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Contents

Introduction

This thesis comprises four essays that belong to different strands of the theoretical economic literature. Both Chapter 1 and Chapter 2 contribute to the study of two-sided one-to-one matching, or assignment, markets with quasi-linear utility and multi-dimensional heterogeneity. Chapter 1 investigates the efficiency of two-sided investments in large (continuum) two-sided economies in which matching and bargaining take place in an endogenously determined market without frictions *after* all agents have made costly, sunk investments. Chapter 2 scrutinizes a novel two-sided matching model with a finite number of agents and two-sided private information about exogenously given attributes. Chapter 3 is a note on the optimal size of fixed-prize research tournaments that seeks to fill two important gaps in an influential paper by Fullerton and McAfee (1999), and Chapter 4 studies the impact of incomplete information on the problem of maximizing revenue in a dynamic version of the knapsack problem, which is a classical combinatorial resource allocation problem with numerous economic applications.

Chapter 2 is based on joint work with Benny Moldovanu, and Chapter 4 is joint work with Alex Gershkov and Benny Moldovanu that has been published (in a slightly different version) as a paper in 2011 (Dizdar, Gershkov and Moldovanu, 2011). For this reason, I use the pronoun "we" in these chapters, whereas I use "I" in Chapter 1 and in Chapter 3.

The analysis of assignment markets with quasi-linear utility has been pioneered by Shapley and Shubik (1971). The basic setting is as follows: there are several heterogeneous agents on each side of the market, e.g. workers and firms, or buyers and sellers, who are characterized by attributes that jointly determine the match value/surplus of each potential partnership (or trade). Monetary transfers among agents are possible. In their famous study, Shapley and Shubik characterized the outcomes of *transferable utility* *assignment games* - assignment economies of the kind just described, in which attributes are exogenously given, and in which matching and bargaining take place under complete information and without any other frictions.

Many papers have varied or extended special cases of the basic assignment game framework, e.g. to investigate effects on prices and matching patterns introduced by private information (in auction or double auction settings, say) or search frictions, or to analyze investment incentives and the efficiency of market outcomes in situations in which attributes, rather than being exogenously given, result from costly investments made by the agents. Much (though not all) of the related literature has built on one particular version of the Shapley-Shubik model that was popularized by Becker (1973).[1] In this kind of model, all agents are completely described by a one-dimensional attribute, representing for example the skill or education of a worker or the quality of the physical capital of a firm. Moreover, attributes satisfy a strong form of complementarity. These restrictive assumptions have strong implications for agents' preferences that are often not tenable (in particular, positive assortative matching is implied in the frictionless model).

A common motivation for the research of Chapters 1 and 2 is to add to the understanding of the economics of assignment markets in which agents on both sides of the market are heterogeneous with respect to *several* relevant characteristics. Therefore, the present work is based on the general Shapley-Shubik model and on parts of the more advanced mathematical literature on optimal transport.

Previous research has noted that in many two-sided economies, agents must make costly investments in physical or human capital before they meet potential partners: agents compete for partners only "ex-post", in a market that is endogenously determined by all agents' (sunk) investments (Acemoglu, 1996; Cole, Mailath and Postlewaite, 2001a). Chapter 1 studies the efficiency of two-sided investments and matching in large economies, when agents are characterized by *multi-dimensional cost/ability types* and must decide about investments in *multi-dimensional attributes.* To this end, I extend the seminal model of Cole, Mailath and Postlewaite (in particular, I maintain the assumptions of complete information and of frictionless ex-post markets), who assumed supermodular match surplus and ordered marginal cost types, to allow for general continuum assignment games. First, I show that there always is an "ex-post contracting" equilibrium (agents

[1] In addition, *linear* versions of the Shapley-Shubik model also play a prominent role in applications to auctions and double auctions.

must not have an incentive to deviate from their investment, given correct anticipation of the post-investment market; see Chapter 1 for the formal definition) that supports ex-ante efficient investment and matching. Hence, the main efficiency result of Cole, Mailath and Postlewaite does not hinge on the single-crossing properties entailed by the Becker framework. The main part of Chapter 1 aims at an in-depth study of the complex interplay of technology and ex-ante heterogeneity that determines whether inefficient equilibria exist as well, or whether these are necessarily ruled out by sufficiently rich attribute markets. I identify a certain form of multiplicity in the technology as the main source of potential coordination failures. Mismatch of agents due to inefficient specialization (which is impossible in Becker-type models) may occur even without such multiplicity, but there are very strong trends towards ex-ante efficiency in these cases. I also illustrate (in a case with multiplicity) that, in contrast to examples given by Cole, Mailath and Postlewaite, it is possible that even arbitrarily high ex-ante heterogeneity does not suffice to rule out inefficiency. The analysis proceeds by means of a combination of general lemmas and several rather involved examples, one of which uses some advanced results from optimal transport theory.

If attributes are private information, then match surplus becomes *informationally interdependent*. In Chapter 2, we study such situations of two-sided private information: a finite number of agents with exogenously given, privately known attributes need to be matched to form productive relationships. We ask whether there are standardized rules for dividing ex-post realized surplus within matched pairs that are compatible with information revelation leading, for each realization of attributes in the economy, to an efficient matching. Maybe surprisingly, we find that for multi-dimensional, complementary attributes, the only robust rules that are compatible with efficient match formation in this sense are those that divide the surplus in each match according to the same *fixed proportion*, independently of the attributes of the pair's members. Such fixed-proportion rules are observed in widely differing circumstances. We interpret our result as highlighting a desirable feature distinguishing such contracts that has previously gone unnoticed, and which complements other rationales that have been given in the literature, based for example on moral hazard or risk-sharing arguments.

Fullerton and McAfee (1999) studied how to design a fixed-prize research tournament in cases where firms/suppliers are heterogeneous with respect to their research costs and where the research technology is stochastic. They focused on how cost asymmetries affect two major issues for the designer (who wants to procure a high quality innovation at low cost): how many contestants should be admitted? How should these be selected from the set of available candidates when costs are *private information* prior to the tournament? The note in Chapter 3 analyzes two open problems with regard to the optimal number of contestants. One problem pertains to the case where costs are common knowledge, the procurer can select participants by charging non-discriminatory entry fees, and no artificial restrictions on asymmetries are imposed. My result generally supports arranging tournaments with the two most efficient firms only, but it also identifies instances of asymmetry for which admitting more contestants is profitable for the procurer. More importantly, I provide a rigorous analysis of optimal tournament size for a case where costs are private information of ex-ante symmetric firms before the tournament. Fullerton and McAfee showed that an all-pay entry auction should then be used to select the most efficient firms from the pool of candidates and to raise money to finance parts of the prize (entry fees, as well as standard discriminatory-price and uniform-price auctions may fail as selection mechanisms). I discuss the procurer's problem of stimulating a given expected aggregate research effort at lowest expected total cost by choosing tournament size optimally, and I derive a closed form solution for the case where marginal costs are uniformly distributed on $[0, \bar{c}]$. The result strongly favors the smallest possible tournament with only two participants.

Chapter 4 analyzes maximization of revenue in the dynamic and stochastic knapsack problem where a given capacity needs to be allocated by a given deadline to sequentially arriving, impatient agents. Each agent is described by a two-dimensional type that reflects his capacity requirement and his willingness to pay per unit of capacity. Types are private information and result from i.i.d. draws. We first characterize all implementable policies that are relevant for the purpose of revenue maximization. A simple characterization of these policies is available (despite two-dimensional private information) since utility functions have a special form. Then we solve the revenue maximization problem for the special case where there is private information about per-unit values, but capacity needs are observable. After that we derive two sets of additional conditions on the joint distribution of values and weights under which the revenue maximizing policy for the case with observable weights is implementable, and thus optimal also for the case with two-dimensional private information. We also construct a simple policy for which per-unit prices vary with requested weight but not with time, and we prove that it is asymptotically revenue maximizing when available capacity and time to the deadline both go to infinity. This highlights the importance of nonlinear as opposed to dynamic pricing.

Chapter 1

Investments and matching with multi-dimensional attributes

This chapter studies the role of multi-dimensional heterogeneity for the efficiency of two-sided investments in large two-sided economies. Heterogeneous agents characterized by multi-dimensional cost/ability types first make sunk investments in multi-dimensional attributes, with which they then compete for partners in a frictionless market with transferable utility. The Kantorovich duality theorem of optimal transport is used to extend the seminal model of Cole, Mailath and Postlewaite (2001a) (CMP) from one-dimensional attributes, supermodular match surplus and ordered marginal cost types (the "1-d supermodular framework") to general continuum assignment games. There always is an ex-post contracting equilibrium that supports ex-ante efficient investment and matching. Hence, the main efficiency result of (CMP) does not hinge on single-crossing conditions. A complex interplay of ex-ante heterogeneity and technology determines whether endogenous attribute markets are necessarily rich enough to rule out inefficient equilibria. Unlike in the 1-d supermodular framework, mismatch of agents due to inefficient specialization may occur. This can happen even if the technology does not feature a kind of multiplicity that is shown to be necessary for inefficient equilibria in the model of (CMP). However, outside options in the endogenous attribute market strongly tend to rule out inefficient investments in this case. The geometric characterization of efficient matchings and other results from the theory of optimal transport are shown to be useful tools for studying such effects. If the technology features multiplicity, then severe coordination failures involving mismatch and/or jointly inefficient investments are often possible. Finally, an *example with simultaneous under- and over-investment shows that, unlike in the examples of (CMP), even extreme ex-ante differentiation of types may not suffice to eliminate inefficient equilibria.*

1.1 Introduction

Many investments in physical and human capital must be made before economic agents compete for complementary partners. For example, individuals make substantial human capital investments before they try to find a job, and firms acquire physical capital before they hire workers (Acemoglu, 1996; Cole, Mailath and Postlewaite, 2001a, 2001b). Similarly, sellers may need to invest in features of their product prior to contracting with buyers. These may, in turn, invest to prepare for optimal usage of the product. In both cases, *two-sided* investments have a strong impact on how productive or profitable potential future relationships can be. However, buyers and sellers (firms and workers) can not contract ex-ante and coordinate their choices directly: at the time investments must be made, the parties have not met each other yet.

What are agents' incentives to invest in view of the subsequent competition for partners?[1] When does future competition trigger efficient two-sided investments, effectively eliminating hold-up problems and coordination failures? Which relationships are formed and how are the profits or productive surpluses shared among partners? These and related questions have been studied both for small, finite and for very large, continuum two-sided economies. In two seminal papers, Cole, Mailath and Postlewaite (2001a, 2001b) examined the case of frictionless (core) bargaining ex-post. In their continuum model (Cole, Mailath and Postlewaite, 2001a, henceforth (CMP)), an equilibrium with efficient investment and matching always exists, but coordination failures may still happen. Other important contributions analyzed the role of search frictions (Acemoglu, 1996) and the impact of non-transferable utility (Peters and Siow, 2002).[2] In all these papers, heterogeneous agents from both sides of a two-sided economy first make costly investments

[1]In the classical hold-up problem due to incomplete contracts, parties are in a relationship when they invest, and the degree to which investments are relationship-specific is determined by exogenous outside options (e.g. Williamson, 1985). Here, like in the more closely related literature that is discussed below, the specificity of any investment is endogenously determined by the investments of all other agents, as well as by properties of the market in which agents compete (see e.g. Cole, Mailath and Postlewaite 2001a, 2001b).

[2]See below for a brief summary of results and of further related literature.

in a simultaneous and non-cooperative manner. They thereby acquire *attributes* with which they then compete for partners, pair off and share a match surplus that depends on the attributes of both parties. Equilibrium requires in particular that agents must not have a profitable deviation at the investment stage. Quasi-linear utility, complete information and a deterministic technology, which turns investments into attributes and matched attributes into match surplus, also belong to the common framework.

The present study is motivated by two central observations. First, investments (and attributes that result from investment) determining match surplus are inherently *multi-dimensional* in most interesting applications. This is important since it implies that, in marked contrast to the situation studied in most of the literature, agents' *preferences over potential partners* are usually *not fully aligned*, or even ordered by standard single-crossing conditions. In particular, investing in multi-dimensional attributes may entail significant specialization: having chosen a particular investment, one may be a suitable partner for some agents but not for others, even if the investment was "high-level". The second observation is that agents are typically very heterogeneous with respect to their cost of acquiring the multi-dimensional attributes. As a simple example, some agent may have low costs for investing in communication and social skills while he has high costs for investing in mathematical skills - but it may be the other way round for another agent from the same side of the market.

In this chapter, I study the implications that multi-dimensional attributes and multi-dimensional cost/ability types have for the efficiency of *two-sided* investments, in a model that generalizes the one of (CMP). In particular, I do neither assume that *gross match surplus* is a strictly *supermodular* function of buyers' and sellers' *one-dimensional* investment levels (strategic complementarity), nor that the heterogeneous agents can be *completely ordered in terms of marginal cost* of investment.[3] In short, I use the general continuous assignment game framework (Shapley and Shubik, 1971; Gretzky, Ostroy and Zame, 1992, 1999) and methods from the closely related mathematical theory of optimal transport (Villani, 2009; Chiappori, McCann and Nesheim, 2010), instead of the Becker (1973) framework of assortative matching. Consequently, the model allows for the representation of general preference relations both before and after investment.

Here is a brief preview of the main results. First, there always is an equilibrium in

[3]Such assumptions à la Becker (1973) have been made by a vast majority of papers that study two-sided matching with quasi-linear utility, and these assumptions have then been adapted by the related literature that includes an ex-ante investment stage.

which all agents invest and match efficiently. Second, unlike in the "1-d supermodular framework" à la (CMP), equilibria featuring a mismatch of buyers and sellers due to inefficient specialization may exist. This is possible even if the economy's net technology does not feature a certain kind of multiplicity that turns out to be necessary for inefficiency in the 1-d supermodular framework. Third, however, without technological multiplicity, outside options in the endogenous attribute market strongly tend to trigger deviations in any hypothesized inefficient equilibrium, so that investments and matching are often forced to be ex-ante efficient. In marked contrast, severe coordination failures involving mismatch and/or jointly inefficient investments in equilibrium partnerships may easily occur when the technology features multiplicity. A very high degree of heterogeneity of ex-ante populations is usually needed to rule out such coordination failures, via sufficiently rich attribute markets. In fact, unlike in the under-/over-investment examples of (CMP), even arbitrarily high ex-ante differentiation may be insufficient. This is true also within the 1-d supermodular framework.

The basic two-stage model is as in (CMP): after all agents have non-cooperatively made their investments, second-stage bargaining leads to a core solution (equivalently, a competitive equilibrium) of the frictionless *continuum transferable utility assignment game* that corresponds to the given gross match surplus function and to the populations of attributes that result from investment. In equilibrium, every agent correctly anticipates the investments of all others, the outcome of the second-stage equilibrium matching market (i.e. allocation and prices/payoffs for existing attributes) and the effect of any deviation from his own equilibrium investment. Given this, agents must not have an incentive for a unilateral deviation. (CMP) called this an *ex-post contracting equilibrium*, and I will follow their nomenclature.

I employ a fundamental theorem from the theory of optimal transport, adapted from Villani (2009), to formalize ex-post contracting equilibrium for general assignment games. Compared to the economic literature on assignment games which has primarily been concerned with *Walrasian prices/core utilities*, the optimal transport result sheds a lot of additional light on the structure of *efficient matchings/Walrasian allocations*.[4] This is important for studying coordination failures in the second part of this chapter.[5] Moreover, the approach also serves to resolve some technical issues with the continuum

[4]See Section 1.2.2.

[5]For the first part of the chapter, one could also build on Gretzky, Ostroy and Zame (1992, 1999).

model that have been discussed at considerable length in (CMP).

Virtually any *ex-ante stable and feasible bargaining outcome*, i.e. pair of efficient matching and core sharing of net surplus in the hypothetical assignment game in which agents can match and write complete contracts prior to investment, can be achieved in ex-post contracting equilibrium.[6] Hence, the main result about the existence of an ex-post contracting equilibrium that is ex-ante efficient (Proposition 3 in (CMP)), does not hinge on the *Spence-Mirrlees single-crossing conditions* which are implied by supermodular match surplus and ordered marginal costs.[7] This is remarkable since the nice explicit proof of (CMP) heavily used single-crossing conditions. Intuitively, with a continuum of buyers and sellers all agents have "essentially perfect substitutes" (Cole, Mailath and Postlewaite, 2001b, pg. 1), and no single individual can affect the market payoffs of others.[8] Transferable utility (TU) and frictionless matching eliminate the remaining potential sources of hold-up, and it turns out that this is sufficient to guarantee the existence of an efficient ex-post contracting equilibrium, regardless of additional structural assumptions about the technology or about the ex-ante populations.[9]

(CMP) gave examples of additional equilibria in which parts of both populations under-invest (over-invest). However, these coordination failures are very special: due to supermodular match surplus and ordered marginal costs, matching is always positively assortative in equilibrium, both ex-post (in investment levels) and ex-ante (in marginal cost types). Even though some buyers and sellers form matches with investments that are not *jointly efficient*, there never is any *mismatch* from an ex-ante perspective.

To organize the analysis of general ex-post contracting equilibria, I first note that (because of no hold-up) any agent's equilibrium investment maximizes net match surplus, *given* the investment of his equilibrium partner. In other words, for all equilibrium matches, investments must form a Nash equilibrium of a hypothetical "full appropriation game". Multiplicity of Nash equilibria of these games corresponds to a multiplicity in the economy's technology. In this case, (generically) only one of the Nash equilibrium profiles maximizes net surplus for the pair. In the framework of (CMP), coordination

[6] This holds true under a very mild technical condition.

[7] The standard formulation of this well known condition may be found for instance in Milgrom and Shannon (1994).

[8] For details, see Cole, Mailath and Postlewaite (2001b), (CMP), Section 1.2.3, and also the discussion of Makowski (2004) below.

[9] Often, though not always, there is a (essentially) unique ex-ante stable and feasible bargaining outcome, see also footnote 28.

failures are possible *only if* such multiplicity in the technology exists (see Proposition 1.5). In contrast, with multi-dimensional cost types and attributes, coordination failures involving inefficient specialization and a mismatch of buyers and sellers may happen even if there is no multiplicity in the technology (see Section 1.5.3.1).

In any case, whether a candidate for an inefficient equilibrium unravels or not is determined by whether the induced attribute market triggers a deviation by some agent (who is better off if he changes his investment and proposes a match with an existing attribute from the other side, given prices for existing attributes). This in turn depends on the exogenous ex-ante heterogeneity. (CMP) used these insights to show that their inefficient equilibria unravel when the ex-ante heterogeneity is sufficiently large.

Once one leaves the 1-d supermodular framework, it is not a priori clear any more who matches with whom in equilibrium. Still, I will argue that in cases without multiplicity in the technology, outside options strongly discipline equilibrium investment and matching towards ex-ante efficiency. To illustrate this in a rigorous way, I analyze one particular *truly multi-dimensional* model in Section 1.5.3. I derive a set of conditions on ex-ante heterogeneity which imply that *any* equilibrium that features a pure, smooth matching of buyers and sellers is ex-ante optimal. This part of the paper uses the geometric characterization of efficient matchings that is implied by the fundamental theorem of optimal transport, along with a more advanced regularity result, applied to the classical case of bilinear surplus.

If technological multiplicity is an issue, then coordination failures may easily occur even for highly differentiated ex-ante populations (as in the examples of (CMP)). Multi-dimensionality adds at least two aggravating factors then: the basic possibility of mismatch substantially weakens unraveling effects, and whether all inefficient candidate equilibria are eliminated depends heavily on the full type distributions rather than just on their supports. Finally, I show that even extreme ex-ante heterogeneity may be insufficient to guarantee that the attribute market is rich enough to rule out coordination failures. The example, in which under- and over-investment occur simultaneously, also adds to the picture of the most interesting inefficiencies in the 1-d supermodular framework.

