \models \neg\mathrm{Sat}_{\mathrm{sl}}(\ulcorner\varphi\urcorner, \ulcorner\overline{a}\urcorner)$.

Therefore for $m \geqslant F(k)$ it holds that $\mathcal{N}_m \models \mathrm{Sat}_{\mathrm{sl}}(\ulcorner\neg\varphi\urcorner, \ulcorner\overline{a}\urcorner) \equiv \neg\mathrm{Sat}_{\mathrm{sl}}(\ulcorner\varphi\urcorner, \ulcorner\overline{a}\urcorner)$. Thus $\mathrm{FM}(\mathcal{N}) \models_{sl} \mathrm{Sat}_{\mathrm{sl}}(\ulcorner\neg\varphi\urcorner, \ulcorner\overline{a}\urcorner) \equiv \neg\mathrm{Sat}_{\mathrm{sl}}(\ulcorner\varphi\urcorner, \ulcorner\overline{a}\urcorner)$.

4. Fix formulae φ, ψ. Let $k \geqslant \ulcorner\varphi \wedge \psi\urcorner > \max\{\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner\}$ and let $n \in [F(k), F(k+1))$. Then for every valuation $\overline{a}$ in $\mathcal{N}_k$ the following are equivalent:

   - $\mathcal{N}_n \models \mathrm{Sat}_{\mathrm{sl}}(\ulcorner\varphi \wedge \psi\urcorner, \ulcorner\overline{a}\urcorner)$,
   - $\mathcal{N}_k \models (\varphi \wedge \psi)[\overline{a}]$,
   - $\mathcal{N}_k \models \varphi[\overline{a}]$ and $\mathcal{N}_k \models \psi[\overline{a}]$,
   - $\mathcal{N}_n \models \mathrm{Sat}_{\mathrm{sl}}(\ulcorner\varphi\urcorner, \ulcorner\overline{a}\urcorner)$ and $\mathcal{N}_n \models \mathrm{Sat}_{\mathrm{sl}}(\ulcorner\psi\urcorner, \ulcorner\overline{a}\urcorner)$,
   - $\mathcal{N}_n \models \mathrm{Sat}_{\mathrm{sl}}(\ulcorner\varphi\urcorner, \ulcorner\overline{a}\urcorner) \wedge \mathrm{Sat}_{\mathrm{sl}}(\ulcorner\psi\urcorner, \ulcorner\overline{a}\urcorner)$.

   Therefore for $m \geqslant F(k)$ it holds that $\mathcal{N}_m \models \mathrm{Sat}_{\mathrm{sl}}(\ulcorner\varphi \wedge \psi\urcorner, \ulcorner\overline{a}\urcorner) \equiv (\mathrm{Sat}_{\mathrm{sl}}(\ulcorner\varphi\urcorner, \ulcorner\overline{a}\urcorner) \wedge \mathrm{Sat}_{\mathrm{sl}}(\ulcorner\psi\urcorner, \ulcorner\overline{a}\urcorner))$. Thus $\mathrm{FM}(\mathcal{N}) \models_{sl} \mathrm{Sat}_{\mathrm{sl}}(\ulcorner\varphi\wedge\psi\urcorner, \ulcorner\overline{a}\urcorner) \equiv (\mathrm{Sat}_{\mathrm{sl}}(\ulcorner\varphi\urcorner, \ulcorner\overline{a}\urcorner) \wedge \mathrm{Sat}_{\mathrm{sl}}(\ulcorner\psi\urcorner, \ulcorner\overline{a}\urcorner))$.

$\square$

Therefore $\mathrm{Sat}_{\mathrm{sl}}$ is an approximate FM–satisfaction definition. As a corollary of Theorem 3.1.2 we get that $\mathrm{Sat}_{\mathrm{sl}}$ under *sl*–semantics defines a Σ_2–complete relation.

We naturally define the notion of a approximate FM–truth definition as follows.

Definition 3.1.3 (Approximate FM–truth Definition)
Let $\psi(x, v)$ be an an approximate FM*–satisfaction definition. We call the predicate $\varphi(x) =_{df} \psi(x, \ulcorner\varepsilon\urcorner) \wedge \mathrm{Sent}(x)$ an approximate* FM*–truth definition.*

Definition 3.1.4 ($\mathrm{Tr}_{\mathrm{sl}}$)

$$\mathrm{Tr}_{\mathrm{sl}}(x) =_{df} \mathrm{Sat}_{\mathrm{sl}}(x, \ulcorner\varepsilon\urcorner) \wedge \mathrm{Sent}(x).$$

Of course since $\mathrm{Sat}_{\mathrm{sl}}$ is an approximate FM–satisfaction definition, $\mathrm{Tr}_{\mathrm{sl}}$ is an approximate FM–truth definition. Note that by Theorem 3.1.2 for every arithmetical sentences ψ, φ the predicate $\mathrm{Tr}_{\mathrm{sl}}$ has the following properties:

- $\mathrm{FM}(\mathcal{N}) \models_{sl} \psi$ if and only if $\mathrm{FM}(\mathcal{N}) \models_{sl} \mathrm{Tr}_{\mathrm{sl}}(\ulcorner\psi\urcorner)$,
- $\mathrm{FM}(\mathcal{N}) \models_{sl} \mathrm{Tr}_{\mathrm{sl}}(\ulcorner\neg\psi\urcorner) \equiv \neg\mathrm{Tr}_{\mathrm{sl}}(\ulcorner\psi\urcorner)$,
- $\mathrm{FM}(\mathcal{N}) \models_{sl} \mathrm{Tr}_{\mathrm{sl}}(\ulcorner\psi \wedge \varphi\urcorner) \equiv (\mathrm{Tr}_{\mathrm{sl}}(\ulcorner\psi\urcorner) \wedge \mathrm{Tr}_{\mathrm{sl}}(\ulcorner\varphi\urcorner))$.

In the following sections we use only the above three properties of the approximate FM–truth definitions.

3.2 Modal logics SL, SL*, $\mathrm{L_{Tr}}$ and $\mathrm{L_{Tr}}^*$

In this section we present the modal logic SL for which we prove a Solovay style completeness theorem for our approximate FM–truth definition $\mathrm{Tr_{sl}}$. The definition of SL mimics the properties of $\mathrm{Tr_{sl}}$ introduced in Section 3.1. Our intention is to interpret $\Box$ as $\mathrm{Tr_{sl}}$ and to add appropriate axioms to the system i.e. translate the properties of $\mathrm{Tr_{sl}}$ to the modal language.

Definition 3.2.1 (SL) *The modal logic* SL *is an extension of the modal logic* K *with the following axioms, for each formula φ,*

$$\Box(\neg\varphi) \equiv \neg\Box\varphi. \tag{3.1}$$

Let us observe, that adding the above axioms to K is enough to make $\Box$ commute with all propositional connectives. The commutativity with conjunction, $\Box(\varphi \wedge \psi) \equiv \Box\varphi \wedge \Box\psi$, is already provable in K. The commutativity with $\Rightarrow$ and $\vee$ follows directly from the commutativity with $\neg$ and $\wedge$.

It follows that for every formula φ, SL $\vdash \Box\varphi \equiv \Diamond\varphi$. For a fixed φ the following are equivalent in SL: $\Box\varphi$, $\Box\neg\neg\varphi$ and, by (3.1), $\neg\Box\neg\varphi$ which is, by the definition, equal to $\Diamond\varphi$.

Now, we define a fixpoint extension SL* of SL.

Definition 3.2.2 (Guarded Propositional Variable)
A propositional variable p is guarded in a modal formula $\varphi(p)$ if each occurrence of p is within the scope of a modal operator.

Firstly, we extend the language of SL. For each p and a formula $\varphi(p)$ such that p is the only propositional variable occurring in φ and p is guarded in φ we add a new propositional constantindex$q_{\langle\varphi,p\rangle}$ $q_{\langle\varphi,p\rangle}$. We call the modal language with those new constants an extended modal language. To contrast the modal language (without propositional constants) with the extended modal language we sometimes refer to modal language as to basic modal language. Note that we do not iterate this language extension – the new propositional constants are added only for basic modal formulae with only one propositional variable in which they are guarded.

We are ready to define the modal logic SL*.

Definition 3.2.3 (SL*) *The logic* SL* *is an extension of* SL *by the following axioms*

$$q_{\langle\varphi,p\rangle} \equiv \varphi(q_{\langle\varphi,p\rangle}/p),$$

where $q_{\langle\varphi,p\rangle}$ is a new propositional constant and $\varphi(q_{\langle\varphi,p\rangle}/p)$ is a result of replacing in $\varphi(p)$ each occurrence of p by $q_{\langle\varphi,p\rangle}$.

The similar fixpoint extensions were introduced by Smoryński, see [Smo85]. The other two logics we define are the modal logic $\mathrm{L_{Tr}}$ of the truth predicate for $\models_{sl}$ and its fixpoint extension $\mathrm{L_{Tr}}^*$.

Definition 3.2.4 (Translation) *Let* PROP *be a set of propositional variables and let* tr *be a function from* PROP *into the set of sentences of the extended arithmetical vocabulary with* MAX.

We denote the value of tr *at a given variable* p *as* p^{tr} *and we extend it inductively on the set of all formulae of modal logic in the following way:*

- $(\neg\varphi)^{\mathrm{tr}} = \neg(\varphi)^{\mathrm{tr}}$,
- $(\varphi \wedge \psi)^{\mathrm{tr}} = \varphi^{\mathrm{tr}} \wedge \psi^{\mathrm{tr}}$,
- $(\Box\varphi)^{\mathrm{tr}} = \mathrm{Tr}_{\mathrm{sl}}(\ulcorner\varphi^{\mathrm{tr}}\urcorner)$,

Any such function is called a translation.

Definition 3.2.5 ($\mathrm{L_{Tr}}$) *The logic* $\mathrm{L_{Tr}}$ *consists of all modal formulae* $\varphi(p_1, \ldots, p_n)$ *such that for any translation* tr *it holds that* $\mathrm{FM}(\mathcal{N}) \models_{sl} \varphi^{\mathrm{tr}}$.

By the properties of $\mathrm{FM}(\mathcal{N})$ under *sl*–semantics the following fact holds.

Fact 3.2.6 *The modal logic* $\mathrm{L_{Tr}}$ *is a consistent normal modal logic.*

We may consider $\mathrm{L_{Tr}}$ the modal logic of the propositional truth predicate for $\models_{sl}$.

We naturally extend translations to the extended modal language in the following way.

Definition 3.2.7 (Extended Translation) *Let* tr *be a translation. We say that* tr^* *is an extended translation if it is an extension of* tr *to the extended modal language, commuting with* $\wedge, \neg, \Box$ *and such that for every propositional constant* $q_{\langle\varphi,p\rangle}$ *if* $(q_{\langle\varphi,p\rangle})^{\mathrm{tr}^*} = \psi$*, then*

$$\mathrm{FM}(\mathcal{N}) \models_{sl} \psi \equiv (\varphi(q_{\langle\varphi,p\rangle}/p))^{\mathrm{tr}^*}.$$

Finally we give an analogous semantic definition of the last modal logic we concern – $\mathrm{L_{Tr}}^*$.

Definition 3.2.8 ($\mathrm{L_{Tr}}^*$) *The logic* $\mathrm{L_{Tr}}^*$ *consists of all modal formulae in an extended modal language* $\varphi(p_1, \ldots, p_n)$ *such that for any extended translation* tr^* *it holds that* $\mathrm{FM}(\mathcal{N}) \models_{sl} \varphi^{\mathrm{tr}^*}$.

By the FM–version of the Diagonal Lemma (2.4.6) for every arithmetical formula $\varphi(x)$ there exists an arithmetical sentence ψ such that

$$\mathrm{FM}(\mathcal{N}) \models_{sl} \psi \equiv \varphi(\ulcorner\psi\urcorner).$$

Therefore, the following holds:

Fact 3.2.9 *Any translation* tr *can be extended to an extended translation* tr^*.

Corollary 3.2.10 *The modal logic* ${\mathrm{L}_{\mathrm{Tr}}}^*$ *is a consistent normal modal logic in an extended modal language. Moreover,* ${\mathrm{L}_{\mathrm{Tr}}}^* \leqslant_{\mathrm{SL}} \mathrm{L}_{\mathrm{Tr}}$.

Our aim is to show that SL and L_{Tr} are equivalent modal logics. The argument proceeds by showing the following dependencies:

$$\mathrm{SL} \leqslant_{\mathrm{SL}} \mathrm{L}_{\mathrm{Tr}} \leqslant_{\mathrm{SL}} {\mathrm{L}_{\mathrm{Tr}}}^* \leqslant_{\mathrm{SL}} \mathrm{SL}.$$

The first two dependencies are proved in Parts 1 and 2 of Theorem 3.2.11. In the main section of this chapter we show the final dependency: ${\mathrm{L}_{\mathrm{Tr}}}^* \leqslant_{\mathrm{SL}} \mathrm{SL}$. This is achieved by showing that for every extended modal logic L such that $\mathrm{SL}^* \leqslant_{\mathrm{SL}^*} \mathrm{L}$ it holds that $\mathrm{L} \leqslant_{\mathrm{SL}} \mathrm{SL}$. Therefore, since by Part 3 of Theorem 3.2.11 it holds that $\mathrm{SL}^* \leqslant_{\mathrm{SL}^*} {\mathrm{L}_{\mathrm{Tr}}}^*$ we get ${\mathrm{L}_{\mathrm{Tr}}}^* \leqslant_{\mathrm{SL}} \mathrm{SL}$. Thus, L_{Tr} is independent from a particular choice of an approximate FM–truth definition for which we translate the $\Box$ operator.

The following theorem is essential for understanding the basic relations between the introduced modal logics.

Theorem 3.2.11

1. $\mathrm{SL} \leqslant_{\mathrm{SL}} \mathrm{L}_{\mathrm{Tr}}$,
2. $\mathrm{L}_{\mathrm{Tr}} \leqslant_{\mathrm{SL}} {\mathrm{L}_{\mathrm{Tr}}}^*$,
3. $\mathrm{SL}^* \leqslant_{\mathrm{SL}^*} {\mathrm{L}_{\mathrm{Tr}}}^*$.

Proof: For the proof of 1 it suffices to observe that the axioms of SL mimics some of the properties of $\mathrm{Tr}_{\mathrm{sl}}(x)$ in $\models_{sl}$.

The proof of 2 is the following. Fix a basic modal formula ψ. Suppose that for every translation tr it holds that $\mathrm{FM}(\mathcal{N}) \models_{sl} \psi^{\mathrm{tr}}$. Let tr^* be an extended translation and tr be its restriction to basic modal formulae. Then $\psi^{\mathrm{tr}^*} = \psi^{\mathrm{tr}}$. It follows that $\mathrm{FM}(\mathcal{N}) \models_{sl} \psi^{\mathrm{tr}^*}$. Therefore, since tr^* was arbitrary, $\psi \in {\mathrm{L}_{\mathrm{Tr}}}^*$.

For the proof of 3 suppose that $\mathrm{SL}^* \vdash \psi$. Observe that under any translation tr^* every fixpoint axiom of SL^* is translated to an *sl*–true sentence. Since ${\mathrm{L}_{\mathrm{Tr}}}^*$ is closed under consequence it follows that $\psi \in {\mathrm{L}_{\mathrm{Tr}}}^*$.

$\Box$

Corollary 3.2.12 *The logics* SL *and* SL^* *are consistent.*

Before proceeding to the proof of the main theorem we need a better understanding of modal logics SL and SL^*.

3.3 Completeness Theorems for SL and SL*

We begin this section with the following remark on SL's models. Let us consider the formula $\Box\bot$ – it is true exactly in those worlds of a given Kripke frame from which there are no accessible worlds – let us call them final. On the other hand $\Diamond\bot$ is obviously equivalent to $\bot$. Since $\mathrm{SL} \vdash \Box\bot \equiv \Diamond\bot$ there are no final points in SL's models.

We call a Kripke frame a *line* if it is of the form $(\{0,\ldots,n\}, S_n \cup \{(n,n)\})$ or (ω, S), where S is the successor relation and S_n is its restriction to the set $\{0,\ldots,n\}$. Thus, a frame is a line if it is a finite initial segment of the successor relation with a loop added at the top or if it is the standard model for arithmetic of the successor relation. We denote the n-th finite line with universe $\{0,\ldots,n\}$ by L_n. The infinite line is denoted by L_ω.

Definition 3.3.1 *A valuation V in a Kripke frame F is admissible for a logic* L *if each axiom of* L *is true in F under V. A modal logic* L *is sound and complete with respect to the family of Kripke frames $\mathcal{F}$ if the set of admissible valuations for* L *is not empty and the following are equivalent:*

1. $\mathrm{L} \vdash \varphi$,
2. *for any $F \in \mathcal{F}$ and for any valuation V in F admissible for* L *it holds that $F \models \varphi[V]$.*

If $\mathcal{F}$ is a singleton $\{F\}$ we say that L *is sound and complete with respect to F.*

We defined provability only for logics, not for theories. Now, we need to fix the notion of a (syntactical) consistency for a given set of formulae.

Definition 3.3.2 *A set of formulae $\mathcal{F}$ is consistent in a logic* L *is there are no $\varphi_1,\ldots,\varphi_n \in \mathcal{F}$ such that $\mathrm{L} \vdash \neg(\varphi_1 \wedge \cdots \wedge \varphi_n)$.*

Definition 3.3.3 *Let $\mathcal{F}$ be a set of modal formulae. By $\Box^{-1}\mathcal{F}$ we denote the following set*

$$\Box^{-1}\mathcal{F} = \{\varphi : \Box\varphi \in \mathcal{F}\}.$$

The main tool for proving a completeness theorem is the following lemma.

Lemma 3.3.4 *Let* L *be a consistent normal modal logic containing* SL *and let $\mathcal{F}$ be a maximal consistent in* L *set of formulae. Then, $\Box^{-1}\mathcal{F}$ is a maximal consistent in* L *set of formulae.*

Proof: Let L and $\mathcal{F}$ satisfy the assumptions of the lemma. One can easily see that $\Box^{-1}\mathcal{F}$ is closed under conjunctions. Therefore, if $\Box^{-1}\mathcal{F}$ was inconsistent in L, then there would be a formula $\varphi \in \Box^{-1}\mathcal{F}$ such that $\mathrm{L} \vdash \neg\varphi$. Then, $\mathrm{L} \vdash \Box\neg\varphi$ and $\mathrm{L} \vdash \neg\Box\varphi$. But $\Box\varphi \in \mathcal{F}$ thus $\mathcal{F}$ would be inconsistent in L.

Now, for the sake of contradiction suppose that $\Box^{-1}\mathcal{F}$ is not maximal with respect to consistency. Thus, there is a formula $\psi \notin \Box^{-1}\mathcal{F}$ such that $\Box^{-1}\mathcal{F} \cup \{\psi\}$ is still consistent. Since $\psi \notin \Box^{-1}\mathcal{F}$, we have that $\Box\psi \notin \mathcal{F}$. By the maximality of $\mathcal{F}$, the set $\mathcal{F}\cup\{\Box\psi\}$ is inconsistent in L. Thus, there exists a formula $\varphi \in \mathcal{F}$ such that $\mathrm{L} \vdash \neg\varphi \vee \neg\Box\psi$. It follows that $\neg\Box\psi \in \mathcal{F}$. Since $\Box\neg\psi$ is equivalent in SL to $\neg\Box\psi$, we have that $\Box\neg\psi \in \mathcal{F}$ and $\neg\psi \in \Box^{-1}\mathcal{F}$. This is a contradiction since $\Box^{-1}\mathcal{F} \cup \{\psi\}$ was assumed to be consistent. $\Box$

Theorem 3.3.5

1. *Modal logics* SL *and* SL* *are sound and complete with respect to the infinite line* L_ω.
2. *Modal logic* SL *is sound and complete with respect to the family of all finite lines.*

Proof: Since the soundness part may be easily verified we concentrate on completeness only. Moreover, in the case of SL it is enough to prove the theorem only for L_ω. Indeed, for a basic modal formula φ, if $L_\omega, 0 \not\models \varphi[V]$ and n is the modal depth of φ, then $L_n, 0 \not\models \varphi[V_n]$, where V_n is a restriction of V to the worlds $0, \ldots, n$.

To prove the completeness we show the following implication: for each formula φ, if $\mathrm{SL}^* \not\vdash \varphi$, then there exists a valuation V admissible for SL* such that $L_\omega, 0 \not\models \varphi[V]$. The same argument works also for SL. The only difference is that in the case of SL we may stop the construction after n steps, where n is the modal depth of φ.

Let us assume that $\mathrm{SL}^* \not\vdash \varphi$. Let $\mathcal{F}_0$ be a set of formulae which contains $\neg\varphi$ and is a maximal consistent in SL*. We construct a sequence of sets $(\mathcal{F}_i)_{i\in\omega}$, such that

$$\mathcal{F}_{i+1} = \Box^{-1}\mathcal{F}_i.$$

By Lemma 3.3.4, each set $\mathcal{F}_i$ is maximal consistent in SL*. Now, we construct a valuation $V\colon \omega \times (\mathrm{PROP} \cup \mathrm{CONSTS}) \longrightarrow \{0,1\}$ in L_ω as follows

$$V(i,p) = 1 \text{ if and only if } p \in \mathcal{F}_i,$$

$$V(i, q_{\langle\psi,p\rangle}) = 1 \text{ if and only if } q_{\langle\psi,p\rangle} \in \mathcal{F}_i,$$

A straightforward proof by induction on the complexity of a formula shows that for all formulae ψ and for all $i \in \omega$,

$$L_\omega, i \models \psi[V] \text{ if and only if } \psi \in \mathcal{F}_i.$$

Moreover, since each $\mathcal{F}_i$ is maximal consistent in SL*, the following equivalence holds, for each $q_{\langle\psi,p\rangle}$,

$$q_{\langle\psi,p\rangle} \in \mathcal{F}_i \text{ if and only if } \psi(q_{\langle\psi,p\rangle}) \in \mathcal{F}_i.$$

Therefore V is admissible for SL*. Thus we constructed a model for SL* which falsifies φ. $\Box$

Let us observe that in the proof above, in the case of SL* we could not restrict ourselves just to a finite number of sets $\mathcal{F}_i$. To the contrary, it can be shown that all sets $\mathcal{F}_i$ are different. This is caused by the fixpoint axioms of SL*. This is why SL* has no finite models while in the case of SL it would be enough to consider sets $\mathcal{F}_0, \ldots, \mathcal{F}_n$, where n is the modal depth of φ.

Let us make one more observation before proceeding to the main section of this chapter. For the approximate FM–truth predicate $\mathrm{Tr}_{\mathrm{sl}}$ we have the following property: for each sentence ψ

$$\mathrm{FM}(\mathcal{N}) \models_{sl} \psi \quad \text{if and only if} \quad \mathrm{FM}(\mathcal{N}) \models_{sl} \mathrm{Tr}_{\mathrm{sl}}(\ulcorner \psi \urcorner).$$

We show that SL is closed under an analogous rule: for each modal formula ψ, SL $\vdash \Box\psi$ if and only if SL $\vdash \psi$. Fix a modal formula ψ. The implication from right to left follows from the necessitation rule. For the other implication assume that SL $\vdash \Box\psi$ and for the sake of contradiction that SL $\nvdash \psi$. Then by the Completeness Theorem (3.3.5) there is a valuation V in L_ω such that $L_\omega, 0 \models \neg\psi[V]$. We construct another valuation V' on L_ω such that for each propositional variable p and $i \in \omega$, $V'(i+1, p) = V(i, p)$ and $V'(0, p) = 0$. Now we have $L_\omega, 1 \models \neg\psi[V']$ and therefore $L_\omega, 0 \models \neg\Box\psi[V']$ which contradicts SL $\vdash \Box\psi$.

3.4 The Main Theorem

In this section we prove the main result of the chapter which characterises the modal logic of the approximate FM–truth predicate.

Definition 3.4.1 *For a formula ψ, we define $(\neg)^0\psi$ as $\neg\psi$ and $(\neg)^1\psi$ as ψ. For a finite set of propositional variables P and a function $\varepsilon\colon \{0, \ldots, k\} \times P \longrightarrow \{0, 1\}$ we define Φ_ε as a formula*

$$\bigwedge_{0 \leqslant i \leqslant k} \bigwedge_{p \in P} \Box^i (\neg)^{\varepsilon(i,p)} p.$$

Any function ε of the above form is called a valuation on variables P and $k+1$ consecutive worlds.

Let P and ε be as above. If the formula Φ_ε is true at a given world a of a model L of SL then it determines completely the values of propositions in P at a and any worlds which can be accessed from a in k steps. Indeed, if $L, a \models \Phi_\varepsilon[V]$ then, for any $p \in P$ and any $0 \leqslant i \leqslant k$,

$$L, a + i \models p[V] \quad \text{if and only if} \quad \varepsilon(i, p) = 1.$$

It follows that Φ_ε determines at a the truth values of formulae with modal depth not greater than k with propositional variables from P. The last observation gives us, by the completeness, the following lemma.

Lemma 3.4.2 *Let $k \in \omega$, let P be a finite set of propositional variables and let $\varepsilon\colon \{0,\ldots,k\}\times P \longrightarrow \{0,1\}$. For any formula φ of modal depth not greater than k with all variables from P, Φ_ε decides φ, that is*

$$\mathrm{SL} \vdash \Phi_\varepsilon \Rightarrow \varphi \quad \textit{or} \quad \mathrm{SL} \vdash \Phi_\varepsilon \Rightarrow \neg\varphi.$$

Now, we show that any valuation on propositional variables is consistent with SL^*.

Lemma 3.4.3 *For every $n > 0$ and $\varepsilon\colon \{0,\ldots,n-1\} \to \{0,1\}$ there is a formula ψ such that*

$$\mathrm{SL}^* \vdash \bigvee_{0\leqslant r<n} \Box^r \left(\bigwedge_{0\leqslant i<n} \Box^i (\neg)^{\varepsilon(i)} \psi \right).$$

Moreover, $\mathrm{SL}^* \vdash (\psi \equiv \Box^n \psi)$.

Proof: For $n = 1$, we put $\psi = \bot$, if $\varepsilon(0) = 0$, and $\psi = \top$ otherwise. For $n > 1$ and a fixed propositional variable p let us consider the following formula

$$\varphi_n =_{df} \bigwedge_{1\leqslant i<n} \neg\Box^i p.$$

Since φ_n is guarded in p there is a propositional constant $q_{\langle\varphi_n,p\rangle}$ such that

$$\mathrm{SL}^* \vdash q_{\langle\varphi_n,p\rangle} \equiv \left(\bigwedge_{1\leqslant i<n} \neg\Box^i q_{\langle\varphi_n,p\rangle} \right).$$

We need the following two properties of $q_{\langle\varphi_n,p\rangle}$: for each valuation V admissible for SL^*, and for each world a:

1. there exists $i \leqslant n-1$ such that $L_\omega, a+i \models q_{\langle\varphi_n,p\rangle}[V]$,
2. if $L_\omega, a \models q_{\langle\varphi_n,p\rangle}[V]$, then for each $1 \leqslant i \leqslant n-1$, $L_\omega, a+i \not\models q_{\langle\varphi_n,p\rangle}[V]$.

It follows that $q_{\langle\varphi_n,p\rangle}$ is true exactly in every n-th world of L_ω. For the first property it suffices to observe that if for all $1 \leqslant i \leqslant n-1$ it holds that $L_\omega, a+i \not\models q_{\langle\varphi_n,p\rangle}[V]$ then, by the fixpoint axiom for $q_{\langle\varphi_n,p\rangle}$, it has to be true at the world a. The second property also follows easily from the fixpoint axiom for $q_{\langle\varphi_n,p\rangle}$.

Now, for a fixed $\varepsilon\colon \{0,\ldots,n-1\} \to \{0,1\}$ we define ψ as follows:

$$\psi = \bigvee_{\substack{0\leqslant j\leqslant n-1\\ \varepsilon(j)=1}} \Box^{n-j} q_{\langle\varphi_n,p\rangle},$$

where the empty disjunction is identified with $\bot$. It is easy to see that the lemma holds for $\psi = \bot$. So, we assume that there is $j \in \{0, \ldots, n-1\}$ such that $\varepsilon(j) = 1$. We show that ψ has desired properties. By the completeness of SL^* with respect to L_ω it is enough to show that for each admissible valuation V there exists $r \leqslant n-1$ such that

$$L_\omega, r \models \bigwedge_{0 \leqslant i \leqslant n-1} \Box^i (\neg)^{\varepsilon(i)} \psi[V].$$

Thus, let V be a valuation admissible for SL^* and let $r \leqslant n-1$ be the smallest world such that

$$L_\omega, r \models q_{\langle \varphi_n, p \rangle}[V].$$

By Properties 1 and 2 of $q_{\langle \varphi_n, p \rangle}$, for each a,

$$L_\omega, a \models q_{\langle \varphi_n, p \rangle}[V]$$

if and only if

$$a = r + ln, \text{ for some } l \in \omega.$$

It follows that for each $0 \leqslant i \leqslant n-1$

$$\begin{aligned} L_\omega, r \models \Box^i \psi[V] &\iff L_\omega, r \models \Box^i \left(\bigvee_{\substack{0 \leqslant j \leqslant n-1 \\ \varepsilon(j)=1}} \Box^{n-j} q_{\langle \varphi_n, p \rangle} \right) \\ &\iff L_\omega, r \models \bigvee_{\substack{0 \leqslant j \leqslant n-1 \\ \varepsilon(j)=1}} \Box^i \Box^{n-j} q_{\langle \varphi_n, p \rangle} \\ &\iff \bigvee_{\substack{0 \leqslant j \leqslant n-1 \\ \varepsilon(j)=1}} (i = j) \\ &\iff \varepsilon(i) = 1. \end{aligned}$$

Similarly, for each $0 \leqslant i \leqslant n-1$,

$$L_\omega, r \models \Box^i \neg \psi[V] \iff \varepsilon(i) = 0.$$

Therefore,

$$L_\omega, r \models \bigwedge_{0 \leqslant i < n} \Box^i (\neg)^{\varepsilon(i)} \psi[V]$$

and since the valuation V is arbitrary and $r \leqslant n-1$, by the completeness of SL^* with respect to L_ω we get

$$\mathrm{SL}^* \vdash \bigvee_{0 \leqslant r < n} \Box^r \left(\bigwedge_{0 \leqslant i < n} \Box^i (\neg)^{\varepsilon(i)} \psi \right).$$

This completes the proof of the first part of the lemma. To prove the "Moreover" part one needs to observe that the only propositional constant used in ψ is $q_{\langle\varphi_n,p\rangle}$ and there are no propositional variables in ψ. But we have $\mathrm{SL}^* \vdash (q_{\langle\varphi_n,p\rangle} \equiv \Box^n q_{\langle\varphi_n,p\rangle})$ and this property easily transfers to all formulae which use only propositional constant $q_{\langle\varphi_n,p\rangle}$ and no propositional variables. $\Box$

Lemma 3.4.4 *Let* $k \in \omega$, *let* P *be a finite set of variables and let* $\varepsilon\colon \{0,\dots,k\} \times P \longrightarrow \{0,1\}$. *If* L *is a consistent, normal modal logic such that* $\mathrm{SL}^* \leqslant_{\mathrm{SL}^*} \mathrm{L}$ *then* Φ_ε *is consistent with* L.

Proof: Let $P = \{p_1,\dots,p_m\}$ and let $n_1,\dots,n_m$ be pairwise coprime natural numbers greater than k. We extend ε to a function from $\bigcup_{1\leqslant t\leqslant m}\{0,\dots,n_t\}\times\{p_t\}$ by putting $\varepsilon(i,p_t) = 0$ for any $i > k$. Now, for $1 \leqslant t \leqslant m$, let ψ_t be an SL^* formula from Lemma 3.4.3 such that

$$\mathrm{SL}^* \vdash \bigvee_{0\leqslant r<n_t} \Box^r \left(\bigwedge_{0\leqslant i<n_t} \Box^i (\neg)^{\varepsilon(i,p_t)} \psi_t \right)$$

and

$$\mathrm{SL}^* \vdash \psi_t \equiv \Box^{n_t}\psi_t.$$

Now, let V be an arbitrary valuation admissible for SL^* and, for $1 \leqslant t \leqslant m$, let $a_t < n_t$ be such that

$$L_\omega, a_t \models \bigwedge_{0\leqslant i<n_t} \Box^i (\neg)^{\varepsilon(i,p_t)} \psi_t[V].$$

We show how to replace all a_t's by a single world b. By the second property of ψ_t's mentioned in the proof of Lemma 3.4.3, for each $l \in \omega$ it holds that

$$L_\omega, a_t + l n_t \models \bigwedge_{0\leqslant i<n_t} \Box^i (\neg)^{\varepsilon(i,p_t)} \psi_t[V].$$

Since n_t's are pairwise coprime, by the Chinese Remainder Theorem, there exists b such that, for each $1 \leqslant t \leqslant m$, the remainder of b modulo n_t is a_t. Therefore it holds that

$$L_\omega, b \models \bigwedge_{1\leqslant t\leqslant m} \bigwedge_{0\leqslant i<n_t} \Box^i (\neg)^{\varepsilon(i,p_t)} \psi_t[V].$$

Moreover, we can choose $b < N = \prod_{1\leqslant t\leqslant m} n_t$.

Since the valuation V was arbitrary, the following formula is provable in SL^*:

$$\bigvee_{0\leqslant j<N} \Box^j (\bigwedge_{1\leqslant t\leqslant m} \bigwedge_{0\leqslant i\leqslant n_t} \Box^i (\neg)^{\varepsilon(i,p_t)} \psi_t).$$

It follows that a weaker formula below is also provable in SL*

$$\bigvee_{0\leqslant j<N} \Box^j(\bigwedge_{1\leqslant t\leqslant m} \bigwedge_{0\leqslant i\leqslant k} \Box^i(\neg)^{\varepsilon(i,p_t)}\psi_t).$$

The last formula is equivalent to

$$\bigvee_{0\leqslant j<N} \Box^j \Phi_\varepsilon(\psi_1/p_1, \dots, \psi_m/p_m). \tag{3.2}$$

Now, if $\neg\Phi_\varepsilon$ is provable in L then, by necessitation and substitution of ψ_i's for p_i's, L proves also

$$\bigwedge_{0\leqslant j<N} \Box^j \neg\Phi_\varepsilon(\psi_1/p_1, \dots, \psi_m/p_m). \tag{3.3}$$

But, (3.2) is equivalent to the negation of (3.3). Since (3.2) is provable in SL* and L is consistent, it follows that Φ_ε has to be consistent with L. □

Lemma 3.4.4 shows that if L is a consistent normal modal logic such that $\mathrm{SL}^* \leqslant_{\mathrm{SL}^*} \mathrm{L}$, then L has to be consistent with any valuation described by a function $\varepsilon\colon \{1, \dots, k\} \times P \longrightarrow \{0, 1\}$. We use this fact to show that $\mathrm{L_{Tr}}^*$ is conservative over SL in the language of SL i.e. to show that $\mathrm{L_{Tr}}^* \leqslant_{\mathrm{SL}} \mathrm{SL}$.

Lemma 3.4.5 *For each consistent normal modal logic* L *such that* $\mathrm{SL}^* \leqslant_{\mathrm{SL}^*} \mathrm{L}$ *it holds that* $\mathrm{L} \leqslant_{\mathrm{SL}} \mathrm{SL}$.

Proof: Let L be a consistent modal logic such that $\mathrm{SL}^* \leqslant_{\mathrm{SL}^*} \mathrm{L}$ and let $\varphi(p_1, \dots, p_n)$ be a formula in the language of SL such that $\mathrm{SL} \nvdash \varphi$. We show that $\mathrm{L} \nvdash \varphi$. Let us assume that all variables of φ are among $p_1, \dots, p_n$. Let $P = \{p_1, \dots, p_n\}$ and let $V\colon \omega \times P \longrightarrow \{0, 1\}$ be a valuation witnessing that $\mathrm{SL}^* \nvdash \varphi$. We may assume that $L_\omega, 0 \not\models \varphi[V]$. In order to determine the value of φ we need only to consider V restricted to the set $\{0, \dots, k\} \times P$, for k being the modal depth of φ. Let ε be V restricted to this set. Since Φ_ε is consistent with $\neg\varphi$, by Lemma 3.4.2 it implies $\neg\varphi$. By Lemma 3.4.4, Φ_ε is consistent with L. Thus, the formula $\neg\varphi$ has to be consistent with L, too. □

Theorem 3.2.11 and Lemma 3.4.5 allow us to establish that

$$\mathrm{SL} \leqslant_{\mathrm{SL}} \mathrm{L_{Tr}} \leqslant_{\mathrm{SL}} \mathrm{L_{Tr}}^* \leqslant_{\mathrm{SL}} \mathrm{SL},$$

where the last $\leqslant_{\mathrm{SL}}$ follows from Lemma 3.4.5 and Theorem 3.2.11(3). Thus, we obtain the following theorem.

Theorem 3.4.6 *The logic* SL *is the modal logic of the approximate* FM*–truth predicate.*

Corollary 3.4.7 *The modal logic* $\mathrm{L_{Tr}}$ *does not depend on the choice of an underlying approximate* FM*–truth predicate.*

3.5 Summary

In this chapter we have investigated how we can approximate FM–truth definitions. Our construction of the approximate FM–satisfaction definition $\mathrm{Sat}_{\mathrm{sl}}$ and the approximate FM–truth definition $\mathrm{Tr}_{\mathrm{sl}}$ shows that there are arithmetical predicates which posses certain properties of the FM–truth definitions.

We show that $\mathrm{Tr}_{\mathrm{sl}}$ have all of the properties expressible in the modal logic which are consistent with the properties of FM–truth definitions. This was achieved by proving the equivalence between the modal logic L_{Tr} of $\mathrm{Tr}_{\mathrm{sl}}$ and the modal logic SL. As a corollary we have obtained that SL is the modal logic of an arbitrary approximate FM–truth definition. It follows that there is no stronger approximate FM–truth definition with respect to the properties expressible in the modal logic.

Chapter 4

The Concrete Foundations of Mathematics

In this chapter we focus on model theory. By model theory we do not mean here the axiomatic model theory that is performed in some fixed set theory i.e. what currently is understood as model theory. Our aim is to develop *concrete* model theory – a theory about *real* structures and their properties. We introduce the notion of a concrete model for which we require both the structure and the satisfaction relation to be representable by finitistic means, only with the use of potential infinity. We focus on model–theoretic constructions known from the axiomatic model theory to identify which of them remain valid in the *concrete* framework and which include some steps essentially requiring the use of actual infinity. We show both positive and negative examples of such constructions. For the latter we identify those steps which are not acceptable in this new framework i.e. those that applied to concrete models may result in obtaining non–concrete models. Most of the model–theoretic constructions considered in this chapter comes from the book by Chang and Keisler's [CK73], but they are performed in the concrete models context. The essential steps of those model–theoretic constructions making them *concrete* are highlighted by a vertical line on the margin.

Willing to get rid of actual infinity we need to avoid referring to set theory in its full extent. As a starting point we employ Mostowski's FM–domains as *models* of potentially infinite worlds and the notion of FM–representability as an explication of representability without actual infinity. We intend to sketch a border between the concrete and the non–concrete part of model theory i.e. between the *safe*, *finitistic* one and the other, somehow transcendent, proper understanding of which requires a satisfactory theory of the actually infinite.

4.1 Basic Definitions

In this section we introduce concrete models, their basic properties and relations between them. We also prove some elementary lemmata and theorems concerning feasibility and impossibility of performing certain arguments in the concrete models context. Our first aim is to define concrete models for which we require both the model and its satisfaction relation to be FM–representable. We begin with the notion of a concrete vocabulary.

Definition 4.1.1 (Concrete Vocabulary) *Let $P_1, \ldots, P_m$ be predicates, let $ar\colon \{P_1, \ldots, P_m\} \to \omega$ be the finite arity function and let C be a set of constants FM–represented by φ_C. We say that a sequence $\sigma = (P_1, \ldots, P_m, ar, \varphi_C)$ is a concrete vocabulary.*

Note that we restrict our attention to vocabularies with only finitely many predicates allowing infinite, but concrete sets of constants. For a concrete vocabulary $\sigma = (P_1, \ldots, P_m, ar, \varphi_C)$ we intuitively denote by C the set FM–represented by φ_C and we refer to its elements as to constants from σ. We write $c \in \sigma$ for $c \in C$. Similarly we say that $P_1, \ldots, P_m$ are predicates from σ and write $P_i \in \sigma$ for $i = 1, \ldots, m$. The computational complexity of a concrete vocabulary σ is defined as the computational complexity of the set of constants from σ. Every concrete vocabulary is associated with a vocabulary in the standard sense and therefore σ–terms (Term_σ), σ–formulae (Form_σ) and σ–sentences (Sent_σ) are defined naturally and are recursive in σ.

