SIGNAL SEPARATION IN THE WIGNER DISTRIBUTION DOMAIN USING FRACTIONAL FOURIER TRANSFORM

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ABSTRACT

In this paper we propose an algorithm based on the fractional Fourier transform to separate the different components of a signal in the Wigner time-frequency domain. The aim is to obtain a compressed representation for such a signal containing a minimal number of parameters. The proposed procedure gets rid of the noise and the cross-terms after separating the signal components. Assuming the signals under consideration have chirps and sinusoids, the fractional Fourier transform is used to rotate the components to obtain a sinusoidal or impulsive sparse representation. The procedure relies on filtering or windowing after obtaining the order of the fractional Fourier transform for each of the components. Simulation results show the effectiveness of this approach in extracting the linear chirps and sinusoids from the noise and in eliminating the cross-terms from the Wigner distribution.

1.   INTRODUCTION

Signals that display a minimal time-frequency product are sparse in either time or frequency. Such is the case of sinusoids and impulses, and of signals that can be represented exactly by these functions. A signal that is composed of chirps is thus not sparse, as its time-frequency product is not minimal. The Wigner distribution is appropriate for a joint time-frequency representation of a signal consisting of linear chirps and sinusoids, although it suffers from cross-terms [1,2]. Although, methods using the fractional Fourier transform have been proposed (see for instance [3]) to isolate and subtract these cross terms, such methods are computationally expensive. The sparseness of the signal is made less clear by the additive noise. Thus noise and cross-terms need to be eliminated before achieving a sparse representation of a noisy signal composed of sinusoids and linear chirps.

The separation and denoising of linear chirps with the fractional Fourier transform (FrFT) it is not a new problem and a lot of work has been done on this subject [4-7]. A recent work, [6], uses the FrFT and windowing to separate overlapped linear chirps by transforming them into impulses.

An issue of interest that was not considered is whether the chirp components should be transformed into impulses or into sinusoids. Considering a signal that is composed of sinusoids and linear chirps and is affected by additive white noise, we propose a procedure based on time-frequency using the Wigner distribution and the FrFT. To separate the sinusoidal and the linear chirp components we need to get rid of the noise and the cross-terms in the Wigner time-frequency representation of the signal. If the signal has a chirp representation, the FrFT can be used to determine the number of chirps and sinusoids present in the signal and to find the corresponding order that would allow us to change the signal into a sparse representation of it. Depending on the signal, it might be more appropriate to obtain an impulse or a sinusoidal sparse representation.

The rest of this paper is organized as follows. In section 2 background information about the fractional Fourier transform is given. The proposed scenario and simulation results are illustrated in section 3. Finally, the paper is concluded in section 4.

2.   FRACTIONAL FOURIER TRANSFORM

The fractional Fourier transform (FrFT) is defined as [8]:

$$X_\alpha(u) = \int_{-\infty}^{\infty} x(t) K_\alpha(t,u) dt$$

where

$$K_\alpha(t,u) = \sqrt{\frac{1 - j \cot \alpha}{2\pi}} \, exp\left(j \frac{t^2 + u^2}{2\pi} \cot \alpha - jut \csc \alpha \right)$$

and
$$0 < \alpha < \pi/2$$

When $\alpha = 0$ the FrFT of the signal $x(t)$ is the signal itself, and if $\alpha = \pi/2$ the FrFT becomes the Fourier transform of the signal. That is why it is considered a generalization of the Fourier transform. Similar to the classical Fourier transform which has a discrete version, the discrete Fourier transform (DFT), there are many representations in literature that have been proposed to implement the FrFT in the discrete domain. In this analysis we choose the one that depends on the DFT because it satisfies all the desired properties of the continuous FrFT [9]. The discrete fractional Fourier transform (DFrFT) is shown below:

$$X^a(m) = \sum_{n=0}^{N-1} K(n,m)x(n)$$

where

$$K(n,m) = \sum_{k=0}^{N-1} u_k(m) e^{-j\frac{\pi k a}{2}} u_k(n)$$

and u_k is the eigenvector of S which is given below:

$$S = \begin{bmatrix} c_0 & 1 & 0 & 0 & \cdots & 1 \\ 1 & c_1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & c_2 & 1 & \cdots & 0 \\ 0 & 0 & 1 & c_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & c_{N-1} \end{bmatrix}$$