1.1.1 Related literature

The most closely related paper (CMP) has already been discussed above. In the case of finitely many buyers and sellers, an efficient ex-post contracting equilibrium that is robust with respect to specifications of the off-equilibrium (core) bargaining outcome exists whenever a, non-generic, “double-overlap” condition is satisfied (Cole, Mailath and Postlewaite, 2001b). Generically, full ex-ante efficiency may be achieved only if off-equilibrium outcomes punish deviations, which requires unreasonable sensitivity to whether the deviating agent is a buyer or a seller. A particular and limited form of mismatch is sometimes possible due to the allocative externality that a single agent can exert on others by “taking away a better partner” through an aggressive investment. This additional form of coordination failure was first identified by Felli and Roberts (2001).[10] Makowski (2004) analyzed a continuum model with general assignment games in which single agents are - and expect to be - pivotal for aggregate market outcomes whenever the endogenous attribute economy has a non-singleton core. He showed that results similar to those of Cole, Mailath and Postlewaite (2001b) hold in this case: in particular, hold-up and Felli-Roberts type inefficiencies are possible. I prefer to follow (CMP) and assume that a single agent is not - and does not expect to be - pivotal for aggregate market outcomes in a very large economy. For example, in contrast to Makowski's model, such assumptions are in principle consistent with the introduction of small amounts of uncertainty, e.g. in the form of a small “probability of death” between investment and market participation.[11] Moreover, the main focus of the present chapter is on whether outside options in endogenous markets necessarily rule out coordination failures - a question that Makowski does not study.

For a particular form of non-transferable utility (NTU, fifty-fifty sharing of an additive match surplus), ordered cost types and continuum populations, Peters and Siow (2002) showed that there is an equilibrium that is ex-ante efficient. Acemoglu (1996) formalized the hold-up problem associated with search frictions in the second-stage matching market. He also demonstrated how such frictions and a resulting “pecuniary externality” (Acemoglu, 1996, pg. 1) may explain social increasing returns in human (and physical) capital accumulation, in a model without technological externalities. Mailath, Postlewaite and Samuelson (2012a, 2012b) introduced another friction, namely that sellers can not observe buyers' attributes and may only use uniform pricing. They studied the impact

[10] They studied the interplay of hold-up and coordination failure when double-overlap does not hold, and when buyers bid for sellers in a particular non-cooperative game.

[11] See Gall, Legros, and Newman (2012) and Bhaskar and Hopkins (2011) for models with a noisy investment technology. Gall, Legros and Newman also obtained a result of “over-investment at the top, under-investment at the bottom”(compare the result of Section 1.5.5), albeit for very different reasons, in a model with non-transferable utility.

that *premuneration values* have for the efficiency of investments in this case. Except for Makowski (2004), all of the above papers assume one-dimensional investments and some kind of single-crossing.

The transferable utility assignment game with *exogenously given attributes* has received considerable attention in the economics literature. In their seminal paper on the "housing market" with finitely many buyers and sellers, Shapley and Shubik (1971) proved that the core of the assignment game is equivalent to the set of Walrasian equilibria, and to the solutions of a linear program. More precisely, "solutions to the linear programming problem [of maximizing aggregate surplus, D. D.] are Walrasian allocations, and solutions to the dual linear programming problem are core utilities and correspond to Walrasian prices", as Gretzky, Ostroy and Zame (1999, pg. 66) succinctly put it. In addition, core utilities are stable and feasible surplus shares for two-sided matching.

Gretzky, Ostroy and Zame (1992) (henceforth (GOZa)) extended these equivalences to the continuum model, for which the heterogeneous populations of buyers and sellers are described by non-negative Borel measures on the spaces of possible attributes. Gretzky, Ostroy and Zame (1999) (henceforth (GOZb)) identified several equivalent conditions for perfect competitiveness of an assignment economy with continuous match surplus (as is the case in this chapter), in the sense that individuals (in the continuum model, infinitesimal individuals) are unable to manipulate prices in their favor. Among these equivalent conditions are that the core is a singleton, that all agents fully appropriate their marginal products, and that the social gains function (i.e. the optimal value of the linear program) is differentiable with respect to the population measure. (GOZb) showed that perfect competition is a generic property for continuum assignment economies with continuous match surplus,[12] and that most large finite assignment economies are "approximately perfectly competitive".

The linear program associated with the transferable utility assignment game, i.e. the optimal transport problem, is also the subject of study of an extensive mathematical literature. An excellent reference is the book by Villani (2009). It surveys a multitude of results, including the fundamental duality theorem about the existence and structure of optimal transports that I use in this chapter. More advanced topics include sufficient conditions for uniqueness of optimal transports/ Walrasian allocations ((GOZb) were

[12]Intuitively, the existence of a long side and a short side of the market as well as "overlaps" of matched agent types are generic and pin down core utilities uniquely.

concerned with uniqueness of prices) and for pure optimal assignments (i.e. each type of agent is matched to exactly one type of agent from the other side), a delicate regularity theory,[13] and many other things.

Chiappori, McCann and Nesheim (2010) related the optimal transport problem to hedonic pricing and to stable matching.[14] Using advanced techniques, they established sufficient conditions for uniqueness and purity of an optimal matching (including a generalized single-crossing condition), as well as a weaker condition that is sufficient for uniqueness. They also stated a condition that implies that derivatives of core utilities are unique, thereby complementing the results of (GOZb). Figalli, Kim and McCann (2011) used advanced techniques from the regularity theory for optimal transport problems to provide necessary and sufficient conditions for monopolistic screening to be a convex program. This substantial achievement sheds light on important earlier contributions to multi-dimensional monopolistic screening, such as Rochet and Choné (1998).

Some less closely related papers analyzed how heterogeneous agents compete for partners through costly *signals* in the assortative framework. In Hoppe, Moldovanu and Sela (2009), investments are wasteful and may be used to signal private information about characteristics that determine match surplus (which is shared fifty-fifty). They studied how the heterogeneity of (finite or infinite) agent populations determines the amount of wasteful signalling. Among other things, they identified conditions such that random matching is welfare superior to assortative matching based on costly signalling. Hopkins (2012) studied a model in which investments signal private information about productive characteristics but also contribute to match surplus (i.e. they are only partially wasteful). His main results nicely identify comparative statics effects associated with changes in the populations, both under NTU and under TU.

The plan of the chapter is as follows: Section 1.2 explains the primitives of the model, the structure of market outcomes, and the two-stage equilibrium concept. Section 1.3 lays out the efficiency benchmark. Section 1.4 contains the result about existence of an ex-ante efficient equilibrium. Section 1.5 studies the interplay of technology and ex-ante heterogeneity of types that determines whether coordination failures may happen, and if so, what they look like. All proofs may be found in Section 1.6.

[13] I use a classical regularity result in Section 1.5.3.3.

[14] Their work is partly inspired by two earlier papers by Ekeland (2005, 2010), who used a convex programming approach.

1.2 Model

1.2.1 Agent populations, costs, and match surplus

There is a continuum of buyers and sellers. All agents have quasi-linear utility functions and utility is transferable in the form of monetary payments. At an ex-ante stage $t = 0$, all buyers and sellers simultaneously and non-cooperatively choose costly investments. Agents may be heterogeneous with respect to costs. Formally, a buyer of *type* $b \in B$ who invests into an *attribute* $x \in X$ incurs a cost $c(x, b)$. Similarly, a seller of type $s \in S$ can invest into an attribute $y \in Y$ at cost $d(y, s)$. B, S, X and Y are compact metric spaces (metrics and induced topologies are suppressed in the notation),[15] and $c : X \times B \to \mathbb{R}_+$ and $d : Y \times S \to \mathbb{R}_+$ are continuous functions.

If a buyer with attribute x and a seller with attribute y match, they generate *gross match surplus* $v(x, y)$. The function $v : X \times Y \to \mathbb{R}_+$ is continuous, and unmatched agents obtain zero surplus.[16]

The continuum populations of buyers and sellers are described by Borel probability measures μ on B and ν on S.[17] The "generic" case with a long side and a short side of the market (that is, more buyers than sellers or vice versa) is easily included by adding (topologically isolated) "dummy" types on the short side. Dummy types $b_\varnothing \in B$ and $s_\varnothing \in S$ always choose dummy investments $x_\varnothing \in X$ and $y_\varnothing \in Y$ at zero cost. $x_\varnothing$ and $y_\varnothing$ are assumed to be prohibitively costly for all $b \neq b_\varnothing$, $s \neq s_\varnothing$, so that no real agent ever chooses them. The assumption that unmatched agents create no surplus translates into $v(x_\varnothing, \cdot) \equiv 0$ and $v(\cdot, y_\varnothing) \equiv 0$.

[15] The basic theory of optimal transport has been developed for general Polish spaces, and some results could be obtained in that setting. Such a gain in generality seems to be of very minor economic importance but would require additional technical assumptions. Thus, I stick to compact metric type- and attribute spaces.

[16] This latter assumption is made merely for simplicity. A model in which it may be a socially valuable option to leave some agents unmatched even though potential partners are still available is ultimately equivalent.

[17] I use normalized measures which is common in optimal transport. (GOZa) and (GOZb) use non-negative Borel measures, which is useful for analyzing the "social gains function" that plays a key role in their work.

1.2.2 The transferable utility assignment game

The two-sided market in which agents compete for partners at the ex-post stage $t = 1$, given the attributes that result from sunk investments, is modeled as a continuum transferable utility assignment game. The basic data are two population measures of attributes, $\widetilde{\mu}$ on X and $\widetilde{\nu}$ on Y, along with the gross match surplus function v.[18] I focus on the linear programming formulation of the assignment game. Proposition 1.1 below, which is adapted from Theorem 5.10 of Villani (2009) suggests a natural definition of a *stable and feasible bargaining outcome* for a given assignment game, as a pair of i) an efficient matching/coupling (a primal solution) and ii) a stable and feasible (pointwise, for all matches that are formed) sharing of match surplus (a dual solution). Given the equivalences established by (GOZa) (compare Section 1.1.1), these solutions also correspond to Walrasian equilibria and to pairs of efficient allocations and core utilities. The exposition of material in this section is deliberately concise. For additional details, Chapters 4 and 5 of Villani (2009) and potentially also (GOZa) and (GOZb) should be consulted.

The feasible allocations (i.e. matchings of attributes) are the so-called *couplings* of $\widetilde{\mu}$ and $\widetilde{\nu}$, i.e. the measures $\widetilde{\pi}$ on $X \times Y$ with marginal measures $\widetilde{\mu}$ and $\widetilde{\nu}$.[19] I write $\Pi(\widetilde{\mu}, \widetilde{\nu})$ for the set of all these couplings. Thus, the linear program of finding an efficient matching/ a Walrasian allocation is to find a $\widetilde{\pi} \in \Pi(\widetilde{\mu}, \widetilde{\nu})$ that attains

$$\sup_{\widetilde{\pi}' \in \Pi(\widetilde{\mu}, \widetilde{\nu})} \int_{X \times Y} v \, d\widetilde{\pi}'.$$

The dual linear program is to minimize aggregate payoffs among all attribute payoff functions that satisfy a pointwise stability requirement (but no feasibility, there is no matching in the dual problem): find functions $\widetilde{\psi} \in L^1(\widetilde{\mu})$ and $\widetilde{\phi} \in L^1(\widetilde{\nu})$ from the constraint set specified below (the constraint qualification must hold for a pair of representatives from the L^1-equivalence classes) which attain

$$\inf_{\{(\widetilde{\psi}', \widetilde{\phi}') \in L^1(\widetilde{\mu}) \times L^1(\widetilde{\nu}) \mid \widetilde{\phi}'(y) + \widetilde{\psi}'(x) \geq v(x,y) \text{ for all } (x,y) \in \mathrm{Supp}(\widetilde{\mu}) \times \mathrm{Supp}(\widetilde{\nu})\}} \left(\int_Y \widetilde{\phi}' \, d\widetilde{\nu} + \int_X \widetilde{\psi}' \, d\widetilde{\mu} \right).$$

[18]Since v is continuous, the framework is equivalent to that of (GOZb) and of Section 3.5 in (GOZa).

[19]Note that since match surplus is non-negative and unmatched agents create zero surplus, there is no need to explicitly consider the possibility that agents remain unmatched. Those agents who match with a dummy type are of course de facto unmatched.

Note that the measure supports $\mathrm{Supp}(\widetilde{\mu})$ and $\mathrm{Supp}(\widetilde{\nu})$ describe the sets of existing attributes, i.e. attributes for which there are agents with that attribute.[20] When searching for optimal stable payoffs, one may restrict attention to functions $\widetilde{\psi}$ that are *v-convex* as defined below and set $\widetilde{\phi} := \widetilde{\psi}^v$, the so-called *v-transform* of $\widetilde{\psi}$.

Definition 1.1. *A function* $\widetilde{\psi} : \mathrm{Supp}(\widetilde{\mu}) \to \mathbb{R}$ *is called* v-convex *(w.r.t. the sets* $\mathrm{Supp}(\widetilde{\mu})$ *and* $\mathrm{Supp}(\widetilde{\nu})$*) if there is a function* $\widetilde{\zeta}$*:* $\mathrm{Supp}(\widetilde{\nu}) \to \mathbb{R} \cup \{+\infty\}$ *such that*

$$\widetilde{\psi}(x) = \sup_{y \in \mathrm{Supp}(\widetilde{\nu})} \Big(v(x,y) - \widetilde{\zeta}(y) \Big) =: \widetilde{\zeta}^v(x).$$

In this case, $\widetilde{\psi}^v(y) := \sup_{x \in \mathrm{Supp}(\widetilde{\mu})} \Big(v(x,y) - \widetilde{\psi}(x) \Big)$ *is called the* v-transform *of* $\widetilde{\psi}$*, and the* v-subdifferential *of* $\widetilde{\psi}$*,* $\partial_v \widetilde{\psi}$ *is defined as*

$$\partial_v \widetilde{\psi} \;:=\; \Big\{ (x,y) \in \mathrm{Supp}(\widetilde{\mu}) \times \mathrm{Supp}(\widetilde{\nu}) | \, \widetilde{\psi}^v(y) + \widetilde{\psi}(x) \;=\; v(x,y) \Big\}.$$

Remark 1.1. *i) A function* $\widetilde{\psi} : \mathrm{Supp}(\widetilde{\mu}) \to \mathbb{R}$ *is v-convex if and only if* $\widetilde{\psi} = (\widetilde{\psi}^v)^v$ *(Proposition 5.8 in Villani (2009)).*[21]

ii) $\widetilde{\psi}$ *and* $\widetilde{\phi} = \widetilde{\psi}^v$ *are pinned down for all* $x \in \mathrm{Supp}(\widetilde{\mu})$ *and* $y \in \mathrm{Supp}(\widetilde{\nu})$*, not just almost surely.*

iii) The relation $\widetilde{\psi}(x) = \sup_{y \in \mathrm{Supp}(\widetilde{\nu})} \Big(v(x,y) - \widetilde{\phi}(y) \Big)$ *reflects price-taking behavior of a single buyer. Given payoffs* $\widetilde{\phi}$ *for existing seller attributes, a buyer with attribute* x *can claim the gross match surplus net of the seller's payoff in any relationship, and he may optimize over all* $y \in \mathrm{Supp}(\widetilde{\nu})$ *(an analogous remark applies for sellers).*

iv) Since v is continuous and $\mathrm{Supp}(\widetilde{\mu})$ *and* $\mathrm{Supp}(\widetilde{\nu})$ *are compact, any v-convex function is continuous and so is its v-transform.*[22] *In particular, v-subdifferentials are closed.*

v) The v-subdifferential $\partial_v \widetilde{\psi}$ *is precisely the set of* (x,y) *for which the payoffs* $(\widetilde{\psi}, \widetilde{\phi})$ *are not only stable but also feasible, and hence may truly be interpreted as surplus shares for the given pair.*

Definition 1.2. *A set* $A \subset X \times Y$ *is called* v-cyclically monotone *if for all* $K \in \mathbb{N}$*,*

[20] More precisely, for any $x \in \mathrm{Supp}(\widetilde{\mu})$, every neighborhood of attributes containing x has strictly positive mass.

[21] This is a generalization of the usual Legendre duality for convex functions.

[22] The proofs of these claims are straightforward. They also follow immediately from the proof of Theorem 6 in (GOZa) who "v-convexify" a given dual solution to extract a continuous representative in the same L^1-equivalence class.

$(x_1, y_1), ..., (x_K, y_K) \in A$ *and* $y_{K+1} = y_1$, *it holds that*

$$\sum_{i=1}^{K} v(x_i, y_i) \geq \sum_{i=1}^{K} v(x_i, y_{i+1}).$$

It is easy to see that the v-subdifferential $\partial_v \widetilde{\psi}$ of a v-convex $\widetilde{\psi}$ is a v-cyclically monotone set. The following proposition is adapted from the fundamental Theorem 5.10 on Kantorovich duality in Villani (2009). It has two parts. The first part assures the existence of both primal and dual solutions (note that "sup" has turned into "max" and "inf" has turned into "min" in the equation below) and the equality of optimal values. The second part makes a statement about the structure of all optimal primal and dual solutions: couplings concentrated on v-cyclically monotone sets are optimal, and the support of *any* optimal coupling is contained in the v-subdifferential of *any* optimal buyer payoff function.

Proposition 1.1. *It holds*

$$\max_{\widetilde{\pi} \in \Pi(\widetilde{\mu}, \widetilde{\nu})} \int_{X \times Y} v \, d\widetilde{\pi} = \min_{\{\widetilde{\psi} | \widetilde{\psi} \, is \, v-convex \; w.r.t. \, \mathrm{Supp}(\widetilde{\mu}) \, and \, \mathrm{Supp}(\widetilde{\nu})\}} \left(\int_Y \widetilde{\psi}^v \, d\widetilde{\nu} + \int_X \widetilde{\psi} \, d\widetilde{\mu} \right).$$

If $\widetilde{\pi} \in \Pi(\widetilde{\mu}, \widetilde{\nu})$ *is concentrated on a v-cyclically monotone set then it is optimal. Moreover, there is a closed set* $\widetilde{\Gamma} \subset \mathrm{Supp}(\widetilde{\mu}) \times \mathrm{Supp}(\widetilde{\nu})$ *such that*

$$\begin{cases} \widetilde{\pi} \textit{ is optimal in the primal problem if and only if } \mathrm{Supp}(\widetilde{\pi}) \subset \widetilde{\Gamma}, \\ a \; v\textit{-convex } \widetilde{\psi} \textit{ is optimal in the dual problem if and only if } \widetilde{\Gamma} \subset \partial_v \widetilde{\psi}. \end{cases}$$

Thus, the dual solutions (attribute payoffs, prices, core utilities) are defined for all existing attributes and constitute a stable sharing of surplus that is (pointwise!) feasible for all matches that are part of an optimal coupling.[23] One may therefore define stable and feasible bargaining outcomes of a given transferable utility assignment game as follows.

Definition 1.3. *A* stable and feasible bargaining outcome *for the assignment game* $(\widetilde{\mu}, \widetilde{\nu}, v)$ *is a pair* $(\widetilde{\pi}, \widetilde{\psi})$, *such that* $\widetilde{\pi} \in \Pi(\widetilde{\mu}, \widetilde{\nu})$ *is an optimal solution for the primal linear*

[23](CMP) had to invest some effort to deal with functions that are defined only almost surely and to define feasibility appropriately, even in the special assortative framework. Proposition 1.1 immediately resolves such issues.

program, and the v-convex function $\widetilde{\psi}$ *is an optimal solution for the dual linear program.*

Proposition 1.1 ensures the existence of a stable and feasible bargaining outcome.

Remark 1.2. *The fact that the support of any optimal coupling is a v-cyclically monotone set hints at the fundamental connection between optimal transport and multi-dimensional monopolistic screening and mechanism design, which underlies the recent work of Figalli, Kim and McCann (2011).*

1.2.3 Ex-post contracting equilibria

Agents' non-cooperative equilibrium investments at $t = 0$ will be described by measurable functions $\beta : B \times S \to X$ and $\sigma : B \times S \to Y$, along with a "pre-assignment" $\pi \in \Pi(\mu, \nu)$ of buyers and sellers. The push-forwards (i.e. the image measures) $\beta_{\#}\pi$ and $\sigma_{\#}\pi$ then describe the resulting populations of attributes. I allow here for the possibility that an agent's investment depends explicitly both on his own type and on the type of agent from the other side that he *plans* to match with. This is needed to include cases in which agents of the same type make different investments, which often happens for instance when type spaces are discrete and, accordingly, there are continua of agents of the same type. However, in many cases it is possible to describe equilibrium investments by measurable functions $\beta : B \to X$ and $\sigma : S \to Y$.[24] Attribute populations are given by $\beta_{\#}\mu$ and $\sigma_{\#}\nu$ then,[25] and one may effectively drop π from the description.[26]

I call a tuple (β, σ, π) an *investment profile*, and I impose an innocuous regularity condition that corresponds to the "no isolated points" condition of (CMP).