We proceed with a definition of a concrete structure.

Definition 4.1.2 (Concrete Structure) *Let $\sigma = (P_1, \ldots, P_m, ar, \varphi_C)$ be a concrete vocabulary. A sequence of formulae $\mathcal{F} = (\varphi_U, \varphi_{P_1}, \ldots, \varphi_{P_m}, \varphi_{C,U})$ is a concrete σ–structure if:*

- *φ_U FM–represents a non-empty set U (the universe),*
- *φ_{P_i} FM–represents a relation on U with arity $ar(P_i)$, for $i = 1, \ldots, m$ (the relations),*
- *$\varphi_{C,U}$ FM–represents a function from C to U (the interpretation of constants).*

Whenever a concrete vocabulary σ is clear from the context (or not important) we call concrete σ–structures simply concrete structures. Similarly we call σ–formulae and sentences simply formulae and sentences.

Concrete structures are arithmetical definitions of FM–representable models, understood in the standard sense. If $\varphi_{C,U}^{\mathrm{FM}(\mathcal{N})}(\ulcorner c \urcorner, n)$ holds, we say n is the interpretation of c in $\mathcal{F}$.

The computational complexity of a concrete structure $\mathcal{F} = (\varphi_U, \varphi_{P_1}, \dots, \varphi_{P_m}, \varphi_{C,U})$ is defined as the complexity of the model it FM–represents. We write $\deg(\mathcal{F})$ for the Turing degree of $\varphi_U^{\mathrm{FM}(\mathcal{N})} \oplus \varphi_{P_1}^{\mathrm{FM}(\mathcal{N})} \oplus \dots \oplus \varphi_{P_m}^{\mathrm{FM}(\mathcal{N})} \oplus \varphi_{C,U}^{\mathrm{FM}(\mathcal{N})}$.

Let $V = \{v_0, \dots, v_k\}$ and let $\mathcal{F}$ be a concrete structure with the universe U. Every finite function $\overline{a}\colon V \to U$ is a valuation in $\mathcal{F}$. We naturally identify valuations with finite sequences of elements of U. Therefore, we write a_i for $\overline{a}(v_i)$ and $\mathrm{lh}(\overline{a})$ for $|V|$. Since valuations are finite objects, we encode them with Gödel numbers.

We want to define what it means for a binary relation to be satisfaction relation in a concrete structure. The following definition captures basic syntactic requirements for satisfaction relations.

Definition 4.1.3 (Pre–satisfaction Relation) *Let σ be a concrete vocabulary and let $\mathcal{F}$ be a concrete σ–structure. An arithmetical binary relation R is a pre–satisfaction in $\mathcal{F}$ if R is FM–representable and for every $n, m \in \omega$, if $R(n,m)$ then:*

- *$n = \ulcorner\theta\urcorner$, for some σ–formula θ,*
- *$m = \ulcorner\overline{a}\urcorner$, for some valuation $\overline{a}$ on $\mathcal{F}$,*
- $\max\{i \in \omega \colon v_i \in \mathrm{FV}(\theta)\} < \mathrm{lh}(\overline{a})$.

We naturally transfer the notion of valuation to terms.

Definition 4.1.4 (Valuation on Terms) *Let σ be a concrete vocabulary and let $\mathcal{F} = (\varphi_U, \varphi_{P_1}, \dots, \varphi_{P_m}, \varphi_{C,U})$ be a concrete σ–structure. We define a valuation on terms in $\mathcal{F}$ in the following way. For constants $c \in \sigma$, every valuation $\overline{a}$ in $\mathcal{F}$ and $n \in \omega$, $\mathrm{Val}^{\mathcal{F}}(\ulcorner c\urcorner, \ulcorner\overline{a}\urcorner) = n$ if $\varphi_{C,U}^{\mathrm{FM}(\mathcal{N})}(\ulcorner c\urcorner, n)$. For every variable v_i and valuation $\overline{a}$ such that $\mathrm{lh}(\overline{a}) > i$, $\mathrm{Val}^{\mathcal{F}}(\ulcorner v_i\urcorner, \ulcorner\overline{a}\urcorner) = a_i$.*

For every concrete structure $\mathcal{F}$ the function $\mathrm{Val}^{\mathcal{F}}(\ulcorner t\urcorner, \ulcorner\overline{a}\urcorner)$ is recursive in $\mathcal{F}$.

We are ready to define a satisfaction relation for concrete structures.

Definition 4.1.5 (Satisfaction Relation) *Let $\sigma = (P_1, \dots, P_m, ar, \varphi_C)$ be a concrete vocabulary and $\mathcal{F} = (\varphi_U, \varphi_{P_1}, \dots, \varphi_{P_m}, \varphi_{C,U})$ be a concrete σ–structure. A binary relation $R_{\models}$ is a satisfaction relation in $\mathcal{F}$ if $R_{\models}$ is a pre–satisfaction relation in $\mathcal{F}$ and it satisfies the following Tarski's conditions. For every σ–formulae ψ, ψ_1, ψ_2, every σ–terms $t_1, \dots, t_k$ and every valuation $\overline{a}$ in $\mathcal{F}$ of length greater than the maximal index of a free variable in those formulae and terms, the following holds:*

- $R_{\models}(\ulcorner t_1 = t_2\urcorner, \ulcorner\overline{a}\urcorner)$ *if and only if* $\mathrm{Val}^{\mathcal{F}}(\ulcorner t_1\urcorner, \ulcorner\overline{a}\urcorner) = \mathrm{Val}^{\mathcal{F}}(\ulcorner t_2\urcorner, \ulcorner\overline{a}\urcorner)$,

- *For* $j = 1, \ldots, m$, $R_{\models}(\ulcorner P_j(t_1, t_2, \ldots, t_{ar(P_j)})\urcorner, \ulcorner \overline{a} \urcorner)$ *if and only if* $\varphi_{P_j}^{\mathrm{FM}(\mathcal{N})}(\mathrm{Val}^{\mathcal{F}}(\ulcorner t_1 \urcorner, \ulcorner \overline{a} \urcorner), \ldots, \mathrm{Val}^{\mathcal{F}}(\ulcorner t_{ar(P_j)} \urcorner, \ulcorner \overline{a} \urcorner))$,
- $R_{\models}(\ulcorner \neg\psi \urcorner, \ulcorner \overline{a} \urcorner)$ *if and only if it is not the case that* $R_{\models}(\ulcorner \psi \urcorner, \ulcorner \overline{a} \urcorner)$,
- $R_{\models}(\ulcorner \psi_1 \wedge \psi_2 \urcorner, \ulcorner \overline{a} \urcorner)$ *if and only if both* $R_{\models}(\ulcorner \psi_1 \urcorner, \ulcorner \overline{a} \urcorner)$ *and* $R_{\models}(\ulcorner \psi_2 \urcorner, \ulcorner \overline{a} \urcorner)$,
- $R_{\models}(\ulcorner \exists v_k\, \psi \urcorner, \ulcorner \overline{a} \urcorner)$ *if and only if there is* $n \in \varphi_U^{\mathrm{FM}(\mathcal{N})}$ *such that* $R_{\models}(\ulcorner \psi \urcorner, \ulcorner \overline{a}[v_k := n] \urcorner)$, *where*

$$\overline{a}[v_k := n]_i = \begin{cases} a_i & \text{if } i < \mathrm{lh}(\overline{a}) \text{ and } i \neq k \\ n & \text{if } i = k \\ \mu\varphi_U^{\mathrm{FM}(\mathcal{N})} & \text{if } i \geqslant \mathrm{lh}(\overline{a}) \text{ and } i < k \end{cases}$$

We are ready to define concrete models.

Definition 4.1.6 (Concrete Model) *Let σ be a concrete vocabulary. A pair $(\mathcal{F}, \varphi_{\models})$ is a concrete σ–model if:*

- $\mathcal{F}$ *is a concrete σ–structure,*
- $\varphi_{\models}$ *FM–represents the satisfaction relation in $\mathcal{F}$.*

For a concrete model $\mathcal{A} = (\mathcal{F}, \varphi_{\models})$, the Turing degree $\deg(\mathcal{A})$ of $\mathcal{A}$ is defined as the Turing degree of $\mathcal{F} \oplus \varphi_{\models}^{\mathrm{FM}(\mathcal{N})}$. Note that since the structure $\mathcal{F}$ can be computed from the satisfaction relation in $\mathcal{F}$, the computational complexity of a concrete model can be equivalently defined as the computational complexity of its satisfaction relation. We therefore consider notions recursive in concrete models and halting problems for concrete models in an obvious meaning.

Now let us transfer the basic notions of model theory to concrete models framework. We want to use standard symbols known from the axiomatic model theory for their concrete analogues. Therefore, we use symbols $\mathcal{A}, \mathcal{B}$ etc. for concrete models and $\mathcal{F}, \mathcal{G}$ etc. for concrete structures.

We use the standard notation $|\mathcal{A}|$ for the universe of a concrete model $\mathcal{A} = (\mathcal{F}, \varphi_{\models})$ i.e. the universe of $\mathcal{F}$ (also abbreviated as $|\mathcal{F}|$). A concrete model is said to be finite if the set $|\mathcal{A}|$ is finite and it is said to be infinite otherwise. Similarly we write $P_j^{\mathcal{A}}$ (or $P_j^{\mathcal{F}}$) to denote the relation $\varphi_{P_j}^{\mathrm{FM}(\mathcal{N})}$ and $c^{\mathcal{A}}$ (or $c^{\mathcal{F}}$) to denote the interpretation of c in $\mathcal{A}$ (or in $\mathcal{F}$). For a concrete model $\mathcal{A} = (\mathcal{F}, \varphi_{\models})$, a formula ψ and a valuation $\overline{a}$ on $\mathcal{A}$ we write $\mathcal{A} \models \psi[\overline{a}]$ if $\varphi_{\models}^{\mathrm{FM}(\mathcal{N})}(\ulcorner \psi \urcorner, \ulcorner \overline{a} \urcorner)$ and we say that $\overline{a}$ satisfies ψ in $\mathcal{A}$. We also write $\mathcal{A} = (\varphi_U, \varphi_{P_1}, \ldots, \varphi_{P_m}, \varphi_{C,U}, \varphi_{\models})$ to abbreviate that $(\varphi_U, \varphi_{P_1}, \ldots, \varphi_{P_m}, \varphi_{C,U})$ is a concrete structure $\mathcal{F}$ and $\mathcal{A} = (\mathcal{F}, \varphi_{\models})$ is a concrete model.

We say that a sentence ψ is true in a concrete model $\mathcal{A}$, in symbols $\mathcal{A} \models \psi$, if $\mathcal{A} \models \psi[\overline{a}]$ for some sequence $\overline{a}$, which is equivalent to $\mathcal{A} \models \psi[\varepsilon]$.

We use standard abbreviations $\mathcal{A} \models T$ to say that for every sentence $\psi \in T$, $\mathcal{A} \models \psi$ i.e. when $\mathcal{A}$ is a concrete model of theory T. By $\mathrm{Th}(\mathcal{A})$ we denote the theory of $\mathcal{A}$ i.e. the set of sentences true in $\mathcal{A}$. Note that $\mathrm{Th}(\mathcal{A})$ is recursive in $\mathcal{A}$, consistent and complete in the vocabulary of $\mathcal{A}$.

In our considerations we also need notions concerning concrete mappings between structures and models.

Definition 4.1.7 (Concrete Morphisms) *Let $\mathcal{F}$ and $\mathcal{G}$ be concrete structures of the same vocabulary σ and let R be a concrete binary relation. Restriction $h = R \restriction_{|\mathcal{F}| \times |\mathcal{G}|}$ of R to the cartesian product of domains of concrete structures $\mathcal{F}$ and $\mathcal{G}$ is a concrete homomorphism if:*

- *h is a function from $|\mathcal{F}|$ into $|\mathcal{G}|$,*
- *for every predicate $P \in \sigma$ and elements $a_1, \ldots, a_{ar(P)} \in |\mathcal{F}|$ if $P^{\mathcal{F}}(a_1, \ldots, a_{ar(P)})$, then $P^{\mathcal{G}}(h(a_1), \ldots, h(a_{ar(P)}))$,*
- *for every constant $c \in \sigma$ it holds that $h(c^{\mathcal{F}}) = c^{\mathcal{G}}$.*

We say that h is a concrete embedding of $\mathcal{F}$ into $\mathcal{G}$ if h is an injective homomorphism and for every predicate $P \in \sigma$ and elements $a_1, \ldots, a_{ar(P)} \in |\mathcal{F}|$, $P^{\mathcal{F}}(a_1, \ldots, a_{ar(P)})$ if and only if $P^{\mathcal{G}}(h(a_1), \ldots, h(a_{ar(P)}))$.

We say that h is onto $\mathcal{G}$ if h is surjective.

We say that h is an isomorphism if h is an embedding and onto $\mathcal{G}$.

We naturally transfer notions of concrete morphisms from concrete structures to concrete models. Let $\mathcal{A} = (\mathcal{F}, \varphi_{\models})$ and $\mathcal{B} = (\mathcal{G}, \psi_{\models})$ be concrete models and let h be a concrete morphism from $\mathcal{F}$ to $\mathcal{G}$. Then we say that h is a concrete morphism from $\mathcal{A}$ to $\mathcal{B}$.

The notion of elementary embedding requires concerning satisfaction relations in concrete models, therefore we define it for concrete models (not concrete structures).

Definition 4.1.8 (Concrete Elementary Embedding) *Let $\mathcal{A}$, $\mathcal{B}$ be concrete models and let h be a concrete morphism from $\mathcal{A}$ to $\mathcal{B}$. We say that h is a concrete elementary embedding of $\mathcal{A}$ into $\mathcal{B}$ if h is a concrete embedding and for all formulae ψ and for all elements $a_1, \ldots, a_k \in |\mathcal{A}|$ it holds that $\mathcal{A} \models \psi[a_1, \ldots, a_k]$ if and only if $\mathcal{B} \models \psi[h(a_1), \ldots, h(a_k)]$.*

Lemma 4.1.9 (On Concrete Isomorphisms) *Let $\mathcal{F}, \mathcal{G}$ be concrete structures. Let $\mathcal{B} = (\mathcal{G}, \varphi_{\models})$. Let $f \colon \mathcal{F} \to \mathcal{G}$ be a concrete isomorphism. The relation:*

$$R_{\models}(\ulcorner \varphi \urcorner, \ulcorner a_0, \ldots, a_k \urcorner) \textit{ if and only if}$$
$$\bigwedge_{i=0,\ldots,k} a_i \in |\mathcal{F}| \wedge \mathcal{B} \models \varphi[f(a_0), \ldots, f(a_k)].$$

is the satisfaction relation in $\mathcal{F}$ and is recursive in $|\mathcal{F}| \oplus f \oplus \mathcal{B}$.

Proof: Let $\mathcal{F}, \mathcal{G}$, $\varphi_{\models}$, $\mathcal{B}$, f and $R_{\models}$ be as in the assumptions of the lemma. For $\overline{a} = a_0, \ldots, a_k$, by $f(\overline{a})$ we denote $f(a_0), \ldots, f(a_k)$.

The relation $R_{\models}$ is recursive in $|\mathcal{F}| \oplus f \oplus \varphi_{\models}^{\mathrm{FM}(\mathcal{N})}$ i.e. in $|\mathcal{F}| \oplus f \oplus \mathcal{B}$. We show that it is the satisfaction relation in $\mathcal{F}$. The relation $R_{\models}$ is obviously a pre–satisfaction relation in $\mathcal{F}$.

We show that $R_{\models}$ satisfies Tarski's conditions. The cases for atomic formulae are satisfied since f is an embedding from $\mathcal{F}$ into $\mathcal{G}$. Let $\overline{a}$ be a valuation in $\mathcal{F}$ and let formulae φ, ψ be such that $\max\{i \colon v_i \in \mathrm{FV}(\varphi)\} < \mathrm{lh}(\overline{a})$ and $\max\{i \colon v_i \in \mathrm{FV}(\psi)\} < \mathrm{lh}(\overline{a})$.

In the case of negation the following are equivalent:

- $R_{\models}(\ulcorner \neg\varphi \urcorner, \ulcorner \overline{a} \urcorner)$,
- $\mathcal{B} \models \neg\varphi[f(\overline{a})]$,
- $\mathcal{B} \not\models \varphi[f(\overline{a})]$,
- it is not the case that $R_{\models}(\ulcorner \varphi \urcorner, \ulcorner \overline{a} \urcorner)$.

In the case of conjunction the following are equivalent:

- $R_{\models}(\ulcorner \varphi \wedge \psi \urcorner, \ulcorner \overline{a} \urcorner)$,
- $\mathcal{B} \models (\varphi \wedge \psi)[f(\overline{a})]$,
- $\mathcal{B} \models \varphi[f(\overline{a})]$ and $\mathcal{B} \models \psi[f(\overline{a})]$,
- $R_{\models}(\ulcorner \varphi \urcorner, \ulcorner \overline{a} \urcorner)$ and $R_{\models}(\ulcorner \psi \urcorner, \ulcorner \overline{a} \urcorner)$.

In the case of the existential quantifier the following are equivalent:

- $R_{\models}(\ulcorner \exists v_k\, \varphi \urcorner, \ulcorner \overline{a} \urcorner)$,
- $\mathcal{B} \models \exists v_k\, \varphi[f(\overline{a})]$,
- there is $n \in |\mathcal{B}|$ such that $\mathcal{B} \models \varphi[f(\overline{a})[v_k := n]]$,
- there is $m \in |\mathcal{F}|$ such that $f(m) = n$ and $\mathcal{B} \models \varphi[f(\overline{a}[v_k := m])]$,
- there is $m \in |\mathcal{F}|$ such that $R_{\models}(\ulcorner \varphi \urcorner, \ulcorner \overline{a}[v_k := m] \urcorner)$.

Therefore $R_{\models}$ is the satisfaction relation in $\mathcal{F}$. $\square$

Lemma 4.1.9 can be strengthen to to the following form.

Lemma 4.1.10 (Co–image of a Concrete Bijection)
Let A be a concrete set and let $\mathcal{B}$ be a concrete σ–model. Let $f \colon A \to |\mathcal{B}|$ be

a concrete bijection. Then there is a concrete σ–model $\mathcal{A}$ such that $A = |\mathcal{A}|$ and it holds that

$$\mathcal{A} \models \psi[a_1, \ldots, a_k] \text{ if and only if } \bigwedge_{i=1,\ldots,k} a_i \in A \text{ and } \mathcal{B} \models \psi[f(a_1), \ldots, f(a_k)].$$

Moreover, f is a concrete isomorphism from $\mathcal{A}$ to $\mathcal{B}$ and $\mathcal{A}$ is recursive in $A \oplus f \oplus \mathcal{B}$.

The concrete model $\mathcal{A}$ from the above lemma is called a co–image of a concrete model $\mathcal{B}$ under the concrete bijection f. The proof of its existence is similar to the proof of Lemma 4.1.9. It additionally requires only to compute the relations and the interpretation of constants in $\mathcal{A}$. This is easily computable in the satisfaction relation.

Let us note that up to a concrete isomorphism all infinite concrete models can be thought of as models on natural numbers i.e. their universes are exactly natural numbers and all finite models can be thought of as models on some initial segment of natural numbers. However, since we are particularly interested in model–theoretic constructions, in order to get a new concrete model from a given one, we often need some *space* for adding new elements. This, however, always can be achieved via some very simple recursive functions, e.g. $i \mapsto 2i$ or $i \mapsto p_j^{1+i}$ which induce concrete isomorphisms.

The expansion and reduction of models are very natural in axiomatic model theory and are used frequently in model–theoretic constructions.

Definition 4.1.11 (Concrete Vocabulary Restriction and Extension)

Let $\sigma = (P_1, \ldots, P_m, ar, \varphi_C)$ be a concrete vocabulary and let φ_D FM–represent a set of constants $D \subseteq C$.

Then a concrete vocabulary $\sigma' = (P_{i_1}, \ldots, P_{i_k}, ar', \varphi_D)$ is called a restriction of the concrete vocabulary σ if $\{P_{i_1}, \ldots, P_{i_k}\} \subseteq \{P_1, \ldots, P_m\}$ and ar' is a restriction of ar to the set $\{P_{i_1}, \ldots, P_{i_k}\}$.

If a concrete vocabulary σ' is an restriction of σ, then σ is called an extension of σ'.

Definition 4.1.12 (Reduction and Expansion) *Let σ be a concrete vocabulary, let σ' be a restriction of σ and let $\mathcal{F} = (\varphi_U, \varphi_{P_1}, \ldots, \varphi_{P_m}, \varphi_{C,U})$ be a concrete σ–structure. Let $\mathcal{G} = (\varphi_U, \varphi_{P_{i_1}}, \ldots, \varphi_{P_{i_k}}, \varphi_{D,U})$ be a concrete σ'–structure.*

We say that $\mathcal{G}$ is a reduction (to σ') of $\mathcal{F}$ if $\varphi_{D,U}^{\mathrm{FM}(\mathcal{N})}$ is a restriction of the function $\varphi_{C,U}^{\mathrm{FM}(\mathcal{N})}$.

If $\mathcal{A} = (\mathcal{F}, \varphi_{\models})$ is a concrete σ–model and $\mathcal{G}$ is a reduction (to σ') of $\mathcal{F}$, then the concrete σ'–model $\mathcal{B} = (\mathcal{G}, \varphi'_{\models})$ is called a reduction (to σ') of $\mathcal{A}$.

If $\mathcal{G}$ is a reduction (to σ') of $\mathcal{F}$, then $\mathcal{F}$ is called an expansion (to σ) of $\mathcal{G}$.

If $\mathcal{B}$ is a reduction (to σ') of $\mathcal{A}$, then $\mathcal{A}$ is called an expansion (to σ) of $\mathcal{B}$.

Let $\sigma = (P_1, \ldots, P_m, ar, \varphi_C)$ be a concrete vocabulary and let $\sigma' = (P_{i_1}, \ldots, P_{i_k}, ar', \varphi_D)$ be a restriction of σ. It is easy to obtain a reduction to σ' of a given concrete σ–model. Let $\mathcal{A} = (\varphi_U, \varphi_{P_1}, \ldots, \varphi_{P_m}, \varphi_{C,U}, \varphi_{\models})$ be a concrete σ–model. Let $\varphi_{D,U}(x, y) =_{df} (\varphi_{C,U}(x, y) \wedge \varphi_D(x))$ and let $\varphi'_{\models}(x, y) =_{df} (\varphi_{\models}(x, y) \wedge \mathrm{Form}_{\sigma'}(x))$. Then $\mathcal{B} = (\varphi_U, \varphi_{P_{i_1}}, \ldots, \varphi_{P_{i_k}}, \varphi_{D,U}, \varphi'_{\models})$ is a concrete σ'–model which is a reduction of $\mathcal{A}$ to σ'. It is, however, not obvious that an expansion of a concrete model is also a concrete model – the key difficulty is to define the satisfaction relation in the extended language. The following lemma shows, that if we want to obtain an expansion by constants only, it is possible to express it using the satisfaction relation from the original concrete model and an interpretation of the new constants.

Lemma 4.1.13 (On Expansions) *Let $\sigma' = (P_1, \ldots, P_m, ar, \varphi_D)$ be a concrete vocabulary and let $\sigma = (P_1, \ldots, P_m, ar, \varphi_C)$ be an extension of σ'. Let $\mathcal{G} = (\varphi_U, \varphi_{P_1}, \ldots, \varphi_{P_m}, \varphi_{D,U})$ be a concrete σ'–structure and let $\mathcal{B} = (\mathcal{G}, \varphi_{\models})$ be a concrete σ'–model. Let $\mathcal{F} = (\varphi_U, \varphi_{P_1}, \ldots, \varphi_{P_m}, \varphi_{C,U})$ be an expansion of $\mathcal{G}$ to σ.*

Then the satisfaction relation in $\mathcal{F}$ is recursive in $\varphi_{C,U}^{\mathrm{FM}(\mathcal{N})} \oplus \varphi_{\models}^{\mathrm{FM}(\mathcal{N})}$ and therefore there is an expansion $\mathcal{A} = (\mathcal{F}, \varphi'_{\models})$ of $\mathcal{B}$ to σ.

Proof: Let σ, σ', $\mathcal{F}$, $\mathcal{B}$ and $\mathcal{G}$ be as in the assumptions of the lemma.

Observe that the satisfaction relation $R_{\models}$ in $\mathcal{F}$ can be defined as follows:

$$R_{\models}(\ulcorner\psi(c_0, \ldots, c_k)\urcorner, \ulcorner\overline{a}\urcorner) \equiv \exists b\, (\mathrm{Seq}(b) \wedge \bigwedge_{i=0}^{k} \varphi_{C,U}^{\mathrm{FM}(\mathcal{N})}(\ulcorner c_i\urcorner, b_i) \wedge \varphi_{\models}^{\mathrm{FM}(\mathcal{N})}(\ulcorner\psi(v_{\mathrm{lh}(\overline{a})}, \ldots, v_{\mathrm{lh}(\overline{a})+k})\urcorner, \ulcorner\overline{a}, b_0, \ldots, b_k\urcorner)).$$

Similarly $R_{\models}$ can be defined also as follows:

$$R_{\models}(\ulcorner\psi(c_0, \ldots, c_k)\urcorner, \ulcorner\overline{a}\urcorner) \equiv \forall b\, ((\mathrm{Seq}(b) \wedge \bigwedge_{i=0}^{k} \varphi_{C,U}^{\mathrm{FM}(\mathcal{N})}(\ulcorner c_i\urcorner, b_i)) \Rightarrow \varphi_{\models}^{\mathrm{FM}(\mathcal{N})}(\psi(v_{\mathrm{lh}(\overline{a})}, \ldots, v_{\mathrm{lh}(\overline{a})+k}), \ulcorner\overline{a}, b_0, \ldots, b_k\urcorner)).$$

Since $\varphi_{\models}^{\mathrm{FM}(\mathcal{N})}$ and $\varphi_{C,U}^{\mathrm{FM}(\mathcal{N})}$ are FM–representable, they can be defined by Σ_2 and Π_2 formulae. It follows that $R_{\models}$ can be defined by a Σ_2 formula, as well as by a Π_2 formula. Therefore by the generalised Post's theorem $R_{\models}$ is Δ^0_2 i.e. concrete and more precisely it is recursive in $\varphi_{C,U}^{\mathrm{FM}(\mathcal{N})} \oplus \varphi_{\models}^{\mathrm{FM}(\mathcal{N})}$. Let $\varphi'_{\models}$ FM–represent $R_{\models}$. Then $\mathcal{A} = (\mathcal{F}, \varphi'_{\models})$ is a concrete σ–model which is an expansion of $\mathcal{B}$ to σ. $\square$

In the axiomatic model theory we often consider expansions of a model by constants naming some elements of some subset of the universe. We show how this is done in the concrete context. Let $\sigma = (P_1, \dots, P_m, ar, \varphi_C)$ be concrete vocabulary, and let $\mathcal{F} = (\varphi_U, \varphi_{P_1}, \dots, \varphi_{P_m}, \varphi_{C,U})$ be a concrete σ–structure. Let D be a recursive set of new constants with an effective presentation $d_0, d_1, \dots$. Let $A \subseteq |\mathcal{F}|$ be a concrete set, let $E = \{d_a \colon a \in A\}$ and let φ_E FM–represent E. Then $\sigma' = (P_1, \dots, P_m, ar, \varphi_C \vee \varphi_E)$ is an extension of σ and we write $\sigma' = \sigma \cup E$. The function $I_{E,A} \colon E \to A$ such that for each $a \in A$ it holds that $I_{E,A}(\ulcorner d_a \urcorner) = a$ is recursive in A. Let $\varphi_{E,A}$ FM–represent $I_{E,A}$. Then $\mathcal{F}' = (\varphi_U, \varphi_{P_1}, \dots, \varphi_{P_m}, \varphi_{C,U} \vee \varphi_{E.A})$ is a concrete σ'–structure. Using classical model–theoretic nomenclature we write $\mathcal{F}' = (\mathcal{F}, a)_{a \in A}$. Similarly we write $(\mathcal{A}, a)_{a \in A}$ to denote a concrete σ'–model $(\mathcal{F}', \varphi_{\models})$. If $\mathcal{A}$ is a concrete model we say that $(\mathcal{A}, a)_{a \in |\mathcal{A}|}$ is a natural expansion of $\mathcal{A}$.

For concrete vocabularies σ, τ we also use standard notation $\sigma \cup \tau$, $\sigma \cap \tau$ and $\sigma \subseteq \tau$ in the intuitive meaning.

Being able to expand concrete models to new concrete vocabularies we can define such basic concepts as a diagram $\mathrm{Diag}(\mathcal{A})$ or an elementary diagram $\mathrm{ElDiag}(\mathcal{A})$ of a concrete model $\mathcal{A}$. Recall that by Lit_σ we denote the set of literals in vocabulary σ.

Definition 4.1.14 ((Elementary) Diagram of a Concrete Model)
Let $\mathcal{A}$ be a concrete σ–model and let $\mathcal{A}' = (\mathcal{A}, a)_{a \in |\mathcal{A}|}$ be its natural expansion to σ'.

The elementary diagram ElDiag($\mathcal{A}$) of $\mathcal{A}$ is defined as $\mathrm{Th}(\mathcal{A}')$.

The diagram Diag($\mathcal{A}$) of $\mathcal{A}$ is defined as $\mathrm{Lit}_{\sigma'} \cap$ *ElDiag($\mathcal{A}$).*

Note that the diagram and the elementary diagram of a concrete model is relative to its natural expansion which depends on the underlying set of new constants naming the elements of the concrete model. Observe also that the diagram and the elementary diagram of a concrete model $\mathcal{A}$ are recursive in $\mathcal{A}$ and are consistent. Moreover elementary diagrams are complete and have the witness property.

Now we transfer basic relations between models in the context of the concrete models.

Definition 4.1.15 (Concrete Submodel)
Let $\mathcal{A} = (\varphi_U, \varphi_{P_1}, \dots, \varphi_{P_m}, \varphi_{C,U}, \varphi_{\models})$ and $\mathcal{B} = (\psi_V, \psi_{P_1}, \dots, \psi_{P_m}, \psi_{C,V}, \psi_{\models})$ be concrete σ–models. We say that $\mathcal{A}$ is a submodel of $\mathcal{B}$, in symbols $\mathcal{A} \subseteq \mathcal{B}$, when:

- $|\mathcal{A}| \subseteq |\mathcal{B}|$,
- *for $i = 1, \dots m$, $P_i^{\mathcal{A}} = P_i^{\mathcal{B}} \cap |\mathcal{A}|^{ar(P_i)}$,*

- *$\varphi_{C,U}$ and $\psi_{C,V}$ FM–represent the same function.*

When $\mathcal{A} \subseteq \mathcal{B}$ we also say that model $\mathcal{B}$ is a concrete extension of a concrete model $\mathcal{A}$. Note that if $\mathcal{A} \subseteq \mathcal{B}$, then the identity function restricted to $|\mathcal{A}|$ is an embedding from $\mathcal{A}$ into $\mathcal{B}$.

Definition 4.1.16 (Concrete Elementary Submodel)
Let $\mathcal{A} = (\varphi_U, \varphi_{P_1}, \ldots, \varphi_{P_m}, \varphi_{C,U}, \varphi_{\models})$ and $\mathcal{B} = (\psi_V, \psi_{P_1}, \ldots, \psi_{P_m}, \psi_{C,V}, \psi_{\models})$ be concrete models. We say that $\mathcal{A}$ is an concrete elementary submodel of $\mathcal{B}$, in symbols $\mathcal{A} \preccurlyeq \mathcal{B}$, if:

- *$\mathcal{A} \subseteq \mathcal{B}$,*
- *for every formula $\vartheta(\overline{x})$ and valuation $\overline{a}$ in $|\mathcal{A}|$ if $\mathcal{A} \models \vartheta[\overline{a}]$, then $\mathcal{B} \models \vartheta[\overline{a}]$.*

When $\mathcal{A} \preccurlyeq \mathcal{B}$ we also say that model $\mathcal{B}$ is a concrete elementary extension of model $\mathcal{A}$. Note that if $\mathcal{A} \preccurlyeq \mathcal{B}$, then the identity function restricted to $|\mathcal{A}|$ is an elementary embedding from $\mathcal{A}$ into $\mathcal{B}$.

In the axiomatic model theory it is a basic fact that if for models $\mathcal{A}'$, $\mathcal{B}$ we have $\mathcal{A}' \models \mathrm{Diag}(\mathcal{B})$, then there is a *canonical* embedding f from $\mathcal{B}$ into the reduction $\mathcal{A}$ of $\mathcal{A}'$ to the common vocabulary and a *canonical* embedding f^* from $(\mathcal{B}, b)_{b \in |\mathcal{B}|}$ into $\mathcal{A}'$. The following lemma shows that those *canonical* embeddings are recursive in $\mathcal{A} \oplus \mathcal{B}$, therefore concrete.

Lemma 4.1.17 (On Embeddings) *Let σ be a concrete vocabulary and let $\mathcal{B}$ be a concrete σ–model. Let $\mathcal{B}' = (\mathcal{B}, b)_{b \in |\mathcal{B}|}$ be a natural expansion of $\mathcal{B}$ and let σ' be the concrete vocabulary of $\mathcal{B}'$. Let $\mathcal{A}'$ be a concrete σ'–model such that $\mathcal{A}' \models Diag(\mathcal{B})$ and let $\mathcal{A}$ be a reduction of $\mathcal{A}'$ to σ.*

Then there is a concrete embedding f of $\mathcal{B}'$ into $\mathcal{A}'$ and therefore also f is the concrete embedding of $\mathcal{B}$ into $\mathcal{A}$. Moreover f is recursive in $\mathcal{A} \oplus \mathcal{B}$.

Proof: Let σ, σ', $\mathcal{A}$, $\mathcal{A}'$, $\mathcal{B}$, $\mathcal{B}'$ be as in the assumptions of the lemma. Therefore $\sigma = (P_1, \ldots, P_m, ar, \varphi_C)$, and $\sigma' = (P_1, \ldots, P_m, ar, \varphi_C \vee \varphi_D)$, where $D = \{d_b \colon b \in |\mathcal{B}|\}$.

Let $I_{\mathcal{B}'}$ be the interpretation of constants in $\mathcal{B}'$. Then by the definition of the natural expansion $I_{\mathcal{B}'}(\ulcorner d_b \urcorner) = b$ for all $b \in |\mathcal{B}|$. Let $I_{\mathcal{A}'}$ be the interpretation of constants in $\mathcal{A}'$.

We define:

$$f(b) =_{df} I_{\mathcal{A}'}(\ulcorner d_b \urcorner).$$

The function $f \colon |\mathcal{B}'| \to |\mathcal{A}'|$ for an element $b \in |\mathcal{B}'|$ returns the interpretation of constant d_b in $\mathcal{A}'$. Since $\mathcal{A}' \models \mathrm{Diag}(B)$, the function f is a concrete homomorphism and is injective, since $I_{\mathcal{B}'}$ is injective.

It remains to show that for every $P \in \sigma$ and every elements $b_1, \ldots, b_{ar(P)} \in |\mathcal{B}'|$ if $\mathcal{A}' \models P[f(b_1), \ldots, f(b_{ar(P)})]$, then $\mathcal{B} \models P(b_1, \ldots, b_{ar(P)})$. Fix $P \in \sigma$

and $b_1, \dots, b_{ar(P)}$ and suppose that $\mathcal{A}' \models P[f(b_1), \dots, f(b_{ar(P)})]$. Then $\mathcal{A}' \models P(d_{b_1}, \dots, d_{b_{ar(P)}})$ and therefore, since $\mathcal{A}' \models \mathrm{Diag}(B)$, $\mathcal{B}' \models P(d_{b_1}, \dots, d_{b_{ar(P)}})$. Thus $\mathcal{B}' \models P[b_1, \dots, b_{ar(P)}]$.

Moreover, f is recursive in $\mathcal{A}' \oplus \mathcal{B}'$ and therefore in $\mathcal{A} \oplus \mathcal{B}$. This ends the proof that f is a concrete embedding of $\mathcal{B}'$ into $\mathcal{A}'$ and therefore that it is also a concrete embedding of $\mathcal{B}$ into $\mathcal{A}$. $\square$

We therefore can construct a concrete canonical embedding of $(\mathcal{B}, b)_{b \in |\mathcal{B}|}$ into $\mathcal{A}'$ knowing that $\mathcal{A}' \models \mathrm{Diag}(\mathcal{B})$. However, in the model–theoretic constructions of the axiomatic model theory the fact that $\mathcal{A}' \models \mathrm{Diag}(\mathcal{B})$ and that $(\mathcal{B}, b)_{b \in |\mathcal{B}|}$ is a submodel of $\mathcal{A}'$ are treated as equivalent, since up to isomorphism they are equivalent. If $f\colon (\mathcal{B}, b)_{b \in |\mathcal{B}|} \to \mathcal{A}'$ is an embedding, then the image $f[(\mathcal{B}, b)_{b \in |\mathcal{B}|}]$ of $(\mathcal{B}, b)_{b \in |\mathcal{B}|}$ under f is isomorphic to $(\mathcal{B}, b)_{b \in |\mathcal{B}|}$ and $f[(\mathcal{B}, b)_{b \in |\mathcal{B}|}] \subseteq \mathcal{A}'$. Or, the other way round, $(\mathcal{B}, b)_{b \in |\mathcal{B}|} \subseteq \mathcal{C}$, where $\mathcal{C}$ is the model obtained by *replacing* the submodel $f[(\mathcal{B}, b)_{b \in |\mathcal{B}|}]$ of $\mathcal{A}'$ by $(\mathcal{B}, b)_{b \in |\mathcal{B}|}$. Those are two canonical ways to justify inferring $(\mathcal{B}, b)_{b \in |\mathcal{B}|} \subseteq \mathcal{A}'$ from $\mathcal{A}' \models \mathrm{Diag}(\mathcal{B})$. As we show they both fail in the context of concrete models. The reason is that the image of a function is defined as $im(f) = \{j \colon \exists i\, f(i) = j\}$, therefore it requires an unbounded quantification which raises the computational complexity. In the worst case we may have that $\deg(im(f)) = \deg(f)'$ which could be a non–concrete set. The following theorem provides an example of such a situation – the image of the universe of a concrete model under a concrete injective function is Σ^0_2–complete.

Theorem 4.1.18 *There are concrete sets A, B and injective concrete function $f\colon B \to A$ such that $f[B]$ is Σ^0_2–complete and $(A - f[B]) \cup B$ is Π^0_2–complete.*

Proof: The proof bases on the fact that every infinite recursively enumerable set is an image of an injective total recursive function. This fact can be relativised to infinite Σ^0_2 sets and functions recursive in the halting problem K. Therefore a Σ^0_2–complete set (e.g. $K^* = \{i \in \omega \colon \Phi^K_i(i)\downarrow\}$) is an image of an injective total concrete function.

Let $B = 2K + 1 = \{2k + 1 \colon k \in K\}$, let $A = \{2i \colon i \in \omega\}$ and let $C = 2K^* = \{2k^* \colon k^* \in K^*\}$. There is a recursive in K (thus concrete) injective function $f\colon B \to A$ such that $f[B] = C$.

Both sets A, B are concrete and the injection f is also concrete. The set $C = f[B]$ is Σ^0_2–complete. Since $f[B] \subseteq A$ and $A \cap B$ are disjoint $(A - f[B]) \cup B$ is Π^0_2–complete. $\square$

By Theorem 4.1.18, in the concrete models context we cannot perform model–theoretic constructions as freely as in the axiomatic model theory. In particular, the constructions of chains of models may fail since we cannot infer $(\mathcal{B}, b)_{b \in |\mathcal{B}|} \subseteq \mathcal{A}'$ from $\mathcal{A}' \models \mathrm{Diag}(\mathcal{B})$.

We define what we mean by sequences and chains of models in the concrete case.