and

$$c_n = 2\left(\cos\left(\frac{2\pi}{N}n\right) - 2\right)$$

3. SYSTEM MODEL AND SIMULATION

The basic idea is to represent the signal $x(t)$ as a summation of linear chirps and then to look for values of α that convert each chirp into a sinusoidal or an impulse signal. The benefit of going through this process is to get the minimum time-frequency product as shown in Fig. 1(a) and (b). The proposed system is shown in Fig. 1(c) and (d). If the signal $x(t)$ is a combination of chirps each with a slope β_i, then each of them will have an optimal order α_i for the FrFT that rotates it to a sinusoidal or an impulse while the other components are ignored. Therefore, we can choose the chirp that corresponds to that optimal order and separate it from the others. By following the same procedure with the rest of the chirps we can separate all of them from each other.

(a)

(b)

(c)

(d)
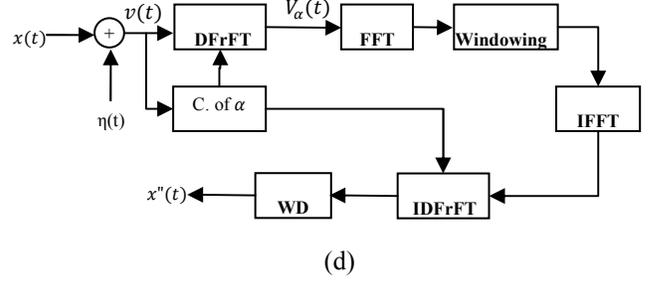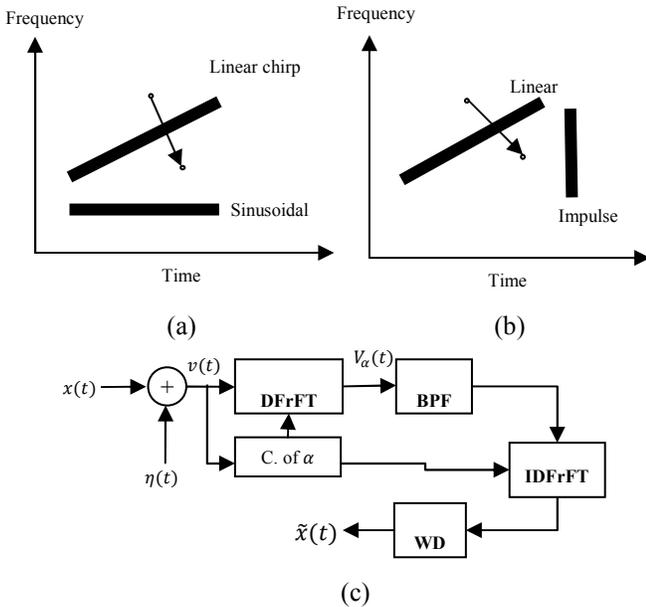
Fig. 1: Transforming the linear chirp using the FrFT to, (a) sinusoidal, (b) impulse representation, (c) the proposed filtering approach, and (d) the proposed windowing approach.

Let $x(t)$ be a linear chirp and $\eta(t)$ additive white Gaussian noise. Then, the noisy signal $v(t)$ is given in (1). As an example, consider the signals $x(t)$ and $v(t)$ are shown in Fig. 2(a) and (b) respectively

$$v(t) = x(t) + \eta(t) \tag{1}$$

where

$$x(t) = \exp\left(-ct^2 + j\left(\frac{\beta t^2}{2} + \omega_0 t\right)\right)$$

(a)

(b)

(c)

(d)
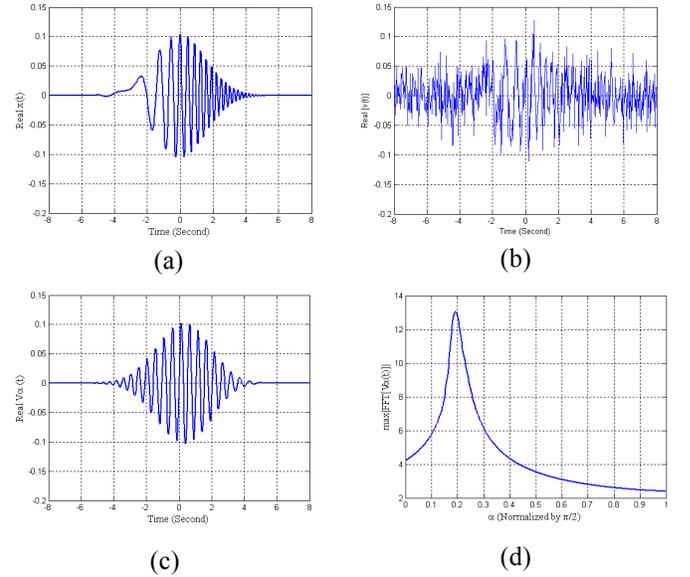
Fig. 2: The real part of linear chirp and a white noise with $S/N = 0dB$, (a) signal, (b) noisy signal, (c) the real part of $V_\alpha(t)$ where $\alpha = \pi/10$, (d) the peak values for FFT of $V_\alpha(t), 0 \le \alpha \le \pi/2$.