Definition 1.4. *An investment profile* (β, σ, π) *is said to be* regular *if it holds for all* $(b, s) \in \mathrm{Supp}(\pi)$ *that* $\beta(b, s) \in \mathrm{Supp}(\beta_{\#}\pi)$ *and* $\sigma(b, s) \in \mathrm{Supp}(\sigma_{\#}\pi)$*.*

For a regular investment profile, there are no buyers and no sellers whose attributes get lost in the description $(\beta_{\#}\pi, \sigma_{\#}\pi, v)$ of the attribute assignment game. Moreover,

[24] (CMP) considered a special case of this. In their model, $B = S = [0, 1]$, $\mu = \nu = U[0, 1]$, $X = Y = \mathbb{R}_+$, and β and σ are "well-behaved", i.e. strictly increasing with finitely many discontinuities, Lipschitz on intervals of continuity points, and without isolated values.

[25] If $\hat{\beta} : B \to X$ is measurable, then $\beta(b, s) := \hat{\beta}(b)$ is product measurable, and $\beta_{\#}\pi = \hat{\beta}_{\#}\mu$ for any coupling $\pi \in \Pi(\mu, \nu)$.

[26] The literature on optimal transport has found important combinations of generalized single-crossing conditions for match surplus and mild conditions on type distributions that jointly ensure "pure" optimal matchings, see e.g. Villani (2009), Chiappori, McCann, and Nesheim (2010).

there are infinitely many "equivalent" outside options for each agent's investment.[27]

I follow (CMP) then and assume that a deviation by a single agent does not affect the payoff of any agent other than himself: given a stable and feasible bargaining outcome $(\tilde{\pi}, \tilde{\psi})$ for $(\beta_{\#}\pi, \sigma_{\#}\pi, v)$, a buyer who chooses attribute $x \in X$ (following a deviation this may be any attribute in X) can get a *gross payoff* of

$$\tilde{r}_B(x) \;=\; \sup_{y \in \operatorname{Supp}(\sigma_{\#}\pi)} \Big(v(x,y) - \tilde{\psi}^v(y) \Big)\,.$$

Since $\tilde{\psi}^v$ is continuous, the same is true for $\tilde{r}_B$ (by the Maximum Theorem), and $\tilde{r}_B(x) = \tilde{\psi}(x)$ holds for all $x \in \operatorname{Supp}(\beta_{\#}\pi)$. Similarly, sellers with attribute $y \in Y$ obtain

$$\tilde{r}_S(y) \;=\; \sup_{x \in \operatorname{Supp}(\beta_{\#}\pi)} \Big(v(x,y) - \tilde{\psi}(x) \Big)\,,$$

which coincides with $\tilde{\psi}^v(y)$ on $\operatorname{Supp}(\sigma_{\#}\pi)$.

Definition 1.5. *An* ex-post contracting equilibrium *is a tuple* $((\beta, \sigma, \pi), (\tilde{\pi}, \tilde{\psi}))$*, where* (β, σ, π) *is a regular investment profile and* $(\tilde{\pi}, \tilde{\psi})$ *is a stable and feasible bargaining outcome for* $(\beta_{\#}\pi, \sigma_{\#}\pi, v)$*, such that it holds for all* $(b, s) \in \operatorname{Supp}(\pi)$ *that*

$$\tilde{\psi}(\beta(b,s)) - c(\beta(b,s), b) \;=\; \sup_{x \in X} \left(\tilde{r}_B(x) - c(x,b)\right) \;=:\; r_B(b),$$

and

$$\tilde{\psi}^v(\sigma(b,s)) - d(\sigma(b,s), s) \;=\; \sup_{y \in Y} \left(\tilde{r}_S(y) - d(y,s)\right) \;=:\; r_S(s).$$

The *equilibrium net payoff* functions r_B and r_S are continuous (by the Maximum Theorem again).

[27] In the Appendix, I show that $\beta(\operatorname{Supp}(\pi))$ and $\sigma(\operatorname{Supp}(\pi))$ are contained and dense in $\operatorname{Supp}(\beta_{\#}\pi)$ and $\operatorname{Supp}(\sigma_{\#}\pi)$ (Lemma 1.5). As a consequence, the fact that $\beta(\operatorname{Supp}(\pi))$ and $\sigma(\operatorname{Supp}(\pi))$ are not necessarily closed or even merely measurable does not cause problems, and one may use the stable and feasible bargaining outcomes for $(\beta_{\#}\pi, \sigma_{\#}\pi, v)$ as defined in Section 1.2.2 to formulate agents' investment problems. Compare also Lemma 1.6 which formally shows how to complete a "stable and feasible bargaining outcome with respect to the sets $\beta(\operatorname{Supp}(\pi))$ and $\sigma(\operatorname{Supp}(\pi))$".

1.3 Ex-ante contracting equilibria

Just as in (CMP), the hypothetical case in which agents can match and bargain without frictions at $t = 0$ and write binding contracts provides the ex-ante efficiency benchmark. Define

$$h(x, y|b, s) := v(x, y) - c(x, b) - d(y, s).$$

The *net match surplus* that a buyer of type b and a seller of type s can generate is

$$w(b, s) := \max_{x \in X, y \in Y} h(x, y|b, s). \tag{1.1}$$

Jointly optimal investments $(x^*(b, s), y^*(b, s))$ maximizing $h(\cdot, \cdot|b, s)$ exist for all $(b, s) \in B \times S$ since X and Y are compact, and since v, c and d are continuous. These solutions need not be unique. By the Maximum Theorem, w is continuous. Applying Proposition 1.1 to (μ, ν, w) thus yields the existence of an *ex-ante stable and feasible bargaining outcome* (π^*, ψ^*), along with a closed set $\Gamma \subset B \times S$ that contains the support of any ex-ante optimal coupling, and which is contained in the w-subdifferential of any optimal w-convex buyer net payoff function.

Definition 1.6. *An* ex-ante contracting equilibrium *for* (μ, ν, w) *is a tuple* $((\pi^*, \psi^*), (x^*, y^*))$*, such that* (π^*, ψ^*) *is a stable and feasible bargaining outcome for* (μ, ν, w)*, and* $(x^*(b, s), y^*(b, s))$ *is a solution to* (1.1) *for all* $(b, s) \in \mathrm{Supp}(\pi^*)$*.*

Let me recall two immediate consequences for future reference. For any ex-ante stable and feasible bargaining outcome (π^*, ψ^*), it holds

$$\begin{aligned} (\psi^*)^w(s) + \psi^*(b) \ &= \ w(b, s) \quad \text{for all } (b, s) \in \mathrm{Supp}(\pi^*) \\ (\psi^*)^w(s) + \psi^*(b) \ &\geq \ w(b, s) \quad \text{for all } b \in \mathrm{Supp}(\mu), s \in \mathrm{Supp}(\nu). \end{aligned} \tag{1.2}$$

1.4 Efficient ex-post contracting equilibria

Theorem 1.1 below shows that any ex-ante stable and feasible bargaining outcome can be achieved in ex-post contracting equilibrium, provided that a very mild technical condition is satisfied.[28] Consequently, the main efficiency result of (CMP) does not hinge on supermodularity of match surplus and cost functions, which imply ordered preferences and assortative matching. This is particularly remarkable since single-crossing conditions took center stage in the proof of (CMP).

By the Maximum Theorem, the solution correspondence for the problem (1.1) is upper-hemicontinuous (and hence a measurable selection always exists). Any ex-ante stable and feasible bargaining outcome (π^*, ψ^*) that satisfies the following mild condition can be achieved as an ex-post contracting equilibrium.

Condition 1.1. *There is a selection (β^*, σ^*) from the solution correspondence for* (1.1) *such that $(\beta^*, \sigma^*, \pi^*)$ is a regular investment profile.*

Theorem 1.1. *Let (π^*, ψ^*) be an ex-ante stable and feasible bargaining outcome for (μ, ν, w) that satisfies Condition 1.1, and let (β^*, σ^*) be the corresponding selection. Then the regular investment profile $(\beta^*, \sigma^*, \pi^*)$ is part of an ex-post contracting equilibrium $((\beta^*, \sigma^*, \pi^*), (\widetilde{\pi}^*, \widetilde{\psi}^*))$ with $\widetilde{\pi}^* = (\beta^*, \sigma^*)_\# \pi^*$.*[29]

1.5 Inefficient equilibria

1.5.1 Two kinds of inefficiency: mismatch and inefficiency of joint investments

A pair of equilibrium investment profile and equilibrium coupling of attributes is compatible with at least one interpretation as an induced coupling of buyers and sellers. For example, if investments are given by injective maps from types to attributes, then

[28] One might ask when ex-ante stable and feasible bargaining outcomes are unique. This (very delicate) question about uniqueness of primal and dual solutions of the optimal transport problem is not peculiar to the two-stage model of this chapter which aims at analyzing the equilibrium investments of agents who subsequently enter a large and frictionless two-sided market. The question has of course been studied previously. For instance, (GOZb) have shown that for a given continuous match surplus function w and *generic* population measures μ and ν, dual solutions are unique. On the other hand, a substantial amount of research in optimal transport has been devoted to establishing sufficient conditions for unique, or more often unique and pure, optimal couplings.

[29] Remember that investments do not depend explicitly on the pre-assignment π^* whenever this coupling is pure! Still, π^* is needed to describe the coupling of attributes $(\beta^*, \sigma^*)_\# \pi^*$ that supports the ex-ante efficient matching of agents.

the interpretation is unique.[30]

It is useful to distinguish two kinds of inefficiency that are conceptually different. First, agents might prepare for the wrong partners and opt for an inefficient specialization for that reason. I will say that an ex-post contracting equilibrium exhibits *mismatch inefficiency* if it is not compatible with any ex-ante optimal coupling of buyers and sellers. Secondly, for a given coupling of buyers and sellers that is compatible with the equilibrium, it might be that some agents match with attributes that are not jointly optimal (i.e. the attributes do not maximize (1.1)). I will call this *inefficiency of joint investments*. In particular, an equilibrium is ex-ante efficient if and only if it has the following two properties (like the equilibrium constructed in Theorem 1.1): it does not exhibit mismatch inefficiency, and there is a compatible ex-ante optimal coupling of buyers and sellers for which there is no inefficiency of joint investments.

To avoid repetition, I would like to refer the reader to Section 1.1 for a preview of the analysis and results of this section. Sections 1.5.1.1 - 1.5.2 should be viewed as preparatory, while Sections 1.5.3 - 1.5.5 contain the main examples and results. Let me stress again though that in (CMP) mismatch is impossible: due to one-dimensional types and attributes and single-crossing conditions, every equilibrium is compatible with the positively assortative coupling of buyer types and seller types, and this coupling is ex-ante optimal. The examples of under-investment (over-investment) are characterized by inefficiency of joint investments for types with low (high) costs. Moreover, these equilibria always cease to exist when the ex-ante populations are sufficiently heterogeneous, since the (endogenous) attribute populations are then *necessarily* so rich that some agent wants to deviate.

1.5.1.1 Technological multiplicity and inefficiency of joint investments

As an expression of "no hold-up", any agent's equilibrium investment must maximize net match surplus, *conditional on* the attribute of his equilibrium partner.

Lemma 1.1. *In an ex-post contracting equilibrium* $((\beta,\sigma,\pi),(\tilde{\pi},\tilde{\psi}))$*, any* $(\beta(b,s'),y)\in \mathrm{Supp}(\tilde{\pi})$*, where* $(b,s')\in\mathrm{Supp}(\pi)$*, satisfies* $\beta(b,s')\in \operatorname{argmax}_{x\in X}\left(v(x,y)-c(x,b)\right)$*. Similarly, for any* $(b',s)\in\mathrm{Supp}(\pi)$ *and any* $(x,\sigma(b',s))\in\mathrm{Supp}(\tilde{\pi})$*, it holds that*

[30] On the other hand, if buyers of different types choose the same attribute and if this type of attribute is matched with investments that stem from different seller types, then the interpretation as a coupling of buyers and sellers is non-unique.

$\sigma(b', s) \in \operatorname{argmax}_{y \in Y} (v(x, y) - d(y, s))$.

In particular, the investments of a buyer b and a seller s who match in equilibrium must form a Nash equilibrium of a hypothetical complete information game between b and s with strategy spaces X and Y and payoffs $v(x, y) - c(x, b)$ and $v(x, y) - d(y, s)$. I will refer to this game as a "full appropriation" (FA) game between b and s.

Corollary 1.1. *If* $(\beta(b, s'), \sigma(b', s)) \in \operatorname{Supp}(\tilde{\pi})$ *for some* $(b, s'), (b', s) \in \operatorname{Supp}(\pi)$*, then* $(\beta(b, s'), \sigma(b', s))$ *is a Nash equilibrium (NE) of the FA game between* b *and* s*.*

Jointly optimal investments always form a Nash equilibrium of the FA game between b and s. When FA games have more than one pure strategy NE (in accordance with the deterministic investment functions β and σ, I consider only pure strategy NE without further mentioning it), this corresponds to a multiplicity in the economy's technology. Such multiplicity is a necessary condition for inefficiency of joint investments:

Corollary 1.2. *Assume that for all* $b \in \operatorname{Supp}(\mu)$*,* $s \in \operatorname{Supp}(\nu)$*, the FA game between* b *and* s *has a unique NE (which then coincides with the unique pair* $(x^*(b, s), y^*(b, s))$ *of jointly optimal investments). Then there is no inefficiency of joint investments in ex-post contracting equilibrium.*

In particular, without technological multiplicity, ex-post contracting equilibria are ex-ante efficient in the framework of (CMP). In the Appendix, I prove the claims made so far about that framework under Assumption 1.1 below, which slightly generalizes the basic model of (CMP) in three respects: no smoothness is assumed, costs need not be convex in attribute choice, and types need not be uniformly distributed on intervals.

Assumption 1.1. *Let* $X, Y, B, S \subset \mathbb{R}_+$*, and assume that* v *is strictly supermodular in* (x, y)*,* c *is strictly submodular in* (x, b)*, and* d *is strictly submodular in* (y, s)*.*[31]

1.5.1.2 The "constrained efficiency" property

(CMP) observed a useful indirect "constrained efficiency" property that quickly follows from the definition of ex-post contracting equilibrium: if there is a pair of agents that would ex-ante block the equilibrium outcome, then no attribute they could use for

[31] See e.g. Milgrom and Roberts (1990) or Topkis (1998) for formal definitions of these very well-known concepts.

blocking may exist in the equilibrium attribute market. Lemma 1.2 below rephrases the result in the current notation. A short proof is included in Section 1.6.

Lemma 1.2 (Lemma 2 of (CMP))**.** *Let* $((\beta, \sigma, \pi), (\tilde{\pi}, \tilde{\psi}))$ *be an ex-post contracting equilibrium. Suppose that there are* $b \in \mathrm{Supp}(\mu)$*,* $s \in \mathrm{Supp}(\nu)$ *and* $(x, y) \in X \times Y$ *such that* $h(x, y|b, s) > r_B(b) + r_S(s)$*. Then,* $x \notin \mathrm{Supp}(\beta_{\#}\pi)$ *and* $y \notin \mathrm{Supp}(\sigma_{\#}\pi)$*.*

1.5.2 Preparation: the basic module for examples

(CMP) constructed their examples in a piecewise manner from two different supermodular surplus functions. This approach can fruitfully be pushed further. I will systematically use a certain one-dimensional basic module that satisfies Assumption 1.1 to construct examples (which, apart from those in Section 1.5.5, do not satisfy Assumption 1.1). For the basic module, let $0 < \alpha < 2$, $\gamma > 0$, $f(z) = \gamma z^\alpha$ for $z \in \mathbb{R}_+$, and set $v(x, y) := f(xy)$ for $x, y \in \mathbb{R}_+$. This defines a strictly supermodular gross match surplus. Costs are given by $c(x, b) = x^4/b^2$ and $d(y, s) = y^4/s^2$ for $b, s \in \mathbb{R}_{++}$. Symmetry is just a trick to keep the analysis reasonably tractable. None of the effects that I illustrate in the following sections depends on symmetry assumptions.

For all (b, s), there is a trivial NE of the FA game, namely $(x, y) = (0, 0)$. However, this stationary point, which is not even a *local* maximizer of $h(x, y|b, s)$, should be viewed as the only unpleasant feature of an otherwise very convenient example.

Therefore, throughout Section 1.5, I will focus on non-trivial *equilibria, in which agents who get a (non-dummy) match partner do not make zero investments. Hence, equilibria that arise only because of the pathological stationary point of the basic module will be ignored!*[32]

Non-trivial NE attributes in the basic module are unique for all (b, s) and coincide with the jointly optimal investments:

$$\begin{cases} x^*(b, s) \;=\; \left(\frac{\gamma\alpha}{4}\right)^{\frac{1}{4-2\alpha}} b^{\frac{4-\alpha}{8-4\alpha}} s^{\frac{\alpha}{8-4\alpha}} \\ y^*(b, s) \;=\; \left(\frac{\gamma\alpha}{4}\right)^{\frac{1}{4-2\alpha}} s^{\frac{4-\alpha}{8-4\alpha}} b^{\frac{\alpha}{8-4\alpha}}. \end{cases} \tag{1.3}$$

Net match surplus is

$$w(b, s) \;=\; \kappa(\alpha, \gamma)(bs)^{\frac{\alpha}{2-\alpha}}, \tag{1.4}$$

[32]Eliminating the trivial NE explicitly by modifying the functions is possible but this would make the subsequent analysis a lot messier, which is not worth the effort.

where

$$\kappa(\alpha,\gamma) \;=\; \gamma^{\frac{2}{2-\alpha}} \left(\frac{\alpha}{4}\right)^{\frac{\alpha}{2-\alpha}} \left(1-\frac{\alpha}{2}\right). \tag{1.5}$$

The maximal surplus net of own costs that seller s can attain when he matches with an investment that buyer b makes for seller s' is

$$\max_{y\in Y}\left(\gamma(x^*(b,s')y)^\alpha - \frac{y^4}{s^2}\right) \;=\; b^{\frac{\alpha}{2-\alpha}} s^{\frac{2\alpha}{4-\alpha}} (s')^{\frac{\alpha^2}{(4-\alpha)(2-\alpha)}} \gamma^{\frac{2}{2-\alpha}} \left(\frac{\alpha}{4}\right)^{\frac{\alpha}{2-\alpha}} \left(1-\frac{\alpha}{4}\right). \tag{1.6}$$

If $x^*(b,s')$ is an equilibrium investment of buyer b then, the net payoff for seller s from the above investment and match, after leaving the equilibrium market gross payoff to buyer b, is

$$\max_{y\in Y}\left(v(x^*(b,s'),y) - d(y,s)\right) - c(x^*(b,s'),b) - r_B(b). \tag{1.7}$$

Symmetric formulae apply for buyers.

1.5.3 A multi-dimensional model without technological multiplicity

In the model of this section, buyers and sellers have multi-dimensional cost types, and they can invest in multi-dimensional attributes. The technology does not feature multiplicity, so that inefficiency of joint investments is impossible: non-trivial NE of FA games are unique. The goal is to shed light on the extent to which mismatch may occur in this situation. Is it possible that agents invest for the wrong partners in equilibrium? Or do the resulting attribute markets necessarily violate Definition 1.5 (or, alternatively, Lemma 1.2), so that only the ex-ante efficient equilibrium survives? The tentative lesson from the analysis below is the following: mismatch is possible in principle and may sometimes occur, but with heterogeneous, differentiated ex-ante populations, the attribute markets that result from mismatch inefficient situations often necessarily cause deviations, so that matching and equilibrium are forced to be ex-ante efficient.