Definition 4.1.19 (Concrete Sequence of Concrete Models)
A formula $\varphi_{Mod}(x, y)$ is a concrete sequence of concrete models if $\varphi_{Mod}(x, y)$ FM–represents a function Mod such that for all $i \in \omega$, $Mod(i)$ is Gödel number of a concrete model.

We use the standard notation for concrete sequences of concrete models. By $(\mathcal{A}_i)_{i\in\omega}$ we abbreviate that $\varphi_{Mod}(x, y)$ is a concrete sequence of concrete models such that for each $i \in \omega$, $Mod(i) = \ulcorner \mathcal{A}_i \urcorner$.

We can require a concrete sequence of low concrete models to satisfy certain additional condition – that the halting problem for i–th concrete model in the sequence is concretely computable from i. This motivates the following definition.

Definition 4.1.20 (Jump–Concrete Sequence of Concrete Models)
Let $(\mathcal{A}_i)_{i\in\omega}$ be a concrete sequence of low concrete models. For $i \in \omega$ let φ_i FM–represent the halting problem for $\mathcal{A}_i$. We say that $(\mathcal{A}_i)_{i\in\omega}$ is a jump–concrete sequence of concrete models if the map $i \mapsto \ulcorner \varphi_i \urcorner$ is concrete.

Definition 4.1.21 (Concrete Chain of Concrete Models)
Let $(\mathcal{A}_i)_{i\in\omega}$ be a concrete sequence of concrete models. We say that $(\mathcal{A}_i)_{i\in\omega}$ is a concrete chain of concrete models if:

- *for $i \in \omega$, $\mathcal{A}_i \subseteq \mathcal{A}_{i+1}$,*
- *$\bigcup_{i\in\omega} |\mathcal{A}_i|$ is a concrete set.*

We say that a concrete chain of concrete models $(\mathcal{A}_i)_{i\in\omega}$ is a concrete elementary chain of concrete models if for each $i \in \omega$ it holds that $\mathcal{A}_i \preccurlyeq \mathcal{A}_{i+1}$.

In the model–theoretic constructions presented further in this chapter we define concrete sequences of concrete models $(\mathcal{A}_i)_{i\in\omega}$ such that for $i \in \omega$ it holds that $\mathcal{A}_{i+1} \models \mathrm{Diag}(\mathcal{A}_i)$. The following theorem is essential for us to perform constructions of concrete chains of concrete models. It states under what circumstances it is possible for a concrete sequence of concrete models $(\mathcal{A}_i)_{i\in\omega}$ to find a concrete chain of concrete models $(\mathcal{B}_i)_{i\in\omega}$ such that for each $i \in \omega$ there is a concrete isomorphism from $\mathcal{B}_i$ into a reduction $\mathcal{A}'_i$ of $\mathcal{A}_i$.

Theorem 4.1.22 (On Concrete Chain Constructions) *Let $\mathcal{A}_0$ be a low concrete σ–model. Let $(\mathcal{A}_i)_{i\in\omega}$ be a jump–concrete sequence of low concrete models. Suppose that for every $i \in \omega$ it holds that:*

- *$\bigoplus_{j\leqslant i} \mathcal{A}_j$ is low,*
- *$\mathcal{A}_{i+1} \models Diag(\mathcal{A}_i)$.*

For $i \in \omega$, let R_i be recursive pairwise disjoint sets such that $i \mapsto \ulcorner R_i \urcorner$ is recursive such that $|\mathcal{A}_i| \subseteq R_i$.

Then there is a concrete chain of concrete σ–models $(\mathcal{B}_i)_{i\in\omega}$ such that for $i \in \omega$ there is a concrete isomorphism $g_i \colon \mathcal{B}_i \to \mathcal{A}_i^\sigma$, where $\mathcal{A}_i^\sigma$ is the reduction to σ of $\mathcal{A}_i$.

Proof: Let $(\mathcal{A}_i)_{i\in\omega}$ and R_i, for $i \in \omega$ be as in the assumptions of the theorem.

Observe that for $i \neq j$ it holds that $|\mathcal{A}_i| \cap |\mathcal{A}_j| = \varnothing$. This follows from the fact that for $i \neq j$, R_i and R_j are disjoint, $|\mathcal{A}_i| \subseteq R_i$ and $|\mathcal{A}_j| \subseteq R_j$.

For $i \in \omega$ let $f_i \colon (\mathcal{A}_i, a)_{a\in|\mathcal{A}_i|} \to \mathcal{A}_{i+1}$ be an embedding provided by Lemma 4.1.17. Note that for each $i \in \omega$ the function f_i is recursive in $\mathcal{A}_i \oplus \mathcal{A}_{i+1}$, thus low. Moreover, from the proof of Lemma 4.1.17, the codes of algorithms computing these embeddings in K are computable from the codes of the algorithms computing $\mathcal{A}_i$ and $\mathcal{A}_{i+1}$ in K. Note also that, since $(\mathcal{A}_i)_{i\in\omega}$ is a jump–concrete sequence of low concrete models, $i \mapsto \ulcorner \mathcal{A}_i^* \urcorner$ is concrete and therefore $i \mapsto \ulcorner f_i^* \urcorner$ is concrete[1].

Define:

$$U_{\mathcal{B}_0} = |\mathcal{A}_0|,$$

$$U_{\mathcal{B}_{i+1}} = U_{\mathcal{B}_i} \cup (|\mathcal{A}_{i+1}| - f_i[|\mathcal{A}_i|]).$$

We show by induction on $i \in \omega$ that $U_{\mathcal{B}_i}$ is a concrete set. The base step is obvious since $|\mathcal{A}_0|$ is concrete. For the induction step suppose that $U_{\mathcal{B}_i}$ is concrete. It holds that $j \in f_i[|\mathcal{A}_i|]$ if and only if $\exists x\, (x \in |\mathcal{A}_i| \wedge f_i(x) = j)$. Since f_i is recursive in the low set $\mathcal{A}_i \oplus \mathcal{A}_{i+1}$, the set defined by the latter formula has the Turing degree $\deg(\mathcal{A}_i \oplus \mathcal{A}_{i+1})' \leqslant \deg(\bigoplus_{j\leqslant i+1} \mathcal{A}_j)' = \mathbf{0}'$. Therefore $f_i[|\mathcal{A}_i|]$ is concrete and $U_{\mathcal{B}_{i+1}}$ is concrete as a boolean combination of concrete sets. Moreover, the algorithm deciding $f_i[|\mathcal{A}_i|]$ with K as an oracle and using $i \mapsto \ulcorner \mathcal{A}_i^* \urcorner$ and $i \mapsto \ulcorner f_i^* \urcorner$ as subroutines can be computed recursively.

Recall that since $i \mapsto \ulcorner R_i \urcorner$ is recursive and for each $i \in \omega$ the set R_i is recursive, $\bigcup_{i\in\omega} R_i$ is concrete. Observe also that $\bigcup_{i\in\omega} U_{\mathcal{B}_i} \subseteq \bigcup_{i\in\omega} |\mathcal{A}_i| \subseteq \bigcup_{i\in\omega} R_i$.

It follows that there is a uniform concrete procedure computing $a \in U_{\mathcal{B}_i}$ given a and i on the input. The algorithm is the following.

[1] Here f_i^* is the halting problem for algorithms with oracle f.

Algorithm 6 Algorithm deciding $a \in U_{\mathcal{B}_i}$

Input: $a, i \in \omega$
Output: truth value of $a \in U_{\mathcal{B}_i}$
  if $i = 0$ **then**
    return truth value of $a \in |\mathcal{A}_0|$
  else
    if $a \in |\mathcal{A}_i|$ and $a \notin f_{i-1}[|\mathcal{A}_{i-1}|]$ **then**
      return true
    else
      return truth value of $a \in U_{\mathcal{B}_{i-1}}$
    end if
  end if

Algorithm 6 uses subroutines which, with K as an oracle, compute $a \in |\mathcal{A}_0|$, $a \in |\mathcal{A}_i|$ and $a \notin f_{i-1}[|\mathcal{A}_{i-1}|]$. It also recursively calls itself (with the second argument decreased).

Observe that for $i \in \omega$ it holds that $U_{\mathcal{B}_i} \subseteq \bigcup_{j \leqslant i} |\mathcal{A}_j|$. It is obvious for $i = 0$. Suppose that $U_{\mathcal{B}_i} \subseteq \bigcup_{j \leqslant i} |\mathcal{A}_j|$. Then $U_{\mathcal{B}_{i+1}} = U_{\mathcal{B}_i} \cup (|\mathcal{A}_{i+1}| - f_i[|\mathcal{A}_i|]) \subseteq \bigcup_{j \leqslant i} |\mathcal{A}_j| \cup |\mathcal{A}_{i+1}| = \bigcup_{j \leqslant i+1} |\mathcal{A}_j|$.

Let g_0 be the identity function restricted to $U_{\mathcal{B}_0}$ and for $i \in \omega$ let:

$$g_{i+1}(j) = \begin{cases} f_i(g_i(j)) & \text{if } j \in U_{\mathcal{B}_i} \\ j & \text{if } j \in U_{\mathcal{B}_{i+1}} - U_{\mathcal{B}_i} \end{cases}$$

We show by induction on $i \in \omega$ that the function g_i is a bijection from $U_{\mathcal{B}_i}$ to $|\mathcal{A}_i|$. The base step for $i = 0$ is obvious. Suppose for the induction hypothesis that g_i is a bijection from $U_{\mathcal{B}_i}$ to $|\mathcal{A}_i|$. We show that g_{i+1} is a bijection from $U_{\mathcal{B}_{i+1}}$ to $|\mathcal{A}_{i+1}|$.

First, we show that g_{i+1} is injective. It is obvious that g_{i+1} is injective on $U_{\mathcal{B}_i}$ and on $|\mathcal{A}_{i+1}| - f_i[|\mathcal{A}_i|]$. For $j \in U_{\mathcal{B}_i}$, $g_{i+1}(j) \in f_i[|\mathcal{A}_i|]$. For $j \in |\mathcal{A}_{i+1}| - f_i[|\mathcal{A}_i|]$, $g_{i+1}(j) \in |\mathcal{A}_{i+1}| - f_i[|\mathcal{A}_i|]$. Therefore g_{i+1} is injective.

Now we show that $g_{i+1}[U_{\mathcal{B}_{i+1}}] = |\mathcal{A}_{i+1}|$. The following sets are equal:

- $g_{i+1}[U_{\mathcal{B}_{i+1}}]$,
- $g_{i+1}[U_{\mathcal{B}_i} \cup (|\mathcal{A}_{i+1}| - f_i[U_{\mathcal{B}_i}])]$,
- $g_{i+1}[U_{\mathcal{B}_i}] \cup g_{i+1}[|\mathcal{A}_{i+1}| - f_i[U_{\mathcal{B}_i}]]$,
- $f_i[g_i[U_{\mathcal{B}_i}]] \cup g_{i+1}[|\mathcal{A}_{i+1}| - f_i[U_{\mathcal{B}_i}]]$,
- $f_i[|\mathcal{A}_i|] \cup (|\mathcal{A}_{i+1}| - f_i[|\mathcal{A}_i|])$,
- $|\mathcal{A}_{i+1}|$.

This ends the proof that for each $i \in \omega$ the function g_i is a bijection from $U_{\mathcal{B}_i}$ to $|\mathcal{A}_i|$.

By induction on $i \in \omega$ we show that g_i is concrete. The base step is obvious since g_0 is a restriction of a recursive function to a concrete set. For the inductive step suppose that g_i is concrete. We show that g_{i+1} is also concrete. Observe that the conditions in the definition of g_{i+1} are both concrete. Moreover, f_i is concrete and by the induction hypothesis g_i is also concrete. Therefore g_i is concrete for each $i \in \omega$.

Since for $i \in \omega$ g_i is a concrete bijection from $U_{\mathcal{B}_i}$ to $|\mathcal{A}_i|$, let $\mathcal{B}'_i$ be the co–image of $\mathcal{A}_i$ under g_i – its existence is provided by Lemma 4.1.10. Therefore $\mathcal{B}'_i$ is a recursive in $U_{\mathcal{B}_i} \oplus g_i \oplus \mathcal{A}_i$ concrete model in the same concrete vocabulary as $\mathcal{A}_i$ and $g_i \colon \mathcal{B}'_i \to \mathcal{A}_i$ is a concrete isomorphism. For $i \in \omega$ let $\mathcal{A}^\sigma_i$ be a reduction of $\mathcal{A}_i$ to σ and $\mathcal{B}_i$ be a reduction of $\mathcal{B}'_i$ to σ.

It is easy to see that for $i \in \omega$ it holds that $\mathcal{B}_i \subseteq \mathcal{B}_{i+1}$. Obviously $|\mathcal{B}_i| \subseteq |\mathcal{B}_{i+1}|$. The interpretation of constants is also obviously the same in $\mathcal{B}_{i+1}$ and $\mathcal{B}_i$. Moreover, for $P \in \sigma$ if $\mathcal{B}_i \models P[a_1, \dots, a_{ar(P)}]$, then $\mathcal{A}^\sigma_i \models P[g_i(a_1), \dots g_i(a_{ar(P)})]$ and $\mathcal{A}^\sigma_{i+1} \models P[f_i(g_i(a_1)), \dots, f_i(g_i(a_{ar(P)}))]$. The last statement is equivalent to $\mathcal{A}^\sigma_{i+1} \models P[g_{i+1}(a_1), \dots, g_{i+1}(a_{ar(P)})]$ by the definition of g_{i+1}. Since $\mathcal{B}'_{i+1}$ is the co–image of $\mathcal{A}_{i+1}$ under g_{i+1} and $\mathcal{B}_{i+1}$ is its reduction to σ it follows that $\mathcal{B}_{i+1} \models P[a_1, \dots, a_{ar(P)}]$.

Therefore, the following diagram is commutative:

$$\begin{tikzcd}
\mathcal{A}^\sigma_0 \arrow[r, hook, "f_0"] & \mathcal{A}^\sigma_1 \arrow[r, hook, "f_1"] & \mathcal{A}^\sigma_2 \arrow[r, hook, "f_2"] & \cdots \\
\mathcal{B}_0 \arrow[u, "g_0"] \arrow[r, phantom, "\subseteq"] & \mathcal{B}_1 \arrow[u, "g_1"] \arrow[r, phantom, "\subseteq"] & \mathcal{B}_2 \arrow[u, "g_2"] \arrow[r, phantom, "\subseteq"] & \cdots
\end{tikzcd}$$

Note that since $i \mapsto \ulcorner A_i \urcorner$ is concrete, then also $i \mapsto \ulcorner f_i \urcorner$ is concrete. Therefore, $i \mapsto \ulcorner U_{\mathcal{B}_i} \urcorner$ is a concrete function. It follows that $i \mapsto \ulcorner g_i \urcorner$ is a concrete function. This implies that $i \mapsto \ulcorner \mathcal{B}'_i \urcorner$ and $i \mapsto \ulcorner \mathcal{B}_i \urcorner$ are also concrete functions.

We now show that $(\mathcal{B}_i)_{i \in \omega}$ is a concrete chain of concrete models. We already know that $i \mapsto \ulcorner \mathcal{B}_i \urcorner$ is concrete, therefore $(\mathcal{B}_i)_{i \in \omega}$ is a concrete sequence of concrete models. We also know that for each $i \in \omega$ it holds that $\mathcal{B}_i \subseteq \mathcal{B}_{i+1}$. It remains to show that $\bigcup_{i \in \omega} |\mathcal{B}_i|$ is concrete. Recall that since $i \mapsto \ulcorner R_i \urcorner$ is recursive and for each $i \in \omega$ the set R_i is recursive, $\bigcup_{i \in \omega} R_i$ is concrete. Observe also that $\bigcup_{j \leqslant i} |\mathcal{B}_j| \subseteq \bigcup_{j \leqslant i} |\mathcal{A}_j| \subseteq \bigcup_{j \leqslant i} R_j$.

Algorithm 7 Algorithm deciding $\bigcup_{i\in\omega}|\mathcal{B}_i|$

Input: $a \in \omega$
Output: truth value of $a \in \bigcup_{i\in\omega}|\mathcal{B}_i|$
  if $a \in \bigcup_{i\in\omega} R_i$ **then**
    $i \leftarrow 0$
    while $a \notin R_i$ **do**
      $i \leftarrow i+1$
    end while
    return truth value of $a \in |\mathcal{B}_i|$
  else
    return false
  end if

Algorithm 7 uses concrete subroutines to compute $a \in \bigcup_{i\in\omega} R_i$, $i \mapsto \ulcorner\mathcal{B}_i\urcorner$ and $a \in |\mathcal{B}_i|$ (Algorithm 6) and it always halts. Therefore the set decided by the algorithm is concrete.

Therefore $(\mathcal{B}_i)_{i\in\omega}$ is a concrete chain of concrete models such that for each $i \in \omega$ there is a concrete isomorphism $g_i\colon \mathcal{B}_i \to \mathcal{A}_i^\sigma$. □

Gerald Sacks in [Sac72] shows an alternative way of dealing with infinite families of structures. He considers directed systems – families of models and embeddings $((\mathcal{A})_{d\in D}, (f_{d,e})_{d,e\in D,\, d\leqslant_D e}))$ indexed by a directed set $\mathcal{D} = (D, \leqslant_D)$ and such that for $d, e \in D$ such that $d \leqslant_D e$, $f_{d,e}$ is an embedding of $\mathcal{A}_d$ into $\mathcal{A}_e$. Such families are natural in the computational complexity aware model–theoretic considerations. For directed systems a direct limit can be defined without the need of reasoning up to isomorphism which, as we shown, may lead to increase of the computational complexity. Such systems can be easily transferred to the concrete models context enabling to define a concrete structures as limits of a concrete directed systems of concrete (not only low) models. We, however, do not elaborate on directed systems, since we are mainly concerned with the classical model–theoretic constructions which deal with chains and elementary chains of models. Translating those constructions to the language of directed systems could reduce the clarity of arguments. In the model–theoretic constructions presented later in this chapter we therefore construct concrete chains of concrete models using Theorem 4.1.22 instead of directed systems and their limits.

In the final paragraph of this introductory section we would like to make a remark on the management of resources in the constructions we perform. In the concrete models context there is only a limited number of constants: $\mathfrak{c}_0, \mathfrak{c}_1, \ldots$. We implicitly assume that in every construction we can always take an infinite recursive set of new constants or even an infinite recursive family of infinite recursive sets of new constants. In the end of this chapter we justify why this assumption is not a great limitation. Moreover, whenever we consider a theory T in a concrete vocabulary σ, we may also think of T as a theory in a recursive concrete vocabulary σ' such that $\sigma \subseteq \sigma'$. Generally every concrete set is treated as contained in some recursive set R, such that there are infinite recursive sets S disjoin from R.

4.2 Model Theory without Actual Infinity

Recall that we say that a theory is $\mathrm{CCW}(\sigma, D)$ if it is consistent, complete in σ and has witness property for σ–formulae in the set of constants D. The following lemma tells us when we may expect concrete models to exist.

Lemma 4.2.1 *Let $\sigma = (P_1, \ldots, P_m, ar, \varphi_C)$ be a concrete vocabulary. Let R be an infinite recursive set and let $D = \{\mathfrak{c}_r : r \in R\}$ be a recursive set of constants. Let $\sigma' = \sigma \cup D$ and let $E = C \cup D$. Let T be a concrete $\mathrm{CCW}(\sigma', D)$. Then there exists a concrete σ'–model $\mathcal{A} \models T$. Moreover, $|\mathcal{A}| \subseteq R$ and $\mathcal{A}$ is recursive in T.*

Proof: Let σ, σ', R, D, E and T be as in the assumptions of the lemma. For $c, d \in E$ we define a relation $\approx$ in the following way: $c \approx d \equiv_{df} c = d \in T$. Observe that since T is consistent and complete in σ', for every $c, d, e \in E$ it holds that:

- $c = c \in T$,
- if $c = d \in T$, then $d = c \in T$,
- if $c = d \in T$ and $d = e \in T$, then $c = e \in T$.

Hence $\approx$ is an equivalence relation. Moreover, for every predicate $P \in \sigma$ of arity k and every sequences of constants $e_1, \ldots, e_k$, $f_1, \ldots, f_k$ from E if for every $i = 1, \ldots, k$ it holds that $e_i = f_i \in T$ and $P(e_1, \ldots, e_k) \in T$, then by completeness of T, it holds that $P(f_1, \ldots, f_k) \in T$. Therefore $\approx$ is a congruence.

The set of equivalence classes of $\approx$ can be identified with

$$U =_{df} \{i \in \omega : i \in R \wedge \forall j < i\, (j \in R \Rightarrow \mathfrak{c}_i \neq \mathfrak{c}_j \in T)\},$$

the set of indices of the least in R representatives of all equivalence classes. Note that U is recursive in T, thus FM–representable, and $U \subseteq R$. Let φ_U be a formula FM–representing U.

For $j = 1, \ldots, m$ we define relations:

$$R_j(i_1, \ldots, i_{ar(P_j)}) \equiv_{df} \bigwedge_{k=1,\ldots,ar(P_j)} i_k \in U \wedge P_j(\mathfrak{c}_{i_1}, \ldots, \mathfrak{c}_{i_{ar(P_j)}}) \in T.$$

Those relations are well defined since $\approx$ is a congruence. They are also recursive in T, thus FM–representable – let φ_{P_j} be formulae representing R_j for $j = 1, \ldots, m$.

We define the interpretation of constants:

$$I_{E,U}(\ulcorner c \urcorner) = i \equiv_{df} c \in E \wedge i \in U \wedge c = \mathfrak{c}_i \in T.$$

$I_{E,U}$ is a function from E to U since U is the set of representatives of equivalence classes of $\approx$. The function $I_{E,U} : E \to U$ is recursive in T. It picks for $c \in E$ the index of the representative from U of the equivalence class of congruence $\approx$ to which c belongs. Let $\varphi_{E,U}$ FM–represent $I_{E,U}$.

We have thus defined a concrete σ'–structure $\mathcal{F} = (\varphi_U, \varphi_{P_1}, \ldots, \varphi_{P_m}, \varphi_{E,U})$. To complete the proof we also need to define satisfaction relation $R_{\models}$ in $\mathcal{F}$.

We define:

$$R_{\models} = \{(\ulcorner \psi(v_{i_1}, \ldots, v_{i_k}) \urcorner, \ulcorner \overline{a} \urcorner) : \psi(v_{i_1}, \ldots, v_{i_k}) \in \mathrm{Form}_{\sigma'} \wedge \overline{a} \in U^{<\omega} \wedge \\ \wedge \max\{i_1, \ldots, i_k\} < \mathrm{lh}(\overline{a}) \wedge \psi(\mathfrak{c}_{a_{i_1}}, \ldots, \mathfrak{c}_{a_{i_k}}) \in T\}.$$

Note that $R_{\models}$ is recursive in T, thus FM–representable. Let $\varphi_{\models}$ FM–represent $R_{\models}$.

We show that relation $R_{\models}$ has all desired properties i.e. it is the satisfaction relation in $\mathcal{F}$. It is obvious by the definition of $R_{\models}$ that it is a pre–satisfaction relation in $\mathcal{F}$. It suffices to show that $R_{\models}$ satisfies Tarski's conditions. In the following reasoning we assume that $\overline{a}$ is a sequence of elements of U such that $\mathrm{lh}(\overline{a})$ is greater then the greatest index of a free variable of considered terms and formulae.

It is easy to see by the definition of $R_{\models}$ and $\mathrm{Val}^{\mathcal{F}}$ that for terms t_1, t_2 it holds that $R_{\models}(\ulcorner t_1 = t_2 \urcorner, \ulcorner \overline{a} \urcorner)$ if and only if $\mathrm{Val}^{\mathcal{F}}(\ulcorner t_1 \urcorner, \ulcorner \overline{a} \urcorner) = \mathrm{Val}^{\mathcal{F}}(\ulcorner t_2 \urcorner, \ulcorner \overline{a} \urcorner)$.

Let $j = 1, \ldots, m$. Let $t_1, \ldots, t_{ar(P_j)}$ be terms and let for $k = 1, \ldots, ar(P_j)$, $i_k = \mathrm{Val}^{\mathcal{F}}(\ulcorner t_k \urcorner, \ulcorner \overline{a} \urcorner)$. We show that $R_{\models}(\ulcorner P_j(t_1, \ldots, t_{ar(P_j)}) \urcorner, \ulcorner \overline{a} \urcorner)$ if and only if $R_j(i_1, \ldots, i_{ar(P_j)})$. The following are equivalent:

- $R_{\models}(\ulcorner P_j(t_1, \ldots, t_{ar(P_j)}) \urcorner, \ulcorner \overline{a} \urcorner)$ (by definition of $R_{\models}$ and $\mathrm{Val}^{\mathcal{F}}$),
- $P_j(\mathfrak{c}_{i_1}, \ldots, \mathfrak{c}_{i_{ar(P_j)}}) \in T$ (by definition of R_j),
- $R_j(i_1, \ldots, i_{ar(P_j)})$.

Now let us consider the case of the negation – we show that for any formula $\psi(v_{i_1}, \ldots, v_{i_k})$, $R_{\models}(\ulcorner \neg\psi \urcorner, \ulcorner \overline{a} \urcorner)$ if and only if it is not the case that $R_{\models}(\ulcorner \psi \urcorner, \ulcorner \overline{a} \urcorner)$. Fix a formula $\psi(v_{i_1}, \ldots, v_{i_k})$. The following are equivalent:

- $R_{\models}(\ulcorner \neg\psi \urcorner, \ulcorner \overline{a} \urcorner)$ (by definition of $R_{\models}$),
- $\neg\psi(\mathfrak{c}_{a_{i_1}}, \ldots, \mathfrak{c}_{a_{i_k}}) \in T$ (by completeness and consistency of T),
- $\psi(\mathfrak{c}_{a_{i_1}}, \ldots, \mathfrak{c}_{a_{i_k}}) \notin T$ (by definition of $R_{\models}$),
- It is not the case that $R_{\models}(\ulcorner \psi \urcorner, \ulcorner \overline{a} \urcorner)$.

For the case of conjunction, for fixed formulae $\psi_1(v_{i_1}, \ldots, v_{i_k})$ and $\psi_2(v_{j_1}, \ldots, v_{j_l})$, the following are equivalent:

- $R_{\models}(\ulcorner \psi_1 \wedge \psi_2 \urcorner, \ulcorner \overline{a} \urcorner)$ (by definition of $R_{\models}$),
- $\psi_1(\mathfrak{c}_{a_{i_1}}, \ldots, \mathfrak{c}_{a_{i_k}}) \wedge \psi_2(\mathfrak{c}_{a_{j_1}}, \ldots, \mathfrak{c}_{a_{j_l}}) \in T$ (completeness and consistency of T),
- $\psi_1(\mathfrak{c}_{a_{i_1}}, \ldots, \mathfrak{c}_{a_{i_k}}) \in T$ and $\psi_2(\mathfrak{c}_{a_{j_1}}, \ldots, \mathfrak{c}_{a_{j_l}}) \in T$ (by definition of $R_{\models}$),
- $R_{\models}(\ulcorner \psi_1 \urcorner, \ulcorner \overline{a} \urcorner)$ and $R_{\models}(\ulcorner \psi_2 \urcorner, \ulcorner \overline{a} \urcorner)$.

The last step of the proof is the quantifier case. Recall the following definition:

$$\overline{a}[v_k := n]_i = \begin{cases} a_i & \text{if } i < \mathrm{lh}(\overline{a}) \text{ and } i \neq k \\ n & \text{if } i = k \\ \mu\varphi_U^{\mathrm{FM}(\mathcal{N})} & \text{if } i \geqslant \mathrm{lh}(\overline{a}) \text{ and } i < k \end{cases}$$

Now fix $\psi(v_k, v_{i_1}, \ldots, v_{i_l})$. The following are equivalent:

- $R_{\models}(\ulcorner \exists v_k\, \psi \urcorner, \ulcorner \overline{a} \urcorner)$ (by definition of $R_{\models}$),
- $\exists v_k\, \psi(v_k, \mathfrak{c}_{a_{i_1}}, \ldots, \mathfrak{c}_{a_{i_l}}) \in T$ (by witness property for T),
- $\psi(\mathfrak{c}_n, \mathfrak{c}_{a_{i_1}}, \ldots, \mathfrak{c}_{a_{i_l}}) \in T$, for some $n \in U$ (by definition of $R_{\models}$),
- $R_{\models}(\ulcorner \psi \urcorner, \ulcorner \overline{a}[v_k := n] \urcorner)$, for some $n \in U$.

Therefore $R_{\models}$ is a satisfaction relation in $\mathcal{F}$ and $\mathcal{A} = (\mathcal{F}, \varphi_{\models})$ is a concrete σ'–model. It is easy to see that if ψ is a σ'–sentence such that $\psi \in T$, then $R_{\models}(\ulcorner \psi \urcorner, \ulcorner \varepsilon \urcorner)$ i.e. $\mathcal{A} \models \psi$.

Recall that $U = |\mathcal{A}| \subseteq R$. Moreover, as it is shown above, relations FM–represented by formulae defining the $\mathcal{A}$ are all recursive in T. Thus $\mathcal{A}$ is recursive in T. $\square$

By Lemma 4.2.1 we know which theories have concrete models and that to construct a concrete model of a theory we need only to find its extension satisfying the assumptions of the lemma. Is it possible for reasonable theories? The answer is positive and is given in next section where we prove completeness theorems.

4.2.1 Completeness Theorems

First let us state and prove a lemma which will help us constructing theories with witness properties.

Lemma 4.2.2 *Let T be a consistent theory in σ, let c be a new constant and let $\varphi(x)$ be a σ–formula. Then T is consistent with the sentence $\exists x\, \varphi(x) \Rightarrow \varphi(c)$.*

Proof: Let T, c and φ be as in the assumptions of the lemma. Suppose for the sake of contradiction that $T \vdash \neg(\exists x\, \varphi(x) \Rightarrow \varphi(c))$. Then, by the lemma on constants, since c does not occur in T, $T \vdash \forall y\, \neg(\exists x\, \psi(x) \Rightarrow \psi(y))$ which is equivalent to $T \vdash \forall y\, (\exists x\, \psi(x) \wedge \neg\psi(y))$. Then $T \vdash \exists x\, \psi(x) \wedge \forall y\, \neg\psi(y)$ which means T is inconsistent which contradicts the hypothesis. $\square$

Recall that for a concrete vocabulary σ the set of all σ–sentences and the set of all σ–formulae with only one free variable v_0 are recursive in σ – thus they have effective in σ presentations. Let $\psi_0, \psi_1, \ldots$ be an effective in σ presentation of σ–sentences and let $\varphi_0, \varphi_1, \ldots$ be an effective in σ presentation of σ–formulae with only one free variable v_0.

We work further with a fixed recursive concrete vocabulary σ. Let C be the set of constants of σ and let R be an infinite recursive set such that $D = \{\mathfrak{c}_r \colon r \in R\}$ is a recursive set of new constants. Let $d_0, d_1, \ldots$ be an effective presentation of D. Let $\sigma' = \sigma \cup D$.

Our aim is to prove the first completeness theorem i.e. to show when we can extend a σ theory T to a concrete $\mathrm{CCW}(\sigma', D)$. Having such an extension of T we can obtain a concrete σ'–model $\mathcal{A}'$ of T, by Lemma 4.2.1. Then it suffices to take a reduction $\mathcal{A}$ of $\mathcal{A}'$ to σ.

The construction mimics the classical Henkin's argument on construction of a model of a consistent theory – our main aim is to show that for consistent theories which have concrete consequences Henkin's construction gives us a concrete model.

First let us define the Henkin's completion of a consistent theory.

Definition 4.2.3 (Henkin's Completion) *Let T be a consistent theory. Define a sequence α_n of formulae as follows:*

- *For $n = 2k$*
$$\alpha_n = \begin{cases} \psi_k & \text{if } T \nvdash \neg(\alpha_0 \wedge \cdots \wedge \alpha_{n-1} \wedge \psi_k) \\ \neg\psi_k & \text{else} \end{cases}$$
- *For $n = 2k+1$*
$$\alpha_n = \exists v_0\, \varphi_k(v_0) \Rightarrow \varphi_k(d_i),$$
for i being the least natural number such that d_i does not occur in $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}, \varphi_k$.

The Henkin's completion of T (with respect to constants D) is the set $\mathrm{HC}(T) = \{\alpha_n \colon n \in \omega\}$. For $n \in \omega$ the n–th cut of the Henkin's completion of T (with respect to constants D) is the set $\mathrm{HC}_n(T) = \{\alpha_i \colon i < n\}$.

We proceed with a series of lemmata which present properties of Henkin's completions. Since we defined Henkin's completions only for consistent theories, let us fix a consistent theory T in σ for which we prove the following lemmata.

First, we show that the underlying theory is contained within its Henkin's completion.

Lemma 4.2.4 $T \subseteq \mathrm{HC}(T)$.

Proof: Let ψ be a σ–sentence such that $\psi \in T$. Then $\psi = \psi_m$ for some $m \in \omega$. Since $\psi \in T$, then $T \vdash \psi$ and also $T \cup \mathrm{HC}_{2m}(T) \vdash \psi$ and since $T \cup \mathrm{HC}_{2m}(T)$ is consistent, $T \cup \mathrm{HC}_{2m}(T) \nvdash \neg\psi$. Then by the deduction theorem $T \nvdash \neg(\alpha_0 \wedge \cdots \wedge \alpha_{2m-1} \wedge \psi)$ and by the construction of the theory $\mathrm{HC}(T)$: $\alpha_{2m} = \psi$ and therefore $\psi \in \mathrm{HC}(T)$. $\square$

Now we show that $\mathrm{HC}(T)$ is indeed a completion i.e. not only $T \subseteq \mathrm{HC}(T)$ but also $\mathrm{HC}(T)$ is complete in σ'.

Lemma 4.2.5 $\mathrm{HC}(T)$ *is complete in* σ'.

Proof: Let ψ be a σ'–sentence. Then $\psi = \psi_m$ for some $m \in \omega$. If $T \nvdash \neg(\alpha_0 \wedge \cdots \wedge \alpha_{2m-1} \wedge \psi_m)$, then $\alpha_{2m} = \psi$ and $\psi \in \mathrm{HC}_{2m+1}(T) \subseteq \mathrm{HC}(T)$. Otherwise $\alpha_{2m} = \neg\psi$ and $\neg\psi \in \mathrm{HC}_{2m+1}(T) \subseteq \mathrm{HC}(T)$.

Therefore for every σ'–sentence ψ, either ψ or its negation is in $\mathrm{HC}(T)$. By the definition this means that $\mathrm{HC}(T)$ is complete in σ'. $\square$

The following lemma shows that the Henkin's completion is also consistent.

Lemma 4.2.6 $\mathrm{HC}(T)$ *is consistent.*

Proof: We show by induction that for each $n \in \omega$ theory $\mathrm{HC}_n(T)$ is consistent. In fact we prove that for each $n \in \omega$ theory $T \cup \mathrm{HC}_n(T)$ is consistent. For the base step of induction let $n = 0$. Since $T \cup \mathrm{HC}_0(T)$ is just T, $T \cup \mathrm{HC}_0(T)$ is consistent by the assumption.

Now for the inductive step suppose that $T \cup \mathrm{HC}_n(T)$ is consistent. There are two cases – n being even or odd.

Suppose $n = 2k$ for some $k \in \omega$.

If $T \nvdash \neg(\alpha_0 \wedge \cdots \wedge \alpha_{n-1} \wedge \psi_k)$ then $\alpha_n = \psi_k$ and by the deduction theorem $T \cup \mathrm{HC}_n(T) \nvdash \neg\alpha_n$ since $\neg(\alpha_0 \wedge \cdots \wedge \alpha_{n-1} \wedge \alpha_n)$ is equivalent to $(\alpha_0 \wedge \cdots \wedge \alpha_{n-1}) \Rightarrow \neg\alpha_n$. By the induction hypothesis $T \cup \mathrm{HC}_n(T)$ is consistent, thus $T \cup \mathrm{HC}_n(T) \cup \{\alpha_n\} = T \cup \mathrm{HC}_{n+1}(T)$ is also consistent.

Suppose now that $T \vdash \neg(\alpha_0 \wedge \cdots \wedge \alpha_{n-1} \wedge \psi_k)$ and $\alpha_n = \neg\psi_k$. Then $T \cup \mathrm{HC}_n(T) \vdash \alpha_n$ and by the induction hypothesis $T \cup \mathrm{HC}_n(T) \cup \{\alpha_n\}$ is consistent. This ends the case of n being even.

Now suppose $n = 2k+1$ for some $k \in \omega$. Then $\alpha_n = \exists v_0\, \psi_k(v_0) \Rightarrow \psi_k(d_j)$ for some j such that d_j does not occur in $\alpha_0, \ldots, \alpha_{n-1}, \psi_k$. Thus by Lemma 4.2.2 $T \cup \mathrm{HC}_{n+1}(T)$ is consistent.

Since every finite subset of $\mathrm{HC}(T)$ is contained in some consistent theory $T \cup \mathrm{HC}_n(T)$, $\mathrm{HC}(T)$ is also consistent. $\square$

We therefore know that the Henkin's completion does not spoil the consistency of a given theory. We now show that the Henkin's completion has the witness property.

Lemma 4.2.7 $\mathrm{HC}(T)$ *has witness property for* σ' *sentences in* D*.*

Proof: Fix a σ'–formula $\varphi(v_i)$. Then there are $k, l \in \omega$ such that $\varphi = \varphi_k$ and $\exists v_0\, \varphi_k(v_0) \Rightarrow \varphi_k(d_l) \in \mathrm{HC}_{2k+2}(T)$. By Lemma 4.2.5 theory $\mathrm{HC}(T)$ is complete. Since $\exists v_i\, \varphi(v_i)$ is equivalent to $\exists v_0\, \varphi(v_0)$, then $\exists v_i\, \varphi(v_i) \Rightarrow \varphi(d_l) \in \mathrm{HC}(T)$. Thus $d_l \in D$ is a witness for $\varphi(v_i)$. $\square$

We know therefore that the Henkin's completion of T is $\mathrm{CCW}(\sigma', D)$. We need now to focus on the complexity of $\mathrm{HC}(T)$ as we want to use the Lemma 4.2.1 to obtain a concrete σ–model of T. We also need to restrict the Henkin's completion's $\mathrm{HC}(T)$ complexity in order to guarantee that it is not too complicated to apply Lemma 4.2.1 to it. Further we show that it is enough to assume that $\mathrm{Cn}(T)$ is concrete.

Now let us additionally suppose that $\mathrm{Cn}(T)$ is concrete.

First, we show an algorithm with recursively enumerable oracle which, given $n \in \omega$ as an input, computes the Gödel number of the n–th member of sequence $\alpha_0, \alpha_1, \ldots$ i.e. the n–th sentence in the construction of $\mathrm{HC}(T)$.

Lemma 4.2.8 *The map* $n \mapsto \ulcorner \alpha_n \urcorner$ *is recursive in* $\mathrm{Cn}(T)$*, therefore concrete.*

Proof: The algorithm is using two data structures: A and B. A is a (dynamic) array which is holding already computed sentences from the sequence $\alpha_0, \alpha_1, \ldots$. For $i \in \omega$, by $A[i]$ we mean the i–th sentence in the sequence A and $A[i] \leftarrow \psi$ means that the i–th member of the array A becomes ψ. The structure B is a set of indices (i.e. natural numbers), which is holding indices of all constants occurring in already computed sentences α_i. The set B has a method *insert* which takes a finite set of natural numbers as an argument and adds all its elements to B.

The general algorithm is presented below. It proceeds exactly as the definition specifies.

Algorithm 8 Algorithm enumerating the Henkin's completion of T

Input: $n \in \omega$
Output: $\ulcorner \alpha_n \urcorner$

```
A ← Array()
B ← Set()
i ← 0
while i ⩽ n do
  if i > 0 then
    B.insert(Consts(A[i − 1]))
  end if
  k ← ⌊i/2⌋
  if 2|i then
    if T ⊢ ¬(A[0] ∧ ··· ∧ A[k − 1] ∧ ψ_k) then
      A[i] ← ¬ψ_k
    else
      A[i] ← ψ_k
    end if
  else
    j ← 0
    while true do
      if j ∉ B and j ∉ Consts(φ_k) then
        break
      end if
      j ← j + 1
    end while
    A[i] ← ∃v_0 φ_k(v_0) ⇒ φ_k(d_j)
  end if
  return ⌜A[n]⌝
end while
```

Algorithm 8 uses $\mathrm{Cn}(T)$ as an oracle for computing $T \vdash \psi$. Therefore the map $n \mapsto \ulcorner \alpha_n \urcorner$ is recursive in $\mathrm{Cn}(T)$ – thus concrete. □

It now remains to show that $\mathrm{HC}(T)$ is also concrete. This result is due to the following lemma.