The FrFT of order α for $v(t)$ is $V_\alpha(t)$ which is shown in Fig. 2(c). The order of the FrFT that transforms the linear chirp $x(t)$ into a sinusoid is $\alpha = \pi/10$. The computation of α (C. of α) can be obtained by taking the FrFT of the signal $x(t)$ for $0 \le \alpha \le \pi/2$ and then taking the FFT of the resulted signal. This signal is a two dimensional and its marginal is

shown in Fig. 2(d). Then the value of α can be obtained by finding the value that corresponds to the $max|FFT\{V_\alpha(t)\}|$. These results were computed when there is no noise. In general, the optimal order can be obtained analytically according to [4]:

$$\alpha = \tan^{-1}\left(\frac{N\beta}{2\pi f_s^2}\right)$$

where: N is the number of samples,
β is the slope of chirp,
f_s is the sampling frequency.

This angle, between the chirp and the horizontal axis, corresponds to the slope of instantaneous frequency of the chirp. The clean signal $x(t)$ and its Wigner distribution are shown in Figs 3(a) and (b). Figs. 3(c) and (d) show the signal $v(t)$ with signal-to-noise ratio ($S/N = 0\ dB$) and its Wigner distribution while Figs. 5(e) and (f) present the filtered signal $\tilde{x}(t)$ with its Wigner distribution. The windowed signal $x''(t)$ and its Wigner distribution are given in Figs. 5(g) and (h). The error of the reconstructed signals $\tilde{x}(t)$ and $x''(t)$ is shown in Figs 3(i) and (j)

(g)

(h)

(i)