Here is the model: the supports of non-dummy buyer types $b=(b_1,b_2)$ and seller types $s=(s_1,s_2)$ are contained in $\mathbb{R}^2_+ - \{0\}$, and agents can invest in attributes $x=(x_1,x_2)\in\mathbb{R}^2_+$ and $y=(y_1,y_2)\in\mathbb{R}^2_+$. Match surplus is bilinear: $v(x,y)=x_1y_1+x_2y_2$. This form has been used in many papers on screening and mechanism design, and with regard to optimal transport, it corresponds to the classical case of quadratic transportation cost.

Costs are $c(x,b) = \frac{x_1^4}{b_1^2} + \frac{x_2^4}{b_2^2}$ and $d(y,s) = \frac{y_1^4}{s_1^2} + \frac{y_2^4}{s_2^2}$. Hence, surplus and costs are additively separable in the two dimensions, and the specification corresponds to setting $\gamma = \alpha = 1$ in the basic module of Section 1.5.2, for both dimensions. The model is constructed such that net surplus is also bilinear. It holds $w(b,s) = \frac{1}{8}(b_1 s_1 + b_2 s_2)$, according to (1.4) and (1.5). From (1.6) it follows that $\max_{y \in Y}(v(x^*(b,s'),y) - d(y,s)) = \sum_{i=1}^{2} \frac{3}{16} s_i^{\frac{2}{3}} (s_i')^{\frac{1}{3}} b_i$. Moreover, $c(x^*(b,s),b) = d(y^*(b,s),s) = \frac{1}{16}(b_1 s_1 + b_2 s_2)$.

I allow that one type parameter of an agent is equal to zero, meaning that any strictly positive investment in the corresponding dimension is infinitely costly (and that, consequently, the agent makes zero investments in that dimension). This assumption is not fully in line with the general model of Section 1.2.1 but it does not cause any problems. It merely serves to avoid unnecessary ε-arguments in Sections 1.5.3.1 and 1.5.3.2. The examples of Sections 1.5.3.1 and 1.5.3.2 are still close to the one-dimensional supermodular framework. The analysis in Section 1.5.3.3 is substantially more involved and uses Proposition 1.1.

1.5.3.1 An example of mismatch inefficiency

In this example, sellers (workers) are generalists who can invest in both dimensions. There are only two possible types, high and low, which are completely ordered. Moreover, sellers are on the long side of the market. Formally, $\nu = a_H \delta_{(s_H,s_H)} + (1-a_H)\delta_{(s_L,s_L)}$, where $0 < s_L < s_H$ and $0 < a_H < 1$. It is easy to derive the unique ex-ante optimal coupling then, irrespectively of the distribution of buyer types. Indeed, $w((b_1,b_2),(s_1,s_1)) = \frac{1}{8}(b_1+b_2)s_1$, so that s_1 and b_1+b_2 are sufficient statistics to determine net match surplus. Since w is strictly supermodular in these sufficient statistics, the unique ex-ante optimal coupling is positively assortative with respect to them.

There are two types/sectors of specialized buyers (employers): with a slight abuse of notation (namely, (b_1,b_2) does not denote a generic buyer type here), let $\mu = a_1\delta_{(b_1,0)} + a_2\delta_{(0,b_2)} + (1-a_1-a_2)\delta_{b_\varnothing}$, where $0 < a_1, a_2, b_1, b_2$ and $a_1 + a_2 < 1$. To make the problem interesting, sector 1 is more productive ex-ante, i.e. $b_1 > b_2$, and not all buyers can get high type sellers, i.e. $a_H < a_1 + a_2$. The ex-ante efficient equilibrium constructed in Theorem 1.1 always exists. What about other, mismatch inefficient equilibria?

Case $\mathbf{a_H > a_2}$: there is exactly one additional, mismatch inefficient equilibrium if and only if

$$\frac{b_2}{b_1} \geq 1 - \frac{s_L}{s_H} \quad \text{and} \quad \frac{3}{2}\frac{b_2}{b_1} \geq \frac{1 - \frac{s_L}{s_H}}{1 - (\frac{s_L}{s_H})^{\frac{2}{3}}}.$$

Otherwise, only the ex-ante efficient equilibrium exists.

In the mismatch inefficient equilibrium, all $(0, b_2)$-buyers match with (s_H, s_H)-sellers, and sector 1 attracts both high and low type sellers. Both conditions put lower bounds on the ratio $\frac{b_2}{b_1}$. The first one is a participation constraint for $(0, b_2)$-buyers. Given the payoff they have to leave to (s_H, s_H)-sellers, it must be weakly profitable for them to invest and enter the market. The second condition makes sure that low-type sellers do not want to deviate and match with $x^*((0, b_2), (s_H, s_H))$-attributes, given the payoff that must be left to buyers from sector 2. The first condition is the more stringent one for small values of $\frac{s_L}{s_H}$ (in which case (s_H, s_H)-sellers capture a lot of surplus), while it is the other way round for $\frac{s_L}{s_H}$ close to one.

Case $\mathbf{a_H} < \mathbf{a_2}$: there is exactly one additional, mismatch inefficient equilibrium if and only if

$$\frac{2}{3}\frac{b_2}{b_1} \geq \frac{\left(\frac{s_H}{s_L}\right)^{\frac{2}{3}} - 1}{\frac{s_H}{s_L} - 1}.$$

Otherwise, only the ex-ante efficient equilibrium exists.

In the mismatch inefficient equilibrium, all (s_H, s_H)-sellers are "depleted" by $(0, b_2)$-buyers. The lower bound on $\frac{b_2}{b_1}$ makes sure that (s_H, s_H)-sellers do not wish to deviate and match with $x^*((b_1, 0), (s_L, s_L))$-attributes. It is most stringent if $\frac{s_H}{s_L}$ is close to 1, in which case the investments made by the more productive sector of buyers are very suitable also for (s_H, s_H)-sellers.

1.5.3.2 The limits of mismatch: a simple example

The example of this section is a variation of the previous one. The population of generalist sellers is more differentiated now: ν is concentrated on $\{(s_1, s_1) | s_L \leq s_1 \leq s_H\}$, where $s_L < s_H$. For simplicity, I assume that ν has a bounded density, uniformly bounded away from zero, with respect to Lebesgue measure on that interval.

Buyers again belong to one of two specialized sectors. Formally, μ is compactly supported in $\mathbb{R}_{++} \times \{0\} \cup \{0\} \times \mathbb{R}_{++} \cup \{b_\varnothing\}$. I assume that μ restricted to $\mathbb{R}_{++} \times \{0\}$ and $\{0\} \times \mathbb{R}_{++}$ also has interval support, with a density as above. In particular, both sectors are heterogeneous and it need not be the case that one sector is uniformly more productive than the other one. It turns out that the only non-trivial ex-post contracting equilibrium is the ex-ante optimal one. The diversity of seller types and their ability to deviate to the other sector suffices to rule out any mismatch inefficient situation.

1.5.3.3 The limits of mismatch continued: a fully multi-dimensional case

The ex-ante populations in this example are of a form for which the regularity theory for optimal transport (for bilinear surplus, recall that $w(b,s) = \frac{1}{8}(b_1 s_1 + b_2 s_2)$) makes sure that the unique ex-ante optimal matching is deterministic and smooth.

Assumption 1.2. $\mathrm{Supp}(\mu), \mathrm{Supp}(\nu) \subset \mathbb{R}^2_{++}$ *are closures of bounded, open and uniformly convex sets with smooth boundaries. Moreover, both measures are absolutely continuous with respect to Lebesgue measure, with smooth densities bounded from above and below on the supports.*

Theorem 12.50 and Theorem 10.28 of Villani (2009) yield:

Proposition 1.2. *Under the conditions of Assumption 1.2, the ex-ante assignment game has a, up to an additive constant* unique, *(w-)* convex *dual solution ψ^* which is* smooth. *Moreover, the unique ex-ante optimal coupling π^* is given by a smooth bijection $T^* : \mathrm{Supp}(\mu) \to \mathrm{Supp}(\nu)$ satisfying $\frac{1}{8}T^*(b) = \nabla\psi^*(b)$.*

I will show that under very mild additional assumptions on the supports, *any* smooth and deterministic matching/coupling of buyers and sellers that is compatible with an arbitrary ex-post contracting equilibrium is necessarily ex-ante optimal. In this sense, the differentiated ex-ante populations and the rich endogenous attribute markets leave no room for mismatch!

The idea of proof is as follows: I first note that whenever a coupling of buyers and sellers that is induced by an ex-post contracting equilibrium is locally given by a smooth map T, then T corresponds to the gradient of the equilibrium buyer net payoff function, $\nabla r_B(b) = \frac{1}{8}T(b)$. Then, I use a local version of the fact that both buyers and sellers must not have incentives to change investments and match with other equilibrium attributes from the other side. This yields bounds both on $DT(b) = 8\,\mathrm{Hess}\, r_B(b)$ (the Hessian) and on the inverse of this matrix. Taken together, these bounds force $\mathrm{Hess}\, r_B$ to be positive semi-definite under mild assumptions on the type supports. It then follows that r_B is convex, so that the coupling associated with T is concentrated on the subdifferential of a convex function. This is a w-cyclically monotone set, and hence T is ex-ante optimal by Proposition 1.1.

Theorem 1.2. *Let Assumption 1.2 hold, and assume that* $\left(\frac{s_1}{b_1}\frac{b_2}{s_2} + \frac{s_2}{b_2}\frac{b_1}{s_1}\right) < 32$ *for all* $b \in \mathrm{Supp}(\mu)$, $s \in \mathrm{Supp}(\nu)$. *Let* $T : \mathrm{Supp}(\mu) \to \mathrm{Supp}(\nu)$ *be a smooth one-to-one onto deterministic coupling of buyers and sellers that is compatible with an ex-post contracting equilibrium. Then* T *is ex-ante efficient.*

The rest of this section contains the sequence of preliminary results leading to Theorem 1.2. Throughout, η will denote a *direction*: $\eta \in \mathbb{R}^2$ and $|\eta| = 1$. Moreover, T always stands for a deterministic, one-to-one onto, coupling of buyers and sellers that is compatible with an ex-post contracting equilibrium. Finally, $\cdot$ denotes the standard inner product on $\mathbb{R}^2$.

Lemma 1.3. *Let* T *be smooth in a neighborhood of* $b \in \mathrm{Supp}(\mu)$. *Then it holds for all admissible directions* η*:*

$$\frac{1}{8}T(b) \cdot \eta = \lim_{t \to 0} \frac{r_B(b + t\eta) - r_B(b)}{t}.$$

Corollary 1.3. *Let* T *be smooth on an open set* $U \subset \mathrm{Supp}(\mu)$. *Then* r_B *is smooth on* U *and satisfies* $\nabla r_B(b) = \frac{1}{8}T(b)$ *for all* $b \in U$.

Proposition 1.3. *Let* T *be smooth on an open set* $U \subset \mathrm{Supp}(\mu)$ *and* $b \in U$. *Then, for* $T(b)$ *and the symmetric, non-singular linear map* $DT(b) = 8\,\mathrm{Hess}\, r_B(b)$ *it holds: both* $3DT(b) + \begin{pmatrix} \frac{T(b)_1}{b_1} & 0 \\ 0 & \frac{T(b)_2}{b_2} \end{pmatrix}$ *and* $3DT(b)^{-1} + \begin{pmatrix} \frac{b_1}{T(b)_1} & 0 \\ 0 & \frac{b_2}{T(b)_2} \end{pmatrix}$ *are positive semi-definite.*

Proposition 1.4. *Let* T *be smooth on an open set* $U \subset \mathrm{Supp}(\mu)$. *For* $b \in U$, *if* $\left(\frac{T(b)_1}{b_1}\frac{b_2}{T(b)_2} + \frac{T(b)_2}{b_2}\frac{b_1}{T(b)_1}\right) < 32$, *then* $DT(b) = 8\,\mathrm{Hess}\, r_B(b)$ *is positive semi-definite.*

1.5.4 Technological multiplicity and severe coordination failures

The under- and over-investment examples of (CMP) show that severe coordination failures may be possible when technological multiplicity is an issue for the given populations. The simple example of this section illustrates how this problem is aggravated outside of the one-dimensional supermodular framework. Coordination failures then typically involve mismatch, which makes it less likely that the endogenous markets contain attributes that destroy the inefficient equilibrium. In particular, whether coordination failures are ruled out or not depends in a complex way on the full ex-ante populations (not just on supports as in (CMP)), and there may well be mismatch inefficient equilibria, potentially without any inefficiency of joint investments, even for rich, differentiated populations.

The example combines the one of Section 1.5.3.2 with an under-investment example à la (CMP). Population measures are as in Section 1.5.3.2, with support $\{(s_1, s_1)|s_L \leq s_1 \leq s_H\}$ $(s_L < s_H)$ for ν, $\{(0, b_2)|b_{2,L} \leq b_2 \leq b_{2,H}\}$ $(b_{2,L} < b_{2,H})$ for the sector 2 population of buyers, and $\{(b_1, 0)|b_{1,L} \leq b_1 \leq b_{1,H}\}$ $(b_{1,L} < b_{1,H})$ for the sector 1 population of buyers. Moreover, the technology for sector 1 remains unchanged. In sector 2 however, match surplus has an additional regime of complementarity for high attributes: $v(x, y) = x_1y_1 + \max(f_1, f_2)(x_2y_2)$, where $f_1(z) = z$ and $f_2(z) = \frac{1}{2}z^{\frac{3}{2}}$. Lemma 1.9 in Section 1.6 shows that the match surplus for sector 2 is strictly supermodular.[33] If the technology for sector 2 were globally given by f_1, then the unique non-trivial NE of the FA game between $(0, b_2)$ and (s_1, s_1) would be $(x, y) = \left(\left(0, \frac{1}{2}b_2^{\frac{3}{4}}s_1^{\frac{1}{4}}\right), \left(0, \frac{1}{2}b_2^{\frac{1}{4}}s_1^{\frac{3}{4}}\right)\right)$, yielding net surplus $\frac{1}{8}b_2s_1$. The corresponding expressions for f_2 are $(x, y) = \left(\left(0, \frac{3}{16}b_2^{\frac{5}{4}}s_1^{\frac{3}{4}}\right), \left(0, \frac{3}{16}b_2^{\frac{3}{4}}s_1^{\frac{5}{4}}\right)\right)$ and $\kappa\left(\frac{3}{2}, \frac{1}{2}\right)(b_2s_1)^3 = \frac{3^3}{2^{15}}(b_2s_1)^3$. Hence, couples with $b_2s_1 < \frac{2^6}{3^{\frac{3}{2}}} =: \tau$ are better off with the f_1-technology, and couples with $b_2s_1 > \tau$ are better off with the f_2-technology. The true technology for sector 2 is defined via f_1 for $x_2y_2 < z_{12} = 4$ and via f_2 for $x_2y_2 > 4$. Still, the identified attributes are the jointly efficient choices for all b_2 and s_1, since $x_2y_2 = \frac{1}{4}b_2s_1$ and $x_2y_2 = \frac{3^2}{2^8}(b_2s_1)^2$ evaluated at the indifference couples $b_2s_1 = \tau$ are equal to $\frac{2^4}{3^{\frac{3}{2}}} < 4$ and $\frac{2^4}{3} > 4$ respectively.

Consider now a situation in which ex-ante efficiency requires that high cost investments are made in sector 2. This is the case iff $(0, b_{2,H})$ is matched to a (s_1^*, s_1^*) satisfying $b_{2,H}s_1^* > \tau$ in the ex-ante efficient equilibrium. As a side remark, note that since there is no 1-d sufficient statistic in which net surplus is supermodular (in contrast to Section 1.5.3.2), the problem of finding the ex-ante optimal coupling is actually non-local and difficult, with potentially very complicated solutions.[34] However, for the present purposes,

[33] As (CMP) noted, the piecewise construction matters only for analytical convenience. One could smooth out kinks without affecting any results.

[34] I briefly illustrate this difficulty by spelling out the 2-cycle condition in the Appendix.

it is not necessary to solve the ex-ante assignment problem explicitly.

If all sector 2 couples invest according to the low technology regime (which is inefficient by assumption), then the analysis of Section 1.5.3.2 implies that $(0, b_{2,H})$ is matched to the seller $(s_{1,q}, s_{1,q})$ who satisfies $\nu(\{(s_1, s_1)|s_1 \geq s_{1,q}\}) = q$, for $q = \mu(\{b|b_1 + b_2 \geq b_{2,H}\})$. This inefficient situation (in which efficient investment opportunities in sector 2 are missed, and some high type sellers invest for sector 1 while they should invest for sector 2) is ruled out if and only if the low regime investments made by $(0, b_{2,H})$ and $(s_{1,q}, s_{1,q})$ are sufficiently high to trigger an upward deviation by at least one of the two parties. This in turn depends crucially on q, and hence on sector 1 of the buyer population. Note finally that if the inefficient equilibrium survives, it exhibits inefficiency of joint investments if $b_{2,H}s_{1,q} > \tau$, while all agents make jointly efficient investments if $b_{2,H}s_{1,q} \leq \tau$.

1.5.5 Simultaneous under- and over-investment: the case of missing middle sectors

In this section, I show that, in contrast to the examples of (CMP) and to the examples given so far in this chapter, even extreme ex-ante heterogeneity may be insufficient to rule out inefficient equilibria. To this end, I return to the 1-d supermodular framework to construct examples with simultaneous under- and over-investment. Combined with the results of (CMP), the current section yields a comprehensive picture of the most interesting coordination failures in the 1-d supermodular framework. In the examples, "lower middle" types under-invest and bunch with low types who invest efficiently, while "upper middle" types over-invest and bunch with high types who invest efficiently. In particular, *the attribute economy lacks an efficient middle sector*. As in (CMP), there is no bunching in a literal sense since attributes are fully differentiated. It should rather be understood as bunching in the same connected component of the attribute economy.

I employ the construction of Lemma 1.9, as well as the notation introduced there. v has three different regimes of complementarity, i.e. $K = 3$, $z_{12} < z_{23}$ and $v(x, y) = (\max_{i=1,\dots,3} f_i)(xy)$. Population measures are absolutely continuous with respect to Lebesgue measure, have interval support, and - for simplicity - they are symmetric, i.e. $\mu = \nu$. I denote the interval support by $I \subset \mathbb{R}_{++}$. By Corollary 1.4 in Section 1.6, b is matched to $s = b$ in any ex-post contracting equilibrium.