Lemma 4.2.9 *Theory* $\mathrm{HC}(T)$ *is recursive in* $\mathrm{Cn}(T)$.

Proof: Let ψ be a σ'–sentence. To decide whether ψ is in $\mathrm{HC}(T)$ it is sufficient to check if ψ is α_n for some $n \in \omega$. Since $\mathrm{HC}(T)$ is complete in σ' and consistent, we only need to generate the sequence $(\alpha_n)_{n\in\omega}$ until $\alpha_k = \psi$ or $\alpha_k = \neg\psi$ for some k. The algorithm deciding $\mathrm{HC}(T)$ is defined as follows:

Algorithm 9 Algorithm deciding the Henkin's completion of T.

Input: $\ulcorner\psi\urcorner$

Output: truth value of $\psi \in \mathrm{HC}(T)$

```
i ← 0
while ⌜α_i⌝ ≠ ⌜ψ⌝ and ⌜α_i⌝ ≠ ⌜¬ψ⌝ do
  i ← i + 1
end while
if ⌜α_i⌝ = ⌜ψ⌝ then
  return true
else
  return false
end if
```

The algorithm halts for every ψ since $\mathrm{HC}(T)$ is complete in σ. It uses Algorithm 8 to compute $\ulcorner\alpha_i\urcorner$ from i. Thus $\mathrm{HC}(T)$ is recursive in $\mathrm{Cn}(T)$ – thus concrete. □

We are finally ready to make use of the lemmata proved above to obtain the Concrete Completeness Theorem.

Theorem 4.2.10 (Concrete Completeness Theorem) *Let σ be a recursive concrete vocabulary. Let T be a consistent theory in σ, such that* $\mathrm{Cn}(T)$ *is concrete. Let R be an infinite recursive set such that $D = \{\mathfrak{c}_r : r \in R\}$ is a recursive set of new constants. Then there is a concrete σ–model $\mathcal{A}$ such that $\mathcal{A} \models T$. Moreover $|\mathcal{A}| \subseteq R$ and $\mathcal{A}$ is recursive in* $\mathrm{Cn}(T)$.

Proof: Let σ, R, D and T be as in the assumptions of the theorem. Let C be the set of constants from σ. Let $\sigma' = \sigma \cup D$.

By Lemma 4.2.9 Henkin's completion $\mathrm{HC}(T)$ of theory T is recursive in $\mathrm{Cn}(T)$ and by Lemmata 4.2.5, 4.2.6 and 4.2.7 it is $\mathrm{CCW}(\sigma', D)$. By Lemma 4.2.1 applied to $\mathrm{HC}(T)$ we get a recursive in $\mathrm{Cn}(T)$ concrete σ'–model $\mathcal{A}'$ such that $\mathcal{A}' \models \mathrm{HC}(T)$ and $|\mathcal{A}'| \subseteq R$. Therefore by Lemma 4.2.4 $\mathcal{A}' \models T$. The reduction $\mathcal{A}$ to σ of $\mathcal{A}'$ is a concrete model of T which is recursive in $\mathrm{Cn}(T)$. □

Corollary 4.2.11 *If T is a consistent, recursively axiomatisable theory, then T has a concrete model.*

By examining the proof of the Concrete Completeness Theorem we can see that, for a given consistent theory, it is sufficient to construct its maximal consistent extension using *easy enough* steps during the construction, to get its concrete model. If each step is recursive with recursively enumerable oracle, then the entire construction is so and it leads to obtaining a concrete model.

Theorem 4.2.10 shows that wide range of natural theories have concrete models. Thus for recursively enumerable and low theories there are concrete models. However, the resulting concrete model may be Δ^0_2–hard. Such a model turns out to be too hard to use in some interesting cases. Suppose we have a concrete model $\mathcal{A}$ and we wish to find a concrete model $\mathcal{B}$ with an elementary embedding $f\colon \mathcal{A} \to \mathcal{B}$. The classical approach is to take a new constant c and consider the theory $T = \mathrm{ElDiag}(A) \cup \{c \neq c_a \colon a \in |\mathcal{A}|\}$ and the Henkin's completion of T. But if $\mathcal{A}$ is Δ^0_2–hard, then $\mathrm{Cn}(T)$ may not be concrete and $\mathrm{HC}(T)$ may even be Δ^0_3–hard – too complicated to have concrete models. This shows that the completeness in the concrete context is not as universal as in the axiomatic one. There exist consistent concrete theories with no concrete models. This is shown more clearly in Section 4.2.5 where we show that syntactic version of the Robinson's joint consistency theorem holds in the concrete models context, while the semantic version fails.

If we want to perform certain operations on concrete models, we need them to be easy enough to apply the Henkin's completion to their elementary diagrams. But the Henkin's completion fails to provide us with such concrete models. Thus, we need to improve the construction. The improvement can be done with the help of the Low Basis Theorem. The Low Basis Theorem can be applied to any low infinite tree, so we need to transfer Henkin's construction to such trees.

Let σ be a recursive concrete vocabulary and let C be the set of constants from σ. Let R be an infinite recursive set such that $D = \{\mathfrak{c}_r \colon r \in R\}$ is a recursive set of new constants with effective presentation $d_0, d_1, \ldots$. Let $\sigma' = \sigma \cup D$. In order to mimic the Henkin's construction, we are going to label the full binary tree with theories. Recall that $\psi_0, \psi_1, \ldots$ is an effective presentation of σ'–sentences and $\varphi_0, \varphi_1, \ldots$ is an effective presentation of σ–formulae with only one free variable v_0.

Let T be a theory in σ. We define $\mathrm{Th}^{\sigma',D}_T(\rho)$ inductively on $\rho \in 2^{<\omega}$ as follows:

1. If $\rho = \varepsilon$, then $\mathrm{Th}^{\sigma',D}_T(\rho) = T$,
2. If $\rho = \tau 0$ and $\mathrm{lh}(\rho) = 2i+1$, then $\mathrm{Th}^{\sigma',D}_T(\rho) = \mathrm{Th}^{\sigma',D}_T(\tau) \cup \{\psi_i\}$,
3. If $\rho = \tau 1$ and $\mathrm{lh}(\rho) = 2i+1$, then $\mathrm{Th}^{\sigma',D}_T(\rho) = \mathrm{Th}^{\sigma',D}_T(\tau) \cup \{\neg\psi_i\}$,
4. If ($\rho = \tau 0$ or $\rho = \tau 1$) and $\mathrm{lh}(\rho) = 2i+2$, then $\mathrm{Th}^{\sigma',D}_T(\rho) = \mathrm{Th}^{\sigma',D}_T(\tau) \cup \{\exists v_0\, \varphi_i(v_0) \Rightarrow \varphi_i(d_k)\}$, where k is the least index such that d_k does not occur neither in $\mathrm{Th}^{\sigma',D}_T(\tau)$ nor φ_i.

Note that for every ρ, theory $\mathrm{Th}^{\sigma',D}_T(\rho)$ is recursive in T, since sequences $(\psi_i)_{i\in\omega}$ and $(\varphi_i)_{i\in\omega}$ are recursive. Since we work with fixed vocabularies σ

and σ' and fixed set of new constants D in further considerations we omit superscripts in $\mathrm{Th}_T^{\sigma',D}(\rho)$ and write $\mathrm{Th}_T(\rho)$ instead.

We define the theory $\mathrm{Th}_{T,f} = \bigcup_{i\in\omega} \mathrm{Th}_T(f{\restriction}i)$ for each set $f \in 2^\omega$. Obviously, not every theory $\mathrm{Th}_{T,f}$ is interesting. One of them is exactly the one obtained from the Henkin's construction applied to T, but there is also another one which will open our way to perform various model–theoretic constructions.

Recall that for a tree $\mathcal{T}$, by $[\mathcal{T}]$ we mean the class of infinite branches trough $\mathcal{T}$. We are mainly interested in consistent theories $\mathrm{Th}_{T,f}$. Our target is to define a subtree $\mathrm{CON}(\mathrm{Th}_T)$ of the full tree Th_T such that for every $f \in 2^\omega$ it holds that $f \in [\mathrm{CON}(\mathrm{Th}_T)]$ if and only if $\mathrm{Th}_{T,f}$ is consistent. This should be done without increasing the computational complexity of the tree.

Let us note that the relation of provability in the theory T is recursively enumerable in T since it can be defined in the following way: $T \vdash \varphi \equiv_{df} \exists k\, \mathrm{Prov}_T(\ulcorner\varphi\urcorner, k)$, where Prov_T represents a relation which is recursive in T. Using provability in the theory T we define consistency of T in the standard way by putting $\mathrm{Con}_T \equiv_{df} T \nvdash \bot$. However we can not apply the consistency requirement, as it is, directly to every node without increasing the complexity of the tree. Therefore, we introduce the notion of n–provability as follows $T \vdash_n \varphi \equiv_{df} \exists k \leqslant n\, \mathrm{Prov}_T(\ulcorner\varphi\urcorner, k)$ – observe that for every $n \in \omega$, n–provability is recursive in T. Note that a sentence ψ is provable in T if and only if there is n such that ψ is n–provable in T. Now we can define a subtree $\mathrm{CON}(\mathrm{Th}_T) =_{df} \{\tau \in 2^{<\omega} \colon \mathrm{Th}_T(\tau) \nvdash_{\mathrm{lh}(\tau)} \bot\}$. Note that for any theory T if $k < n$, then $T \nvdash_n \bot$ implies $T \nvdash_k \bot$, hence $\mathrm{CON}(\mathrm{Th}_T)$ is indeed closed downwards therefore it is a tree.

Now let us establish certain facts about $\mathrm{CON}(\mathrm{Th}_T)$.

Lemma 4.2.12 *The tree* $\mathrm{CON}(\mathrm{Th}_T)$ *is infinite i.e.* $[\mathrm{CON}(\mathrm{Th}_T)]$ *is not empty.*

Proof: Let $f \in 2^\omega$ be such that $\mathrm{Th}_{T,f} = \mathrm{HC}(T)$. Then $f \in [\mathrm{CON}(\mathrm{Th}_T)]$. $\square$

Recall that $f \in [\mathrm{CON}(\mathrm{Th}_T)]$ if and only if for every $i \in \omega$ it holds that $\mathrm{Th}_T(f{\restriction}i) \nvdash_i \bot$. The next lemma shows that theories for infinite branches of $\mathrm{CON}(\mathrm{Th}_T)$ are all consistent.

Lemma 4.2.13 *For every* $f \in 2^\omega$ *it holds that* $f \in [\mathrm{CON}(\mathrm{Th}_T)]$ *if and only if* $\mathrm{Th}_{T,f}$ *is consistent.*

Proof: Fix $f \in 2^\omega$.

Suppose that $f \in [\mathrm{CON}(\mathrm{Th}_T)]$ and for the sake of contradiction that $\mathrm{Th}_{T,f}$ is inconsistent. Then there are $i, j \in \omega$ such that $\mathrm{Th}_T(f{\restriction}j) \vdash_i \bot$. Therefore for $k = \max i, j$ it follows that $\mathrm{Th}_T(f{\restriction}k) \vdash_k \bot$, and hence it is not the case that $\forall i \in \omega\, \mathrm{Th}_T(f{\restriction}i) \nvdash_i \bot$, thus $f \notin [\mathrm{CON}(\mathrm{Th}_T)]$ which is a contradiction.

Suppose now that $\mathrm{Th}_{T,f}$ is consistent. Then there is no $i \in \omega$ such that $\mathrm{Th}_T(f\restriction i) \vdash_i \perp$ which means that for all $i \in \omega$ it holds that $f\restriction i \in \mathrm{CON}(\mathrm{Th}_T)$ which means $f \in [\mathrm{CON}(\mathrm{Th}_T)]$ by the definition. $\square$

The following lemma shows that theories for the infinite branches of $\mathrm{CON}(\mathrm{Th}_T)$ are also complete in σ'.

Lemma 4.2.14 *For every $f \in 2^\omega$ if $f \in [\mathrm{CON}(\mathrm{Th}_T)]$, then $\mathrm{Th}_{T,f}$ is complete in σ'.*

Proof: Let T be as in the assumptions of the lemma. Let $f \in 2^\omega$ be such that $f \in [\mathrm{CON}(\mathrm{Th}_T)]$.

Let ψ be an arbitrary σ'–sentence. Then $\psi = \psi_k$ for some $k \in \omega$. If $f(2k+1) = 0$, then $\psi \in \mathrm{Th}_T(f\restriction k) \subseteq \mathrm{Th}_{T,f}$. Otherwise $f(2k+1) = 1$ and then $\neg\psi \in \mathrm{Th}_T(f\restriction k) \subseteq \mathrm{Th}_{T,f}$. $\square$

The last lemma to be proved before the key theorem of this section states that theories of infinite branches of $[\mathrm{CON}(\mathrm{Th}_T)]$ have witness property.

Lemma 4.2.15 *For every $f \in 2^\omega$ if $f \in [\mathrm{CON}(\mathrm{Th}_T)]$, then $\mathrm{Th}_{T,f}$ have witnesses for σ' in D.*

Proof: Let $f \in 2^\omega$ be such that $f \in [\mathrm{CON}(\mathrm{Th}_T)]$. We show that $\mathrm{Th}_{T,f}$ has the witness property for σ' in D.

Fix a σ'–formula $\varphi(v_i)$. Then there are $k, l \in \omega$ such that $\varphi = \varphi_k$ and $\exists v_0\, \varphi_k(v_0) \Rightarrow \varphi_k(d_l) \in \mathrm{Th}_T(f\restriction 2k+2)$. By Lemma 4.2.14 theory $\mathrm{Th}_{T,f}$ is complete. Since $\exists v_i\, \varphi(v_i)$ is equivalent to $\exists v_0\, \varphi(v_0)$, then $\exists v_i\, \varphi(v_i) \Rightarrow \varphi(d_l) \in \mathrm{Th}_{T,f}$. Thus, d_l is a witness for $\varphi(v_i)$. $\square$

We are ready to state and prove one of the most important theorems of this chapter.

Theorem 4.2.16 (Low Completeness Theorem) *Let σ be a recursive concrete vocabulary and let T be a consistent low theory. Let R be a recursive set such that $D = \{\mathfrak{c}_r : r \in R\}$ is a recursive set of new constants – the witnesses. Then there is a low concrete σ–model $\mathcal{A}$ such that $|\mathcal{A}| \subseteq R$, $\mathcal{A} \models T$ and $\mathcal{A} \oplus T$ is low.*

Proof: Let σ, R, D and T be a as in the assumptions of the theorem.

Since T is low, the tree $\mathrm{CON}(\mathrm{Th}_T^{\sigma',D})$ is recursive in T, thus low. By Lemma 4.2.12 the tree $\mathrm{CON}(\mathrm{Th}_T^{\sigma',D})$ is infinite. Then by the Low Basis Theorem (2.3.13) there is a low set $f \in [\mathrm{CON}(\mathrm{Th}_T^{\sigma',D})]$ such that $f \oplus T$ is low. Since $f \in [\mathrm{CON}(\mathrm{Th}_T^{\sigma',D})]$, by Lemmata 4.2.13, 4.2.14 and 4.2.15, it holds that $\mathrm{Th}_{T,f}^{\sigma',D}$ is low and a $\mathrm{CCW}(\sigma', D)$. Therefore by Lemma 4.2.1 there is a concrete σ'–model $\mathcal{A}'$ such that $|\mathcal{A}| \subseteq R$, $\mathcal{A}' \models \mathrm{Th}^{\sigma',D}_{T,f}$ and $\mathcal{A}'$ is recursive in f. Therefore $\mathcal{A}' \models T$. Let $\mathcal{A}$ be a reduction of $\mathcal{A}'$ to σ. Then $\mathcal{A}$ is a recursive in f concrete σ–model such that $\mathcal{A} \models T$. Moreover $\mathcal{A} \oplus T$ is low. $\Box$

Note that the proof of the Low Basis Theorem is effective (see Theorem 2.3.14). Consider the function $(\ulcorner T^{*} \urcorner, \ulcorner D \urcorner) \mapsto \ulcorner \mathcal{A} \urcorner$ such that given the algorithm deciding the halting problem of a consistent theory T and the set of witnesses D it computes by means of the Low Completeness Theorem (4.2.16) the Gödel number $\ulcorner \mathcal{A} \urcorner$ of a low concrete model such that $\mathcal{A} \models T$ and such that $\mathcal{A} \oplus T$ is low. Then $(\ulcorner T^{*} \urcorner, \ulcorner D \urcorner) \mapsto \ulcorner \mathcal{A} \urcorner$ is concrete. Moreover, by Theorem 2.3.14 the function $(\ulcorner T^{*} \urcorner, \ulcorner D \urcorner) \mapsto \ulcorner \mathcal{A}^{*} \urcorner$, outputting the code of an algorithm which decides in K the halting problem for $\mathcal{A}$, is also concrete.

As usual, by completeness we obtain compactness. The following two forms of those holds.

Theorem 4.2.17 (Concrete Compactness Theorem) *Let T be a theory in a recursive concrete vocabulary σ such that* $\mathrm{Cn}(T)$ *is concrete and every finite subset T_0 of T is consistent. Then T has a concrete model.*

Theorem 4.2.18 (Low Compactness Theorem) *Let T be a low theory in a recursive concrete vocabulary σ such that every finite subset T_0 of T is consistent. Then T has a low concrete model.*

4.2.2 Omitting Types

One of the very important theorems of model theory is the omitting types theorem. It characterises theories, models of which are not consistent with given sets of formulae. With the help of the concrete version of the omitting types theorem we show that for every low concrete model $\mathcal{A}$ of Zermelo–Fraenkel set theory there exists a concrete model $\mathcal{B}$ and concrete elementary end embedding $f : \mathcal{A} \to \mathcal{B}$.

We begin with some basic definitions.

Definition 4.2.19 *Let $\Sigma = \Sigma(x_1, \ldots, x_m)$ be a set of formulae with all free variables contained in $\{x_1, \ldots, x_m\}$, let $\mathcal{A}$ be a concrete model and let $a_1, \ldots, a_m \in |\mathcal{A}|$.*

We say that elements $a_1, \ldots, a_m$ realise Σ in $\mathcal{A}$ if for every $\sigma \in \Sigma$, $\mathcal{A} \models \sigma[a_1, \ldots, a_m]$.

We say that $\mathcal{A}$ realises Σ if there are elements $a_1, \ldots, a_m$ which realise Σ in $\mathcal{A}$.

If $\mathcal{A}$ does not realise Σ we say that $\mathcal{A}$ omits Σ.

An interesting question is whether a theory T has a concrete model which realises or omits a given set of formulae Σ. We have the following characterisation of theories realising sets of formulae.

Theorem 4.2.20 (Realising Types Theorem) *Let T be a theory and let $\Sigma(x_1, \ldots, x_m)$ be a set of formulae. Assume that $T \oplus \Sigma$ is low. The following are equivalent:*

1. *T has a concrete model realising Σ,*
2. *Every finite subset of Σ_0 of Σ is realised in some concrete model of T*
3. *$T \cup \{\exists x_1, \ldots, x_m\, \sigma_1 \wedge \ldots \wedge \sigma_n : n \in \omega, \sigma_1, \ldots, \sigma_n \in \Sigma\}$ is consistent.*

Proof: The implication from 1 to 2 is obvious.

For the implication from 2 to 3 suppose that every finite subset Σ_0 of Σ is realised in some concrete model of T. Suppose for the sake of contradiction that $T \cup \{\exists x_1, \ldots, x_m\, \sigma_1 \wedge \ldots \wedge \sigma_n : n \in \omega, \sigma_1, \ldots, \sigma_n \in \Sigma\}$ inconsistent. Then there exists a finite set $S_0 \subseteq \{\exists x_1, \ldots, x_m\, \sigma_1 \wedge \ldots \wedge \sigma_n : n \in \omega, \sigma_1, \ldots, \sigma_n \in \Sigma\}$ such that $T \vdash \neg \bigwedge S_0$. Let Σ_0 be the set of all $\sigma \in \Sigma$ occurring in S_0. The set Σ_0 is finite and it is contained in Σ, therefore by 2 there is a concrete model $\mathcal{A}$ and $\overline{a} \in |\mathcal{A}|$ such that $\mathcal{A} \models T$ and $\mathcal{A} \models \Sigma_0[\overline{a}]$. Hence $\mathcal{A} \models S_0$, which is a contradiction.

For the implication from 3 to 1 suppose that theory $T \cup \{\exists x_1, \ldots, x_m\, \sigma_1 \wedge \ldots \wedge \sigma_n : n \in \omega, \sigma_1, \ldots, \sigma_n \in \Sigma\}$ is consistent. Let $c_1, \ldots, c_m$ be new constants. Consider theory $S = T \cup \Sigma(c_1, \ldots, c_m)$. If S was not consistent, then there would be finite $\Sigma_0(c_1, \ldots, c_m)$ such that $T \vdash \neg \bigwedge \Sigma_0(c_1, \ldots, c_m)$. Since constants $c_1, \ldots, c_m$ do not occur in T, by lemma on constants, we have $T \vdash \forall x_1, \ldots, x_m \neg \bigwedge \Sigma_0(x_1, \ldots, x_m)$. But by the assumption $T \cup \{\exists x_1, \ldots, x_m \bigwedge \Sigma_0(x_1, \ldots, x_m)\}$ is consistent, which means that $T \nvdash \neg\exists x_1, \ldots, x_m \bigwedge \Sigma_0(x_1, \ldots, x_m)$ which is a contradiction. Therefore S is consistent.

Since $T \oplus \Sigma$ is low, S is low and $\mathrm{Cn}(S)$ is concrete.

Hence by the Concrete Completeness Theorem (4.2.10) there is a concrete model $\mathcal{A}$ of S. Therefore $\mathcal{A} \models T$ and there are $a_1, \ldots a_m \in |\mathcal{A}|$ such that $\mathcal{A} \models \Sigma[a_1, \ldots, a_m]$. □

In fact in the proof of the implication from 3 to 1, we could use the Low Completeness Theorem to S to obtain a low concrete model $\mathcal{A}$ realising Σ.

Definition 4.2.21 *Let $\Sigma = \Sigma(x_1, \ldots, x_m)$ be a concrete set of formulae, let $\varphi \in \Sigma$ and let T be such a theory that $\mathrm{Cn}(T)$ is concrete.*

We say that φ is consistent with T if there is a concrete model of T realising $\{\varphi\}$.

We say that Σ is consistent with T if there is a concrete model of T realising Σ.

Definition 4.2.22 *Let T be such a theory that* $\mathrm{Cn}(T)$ *is concrete and let* $\Sigma = \Sigma(x_1, \ldots, x_m)$ *be a concrete set of formulae.*

We say that T locally realises Σ if there exists a formula $\varphi = \varphi(x_1, \ldots, x_m)$ such that φ is consistent with T and for all $\sigma \in \Sigma$ it holds that $T \vdash \forall x_1, \ldots, x_m\, (\varphi \Rightarrow \sigma)$.

We say that T locally omits Σ if T does not locally realise Σ.

Note that verifying whether even a recursive theory locally omits a recursive set of formulae is computationally hard – in general it is expressed by a Π_3 sentence – standard algorithms to decide the truth value for this class of sentences are too complex for the concrete models framework. On the other hand, as we show below, if a theory T is such that $\mathrm{Cn}(T)$ is concrete and T locally omits a set $\Sigma(x_1, \ldots, x_m)$, then there is a concrete model of T which omits Σ. We may therefore restrict our interest to pairs of theories and sets of formulae for which checking whether locally omitting holds is Δ^0_2 or treat it as an external condition for the existence of a concrete model of a theory omitting a given set of formulae.

We want to prove a version of the Omitting Types Theorem for the concrete models framework. One of the proofs of the standard version of the theorem is based on a construction of a model on constants and is a generalisation of the proof of the completeness theorem. We have transferred the latter to concrete models as the Concrete Completeness Theorem (4.2.10). The proof goes by constructing such a Henkin's completion of a given theory that satisfies the witness property, completeness and consistency of the resulting theory. Moreover, there is another step in this Henkin's completion which guarantees that the resulting model omits the given set of formulae.

Obviously in the concrete models framework we cannot consider arbitrary theories since some of them may be computationally too hard to have concrete models, but it is sufficient to restrict our attention to theories satisfying the assumptions of the Concrete Completeness Theorem (4.2.10) i.e. to such theories T for which $\mathrm{Cn}(T)$ is a concrete set.

Let σ be a recursive concrete vocabulary and let C be the set of constants from σ. Let D be a recursive set of new constants with an effective presentation $d_0, d_1, \ldots$, let $\sigma' = \sigma \cup D$. Let $\psi_0, \psi_1, \ldots$ be an effective presentation of σ'–sentences and let $\varphi_0, \varphi_1, \ldots$ be an effective presentation of σ'–formulae with only one free variable v_0. Let $\overline{d}_0, \overline{d}_1, \ldots$ be an effective presentation of sequences of indices of constants from D such that for $i \in \omega$, $\mathrm{lh}(\overline{d}_i) = m$. For $k \in \omega$, if $\overline{d}_k = j_1, \ldots, j_m$, then by $\overline{v}_k$ we denote $v_{j_1}, \ldots, v_{j_m}$ and $\overline{d}_k = \mathfrak{c}_{j_1}, \ldots, \mathfrak{c}_{j_m}$.

Definition 4.2.23 (Typification of a Sentence) *Let $\psi(d_{i_1}, \ldots, d_{i_n})$ be a σ'–sentence and $j_1, \ldots, j_m \in \omega$. A typification $\beta_\psi^{j_1, \ldots, j_m}$ is a formula obtained by:*

1. *replacing every constant $\mathfrak{c}_k$ in ψ by a variable v_k and renaming bounded variables if necessary – denote this formula by $\gamma_\psi^{j_1,\dots,j_m}$,*

2. *quantifying existentially every free variable occurring in $\gamma_\psi^{j_1,\dots,j_m}$ with indices not in $\{j_1,\dots,j_m\}$.*

Note that $\mathrm{FV}(\beta_\psi^{j_1,\dots,j_m}) \subseteq \{v_{j_1},\dots,v_{j_m}\}$.

Lemma 4.2.24 *Let T be a consistent theory and $\alpha(c_{i_1},\dots,c_{i_n})$ a sentence consistent with T. Then $\alpha \wedge \beta_\alpha^{\overline{d}}$ is consistent with T.*

Proof: Let T and α be as in the assumptions of the lemma. Then $T \cup \{\alpha\}$ is consistent. Note that by the construction of $\beta_\alpha^{\overline{d}}$ it holds that $\alpha \vdash \exists\overline{v}\,\beta_\alpha^{\overline{d}}$. Therefore $T \cup \{\alpha\} \vdash \exists\overline{v}\,\beta_\alpha^{\overline{d}}$ and since it is consistent, $T \cup \{\alpha\} \nvdash \forall\overline{v}\,\neg\beta_\alpha^{\overline{d}}$. The last statement is equivalent to $T \nvdash \forall\overline{v}\,(\alpha \Rightarrow \neg\beta_\alpha^{\overline{d}})$, thus $\alpha \wedge \beta_\alpha^{\overline{d}}$ is consistent with T. □

Lemma 4.2.25 *Let T be a consistent theory, $\Sigma(x_1,\dots,x_m)$ be a set of formulae, and α be a sentence consistent with T. If T locally omits Σ, then there exists $\sigma \in \Sigma$ such that $T \nvdash \forall\overline{v}\,((\alpha \wedge \beta_\alpha^{\overline{d}}) \Rightarrow \sigma(\overline{v}))$.*

Proof: Let T, Σ and α be as in the assumptions of the lemma. Then by Lemma 4.2.24 $\alpha \wedge \beta_\alpha^{\overline{d}}$ is consistent with T and by the fact that T locally omits Σ, there exists $\sigma \in \Sigma$ such that $T \nvdash \forall\overline{v}\,((\alpha \wedge \beta_\alpha^{\overline{d}}) \Rightarrow \sigma(\overline{v}))$. □

Definition 4.2.26 (Type Omitting Henkin's Completion)
Let $\Sigma = \Sigma(x_1,\dots,x_m)$ be a concrete set and let T be a consistent theory locally omitting Σ such that $\mathrm{Cn}(T)$ is concrete.

Define a sequence α_n of formulae as follows:

- *For $n = 3k$*
$$\alpha_n = \begin{cases} \psi_k & \text{if } T \nvdash \neg(\alpha_0 \wedge \dots \wedge \alpha_{n-1} \wedge \psi_k) \\ \neg\psi_k & \text{else} \end{cases}$$

- *For $n = 3k+1$*
$$\alpha_n = \exists v_0\,\varphi_k(v_0) \Rightarrow \varphi_k(d_i),$$
for i being the least natural number such that d_i does not occur in $\alpha_0, \alpha_1, \dots, \alpha_{n-1}, \varphi_k$.

- *For $n = 3k+2$ let $\beta_{\alpha_0\wedge\dots\wedge\alpha_{n-1}}^{\overline{d}_k}$ be a typification of $\alpha_0 \wedge \dots \wedge \alpha_{n-1}$.*
$$\alpha_n = \neg\sigma(\overline{d}_k),$$
where $\sigma \in \Sigma$ is a formula with the least Gödel number such that $T \nvdash \forall\overline{v}_k\,((\alpha_0 \wedge \ldots \wedge \alpha_{n-1} \wedge \beta_{\alpha_0\wedge\ldots\wedge\alpha_{n-1}}^{\overline{d}_k}) \Rightarrow \sigma(\overline{v}_k))$. Existence of such σ is guaranteed by Lemma 4.2.25.

We call the set $\mathrm{TOHC}(T) = \{\alpha_n \colon n \in \omega\}$ *the type omitting Henkin's completion of* T *(with respect to constants* D*). For* $n \in \omega$ *we call the set* $\mathrm{TOHC}_n(T) = \{\alpha_i \colon i < n\}$ *the* n*–th cut of type omitting Henkin's completion of* T *(with respect to constants* D*).*

From now on, we work with a fixed concrete set of formulae $\Sigma = \Sigma(x_1, \ldots, x_m)$ and a consistent theory T in σ such that $\mathrm{Cn}(T)$ is concrete and which locally omits Σ. Our aim is to construct a concrete model of T by means of Lemma 4.2.1. We therefore need a concrete $\mathrm{CCW}(\sigma', D)$ containing T and such that the concrete model obtained by means of Lemma 4.2.1 omits Σ. Some part of the work has been already done in the section on completeness theorems. Let us proceed to lemmata about $\mathrm{TOHC}(T)$.

Lemma 4.2.27 $T \subseteq \mathrm{TOHC}(T)$.

Lemma 4.2.28 $\mathrm{TOHC}(T)$ *is complete in* σ'.

The two lemmata above state that $\mathrm{TOHC}(T)$ is a completion of T and their proofs are analogous to the proofs of Lemmata 4.2.4 and 4.2.5.

The proof of consistency of the type omitting Henkin's completion is harder than the regular Henkin's completion.

Lemma 4.2.29 $\mathrm{TOHC}(T)$ *is consistent.*

Proof: The proof structure is analogous to the proof of Lemma 4.2.6 – by induction on n we show that for each $n \in \omega$ theory $T \cup \mathrm{TOHC}_n(T)$ is consistent.

The base step for $n = 0$ is obvious since T is consistent.

For the induction hypothesis suppose that $T \cup \mathrm{TOHC}_n(T)$ is consistent. There are three cases: $n = 3k$, $n = 3k + 1$ and $n = 3k + 2$ but proofs for $n = 3k$ and $n = 3k + 1$ are analogous to proofs of $n = 2k$ and $n = 2k + 1$ cases in the proof of Lemma 4.2.13.

It remains to show the lemma in the case of $n = 3k + 2$, when $\alpha_n = \neg\sigma(\overline{d}_k)$, where $\sigma \in \Sigma$ is the least formula in sense of Gödel number such that $T \not\vdash \forall \overline{v}_k\, ((\alpha_0 \wedge \ldots \wedge \alpha_{n-1} \wedge \beta^{\overline{d}_k}_{\alpha_0 \wedge \ldots \wedge \alpha_{n-1}}) \Rightarrow \sigma(\overline{v}_k))$. Suppose for a contradiction that $T \vdash (\alpha_0 \wedge \ldots \wedge \alpha_{n-1}) \Rightarrow \sigma(\overline{d}_k)$. Then by the lemma on constants $T \vdash \forall \overline{x}, \overline{y}\, \gamma^{\overline{d}_k}_{\alpha_0 \wedge \ldots \wedge \alpha_{n-1}}(\overline{x}, \overline{y}) \Rightarrow \sigma(\overline{y})$. But on the other hand it holds that $T \cup \{\alpha_0 \wedge \ldots \wedge \alpha_{n-1}\} \not\vdash \forall \overline{v}_k\, (\exists \overline{x}\, \gamma^{\overline{d}_k}_{\alpha_0 \wedge \ldots \wedge \alpha_{n-1}}) \Rightarrow \sigma(\overline{v}_k)$, which is equivalent to $T \cup \{\alpha_0 \wedge \ldots \wedge \alpha_{n-1}\} \not\vdash \forall \overline{v}_k\, \forall \overline{x}\, (\gamma^{\overline{d}_k}_{\alpha_0 \wedge \ldots \wedge \alpha_{n-1}} \Rightarrow \sigma(\overline{v}_k))$. This contradicts the assumption that $\sigma(\overline{d}_k)$ is inconsistent with $T \cup \{\alpha_0 \wedge \ldots \wedge \alpha_{n-1}\}$. Therefore $\mathrm{TOHC}_{n+1}(T) = \mathrm{TOHC}_n(T) \cup \{\neg\sigma(\overline{d}_k)\}$ is consistent.

Since every finite subset of $\mathrm{TOHC}(T)$ is contained in some consistent theory $T \cup \mathrm{TOHC}_n(T)$, $\mathrm{TOHC}(T)$ is also consistent. $\square$

To prove that TOHC(T) is CCW(σ', D) it remains to show that it has witness property. This is shown by the following lemma.

Lemma 4.2.30 TOHC(T) *has witness property for* σ'*–sentences in* D*.*

This is guaranteed by the step $3k + 1$ of the construction of TOHC(T).

We proceed with two lemmata showing that the type omitting Henkin's completion of T is a concrete theory. This is shown by presenting algorithms with recursively enumerable oracle one of which computes the map $n \mapsto \ulcorner \alpha_n \urcorner$ and the other decides TOHC(T). The algorithms are similar to ones presented in the proofs of Lemmata 4.2.8 and 4.2.9 but they have to deal with additional case in the definition of sequence $(\alpha_n)_{n\in\omega}$.

Lemma 4.2.31 *The map* $n \mapsto \ulcorner \alpha_n \urcorner$ *is recursive in* $\mathrm{Cn}(T) \oplus \Sigma$*, therefore concrete.*

Proof: The algorithm uses the same data structures as Algorithm 8 presented in the proof of Lemma 4.2.8. The cases of $n = 3k$ and $n = 3k + 1$ are dealt with in an analogous way as the cases $n = 2k$ and $n = 2k + 1$ in the above–mentioned proof, thus it remains to show how to compute $\ulcorner \alpha_n \urcorner$ in the case of $n = 3k + 2$.

The general algorithm is presented below. Let $\sigma_0, \sigma_1, \ldots$ be an effective in Σ presentation of Σ.

Algorithm 10 Algorithm enumerating TOHC(T)

Input: $n \in \omega$
Output: $\ulcorner \alpha_n \urcorner$

$A \leftarrow Array()$
$B \leftarrow Set()$
$i \leftarrow 0$
while $i \leqslant n$ **do**
  if $i > 0$ **then**
    $B.insert(Consts(A[i-1]))$
  end if
  $k \leftarrow \lfloor i/3 \rfloor$
  if $3|i$ **then**
    if $T \vdash \neg(A[0] \wedge \cdots \wedge A[k-1] \wedge \psi_k)$ **then**
      $A[i] \leftarrow \neg\psi_k$
    else
      $A[i] \leftarrow \psi_k$
    end if
  end if
  if $3|i+2$ **then**
    $j \leftarrow 0$
    while true do
      if $j \notin B$ and $j \notin Consts(\varphi_k)$ **then**
        break
      end if
      $j \leftarrow j+1$
    end while
    $A[i] \leftarrow \exists v_0\, \varphi_k(v_0) \Rightarrow \varphi_k(d_j)$
  end if
  if $3|i+1$ **then**
    $j \leftarrow 0$
    while $T \vdash \forall \overline{v}_k\, ((A[0] \wedge \ldots \wedge A[n-1] \wedge \beta^{\overline{d}_k}_{A[0]\wedge\ldots\wedge A[i-1]}) \Rightarrow \sigma_j(\overline{v}_k))$ **do**
      $j \leftarrow j+1$
    end while
    $A[i] \leftarrow \neg\sigma_j(\overline{d}_k)$
  end if
  return $\ulcorner A[n] \urcorner$
end while

Note that computing $\beta^{\overline{d}_k}_{A[0]\wedge\ldots\wedge A[i-1]}$ in the case of $n = 3k+2$ requires only syntactic transformations i.e. is recursive. Similarly, by the facts that T locally omits Σ and T is consistent with $\alpha_0 \wedge \ldots \wedge \alpha_{n-1} \wedge \beta^{\overline{d}_k}_{A[0]\wedge\ldots\wedge A[i-1]}$ it holds that there exists $\sigma \in \Sigma$ such that $\neg\sigma$ is consistent with $T \cup \mathrm{TOHC}_n(T)$.

Therefore, the while–loop in this case always ends and thus the algorithm always halts.

Since $T \vdash \psi$ is recursive with recursively enumerable oracle as $\mathrm{Cn}(T)$ is concrete, Algorithm 10 is recursive with recursively enumerable oracle that is $\mathrm{Cn}(T) \oplus \Sigma$ and therefore the map $n \mapsto \ulcorner \alpha_n \urcorner$ is concrete. $\Box$

Lemma 4.2.32 $\mathrm{TOHC}(T)$ *is recursive in* $\mathrm{Cn}(T) \oplus \Sigma$*, therefore concrete.*

To decide $\mathrm{TOHC}(T)$ Algorithm 9 presented in the proof of Lemma 4.2.9 is sufficient.

Theorem 4.2.33 (Concrete Omitting Types Theorem) *Let σ be a recursive concrete vocabulary. Let T be a consistent theory in σ such that $\mathrm{Cn}(T)$ is concrete and let $\Sigma(x_1, \ldots, x_m)$ be a concrete set of formulae. Let R be a recursive set such that $D = \{\mathfrak{c}_r : r \in R\}$ be a recursive set of new constants and let $\sigma' = \sigma \cup D$. If T locally omits Σ, then there is a concrete σ'–model $\mathcal{A} \models T$ omitting Σ. Moreover $\mathcal{A}$ is recursive in $\mathrm{Cn}(T) \oplus \Sigma$ and $|\mathcal{A}| \subseteq R$.*

Proof: Let σ, D, σ', T and Σ be as in the assumptions of the theorem.

Then, by Lemma 4.2.32 type omitting Henkin's completion $\mathrm{TOHC}(T)$ of theory T is recursive in $\mathrm{Cn}(T) \oplus \Sigma$ and by Lemmata 4.2.28, 4.2.29 and 4.2.30 it is $\mathrm{CCW}(\sigma', D)$. By Lemma 4.2.1 applied to $\mathrm{TOHC}(T)$ we get a recursive in $\mathrm{Cn}(T) \oplus \Sigma$ concrete σ'–model $\mathcal{A}$ such that $\mathcal{A} \models \mathrm{TOHC}(T)$ and $|\mathcal{A}| \subseteq R$. Therefore by Lemma 4.2.27 $\mathcal{A}' \models T$.