(j)
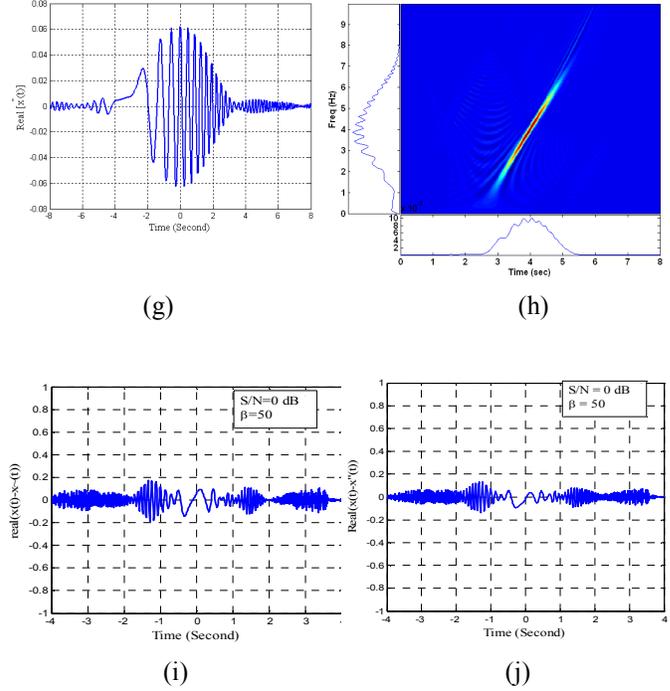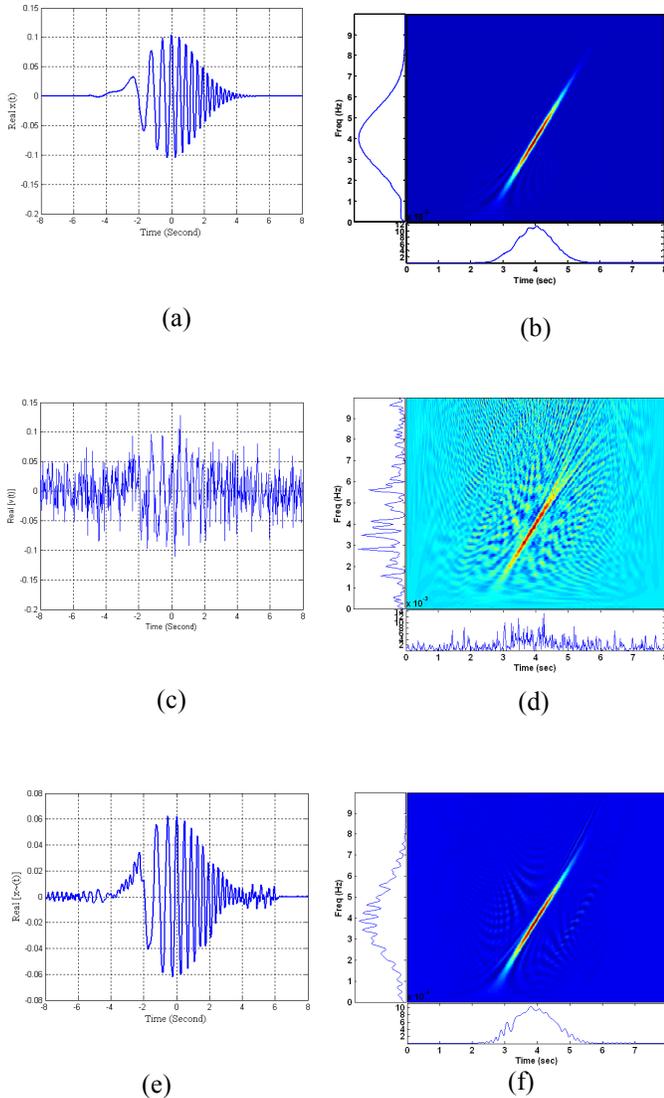
Fig. 3: (a) The signal $x(t)$, (b) the Wigner distribution of $x(t)$, (c) the signal $v(t)$ with $S/N = 0dB$, (d) the Wigner distribution of $v(t)$, (e) the output signal $\tilde{x}(t)$, (f) the Wigner distribution of $\tilde{x}(t)$, (g) the output signal $x''(t)$, (h) the Wigner distribution of $x''(t)$, (i) $\tilde{E}$, and (j) E''.

It is not clear from Figs 3(i) and (j) which approach gives better results. Figures 4(a,b,c, and d) explain the performance of our approach (filtering) with the approach (windowing). A linear chirp that has different values of β is used for this comparison. The average linear error is calculated as shown below:

$$\tilde{E} = avg(|x(t) - \tilde{x}(t)|)$$

and

$$E'' = avg(|x(t) - x''(t)|)$$

From these Figures we can conclude that the intersection of the error of the two approaches is a function of β (in more general is a function of the slope of the linear chirp, meaning a function of $(N, \beta, and\ f_s)$). If β has a small value (i. e. $\alpha < \pi/4$), then the filtering approach gives better results than windowing approach and vice versa.

(a)

(b)

(c)

(d)

(e)

(f)

(a)

(b)

1881

(e)

(f)
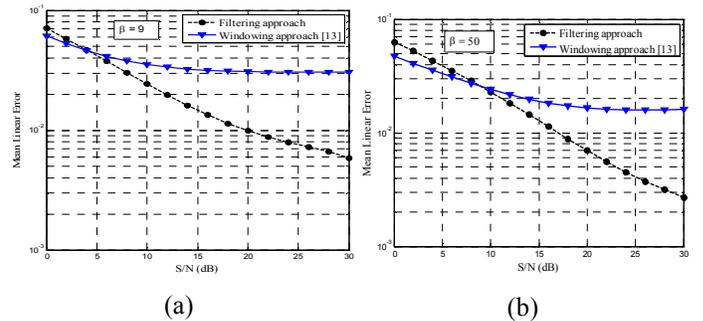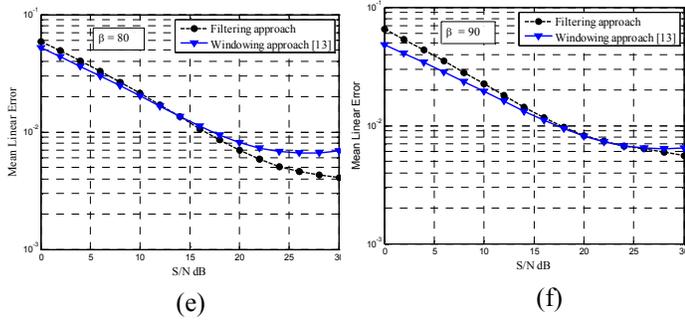
Fig. 4. (a) The error of $x(t)$ and $\tilde{x}(t)$, $S/N = 0\ dB$, (b) the error of $x(t)$ and $x''(t)$, $S/N = 0\ dB$, (c) mean linear error $\beta = 9$, (d) mean linear error $\beta = 50$, (e) Mean linear error $\beta = 80$, (f) mean linear error $\beta = 90$.