If surplus were globally given by $f_i(xy)$ ($i \in \{1, 2, 3\}$) rather than v, then the non-trivial

NE of the FA game for (b, b) would be unique and given by

$$x_i^*(b,b) = y_i^*(b,b) = \left(\frac{\gamma_i \alpha_i}{4}\right)^{\frac{1}{4-2\alpha_i}} b^{\frac{1}{2-\alpha_i}}. \tag{1.8}$$

The net surplus that the couple (b, b) would generate according to f_i is $w_i(b) = \kappa_i b^{\frac{2\alpha_i}{2-\alpha_i}}$, where $\kappa_i = \kappa(\alpha_i, \gamma_i)$. For $i < j$, w_j crosses w_i exactly once in $\mathbb{R}_{++}$ and this crossing is from below (like for the functions f_i, f_j). The type at which this crossing occurs is

$$b_{ij} = \left(\frac{\kappa_i}{\kappa_j}\right)^{\frac{1}{2\alpha_j/(2-\alpha_j) - 2\alpha_i/(2-\alpha_i)}} = \left(\frac{\kappa_i}{\kappa_j}\right)^{\frac{(2-\alpha_i)(2-\alpha_j)}{4(\alpha_j-\alpha_i)}}.$$

I write x_{iij} for the attributes that the indifference types b_{ij} would use under f_i, and x_{jij} for those attributes they would use under f_j. These are given by

$$x_{iij} = z_{ij}^{\frac{1}{2}} \left(\frac{\alpha_i}{4}\right)^{\frac{1}{4-2\alpha_i}} \left(\frac{\left(\frac{\alpha_i}{4}\right)^{\frac{\alpha_i}{2-\alpha_i}}(2-\alpha_i)}{\left(\frac{\alpha_j}{4}\right)^{\frac{\alpha_j}{2-\alpha_j}}(2-\alpha_j)}\right)^{\frac{2-\alpha_j}{4(\alpha_j-\alpha_i)}},$$

and

$$x_{jij} = z_{ij}^{\frac{1}{2}} \left(\frac{\alpha_j}{4}\right)^{\frac{1}{4-2\alpha_j}} \left(\frac{\left(\frac{\alpha_i}{4}\right)^{\frac{\alpha_i}{2-\alpha_i}}(2-\alpha_i)}{\left(\frac{\alpha_j}{4}\right)^{\frac{\alpha_j}{2-\alpha_j}}(2-\alpha_j)}\right)^{\frac{2-\alpha_i}{4(\alpha_j-\alpha_i)}}.$$

Thus, x_{iij} and x_{jij} depend on γ_i, γ_j only through γ_i/γ_j, and moreover, x_{jij}/x_{iij} depends only on α_i and α_j. It follows that

$$\frac{x_{jij}}{x_{iij}} = \left(\frac{\alpha_j}{4}\right)^{\frac{1}{4-2\alpha_j}} \left(\frac{\alpha_i}{4}\right)^{-\frac{1}{4-2\alpha_i}} \left(\frac{\left(\frac{\alpha_i}{4}\right)^{\frac{\alpha_i}{2-\alpha_i}}(2-\alpha_i)}{\left(\frac{\alpha_j}{4}\right)^{\frac{\alpha_j}{2-\alpha_j}}(2-\alpha_j)}\right)^{\frac{1}{4}} = \left(\frac{\alpha_j(2-\alpha_i)}{\alpha_i(2-\alpha_j)}\right)^{\frac{1}{4}}.$$

This ratio is greater than one (since $0 < \alpha_i < \alpha_j < 2$), so that there is an upward jump in attribute choice where the indifferent types would like to switch from the f_i to the f_j surplus function.

If parameters are such that $b_{12} < b_{23}$, then f_1 would be the best surplus function for $b < b_{12}$, f_2 would be best for $b_{12} < b < b_{23}$, and f_3 would be best for $b_{23} < b$.

However, the true gross surplus function is v with its three different regimes. The above comparison of net surplus from optimal choices for globally valid f_1, f_2 and f_3 is sufficient to find the ex-ante efficient ex-post contracting equilibrium if and only if the "jump attributes" actually lie in the valid regimes. Formally, this requires

$$x_{112}^2 < z_{12} < x_{212}^2 < x_{223}^2 < z_{23} < x_{323}^2. \tag{1.9}$$

If $b_{12} < b_{13} < b_{23}$, it is clear from (1.8) that $x_{112} < x_{113}$, $x_{212} < x_{223}$ and $x_{313} < x_{323}$. I show next that the following two conditions may simultaneously be satisfied:

a) (1.9) holds

b) the jump from x_{113} to x_{313} (which is not part of the efficient equilibrium!) is also between valid regimes, that is $x_{113}^2 < z_{12}$ and $z_{23} < x_{313}^2$.

Indeed, as an example, let $\alpha_1 = 0.1$, $\alpha_2 = 0.6$, $\alpha_3 = 1.6$, $\gamma_1 = 1$, $\gamma_2 = 1.5$ and $\gamma_3 = 1$. Then $z_{12} = 4/9$, $z_{23} = 3/2$, $b_{12} \approx 1.5823$, $b_{13} \approx 1.8908$, $b_{23} \approx 1.9266$ and jump attributes for all three possible jumps lie in the valid regimes: $x_{112}^2 \approx 0.2326$, $x_{113}^2 \approx 0.2806$, $x_{212}^2 \approx 0.6637$, $x_{223}^2 \approx 0.8793$, $x_{313}^2 \approx 2.4459$ and $x_{323}^2 \approx 2.6863$.

In the Appendix, I show that for the above parameters and any symmetric populations with interval support I and $b_{13} \in I$, the inefficient outcome in which types $b < b_{13}$ make investments $\beta(b) = \sigma(b) = \left(\frac{\gamma_1\alpha_1 b^2}{4}\right)^{\frac{1}{4-2\alpha_1}}$ and types $b > b_{13}$ make investments $\beta(b) = \sigma(b) = \left(\frac{\gamma_3\alpha_3 b^2}{4}\right)^{\frac{1}{4-2\alpha_3}}$ can be sustained as an ex-post contracting equilibrium with symmetric payoffs $\widetilde{\psi}(x) = \widetilde{\phi}(x) = v(x,x)/2$ on $\mathrm{cl}(\beta(I)) = \mathrm{cl}(\sigma(I))$ ($\mathrm{cl}(\cdot)$ denotes the closure of a set).

1.6 Appendix for Chapter 1

The next lemma shows that the sets of existing buyer and seller types, $\mathrm{Supp}(\mu)$ and $\mathrm{Supp}(\nu)$, may equivalently be described by $\mathrm{Supp}(\pi)$ for any $\pi \in \Pi(\mu,\nu)$.

Lemma 1.4. *Let $P_B(b,s) = b$ and $P_S(b,s) = s$ be the coordinate projections. For any $\pi \in \Pi(\mu,\nu)$ it holds $\mathrm{Supp}(\mu) = \mathrm{P}_B(\mathrm{Supp}(\pi))$ and $\mathrm{Supp}(\nu) = \mathrm{P}_S(\mathrm{Supp}(\pi))$.*

Proof of Lemma 1.4. I prove the claim for μ only and first show $\mathrm{P}_B(\mathrm{Supp}(\pi)) \subset \mathrm{Supp}(\mu)$. So take any $(b,s) \in \mathrm{Supp}(\pi)$. Then, for any open neighborhood U of b, $\pi(U \times S) > 0$ and hence $\mu(U) > 0$. Thus, $b \in \mathrm{Supp}(\mu)$.

I next prove the slightly less trivial inclusion $\mathrm{Supp}(\mu) \subset \mathrm{P}_B(\mathrm{Supp}(\pi))$. Assume to the contrary that there is some $b \in \mathrm{Supp}(\mu)$ that is not contained in $\mathrm{P}_B(\mathrm{Supp}(\pi))$. The latter assumption implies that for all $s \in S$ there are open neighborhoods $U_s \subset B$ of b and $V_s \subset S$ of s such that $\pi(U_s \times V_s) = 0$. Since S is compact, the open cover $\{V_s\}_{s\in S}$ of S contains a finite subcover $\{V_{s_1}, ..., V_{s_k}\}$. Moreover, $U := \bigcap_{i=1}^k U_{s_i}$ is an open neighborhood of b and $U \times S \subset \bigcup_{i=1}^k U_{s_i} \times V_{s_i}$. This leads to the contradiction $0 < \mu(U) = \pi(U \times S) \leq \pi(\bigcup_{i=1}^k U_{s_i} \times V_{s_i}) = 0$. $\square$

Lemma 1.5. *Let (β, σ, π) be a regular investment profile. Then $\beta(\mathrm{Supp}(\pi))$ is dense in $\mathrm{Supp}(\beta_{\#}\pi)$, $\sigma(\mathrm{Supp}(\pi))$ is dense in $\mathrm{Supp}(\sigma_{\#}\pi)$, and $(\beta, \sigma)(\mathrm{Supp}(\pi))$ is dense in $\mathrm{Supp}((\beta, \sigma)_{\#}\pi)$.*

Proof of Lemma 1.5. I prove the claim for $\beta(\mathrm{Supp}(\pi))$. Assume to the contrary that there is a point $x \in \mathrm{Supp}(\beta_{\#}\pi)$ and an open neighborhood U of x such that $U \cap \beta(\mathrm{Supp}(\pi)) = \emptyset$. Then $\beta_{\#}\pi(U) > 0$ (by definition of the support) and on the other hand $\beta_{\#}\pi(U^c) \geq \pi(\mathrm{Supp}(\pi)) = 1$. Contradiction. $\square$

Lemma 1.6. *Let (β, σ, π) be a regular investment profile. Let $\widetilde{\psi} : \beta(\mathrm{Supp}(\pi)) \to \mathbb{R}$ be v-convex with respect to the (not necessarily closed) sets $\beta(\mathrm{Supp}(\pi))$ and $\sigma(\mathrm{Supp}(\pi))$, let $\widetilde{\psi}^v$ be its v-transform, and let $\widetilde{\pi} \in \Pi(\beta_{\#}\pi, \sigma_{\#}\pi)$ be such that $\widetilde{\psi}^v(y) + \widetilde{\psi}(x) = v(x, y)$ on a dense subset of $\mathrm{Supp}(\widetilde{\pi})$. Then there is a unique extension of $(\widetilde{\psi}, \widetilde{\psi}^v)$ to a v-dual pair with respect to the compact metric spaces $\mathrm{Supp}(\beta_{\#}\pi)$ and $\mathrm{Supp}(\sigma_{\#}\pi)$, and with this extension $(\widetilde{\pi}, \widetilde{\psi})$ becomes a stable and feasible bargaining outcome in the sense of Definition 1.3.*

Proof of Lemma 1.6. First define for all $y \in \mathrm{Supp}(\sigma_{\#}\pi)$,

$$\widetilde{\phi}_0(y) := \sup_{x \in \beta(\mathrm{Supp}(\pi))} (v(x, y) - \widetilde{\psi}(x)).$$

By definition, $\widetilde{\phi}_0$ coincides with $\widetilde{\psi}^v$ on the set $\sigma(\mathrm{Supp}(\pi)) \subset \mathrm{Supp}(\sigma_{\#}\pi)$, which is a dense subset by Lemma 1.5. Next, set for all $x \in \mathrm{Supp}(\beta_{\#}\pi)$,

$$\widetilde{\psi}_1(x) := \sup_{y \in \mathrm{Supp}(\sigma_{\#}\pi)} (v(x, y) - \widetilde{\phi}_0(y)),$$

and finally for all $y \in \mathrm{Supp}(\sigma_{\#}\pi)$,

$$\widetilde{\phi}_1(y) := \sup_{x \in \mathrm{Supp}(\beta_{\#}\pi)} (v(x, y) - \widetilde{\psi}_1(x)).$$

By definition, $\widetilde{\psi}_1$ is a v-convex function with respect to the compact metric spaces $\mathrm{Supp}(\beta_\# \pi)$ and $\mathrm{Supp}(\sigma_\# \pi)$, and $\widetilde{\phi}_1$ is its v-transform. Observe that $\widetilde{\psi}_1$ coincides with $\widetilde{\psi}$ on $\beta(\mathrm{Supp}(\pi))$ and that $\widetilde{\phi}_1$ coincides with $\widetilde{\psi}^v$ on $\sigma(\mathrm{Supp}(\pi))$. Indeed, for any $x = \beta(b, s)$ with $(b, s) \in \mathrm{Supp}(\pi)$, the set of real numbers used to define the supremum $\widetilde{\psi}(x)$ is contained in the one used to define $\widetilde{\psi}_1(x)$. Assume then for the sake of deriving a contradiction that $\widetilde{\psi}_1(\beta(b, s)) > \widetilde{\psi}(\beta(b, s))$. Then there must be some $y \in \mathrm{Supp}(\sigma_\# \pi)$, such that $v(\beta(b, s), y) > \widetilde{\psi}(\beta(b, s)) + \widetilde{\phi}_0(y)$ and hence in particular $v(\beta(b, s), y) > \widetilde{\psi}(\beta(b, s)) + v(\beta(b, s), y) - \widetilde{\psi}(\beta(b, s))$, which yields a contradiction. A completely analogous argument shows that $\widetilde{\phi}_1(\sigma(b, s)) = \widetilde{\psi}^v(\sigma(b, s))$ for all $(b, s) \in \mathrm{Supp}(\pi)$. So $\widetilde{\psi}(x) := \widetilde{\psi}_1(x)$ and $\widetilde{\psi}^v(y) := \widetilde{\phi}_1(y)$ are well-defined (and unique) extensions to a v-dual pair with respect to $\mathrm{Supp}(\beta_\# \pi)$ and $\mathrm{Supp}(\sigma_\# \pi)$.

Since, for the extended $\widetilde{\psi}$, $\partial_v \widetilde{\psi}$ is closed, it follows that $\mathrm{Supp}(\widetilde{\pi}) \subset \partial_v \widetilde{\psi}$. Hence $(\widetilde{\pi}, \widetilde{\psi})$ is a stable and feasible bargaining outcome. □

Proof of Theorem 1.1. By assumption, it holds for all $b \in B, s \in S$ that

$$v(\beta^*(b, s), \sigma^*(b, s)) - c(\beta^*(b, s), b) - d(\sigma^*(b, s), s) \;=\; w(b, s),$$

and moreover $(\beta^*, \sigma^*, \pi^*)$ is a regular investment profile. The measure $\widetilde{\pi}^* := (\beta^*, \sigma^*)_\# \pi^*$ couples $\beta^*_\# \pi^*$ and $\sigma^*_\# \pi^*$, and it is intuitively quite clear that this must be an optimal coupling. Indeed, from a social planner's point of view, and modulo technical details, the problem of finding an ex-ante optimal coupling with corresponding mutually optimal investments is equivalent to a two-stage optimization problem where he must first decide on investments for all agents and then match the two resulting populations optimally.

I next construct the v-convex buyer payoff function $\widetilde{\psi}^*$ that will be the other part of the stable and feasible bargaining outcome in the ex-post contracting equilibrium. Optimality of $\widetilde{\pi}^*$ will be proven along the way.[35] For any x for which there is some $(b, s) \in \mathrm{Supp}(\pi^*)$ such that $x = \beta^*(b, s)$, set

$$\widetilde{\psi}^*(x) := \psi^*(b) + c(x, b).$$

[35] It should be kept in mind that there may be other stable and feasible bargaining outcomes for $(\beta^*_\# \pi^*, \sigma^*_\# \pi^*, v)$. These other outcomes are incompatible with (two-stage) ex-post contracting equilibrium however.

This is well-defined. Indeed, take any other $(b', s') \in \mathrm{Supp}(\pi^*)$ with $x = \beta^*(b', s')$. Since ψ^* is a w-convex dual solution, it holds $\mathrm{Supp}(\pi^*) \subset \partial_w \psi^*$. Thus $v(x, \sigma^*(b,s)) - c(x,b) - d(\sigma^*(b,s), s) = w(b,s) = (\psi^*)^w(s) + \psi^*(b)$. Moreover, $v(x, \sigma^*(b,s)) - c(x,b') - d(\sigma^*(b,s), s) \leq w(b', s) \leq (\psi^*)^w(s) + \psi^*(b')$, where the first inequality follows from the definition of w and the second follows from (1.2). This implies $c(x,b) - c(x,b') \leq \psi^*(b') - \psi^*(b)$, and hence $\psi^*(b) + c(x,b) \leq \psi^*(b') + c(x,b')$. Reversing roles in the above argument shows that $\widetilde{\psi}^*(x)$ is well-defined.

Similarly, for any y for which there is some $(b,s) \in \mathrm{Supp}(\pi^*)$ such that $y = \sigma^*(b,s)$,

$$\widetilde{\phi}^*(y) := (\psi^*)^w(s) + d(y,s)$$

is well-defined. $\widetilde{\psi}^*(x)$ and $\widetilde{\phi}^*(y)$ are the gross payoffs that agents get in their ex-ante efficient matches if the net payoffs are ψ^* and $(\psi^*)^w$ respectively.

From the equality in (1.2) and from the definitions of $\widetilde{\psi}^*$ and $\widetilde{\phi}^*$ it follows that for all $(b,s) \in \mathrm{Supp}(\pi^*)$,

$$\begin{aligned} v(\beta^*(b,s), \sigma^*(b,s)) &= w(b,s) + c(\beta^*(b,s), b) + d(\sigma^*(b,s), s) \\ &= \psi^*(b) + (\psi^*)^w(s) + c(\beta^*(b,s), b) + d(\sigma^*(b,s), s) \\ &= \widetilde{\psi}^*(\beta^*(b,s)) + \widetilde{\phi}^*(\sigma^*(b,s)). \end{aligned} \tag{1.10}$$

Moreover, the inequality in (1.2) implies for any $x = \beta^*(b,s)$ and $y = \sigma^*(b', s')$ with $(b,s), (b', s') \in \mathrm{Supp}(\pi^*)$,

$$\begin{aligned} \widetilde{\psi}^*(x) + \widetilde{\phi}^*(y) &= \psi^*(b) + c(x,b) + (\psi^*)^w(s') + d(y, s') \\ &\geq w(b, s') + c(x,b) + d(y,s') \geq v(x,y). \end{aligned} \tag{1.11}$$

(1.10) and (1.11) imply that with respect to the sets $\beta^*(\mathrm{Supp}(\pi^*))$ and $\sigma^*(\mathrm{Supp}(\pi^*))$, $\widetilde{\psi}^*$ is a v-convex function, and $\widetilde{\phi}^*$ is its v-transform. Furthermore (by (1.10)), the set $(\beta^*, \sigma^*)(\mathrm{Supp}(\pi^*))$, which by Lemma 1.5 is dense in $\mathrm{Supp}(\widetilde{\pi}^*)$, is contained in the v-subdifferential of $\widetilde{\psi}^*$. Completing $\widetilde{\psi}^*$ as in Lemma 1.6 yields the stable and feasible bargaining outcome $(\widetilde{\pi}^*, \widetilde{\psi}^*)$ for $(\beta^*_{\#}\pi^*, \sigma^*_{\#}\pi^*, v)$.

It remains to be shown that no agent has an incentive to deviate. So assume that there is a buyer of type $b \in \mathrm{Supp}(\mu)$ for whom it is profitable to deviate. Then there must be some $x \in X$ such that

$$\sup_{y \in \mathrm{Supp}(\sigma^*_{\#}\pi^*)} \Big(v(x,y) - (\widetilde{\psi}^*)^v(y)\Big) - c(x,b) > \psi^*(b),$$

and hence there is some $y \in \mathrm{Supp}(\sigma^*_{\#}\pi^*)$ for which

$$v(x,y) - (\widetilde{\psi}^*)^v(y) - c(x,b) \;>\; \psi^*(b).$$

Since $\sigma^*(\mathrm{Supp}(\pi^*))$ is dense in $\mathrm{Supp}(\sigma^*_{\#}\pi^*)$ and by continuity of v and $(\widetilde{\psi}^*)^v$, it follows that there is some $(b', s') \in \mathrm{Supp}(\pi^*)$ such that

$$v(x, \sigma^*(b', s')) - (\psi^*)^w(s') - d(\sigma^*(b', s'), s') - c(x,b) \;>\; \psi^*(b).$$

Hence in particular $w(b, s') \;>\; (\psi^*)^w(s') + \psi^*(b)$, which contradicts (1.2). The argument for sellers is analogous. □

Proof of Lemma 1.1. Assume to the contrary that there is some x such that $v(x,y) - c(x,b) \;>\; v(\beta(b,s'),y) - c(\beta(b,s'),b)$. $(\beta(b,s'),y) \;\in\; \mathrm{Supp}(\widetilde{\pi})$ implies $\widetilde{\psi}(\beta(b,s')) \;=\; v(\beta(b,s'),y) - \widetilde{\psi}^v(y)$. Hence

$$\begin{aligned}\widetilde{\psi}(\beta(b,s')) - c(\beta(b,s'),b) &= v(\beta(b,s'),y) - \widetilde{\psi}^v(y) - c(\beta(b,s'),b)\\ &< v(x,y) - \widetilde{\psi}^v(y) - c(x,b) \;\le\; r_B(b),\end{aligned}$$

which contradicts the assumption that $\beta(b,s')$ is an equilibrium choice of buyer b. The proof for sellers is analogous. □

Proof of Lemma 1.2. Assume to the contrary that $x \in \mathrm{Supp}(\beta_{\#}\pi)$. Then, from the definition of r_S, and by assumption,

$$r_S(s) + \widetilde{\psi}(x) - c(x,b) \;\ge\; v(x,y) - \widetilde{\psi}(x) - d(y,s) + \widetilde{\psi}(x) - c(x,b) \;>\; r_B(b) + r_S(s).$$

Hence $\widetilde{\psi}(x) - c(x,b) > r_B(b)$, a contradiction (formally, $\widetilde{\psi}(x) = v(x,y') - \widetilde{\psi}^v(y')$ for some $y' \in \mathrm{Supp}(\sigma_{\#}\pi)$ matched with x under $\widetilde{\pi}$ and this leads to a contradiction to the definition of r_B). The proof for $y \notin \mathrm{Supp}(\sigma_{\#}\pi)$ is analogous. □

Basic facts about the 1-d supermodular model

As is well known, strict supermodularity of v forces optimal couplings to be positively assortative for any attribute economy. The Kantorovich duality theorem can be used for a very short proof.