It remains to show that $\mathcal{A}$ omits Σ i.e. for every $a_1, \ldots, a_m \in |\mathcal{A}|$ there exists $\tau \in \Sigma$ such that $\mathcal{A} \models \neg\tau[a_1, \ldots, a_m]$. Fix a tuple $a_1, \ldots, a_m \in |\mathcal{A}|$. Then a sequence $d_{a_1}, \ldots, d_{a_m}$ occurs in the presentation $\overline{d}_0, \overline{d}_1, \ldots$ of all m–tuples of constants from D – let l be the index of $d_{a_1}, \ldots, d_{a_m}$ in this sequence. Then $\alpha_{3l+2} = \neg\tau(d_{a_1}, \ldots, d_{a_m})$ for some $\tau \in \Sigma$. Therefore $\neg\tau(d_{a_1}, \ldots, d_{a_m}) \in \mathrm{TOHC}(T)$ and $\mathcal{A} \models \neg\tau[a_1, \ldots, a_m]$. Thus $a_1, \ldots, a_m$ does not realise Σ in $\mathcal{A}$ and since the choice of $a_1, \ldots, a_m \in |\mathcal{A}|$ was arbitrary – $\mathcal{A}$ omits Σ. $\Box$

The Concrete Omitting Types Theorem can be strengthened to the version with countable number of types. This is shown by the following theorem which helps us prove the main theorem of this section stating that for every low concrete model $\mathcal{A}$ of ZF there exists a concrete model $\mathcal{B}$ and a concrete elementary end embedding $f\colon (\mathcal{A}, a)_{a \in \mathcal{A}} \to \mathcal{B}$.

Theorem 4.2.34 (Concrete Countable Omitting Types Theorem) *Let σ be a recursive concrete vocabulary and let T be a consistent σ–theory such that $\mathrm{Cn}(T)$ is concrete. For each $i \in \omega$ let the set of formulae $\Sigma_i(x_1, \ldots, x_{m_i})$ be concrete. Let the map $i \mapsto \ulcorner \Sigma_i \urcorner$ be concrete. For $i \in \omega$, let R_i be a recursive set such that $D_i = \{\mathfrak{c}_r : r \in R_i\}$ is a recursive set of new constants such that $\bigcup_{i \in \omega} D_i$ is recursive. Let $\sigma' = \sigma \cup D$. If T locally omits*

Σ_i *for each* $i \in \omega$*, then there is a concrete* σ'*–model* $\mathcal{A} \models T$ *omitting* Σ_i *for each* $i \in \omega$*.*

Proof: The proof is similar to the proof of the Concrete Omitting Types Theorem (4.2.33). Suppose the assumptions of the theorem hold.

Let $\overline{d}^i_0, \overline{d}^i_1, \ldots$ be an effective presentation of m_i–tupes of constants from D. We change the definition of α_n in the case of $n = 3k + 2$ – we put $\alpha_n = \bigwedge_{i=0,\ldots,k} \neg\sigma^i(\overline{d}^k_{k-i})$, where $\sigma^i \in \Sigma_i$ is a formula with the least Gödel number such that for $i = 0, \ldots, k$ it holds that

$$T \not\vdash \forall \overline{v}_k \left((\alpha_0 \wedge \ldots \wedge \alpha_{n-1} \wedge \bigwedge_{j<i} \neg\sigma^j(\overline{d}^j_{k-j}) \wedge \beta^{\overline{d}^j_{k-j}}_{\alpha_0 \wedge \ldots \wedge \alpha_{n-1}}\right) \Rightarrow \sigma^i(\overline{v}^i_{k-i})).$$

Existence of such σ^i is guaranteed by Lemma 4.2.25. Thus, $\mathrm{TOHC}'(T)$, defined in such a way has the property that for every $i \in \omega$ and every m_i–tuple of constants $\overline{d}^i_j$ there exists $\sigma^i \in \Sigma_i$ such that $\neg\sigma^i(\overline{d}^i_j) \in \mathrm{TOHC}(T)$. By the argument analogous to one from proof of Theorem 4.2.33 we obtain a concrete σ'–model $\mathcal{A} \models T$ and $\mathcal{A}$ omits Σ_i for all $i \in \omega$. □

Now, using the Concrete Omitting Types Theorem we will show that for every low concrete model $\mathcal{A}$ of Zermelo–Fraenkel set theory there exists a concrete model $\mathcal{B}$ and a concrete elementary end embedding $f\colon \mathcal{A} \to \mathcal{B}$. First we define a concrete end embedding.

Definition 4.2.35 *Let* $\mathcal{A}$*,* $\mathcal{B}$ *be concrete models of Zermelo–Fraenkel set theory. An embedding* $f\colon \mathcal{A} \to \mathcal{B}$ *is a concrete end embedding, in symbols* $\mathcal{A} \hookrightarrow_{end} \mathcal{B}$ *if* f *is a concrete embedding and for every* $a \in |\mathcal{A}|, b \in |\mathcal{B}|$ *if* $\mathcal{B} \models b \in f(a)$*, then* $b = f(d)$*, for some* $d \in |\mathcal{A}|$*.*

A concrete end embedding is proper if it is not onto.

Theorem 4.2.36 *Let* $\mathcal{A}$ *be a low concrete model of Zermelo–Fraenkel set theory. Then there is a concrete model* $\mathcal{B}$*, and a proper elementary end embedding* f *from* $\mathcal{A}$ *to* $\mathcal{B}$*.*

Proof: Fix a low concrete σ–model $\mathcal{A} \models ZF$. Let $C_A = \{c_a\colon a \in |\mathcal{A}|\}$ be new constants from $\mathrm{ElDiag}(\mathcal{A})$ and let c be a constant such that $c \notin C_A$ and $c \notin \sigma$. Let $\sigma' = \sigma \cup C_A \cup \{c\}$. Let $T = \mathrm{ElDiag}(\mathcal{A}) \cup \{c \notin c_a\colon a \in |\mathcal{A}|\}$ be a theory in σ'.

First we show that T is consistent. Suppose for the sake of contradiction that T is not consistent. Then there is a finite sequence $a_1, \ldots, a_n \in |\mathcal{A}|$ such that $\mathrm{ElDiag}(\mathcal{A}) \vdash \neg \bigwedge_{i=1,\ldots,n} c \notin c_{a_i}$. By the lemma on constants, since c does not occur in $\mathrm{ElDiag}(\mathcal{A})$, the following holds $\mathrm{ElDiag}(\mathcal{A}) \vdash \forall x \neg \bigwedge_{i=1,\ldots,n} x \notin c_{a_i}$. The last statement implies that $\mathrm{ElDiag}(\mathcal{A}) \vdash \forall x \bigvee_{i=1,\ldots,n} x \in c_{a_i}$. This means that every set (as an element of the universe) in $(\mathcal{A}, a)_{a\in|\mathcal{A}|}$ is contained in $\bigcup_{i=1,\ldots,n} c_{a_i}$. We have $ZF \vdash \forall x\, x \notin x$, therefore also $\mathrm{ElDiag}(\mathcal{A}) \vdash$

$\forall x\, x \notin x$. But we also have $\mathrm{ElDiag}(\mathcal{A}) \vdash \bigcup_{i=1,\dots,n} c_{a_i} \in \bigcup_{i=1,\dots,n} c_{a_i}$ which is a contradiction, since $\mathrm{ElDiag}(\mathcal{A})$ is consistent.

We claim that a formula $\varphi(x,c)$ is consistent with T if and only if $\mathrm{ElDiag}(\mathcal{A}) \vdash \forall y\, \exists x\, \exists z\, (z \notin y \wedge \varphi(x,z))$.

For the right–to–left implication suppose that

$$\mathrm{ElDiag}(\mathcal{A}) \vdash \forall y\, \exists x\, \exists z\, (z \notin y \wedge \varphi(x,z)).$$

Suppose for the sake of contradiction that $\varphi(x,c)$ is not consistent with T. By arguments similar to those used in the proof of consistency of T, there is a finite sequence $a_1,\dots,a_n \in |\mathcal{A}|$ such that it holds that

$$\mathrm{ElDiag}(\mathcal{A}) \vdash \forall x\, \forall z\, (z \in \bigcup_{i=1,\dots,n} c_{a_i} \vee \neg\varphi(x,z)).$$

Therefore $\mathrm{ElDiag}(\mathcal{A}) \vdash \exists y\, \forall x\, \forall z\, (z \in y \vee \neg\varphi(x,z))$ by the completeness of $\mathrm{ElDiag}(\mathcal{A})$. This implies that $\mathrm{ElDiag}(\mathcal{A}) \vdash \neg\forall y\, \exists x\, \exists z\, (z \notin y \wedge \varphi(x,z))$. This contradicts the consistency of $\mathrm{ElDiag}(\mathcal{A})$.

For the converse suppose that

$$\mathrm{ElDiag}(\mathcal{A}) \nvdash \forall y\, \exists x\, \exists z\, (z \notin y \wedge \varphi(x,z)).$$

Then, by the completeness of $\mathrm{ElDiag}(\mathcal{A})$ it holds that

$$\mathrm{ElDiag}(\mathcal{A}) \vdash \exists y\, \forall x\, \forall z\, (\varphi(x,z) \Rightarrow z \in y).$$

Therefore there is $a \in |\mathcal{A}|$ such that

$$\mathrm{ElDiag}(\mathcal{A}) \vdash \forall x\, \forall z\, (\varphi(x,z) \Rightarrow z \in c_a).$$

This entails that $\mathrm{ElDiag}(\mathcal{A}) \vdash \forall x\, (\varphi(x,c) \Rightarrow c \in c_a)$. On the other hand, we have $T \vdash c \notin c_a$, thus $T \vdash \forall x\, \neg\varphi(x,c)$ and therefore $\varphi(x,c)$ is inconsistent with T. This ends the proof of the claim.

For $a \in |\mathcal{A}|$ we define:

$$\Sigma_a(x) = \{x \in c_a\} \cup \{x \neq c_b \colon \mathrm{ElDiag}(\mathcal{A}) \vdash c_b \in c_a\}.$$

We show that for every $a \in |\mathcal{A}|$ the theory T locally omits $\Sigma_a(x)$. Fix $a \in |\mathcal{A}|$. Let $\varphi(x,c)$ be an arbitrary formula consistent with T. If $\varphi(x,c) \wedge x \notin c_a$ is consistent with T then we are done. Suppose then that $\varphi(x,c) \wedge x \notin c_a$ is not consistent with T. Then $\varphi(x,c) \wedge x \in c_a$ is consistent with T i.e. $\mathrm{ElDiag}(\mathcal{A}) \vdash \forall y\, \exists x\, \exists z\, (z \notin y \wedge \varphi(x,z) \wedge x \in c_a)$.

We want to show that $\mathrm{ElDiag}(\mathcal{A}) \vdash \exists x\, \forall y\, \exists z\, (z \notin y \wedge \varphi(x,z) \wedge x \in c_a)$. In order to do so, we show that

$$ZF \vdash \neg\exists x\, \forall y\, \exists z\, (z \notin y \wedge \varphi(x,z) \wedge x \in c_a) \Rightarrow \neg\forall y\, \exists x\, \exists z\, (z \notin y \wedge \varphi(x,z) \wedge x \in c_a).$$

We perform the argument in ZF. Fix some set A. Suppose that

$$\forall x\, \exists y\, \forall z\, (z \in y \vee \neg\varphi(x,z) \vee x \notin A).$$

This is equivalent to

$$\forall x \in A\, \exists y\, \forall z\, (\varphi(x,z) \Rightarrow z \in y).$$

For $x \in A$ let

$$Y_x = \bigcap\{y \colon \forall z\, (\varphi(x,z) \Rightarrow z \in y)\},$$

which is a set since Y_x is not empty by the assumption. Now let $Y = \bigcup_{x\in A} Y_x$. We show that for every $x \in A$ and z such that $\varphi(x,z)$, $z \in Y$ hold. Fix $x \in A$ and z such that $\varphi(x,z)$ holds. Let y be such that $\forall w\, (\varphi(x,w) \Rightarrow w \in y)$. Then $z \in y$ and therefore also $z \in Y_x \subseteq Y$. Therefore,

$$\exists y\, \forall z\, \forall x \in A\, (\varphi(x,z) \Rightarrow z \in y)$$

which is equivalent to

$$\exists y\, \forall z\, \forall x\, (z \in y \vee \neg\varphi(x,z) \vee x \notin A).$$

Therefore, the following implication holds: if $\mathrm{ElDiag}(\mathcal{A}) \vdash \forall x\, \exists y\, \forall z\, (z \in y \vee \neg\varphi(x,z) \vee x \notin c_a)$, then $\mathrm{ElDiag}(\mathcal{A}) \vdash \exists y\, \forall z\, \forall x\, (z \in y \vee \neg\varphi(x,z) \vee x \notin c_a)$. But the negation of the consequent of this implication holds. Hence, $\mathrm{ElDiag}(\mathcal{A}) \vdash \exists x\, \forall y\, \exists z\, (z \notin y \wedge \varphi(x,z) \wedge x \in c_a)$.

Therefore there is $b \in |\mathcal{A}|$ such that $\varphi(c_b, c) \wedge c_b \in c_a$ is consistent with T and thus $\varphi(x,c) \wedge x \in c_a$ is also consistent with with T. Therefore for each $a \in |\mathcal{A}|$ theory T locally omits Σ_a.

For every $a \in |\mathcal{A}|$ the set $\Sigma_a(x)$ is recursive in $\mathcal{A}$, thus low. Moreover, there is an effective in $\mathcal{A}$ presentation of $\{\ulcorner\Sigma_a(x)\urcorner \colon a \in |\mathcal{A}|\}$.

Since T is a low theory, $\mathrm{Cn}(T)$ is concrete. The set $\{\ulcorner\Sigma_a(x)\urcorner \colon a \in |\mathcal{A}|\}$ can be enumerated concretely and for each $a \in |\mathcal{A}|$ the theory T locally omits Σ_a. Therefore, by the Concrete Countable Omitting Types Theorem (4.2.34), there exists a concrete σ'–model $\mathcal{B}' \models T$ such that $\mathcal{B}'$ omits Σ_a for each $a \in |\mathcal{A}|$.

Observe that by Lemma 4.1.17 there exists a concrete embedding f of the concrete σ'–model $(\mathcal{A}, a)_{a\in|\mathcal{A}|}$ to $\mathcal{B}'$, since $\mathcal{B}' \models \mathrm{Diag}(\mathcal{A})$. This concrete embedding is elementary since $\mathcal{B}' \models \mathrm{ElDiag}(\mathcal{A})$. Since $\mathcal{B}' \models c \notin c_a$ for every $a \in |\mathcal{A}|$, it holds that $\mathcal{B}' \models c \neq c_a$, therefore f is not onto $\mathcal{B}'$.

For every $a \in |\mathcal{A}|$, $\mathcal{B}'$ omits Σ_a. Thus, f is an end embedding. Let $a \in |\mathcal{A}|$ and $b \in |\mathcal{B}|$ and suppose that $\mathcal{B}' \models b \in f(a)$. We want to find $d \in |\mathcal{A}|$ such that $b = f(d)$. By the construction of f, $f(a) = c_a^{\mathcal{B}'}$. Therefore $\mathcal{B}' \models b \in c_a$. Since $\mathcal{B}'$ omits Σ_a there is $\sigma_{a,b} \in \Sigma_a$ such that $\mathcal{B}' \models \neg\sigma_{a,b}[b]$. By the definition of Σ_a, $\sigma_{a,b}$ is either of the form $x \notin c_a$ or $x \neq c_d$ for some $d \in |\mathcal{A}|$ such that $\mathrm{ElDiag}(\mathcal{A}) \vdash c_d \in c_a$. The first case is impossible since $\mathcal{B}' \models b \in c_a$. Therefore let $d \in |\mathcal{A}|$ be such that $\mathcal{B}' \models b = c_d$. This is equivalent to $\mathcal{B}' \models b = f(d)$. Therefore, f is an end embedding.

□

Let us end this section with the following example. Let T be Peano arithmetic (and suppose that T is consistent) and let $\Sigma = \{x > \overline{n} \colon n \in \omega\}$. Both T and Σ are recursive and the only model of T which omits Σ is necessarily the standard model of arithmetic – $\mathcal{N}$, which is obviously not a concrete model, since the satisfaction relation in this model is not even arithmetical. Therefore T cannot locally omit Σ as otherwise by the Concrete Omitting Types Theorem (4.2.33) there would be a concrete model T which omits Σ. This is a very non–constructive proof of the fact that T locally realises Σ. To get a constructive proof of this fact one should show a formula $\varphi(x)$ consistent with T such that for all $\sigma \in \Sigma$ it holds that $T \vdash \forall x\, (\varphi(x) \Rightarrow \sigma(x))$ i.e. $\varphi(x)$ forces that x is not a standard element. The existence of such $\varphi(x)$ follows from the argument from the proof of the second Gödel's incompleteness theorem which states that $T \nvdash \mathrm{Con}_T$ i.e. $T \nvdash \forall x\, \neg \mathrm{Prov}_T(x, \ulcorner \bot \urcorner)$. Therefore $\mathrm{Prov}_T(x, \ulcorner \bot \urcorner)$ is consistent with T, but since for every $n \in \omega$ we have $T \nvdash \mathrm{Prov}_T(\ulcorner \bar{n} \urcorner, \ulcorner \bot \urcorner)$ it holds that $T \vdash \forall x\, (\mathrm{Prov}_T(x, \ulcorner \bot \urcorner) \Rightarrow x > \bar{n})$. This proves that for each $\sigma \in \Sigma$ it holds that $T \vdash (\mathrm{Prov}_T(x, \ulcorner \bot \urcorner) \Rightarrow \sigma(x))$ which literally means that T locally realises Σ.

4.2.3 Σ_n Chains of Concrete Models and Applications

First, recall an interesting result from the axiomatic model theory.

Theorem 4.2.37 (Chang–Keisler) *Let $n > 0$ and let ψ be a sentence. The following are equivalent:*

- *ψ is equivalent to a Σ_{n+1} sentence and to a Π_{n+1} sentence,*
- *ψ is equivalent to a Boolean combination of Σ_n sentences.*

First let us show that there is a recursively enumerable (with an empty oracle) algorithm such that given logically equivalent Σ_{n+1} sentence φ and Π_{n+1} sentence ψ, it browses all pairs (β, p), where β is a boolean combination of Σ_n sentences and p is a purely logical proof, and halts if and only if p is the proof of $\varphi \equiv \beta \wedge \psi \equiv \beta$. Let $\beta_0, \beta_1, \ldots$ be an effective presentation of boolean combinations of Σ_n sentences. Let $p_0, p_1, \ldots$ be an effective presentation of all purely logical proofs. The algorithm is presented below.

Algorithm 11 Algorithm searching for a boolean combination of Σ_n sentences

Input: sentences $\psi \in \Sigma_{n+1}, \varphi \in \Pi_{n+1}$ such that $\psi \equiv \varphi$
Output: $\theta \in \mathrm{Bool}(\Sigma_n)$ such that $\theta \equiv \varphi \equiv \psi$

```
i ← 0
while true do
  for j ← 0 to i do
    for k ← 0 to i do
      if p_k is a proof of (φ ≡ β_i ∧ ψ ≡ β_i) then
        return β_i
      end if
    end for
  end for
  i ← i + 1
end while
```

The algorithm halts on every proper ($\varphi \equiv \psi$, $\psi \in \Sigma_{n+1}$ and $\varphi \in \Pi_{n+1}$) input if and only if Theorem 4.2.37 holds. Moreover using the halting problem K as an oracle we can decide whether the algorithm halts on a given input and if the input is proper. Therefore for every pair of equivalent sentences ψ, φ such that ψ is Σ_{n+1} and φ is Π_{n+1} we can *concretely* find a Boolean combination θ of Σ_n sentences equivalent to both φ and ψ or decide that there is no such θ.

The classical proof of Theorem 4.2.37 in the axiomatic model theory goes through Σ_n chains of models. The aim of this section is to analyse the model–theoretic construction in this proof of the theorem. We show that the construction fails is the final step. We cannot expect the sum of a concrete Σ_n chain of concrete models to be a concrete model. In fact we only need the Π_{n+1} theory of such a sum to be consistent. However such Π_{n+1} theory is not necessarily concrete and moreover, there is no obvious way for showing its consistency. In the axiomatic model–theoretic version of the proof of the theorem this is provided by the existence of the sum of a Σ_n chain of models. However, we believe that the part of the construction up to the infelicitous step is itself interesting and we present it below.

First, let us define what we mean by a Σ_n chain in concrete models framework.

Definition 4.2.38 (Concrete Σ_n Chain of Concrete Models)
A concrete chain of concrete models $(\mathcal{A}_i)_{i\in\omega}$ is a concrete Σ_n chain if for every Σ_n formula $\psi(\overline{x})$, $i \in \omega$ and a finite sequence $\overline{a} \in |\mathcal{A}_i|$ if $\mathcal{A}_i \models \psi[\overline{a}]$, then $\mathcal{A}_{i+1} \models \psi[\overline{a}]$.

As we mentioned above, we cannot expect the sum of a concrete Σ_n chain of concrete models to be a concrete model. This is shown by the following theorem.

Theorem 4.2.39 *There is a recursive concrete Σ_1 chain $(\mathcal{A}_i)_{i\in\omega}$ of recursive concrete models such that $\bigcup_{i\in\omega}\mathcal{A}_i$ is not a concrete model.*

Proof: The proof goes by showing a chain of recursive models whose sum is the standard model of arithmetic. For $i \in \omega$ let φ_i FM–represent the set $\{0,\ldots,n\}$, φ^i_+ FM–represent the the $R_+ \cap \{0,\ldots,n\}^3$ and $\varphi^i_\times$ FM–represent $R_\times \cap \{0,\ldots,n\}^3$. We put $\mathcal{F}_i = (\varphi_i, \varphi^i_+, \varphi^i_\times)$ i.e. the i–th concrete structure FM–represents the i–th initial segment of the standard model of arithmetic $\mathcal{N}$. Each $\mathcal{F}_i$ is finite so there are recursive concrete models $\mathcal{A}_i = (\mathcal{F}_i, \varphi^i_{\models})$. Moreover, the map $i \mapsto \ulcorner\mathcal{A}_i\urcorner$ is recursive and $\bigcup_{i\in\omega}|\mathcal{A}_i| = \omega$ is also recursive. Therefore $(\mathcal{A}_i)_{i\in\omega}$ is a concrete chain of concrete models. Since concrete Σ_1 chains of concrete models are just concrete chains of concrete models, $(\mathcal{A}_i)_{i\in\omega}$ is a concrete Σ_1 chain of concrete models.

On the other hand $\mathcal{F} = \bigcup_{i\in\omega}\mathcal{F}_i = (\varphi_\omega, \varphi_+, \varphi_\times)$, where φ_ω FM–represents ω, φ_+ FM–represents R_+ and $\varphi_\times$ FM–represents $R_\times$. Therefore $\mathcal{F}$ FM–represents the standard model of arithmetic $\mathcal{N}$. Since the satisfaction relation in $\mathcal{N}$ is not even arithmetical there is no formula $\varphi_{\models}$ which FM–represents it. Therefore there is no concrete model $\mathcal{A} = (\mathcal{F}, \varphi_{\models})$. □

Since by Theorem 4.2.39 we cannot expect the sum of a concrete Σ_n chain of concrete models to be a concrete model, we want to avoid using such sums in our construction. In fact we only need to consider Π_{n+1} theory of such a sum. Therefore we introduce the following definition.

Definition 4.2.40 ($(n+1)$–th Universal Sentences of a Σ_n Chain) *Let $(\mathcal{A}_i)_{i\in\omega}$ be a concrete Σ_n chain of concrete models. The $(n+1)$–th set of universal sentences of $(\mathcal{A}_i)_{i\in\omega}$ is defined as follows:*

$$\Pi_{n+1}((\mathcal{A}_i)_{i\in\omega}) = \{\forall\overline{x}\,\psi(\overline{x})\colon \psi(\overline{x}) \in \Sigma_n \wedge \\ \forall\overline{a} \in \bigcup_{i\in\omega}|\mathcal{A}_i|\,\psi(\overline{a}) \in \bigcup_{i\in\omega}\Sigma_n\text{–}\mathit{Diag}(\mathcal{A}_i)\}.$$

Note that the $(n+1)$–th universal sentences of a concrete Σ_n chain of concrete models may be not a concrete set. We introduce it just for the presentation of the construction.

Lemma 4.2.41 *Let $(\mathcal{A}_i)_{i\in\omega}$ be a concrete Σ_n chain of concrete models. Then for every Π_{n+1} sentence ψ, if $\forall i \in \omega\, \mathcal{A}_i \models \psi$, then $\psi \in \Pi_{n+1}((\mathcal{A}_i)_{i\in\omega})$.*

Proof: Let $(\mathcal{A}_i)_{i\in\omega}$ be a concrete Σ_n chain of concrete models.

Let ψ be a Π_{n+1} sentence and suppose that for every $i \in \omega$ it holds that $\mathcal{A}_i \models \psi$. Then $\psi = \forall\overline{x}\,\psi'(\overline{x})$, for some Σ_n formula $\psi'(\overline{x})$.

For every $i \in \omega$ and every $\overline{a}_i \in |\mathcal{A}_i|$ it holds that $\mathcal{A}_i \models \psi'(\overline{a}_i)$ i.e. $\psi'(\overline{a}_i) \in \Sigma_n$–Diag$(\mathcal{A}_i)$. We show that $\psi \in \Pi_{n+1}((\mathcal{A}_i)_{i\in\omega})$. It is sufficient to show that for every $\overline{a} \in \bigcup_{i\in\omega}|\mathcal{A}_i|$ it holds that $\psi'(\overline{a}) \in \bigcup_{i\in\omega}\Sigma_n$–Diag$(\mathcal{A}_i)$. Fix $\overline{a} \in$

$\bigcup_{i \in \omega} |\mathcal{A}_i|$. Then there is $k \in \omega$ such that $\overline{a} \in |\mathcal{A}_k|$. But $\psi'(\overline{a}) \in \Sigma_n$–Diag$(\mathcal{A}_i) \subseteq \bigcup_{i \in \omega} \Sigma_n$–Diag$(\mathcal{A}_i)$, thus indeed $\psi'(\overline{a}) \in \bigcup_{i \in \omega} \Sigma_n$–Diag$(\mathcal{A}_i)$ and therefore $\psi \in \Pi_{n+1}((\mathcal{A}_i)_{i \in \omega})$.

□

The main lemma in the proof of Theorem 4.2.37 translated to the concrete models framework is presented below.

Lemma 4.2.42 *Let ϑ be a sentence equivalent to both Σ_{n+1} and Π_{n+1} sentences and let $\mathcal{A}, \mathcal{B}$ be concrete models computable in C, for some low set C. If every Σ_n sentence holds in $\mathcal{A}$ if and only if it holds in $\mathcal{B}$, then $\mathcal{A} \models \vartheta$ if and only if $\mathcal{B} \models \vartheta$.*

It is the model–theoretic construction from the proof of Lemma 4.2.42 which fails and makes the entire proof of the Theorem 4.2.37 fail. However if Lemma 4.2.42 was proven the proof of Theorem 4.2.37 would go as follows.

Let σ be a recursive concrete vocabulary. Let φ be a σ–sentence. The easy implication of Theorem 4.2.37 is from the bottom up and can be easily proven by putting φ to prenex normal form in obvious two ways to obtain equivalent Σ_{n+1} and Π_{n+1} sentences.

For the harder implication suppose that φ is equivalent to both Σ_{n+1} sentence and Π_{n+1} sentence and for the sake of contradiction suppose that it is not equivalent to any Boolean combination of Σ_n sentences.

Let $\sigma_1, \ldots, \sigma_m$ be a finite sequence of Σ_n sentences. There are 2^m conjunctions of the form $\rho = \sigma'_1 \wedge \ldots \wedge \sigma'_m$, where σ'_i is σ_i or $\neg\sigma_i$ for $i = 1, \ldots, m$. Some of those conjunctions are consistent with both φ and $\neg\varphi$ as otherwise for every such conjunction ρ there would be $\vdash \rho \Rightarrow \varphi$ or $\vdash \rho \Rightarrow \neg\varphi$ and therefore by taking $\xi = \bigvee\{\rho\colon \vdash \rho \Rightarrow \varphi\} \vee \bigvee\{\neg\rho\colon \vdash \rho \Rightarrow \neg\varphi\}$ we could easily show via propositional calculus that φ is equivalent to ξ and thus, contrary to our assumptions, that φ is equivalent to a Boolean combination of Σ_n sentences.

There are, therefore, low concrete models $\mathcal{A}, \mathcal{B}$ such that for $i = 1, \ldots, m$ it holds that $\mathcal{A} \models \sigma_i$ if and only if $\mathcal{B} \models \sigma_i$, $\mathcal{A} \models \varphi$ and $\mathcal{B} \models \neg\varphi$. Since the choice of the sequence $\sigma_1, \ldots, \sigma_m$ was arbitrary, for every finite set $S = \{\sigma_1, \ldots, \sigma_m\}$ of Σ_n sentences there are concrete models $\mathcal{A} \models S \cup \{\varphi\}$ and $\mathcal{B} \models S \cup \{\neg\varphi\}$.

Now let $\alpha_0, \alpha_1, \ldots$ be an effective presentation of Σ_n σ–sentences. Consider full binary trees $Tree^n_T$ labelled with theories, defined inductively on $\rho \in 2^{<\omega}$ as follows:

1. If $\rho = \varepsilon$, then $Tree^n_T(\rho) = T$,
2. If $\rho = \tau 0$, then $Tree^n_T(\rho) = Tree^n_T(\tau) \cup \{\alpha_{\mathrm{lh}(\tau)}\}$,

3. If $\rho = \tau 1$, then $Tree^n_T(\rho) = Tree^n_T(\tau) \cup \{\neg\alpha_{\mathrm{lh}(\tau)}\}$.

Consider recursive trees $T_1 = \mathrm{CON}(Tree^n_{\{\varphi\}})$ and $T_2 = \mathrm{CON}(Tree^n_{\{\neg\varphi\}})$. By our previous investigations, for every $n \in \omega$ there is $\rho \in 2^{<\omega}$ with $\mathrm{lh}(\rho) = n$ such that $\rho \in T_1$ and $\rho \in T_2$. Thus $T_1 \cap T_2$ is an infinite recursive tree. By the Low Basis Theorem (2.3.13) there is a low set $f \in 2^\omega$ such that $S_1 = \bigcup_{i\in\omega} Tree^n_{\{\varphi\}}(f{\restriction}i)$ and $S_2 = \bigcup_{i\in\omega} Tree^n_{\{\neg\varphi\}}(f{\restriction}i)$ are recursive in f and consistent theories which agree on all Σ_n sentences. Let $\psi_0, \psi_1, \ldots$ and $\varphi_0, \varphi_1, \ldots$ be effective presentations of sentences and formulae with v_0 as the only free variable respectively. We construct a full binary tree T labelled by pairs of theories inductively on $\rho \in 2^{<\omega}$ as follows:

1. If $\rho = \varepsilon$, then $T(\rho) = (S_1, S_2)$,
2. If $\rho = \tau 0$ and $\mathrm{lh}(\rho) = 4i + 1$, then $T(\rho) = (T(\tau)_1 \cup \{\psi_i\}, T(\tau)_2)$,
3. If $\rho = \tau 1$ and $\mathrm{lh}(\rho) = 4i + 1$, then $T(\rho) = (T(\tau)_1 \cup \{\neg\psi_i\}, T(\tau)_2)$,
4. If ($\rho = \tau 0$ or $\rho = \tau 1$) and $\mathrm{lh}(\rho) = 4i + 2$, then $T(\rho) = (T(\tau)_1 \cup \{\exists v_0\, \varphi_i(v_0) \Rightarrow \varphi_i(d_{k_1})\}, T(\tau)_2)$, where k is the least index such that d_k does not occur in $T(\tau)_1$ and φ_i.
5. If $\rho = \tau 0$ and $\mathrm{lh}(\rho) = 4i + 3$, then $T(\rho) = (T(\tau)_1, T(\tau)_2 \cup \{\psi_i\})$,
6. If $\rho = \tau 1$ and $\mathrm{lh}(\rho) = 4i + 3$, then $T(\rho) = (T(\tau)_1, T(\tau)_2) \cup \{\neg\psi_i\}$,
7. If ($\rho = \tau 0$ or $\rho = \tau 1$) and $\mathrm{lh}(\rho) = 4i + 4$, then $T(\rho) = (T(\tau)_1, T(\tau)_2 \cup \{\exists v_0\, \varphi_i(v_0) \Rightarrow \varphi_i(d_{k_2})\})$, where k is the least index such that d_k does not occur in $T(\tau)_2$ and φ_i.

We define the subtree $\mathrm{CON}_2(T)$ of T in the following way: for every $\rho \in 2^{<\omega}$, $\mathrm{CON}_2(T)(\rho)$ if and only if $T_1(\rho) \nvdash_{\mathrm{lh}(\rho)} \bot$ and $T_2(\rho) \nvdash_{\mathrm{lh}(\rho)} \bot$.

Observe that $\mathrm{CON}_2(T)$ is recursive in f, and therefore low. Since both S_1 and S_2 are consistent, by the Low Basis Theorem (2.3.13) there is a low $g \in [\mathrm{CON}_2(T)]$. Let $T^g = (\bigcup_{i\in\omega} T(g{\restriction}i)_1, \bigcup_{i\in\omega} T(g{\restriction}i)_2)$. By arguments similar to those used in the proof of the Low Completeness Theorem (4.2.16) there are recursive in g, thus low, models $\mathcal{A}, \mathcal{B}$ such that $\mathcal{A} \models S_1$ and $\mathcal{B} \models S_2$. But this contradicts result from Lemma 4.2.42 since for every Σ_n sentence ξ it holds that $\mathcal{A} \models \xi$ if and only if $\mathcal{B} \models \xi$ but $\mathcal{A} \models \varphi$ and $\mathcal{B} \models \neg\varphi$. This would complete the proof of Theorem 4.2.37.

However, as we mentioned before the construction from the proof of Lemma 4.2.42 fails. We present below the construction from the proof the lemma.

Let ϑ be a sentence equivalent to both Σ_{n+1} and Π_{n+1} sentences. Suppose for the sake of contradiction that for some low set g there are recursive in g concrete models $\mathcal{A}, \mathcal{B}$ which satisfy the same Σ_n sentences, but $\mathcal{A} \models \vartheta$ and

$\mathcal{B} \models \neg\vartheta$. Our aim is to build a tower of concrete models as in the following diagram.

$$\begin{tikzcd}
\mathcal{A} \arrow[r, phantom, "\cong"] & \mathcal{A}_0 \arrow[r, phantom, "\equiv"] \arrow[d, phantom, sloped, "\subseteq_n"] & \mathcal{A}_1 \arrow[r, phantom, "\equiv"] \arrow[d, phantom, sloped, "\subseteq_n"] & \mathcal{A}_2 \arrow[r, phantom, "\equiv"] \arrow[d, phantom, sloped, "\subseteq_n"] & \cdots \\
\mathcal{B} \arrow[r, phantom, "\equiv"] & \mathcal{B}_0 \arrow[r, phantom, "\equiv"] \arrow[ur, phantom, sloped, "\subseteq_n"] & \mathcal{B}_1 \arrow[r, phantom, "\equiv"] \arrow[ur, phantom, sloped, "\subseteq_n"] & \mathcal{B}_2 \arrow[r, phantom, "\equiv"] & \cdots
\end{tikzcd}$$

Before proceeding to the construction of concrete models $\mathcal{A}_i$, $\mathcal{B}_i$ fitting the above diagram, we construct concrete models $\mathcal{C}_i$, $\mathcal{D}_i$ as in the following diagram:

$$\begin{tikzcd}
\mathcal{A} \arrow[r, phantom, "\cong"] & \mathcal{C}_0 \arrow[r, phantom, "\equiv"] \arrow[d, hook] & \mathcal{C}_1 \arrow[r, phantom, "\equiv"] \arrow[d, hook] & \mathcal{C}_2 \arrow[r, phantom, "\equiv"] \arrow[d, hook] & \cdots \\
\mathcal{B} \arrow[r, phantom, "\equiv"] & \mathcal{D}_0 \arrow[r, phantom, "\equiv"] \arrow[ur, hook] & \mathcal{D}_1 \arrow[r, phantom, "\equiv"] \arrow[ur, hook] & \mathcal{D}_2 \arrow[r, phantom, "\equiv"] & \cdots
\end{tikzcd}$$

Let $V, R_0, R_1, \ldots, V_0, V_1, \ldots$ be pairwise disjoint recursive sets such that:

- $i \mapsto \ulcorner R_i \urcorner$ is recursive,
- $i \mapsto \ulcorner V_i \urcorner$ is recursive,
- for $i \in \omega$, $D_i = \{\mathfrak{c}_r : r \in R_i\}$ is a set of new constants,
- for $i \in \omega$, $W_i = \{\mathfrak{c}_r : r \in V_i\}$ is a set of new constants.

In the construction below we define Σ_n–Diag$(\mathcal{C}_i)$ using constants from D_{2i}, and Σ_n–Diag$(\mathcal{D}_i)$ using constants from D_{2i+1}. Constants from sets W_i are witnesses used while applying the Low Completeness Theorem (4.2.16).

Let $a_0, a_1, \ldots$ be an effective presentation of V and let $W = \{a_i \in V : i \in |\mathcal{A}|\}$. Then the function $iso \colon W \to |\mathcal{A}|$ such that $iso(a_i) = i$ is a concrete bijection. Let $\mathcal{C}_0$ be the co–image of $\mathcal{A}$ under iso. Then, by Lemma 4.1.10, $\mathcal{C}_0$ is a concrete model. Observe that $\mathcal{C}_0$ is low since it is recursive in $\mathcal{A}$. Indeed, Lemma 4.1.10 guarantees that $\mathcal{C}_0$ is recursive in $W \oplus \mathcal{A} \oplus iso$. Since W is recursive, $a_0, a_1, \ldots$ is an effective presentation of W, iso is recursive in $\mathcal{A}$. Moreover, $|\mathcal{C}_0| = W \subseteq V$.

For $k \in \omega$, let $\sigma^k = \sigma \cup \bigcup_{i<k} D_i$.

We inductively construct a tower of low concrete models in the following way, for $i \in \omega$:

- $\mathcal{C}_0 \overset{iso}{\cong} \mathcal{A}$,
- $T_i = \mathrm{Th}(\mathcal{B}) \cup \Sigma_n$–Diag$(\mathcal{C}_i)$,
- $\mathcal{D}_i$ is the concrete σ^{2i+1}–model obtained by the Low Completeness Theorem (4.2.16) applied to T_i and to the set of witnesses W_{2i},

- $S_{i+1} = \mathrm{Th}(\mathcal{A}) \cup \Sigma_n$–Diag$(\mathcal{D}_i)$,
- $\mathcal{C}_{i+1}$ is the concrete σ^{2i+2}–model obtained by the Low Completeness Theorem (4.2.16) applied to S_{i+1} and to the set of witnesses W_{2i+1}.

By induction on $i \in \omega$ we show that the construction is proper. It is sufficient to show that theories T_i and S_i are consistent and low.

For $i \in \omega$, let $L_i = \bigoplus_{m \leqslant i}(\mathcal{C}_m \oplus \mathcal{D}_m)$.

For the base step we show that $T_0 = \mathrm{Th}(\mathcal{B}) \cup \Sigma_n$–Diag$(\mathcal{C}_0)$ is consistent and low σ^1–theory.

First, observe that T_0 is recursive in $\mathcal{B} \oplus \mathcal{C}_0$, therefore it is recursive in g, thus low.

If T_0 was inconsistent, then there would be a finite set of Σ_n σ^1–sentences $F(\overline{d}) \subseteq \Sigma_n$–Diag$(\mathcal{C}_0)$ such that $\mathrm{Th}(\mathcal{B}) \vdash \neg \bigwedge F(\overline{d})$, where $\overline{d}$ denotes all the constants in F that are not in σ. Then by the lemma on constants it would hold that $\mathrm{Th}(\mathcal{B}) \vdash \forall \overline{x} \neg \bigwedge F(\overline{x})$. Observe that $\bigwedge F$ is equivalent to a Σ_n sentence and therefore $\forall \overline{x} \neg \bigwedge F(\overline{x})$ is equivalent to a Π_n σ–sentence. Therefore $\mathcal{B} \models \forall \overline{x} \neg \bigwedge F(\overline{x})$. Since $\mathcal{B}$ and $\mathcal{A}$ satisfy the same Σ_n σ–sentences, and $\mathcal{C}_0$ is isomorphic to $\mathcal{A}$ it holds that $\mathcal{C}_0 \models \forall \overline{x} \neg \bigwedge F(\overline{x})$. But $\bigwedge F \in \Sigma_n$–Diag$(\mathcal{C}_0)$ and therefore $\mathcal{C}_0 \models \exists \overline{x} \bigwedge F(\overline{x})$ which is a contradiction. Therefore T_0 is consistent.