The proposed approaches attenuate the cross-terms on the Wigner distribution as given in Fig. 5. This can be done by taking the Wigner distribution of each chirp after separating them using our filtering/windowing approaches and then synthesizing the results together. The resolution of the time-frequency distribution of the proposed model is better than the Wigner distribution of the original signal, but the cost for this improvement will be the marginals are not satisfied anymore.

(a)

(b)

(c)

(d)
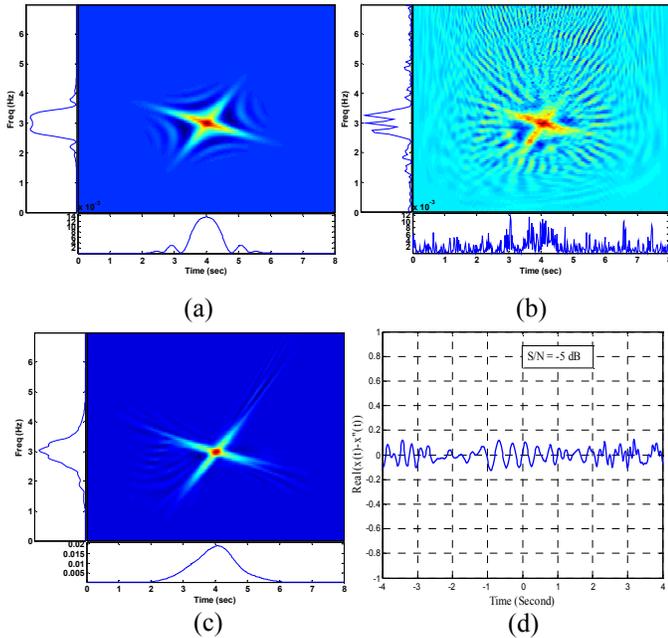
Fig. 5. The multi-component signal with $S/N = -5\ dB$, (a) the Wigner distribution of $x(t)$, (b) the Wigner distribution of $v(t)$, (c) the Wigner distribution of the separated chirps, (d) E''.

The process of separation of the cross-components is illustrated in Fig. 6. The optimal orders are $\alpha_1 = \pi/21$ and $\alpha_2 = \pi/93$ as shown in Figs 6 (a) and (b). The resulted signals from the rotation that corresponds to each order are shown in Figs. 6 (c) and (d). This shows clearly how the FrFT

can detect the signal and denoise it. The separated signals are shown in Figs. 6 (e) and (f).

(a)

(b)

(c)

(d)

(e)