Lemma 1.7. *Let Assumption 1.1 hold. Then, for any attribute economy* $(\tilde{\mu}, \tilde{\nu}, v)$*, the unique optimal coupling is the positively assortative one.*

Proof of Lemma 1.7. By Kantorovich duality, the support of any optimal coupling $\tilde{\pi}$ is a v-cyclically monotone set. In particular, for any $(x, y), (x', y') \in \operatorname{Supp}(\tilde{\pi})$ with $x > x'$, it holds $v(x, y)+v(x', y') \geq v(x, y')+v(x', y)$ and hence $v(x, y)-v(x', y) \geq v(x, y')-v(x', y')$. Since v has strictly increasing differences, it follows that $y \geq y'$. □

Lemma 1.8. *Let Assumption 1.1 hold. Then, in any ex-post contracting equilibrium, attribute choices are non-decreasing with respect to agents' own type.*

Proof of Lemma 1.8. From Definition 1.5, $\beta(b, s) \in \operatorname{argmax}_{x \in X}(\tilde{r}_B(x) - c(x, b))$. The objective is strictly supermodular in (x, b). By Theorem 2.8.4 from Topkis (1998), all selections from the solution correspondence are non-decreasing in b. The argument for sellers is analogous. □

Corollary 1.4. *Let Assumption 1.1 hold. Then every ex-post contracting equilibrium* $((\beta, \sigma, \pi), (\tilde{\pi}, \tilde{\psi}))$ *is compatible with the positively assortative coupling of buyers and sellers.*

Note that the positively assortative coupling may of course feature matching buyers of the same type to different seller types, and vice versa, whenever the distributions have atoms. This does not matter for the result.

Proposition 1.5. *Let Assumption 1.1 hold and assume that Corollary 1.2 applies. Then every ex-post contracting equilibrium is ex-ante efficient.*

Proof of Proposition 1.5. By Corollary 1.4, buyer and seller types can ex-post be interpreted to be coupled in the same way (positively assortative) in any equilibrium. In particular, this is true for the ex-ante efficient equilibrium that was constructed in Section 1.4 (note that, by (upper hemi-) continuity, Condition 1.1 is automatically satisfied if $(x^*(b, s), y^*(b, s))$ is unique for all (b, s)). By Corollary 1.2, all agents make jointly optimal investments for their equilibrium partnership in every ex-post contracting equilibrium. This proves the claim. □

Formulae and proofs for the basic module

NE of the FA game for (b, s) necessarily are stationary points of $h(x, y|b, s) = \gamma(xy)^\alpha - \frac{x^4}{b^2} - \frac{y^4}{s^2}$. By behavior of this objective function on the main diagonal $x = y$ for small x, as well as by the asymptotic behavior as $x \to \infty$ or $y \to \infty$, there is an interior global maximum. Necessary first order conditions are

$$\begin{cases} \gamma\alpha x^{\alpha-1}y^\alpha &= \frac{4}{b^2}x^3 \\ \gamma\alpha x^\alpha y^{\alpha-1} &= \frac{4}{s^2}y^3 \end{cases} \Rightarrow \begin{cases} y &= (\frac{4}{\gamma\alpha b^2})^{1/\alpha}x^{(4-\alpha)/\alpha} \\ x &= (\frac{4}{\gamma\alpha s^2})^{1/\alpha}y^{(4-\alpha)/\alpha}. \end{cases}$$

Plugging in yields a unique stationary point apart from $(0, 0)$, given by

$$\begin{cases} x^{(4-\alpha)^2/\alpha^2-1} = (\frac{\gamma\alpha s^2}{4})^{1/\alpha}(\frac{\gamma\alpha b^2}{4})^{(4-\alpha)/\alpha^2} \\ y^{(4-\alpha)^2/\alpha^2-1} = (\frac{\gamma\alpha b^2}{4})^{1/\alpha}(\frac{\gamma\alpha s^2}{4})^{(4-\alpha)/\alpha^2}. \end{cases} \Rightarrow \begin{cases} x &= (\frac{\gamma\alpha}{4})^{1/(4-2\alpha)}b^{(4-\alpha)/(8-4\alpha)}s^{\alpha/(8-4\alpha)} \\ y &= (\frac{\gamma\alpha}{4})^{1/(4-2\alpha)}s^{(4-\alpha)/(8-4\alpha)}b^{\alpha/(8-4\alpha)}. \end{cases}$$

This proves (1.3). Net match surplus is

$$\begin{aligned} w(b, s) &= \gamma(x^*(b, s)y^*(b, s))^\alpha - \frac{x^*(b, s)^4}{b^2} - \frac{y^*(b, s)^4}{s^2} \\ &= \gamma\left(\frac{\gamma\alpha}{4}\right)^{\frac{\alpha}{2-\alpha}}(bs)^{\frac{\alpha}{2-\alpha}} - \frac{1}{b^2}\left(\frac{\gamma\alpha}{4}\right)^{\frac{2}{2-\alpha}} b^{\frac{4-\alpha}{2-\alpha}}s^{\frac{\alpha}{2-\alpha}} - \frac{1}{s^2}\left(\frac{\gamma\alpha}{4}\right)^{\frac{2}{2-\alpha}} s^{\frac{4-\alpha}{2-\alpha}}b^{\frac{\alpha}{2-\alpha}} \\ &= \kappa(\alpha, \gamma)(bs)^{\frac{\alpha}{2-\alpha}}, \end{aligned}$$

where

$$\kappa(\alpha, \gamma) = \gamma^{\frac{2}{2-\alpha}}\left(\left(\frac{\alpha}{4}\right)^{\frac{\alpha}{2-\alpha}} - 2\left(\frac{\alpha}{4}\right)^{\frac{2}{2-\alpha}}\right) = \gamma^{\frac{2}{2-\alpha}}\left(\frac{\alpha}{4}\right)^{\frac{\alpha}{2-\alpha}}\left(1 - \frac{\alpha}{2}\right).$$

This proves (1.4) and (1.5).

Now, let $x = x^*(b, s')$. To derive (1.6), note that from the first order condition it follows that $y = \left(\frac{\gamma\alpha s^2}{4}\right)^{\frac{1}{4-\alpha}} x^{\frac{\alpha}{4-\alpha}}$. Hence

$$\begin{aligned}
\gamma(xy)^\alpha - \frac{y^4}{s^2} &= \gamma x^{\frac{4\alpha}{4-\alpha}} \left(\frac{\gamma\alpha}{4}\right)^{\frac{\alpha}{4-\alpha}} s^{\frac{2\alpha}{4-\alpha}} - \frac{1}{s^2}\left(\frac{\gamma\alpha s^2}{4}\right)^{\frac{4}{4-\alpha}} x^{\frac{4\alpha}{4-\alpha}} \\
&= s^{\frac{2\alpha}{4-\alpha}} \left(\left(\frac{\gamma\alpha}{4}\right)^{\frac{1}{4-2\alpha}} b^{\frac{4-\alpha}{8-4\alpha}} (s')^{\frac{\alpha}{8-4\alpha}}\right)^{\frac{4\alpha}{4-\alpha}} \left(\gamma\left(\frac{\gamma\alpha}{4}\right)^{\frac{\alpha}{4-\alpha}} - \left(\frac{\gamma\alpha}{4}\right)^{\frac{4}{4-\alpha}}\right) \\
&= b^{\frac{\alpha}{2-\alpha}} s^{\frac{2\alpha}{4-\alpha}} (s')^{\frac{\alpha^2}{(4-\alpha)(2-\alpha)}} \gamma^{\frac{2}{2-\alpha}} \left(\frac{\alpha}{4}\right)^{\frac{\alpha}{2-\alpha}} \left(1-\frac{\alpha}{4}\right).
\end{aligned}$$

For $s = s'$, this coincides with $w(b,s) + c(x^*(b,s), b)$.

Lemma 1.9. *Let $K \in \mathbb{N}$, $0 < \alpha_1 < ... < \alpha_K < 2$, $\gamma_1, ..., \gamma_K > 0$ and $f_i(z) = \gamma_i z^{\alpha_i}$ for $i = 1, ..., K$. For $i < j$, there is a unique $z_{ij} \in \mathbb{R}_{++}$ in which f_j crosses f_i (from below). z_{ij} is given by*

$$z_{ij} = \left(\frac{\gamma_i}{\gamma_j}\right)^{\frac{1}{\alpha_j - \alpha_i}}.$$

Consider parameter constellations for which $z_{12} < z_{23} < ... < z_{(K-1)K}$. In this case, $(\max_{i=1,...,K} f_i)(xy)$ defines a strictly supermodular function in $(x,y) \in \mathbb{R}^2_+$.

Proof of Lemma 1.9. Note that $f_1(xy)$ is strictly increasing and strictly supermodular in (x,y), and that $(\max_{i=1,...,K} f_i)(xy) = g(f_1(xy))$ for the strictly increasing, convex function

$$g(t) = \begin{cases} t & \text{for } t \le \gamma_1 z_{12}^{\alpha_1} \\ \gamma_1^{-\alpha_i/\alpha_1} \gamma_i t^{\alpha_i/\alpha_1} & \text{for } \gamma_1 z_{(i-1)i}^{\alpha_1} < t \le \gamma_1 z_{i(i+1)}^{\alpha_1},\ i = 2, ..., K-1 \\ \gamma_1^{-\alpha_K/\alpha_1} \gamma_K t^{\alpha_K/\alpha_1} & \text{for } t > \gamma_1 z_{(K-1)K}^{\alpha_1}. \end{cases}$$

The claim thus follows, e.g from an adaptation of Lemma 2.6.4 in Topkis (1998). □

Proofs for Section 1.5.3.1

Note first that it is not possible that (s_L, s_L)-sellers are matched while some (s_H, s_H)-sellers remain unmatched. This follows immediately from (1.6) and (1.7) (the net return from making zero investments and staying unmatched is zero for both types of seller).

Case $\mathbf{a_H} > \mathbf{a_2}$: some (s_H, s_H)-sellers must match with $(b_1, 0)$-buyers then. The equilibrium is not compatible with the ex-ante optimal coupling if and only if some $((0, b_2), (s_H, s_H))$-couples *and* $((b_1, 0), (s_L, s_L))$-couples exist as well, in any compatible coupling. In particular, by uniqueness of non-trivial NE of the FA games, $r_B(0, b_2) + r_S(s_H, s_H) = \frac{1}{8} b_2 s_H$ and $r_B(b_1, 0) + r_S(s_L, s_L) = \frac{1}{8} b_1 s_L$. Thus $r_B(0, b_2) + r_S(s_L, s_L) + r_B(b_1, 0) + r_S(s_H, s_H) < \frac{1}{8} b_2 s_L + \frac{1}{8} b_1 s_H$ (by strict supermodularity). If $((0, b_2), (s_L, s_L))$-couples existed as well, then all attributes needed for efficient matches would exist in the attribute market, and at least one of the couples $((0, b_2), (s_L, s_L))$, $((b_1, 0), (s_H, s_H))$ would violate Lemma 1.2.

Hence, the only candidate for an inefficient ex-post contracting equilibrium is the one in which only $((0, b_2), (s_H, s_H))$-, $((b_1, 0), (s_H, s_H))$- and $((b_1, 0), (s_L, s_L))$-couples exist. As sellers are on the long side, some (s_L, s_L)-types remain unmatched and make zero investments, so that $r_S(s_L, s_L) = 0$. Thus, $r_B(b_1, 0) = \frac{1}{8} b_1 s_L$, $r_S(s_H, s_H) = \frac{1}{8} b_1 (s_H - s_L)$ and $r_B(0, b_2) = \frac{1}{8} b_2 s_H - r_S(s_H, s_H)$. In particular, neither $(b_1, 0)$ nor (s_H, s_H) have profitable deviations. The remaining equilibrium conditions are that $(0, b_2)$-types do not want to deviate to zero investments, i.e. $r_B(0, b_2) \geq 0$ (there is only one suitable attribute to match with for them in the candidate equilibrium, the one chosen by (s_H, s_H)-types for sector 2), and that (s_L, s_L)-types can not get a strictly positive net return from investing to match with $x^*((0, b_2), (s_H, s_H))$. According to (1.6) and (1.7), the latter condition is equivalent to

$$\frac{3}{16} b_2 s_L^{\frac{2}{3}} s_H^{\frac{1}{3}} - \frac{1}{16} b_2 s_H - r_B(0, b_2) \leq 0.$$

Plugging in $r_B(0, b_2)$ and rearranging terms yields

$$\frac{3}{2} \frac{b_2}{b_1} \geq \frac{1 - \frac{s_L}{s_H}}{1 - (\frac{s_L}{s_H})^{\frac{2}{3}}}.$$

Finally, $r_B(0, b_2) \geq 0$ may be rewritten as $\frac{b_2}{b_1} \geq 1 - \frac{s_L}{s_H}$.

Case $\mathbf{a_H} < \mathbf{a_2}$: as before, inefficiency requires the existence of both $((0, b_2), (s_H, s_H))$- and $((b_1, 0), (s_L, s_L))$-couples. Some $((0, b_2), (s_L, s_L))$-couples necessarily exist as well. As in the previous case, the existence of $((b_1, 0), (s_H, s_H))$-couples would then contradict Lemma 1.2. So, the only possibility is that all (s_H, s_H)-sellers are depleted by sector 2. It follows that $r_S(s_L, s_L) = 0$, $r_B(0, b_2) = \frac{1}{8} b_2 s_L$, $r_S(s_H, s_H) = \frac{1}{8} b_2 (s_H - s_L)$

and $r_B(b_1, 0) = \frac{1}{8}b_1 s_L$. Buyers and (s_L, s_L)-sellers have no profitable deviations. The remaining equilibrium condition for (s_H, s_H) is

$$\frac{1}{8}b_2\,(s_H - s_L) \;\geq\; \frac{3}{16}b_1\, s_H^{\frac{2}{3}} s_L^{\frac{1}{3}} - \frac{3}{16}b_1\, s_L,$$

which may be rewritten as

$$\frac{2}{3}\frac{b_2}{b_1} \;\geq\; \frac{\left(\frac{s_H}{s_L}\right)^{\frac{2}{3}} - 1}{\frac{s_H}{s_L} - 1}.$$

Proofs for Section 1.5.3.2

Assume that there is an ex-post contracting equilibrium that is not ex-ante efficient. So, in any compatible matching of buyers and sellers there exist $(s_1', s_1'), (s_1'', s_1'')$ with $s_1' < s_1''$ and b', b'' with $|b'| > |b''|$ such that (b', s') and (b'', s'') match (with their jointly efficient investments). Equilibrium matching is positively assortative within each sector (according to Corollary 1.4), so that b' and b'' must be from different sectors. W.l.o.g. $b' = (b_1', 0), b'' = (0, b_2'')$. Define open right-neighborhoods $R_\varepsilon(s_1) := \{t_1 | s_1 < t_1 < s_1 + \varepsilon\}$, and

$$\hat{s}_1 := \inf\{s_1 \geq s_1' | \text{for all } \varepsilon > 0 \text{ there are } t_1 \in R_\varepsilon(s_1) \text{ with investments } y^*((0,\cdot),(t_1,t_1))\}.$$

The set used to define the infimum is non-empty since (s_1'', s_1'') makes investment $y^*((0, b_2''), (s_1'', s_1''))$, ν is absolutely continuous w.r.t. Lebesgue measure and investment profiles are regular (Definition 1.4). Hence, $\hat{s}_1$ exists and satisfies $s_1' \leq \hat{s}_1 \leq s_1''$. If $\hat{s}_1 > s_1'$, then every left-neighborhood of $\hat{s}_1$ contains sellers investing for sector 1. If $\hat{s}_1 = s_1'$, then $(\hat{s}_1, \hat{s}_1)$ invests for sector 1 by assumption. In either case, regularity (and completion) implies that there are suitable attributes for $(\hat{s}_1, \hat{s}_1)$ in both sectors: there are $(\hat{b}_1, 0)$, $\hat{b}_1 \geq b_1'$ and $(0, \hat{b}_2)$, $\hat{b}_2 \leq b_2''$ (in particular $\hat{b}_2 < \hat{b}_1$) such that $x^*((0, \hat{b}_2), (\hat{s}_1, \hat{s}_1)), x^*((\hat{b}_1, 0), (\hat{s}_1, \hat{s}_1)) \in \mathrm{Supp}(\beta_{\#}\pi)$. $(\hat{s}_1, \hat{s}_1)$ must be indifferent between the two corresponding equilibrium matches. This implies $r_S(\hat{s}_1, \hat{s}_1) = \frac{1}{8}\hat{b}_1\hat{s}_1 - r_B(\hat{b}_1, 0) = \frac{1}{8}\hat{b}_2\hat{s}_1 - r_B(0, \hat{b}_2)$. By construction, there are buyers from sector 2 just above $\hat{b}_2$ who invest for seller types just above $\hat{s}_1$ and vice versa. I will show now that these seller types can profitably deviate to match with $x^*((\hat{b}_1, 0), (\hat{s}_1, \hat{s}_1))$. This will be the desired contradiction. Indeed, r_S must be right-differentiable at $\hat{s}_1$ with derivative $\frac{1}{8}\hat{b}_2$. However, if $s_1 > \hat{s}_1$ invests for and matches with $x^*((\hat{b}_1, 0), (\hat{s}_1, \hat{s}_1))$, this type can claim

$$\frac{3}{16}\hat{b}_1 s_1^{\frac{2}{3}} \hat{s}_1^{\frac{1}{3}} - \frac{1}{16}\hat{b}_1\hat{s}_1 - r_B(\hat{b}_1, 0) = \frac{3}{16}\hat{b}_1 s_1^{\frac{2}{3}} \hat{s}_1^{\frac{1}{3}} - \frac{3}{16}\hat{b}_1\hat{s}_1 + r_S(\hat{s}_1, \hat{s}_1).$$

The leading order term in the expansion of the first two terms on the right hand side (around $\hat{s}_1$) is $\frac{1}{8}\hat{b}_1(s_1 - \hat{s}_1)$. This contradicts the conclusion about the derivative of r_S obtained from sector 2 (since $\hat{b}_2 < \hat{b}_1$).