Let $\mathcal{D}_0$ be the low concrete σ^1–model obtained by applying the Low Completeness Theorem (4.2.16) to T_0 and the set of witnesses W_0. Then $L_0 = \mathcal{D}_0 \oplus \mathcal{C}_0$ is low, $|\mathcal{D}_0| \subseteq V_0$. Moreover, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}_0$ and $\mathcal{D}_0$ satisfy the same Σ_n σ–sentences.

For the induction hypothesis suppose that low concrete models $\mathcal{C}_i$ and $\mathcal{D}_i$ are constructed and that L_i is low. Moreover let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}_i$ and $\mathcal{D}_i$ satisfy the same Σ_n σ–sentences.

Observe that $S_{i+1} = \mathrm{Th}(\mathcal{A}) \cup \Sigma_n$–Diag$(\mathcal{D}_i)$ is recursive in L_i, therefore low.

We show that it is also consistent. Otherwise there would be a finite set of Σ_n σ^{2i+2}–sentences $F \subseteq \Sigma_n$–Diag$(\mathcal{D}_i)$ such that $\mathrm{Th}(A) \vdash \neg \bigwedge F(\overline{d})$, where $\overline{d}$ are all the constants from F not occurring in σ. By the similar arguments to those presented in the base step, we have $\mathcal{A} \models \forall \overline{x} \neg \bigwedge F(\overline{x})$ and $\mathcal{D}_i \models \exists \overline{x} \bigwedge F(\overline{x})$. This is a contradiction, since $\exists \overline{x} \bigwedge F(\overline{x})$ is equivalent to a Σ_n σ–sentence and $\mathcal{A}$ and $\mathcal{D}_i$ satisfy the same Σ_n σ–sentences.

Let $\mathcal{C}_{i+1}$ be the low concrete σ^{2i+2}–model obtained by applying the Low Completeness Theorem (4.2.16) to T_{i+1} and the set of witnesses W_{2i+1}. Then $\mathcal{C}_{i+1} \oplus L_i$ is low, $|\mathcal{C}_{i+1}| \subseteq V_{2i+1}$ and $\mathcal{C}_{i+1}$ satisfies the same Σ_n σ–sentences as $\mathcal{D}_i$.

We show that T_{i+1} is consistent and low by an analogous arguments. By the construction we get a low concrete σ^{2i+2} model $\mathcal{D}_{i+1} \models S_{i+1}$, such that $L_{i+1} = L_i \oplus \mathcal{C}_{i+1} \oplus \mathcal{D}_{i+1}$ is low. Moreover, $|\mathcal{D}_{i+1}| \subseteq V_{2i+2}$ and $\mathcal{D}_{i+1}$ satisfy the same Σ_n σ–sentences as $\mathcal{C}_{i+1}$.

We show that $\mathcal{C}_0, \mathcal{D}_0, \mathcal{C}_1, \mathcal{D}_1, \ldots$ is a jump–concrete sequence of low concrete models. For $i \in \omega$, let $\mathcal{C}_i^*$ be the halting problem for $\mathcal{C}_i$ and let $\mathcal{D}_i^*$ be the halting problem for $\mathcal{D}_i$. We show that $i \mapsto \ulcorner \mathcal{D}_i^* \urcorner$ and $i \mapsto \ulcorner \mathcal{C}_i^* \urcorner$ are concrete maps. The halting problem for $\mathcal{A} \oplus \mathcal{B}$ is concrete, since $\mathcal{A} \oplus \mathcal{B}$ is low. The low concrete model $\mathcal{C}_0$ is recursive in $\mathcal{A}$, therefore $\mathcal{C}_0^*$ is concrete. Observe that T_0 is a sum of the theory of a low concrete model $\mathcal{B}$ and Σ_n–Diag($\mathcal{C}_0$) which is low. Computing Σ_n diagrams and theories of concrete models is recursive in the complexity these concrete models. Therefore, T_0 and the halting problem for T_0 are concrete. According to the construction, we apply the Low Completeness Theorem (4.2.16) to T_0 and the to set of witnesses W_0 to obtain $\mathcal{D}_0$. Therefore, as was already discussed (see 2.3.14, 4.2.16 and the discussion directly after the latter), $\mathcal{D}_0$ is concretely computable from T_0^* and W_0, and d_0 such that $\Phi_{d_0}^K$ decides the halting problem for $\mathcal{D}_0$ can be computed from T_0^* and W_0.

General algorithms for $i \mapsto \ulcorner \mathcal{C}_i^* \urcorner$ and $i \mapsto \ulcorner \mathcal{D}_i^* \urcorner$ proceed along the construction of the tower of concrete models. They repetitively use the algorithm from Theorem 2.3.14 to obtain codes of algorithms which, in K, decide the next concrete model and its halting problem. They also need to compute Σ_n diagrams and theories of concrete models as it is discussed in the previous paragraph.

Therefore the maps $i \mapsto \ulcorner \mathcal{C}_i^* \urcorner$ and $i \mapsto \ulcorner \mathcal{D}_i^* \urcorner$ are concrete.

Henceforth:

- $\mathcal{C}_0, \mathcal{D}_0, \mathcal{C}_1, \mathcal{D}_1, \ldots$ is a jump–concrete sequence of low concrete models,
- for every $i \in \omega$ it holds that $\bigoplus_{j \leqslant i} \mathcal{C}_j \oplus \mathcal{D}_j$ is low,
- for every $i \in \omega$ it holds that $\mathcal{D}_i \models \Sigma_n$–Diag($\mathcal{C}_i$) and $\mathcal{C}_{i+1} \models \Sigma_n$–Diag($\mathcal{D}_i$).

Moreover, sets $V, V_0, V_1, \ldots$ are disjoint and such that $i \mapsto \ulcorner V_i \urcorner$ is recursive, $|\mathcal{C}_0| \subseteq V$, $|\mathcal{D}_i| \subseteq V_{2i}$ and $|\mathcal{C}_i| \subseteq V_{2i+1}$.

Therefore, by Theorem 4.1.22, for every $i \in \omega$ there is a concrete σ–model $\mathcal{A}_i$ and a concrete isomorphism g_i from $\mathcal{A}_i$ to the reduction of $\mathcal{C}_i$ to σ. Similarly, for $i \in \omega$ there is a concrete σ–model $\mathcal{B}_i$ and a concrete isomorphism h_i from $\mathcal{B}_i$ to a reduction of $\mathcal{D}_i$ to σ.

We therefore have the following:

- $\mathcal{A}_0 \subseteq_n \mathcal{B}_0 \subseteq_n \mathcal{A}_1 \subseteq_n \mathcal{B}_1 \subseteq_n \ldots,$
- $\forall i \in \omega\, \mathrm{Th}(\mathcal{A}_i) \equiv \mathrm{Th}(\mathcal{A})$ and $\forall i \in \omega\, \mathrm{Th}(\mathcal{B}_i) \equiv \mathrm{Th}(\mathcal{B})$,

which is illustrated by the following diagram.

$$\begin{tikzcd}
\mathcal{A} \arrow[r, phantom, "\cong"] & \mathcal{A}_0 \arrow[r, phantom, "\equiv"] & \mathcal{A}_1 \arrow[r, phantom, "\equiv"] & \mathcal{A}_2 \arrow[r, phantom, "\equiv"] & \cdots \\
\\
\mathcal{B} \arrow[r, phantom, "\equiv"] & \mathcal{B}_0 \arrow[uu, phantom, "\rotatebox{90}{$\subseteq_n$}"] \arrow[uur, phantom, "\rotatebox{45}{$\subseteq_n$}"] \arrow[r, phantom, "\equiv"] & \mathcal{B}_1 \arrow[uu, phantom, "\rotatebox{90}{$\subseteq_n$}"] \arrow[uur, phantom, "\rotatebox{45}{$\subseteq_n$}"] \arrow[r, phantom, "\equiv"] & \mathcal{B}_2 \arrow[uu, phantom, "\rotatebox{90}{$\subseteq_n$}"] \arrow[r, phantom, "\equiv"] & \cdots
\end{tikzcd}$$

Observe that there are two concrete chains of concrete models $(\mathcal{A}_i)_{i\in\omega}$ and $(\mathcal{B}_i)_{i\in\omega}$. Since $\mathcal{A}_i \subseteq_n \mathcal{B}_i \subseteq_n \mathcal{A}_{i+1}$ for each $i \in \omega$ it holds that $\Pi_{n+1}((\mathcal{A}_i)_{i\in\omega}) = \Pi_{n+1}((\mathcal{B}_i)_{i\in\omega})$.

By the assumption ϑ is equivalent to a Π_{n+1} sentence and to a Σ_{n+1} sentence. Let ψ be a Π_{n+1} sentence equivalent to ϑ and let φ be a Π_{n+1} sentence equivalent to $\neg\vartheta$.

Each $\mathcal{A}_i$ is elementarily equivalent to $\mathcal{A}$, $\mathcal{A} \models \vartheta$ and ϑ is equivalent to ψ. Therefore, by Lemma 4.2.41, it follows that $\psi \in \Pi_{n+1}((\mathcal{A}_i)_{i\in\omega})$. On the other hand, each $\mathcal{B}_i$ is elementarily equivalent to $\mathcal{B}$, $\mathcal{B} \models \neg\vartheta$ and $\neg\vartheta$ is equivalent to φ. Then, by Lemma 4.2.41, it holds that $\varphi \in \Pi_{n+1}((\mathcal{A}_i)_{i\in\omega})$. Therefore, since ψ is equivalent to $\neg\varphi$, the set $\Pi_{n+1}((\mathcal{A}_i)_{i\in\omega})$ is inconsistent.

Here is where the construction fails for concrete models – we do not know whether $\Pi_{n+1}((\mathcal{A}_i)_{i\in\omega})$ is consistent or not and the contradiction in the original proof of the lemma comes from the fact that $\Pi_{n+1}((\mathcal{A}_i)_{i\in\omega})$ is consistent as a subset of a theory $\mathrm{Th}(\bigcup_{i\in\omega}\mathcal{A}_i)$.

This, of course, does not imply that Theorem 4.2.37 is false in the concrete models framework. It just means that the construction by Chang and Keisler does not work in this context.

4.2.4 Preservation Theorems

We already know that a sum of a recursive concrete chain of recursive concrete models may not be a concrete model. In this section we focus on concrete elementary chains of concrete models in the case of which the situation is different. The satisfaction relation in the sum of a concrete elementary chain of concrete models can be computed from satisfaction relations in concrete models in the concrete elementary chain. In fact the sum of elementary diagrams of concrete models from the concrete elementary chain converges to the elementary diagram of the sum of the chain. Being able to sum elementary chains, we can perform model–theoretic constructions employing elementary chains. In this section we show two results achieved by the use of elementary chains and their sums – the so called preservation theorems.

First, let us prove that summing concrete elementary chains is admissible in concrete models framework.

Theorem 4.2.43 *Let σ be a concrete vocabulary. Let $((\mathcal{F}, \varphi^i_{\models}))_{i\in\omega}$ be a concrete elementary chain of concrete σ–models. Then there is a concrete model $\mathcal{A} = \bigcup_{i\in\omega}(\mathcal{F}_i, \varphi^i_{\models})$.*

Moreover for every $k \in \omega$ it holds that $\mathcal{A}_k \preccurlyeq \mathcal{A}$.

Proof: For every $i \in \omega$ we denote the concrete model $(\mathcal{F}_i, \varphi^i_{\models})$ by $\mathcal{A}_i$. Let σ be a concrete vocabulary and let $(\mathcal{A}_i)_{i\in\omega}$ be such that for every $i \in \omega$ the model $\mathcal{A}_i$ is a concrete model. Moreover, let the set $\bigcup_{i\in\omega} |\mathcal{A}_i|$ and the map $i \mapsto \ulcorner \mathcal{A}_i \urcorner$ be also concrete.

It is easy to see that since $((\mathcal{F}_i, \varphi^i_{\models}))_{i\in\omega}$ is a concrete chain of concrete models, $\bigcup_{i\in\omega} |\mathcal{F}_i|$ is concrete and the structure $\mathcal{F} = \bigcup_{i\in\omega} \mathcal{F}_i$ is a concrete structure. It remains to show that the satisfaction relation $R_{\models}$ in $\mathcal{F}$ is concrete.

The following algorithm decides the satisfaction relation in $\mathcal{F}$.

Algorithm 12 Algorithm deciding the satisfaction relation $R_{\models}$ in $\mathcal{F}$

Input: $\ulcorner\psi\urcorner, \ulcorner a_1, \ldots, a_k\urcorner$
Output: truth value of $R(\ulcorner\psi\urcorner, \ulcorner a_1, \ldots, a_k\urcorner)$
  if $\psi \notin \mathrm{Form}_\sigma$ or $\neg \bigwedge_{j=1,\ldots,k} a_j \in |\mathcal{F}|$ or $k \leqslant \max\{i\colon v_i \in \mathrm{FV}(\psi)\}$ **then**
    return false
  else
    $i \leftarrow 0$
    while $\neg \bigwedge_{j=1,\ldots,k} a_j \in |\mathcal{A}_i|$ **do**
      $i \leftarrow i + 1$
    end while
    if $\mathcal{A}_i \models \psi[a_1, \ldots, a_k]$ **then**
      return true
    else
      return false
    end if
  end if

Note that Algorithm 12 uses only concrete queries: the membership in $|\mathcal{F}|$, membership in Form_σ, membership in $|\mathcal{A}_i|$ and satisfaction in $\mathcal{A}_i$. It also uses the concrete map $i \mapsto \ulcorner \mathcal{A}_i \urcorner$. Moreover, the algorithm always halts – the while–loop must eventually stop, since for $j = 1, \ldots, k$ it holds that there exists $i \in \omega$ such that $a_j \in |\mathcal{A}_i|$. Algorithm 12 computes the least index i such that each a_j is in the universe of $\mathcal{A}_i$ and checks whether ψ is satisfied by $a_1, \ldots, a_k$ is $\mathcal{A}_i$.

Let $R_{\models}(\ulcorner\psi\urcorner, \ulcorner\overline{a}\urcorner)$ be the relation decided by Algorithm 12. As we have shown $R_{\models}$ is concrete. It is easy to see that $R_{\models}$ is a pre–satisfaction in $\mathcal{F}$. It remains to show that it satisfies the Tarski's conditions. The case of atomic formulae is obvious by the definition of $\mathcal{F}$. Let φ, ψ be σ–formulae and let $a_1, \ldots, a_k$ be elements of $\mathcal{F}$ such that k is greater the the highest index of free variables occurring in φ and ψ.

For the case of negation, observe that the following are equivalent:

- $R_{\models}(\ulcorner\neg\psi\urcorner, \ulcorner a_1, \ldots, a_k\urcorner)$,
- $\mathcal{A}_i \models \neg\psi[a_1, \ldots, a_k]$, where i is the least index such that $a_1, \ldots, a_k \in |\mathcal{A}_i|$,
- $\mathcal{A}_i \not\models \psi[a_1, \ldots, a_k]$, where i is the least index such that $a_1, \ldots, a_k \in |\mathcal{A}_i|$,
- it is not the case that $R_{\models}(\ulcorner\psi\urcorner, \ulcorner a_1, \ldots, a_k\urcorner)$.

For the case of conjunction, observe that the following are equivalent:

- $R_{\models}(\ulcorner\psi \wedge \varphi\urcorner, \ulcorner a_1, \ldots, a_k\urcorner)$,
- $\mathcal{A}_i \models (\psi \wedge \varphi)[a_1, \ldots, a_k]$, where i is the least index such that $a_1, \ldots, a_k \in |\mathcal{A}_i|$,
- $\mathcal{A}_i \models \psi[a_1, \ldots, a_k]$ and $\mathcal{A}_i \models \varphi[a_1, \ldots, a_k]$, where i is the least index such that $a_1, \ldots, a_k \in |\mathcal{A}_i|$,
- $R_{\models}(\ulcorner\psi\urcorner, \ulcorner a_1, \ldots, a_k\urcorner)$ and $R_{\models}(\ulcorner\varphi\urcorner, \ulcorner a_1, \ldots, a_k\urcorner)$.

For the case of existential quantification, observe that the following are equivalent:

- $R_{\models}(\ulcorner\exists v_m\, \psi\urcorner, \ulcorner a_1, \ldots, a_k\urcorner)$,
- $\mathcal{A}_i \models \exists v_m\, \psi[a_1, \ldots, a_k]$, where i is the least index such that $a_1, \ldots, a_k \in |\mathcal{A}_i|$,
- there is $n \in |\mathcal{A}_i|$ such that $\mathcal{A}_i \models \psi[(a_1, \ldots, a_k)[v_m := n]]$, where i is the least index such that $a_1, \ldots, a_k \in |\mathcal{A}_i|$,
- there is $n \in |\mathcal{F}|$ such that $R_{\models}(\ulcorner\psi\urcorner, \ulcorner(a_1, \ldots, a_k)[v_m := n]\urcorner)$.

Therefore, $R_{\models}$ is the satisfaction relation in $\mathcal{F}$. Let $\varphi_{\models}$ FM–represent $R_{\models}$. Then $\mathcal{A} = (\mathcal{F}, \varphi_{\models})$ is a concrete σ–model and $\mathcal{A} = \bigcup_{i\in\omega} \mathcal{A}_i$.

It remains to show that for each $i \in \omega$ it holds that $\mathcal{A}_i \preccurlyeq \mathcal{A}$. Fix $i \in \omega$. It is obvious that $\mathcal{A}_i \subseteq \mathcal{A}$. Suppose that $\mathcal{A}_i \models \psi[a_1, \ldots, a_k]$, then let l be the least index such that $\mathcal{A}_l \models \psi[a_1, \ldots, a_k]$. Since $l \leqslant i$, it holds that $\mathcal{A}_l \preccurlyeq \mathcal{A}_i$. Therefore, by the definition of satisfaction relation in $\mathcal{A}$, $\mathcal{A} \models \psi[a_1, \ldots, a_k]$

holds. This ends the proof. □

We have already shown that the sum of a concrete elementary chain of concrete models is also a concrete model. This leads us to a search for applications of elementary chains in the axiomatic model theory. Some of those applications are known as preservation theorems – they establish a connection between existence of certain axiomatisations of a given theory and closures of class of its models on certain operations.

In this section we work with a fixed recursive vocabulary σ i.e. all formulae, theories and sets of formulae are σ–formulae and sets of σ–formulae. We begin with a useful lemma on axiomatisations. Both the lemma and its proof come from [CK73], however we have to restrict the complexity of sets appearing there.

Lemma 4.2.44 *Let T be a consistent recursive theory and let Δ be a recursive set of sentences closed under finite disjunctions. Then the following are equivalent:*

- *T has a concrete set of axioms $\Gamma \subseteq \Delta$,*
- *for all low concrete models $\mathcal{A}, \mathcal{B}$ such that $\mathcal{A} \oplus \mathcal{B}$ is low it holds that if*
  - *$\mathcal{A} \models T$ and*
  - *for all $\delta \in \Delta$ if $\mathcal{A} \models \delta$, then $\mathcal{B} \models \delta$,*

  then $\mathcal{B} \models T$.

Proof: Let T and Δ be as in the assumptions of the lemma.

For the easy left–to–right, implication let $\Gamma \subseteq \Delta$ be a concrete set of axioms for T.

Fix low concrete models $\mathcal{A}$, $\mathcal{B}$ such that $\mathcal{A} \oplus \mathcal{B}$ is low, $\mathcal{A} \models T$ and for each $\delta \in \Delta$ if $\mathcal{A} \models \delta$, then $\mathcal{B} \models \delta$.

Then, since Γ is a set of axioms for T and $\Gamma \subseteq \Delta$, from $\mathcal{A} \models T$ we infer that $\mathcal{A} \models \Gamma$ and $\mathcal{B} \models \Gamma$. Finally, since Γ is an axiomatisation of T it holds that $\mathcal{B} \models T$.

For the other implication suppose that for all low models $\mathcal{A}$, $\mathcal{B}$ such that if:

- $\mathcal{A} \oplus \mathcal{B}$ is low,
- $\mathcal{A} \models T$,
- for each $\delta \in \Delta$ if $\mathcal{A} \models \delta$, then $\mathcal{B} \models \delta$

then $\mathcal{B} \models T$.

We define the set $\Gamma = \{\varphi \colon T \vdash \varphi \text{ and } \varphi \in \Delta\}$. Note that since T and Δ are recursive, Γ is recursively enumerable and by Craig's theorem it has a recursive axiomatisation Γ_0. Of course $T \vdash \Gamma$. We show that $\Gamma \vdash T$ i.e. that Γ is a concrete axiomatisation of T.

By the Low Completeness Theorem (4.2.16) it is sufficient to show that for every low model $\mathcal{B}$ such that $\mathcal{B} \models \Gamma_0$ it holds that $\mathcal{B} \models T$.

Fix a low model $\mathcal{B}$ such that $\mathcal{B} \models \Gamma_0$. Then $\mathcal{B} \models \Gamma$. Our aim is to show that $\mathcal{B}$ is also a model of T. We define $\Sigma = \{\neg\delta \colon \mathcal{B} \models \neg\delta \text{ and } \delta \in \Delta\}$. It is easy to see that Σ is recursive in $\mathcal{B}$ and therefore $T \cup \Sigma$ is also recursive in $\mathcal{B}$.

We show that $T \cup \Sigma$ is consistent.

Suppose for the sake of contradiction that $T \cup \Sigma$ is inconsistent. Then there are $\neg\delta_1, \ldots, \neg\delta_n \in \Sigma$ such that $T \vdash \neg(\neg\delta_1 \wedge \ldots \wedge \neg\delta_n)$. Therefore $T \vdash \delta_1 \vee \ldots \vee \delta_n$. Note that $\delta_i \in \Delta$ for $i = 1, \ldots, n$ and since Δ is closed under finite disjunctions, $\delta_1 \vee \ldots \vee \delta_n \in \Delta$. Therefore $\delta_1 \vee \ldots \vee \delta_n \in \Gamma$ and thus, since $\mathcal{B} \models \Gamma$, we have $\mathcal{B} \models \delta_1 \vee \ldots \vee \delta_n$ which contradicts the fact that $\mathcal{B} \models \neg\delta_i$ for $i = 1, \ldots, n$.

We therefore know that $T \cup \Sigma$ is recursive in $\mathcal{B}$ and consistent. Let $\mathcal{A}$ be a low model of $T \cup \Sigma$ obtained by the Low Completeness Theorem (4.2.16). Then $\mathcal{A} \oplus \mathcal{B}$ is low.

Observe that $\mathcal{A} \models T \cup \Sigma$, hence also $\mathcal{A} \models T$. Fix $\delta \in \Delta$ such that $\mathcal{A} \models \delta$. If it was the case that $\mathcal{B} \models \neg\delta$, then by the definition of Σ it would hold that $\neg\delta \in \Sigma$, which is impossible, since $\mathcal{A} \models \Sigma$. Therefore, by the assumption, $\mathcal{B} \models T$. This completes the proof. $\square$

The first preservation theorem which transfers to the concrete models framework is preservation under unions of chains.

Theorem 4.2.45 *Let T be a consistent recursive σ–theory. T is preserved under union of chains if and only if T has a Π_2 axiomatisation.*

Proof: Fix a recursive theory T.

For the easy, right–to–left direction, suppose that there is a set S of Π_2 sentences axiomatising T. Let $(\mathcal{A}_i)_{i\in\omega}$ be a concrete chain of concrete models of T with the union $\mathcal{A} = \bigcup_{i\in\omega} \mathcal{A}_i$ being a concrete model. Consider a sentence $\varphi = \forall\overline{x}\,\exists\overline{y}\,\varphi_0(\overline{x}, \overline{y})$, where φ_0 is quantifier free and such that $\varphi \in S$.

Since $\varphi \in S$, for every $i \in \omega$ it holds that $\mathcal{A}_i \models \varphi$. For every $\overline{a} \in |\mathcal{A}|$ there exists $j \in \omega$ such that $\overline{a} \in |\mathcal{A}_j|$. Then there exists $\overline{b} \in |\mathcal{A}_j|$ such that $\mathcal{A}_j \models \varphi_0[\overline{a}, \overline{b}]$ and therefore also $\mathcal{A} \models \varphi_0(\overline{a}, \overline{b})$, hence $\mathcal{A} \models \varphi$.

Therefore, for every $\varphi \in S$ it holds that $\mathcal{A} \models \varphi$, and thus $\mathcal{A} \models T$, since S is an axiomatisation of T. This ends the proof of the easy direction of the equivalence.

Suppose now that T is preserved under unions of chains i.e. for every concrete chain of concrete models of T its union (if it exists) also satisfies T.

We define the set of sentences Δ as a set of finite disjunctions of the set of all Π_2 sentences. Δ is obviously closed under finite disjunctions and recursive.

By Lemma 4.2.44, it is sufficient to show that for every low concrete models $\mathcal{A}$, $\mathcal{B}$ such that:

- $\mathcal{A} \oplus \mathcal{B}$ is low,
- $\mathcal{A} \models T$,
- for every $\delta \in \Delta$ if $\mathcal{A} \models \delta$, then $\mathcal{B} \models \delta$,

it holds that $\mathcal{B} \models T$.

Let $\mathcal{A}, \mathcal{B}$ be low models such that $\mathcal{A} \oplus \mathcal{B}$ is low. Suppose that $\mathcal{A} \models T$ and for every $\delta \in \Delta$ if $\mathcal{A} \models \delta$, then $\mathcal{B} \models \delta$. We show that $\mathcal{B} \models T$. Our aim is to construct a tower of concrete models as in the following diagram:

$$\begin{tikzcd}
\mathcal{B} \arrow[r, phantom, "\cong"] & \mathcal{B}_0 \arrow[r, phantom, "\preccurlyeq"] \arrow[d, phantom, sloped, "\supseteq"] & \mathcal{B}_1 \arrow[r, phantom, "\preccurlyeq"] \arrow[d, phantom, sloped, "\supseteq"] & \mathcal{B}_2 \arrow[r, phantom, "\preccurlyeq"] \arrow[d, phantom, sloped, "\supseteq"] & \cdots \\
\mathcal{A} \arrow[r, phantom, "\equiv"] & \mathcal{A}_0 \arrow[r, phantom, "\equiv"] \arrow[ur, phantom, sloped, "\subseteq"] & \mathcal{A}_1 \arrow[r, phantom, "\equiv"] \arrow[ur, phantom, sloped, "\subseteq"] & \mathcal{A}_2 \arrow[r, phantom, "\equiv"] & \cdots
\end{tikzcd}$$

Before proceeding to the construction of the concrete models $\mathcal{A}_i$, $\mathcal{B}_i$ as in the diagram above, we construct concrete models $\mathcal{C}_i$, $\mathcal{D}_i$ fitting the following diagram:

$$\begin{tikzcd}
\mathcal{B} \arrow[r, phantom, "\cong"] & \mathcal{D}_0 \arrow[r, hook, "g_{0,1}"] \arrow[d, hook, "f_{0,0}"'] & \mathcal{D}_1 \arrow[r, hook, "g_{1,2}"] \arrow[d, hook, "f_{1,1}"'] & \mathcal{D}_2 \arrow[r, hook, "g_{2,3}"] \arrow[d, hook, "f_{2,2}"'] & \cdots \\
\mathcal{A} \arrow[r, phantom, "\equiv"] & \mathcal{C}_0 \arrow[r, phantom, "\equiv"] \arrow[ur, hook, "f_{0,1}"] & \mathcal{C}_1 \arrow[r, phantom, "\equiv"] \arrow[ur, hook, "f_{1,2}"] & \mathcal{C}_2 \arrow[r, phantom, "\equiv"] & \cdots
\end{tikzcd}$$

The construction below guarantees what follows. For $i \in \omega$, concrete morphisms $f_{i,i}$ are from $(\mathcal{D}_i, d)_{d \in |\mathcal{D}_i|}$ to $\mathcal{C}_i$ and are concrete embeddings. Similarly, concrete morphisms $f_{i,i+1}$ are from $(\mathcal{C}_i, c)_{c \in |\mathcal{C}_i|}$ to $\mathcal{D}_{i+1}$ and are concrete embeddings. Concrete morphisms $g_{i,i+1}$ from $(\mathcal{D}_i, d)_{d \in |\mathcal{D}_i|}$ to the restriction of $\mathcal{D}_{i+1}$ to the common vocabulary are concrete elementary embeddings. We proceed to the construction of concrete models $\mathcal{C}_i$ and $\mathcal{D}_i$.

Let $V, R_0, R_1, \ldots, V_0, V_1, \ldots$ be pairwise disjoint recursive sets such that:

- $i \mapsto \ulcorner R_i \urcorner$ is recursive,
- $i \mapsto \ulcorner V_i \urcorner$ is recursive,
- for $i \in \omega$, $D_i = \{\mathfrak{c}_r : r \in R_i\}$ is a set of new constants,
- for $i \in \omega$, $W_i = \{\mathfrak{c}_r : r \in V_i\}$ is a set of new constants.

In the construction below we define $\mathrm{Diag}(\mathcal{C}_i)$ using new constants from D_{2i+1}, and $\mathrm{ElDiag}(\mathcal{D}_i)$ (and Π_1–$\mathrm{Diag}(\mathcal{D}_i)$) using new constants from D_{2i}. Constants from sets W_i are witnesses used while applying the Low Completeness Theorem (4.2.16).

Let $a_0, a_1, \ldots$ be an effective presentation of V. Then $V^{\mathcal{B}} = \{a_b : b \in |\mathcal{B}|\}$ is recursive in $\mathcal{B}$. The the function $iso : V^{\mathcal{B}} \to |\mathcal{B}|$ such that for each $b \in |\mathcal{B}|$ it holds that $iso(a_b) = b$ is a concrete recursive in $\mathcal{B}$ bijection. Let $\mathcal{D}_0$ be the co–image of $\mathcal{B}$ under iso. Therefore by Lemma 4.1.10 $\mathcal{D}_0$ is a concrete recursive in $\mathcal{B}$ model and iso is a concrete recursive in $\mathcal{B}$ isomorphism. Moreover, $|\mathcal{D}_0| = V^{\mathcal{B}} \subseteq V$.

For $k \in \omega$, let $\sigma^k = \sigma \cup \bigcup_{i<k} D_i$.

The construction is inductive as follows. For $i \in \omega$:

- $\mathcal{D}_0 \overset{iso}{\cong} \mathcal{B}$,

- $T_i = \mathrm{Th}(\mathcal{A}) \cup \Pi_1$–$\mathrm{Diag}(\mathcal{D}_i)$,
- $\mathcal{C}_i$ is the concrete σ^{2i+1}–model obtained by applying the Low Completeness Theorem (4.2.16) to T_i and to the set of witnesses W_{2i},

- $S_{i+1} = \mathrm{Diag}(\mathcal{C}_i) \cup \mathrm{ElDiag}(\mathcal{D}_i)$,
- $\mathcal{D}_{i+1}$ is the concrete σ^{2i+2}–model obtained by applying the Low Completeness Theorem (4.2.16) to S_{i+1} and to the set of witnesses W_{2i+1}.

For $i \in \omega$, let $L_i = \bigoplus_{m<i}(\mathcal{C}_m \oplus \mathcal{D}_m) \oplus \mathcal{D}_i$.

By induction we show that the construction is correct. It suffices to show that for every $i \in \omega$, theories T_i and S_{i+1} are low and consistent.

In the base step of induction we have constructed a low concrete model $\mathcal{D}_0$ by means of Lemma 4.1.10.

Now for the inductive hypothesis suppose that the construction was performed up to a low concrete model $\mathcal{D}_i$ for some $i \in \omega$ and that L_i is low. Consider theories $T_1^i = \mathrm{Th}(\mathcal{A})$ and $T_2^i = \Pi_1$–$\mathrm{Diag}(\mathcal{D}_i)$. Note that they are both recursive in a low set L_i and therefore $T_1^i \cup T_2^i$ is recursive in L_i, thus low.

If $T_1^i \cup T_2^i$ was inconsistent, there would be a finite set $S(\overline{d}) \subseteq T_2^i$ such that $T_1^i \vdash \neg \bigwedge S(\overline{d})$. Then, by lemma on constants, $T_1^i \vdash \forall \overline{x} \neg \bigwedge S(\overline{x})$ and since $T_1^i = \mathrm{Th}(\mathcal{A})$ this would mean that $\mathcal{A} \models \forall \overline{x} \neg \bigwedge S(\overline{x})$. Observe that $\forall \overline{x} \neg \bigwedge S(\overline{x})$ is equivalent to a Π_2 sentence. Since $\mathcal{A} \models \forall \overline{x} \neg \bigwedge S(\overline{x})$ it follows that $\mathcal{B} \models \forall \overline{x} \neg \bigwedge S(\overline{x})$. It follows that $\mathcal{D}_0 \models \forall \overline{x} \neg \bigwedge S(\overline{x})$ and therefore it also holds that $\mathcal{D}_i \models \forall \overline{x} \neg \bigwedge S(\overline{x})$. But $\mathcal{D}_i \models \bigwedge S(\overline{d})$ which is a contradiction.

Therefore $T_1^i \cup T_2^i$ is consistent, recursive in a low set L_i. Let $\mathcal{C}_i$ be the low concrete σ^{2i+1}–model obtained by the Low Completeness Theorem (4.2.16) applied to the theory $T_1^i \cup T_2^i$ and to the set of witnesses W_{2i}. Since $\mathcal{C}_i \models \mathrm{Th}(\mathcal{A})$, it holds that $\mathcal{C}_i \equiv \mathcal{A}$. Moreover, $\mathcal{C}_i \models \mathrm{Diag}(\mathcal{D}_i)$, therefore by Lemma 4.1.17 there is a concrete, recursive in $L_i \oplus \mathcal{C}_i$, embedding $f_{i,i} : (\mathcal{D}_i, d)_{d \in |\mathcal{D}_i|} \to \mathcal{C}_i$. Finally $|\mathcal{C}_i| \subseteq V_{2i}$ and $L_i \oplus \mathcal{C}_i$ is low.

Now consider theories $T_3^i = \mathrm{Diag}(\mathcal{C}_i)$ and $T_4^i = \mathrm{ElDiag}(\mathcal{D}_i)$ – both recursive in $L_i \oplus \mathcal{C}_i$, thus low.

If $T_3^i \cup T_4^i$ was inconsistent, then there would be a finite set of sentences $S(\overline{c}, \overline{d}) \subseteq \mathrm{Diag}(\mathcal{C}_i)$ such that $T_4^i \vdash \neg \bigwedge S(\overline{c}, \overline{d})$. Then, by the lemma on constants, there would be $T_4^i \vdash \forall \overline{x} \neg \bigwedge S(\overline{x}, \overline{d})$, therefore $\forall \overline{x} \neg \bigwedge S(\overline{x}, \overline{d}) \in \mathrm{ElDiag}(\mathcal{D}_i)$. Since $\forall \overline{x} \neg \bigwedge S(\overline{x}, \overline{d})$ is a Π_1 sentence and $\mathcal{C}_i \models \Pi_1$–$\mathrm{Diag}(\mathcal{D}_i)$, it follows that $\forall \overline{x} \neg \bigwedge S(\overline{x}, \overline{d}) \in \mathrm{ElDiag}(\mathcal{C}_i)$. However, we also have that $\bigwedge S(\overline{c}, \overline{d}) \in \mathrm{ElDiag}(\mathcal{C}_i)$ which is a contradiction.

Therefore $T_3^i \cup T_4^i$ is consistent and recursive in $L_i \oplus \mathcal{C}_i$, thus low. Let $\mathcal{D}_{i+1}$ be the low concrete σ^{2i+2}–model obtained by the Low Completeness Theorem (4.2.16) applied to the theory $T_3^i \cup T_4^i$ and the set of witnesses W_{2i+1}. Recall that $L_{i+1} = L_i \oplus \mathcal{C}_i \oplus \mathcal{D}_{i+1}$. Since $\mathcal{D}_{i+1}$ is obtained by the Low Completeness Theorem from a low set recursive in $L_i \oplus \mathcal{C}_i$, the set L_{i+1} is low. Since $\mathcal{D}_{i+1} \models \mathrm{Diag}(\mathcal{C}_i)$, it follows that there is a concrete, recursive in L_{i+1} embedding $f_{i,i+1} : (\mathcal{C}_i, c)_{c \in |\mathcal{C}_i|} \to \mathcal{D}_{i+1}$. Since $\mathcal{D}_{i+1} \models \mathrm{ElDiag}(\mathcal{D}_i)$, there is a concrete, recursive in L_{i+1}, elementary embedding $g_{i,i+1}$ from $(\mathcal{D}_i, d)_{d \in |\mathcal{D}_i|}$ to the reduction of $\mathcal{D}_{i+1}$ to the common vocabulary. This ends the induction step.

Therefore, we have:

$$
\begin{tikzcd}
\mathcal{B} \arrow[r, phantom, "\cong"] & \mathcal{D}_0 \arrow[r, hook, "g_{0,1}"] \arrow[d, hook, "f_{0,0}"'] & \mathcal{D}_1 \arrow[r, hook, "g_{1,2}"] \arrow[d, hook, "f_{1,1}"'] & \mathcal{D}_2 \arrow[r, hook, "g_{2,3}"] \arrow[d, hook, "f_{2,2}"'] & \cdots \\
\mathcal{A} \arrow[r, phantom, "\equiv"] & \mathcal{C}_0 \arrow[r, phantom, "\equiv"] \arrow[ur, hook, "f_{0,1}"] & \mathcal{C}_1 \arrow[r, phantom, "\equiv"] \arrow[ur, hook, "f_{1,2}"] & \mathcal{C}_2 \arrow[r, phantom, "\equiv"] & \cdots
\end{tikzcd}
$$

We show that $\mathcal{D}_0, \mathcal{C}_0, \mathcal{D}_1, \mathcal{C}_1, \ldots$ is a jump–concrete sequence of low concrete models. For $i \in \omega$ let $\mathcal{C}_i^*$ be the halting problem for $\mathcal{C}_i$ and let $\mathcal{D}_i^*$ be the halting problem for $\mathcal{D}_i$. We show that $i \mapsto \ulcorner \mathcal{D}_i^* \urcorner$ and $i \mapsto \ulcorner \mathcal{C}_i^* \urcorner$ are concrete maps. The argument is similar to that presented in the previous section. Algorithms for $i \mapsto \ulcorner \mathcal{C}_i^* \urcorner$ and $i \mapsto \ulcorner \mathcal{D}_i^* \urcorner$ use the algorithm from Theorem 2.3.14 to compute codes of algorithms which decide subsequent concrete models and halting problem of these models in K. The only differences come from the construction, where $\mathrm{Th}(\mathcal{A})$, $\mathrm{Diag}(\mathcal{C}_i)$, Π_1–$\mathrm{Diag}(\mathcal{D}_i)$ and $\mathrm{ElDiag}(\mathcal{D}_i)$ are computed. However, these sets are easily computable in the underlying concrete models. Therefore, $\mathcal{D}_0, \mathcal{C}_0, \mathcal{D}_1, \mathcal{C}_1, \ldots$ is a jump–concrete sequence of low concrete models.

Therefore:

- $\mathcal{D}_0, \mathcal{C}_0, \mathcal{D}_1, \mathcal{C}_1, \ldots$ is a jump–concrete sequence of low concrete models,
- for every $i \in \omega$ it holds that $\bigoplus_{j \leqslant i} \mathcal{C}_j \oplus \mathcal{D}_j$ is low,
- for every $i \in \omega$ it holds that $\mathcal{D}_i \models \mathrm{Diag}(\mathcal{C}_i) \cup \mathrm{ElDiag}(\mathcal{D}_i)$ and $\mathcal{C}_i \models \mathrm{Th}(\mathcal{A}) \cup \Pi_1$–$\mathrm{Diag}(\mathcal{D}_i)$.

Observe that for each $i \in \omega$ it holds that $|\mathcal{C}_i| \subseteq V_{2i}$ and $|\mathcal{D}_i| \subseteq V_{2i+1}$. Therefore, the universes of all constructed models are all disjoint. We assumed also that $i \mapsto \ulcorner V_i \urcorner$ is recursive.