(f)
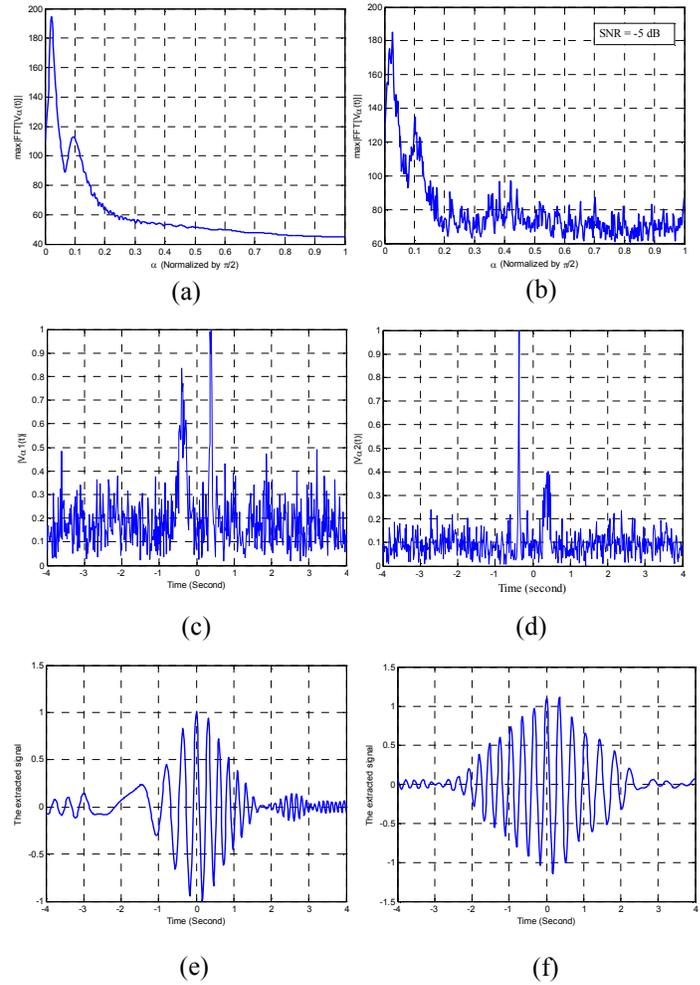
Fig. 6: (a) The peak values of FFT of $V_\alpha(t)$ with no noise, (b) the peak values of FFT of $V_\alpha(t)$ with $S/N = -5\ dB$, (c) The magnitude of $V_{\alpha1}(t)$, (d) the magnitude of $V_{\alpha2}(t)$, (e) the extracted signal at order $\alpha_1 = \pi/21$, (f) the extracted signal $\alpha_2 = \pi/93$.

Finally, an actual noisy sound signal of a canary bird is denoised and separated into its components. The original signal, noisy signal, and denoised signal are shown in Fig. 7.

(a)

(b)

Fig. 7: (a) The canary bird sound signal, (b) the Wigner distribution of the bird signal, (c) the noisy canary bird sound signal with $S/N = 5\,dB$, (d) the Wigner distribution of the noisy bird signal, (e) the denoised canary bird signal, and (f) the Wigner distribution of the denoised signal.

The separation of the components of the sound signal is shown in Fig. 8. The signal consists of three components, two sinusoidal and a chirp with optimal order $\alpha_2 = \pi/60$.
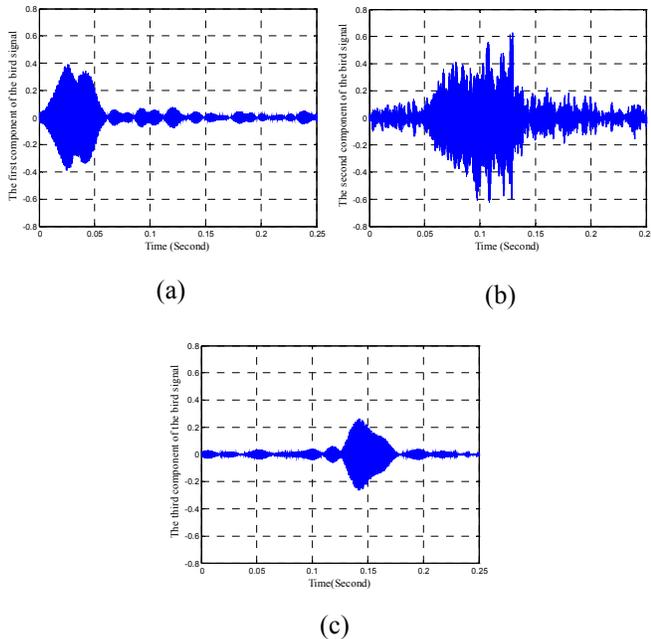
Fig. 8: (a) The first component of the bird sound signal (sinusoidal), (b) the second component of bird sound signal (chirp) with $\alpha_2 = \pi/60$, (c) the third component of the bird sound signal (sinusoidal).

4. CONCLUSIONS AND FUTURE WORK

In this paper, we propose a procedure that combines the Wigner distribution with the fractional Fourier transform to separate signal components, get rid of the cross-terms, and denoise them. The filtering and the windowing procedures are presented and explored to represent the signal by a sinusoidal or an impulse. The simulation results show the advantages of our model in attenuating the noise and the cross terms as well. Also, the filtering scenario gives better performance over the windowing one when the projection of the slope of the linear chirp on the frequency axis is smaller than the projection of the slope of the signal on the time axis (i.e. $\alpha < \pi/4$). An interesting direction for future research could be to determine how we can use this method to improve the theory of compressive sensing especially for signals represented as combination of chirps.

5. REFERENCES

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