Proofs for Section 1.5.3.3

Proof of Lemma 1.3. I show $\limsup_{t\to 0, t>0} \frac{r_B(b+t\eta) - r_B(b)}{t} \leq \frac{1}{8}T(b) \cdot \eta$ and $\liminf_{t\to 0, t>0} \frac{r_B(b+t\eta) - r_B(b)}{t} \geq \frac{1}{8}T(b) \cdot \eta$. Assume to the contrary that $\limsup_{t\to 0, t>0} \frac{r_B(b+t\eta) - r_B(b)}{t} > \frac{1}{8}T(b) \cdot \eta$. Then there is an $a > \frac{1}{8}T(b) \cdot \eta$ and a monotone decreasing sequence (t_n) with $\lim_{n\to\infty} t_n = 0$ such that $r_B(b+t_n\eta) \geq r_B(b)+t_n a$. Consider the sellers $T(b + t_n\eta)$ then. Net payoffs must satisfy

$$\begin{aligned} r_S(T(b+t_n\eta)) &= \frac{1}{8}(b+t_n\eta)\cdot T(b+t_n\eta) - r_B(b+t_n\eta) \\ &\leq \frac{1}{8}(b+t_n\eta)\cdot T(b+t_n\eta) - r_B(b) - t_n a \\ &= r_s(T(b)) + t_n(\frac{1}{8}b \cdot DT(b)\eta + \frac{1}{8}T(b)\cdot\eta - a) + o(t_n). \end{aligned}$$

On the other hand, when $T(b + t_n\eta)$ invests optimally to match with $x^*(b, T(b))$ he gets:

$$\begin{aligned} &\sum_{i=1}^{2} \frac{3}{16} T(b+t_n\eta)_i^{\frac{2}{3}} T(b)_i^{\frac{1}{3}} b_i - \frac{1}{16} b \cdot T(b) - r_B(b) \\ &= \sum_{i=1}^{2} \left(\frac{3}{16}\left(T(b)_i^{\frac{2}{3}} + \frac{2}{3}T(b)_i^{-\frac{1}{3}} t_n (DT(b)\eta)_i\right) T(b)_i^{\frac{1}{3}} b_i - \frac{3}{16} b_i T(b)_i\right) + r_S(T(b)) + o(t_n) \\ &= r_S(T(b)) + \frac{1}{8} t_n b \cdot DT(b)\eta + o(t_n). \end{aligned}$$

It follows that for small t_n, $T(b + t_n\eta)$ has a profitable deviation. This contradicts equilibrium. Thus, $\limsup_{t\to 0, t>0} \frac{r_B(b+t\eta) - r_B(b)}{t} \leq \frac{1}{8}T(b) \cdot \eta$. $\liminf_{t\to 0, t>0} \frac{r_B(b+t\eta) - r_B(b)}{t} \geq$

$\frac{1}{8}T(b) \cdot \eta$ may be shown by an analogous argument, this time using deviations by buyers. $\square$

Proof of Corollary 1.3. At $b \in U$ derivatives in all directions η exist and are given by $\frac{1}{8}T(b) \cdot \eta$ (Lemma 1.3). These are smooth on U since T is smooth. Hence r_B is smooth on U and satisfies $\nabla r_B = \frac{1}{8}T$. $\square$

Proof of Proposition 1.3. Given $b \in U$, an arbitrary direction η and $t > 0$, consider buyers $b + t\eta$ and b. Ex-post contracting equilibrium requires in particular that $b + t\eta$ does not want to deviate from his match with $T(b + t\eta)$ and invest (optimally) for $y^*(b, T(b))$ instead. Moreover, $T(b + t\eta)$ must not want to deviate and match with $x^*(b, T(b))$. The two resulting conditions are:

$$\sum_{i=1}^{2} \frac{3}{16}(b_i + t\eta_i)^{\frac{2}{3}} b_i^{\frac{1}{3}} T(b)_i \leq r_B(b + t\eta) + \frac{1}{2}r_B(b) + \frac{3}{2}r_S(T(b)), \tag{1.12}$$

and

$$\sum_{i=1}^{2} \frac{3}{16}T(b + t\eta)_i^{\frac{2}{3}} T(b)_i^{\frac{1}{3}} b_i \leq r_S(T(b + t\eta)) + \frac{1}{2}r_S(T(b)) + \frac{3}{2}r_B(b). \tag{1.13}$$

I next derive the second order approximations of the left and right hand side of (1.12), using the following identities:

$$\begin{aligned} (b_i + t\eta_i)^{\frac{2}{3}} &= b_i^{\frac{2}{3}} + \frac{2}{3}b_i^{-\frac{1}{3}}t\eta_i - \frac{1}{9}b_i^{-\frac{4}{3}}t^2\eta_i^2 + o(t^2), \\ T(b + t\eta) &= T(b) + tDT(b)\eta + \frac{t^2}{2}D^2T(b)(\eta, \eta) + o(t^2). \end{aligned}$$

$$\begin{aligned} \sum_{i=1}^{2} \frac{3}{16}(b_i + t\eta_i)^{\frac{2}{3}} b_i^{\frac{1}{3}} T(b)_i &= \sum_{i=1}^{2} \frac{3}{16}\left(b_i^{\frac{2}{3}} + \frac{2}{3}b_i^{-\frac{1}{3}}t\eta_i - \frac{1}{9}b_i^{-\frac{4}{3}}t^2\eta_i^2\right) b_i^{\frac{1}{3}} T(b)_i + o(t^2) \\ &= \frac{3}{16}b \cdot T(b) + t\frac{1}{8}\eta \cdot T(b) - t^2\frac{1}{48}\sum_{i=1}^{2} \frac{T(b)_i}{b_i}\eta_i^2 + o(t^2). \end{aligned}$$

$$\begin{aligned} r_B(b+t\eta)+\frac{1}{2}r_B(b)+\frac{3}{2}r_S(T(b)) &= \frac{3}{16}b\cdot T(b)+r_B(b+t\eta)-r_B(b) \\ &= \frac{3}{16}b\cdot T(b)+t\frac{1}{8}\eta\cdot T(b)+t^2\frac{1}{16}\eta\cdot DT(b)\eta+o(t^2). \end{aligned}$$

Thus, inequality (1.12) turns into

$$t^2\eta\cdot\left(3DT(b)+\begin{pmatrix}\frac{T(b)_1}{b_1} & 0\\ 0 & \frac{T(b)_2}{b_2}\end{pmatrix}\right)\eta+o(t^2)\geq o(t^2).$$

Letting $t\to 0$ shows that $3DT(b)+\begin{pmatrix}\frac{T(b)_1}{b_1} & 0\\ 0 & \frac{T(b)_2}{b_2}\end{pmatrix}$ must be positive semi-definite. The second claim follows by symmetry (or from explicitly spelling out the second order approximation of (1.13)). □

Proof of Proposition 1.4. Since $DT(b)=8\,\mathrm{Hess}\,r_B(b)$ is symmetric, there is a basis of $\mathbb{R}^2$ consisting of orthonormal (w.r.t. the standard inner product) eigenvectors. Since $DT(b)$ is non-singular, all eigenvalues differ from zero. For the purpose of deriving a contradiction, assume that $DT(b)$ has an eigenvalue $\lambda<0$, with corresponding eigenvector η. From the first result in Proposition 1.3 it follows that $3\lambda+\eta_1^2\frac{T(b)_1}{b_1}+(1-\eta_1^2)\frac{T(b)_2}{b_2}\geq 0$, i.e. $\eta_1^2\frac{T(b)_1}{b_1}+(1-\eta_1^2)\frac{T(b)_2}{b_2}\geq 3|\lambda|$. The second result of Proposition 1.3 implies $3\lambda^{-1}+\eta_1^2\frac{b_1}{T(b)_1}+(1-\eta_1^2)\frac{b_2}{T(b)_2}\geq 0$, i.e. $\eta_1^2\frac{b_1}{T(b)_1}+(1-\eta_1^2)\frac{b_2}{T(b)_2}\geq\frac{3}{|\lambda|}=\frac{9}{3|\lambda|}$. It follows

$$\begin{aligned} 9 &\leq \left(\eta_1^2\frac{T(b)_1}{b_1}+(1-\eta_1^2)\frac{T(b)_2}{b_2}\right)\left(\eta_1^2\frac{b_1}{T(b)_1}+(1-\eta_1^2)\frac{b_2}{T(b)_2}\right) \\ &\leq 1+\eta_1^2(1-\eta_1^2)\left(\frac{T(b)_1}{b_1}\frac{b_2}{T(b)_2}+\frac{T(b)_2}{b_2}\frac{b_1}{T(b)_1}\right). \end{aligned}$$

Since $\eta_1^2(1-\eta_1^2)\leq\frac{1}{4}$ this requires $32\leq\left(\frac{T(b)_1}{b_1}\frac{b_2}{T(b)_2}+\frac{T(b)_2}{b_2}\frac{b_1}{T(b)_1}\right)$. Contradiction. □

Proof of Theorem 1.2. Since $\mathrm{Supp}(\mu)$ is the closure of an open convex set, Proposition 1.4 implies that T is the gradient of a convex function on $\mathrm{Supp}(\mu)$. This implies that the coupling π_T associated with T is concentrated on a w-cyclically monotone set. Hence, by Proposition 1.1, it is ex-ante optimal. □

Supplement for Section 1.5.4

For $s_1' < s_1''$, $(b_1, 0)$ and $(0, b_2)$ with $b_2 s_1'' > \tau$, the expression that must be analyzed to verify 2-cycle monotonicity is

$$w((0,b_2),(s_1'',s_1'')) + w((b_1,0),(s_1',s_1')) - w((0,b_2),(s_1',s_1')) - w((b_1,0),(s_1'',s_1'')).$$

Two cases should be distinguished. If $s_1' \geq \frac{\tau}{b_2}$, then

$$\begin{aligned}&w((0,b_2),(s_1'',s_1'')) + w((b_1,0),(s_1',s_1')) - w((0,b_2),(s_1',s_1')) - w((b_1,0),(s_1'',s_1''))\\ &= \int_{s_1'}^{s_1''} \kappa\left(\frac{3}{2},\frac{1}{2}\right) 3b_2^3 t^2 - \kappa(1,1) b_1 \, dt > \int_{s_1'}^{s_1''} 3\kappa(1,1) b_2 - \kappa(1,1) b_1 \, dt.\end{aligned}$$

The inequality holds since $b_2 t > \tau$, so that $\kappa\left(\frac{3}{2},\frac{1}{2}\right) b_2^3 t^3 > \kappa(1,1) b_2 t$. In particular, matching $(0, b_2)$ to the higher seller type is definitely in line with 2-cycle monotonicity if $3b_2 \geq b_1$. If $s_1' < \frac{\tau}{b_2}$ however, an additional term with a potentially opposite sign occurs.

$$\begin{aligned}&w((0,b_2),(s_1'',s_1'')) + w((b_1,0),(s_1',s_1')) - w((0,b_2),(s_1',s_1')) - w((b_1,0),(s_1'',s_1''))\\ &= \int_{\tau/b_2}^{s_1''} \kappa\left(\frac{3}{2},\frac{1}{2}\right) 3b_2^3 t^2 - \kappa(1,1) b_1 \, dt + \int_{s_1'}^{\tau/b_2} \kappa(1,1)(b_2 - b_1) dt.\end{aligned}$$

Proofs for Section 1.5.5

$$\begin{aligned}
x_{iij} &= \left(\frac{\gamma_i \alpha_i}{4}\right)^{\frac{1}{4-2\alpha_i}} \left(\frac{\kappa_i}{\kappa_j}\right)^{\frac{2-\alpha_j}{4(\alpha_j-\alpha_i)}} \\
&= \gamma_i^{\frac{1}{4-2\alpha_i} + \frac{2-\alpha_j}{2(2-\alpha_i)(\alpha_j-\alpha_i)}} \gamma_j^{-\frac{1}{2(\alpha_j-\alpha_i)}} \left(\frac{\alpha_i}{4}\right)^{\frac{1}{4-2\alpha_i}} \left(\frac{\left(\frac{\alpha_i}{4}\right)^{\frac{\alpha_i}{2-\alpha_i}}(2-\alpha_i)}{\left(\frac{\alpha_j}{4}\right)^{\frac{\alpha_j}{2-\alpha_j}}(2-\alpha_j)}\right)^{\frac{2-\alpha_j}{4(\alpha_j-\alpha_i)}} \\
&= \left(\frac{\gamma_i}{\gamma_j}\right)^{\frac{1}{2(\alpha_j-\alpha_i)}} \left(\frac{\alpha_i}{4}\right)^{\frac{1}{4-2\alpha_i}} \left(\frac{\left(\frac{\alpha_i}{4}\right)^{\frac{\alpha_i}{2-\alpha_i}}(2-\alpha_i)}{\left(\frac{\alpha_j}{4}\right)^{\frac{\alpha_j}{2-\alpha_j}}(2-\alpha_j)}\right)^{\frac{2-\alpha_j}{4(\alpha_j-\alpha_i)}} \\
&= z_{ij}^{\frac{1}{2}} \left(\frac{\alpha_i}{4}\right)^{\frac{1}{4-2\alpha_i}} \left(\frac{\left(\frac{\alpha_i}{4}\right)^{\frac{\alpha_i}{2-\alpha_i}}(2-\alpha_i)}{\left(\frac{\alpha_j}{4}\right)^{\frac{\alpha_j}{2-\alpha_j}}(2-\alpha_j)}\right)^{\frac{2-\alpha_j}{4(\alpha_j-\alpha_i)}} .
\end{aligned}$$

A similar computation yields

$$\begin{aligned} x_{jij} &= \left(\frac{\gamma_j \alpha_j}{4}\right)^{\frac{1}{4-2\alpha_j}} \left(\frac{\kappa_i}{\kappa_j}\right)^{\frac{2-\alpha_i}{4(\alpha_j-\alpha_i)}} \\ &= \gamma_i^{\frac{1}{2(\alpha_j-\alpha_i)}} \gamma_j^{\frac{1}{4-2\alpha_j}-\frac{2-\alpha_i}{2(2-\alpha_j)(\alpha_j-\alpha_i)}} \left(\frac{\alpha_j}{4}\right)^{\frac{1}{4-2\alpha_j}} \left(\frac{\left(\frac{\alpha_i}{4}\right)^{\frac{\alpha_i}{2-\alpha_i}}(2-\alpha_i)}{\left(\frac{\alpha_j}{4}\right)^{\frac{\alpha_j}{2-\alpha_j}}(2-\alpha_j)}\right)^{\frac{2-\alpha_i}{4(\alpha_j-\alpha_i)}} \\ &= z_{ij}^{\frac{1}{2}} \left(\frac{\alpha_j}{4}\right)^{\frac{1}{4-2\alpha_j}} \left(\frac{\left(\frac{\alpha_i}{4}\right)^{\frac{\alpha_i}{2-\alpha_i}}(2-\alpha_i)}{\left(\frac{\alpha_j}{4}\right)^{\frac{\alpha_j}{2-\alpha_j}}(2-\alpha_j)}\right)^{\frac{2-\alpha_i}{4(\alpha_j-\alpha_i)}} . \end{aligned}$$

It is straightforward to check that $\widetilde{\psi}$ is a v-convex function with respect to the sets $\mathrm{cl}(\beta(I))$ and $\mathrm{cl}(\sigma(I)) = \mathrm{cl}(\beta(I))$, that $\widetilde{\phi}$ is its transform, and that the deterministic coupling of the symmetric attribute measures given by the identity mapping is supported in $\partial_v \widetilde{\psi}$. This yields a stable and feasible bargaining outcome for the attribute economy.

Given $\widetilde{\phi}$, buyer type b_{13} is indifferent between the option (choose $x = x_{113}$, match with $y = x_{113}$) and the option (choose $x = x_{313}$, match with $y = x_{313}$). Indeed, net payoffs from this are $\gamma_1 x_{113}^{2\alpha_1}/2 - c(x_{113}, b_{13}) = w_1(b_{13})/2$ and $\gamma_3 x_{313}^{2\alpha_3}/2 - c(x_{313}, b_{13}) = w_3(b_{13})/2$ which are equal by definition of b_{13}. I show next that these are indeed the optimal choices for buyer type b_{13}. Note that for a given y, the conditionally optimal $x(y, b_{13})$ solves

$$\max_{x \in \mathbb{R}_+} \left(v(x,y) - \frac{v(y,y)}{2} - c(x, b_{13}) \right),$$

where

$$v(x,y) = \begin{cases} \gamma_1 (xy)^{\alpha_1} & \text{for } x \leq z_{12}/y \\ \gamma_2 (xy)^{\alpha_2} & \text{for } z_{12}/y \leq x \leq z_{23}/y \\ \gamma_3 (xy)^{\alpha_3} & \text{for } z_{23}/y \leq x. \end{cases}$$

Let $y \leq x_{113}$. Then, $x(y, b_{13}) \leq x_{113}$. Indeed,

$$\frac{\partial}{\partial x} \left(\gamma_i (xy)^{\alpha_i} - \frac{x^4}{b_{13}^2} \right) = \gamma_i \alpha_i y^{\alpha_i} x^{\alpha_i - 1} - \frac{4x^3}{b_{13}^2}$$

is strictly positive for $x < \left(\frac{\gamma_i \alpha_i y^{\alpha_i} b_{13}^2}{4}\right)^{\frac{1}{4-\alpha_i}}$ and strictly negative for $x > \left(\frac{\gamma_i \alpha_i y^{\alpha_i} b_{13}^2}{4}\right)^{\frac{1}{4-\alpha_i}}$.

For $y \leq x_{113}$, this zero is less than or equal to $\left(\frac{\gamma_i \alpha_i x_{113}^{\alpha_i} b_{13}^2}{4}\right)^{\frac{1}{4-\alpha_i}}$, which for $i = 1$ equals x_{113}. For $i = 2$, $\left(\frac{\gamma_2 \alpha_2 x_{113}^{\alpha_2} b_{13}^2}{4}\right)^{\frac{1}{4-\alpha_2}} = 0.8385 < z_{12}/x_{113} = 0.8391$, so that the derivative is negative on the entire second part of the domain. Similarly, $\left(\frac{\gamma_3 \alpha_3 x_{113}^{\alpha_3} b_{13}^2}{4}\right)^{\frac{1}{4-\alpha_3}} = 0.7599 < z_{23}/x_{113}$. It follows that $\max_{x \in \mathbb{R}_+, y \leq x_{113}} \left(v(x,y) - \frac{v(y,y)}{2} - c(x, b_{13})\right)$ is attained in the domain of definition of v where it coincides with f_1, the first order condition then yields $y = x$ and thus (maximizing $\frac{\gamma_1 x^{2\alpha_1}}{2} - c(x, b_{13})$) $x = y = x_{113}$. A completely analogous reasoning applies for $y \geq x_{313}$ (I omit the details), showing that $\max_{x \in \mathbb{R}_+, y \geq x_{313}} \left(v(x,y) - \frac{v(y,y)}{2} - c(x, b_{13})\right)$ is attained at $x = y = x_{313}$.

Therefore, buyer type b_{13} is indifferent between his two optimal choices (choose x_{113}, match with $y = x_{113}$) and (choose x_{313}, match with $y = x_{313}$). Note next that the buyer objective function $v(x,y) - \frac{v(y,y)}{2} - c(x,b)$ is supermodular in (x,y) on the lattice $\mathbb{R}_+ \times \mathrm{cl}(\beta(I))$ and has increasing differences in $((x,y),b)$. By Theorem 2.8.1 of Topkis (1998), the solution correspondence is increasing w.r.t. b in the usual set order (see Topkis 1998, Chapter 2.4). Hence, for $b < b_{13}$ there must be an optimum in the domain where v is defined via f_1. First order conditions lead to $y = x$, thus to maximization of $\gamma_1 x^{2\alpha_1}/2 - c(x,b)$ and hence to $x = \beta(b)$. The argument for buyer types $b > b_{13}$ is analogous. Since the entire argument applies to sellers as well, this concludes the proof. Q.E.D.

Chapter 2

Surplus division and efficient matching

We study a two-sided matching model with a finite number of agents who are characterized by privately known, multi-dimensional attributes that jointly determine the match surplus of each potential partnership. Utility functions are quasi-linear, and monetary transfers among agents are feasible. We ask the following question: what divisions of surplus within matched pairs are compatible with information revelation leading to the formation of an efficient (surplus-maximizing) matching? Our main result shows that the only robust rules compatible with efficient matching are those that divide realized surplus in a fixed proportion, independently of the attributes of the pair's members. In other words, to enable efficient match formation, it is necessary that each agent expects to get the same fixed percentage of surplus in every conceivable match. A more permissive result is obtained for one-dimensional attributes and supermodular value functions.

2.1 Introduction

We study a two-sided one-to-one matching (or assignment) market with a finite number of privately informed agents that need to be matched to form productive relationships. For ease of reference, we call the two sides of the market "workers" and "employers". Agents are characterized by multi-dimensional attributes which determine the match value / surplus potentially created by each employer-worker pair. Attributes are private information, and our model is a incomplete information version of the famous assignment game due to Shapley and Shubik (1971). In particular, for each pair, match surplus is informationally interdependent, a natural feature in the context of matching models.