By Theorem 4.1.22 it follows that there is a concrete chain of concrete models $\mathcal{B}_0, \mathcal{A}_0, \mathcal{B}_1, \mathcal{A}_1, \ldots$ and for each $i \in \omega$ there are concrete isomorphisms $g_i : \mathcal{A}_i \to \mathcal{C}_i^\sigma$ and $h_i : \mathcal{B}_i \to \mathcal{D}_i^\sigma$, where $\mathcal{C}_i^\sigma$ and $\mathcal{D}_i^\sigma$ are reductions to σ of the underlying concrete models.

Therefore, we have the following diagram:

$$
\begin{array}{ccccccccc}
\mathcal{B} & \cong & \mathcal{B}_0 & \preccurlyeq & \mathcal{B}_1 & \preccurlyeq & \mathcal{B}_2 & \preccurlyeq & \cdots \\
 & & \rotatebox{90}{$\subseteq$} & \rotatebox{45}{$\preccurlyeq$} & \rotatebox{90}{$\subseteq$} & \rotatebox{45}{$\preccurlyeq$} & \rotatebox{90}{$\subseteq$} & & \\
\mathcal{A} & \equiv & \mathcal{A}_0 & \equiv & \mathcal{A}_1 & \equiv & \mathcal{A}_2 & \equiv & \cdots
\end{array}
$$

The sum $\bigcup_{i \in \omega} \mathcal{B}_i$ of the concrete chain of concrete models $(\mathcal{B}_i)_{i \in \omega}$ is a concrete model by Lemma 4.2.43.

Moreover, $\bigcup_{i \in \omega} \mathcal{B}_i$ is equal to $\bigcup_{i \in \omega} \mathcal{A}_i$. It holds that $\mathcal{B}_0 \preccurlyeq \bigcup_{i \in \omega} \mathcal{B}_i$. Since for each $i \in \omega$ it holds that $\mathcal{A}_i \models T$ and T is closed under unions of chains, it follows that $\bigcup_{i \in \omega} \mathcal{B}_i \models T$. Therefore also $\mathcal{B}_0 \models T$ and thus $\mathcal{B} \models T$.

We have shown then that for all low models $\mathcal{A}, \mathcal{B}$ such that $\mathcal{A} \oplus \mathcal{B}$ is low, $\mathcal{A} \models T$ and every Π_2 sentence δ which is true in $\mathcal{A}$ is also true in $\mathcal{B}$ (thus the same holds for sentences from Δ), it holds that $\mathcal{B} \models T$. Therefore by Lemma 4.2.44, there is an axiomatisation $\Gamma \subseteq \Delta$ of T. By putting axioms from Γ to prenex normal forms in an appropriate way, we obtain Γ' which contains only Π_2 sentences and which axiomatises T. $\square$

Another classical preservation result characterises theories closed under homomorphisms.

We introduce the notion of a positive formula.

Definition 4.2.46 (Positive Formula) *A formula ψ is positive if*

- *ψ is an atomic formula,*
- *$\psi = \psi_1 \circ \psi_2$, for positive formulae ψ_1, ψ_2 and $\circ \in \{\wedge, \vee\}$.*
- *$\psi = Qx\,\psi_1$, for a positive formula ψ_1 and $Q \in \{\exists, \forall\}$.*

We state the characterisation theorem for theories closed under concrete homomorphisms.

Theorem 4.2.47 *Let T be a consistent recursive σ–theory. Then T is preserved under concrete homomorphisms if and only if it has a set of positive axioms.*

Proof: For the easy right–to–left implication it is sufficient to show that concrete homomorphisms preserve the truth of positive formulae i.e. if $h\colon \mathcal{A} \to \mathcal{B}$ is a concrete homomorphism of a concrete model $\mathcal{A}$ onto a concrete model $\mathcal{B}$ and $\varphi(x_0, x_1, \ldots, x_n)$ is a positive formula, then $\mathcal{A} \models \varphi[a_0, a_1, \ldots, a_n]$ implies $\mathcal{B} \models \varphi[h(a_0), h(a_1), \ldots, h(a_n)]$.

We prove this by induction on the construction of positive formulae. The base case follows directly from the definition of concrete homomorphism.

Suppose that the truth of formulae φ and ψ is preserved under concrete homomorphisms.

First, we show that the truth of $\varphi \wedge \psi$ is also preserved under concrete homomorphisms. Let $a_1, \ldots, a_k \in |\mathcal{A}|$ and let $\mathcal{A} \models (\varphi \wedge \psi)[a_1, \ldots, a_k]$. Then by the definition of the satisfaction relation we have that $\mathcal{A} \models \varphi[a_1, \ldots, a_k]$ and $\mathcal{A} \models \psi[a_1, \ldots, a_k]$. By the induction hypothesis it holds that $\mathcal{B} \models \varphi[h(a_1), \ldots, h(a_k)]$ and $\mathcal{B} \models \psi[h(a_1), \ldots, h(a_k)]$. Again, by the definition of the satisfaction relation it follows that $\mathcal{B} \models (\varphi \wedge \psi)[h(a_1), \ldots, h(a_k)]$ which completes the proof of the case of $\wedge$. The case of $\vee$ is analogous. Now suppose that $\mathcal{A} \models \exists v_m\, \varphi[a_1, \ldots, a_k]$. By the definition of the satisfaction relation there exists $n \in |\mathcal{A}|$ such that $\mathcal{A} \models \varphi[(a_1, \ldots, a_k)[v_m := n]]$. Using the induction hypothesis we infer that $\mathcal{B} \models \varphi[(h(a_1), \ldots, h(a_k))[v_m := h(n)]]$. Therefore, $\mathcal{B} \models \exists v_m\, \varphi[h(a_1), \ldots, h(a_k)]$. In the proof of the final case – for the $\forall$ quantifier – we use the fact that h is onto $|\mathcal{B}|$ i.e. that for every $b \in |\mathcal{B}|$ there exists $a \in |\mathcal{A}|$ such that $b = h(a)$. Let $b \in |\mathcal{B}|$ and $b = h(a)$ for $a \in |\mathcal{A}|$. Let $\mathcal{A} \models \forall v_m\, \varphi[a_1, \ldots, a_k]$. Then $\mathcal{A} \models \varphi[(a_1, \ldots, a_k)[v_m := a]]$ and by the induction hypothesis $\mathcal{B} \models \varphi[(h(a_1), \ldots, h(a_k))[v_m := b]]$. Since b was arbitrary,

$\mathcal{B} \models \forall x\, \varphi[h(a_1), \ldots, h(a_k)]$ by the definition of the satisfaction relation. This ends the proof of the easy direction of the theorem.

For the proof of the other implication suppose that T is preserved under concrete homomorphisms. Let $\mathcal{A} \operatorname{pos} \mathcal{B}$ mean that for every positive sentence ψ if $\mathcal{A} \models \psi$, then $\mathcal{B} \models \psi$. We want to use Lemma 4.2.44. Let the set Δ be the set of all positive formulae – it is recursive and closed under finite disjunctions. Let Δ^- be the set of negations of positive formulae, obviously Δ^- is also recursive.

We show that for all low concrete models $\mathcal{A}, \mathcal{B}$ such that:

- $\mathcal{A} \oplus \mathcal{B}$ is low,
- $\mathcal{A} \models T$,
- $\mathcal{A} \operatorname{pos} \mathcal{B}$,

it holds that $\mathcal{B} \models T$.

Fix two low models $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A} \oplus \mathcal{B}$ is low, $\mathcal{A} \models T$ and such that $\mathcal{A} \operatorname{pos} \mathcal{B}$. We want to construct a tower of models as in the diagram:

$$\begin{tikzcd}
\mathcal{A} \arrow[r, phantom, "\cong"] & \mathcal{A}_0 \arrow[r, phantom, "\preccurlyeq"] \arrow[dr, "h_{0,1}"'] & \mathcal{A}_1 \arrow[r, phantom, "\preccurlyeq"] \arrow[dr, "h_{1,2}"'] & \mathcal{A}_2 \arrow[r, phantom, "\preccurlyeq"] & \cdots \arrow[r, phantom, "\preccurlyeq"] & \mathcal{A}_\omega \arrow[d, "h_\omega"'] \\
\mathcal{B} \arrow[r, phantom, "\cong"] & \mathcal{B}_0 \arrow[r, phantom, "\preccurlyeq"] & \mathcal{B}_1 \arrow[r, phantom, "\preccurlyeq"] \arrow[u, "h_{0,1}"] & \mathcal{B}_2 \arrow[r, phantom, "\preccurlyeq"] \arrow[u, "h_{1,2}"] & \cdots \arrow[r, phantom, "\preccurlyeq"] & \mathcal{B}_\omega
\end{tikzcd}$$

Similarly to the previous constructions, we start with constructing concrete models $\mathcal{C}_i$, $\mathcal{D}_i$ as in the following diagram:

$$\begin{tikzcd}
\mathcal{A} \arrow[r, phantom, "\cong"] & \mathcal{C}_0 \arrow[r, hook, "e_0^0"] \arrow[dr, "f_0"'] & \mathcal{C}_1 \arrow[r, hook, "e_1^0"] \arrow[dr, "f_1"'] & \mathcal{C}_2 \arrow[r, hook, "e_2^0"] & \cdots \\
\mathcal{B} \arrow[r, phantom, "\cong"] & \mathcal{D}_0 \arrow[r, hook, "e_0^1"] & \mathcal{D}_1 \arrow[r, hook, "e_1^1"] \arrow[u, "g_1"] & \mathcal{D}_2 \arrow[r, hook, "e_2^1"] \arrow[u, "g_2"] & \cdots
\end{tikzcd}$$

Let $R_0, R_1, \ldots, V_0, V_1, \ldots$ be pairwise disjoint recursive sets such that:

- $i \mapsto \ulcorner R_i \urcorner$ is recursive,
- $i \mapsto \ulcorner V_i \urcorner$ is recursive,
- for $i \in \omega$, $D_i = \{\mathfrak{c}_r : r \in R_i\}$ is a set of new constants,

- for $i \in \omega$, $W_i = \{\mathfrak{c}_r : r \in V_i\}$ is a set of new constants.

In the construction below, for $i \in \omega$ we define $\mathrm{ElDiag}(\mathcal{D}_i)$ using constants from D_{2i}, and $\mathrm{ElDiag}(\mathcal{C}_i)$ using constants from D_{2i+1}. Constants from sets W_i are witnesses used while applying the Low Completeness Theorem (4.2.16).

Let $v_0^0, v_1^0, \ldots$ be an effective presentation of V_0 and let $v_0^1, v_1^1, \ldots$ be an effective presentation of V_1. Then $V_0^{\mathcal{A}} = \{v_a^0 : a \in |\mathcal{A}|\}$ is a recursive in $\mathcal{A}$ set and $V_1^{\mathcal{B}} = \{v_b^1 : b \in |\mathcal{B}|\}$ is a recursive in $\mathcal{B}$ set. Then the function $iso_{\mathcal{A}} : V_0^{\mathcal{A}} \to |\mathcal{A}|$ such that for each $a \in |\mathcal{A}|$ it holds that $iso_{\mathcal{A}}(v_a^0) = a$ is a concrete, recursive in $\mathcal{A}$, bijection. Similarly, the function $iso_{\mathcal{B}} : V_1^{\mathcal{B}} \to |\mathcal{B}|$ such that for each $b \in |\mathcal{B}|$ it holds that $iso_{\mathcal{B}}(v_b^1) = b$ is a concrete, recursive in $\mathcal{B}$ bijection. Let $\mathcal{C}_0$ be the co–image of $\mathcal{A}$ under $iso_{\mathcal{A}}$ and let $\mathcal{D}_0$ be the co–image of $\mathcal{B}$ under $iso_{\mathcal{B}}$. Then, by Lemma 4.1.10 $\mathcal{C}_0$ is a recursive in $\mathcal{A}$ concrete model and $\mathcal{D}_0$ is a recursive in $\mathcal{B}$ concrete model. Note that $iso_{\mathcal{A}}$ and $iso_{\mathcal{B}}$ are concrete isomorphisms of the corresponding concrete models. Note also that $|\mathcal{C}_0| \subseteq V_0$ and $|\mathcal{D}_0| \subseteq V_1$.

For $i \in \omega$ let $\sigma^i = \sigma \cup \bigcup_{j<i} D_j$.

The construction goes by induction.

- $\mathcal{C}_0 \overset{iso_{\mathcal{A}}}{\cong} \mathcal{A}$,

- $\mathcal{D}_0 \overset{iso_{\mathcal{B}}}{\cong} \mathcal{B}$,

- $T_{i+1} = (\mathrm{ElDiag}(\mathcal{C}_i) \cap \Delta) \cup \mathrm{ElDiag}(\mathcal{D}_i)$,
- $\mathcal{D}_{i+1}$ is the low concrete σ^{2i+2}–model obtained applying the Low Completeness Theorem to T_{i+1} and to the set of witnesses W_{2i+2},

- $S_{i+1} = (\mathrm{ElDiag}(\mathcal{D}_{i+1}) \cap \Delta^-) \cup \mathrm{ElDiag}(\mathcal{C}_i)$,
- $\mathcal{C}_{i+1}$ is the low concrete σ^{2i+3}–model obtained applying the Low Completeness Theorem to T_{i+1} and to the set of witnesses W_{2i+3}.

For $i \in \omega$, let $L_i = \bigoplus_{m \leqslant i} (\mathcal{C}_m \oplus \mathcal{D}_m)$.

By induction we show that the construction is correct. It is sufficient to show that for each $i \in \omega$ theories T_i and S_i are low and both conditions: $\mathcal{C}_i \operatorname{pos} \mathcal{D}_i$ and $\mathcal{C}_i \operatorname{pos} \mathcal{D}_{i+1}$ hold.

For the base step we put $\mathcal{C}_0 \cong \mathcal{A}$ with the the concrete isomorphism $iso_{\mathcal{A}}$ and $\mathcal{D}_0 \cong \mathcal{B}$ with the concrete isomorphism $iso_{\mathcal{B}}$. It follows that $\mathcal{C}_0 \operatorname{pos} \mathcal{D}_0$. We know that $L_0 = \mathcal{C}_0 \oplus \mathcal{D}_0$ is low. This ends the base step.

For the induction hypothesis suppose that the tower is constructed up to $i \in \omega$ i.e. low concrete models $\mathcal{C}_i$ and $\mathcal{D}_i$ are given, $L_i = \bigoplus_{m \leqslant i} \mathcal{C}_m \oplus \mathcal{D}_m$ is low and $\mathcal{C}_i \operatorname{pos} \mathcal{D}_i$.

First, consider the σ^{2i+2}–theory $T_{i+1} = (\mathrm{ElDiag}(\mathcal{C}_i) \cap \Delta) \cup \mathrm{ElDiag}(\mathcal{D}_i)$. The theory T_{i+1} is recursive in L_i, and therefore low.

We show that T_{i+1} is consistent. Otherwise, there exists a finite set of positive σ^{2i+2} sentences $F(\overline{a}, \overline{b}) \subseteq \mathrm{ElDiag}(\mathcal{C}_i)$ such that $\mathrm{ElDiag}(\mathcal{D}_i) \vdash \neg \bigwedge F(\overline{a}, \overline{b})$, where $\overline{a}$, $\overline{b}$ are all the constants occurring in F, from D_{2i+1} and D_{2i} respectively. In the case of $i = 0$ the proof is easier since $\overline{b}$ is empty. We focus on the case of $i > 0$. By the lemma on constants $\mathrm{ElDiag}(\mathcal{D}_i) \vdash \forall \overline{x} \neg \bigwedge F(\overline{x}, \overline{b})$. It follows that $\mathrm{ElDiag}(\mathcal{D}_i) \vdash \exists \overline{y} \forall \overline{x} \neg \bigwedge F(\overline{x}, \overline{y})$. Therefore $\mathcal{D}_i \not\models \forall \overline{y} \exists \overline{x} \bigwedge F(\overline{x}, \overline{y})$. The sentence $\forall \overline{y} \exists \overline{x} \bigwedge F(\overline{x}, \overline{y})$ is positive and in the common language of $\mathcal{C}_i$ and $\mathcal{D}_i$. Since $\mathcal{C}_i \operatorname{pos} \mathcal{D}_i$ it follows that $\mathcal{C}_i \not\models \forall \overline{y} \exists \overline{x} \bigwedge F(\overline{x}, \overline{y})$. Therefore $\mathcal{C}_i \not\models \exists \overline{x} \bigwedge F(\overline{x}, \overline{b})$. But $\bigwedge F(\overline{a}, \overline{b}) \in \mathrm{ElDiag}(\mathcal{C}_i)$ which is a contradiction.

Therefore T_{i+1} is low and consistent σ^{2i+2} theory. Let $\mathcal{D}_{i+1}$ be the low concrete σ^{2i+2}–model obtained by applying the Low Completeness Theorem (4.2.16) to T_{i+1} and to the set of witnesses W_{2i+2}. Then $\mathcal{C}_i \operatorname{pos} \mathcal{D}_{i+1}$, since $\mathcal{D}_{i+1} \models \mathrm{ElDiag}(\mathcal{C}_i) \cap \Delta$. The set $L_i \oplus \mathcal{D}_{i+1}$ is low. It also holds that $|\mathcal{D}_{i+1}| \subseteq V_{2i+2}$.

Now consider the σ^{2i+3}–theory $S_{i+1} = (\mathrm{ElDiag}(\mathcal{D}_{i+1}) \cap \Delta^-) \cup \mathrm{ElDiag}(\mathcal{C}_i)$. S_{i+1} is recursive in $L_i \oplus \mathcal{D}_{i+1}$, thus low.

Suppose for the sake of contradiction that S_{i+1} is inconsistent. Then there exists a finite set of sentences $F(\overline{c}, \overline{d}) \subseteq \mathrm{ElDiag}(\mathcal{D}_{i+1}) \cap \Delta^-$ such that $\mathrm{ElDiag}(\mathcal{C}_i) \vdash \neg \bigwedge F(\overline{c}, \overline{d})$, $\overline{c}$ are all the constants from D_{2i+1} occurring in F and $\overline{d}$ are all the constants from D_{2i+2} occurring in F. Let G be the corresponding set of positive sentences whose negations are in F. Then $\mathrm{ElDiag}(\mathcal{C}_i) \vdash \exists \overline{x} \neg \bigwedge F(\overline{x}, \overline{d})$ and by the lemma on constants $\mathrm{ElDiag}(\mathcal{C}_i) \vdash \forall \overline{y} \exists \overline{x} \neg \bigwedge F(\overline{x}, \overline{y})$, and therefore $\mathrm{ElDiag}(\mathcal{C}_i) \vdash \forall \overline{y} \exists \overline{x} \bigvee G(\overline{x}, \overline{y})$. Since $\forall \overline{y} \exists \overline{x} \bigvee G(\overline{x})$ is a positive σ^{2i+1}–sentence and $\mathcal{C}_i \operatorname{pos} \mathcal{D}_{i+1}$, we have that $\mathcal{D}_{i+1} \models \forall \overline{y} \exists \overline{x} \bigvee G(\overline{x}, \overline{y})$. This is a contradiction, since we have $\mathcal{D}_{i+1} \models \exists \overline{y} \exists \overline{x} \bigwedge F(\overline{x}, \overline{y})$.

Let $\mathcal{C}_{i+1}$ be the low concrete σ^{2i+3}–model obtained by applying the Low Completeness Theorem to S_{i+1} and to the set of witnesses W_{2i+3}. Then we have that $L_{i+1} = L_i \oplus \mathcal{C}_{i+1} \oplus \mathcal{D}_{i+1}$ is low. We have $|\mathcal{C}_{i+1}| \subseteq V_{2i+3}$.

We show that $\mathcal{C}_{i+1} \operatorname{pos} \mathcal{D}_{i+1}$. Let ψ be a positive sentence such that $\mathcal{C}_{i+1} \models \psi$. Suppose for the sake of contradiction that $\mathcal{D}_{i+1} \not\models \psi$. Then $\mathcal{D}_{i+1} \models \neg\psi$. Therefore, since $\mathcal{C}_{i+1} \models \mathrm{ElDiag}(\mathcal{D}_{i+1}) \cap \Delta^-$, it holds that $\mathcal{C}_{i+1} \models \neg\psi$. This contradicts the assumption.

Observe that for each $i \in \omega$, since $\mathcal{D}_{i+1} \models \mathrm{ElDiag}(\mathcal{C}_i) \cap \Delta$, there is a canonical concrete homomorphism f_i from $(\mathcal{C}_i, c)_{c \in |\mathcal{C}_i|}$ to $\mathcal{D}_{i+1}$ such that for $c \in |\mathcal{C}_i|$, $f_i(c)$ is the interpretation in $\mathcal{D}_{i+1}$ of the constant naming c in $\mathcal{C}_i$. Dually for $i \in \omega$, since $\mathcal{C}_{i+1} \models \mathrm{ElDiag}(\mathcal{D}_{i+1}) \cap \Delta^-$, there is a canonical concrete function g_{i+1} from $|\mathcal{D}_{i+1}|$ to $|\mathcal{C}_{i+1}|$ such that $g_{i+1}(d)$ is the interpretation in $\mathcal{C}_{i+1}$ of the constant naming d in $\mathcal{D}_{i+1}$.

Moreover, since $\mathcal{C}_{i+1} \models \mathrm{ElDiag}(\mathcal{C}_i)$ and $\mathcal{D}_{i+1} \models \mathrm{ElDiag}(\mathcal{D}_i)$, by Lemma 4.1.17 there are concrete elementary embeddings e_i^0 from $\mathcal{C}_i$ into the reduction of $\mathcal{C}_{i+1}$ to the common vocabulary and e_i^1 from $\mathcal{D}_i$ into the reduction of $\mathcal{D}_{i+1}$ to the common vocabulary.

We have the following diagram:

$$\begin{tikzcd}
\mathcal{A} & \cong & \mathcal{C}_0 \arrow[r, hook, "e_0^0"] \arrow[dr, "f_0"'] & \mathcal{C}_1 \arrow[r, hook, "e_1^0"] \arrow[dr, "f_1"'] & \mathcal{C}_2 \arrow[r, hook, "e_2^0"] & \cdots \\
\mathcal{B} & \cong & \mathcal{D}_0 \arrow[r, hook, "e_0^1"] & \mathcal{D}_1 \arrow[u, "g_1"] \arrow[r, hook, "e_1^1"] & \mathcal{D}_2 \arrow[u, "g_2"] \arrow[r, hook, "e_2^1"] & \cdots
\end{tikzcd}$$

By the arguments analogous to those presented in the previous proof, the entire construction can be performed concretely. $(\mathcal{C}_i)_{i\in\omega}$ and $(\mathcal{D}_i)_{i\in\omega}$ are jump–concrete sequences of concrete models and they satisfy the assumptions of Theorem 4.1.22.

Therefore, by Theorem 4.1.22, for each $i \in \omega$ there are concrete σ–models $\mathcal{A}_i$ and concrete isomorphisms j_i^0 from $\mathcal{A}_i$ into the reduction of $\mathcal{C}_i$ to σ. Similarly, there are also concrete models $\mathcal{B}_i$ and concrete isomorphisms j_i^1 from $\mathcal{B}_i$ to the reduction of $\mathcal{D}_i$ to σ.

Moreover, since concrete embeddings e_i^0 and e_i^1 are elementary, $(\mathcal{A}_i)_{i\in\omega}$ and $(\mathcal{B}_i)_{i\in\omega}$ are concrete elementary chains of concrete models.

Let $\mathcal{A}_\omega$ and $\mathcal{B}_\omega$ be sums of these concrete elementary chains of concrete models (their existence is provided by Theorem 4.2.43).

Then, for $i \in \omega$, $h_{i,i+1} = (j_{i+1}^1)^{-1} \circ f_i \circ j_i^0$ is a concrete homomorphism from $\mathcal{A}_i$ into $\mathcal{B}_{i+1}$. For $i > 0$, $h_{i,i} = (j_i^0)^{-1} \circ g_i \circ j_i^1$ is a concrete function from $|\mathcal{B}_i|$ into $|\mathcal{A}_i|$. It is easy to see that for $i > 0$ it holds that $h_{i+1,i+1}^{-1} \subseteq h_{i+1,i+2}$. Moreover for $i \in \omega$ it holds that $h_{i,i+1} \subseteq h_{i+1,i+2}$.

We define $h_\omega = \bigcup_{i\in\omega} h_{i,i}$. Observe that h_ω is a concrete homomorphism and it is onto.

We have the following diagram:

$$\begin{tikzcd}
\mathcal{A} & \cong & \mathcal{A}_0 \arrow[drr, "h_{0,1}"'] & \preccurlyeq & \mathcal{A}_1 \arrow[drr, "h_{1,2}"'] & \preccurlyeq & \mathcal{A}_2 & \preccurlyeq & \cdots & \preccurlyeq & \mathcal{A}_\omega \arrow[d, "h_\omega"] \\
\mathcal{B} & \cong & \mathcal{B}_0 & \preccurlyeq & \mathcal{B}_1 \arrow[u, "h_{0,1}"] & \preccurlyeq & \mathcal{B}_2 \arrow[u, "h_{1,2}"] & \preccurlyeq & \cdots & \preccurlyeq & \mathcal{B}_\omega
\end{tikzcd}$$

Since $\mathcal{A}_\omega \models T$ and T is preserved under concrete homomorphisms, we have that $\mathcal{B}_\omega \models T$ and therefore $\mathcal{B}_0 \models T$. Since $\mathcal{B}_0$ is concretely isomorphic to $\mathcal{D}_0$ which is concretely isomorphic to $\mathcal{B}$ then also $\mathcal{B} \models T$. Then, by Lemma 4.2.44 T has a set of positive axioms.

$\square$

Note that in this section we considered consistent recursive theories. The results presented above can be naturally strengthened to consistent, recursively enumerable theories. This can be obtained by means of Craig's theorem stating that every recursively enumerable theory has a recursive set of axioms. Moreover the procedure of finding a recursive axiomatisation of a recursively enumerable theory is recursive. Therefore, for every recursively enumerable theory we may take its recursive axiomatisation and prove the theorems presented in this section for this axiomatisation.

4.2.5 Craig's Interpolation Lemma and Robinson's Joint Consistency Theorem

In this section we consider two theorems that are very useful in the axiomatic model theory – Craig's interpolation lemma and Robinson's joint consistency theorem. Let us first state both theorems.

Theorem 4.2.48 (Craig's Interpolation Lemma) *Let φ, ψ be σ_1 and σ_2 sentences respectively and let $\varphi \models \psi$. The there is a $\sigma_1 \cap \sigma_2$ sentence ϑ such that $\varphi \models \vartheta$ and $\vartheta \models \psi$.*

Theorem 4.2.49 (Robinson's Joint Consistency Theorem) *Let T be a consistent and complete theory in vocabulary σ_0. Let σ_1 and σ_2 be vocabularies such that $\sigma_0 = \sigma_1 \cap \sigma_2$. Let T_1, T_2 be consistent σ_1, σ_2 theories respectively such that $T \subseteq T_1 \cap T_2$. Then theory $T_1 \cup T_2$ is consistent.*

Both the above–mentioned theorems give us some insight on how different languages interact. The statement of Craig's interpolation lemma for concrete models framework requires only to understand $\models$ symbol as it was introduced in Section 4.1 instead of the standard model–theoretic interpretation as well as understanding σ_1 and σ_2 as concrete vocabularies.

The original proof of Craig's interpolation lemma (see [Cra57]) was performed in the sequent calculus. It is then purely syntactic and constructive, since it provides us with an algorithm which takes sentences φ, ψ such that $\varphi \models \psi$ as an input and computes the interpolant for them. This tells us that Craig's lemma holds for concrete models framework and so does the closely related to it Robinson's joint consistency theorem.

Note that we have presented the syntactic version of the Robinson's joint consistency theorem stating that the sum $T_1 \cup T_2$ of theories is consistent. In the axiomatic model–theory there is an equivalent formulation of Robinson's joint consistency theorem stating that $T_1 \cup T_2$ has a model. In the concrete models context those two formulation are not equivalent since the Concrete Completeness Theorem (4.2.10) requires that the set of consequences of a theory is concrete. In the case of Robinson's joint consistency theorem, for concrete theories T_1, T_2, consequences of $T_1 \cup T_2$ may be Σ^0_2–complete and therefore there may be no concrete model of $T_1 \cup T_2$.

Since we are especially interested in model–theoretic constructions, we would like to consider Robinson's construction which was originally used in the proof of his theorem (see [CK73]). However we consider this construction in the context of Craig's interpolation lemma.

We show that Robinson's construction fails in the final step i.e. we cannot expect a, so called, glued model to be concrete. First, let us define what we mean by a glued model.

Definition 4.2.50 (Glued Structures and Models)
Let $\sigma_1 = (P_1, \ldots, P_m, ar, \varphi_C)$ and $\sigma_2 = (S_1, \ldots, S_k, ar', \varphi_D)$ be concrete vocabularies. Let $\mathcal{F} = (\varphi_U, \varphi_{P_1}, \ldots, \varphi_{P_m}, \varphi_{U,C})$ and let $\mathcal{G} = (\psi_U, \psi_{S_1}, \ldots, \psi_{S_k}, \psi_{V,D})$ be concrete σ_1 and σ_2 structures respectively. Moreover, let φ_U and ψ_U FM–represent the same set U and the interpretations of constants FM–represented by $\varphi_{C,U}$ and $\varphi_{V,D}$ agree on constants from $\sigma_1 \cap \sigma_2$.

Then $\mathcal{F} * \mathcal{G} = (\varphi_U, \varphi_{P_1}, \ldots, \varphi_{P_m}, \psi_{S_1}, \ldots, \psi_{S_k}, \varphi_{U,C} \vee \psi_{V,D})$ is a concrete–$\sigma_1 \cup \sigma_2$ structure. We say that $\mathcal{F} * \mathcal{G}$ is glued from $\mathcal{F}$ and $\mathcal{G}$.

Let $\mathcal{A} = (\mathcal{F}, \varphi_{\models})$ and $\mathcal{B} = (\mathcal{G}, \psi_{\models})$ be concrete σ_1 and σ_2 models respectively. We say that a concrete $\sigma_1 \cup \sigma_2$–model $\mathcal{C} = (\mathcal{F} * \mathcal{G}, \theta_{\models})$ is glued from $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{C} = \mathcal{A} * \mathcal{B}$.

The failure of the construction is due to the following theorem which tells that a model obtained by glueing even recursive models may not be concrete.

Theorem 4.2.51 *There are concrete models $\mathcal{A}, \mathcal{B}$ such that there is no concrete model $\mathcal{A} * \mathcal{B}$.*

Proof: We prove even stronger fact, namely that there are recursive concrete models $\mathcal{A}$, $\mathcal{B}$, such that $\mathcal{A} * \mathcal{B}$ is not arithmetical.

Let φ_ω FM–represent ω, φ_+ FM–represent the ternary relation of addition on the natural numbers R_+ and $\varphi_\times$ FM–represent the ternary relation of multiplication on the natural numbers $R_\times$.

The concrete structure $\mathcal{F} = (\varphi_\omega, \varphi_+)$ FM–represents the standard model of Presburger's arithmetic. The satisfaction relation $R_{\models}$ in $\mathcal{F}$ is recursive. Let $\varphi_{\models}$ FM–represent $R_{\models}$. Let $\mathcal{A} = (\mathcal{F}, \varphi_{\models})$. Note that $\mathcal{A}$ is recursive.

Similarly the concrete structure $\mathcal{G} = (\varphi_\omega, \varphi_\times)$ FM–represents the standard model of Skolem's arithmetic. The satisfaction relation $S_{\models}$ in $\mathcal{G}$ is also recursive. Let $\psi_{\models}$ FM–represent $S_{\models}$. Let $\mathcal{B} = (\mathcal{G}, \psi_{\models})$. Note that $\mathcal{B}$ is recursive.

Then the concrete structure $\mathcal{F} * \mathcal{G}$ FM–represents the standard model of arithmetic of addition and multiplication. The satisfaction relation in $\mathcal{F} * \mathcal{G}$ is not arithmetical and therefore it cannot be FM–represented by any formula $\theta_{\models}$. Thus there is no concrete model $\mathcal{C} = \mathcal{A} * \mathcal{B}$. $\square$

Similarly as in Section 4.2.3, we present the Robinson's construction used in the proof of Craig's interpolation lemma, since we are interested rather in feasibility of the constructions than the results themselves.

Robinson's construction uses two elementary chains – a tower – of models which are interlaced by the common stem. This common stem is the complete, consistent theory in the common language. The Low Completeness Theorem (4.2.16) and Theorem 4.1.22 enable us to construct the desired tower of concrete models. Lemma 4.2.43 on the existence of the sum of elementary concrete chains proven in the previous section, allows us to sum those chains to get two concrete models in languages σ_1 and σ_2 with the same universe.

However, at the last stage of the proof, a sudden problem appears. In the axiomatic model theory it is sufficient to glue the two models together by simply taking relations from both of them into one common model. Such a construction is not admissible in the concrete models framework as it is shown by Theorem 4.2.51.

Let us now show how the Robinson's construction is used to prove the Craig's lemma. The construction is performed in concrete models until the last step which is shown to be illegal in this framework.

Let us fix φ and ψ such that $\varphi \models \psi$. The model–theoretic proof of Craig's interpolation lemma goes by showing that if there was no interpolant for φ and ψ, then there would be a model of $\varphi \wedge \neg\psi$ which would contradict $\varphi \models \psi$. Let us therefore assume that there is no interpolant for φ and ψ.

Let us introduce a convenient notation: $\sigma_0 = \sigma_1 \cap \sigma_2$, $\sigma_3 = \sigma_1 \cup \sigma_2$.

By the assumption, there is no σ_0 sentence ϑ such that $\varphi \models \vartheta$ and $\vartheta \models \psi$. Therefore there is also no interpolant in $\sigma_0 \cup C$, for any concrete set of constants C. Otherwise there would be a $\sigma_0 \cup \{c_1, \ldots, c_k\}$–sentence $\vartheta(c_1, \ldots, c_k)$ such that $\varphi \models \vartheta(c_1, \ldots, c_k)$ and $\vartheta(c_1, \ldots, c_k) \models \psi$. Since constants $c_1, \ldots, c_k$ do not appear in φ, by the lemma on constants we have that $\varphi \models \forall x_1 \ldots, x_k\, \vartheta(x_1, \ldots, x_k)$. On the other hand, since $\vartheta(c_1, \ldots, c_k) \models \psi$, then also $\forall x_1 \ldots, x_k\, \vartheta(x_1, \ldots, x_k) \models \psi$, since $\forall x_1 \ldots, x_k\, \vartheta(x_1, \ldots, x_k) \models \vartheta(c_1, \ldots, c_k)$. This would mean that $\forall x_1 \ldots, x_k\, \vartheta(x_1, \ldots, x_k)$ is a σ_0 interpolant for φ and ψ which contradicts the assumption.

Before proceeding to the construction itself let us take care of degenerate cases, which in this context are:

1. φ is not satisfiable,

2. ψ is valid.

In the case of 1, the sentence $\exists x\, x \neq x$ would be an interpolant contradicting the assumption. In the case of 2, $\exists x\, x = x$ would be an interpolant yielding, again, a contradiction. We can therefore assume that φ is satisfiable and that ψ is not valid i.e. $\neg\psi$ is satisfiable.

In the construction in axiomatic model theory the aim is to construct a model of $\varphi \wedge \neg\psi$ to get the contradiction with $\varphi \models \psi$. This aim is not achievable in the concrete models framework, but since it fails in the final step, and the previous are feasible in the concrete models framework, we present them below.

We start with a definition of separable theories.

Definition 4.2.52 (Separable Theories) *Theories T_1 in a concrete vocabulary σ_1 and T_2 in a concrete vocabulary σ_2 are separable if there is a $\sigma_1 \cap \sigma_2$ sentence ϑ such that $T_1 \vdash \vartheta$ and $T_2 \vdash \neg\vartheta$.*

If there is no such sentence we call them inseparable.

Therefore $\{\varphi\}$ and $\{\neg\psi\}$ are inseparable.

Let R be a recursive set such that $D = \{\mathfrak{c}_r : r \in R\}$ is a recursive set of new constants. We want to construct a low $\mathrm{CCW}(\sigma_0 \cup D, D)$ theory A such that both theories $A \cup \{\varphi\}$ and $A \cup \{\neg\psi\}$ are inseparable and therefore consistent.

Consider $\mathrm{Th}_{\varnothing}^{\sigma_0 \cup D, D}$ – the family of Henkin–style extensions of the empty theory labelled by nodes of the full binary tree. Such families were useful in the proof of the Low Completeness Theorem (4.2.16). Further we omit the superscripts $\sigma_0 \cup D, D$. Instead of considering $\mathrm{CON}(\mathrm{Th}_{\varnothing})$, we need some other operator such that not only it leaves the infinite branches through $\mathrm{Th}_{\varnothing}$ whose theories are consistent. We want to define an operator such that when applied it leaves only infinite paths $f \in 2^{\omega}$ such that for every $i \in \omega$, the theories $\mathrm{Th}_{\varnothing}(f{\restriction}i) \cup \{\varphi\}$ and $\mathrm{Th}_{\varnothing}(f{\restriction}i) \cup \{\neg\psi\}$ are inseparable. Of course this new operator cannot increase the complexity of the tree.

We use the same trick as with introducing the n–provability notion in the definition of CON. Observe that separability of theories A, B can be expressed as a Σ_1–formula relative to A and B.

$$\mathrm{Sep}(A, B) \equiv_{df} \exists k, l_1, l_2\, \mathrm{Sent}_{\sigma_0}(k) \wedge \mathrm{Prov}_A(k, l_1) \wedge \mathrm{Prov}_B(\mathrm{Neg}(k), l_2).$$

Since in our case $A = \{\varphi\}$ and $B = \{\neg\psi\}$ are recursive theories, their inseparability is expressible by a regular Σ_1–sentence.

We may introduce a recursive notion of n–separability in the following way:

$$n - \mathrm{Sep}(A, B) \equiv_{df} \exists k, l_1, l_2 \leqslant n\, \mathrm{Sent}_{\sigma_0}(k) \wedge \mathrm{Prov}_A(k, l_1) \wedge \mathrm{Prov}_B(\mathrm{Neg}(k), l_2).$$

Of course theories A, B are inseparable, therefore for every $n \in \omega$ they are not n–separable. Finally, we define the operator $\mathrm{INSEP}_{A,B}$:

$$\mathrm{INSEP}_{A,B}(\mathrm{Th}_T) =_{df} \{\tau \in 2^{<\omega}: \\ \text{it is not the case that } \mathrm{lh}(\tau) - \mathrm{Sep}(\mathrm{Th}_T(\tau) \cup A, \mathrm{Th}_T(\tau) \cup B)\}.$$

We consider the recursive tree $\mathrm{INSEP}_{\{\varphi\},\{\neg\psi\}}(\mathrm{Th}_\varnothing)$. First, note that the family $[\mathrm{INSEP}_{\{\varphi\},\{\neg\psi\}}(\mathrm{Th}_\varnothing)]$ of infinite branches through $\mathrm{INSEP}_{\{\varphi\},\{\neg\psi\}}(\mathrm{Th}_\varnothing)$ is not empty.

Lemma 4.2.53 $[\mathrm{INSEP}_{\{\varphi\},\{\neg\psi\}}(\mathrm{Th}_\varnothing)] \neq \varnothing$.

Proof: We show that $\mathrm{INSEP}_{\{\varphi\},\{\neg\psi\}}(\mathrm{Th}_\varnothing)$ has arbitrarily long branches. It is sufficient to show that for every $n \in \omega$ there is τ such that $\mathrm{lh}(\tau) = n$ and theories $\mathrm{Th}_\varnothing(\tau) \cup \{\varphi\}$ and $\mathrm{Th}_\varnothing(\tau) \cup \{\neg\psi\}$ are inseparable. We proceed by induction on $n \in \omega$.

For the base step assume that $n = 0$. Then $\tau = \varepsilon$. Then $\mathrm{Th}_\varnothing(\varepsilon) = \varnothing$ and theories $\{\varphi\}$ and $\{\neg\psi\}$ are inseparable by the assumption.

For the inductive step suppose that for some τ such that $\mathrm{lh}(\tau) = n$ theories $\mathrm{Th}_\varnothing(\tau) \cup \{\varphi\}$ and $\mathrm{Th}_\varnothing(\tau) \cup \{\neg\psi\}$ are inseparable.

Suppose $n = 2i+1$. Then, since theories $\mathrm{Th}_\varnothing(\tau)\cup\{\varphi\}$ and $\mathrm{Th}_\varnothing(\tau)\cup\{\neg\psi\}$ are inseparable ψ_i or $\neg\psi_i$ is consistent with them both. Therefore ether theories $\mathrm{Th}_\varnothing(\tau 0) \cup \{\varphi\}$ and $\mathrm{Th}_\varnothing(\tau 0) \cup \{\neg\psi\}$ or theories $\mathrm{Th}_\varnothing(\tau 1) \cup \{\varphi\}$ and $\mathrm{Th}_\varnothing(\tau 1) \cup \{\neg\psi\}$ are inseparable.

Suppose $n = 2i$. Then, for $k = 0, 1$ the theories $\mathrm{Th}_\varnothing(\tau k)$ and $\mathrm{Th}_\varnothing(\tau k)$ are both equal to $\mathrm{Th}_\varnothing(\tau) \cup \{\exists v_0\, \varphi_i(v_0) \Rightarrow \varphi_i(d)\}$ for some new constant d. If $\mathrm{Th}_\varnothing(\tau k) \cup \{\varphi\}$ and $\mathrm{Th}_\varnothing(\tau k) \cup \{\neg\psi\}$ were separated by a formula θ, then θ would also separate $\mathrm{Th}_\varnothing(\tau) \cup \{\varphi\}$ and $\mathrm{Th}_\varnothing(\tau) \cup \{\neg\psi\}$ which contradicts the assumption. $\square$

Therefore $[\mathrm{INSEP}_{\{\varphi\},\{\neg\psi\}}(\mathrm{Th}_\varnothing)]$ is not empty. Moreover, for every $f \in 2^\omega$ if $f \in [\mathrm{INSEP}_{\{\varphi\},\{\neg\psi\}}(\mathrm{Th}_\varnothing)]$, then theories $\mathrm{Th}_{f,\varnothing}\cup\{\varphi\}$ and $\mathrm{Th}_{f,\varnothing}\cup\{\neg\psi\}$ are inseparable, and therefore consistent. Otherwise, they would be n–separable for some $n \in \omega$. Therefore, for some $i \in \omega$ theories $\mathrm{Th}_\varnothing(f{\restriction}i) \cup \{\varphi\}$ and $\mathrm{Th}_\varnothing(f{\restriction}i) \cup \{\neg\psi\}$ would be n–separable. Therefore, for $l = \max\{i, k\}$ theories $\mathrm{Th}_\varnothing(f{\restriction}l) \cup \{\varphi\}$ and $\mathrm{Th}_\varnothing(f{\restriction}l) \cup \{\neg\psi\}$ would be l–separable. Thus $f{\restriction}l \notin \mathrm{INSEP}_{\{\varphi\},\{\neg\psi\}}(\mathrm{Th}_\varnothing)$. This would contradict the fact that $f \in [\mathrm{INSEP}_{\{\varphi\},\{\neg\psi\}}(\mathrm{Th}_\varnothing)]$.

It is easy to show, by the arguments similar to those from Section 4.2.1, that if $f \in [\mathrm{INSEP}_{\{\varphi\},\{\neg\psi\}}(\mathrm{Th}_\varnothing)]$, then that $\mathrm{Th}_{f,\varnothing}$ is a $\mathrm{CCW}(\sigma_0 \cup D, D)$. By the Low Basis Theorem we obtain a low set f such that $f \in [\mathrm{INSEP}_{\{\varphi\},\{\neg\psi\}}(\mathrm{Th}_\varnothing)]$. Let $A = \mathrm{Th}^{\sigma_0\cup D,D}_{\varnothing,f}$.

Therefore, $A\cup\{\varphi\}$ and $A\cup\{\neg\psi\}$ are inseparable, consistent low theories. Our aim is to build the following tower of concrete models:

$$
\begin{tikzcd}
\mathcal{A}_0 \arrow[d, dashed] & \preccurlyeq & \mathcal{A}_1 \arrow[d, dashed] & \preccurlyeq & \mathcal{A}_2 \arrow[d, dashed] & \preccurlyeq & \cdots & \preccurlyeq & \mathcal{A}_\omega \\
\mathcal{B}_0 \arrow[urr, dashed] & \preccurlyeq & \mathcal{B}_1 \arrow[urr, dashed] & \preccurlyeq & \mathcal{B}_2 & \preccurlyeq & \cdots & \preccurlyeq & \mathcal{B}_\omega
\end{tikzcd}
$$

The dashed arrows above are shown to be elementary embeddings of the reductions of concrete models to the common vocabulary σ_0.

Similarly as in previous constructions, we start with the construction of auxiliary concrete models $\mathcal{C}_i$ and $\mathcal{D}_i$.

Let $R_0, R_1, \ldots, V_0, V_1, \ldots$ be pairwise disjoint recursive sets such that:

- $i \mapsto \ulcorner R_i \urcorner$ is recursive,
- $i \mapsto \ulcorner V_i \urcorner$ is recursive,
- for $i \in \omega$, $D_i = \{\mathfrak{c}_r : r \in R_i\}$ is a set of new constants,
- for $i \in \omega$, $W_i = \{\mathfrak{c}_r : r \in V_i\}$ is a set of new constants.

In the argument presented below, for each $i \in \omega$ we define $\mathrm{ElDiag}(\mathcal{C}_i)$ taking new constants from D_{2i}, and $\mathrm{ElDiag}(D_i)$ taking new constants from D_{2i+1}. Constants from sets W_i are the witnesses used while applying the Low Completeness Theorem (4.2.16).

For $i = 0, 1, 2$ and $k \in \omega$ let $\sigma_i^k = \sigma_i \cup D \cup \bigcup_{j<k} D_j$.

We define the tower of low models as follows. For $i \in \omega$, we want what follows:

- $T_0 = A \cup \{\varphi\}$,
- $\mathcal{C}_i$ is the concrete σ_1^{2i}–model obtained by applying the Low Completeness Theorem (4.2.16) to T_i and to the set of witnesses W_{2i},
- $S_0 = (\mathrm{ElDiag}(\mathcal{C}_0) \cap \mathrm{Sent}_{\sigma_0^1}) \cup \{\neg\psi\}$,
- $\mathcal{D}_i$ is the concrete σ_2^{2i+1}–model obtained by applying the Low Completeness Theorem (4.2.16) to S_i and to the set of witnesses W_{2i+1},
- $T_{i+1} = (\mathrm{ElDiag}(\mathcal{D}_i) \cap \mathrm{Sent}_{\sigma_0^{2i+2}}) \cup \mathrm{ElDiag}(\mathcal{C}_i)$,
- $S_{i+1} = (\mathrm{ElDiag}(\mathcal{C}_{i+1}) \cap \mathrm{Sent}_{\sigma_0^{2i+3}}) \cup \mathrm{ElDiag}(\mathcal{D}_i)$.

For $i \in \omega$, let $L_i = \bigoplus_{m \leqslant i} \mathcal{C}_m \oplus \mathcal{D}_m$.

By induction we show that this construction is proper – it is sufficient to show that for every $i \in \omega$ the sets T_i and S_i are low and consistent. Then, by the Low Completeness Theorem (4.2.16), we obtain the desired low concrete models.

Let $i = 0$. The theory T_0 is low and consistent by the previous arguments. Obviously $A \subseteq T_0$. Let $\mathcal{C}_0$ be the concrete σ_1^0–model obtained by the Low Completeness Theorem (4.2.16) applied to T_0 and to the set of witnesses W_0. Then $A \oplus \mathcal{C}_0$ is low. Moreover, $\mathcal{C}_0 \models A$ and $|\mathcal{C}_0| \subseteq V_0$.

We have $S_0 = (\mathrm{ElDiag}(\mathcal{C}_0) \cap \mathrm{Sent}_{\sigma_0^1}) \cup \{\neg\psi\}$, where new constants to define $\mathrm{ElDiag}(\mathcal{C}_0)$ are from D_0. Therefore S_0 is recursive in $\mathcal{C}_0$, thus low, σ_2^1 theory.

We show that S_0 is consistent. Otherwise, there would be a finite set $F \subseteq \mathrm{ElDiag}(\mathcal{C}_0)$ such that $\neg\psi \vdash \neg \bigwedge F(\overline{d})$, such that $\overline{d}$ are all constants from D_0 appearing in F. Then, by the lemma on constants $\neg\psi \vdash \forall\overline{x} \neg \bigwedge F(\overline{x})$. On the other hand, $\bigwedge F(\overline{d}) \in \mathrm{ElDiag}(\mathcal{C}_0)$, and therefore $\exists\overline{x} \bigwedge F(\overline{x})$ is a σ_0–sentence true in $\mathcal{C}_0$. Then, $\exists\overline{x} \bigwedge F(\overline{x}) \in A$, and thus it is not the case that $\neg\psi \vdash \neg\exists\overline{x} \bigwedge F(\overline{x})$. This contradicts the assumption.

Therefore, S_0 is low and consistent. Let $\mathcal{D}_0$ be the concrete σ_2^1–model obtained by applying the Low Completeness Theorem (4.2.16) to S_0 and to the set of witnesses W_1. Then, $|\mathcal{D}_0| \subseteq V_1$ and $L_0 = \mathcal{D}_0 \oplus \mathcal{C}_0$ is low. Moreover, $\mathcal{D}_0 \models S_0$. This ends the base step.

For the inductive step suppose that the concrete models $\mathcal{C}_i$, $\mathcal{D}_i$ such that $\mathcal{C}_i \models T_i$ and $\mathcal{D}_i \models S_i$ are constructed. Moreover, suppose that L_i is low.

The σ_1^{2i+2}–theory $T_{i+1} = (\mathrm{ElDiag}(\mathcal{D}_i) \cap \mathrm{Sent}_{\sigma_0^{2i+2}}) \cup \mathrm{ElDiag}(\mathcal{C}_i)$ is recursive in $\mathcal{C}_i \oplus \mathcal{D}_i$, and therefore in L_i, thus low.

We show that T_{i+1} is consistent. Otherwise, there would be a finite set of σ_0^{2i+2}–sentences $F \subseteq \mathrm{ElDiag}(\mathcal{D}_i)$ such that $\mathrm{ElDiag}(\mathcal{C}_i) \vdash \neg \bigwedge F(\overline{d})$, where $\overline{d}$ are all the constants from D_{2i+1} appearing in F. By the lemma on constants it would follow that $\mathrm{ElDiag}(\mathcal{C}_i) \vdash \forall\overline{x} \neg \bigwedge F(\overline{x})$. Therefore, $\forall\overline{x} \neg \bigwedge F(\overline{x})$ is a σ_0^{2i+1}–sentence and $\forall\overline{x} \neg \bigwedge F(\overline{x}) \in \mathrm{ElDiag}(\mathcal{C}_i)$. Since $\mathcal{D}_i \models S_i$, we have also $\mathcal{D}_i \models \forall\overline{x} \neg \bigwedge F(\overline{x})$. This contradicts that fact that $F \subseteq \mathrm{ElDiag}(D_i)$.

Therefore, T_{i+1} is a low consistent σ_1^{2i+2}–theory. Let $\mathcal{C}_{i+1}$ be the low concrete σ_1^{2i+2}–model obtained by applying the Low Completeness Theorem (4.2.16) to T_{i+1} and to the set of witnesses W_{2i+2}. Then $L_i \oplus \mathcal{C}_{i+1}$ is low and $|\mathcal{C}_{i+1}| \subseteq V_{2i+1}$. Moreover, $\mathcal{C}_{i+1} \models T_{i+1}$.

The σ_2^{2i+3}–theory $S_{i+1} = (\mathrm{ElDiag}(\mathcal{C}_{i+1}) \cap \mathrm{Sent}_{\sigma_0^{2i+3}}) \cup \mathrm{ElDiag}(\mathcal{D}_i)$ is recursive in $L_i \oplus \mathcal{C}_{i+1}$, and therefore low.

If S_{i+1} was inconsistent, there would be a finite set of σ_0^{2i+3}–sentences $F(\overline{d}) \subseteq \mathrm{ElDiag}(\mathcal{C}_{i+1})$ such that $\mathrm{ElDiag}(\mathcal{D}_i) \vdash \neg \bigwedge F(\overline{d})$ and $\overline{d}$ are all the constants from D_{2i+2} appearing in F. By the similar arguments as in the case of T_{i+1} it would contradict the fact that $\mathcal{C}_{i+1} \models T_{i+1}$.

Therefore, S_{i+1} is a low consistent theory. Let $\mathcal{D}_{i+1}$ be the low concrete σ_2^{2i+3}–model obtained by applying the Low Completeness Theorem (4.2.16) to S_{i+1} and to the set of witnesses W_{2i+2}. Then $L_{i+1} = L_i \oplus \mathcal{D}_{i+1} \oplus \mathcal{C}_{i+1}$ is low and $|\mathcal{C}_{i+1}| \subseteq V_{2i+1}$. Moreover, $\mathcal{C}_{i+1} \models T_{i+1}$.

The construction is concrete by the arguments similar to those presented in the previous sections, since it uses only the Low Completeness Theorem (4.2.16) applied to boolean combinations of concrete theories. Therefore, the maps $i \mapsto \ulcorner \mathcal{C}_i^* \urcorner$ and $i \mapsto \ulcorner \mathcal{D}_i^* \urcorner$ are concrete. Hence $(\mathcal{C}_i)_{i\in\omega}$ and $(\mathcal{D}_i)_{i\in\omega}$ are jump–concrete sequences of low concrete models.

For each $i \in \omega$, $\bigoplus_{j\leqslant i} \mathcal{C}_j \oplus \mathcal{D}_j$ is low. Moreover, the universes of the concrete models are all disjoint and contained in some disjoint recursive sets V_i such that $i \mapsto \ulcorner V_i \urcorner$ is recursive.

Consider reductions $\mathcal{C}_i^0$ and $\mathcal{D}_i^0$ of $\mathcal{C}_i$ to σ_0^{2i} and $\mathcal{D}_i$ to σ_0^{2i+1} respectively. Then, for $i \in \omega$ it holds that $\mathcal{D}_i^0 \models \mathrm{Diag}(\mathcal{C}_i^0)$ and $\mathcal{C}_{i+1}^0 \models \mathrm{Diag}(\mathcal{D}_i^0)$. Then the jump–concrete sequence of concrete models $\mathcal{C}_0^0, \mathcal{D}_0^0, \mathcal{C}_1^0, \mathcal{D}_1, \dots$ satisfies the assumptions of Theorem 4.1.22. Therefore, there is a concrete chain of concrete σ_0–models $\tilde{\mathcal{A}}_0, \tilde{\mathcal{B}}_0, \tilde{\mathcal{A}}_1, \tilde{\mathcal{B}}_1, \dots$ and concrete isomorphisms g_i from $\tilde{\mathcal{A}}_i$ to the reduction of $\mathcal{C}_i^0$ to σ_0 and h_i from $\tilde{\mathcal{B}}_i$ to the reduction of $\mathcal{D}_i^0$ to σ_0. For every $i \in \omega$, let $\mathcal{A}_i$ be the concrete σ_1–model obtained by taking the co–image of the reduction of $\mathcal{C}_i$ to σ_1 under g_i and let $\mathcal{B}_i$ be the concrete σ_2–model obtained by taking the co–image of the reduction of $\mathcal{D}_i$ to σ_2 under h_i. These models are concrete by Lemma 4.1.10.

Note that for $i \in \omega$ it holds that $\mathcal{C}_{i+1} \models \mathrm{ElDiag}(\mathcal{C}_i)$ and $\mathcal{D}_{i+1} \models \mathrm{ElDiag}(\mathcal{D}_i)$. Therefore, $(\mathcal{A}_i)_{i\in\omega}$ and $(\mathcal{B})_{i\in\omega}$ are concrete elementary chains of concrete models. By Theorem 4.2.43 there exists the concrete σ_1–model $\mathcal{A}_\omega = \bigcup_{i\in\omega} \mathcal{A}_i$ and the concrete σ_2–model $\mathcal{B}_\omega = \bigcup_{i\in\omega} \mathcal{B}_i$.

Moreover, by the construction it holds that $|\mathcal{A}_\omega| = |\mathcal{B}_\omega|$. In the axiomatic model theory it is sufficient to perform a glueing of these two models to obtain a model of in the full language $\sigma_1 \cup \sigma_2$ in which $\varphi \wedge \neg\psi$ is true. Such a glueing, however, is not admissible in the concrete models framework. The full picture is as follows.

$$\begin{tikzcd}
\mathcal{A}_0 \arrow[r, phantom, "\preccurlyeq"] \arrow[dd, dashed] & \mathcal{A}_1 \arrow[r, phantom, "\preccurlyeq"] \arrow[dd, dashed] & \mathcal{A}_2 \arrow[r, phantom, "\preccurlyeq"] \arrow[dd, dashed] & \cdots \arrow[r, phantom, "\preccurlyeq"] & \mathcal{A}_\omega \arrow[dr] & \\
 & & & & & \mathcal{A}_\omega * \mathcal{B}_\omega \\
\mathcal{B}_0 \arrow[r, phantom, "\preccurlyeq"] \arrow[uur, dashed] & \mathcal{B}_1 \arrow[r, phantom, "\preccurlyeq"] \arrow[uur, dashed] & \mathcal{B}_2 \arrow[r, phantom, "\preccurlyeq"] & \cdots \arrow[r, phantom, "\preccurlyeq"] & \mathcal{B}_\omega \arrow[ur] &
\end{tikzcd}$$

One may note that that the concrete models $\mathcal{A}_\omega$ and $\mathcal{B}_\omega$ are not necessarily low. If we were able to perform the construction in such a way as to ensure that those concrete models are low, we might be able to take a low $\mathrm{Th}(\mathcal{A}_\omega) \cup \mathrm{Th}(\mathcal{B}_\omega)$ and use the Low Completeness Theorem (4.2.16) to obtain a low concrete $(\sigma_1 \cup \sigma_2)$–model. First, let us note that such a construction is possible. We may use the notion of n–separability and the operator INSEP to directly construct low (with the same Turing degree) theories T_1 and T_2 such that T_1 is a $\mathrm{CCW}(\sigma_1 \cup D, D)$ and T_2 is a $\mathrm{CCW}(\sigma_2 \cup D, D)$. We could obtain low concrete models $\mathcal{A}_\omega \models T_1$ and $\mathcal{B}_\omega \models T_2$ from these theories. We would get that $T_1 \cup T_2$ is low, and therefore its consequences are concrete. But there would be no obvious way to show that $T_1 \cup T_2$ is consistent. The consistency of $T_1 \cup T_2$ for arbitrary theories obtained this way would actually be equivalent to the syntactic formulation of the Robinson's joint consistency theorem and therefore to the Craig's lemma itself, which we would like to prove by this argument. Moreover, such a construction would bypass the Robinson's construction which we were particularly interested in.

4.3 A Note on the Usage of Resources in the Concrete Constructions

As we declared earlier, we want to elaborate on the usage of resources in the concrete model–theoretic constructions. In previous sections we have shown how to build various concrete chains and towers of concrete models. This is done by constructing concrete sequences of low concrete models which satisfy the assumptions of Theorem 4.1.22.

First, observe that these constructions cannot be, in general, extended beyond ω steps. Concrete models obtained by them are not necessarily low whereas all of our constructions require that we start with low concrete models. However, if resulting concrete models are low, there are no contraindications to use them as starting models in further constructions.

We also need to justify our assumption from the beginning of this chapter. Namely we need to justify that we can always assume that we can take a set or a recursive family of sets of new constants. This may seem impossible, since we have only a limited number of constants in the language – namely: $\mathfrak{c}_0, \mathfrak{c}_1, \ldots$.

In our constructions we needed at most $\omega \cdot 2$ infinite sets of new constants. We could have indexed these sets more carefully so that only ω sets of new constants are needed. Then it is simple to devise a notation, with desired properties, for such families of constants.

Let A be a recursive set and let $a_0, a_1, \ldots$ be its effective presentation. First, we take $A_2 = \{a_{2i} \colon i \in \omega\}$. This is to ensure that after performing the construction there is still a recursive set, namely, $\{a_{2i+1} \colon i \in \omega\}$ left for further constructions. For $i \in \omega$ let $b_i = a_{2i}$ and let $B = \{b_i \colon i \in \omega\} = A_2$. For $i \in \omega$ we define recursive subsets C_i of B such that:

1. $\bigcup_{i \in \omega} C_i$ is recursive,

2. $i \mapsto \ulcorner C_i \urcorner$ is recursive,

3. $ind\colon \bigcup_{i\in\omega} C_i \to \omega$, such that $ind(c) = i$ if and only if $c \in C_i$, is recursive.

Note that 1 and 2 imply 3 and we add this last point just for clarity. For $i \in \omega$ we put $C_i = \{b_{p_i^{1+k}} \colon k \in \omega\}$. Each C_i can be used as a set of indices of new constants. It is easy to see that this definition satisfies our requirements. This allows performing each model–theoretic construction presented in this chapter.

In fact, we could devise notations for way more than just ω recursive sets of new numbers, satisfying conditions:

1. $\bigcup_{\beta\in\alpha} C_\beta$ is recursive,

2. $\ulcorner \beta \urcorner \mapsto \ulcorner C_\beta \urcorner$ is recursive,

3. $ind\colon \bigcup_{\beta\in\alpha} C_\beta \to \alpha$, such that $ind(c) = \ulcorner \beta \urcorner$ if and only if $c \in C_\beta$, is recursive.

To be precise, we need a proper notation for ordinal numbers $\leqslant \alpha$ i.e. we need to define $\ulcorner \beta \urcorner$.

In [Kle38] Kleene considered notations for ordinal numbers. He proposed an encoding of ordinal numbers such that the successor and the limit of a recursive increasing ω–sequence of ordinal numbers is easily computable. His notation system captures all recursive ordinal numbers. The limit of Kleene's system – the first non–recursive ordinal number – is denoted by ω_1^{CK}. The class of all recursive ordinal numbers is denoted by $\mathcal{O}$. We can use Kleene's notation for our purposes.

4.4 Beyond Concrete Foundations, Open Problems and Further Work

4.4.1 Summary of the Concrete Model Theory

In Section 4.2 we have investigated well known model–theoretic constructions from the axiomatic model theory. We have shown which of these constructions may be performed in the concrete models framework. Section 4.2 may leave an impression that one needs to deal very cautiously with concrete models. There is a lot of arguments which are natural in the axiomatic model theory, and which lead to obtaining non–concrete models from concrete models. In this section we summarise our knowledge about the concrete model theory. We also state some open problems and further development paths for the concrete model theory.

The first, not very surprising, observation is that all concrete models are at most countable. Therefore, a large part of the axiomatic model theory, concerning higher cardinalities, is automatically excluded from the concrete models framework. This is intended, since concrete models are those which are representable without actual infinity, whereas uncountable cardinalities are certainly not.

We show that some constructions that are natural in the axiomatic model theory are not admissible in the concrete models framework. Theorem 4.2.39 shows that a recursive chain of recursive models may sum to a non–arithmetical model, far beyond the concrete models framework. Another construction which leads from concrete models to non–concrete models is model glueing. Model glueing makes one model from two models which have the same universe (but maybe different relations), by taking all relations from both models altogether. Theorem 4.2.51 shows that a model glued from two concrete models may be even non–arithmetical.

In this chapter we also show a general difficulty in constructing concrete chains and towers of concrete models. In Theorem 4.1.18 we show that we cannot reason *up to an isomorphism* as freely as in the axiomatic model theory. We show that from the fact that $\mathcal{A} \models \mathrm{Diag}(\mathcal{B})$ we cannot, in general, infer that $(\mathcal{B}, b)_{b\in|\mathcal{B}|}$ is concretely isomorphic to a concrete submodel of $\mathcal{A}$. Such inferences are very common in classical versions of model–theoretic constructions which we consider. In the axiomatic model theory such reasonings are justified by the fact that an image of an isomorphism of models is also a model. If $\mathcal{A} \models \mathrm{Diag}(\mathcal{B})$, then there is an isomorphism f such that $f[(\mathcal{B}, b)_{b\in|\mathcal{B}|}] \subseteq \mathcal{A}$. However, the image of a set is defined with the use of an unbounded existential quantification. Taking an image of a set increases Turing degree by a jump. Therefore, an image of a concrete model, under a concrete isomorphism, may be a non–concrete model. This problem is solved by Theorem 4.1.22 which enables us to construct concrete chains and concrete towers of concrete models. Theorem 4.1.22 is essential in our concrete reconstructions of the classical model–theoretic constructions.

It is worth noting that the Concrete Completeness Theorem (4.2.10) is not as universal as the Completeness Theorem from the axiomatic model theory. It requires the consequences $\mathrm{Cn}(T)$ of a theory T to be concrete, to infer that there exists a concrete model of T. Therefore, there are concrete, syntactically consistent theories with no concrete models. Such theories appear in the discussion of Robinson's joint consistency theorem, semantic and syntactic versions of which are not equivalent in the concrete context. This is why the Concrete Completeness Theorem (4.2.10) was not sufficient for performing model–theoretic constructions requiring iterative constructions of concrete models. On the other hand, the Low Completeness Theorem (4.2.16) is perfectly suitable for these purposes, since it produces low concrete models of low theories. Therefore, consistency of a low theory T is equivalent to the existence of a low concrete model of T.

In Section 4.2.4 we have shown that even though we cannot, in general, sum concrete chains of concrete models, by Theorem 4.2.43, if the concrete chain is elementary, then the sum exists. This enables us to perform various model–theoretic constructions (see Sections 4.2.4 and 4.2.5). Being able to sum concrete elementary chains of concrete models, we prove Preservation Theorems (4.2.45, 4.2.47) for recursive (in fact recursively enumerable) theories.

4.4.2 A Comparison with Experimental Logics

An interesting consequence about a large class of concrete models can be drawn from a theorem presented by Jeroslow in [Jer75], where experimental logics are considered. This are trial–and–error systems that *learn* their theorems over time. Experimental logics are just ternary recursive predicates $H(t, x, y)$ with the intended meaning: *at time t, the finite sequence with Gödel number y is accepted as a proof of the sentence with Gödel number x*. The definition is very broad as there are no further requirements for being *accepted as a proof*. There are two important notions regarding experimental logics. The theorems of an experimental logic H are defined by:

$$Rec_H(x) \equiv_{df} \forall t\, \exists s \geqslant t\, \exists y\, H(s, x, y),$$

which means that the sentence with Gödel number x is *provable* in infinite number of points in time. Another very important notion is the notion of stability of a sentence:

$$Stbl_H(x) \equiv_{df} \exists t\, \exists y\, \forall s \geqslant t\, H(s, x, y),$$

which means that there is a point in time after which the sentence with Gödel number x is always *provable*. One may see a similarity between stable sentences in experimental logics and sentences true in FM–domains with respect to *sl*–semantics.

Jeroslow focuses on convergent experimental logics i.e. those H for which it holds that for every x, $Rec_H(x) \equiv Stbl_H(x)$. In his paper the following theorem is shown:

Theorem 4.4.1 ([Jer75]) *The sets of theorems of convergent experimental logics are exactly Δ^0_2 sets.*

For instance, elementary diagrams of concrete models may be seen also as theorems of some convergent experimental logic. The following theorem comes from [Jer75].

Theorem 4.4.2 ([Jer75]) *Let H be a consistent, convergent, experimental logic whose theorems contain PA and are closed under first order reasonings. Then, there is a Π_1–sentence unprovable in H which is true in $\mathcal{N}$.*

As a corollary we obtain that every concrete model $\mathcal{A}$ of PA satisfies a false (in $\mathcal{N}$) Σ_1–sentence. This follows from the fact that the theory $\mathrm{Th}(\mathcal{A})$ satisfies the assumptions of Theorem 4.4.2.

4.4.3 Paths of Further Investigations

In this section we present some remarks on the open problems of concrete model theory and possible directions of further investigations in this field.

One obvious path of further investigations of concrete foundations of mathematics is further recognition of classical model–theoretic constructions and theorems fitting the concrete models framework. Recursively saturated models and their use in model theory are particularly interesting. Another topic concerns transferring the method of ultrafilters to concrete model theory.

Another path of further development we emphasise, concerns low sets. Low sets appear not to be sufficiently well–researched. A comprehensive study on low sets would bring a lot of insight into model–theoretic constructions admissible in concrete models framework. Knowing some *decent* properties of low sets would significantly simplify the work with concrete models. One of the problems concerning low sets that appeared in this chapter concerns with sequences of concrete models. Constructions of sequences of concrete models presented in this chapter have a common stem. They start with low concrete models $\mathcal{A}$, $\mathcal{B}$ and proceed by defining some recursive (or recursive in $\mathcal{A} \oplus \mathcal{B}$) operations to obtain a consistent, recursive in $\mathcal{A} \oplus \mathcal{B}$ theory. Then, the Low Completeness Theorem is applied to this theory to obtain the next low concrete model in a sequence. Theorem 4.1.22 is applied to extract a concrete chain of concrete (not necessarily low) models. We can switch to the approach of Sacks, presented in [Sac72], and consider concrete directed systems of low concrete models instead of concrete chains of concrete models. However, to compute the limit of a concrete directed system of low concrete models $\mathcal{A}_0 \hookrightarrow \mathcal{A}_1 \hookrightarrow \ldots$, one requires $\bigoplus_{i\in\omega} \mathcal{A}_i$ as an oracle. Possibility of continuing model–theoretic constructions after reaching the limit requires that this limit should be a low concrete model. It remains an open question whether the constructions we consider have this property and, if they do not have it, whether they can be improved.

Last but not least, a very interesting path of further study is considering *partial models*. A *partial model* is a model whose structure is concrete, but only a part i.e. some subset of the satisfaction relation is concrete. Relaxing the requirement on the satisfaction relation would be very helpful even in the context of the work presented in this chapter. Recall that in Sections 4.2.3 and 4.2.5 we present some negative results – constructions which fail for concrete models. Observe that in both cases we construct a *partial model* which is not necessarily concrete (by summing a concrete Σ_n chain of concrete models and by glueing two concrete models). Similarly, in the original proof of the Gödel's completeness theorem, we only know that a irrefutable sentence φ is true in the model obtained by the construction, whereas the entire satisfaction relation – firstly: is not of the main concern, and secondly: may be not concrete. *Partial models* would also enable us to consider the standard model of arithmetic – $\mathcal{N}$, while in the concrete models framework $\mathcal{N}$ simply does not exist.

Bibliography

[Aria] Aristotle. *Physics.* Circa 350 BC. English translation in Hardie, R. P. and Gaye, R. K. url: http://classics.mit.edu/Aristotle/physics.html.

[Arib] Aristotle. *Posterior Analytics.* Circa 350 BC. Oxford: B.H. Blackwell. English translation in G. R. G. Mure. url: http://classics.mit.edu/Aristotle/posterior.html.

[AZ11] Z. Adamowicz and P. Zbierski. *Logic of Mathematics: A Modern Course of Classical Logic.* Pure and Applied Mathematics: A Wiley Series of Texts, Monographs and Tracts. Wiley, 2011.

[Boo93] G. Boolos. *The logic of provability.* Cambridge University Press, 1993.

[CK73] C. C. Chang and J. Keisler. *Model Theory.* Studies in Logic and the Foundations of Mathematics. North-Holland, 1973. Third edition, 1990.

[Cra57] W. Craig. Three uses of the Herbrand–Gentzen theorem in relating model theory and proof theory. *The Journal of Symbolic Logic*, 22:269–285, 9 1957.

[CZ10] M. Czarnecki and K. Zdanowski. A modal logic of a truth definition for finite models. In manuscript, 2010.

[Eps79] R. L. Epstein. *Degrees of Unsolvability Structure and Theory.* Lecture Notes in Mathematics. Springer, 1 edition, 1979.

[Ers73] Yu. L. Ershov. Constructive models. In *Selected Problems in Algebra and Logic*, pages 111–130. Nauka, Novosibirsk, 1973. In Russian.

[Ewa05] W. B. Ewald. *From Kant to Hilbert*, volume 2. OUP Oxford, 2005.

[Fra76] A. A. Fraenkel. *Abstract set theory.* Studies in logic and the foundations of mathematics. North-Holland Pub. Co., 1976.

[Fre79] G. Frege. *Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens.* Verlag von Louis Nebert, Halle, 1879.

[GKP94] R. L. Graham, D. E. Knuth, and O. Patashnik. *Concrete Mathematics: A Foundation for Computer Science.* Addison-Wesley Longman Publishing Co., Inc., Boston, MA, USA, 2nd edition, 1994.

[Göd30] K. Gödel. Die Vollständigkeit der Axiome des logischen Funktionenkalküls. *Monatshefte für Mathematik und Physik*, 37(1):349–360, 1930. English translation in [Göd02a].

[Göd31] K. Gödel. Über Formal Unentscheidbare Sätze der Principia Mathematica und Verwandter Systeme, I. *Monatshefte für Math.u.Physik*, 38:173–198, 1931. English translation in [Göd02b].

[Göd02a] K. Gödel. The completeness of the axioms of the functional calculus of logic. In J. van Heijenoort, editor, *From Frege to Gödel: A Source Book in Mathematical Logic, 1879–1931*, pages 582–591. Harvard University Press, 2002.

[Göd02b] K. Gödel. On formally undecidable propositions of Principia mathematica and related systems i. In J. van Heijenoort, editor, *From Frege to Gödel: A Source Book in Mathematical Logic, 1879–1931*, pages 596–617. Harvard University Press, 2002.

[Gol65] E. M. Gold. Limiting recursion. *Journal of Symbolic Logic*, 30:28–48, 1965.

[GSP60] C. F. Gauss, H. C. Schumacher, and C. A. F. Peters. *Briefwechsel zwischen C. F. Gauss und H. C. Schumacher.* Number vol. 2 in Briefwechsel zwischen C. F. Gauss und H. C. Schumacher. 1860.

[Har98] V. Harizanov. Pure computable model theory. In *Handbook of Recursive Mathematics - Volume 1: Recursive Model Theory*, volume 138 of *Studies in Logic and the Foundations of Mathematics*, pages 3–114. Elsevier, 1998.

[HB39] D. Hilbert and P. Bernays. *Grundlagen der Mathematik.* Number vol. 2 in Grundlagen der Mathematik. Springer-Verlag, 1939.

[Hea56] T. L. Heath. *The Thirteen Books of the Elements.* Number vol. 1. Dover Publications, 1956.

[Hil26] D. Hilbert. Über das Unendliche. *Mathematische Annalen*, (95):161–190, 1926. Lecture given in Münster, 4 June 1925. English translation in [Hil02].

[Hil02] D. Hilbert. On the infinite. In J. van Heijenoort, editor, *From Frege to Gödel: A Source Book in Mathematical Logic, 1879–1931*, pages 367–392. Harvard University Press, 2002.

[Hod93] W. Hodges. *Model Theory*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1993.

[HP93] P. Hájek and P. Pudlák. *Metamathematics of First–Order Arithmetic*. Springer Verlag, 1993.

[Jer75] R. G. Jeroslow. Experimental logics and Δ_2^0-theories. *Journal of Philosophical Logic*, 4(3):253–267, 1975.

[JS72] C. G. Jr Jockusch and R. I. Soare. Classes and degrees of theories. *Transactions of the American Mathematical Society*, 173:33–56, 1972.

[Kle38] S. C. Kleene. On notation for ordinal numbers. *J. Symb. Log.*, 3(4):150–155, 1938.

[Kle52] S. Kleene. *Introduction to Metamathematics*. North-Holland, Amsterdam, 1952.

[Koł04] L. A. Kołodziejczyk. Truth definitions in finite models. *J. Symb. Log.*, 69(1):183–200, 2004.

[KP54] S. C. Kleene and E. L. Post. The upper semi-lattice of degrees of recursive unsolvability. *Annals of Mathematics*, 59(3):37–407, 1954.

[Kro87] L. Kronecker. Über den Zahlbegriff. *Journal für die reine und angewandte Mathematik*, 101:337–355, 1887. English translation in [Ewa05].

[KZ05] M. Krynicki and K. Zdanowski. Theories of arithmetics in finite models. *J. Symbolic Logic*, 70(1):1–28, 03 2005.

[Mar54] A. A. Jr Markov. The theory of algorithms. *Trudy Mat. Instituta im. Steklova*, (42), 1954. In Russian.

[Mon14] B. Monin. personal communication, 2014.

[Mos01] M. Mostowski. On representing concepts in finite models. *Mathematical Logic Quarterly*, 47:513–523, 2001.

[Mos03] M. Mostowski. On representing semantics in finite models. In A. Rojszczak†, J. Cachro, and G. Kurczewski, editors, *Philosophical Dimensions of Logic and Science*, pages 15–28. Kluwer Academic Publishers, 2003.

[Mos08] M. Mostowski. On representability in finite arithmetics. In A. Cordon-Franco, A. Fernandez-Margarit, and F. Felix Lara-Martin, editors, *JAF26, 26eme Journees sur les Arithmetiques Faibles (26th Weak Arithmetics Days)*, pages 53–64, Sevilla, 2008. Fenix Editora.

[Mos11] M. Mostowski. Finitistic foundations of mathematics. In manuscript, 2011.

[MW04] M. Mostowski and A. Wasilewska. Elementary properties of divisibility in finite models. *Mathematical Logic Quarterly*, 50:169–174, 2004.

[Myc81] J. Mycielski. Analysis without actual infinity. *Journal of Symbolic Logic*, 46:625–633, 1981.

[Myc86] J. Mycielski. Locally finite theories. *Journal of Symbolic Logic*, 51:59–62, 1986.

[MZ05a] M. Mostowski and K. Zdanowski. Coprimality in finite models. In Luke Ong, editor, *Computer Science Logic*, volume 3634 of *Lecture Notes in Computer Science*, pages 263–275. Springer Berlin Heidelberg, 2005.

[MZ05b] M. Mostowski and K. Zdanowski. FM-representability and beyond. In *Proceedings of the First International Conference on Computability in Europe: New Computational Paradigms*, CiE'05, pages 358–367, Berlin, Heidelberg, 2005. Springer-Verlag.

[Pos44] E. L. Post. Recursively enumerable sets of positive integers and their decision problems. *Bull. Amer. Math. Soc.*, 50(5):284–316, 05 1944.

[Put65] H. Putnam. Trial and error predicates and the solution to a problem of Mostowski. *J. Symbolic Logic*, 30:49–57, 1965.

[Sac72] G. E. Sacks. *Saturated model theory.* Mathematics lecture note series. Benjamin, W. A., 1972.

[Sho59] J. R. Shoenfield. *On degrees of unsolvability*, volume 69. Annals of Mathematics, 1959.

[Sim77] S. G. Simpson. First order theory of the degrees of recursive unsolvability. *Ann. of Math. (2)*, 105:121–139, 1977.

[Sim10] S. G. Simpson. *Subsystems of Second Order Arithmetic.* Perspectives in Logic. Cambridge University Press, 2010.

[Smo85] C. Smoryński. *Self-reference and Modal Logic.* Universitext (1979). Springer–Verlag, 1985.

[Sol76] R. M. Solovay. Provability interpretations of modal logic. *Israel Journal of Mathematics*, 25(3-4):287–304, 1976.

[Tai81] W. W. Tait. Finitism. *Journal of Philosophy*, 78(9):524–546, 1981.

[Tar33] A. Tarski. *Pojęcie Prawdy W Językach Nauk Dedukcyjnych.* Number 34 in Prace Towarzystwa Naukowego Warszawskiego, Wydział III–Nauk matematyczno-fizycznych. 1933. English translation in [Tar56].

[Tar56] A. Tarski. The concept of truth in formalized languages. In ed. J. H. Woodger, editor, *Logic, Semantics, Metamathematics*, pages 152–278. Oxford University Press, 1956. Translated from the German version by J. H. Woodger.

[Tur36] A. Turing. On computable numbers, with an application to the Entscheidungsproblem. In *Proc. of the London Math. Soc.*, volume 42, pages 230–265, 1936.

[Tur39] A. Turing. Systems of logic based on ordinals. In *Proc. of the London Math. Soc.*, 1939.

[TV58] A. Tarski and R. L. Vaught. Arithmetical extensions of relational systems. *Compositio Mathematica*, 13:81–102, 1956–1958.

[WR27] A. N. Whitehead and B. A. W. Russell. *Principia mathematica; 2nd ed.* Cambridge Univ. Press, 1927.

Index