We consider situations in which the ex-post realized match surplus in every partnership is divided according to some standardized contract or "sharing rule". For instance, such a sharing rule might result from specifying claims to various components of joint surplus (or fixed shares of these). If partners' attributes are ex-post verifiable, the rule might also determine shares directly as a function of these attributes. The implied division of surplus determines agents' utilities in every possible matching, net of additional monetary transfers that are decided upon when workers and employers compete for partners under incomplete information, i.e. in the match formation phase. We ask the following question: can we characterize standardized divisions of surplus that are compatible with information revelation leading, for each realization of attributes in the economy, to an efficient matching? Our main result shows that in settings with multi-dimensional, complementary attributes, the only sharing rules that may induce efficient match formation are those that divide the surplus in each match according to a *fixed proportion*, independently of the attributes of the pair's members. Thus, to enable efficient matching it is necessary and sufficient that all workers expect to get the same fixed percentage of surplus in every conceivable match, and the same thing must hold for employers! More flexibility is possible when attributes are one-dimensional and match surplus is supermodular. Efficient matching is then compatible with any division that leaves each partner with a fraction of the surplus that is also supermodular.

The equilibrium notion used here is the *ex-post equilibrium.* This is a generalization of equilibrium in dominant strategies appropriate for settings with interdependent values, and it embodies a notion of no regret: chosen actions must be considered optimal even after the private information of others is revealed. Ex-post equilibrium is a belief-free notion, and our results do not depend in any way on the distribution of attributes in the population.[1]

An interesting illustration for a fixed-proportion rule is offered by the German law governing the sharing of profit among a public sector employer and an employee arising from the employee's invention activity. The law differentiates between universities and all other public institutions.[2] Outside universities - where, presumably, the probability

[1]See also Bergemann and Morris (2005) for the tight connections between ex-post equilibria and "robust design".

[2]Practically all German universities are public.

of an employee making a job-related discovery is either nil or very low - the law allows any *ex-ante* negotiated contract governing profit sharing (see §40-1 in *Bundesgesetzblatt III, 422-1*). In marked contrast, independently of circumstances, any university and any researcher working there must divide the profit from the researcher's invention according to *a fixed 30%-70% rule*, with the employee getting the 30% share (see §42-4). The rigidity of this "no-exception" rule is additionally underlined by an explicit mention that all feasible arrangements under §40-1 are not applicable within universities (see §42-5).

The occurrence of inflexible, fixed-proportion rules for sharing ex-post surplus - shares do not vary with attributes and are not the object of negotiation - is a recurrent theme in several interesting literatures that try to explain this somewhat puzzling phenomenon. For example, Newbery and Stiglitz (1979) and Allen (1985), among many others, noted that sharecropping contracts in many rural economies involve shares of around one half for landlord and tenant. This percentage division is observed in widely differing circumstances and has persisted in many places for a considerable length of time.[3] Another example is the *commenda,* a rudimentary form of company that formed for the duration of a single shipping mission in medieval Venice (for more details and impact on Venice's extraordinary economic expansion see Acemoglu and Robinson, 2012). A commenda involved two partners, one who stayed at home, and one who accompanied the cargo. Only two types of contracts were possible: either the sedentary partner provided 100% of the capital and received 75% of the profits, or he provided 67% of the capital and received 50% of the profits.

Our study is at the intersection of several important strands of the economic literature. We briefly review below some related papers from each of these strands, emphasizing both the existing relations to our work and the present novel aspects.

1. Matching: An overwhelming majority of studies within the large body of work on two-sided markets has assumed either complete information or *private values* models, that is, models where agents' preferences only depend on signals available to them at the time of the decision, but not on signals privately available to others. This holds for both the Gale-Shapley (1962) type of models with ordinal preferences, and for the branch studying variants and applications (to auctions and double auctions, say) of the assignment model with quasi-linear preferences due to Shapley and Shubik (1971).

[3] For example, Chao (1983) noted that a fixed 50-50 ratio was prevalent in China for more than 2000 years. The French and Italian words for "sharecropping" literally mean "50-50 split".

In the Gale-Shapley model, one-sided serial dictatorship where women, say, sequentially choose partners according to their preferences leads to a Pareto-optimal matching. Difficulties occur when the stronger *stability* requirement is invoked: a standard result is that no stable matching can be implemented in dominant strategies if both sides of the market are privately informed (see Roth and Sotomayor, 1990). Chakraborty, Citanna and Ostrovsky (2010) showed that stability may fail even in a one-sided private information model if preferences on one side of the market (colleges, say) depend on information available to agents on the same side of the market.

Becker (1973) has popularized a special case of the complete information Shapley-Shubik model where agents have one-dimensional attributes, and where the match value is a supermodular function of these attributes. In particular, agents are completely ordered according to their marginal productivity, and efficient matching is assortative. If two-sided incomplete information is introduced in the Becker model one immediately obtains *interdependent values*, i.e., agents' preferences also depend on signals available to others. But, somewhat surprisingly, there are only very few such models: most of the literature has assumed either complete information, or one-sided private information (which yields private values), or a continuum of types (so that aggregate uncertainty disappears). An exception is Hoppe, Moldovanu and Sela (2009) who analyzed a two-sided matching model with a finite number of privately informed agents, characterized by complementary one-dimensional attributes. In their model match surplus is divided in a fixed proportion, and they showed that efficient, assortative matching can arise as one of the equilibria of a bilateral signaling game. This finding is consistent with the results of the present chapter.

Another strand using Becker's specification and complete information has combined matching with ex-ante investment: before matching, agents undertake costly investments that affect their attributes and hence, ultimately, their match values. In two recent studies in this vein, Mailath, Postlewaite and Samuelson (2012a, 2012b) focused on the role of what they call "premuneration values", i.e., the surplus accruing to agents from matching, net of additional monetary transfers. They detailed how these values are affected by the specification of property rights.[4] Under personalized pricing - that must finely depend on the attributes of the matched pairs - an equilibrium which entails efficient investment

[4] They also noted that in certain circumstances it may be difficult to adjust premuneration values due to legal restrictions, prevailing social norms, or non-contractible components of match value.

and matching always exists in large (continuum) markets, no matter how surplus is shared.[5] In contrast, when personalized pricing is not feasible, premuneration values affect both the incentives to invest and final payoffs, and under-investment typically occurs.

2. Property rights: Our study is also related to the large literature analyzing the effects of the ex-ante allocation of property rights on bargaining outcomes, following Coase (1960). Traditionally, this literature has not placed the bargaining agents in an explicit market context. The interplay between private information and ex-ante property rights in private value settings was emphasized by Myerson-Satterthwaite (1983) and Cramton-Gibbons-Klemperer (1987) in a buyer-seller framework and a partnership dissolution model, respectively. Fieseler, Kittsteiner and Moldovanu (2003) offered a unified treatment that allows for interdependent values and encompasses both the above private values models and Akerlof's (1970) market for lemons. In all these papers, agents have one-dimensional types and a value maximizing allocation can be implemented via standard Clarke-Groves-Vickrey schemes. Whenever inefficiencies for certain allocations of property rights occur, these stem from the inability to design budget-balanced and individually rational transfers that sustain the value maximizing allocation.[6] Brusco, Lopomo, Robinson and Viswanathan (2007) and Gärtner and Schmutzler (2009) looked at mergers with interdependent values, a setting which is more related to the present study.[7] They focused on the difficulties that privately known stand-alone values pose for designing combinations of property rights and budget-balanced and individually rational transfers that lead to value maximizing mergers.

In marked contrast to all the above papers, our present analysis completely abstracts from budget-balancedness and individual rationality. In our setting, stand-alone values are known and forming a match between two agents is always better than leaving them as singles, but who is matched to whom is crucial for allocative efficiency. The fixed-proportion sharing rules are dictated here by the mere requirement of value maximization together with incentive compatibility.

[5] Thus, as in Cole, Mailath and Postlewaite (2001a), market competition eliminates hold-up problems.

[6] With several buyers and sellers, the Myerson-Satterthwaite model becomes a one-dimensional, linear incomplete information version of the Shapley-Shubik assignment game. Only in the limit, when the market gets very large, one can reconcile, via almost efficient double-auctions, incentives for information revelation with budget-balancedness and individual rationality.

[7] However, at most one match is formed in these models, and private information consists of, or can be reduced to, one-dimensional types.

3. Multi-dimensional attributes and mechanism design: The prevalent assumption that agents can be described by a single trait such as skill, technology, wealth, or education is often not tenable. Workers, for example, have very diverse job-relevant characteristics, which are only partially correlated. Tinbergen (1956) pioneered the analysis of labor markets where jobs and workers are described by several characteristics. The study of complete information assignment models with a continuum of traders and multi-dimensional attributes has been pioneered by Gretsky, Ostroy and Zame (1992, 1999). Chapter 1 of this thesis generalizes the matching cum ex-ante investment model due to Cole, Mailath and Postlewaite (2001a) along this line.

The present combination of multi-dimensional attributes, private information and interdependent values is usually detrimental to efficient implementation. In fact, Jehiel, Meyer-ter-Vehn, Moldovanu and Zame (2006) have shown that, generically, only trivial social choice functions - where the outcome does not depend on the agents' private information - can be ex-post implemented when values are interdependent and types are multi-dimensional. Jehiel and Moldovanu (2001) have shown that, generically, the efficient allocation cannot be implemented even if the weaker Bayes-Nash equilibrium concept is used.

Our present insight can be reconciled with those general negative results by noting that the two-sided matching model is not generic. In particular, we assume here that match surplus has the same functional form for all pairs (as a function of the respective attributes), and that the match surplus of any pair depends neither on how agents outside that pair match, nor on what their attributes are. These features are natural for the matching model but are "non-generic". Moreover, fixed-proportion sharing can be interpreted as using some limited amount of ex-post information (for each pair, the surplus realized by that pair) to determine final payoffs, and it is known that such features may potentially aid implementation (see Mezzetti, 2004 and Remark 2.1 below).

The sufficiency of fixed-proportion sharing for incentive compatibility is related to the presence of individual utilities that admit a cardinal alignment with social welfare: aggregate surplus becomes a *cardinal potential*, as defined by Jehiel, Meyer-ter-Vehn and Moldovanu (2008).[8] By proving necessity of fixed-proportion rules,[9] we identify

[8]They presented several non-generic cases where ex-post implementation is possible. See also Bikhchandani (2006) for other such cases, e.g. certain auction settings.

[9]There is some minor additional flexibility if the rule is not required to be independent of whether employers or workers are on the long side of the market, see Theorem 2.1.

a class of interesting settings for which efficient implementation is possible *only if* social welfare is a cardinal potential. Our result is also reminiscent of Roberts' (1979) characterization of dominant strategy implementation in private values settings, but both technical assumptions and proof are very different here. The analysis of the special case with one-dimensional types and supermodular match surplus is based on an elegant characterization result due to Bergemann and Välimäki (2002), who generalized previous insights due to Jehiel and Moldovanu (2001) and Dasgupta and Maskin (2000).

Finally, in a recent contribution, Che, Kim and Kojima (2012) have shown that efficiency is not compatible with incentive compatibility in a one-sided assignment model where agents' values over objects are allowed to depend on information of other agents. Their signals are one-dimensional but inefficiency occurs there because of the assumed lack of monetary transfers.

The chapter is organized as follows: in Section 2.2 we present the matching model. In Section 2.3 we state our results, both for the multi-dimensional case and for the special case of one-dimensional attributes and supermodular surplus. Section 2.4 concludes. All proofs are in the Appendix.

2.2 The matching model

There are I employers and J workers. Each employer e_i ($i \in \mathcal{I} = \{1, ..., I\}$) privately knows his type $x_i \in X$, and each worker w_j ($j \in \mathcal{J} = \{1, ..., J\}$) privately knows his type $y_j \in Y$. The supports of agents' possible types, X and Y, are open connected subsets of Euclidean space $\mathbb{R}^n$ for some $n \in \mathbb{N}$. If an employer of type x and a worker of type y form a match, they subsequently create a *match surplus* of $v(x, y)$, where $v : X \times Y \to \mathbb{R}_+$ is continuously differentiable. Unmatched agents create zero surplus, and all agents have quasi-linear utilities.

The set of alternatives $\mathcal{M}$ consists of all possible one-to-one matchings of employers and workers. If $I \leq J$, these are the injective maps $m : \mathcal{I} \to \mathcal{J}$. A matching $m \in \mathcal{M}$ will be called *efficient* for a type profile $(x_1, ..., x_I, y_1, ..., y_J)$ if and only if it maximizes aggregate surplus $u_{m'}(x_1, ..., x_I, y_1, ..., y_J) = \sum_{i=1}^{I} v(x_i, y_{m'(i)})$ among all $m' \in \mathcal{M}$. Analogous definitions apply for the case $J \leq I$. Efficient matchings can be obtained as the solutions of a finite linear program (Shapley and Shubik, 1971).

2.2.1 Sharing rules

A sharing rule specifies a standardized division of match surplus. As noted in Section 2.1, such a division might result from defining claims to (fixed shares of) different components of joint surplus (whose relative contribution to full match surplus may depend on attributes), or, if partners' attributes are ex-post verifiable, shares may be defined as a function of both attributes. We introduce the following notation: if e_i and w_j are matched in $m \in \mathcal{M}$, then employer e_i's share of surplus is $v_m^{e_i}(x_1, ..., x_I, y_1, ..., y_J) = \gamma(x_i, y_j)v(x_i, y_j)$, and worker w_j's share is $v_m^{w_j}(x_1, ..., x_I, y_1, ..., y_J) = (1 - \gamma(x_i, y_j))v(x_i, y_j)$, where we assume that $\gamma : X \times Y \to \mathbb{R}$ is continuously differentiable.[10] If e_i remains unmatched in m we have $v_m^{e_i}(x_1, ..., x_I, y_1, ..., y_J) = 0$ (similarly, $v_m^{w_j}(x_1, ..., x_I, y_1, ..., y_J) = 0$ if w_j stays unmatched). Note that, for each realization of attributes, the set of value-maximizing matchings does not depend on γ. It will follow easily that, for efficient matching to be implementable, it is necessary that γ is $[0, 1]$-valued (to provide strict incentives for truth-telling, it is necessary that γ is $(0, 1)$-valued).

Together with a sharing rule, the previously described matching model gives rise to a natural social choice setting with interdependent values. Every agent attaches a value to each possible alternative, i.e. to matchings of employers and workers. This value depends both on the agent's own type and on the type of the partner, but not on the private information of other agents. Moreover, this value does not depend on how other agents match. Thus, there are no allocative externalities, and there are no informational externalities across matched pairs.

2.2.2 Mechanisms

By the *Revelation Principle*, we may restrict attention to direct revelation mechanisms where truthful reporting by all agents forms an ex-post equilibrium. A direct revelation mechanism (mechanism hereafter) is given by functions $\Psi : X^I \times Y^J \to \mathcal{M}$, $t^{e_i} : X^I \times Y^J \to \mathbb{R}$ and $t^{w_j} : X^I \times Y^J \to \mathbb{R}$, for all $i \in \mathcal{I}$, $j \in \mathcal{J}$. Ψ selects a feasible matching as a function of reports, t^{e_i} is the monetary transfer to employer e_i, and t^{w_j} is the monetary transfer to worker w_j, as functions of reports.

Truth-telling is an ex-post equilibrium if for all employers e_i, for all workers w_j, and

[10] In principle, one might allow that shares depend on the identities of the partners. Given that match surplus is determined by productive attributes only, this would not be conducive to efficient implementation. This intuition may easily be formalized using the techniques from the Appendix.

for all type profiles $p = (x_1, ..., x_I, y_1, ..., y_J)$, $p' = (x_1, ..., x_i', ..., x_I, y_1, ..., y_J)$ and $p'' = (x_1, ..., x_I, y_1, ..., y_j'', ..., y_J)$ it holds that

$$\begin{aligned} v^{e_i}_{\Psi(p)}(p) + t^{e_i}(p) &\geq v^{e_i}_{\Psi(p')}(p) + t^{e_i}(p') \\ v^{w_j}_{\Psi(p)}(p) + t^{w_j}(p) &\geq v^{w_j}_{\Psi(p'')}(p) + t^{w_j}(p''). \end{aligned}$$

2.3 The main results

We now turn to our main question: which sharing rules, if any, are compatible with information revelation leading to an efficient matching? In other words, using the mechanism design terminology, we ask for which induced utility functions we can implement the value-maximizing social choice function in ex-post equilibrium.

For our main results we need an assumption known as the *twist condition* in the mathematical literature on optimal transport (see Villani, 2009). This is a multi-dimensional generalization of the well-known Spence-Mirrlees condition. While in optimal transport - where measures of agents are matched - the condition is invoked in order to ensure that the optimal transport, corresponding here to the efficient matching, is unique and deterministic, we use it for quite different, technical reasons (see the lemmas in Section 2.5).

Condition 2.1. *i) For all $x \in X$, the continuous mapping from Y to $\mathbb{R}^n$ given by $y \mapsto (\nabla_X v)(x, y)$ is injective.*

ii) For all $y \in Y$, the continuous mapping from X to $\mathbb{R}^n$ given by $x \mapsto (\nabla_Y v)(x, y)$ is injective.

Match surplus functions that fulfill Condition 2.1 model many interesting complementarities between multi-dimensional types of workers and employers. In particular, v is not additively separable with respect to x and y, so that the precise allocation of match partners really matters for efficiency.[11] As a simple example consider the bilinear match surplus: $v(x, y) = x \cdot y$, where $\cdot$ denotes the standard inner product on $\mathbb{R}^n$. Then $(\nabla_X v)(x, y) = y$ and $(\nabla_Y v)(x, y) = x$, and Condition 2.1 is satisfied.

[11] Note for instance that if v is additively separable and $I = J$, then *all* matchings are efficient, and hence the efficient matching can trivially be implemented, no matter what γ is. This stands in sharp contrast to the result of Theorem 2.1.

We can now state the central results concerning the necessity and sufficiency of fixed-proportion sharing:

Theorem 2.1. *Let $n \geq 2$, $I, J \geq 2$, and assume that Condition 2.1 is satisfied. Then the following are equivalent:*

i) The efficient matching is implementable in ex-post equilibrium.

ii) There is a constant $\lambda_0 \in [0,1]$ and functions $g : X \to \mathbb{R}$ and $h : Y \to \mathbb{R}$ such that for all $x \in X$, $y \in Y$ it holds that

$$(\gamma v)(x, y) \;=\; \lambda_0 v(x, y) + g(x) + h(y),$$

and, moreover, h is constant if $I < J$, and g is constant if $I > J$.

Corollary 2.1. *The only sharing rules that can implement the efficient matching irrespective of whether employers or workers are on the short side of the market are of the form $(\gamma v)(x, y) = \lambda_0 v(x, y) + c$, where $\lambda_0 \in [0,1]$ and c is a constant.*

It is easy to show that under the conditions of Theorem 2.1 ii), it is possible to align all agents' utilities with aggregate surplus (via appropriate Clark-Groves-Vickrey type transfers). When the part of the share that is proportional to match surplus is strictly positive for both sides of the market (i.e. $\lambda_0 \in (0,1)$), then a *strict* cardinal alignment is possible: in this case, aggregate surplus is a *cardinal potential* for the individual utilities (Jehiel, Meyer-ter-Vehn and Moldovanu, 2008).

Proving that Theorem 2.1 ii) is necessary for efficient implementation is much more difficult. The heart of our proof is concerned with situations with two agents on each side, and it exploits the implications of incentive compatibility on the part of employers for varying worker type profiles. Condition 2.1 ensures that the subset of types for which both feasible matchings are efficient is a well-behaved manifold. As mentioned earlier, our result is reminiscent of Roberts' (1979) Theorem that shows (under some relatively strong technical conditions) that any dominant-strategy implementable social choice function must maximize a weighted sum of individual utilities plus some alternative-specific constants. Both present assumptions and proof are quite different from Roberts'.[12]

[12] Our main technical result is derived by varying a social choice setting with only two alternatives (Roberts studied a single setting with at least three alternatives), surplus may take here general functional forms, and type spaces are arbitrary connected open sets (Roberts has linear utilities and needs an unbounded type space).

Remark 2.1. *Mezzetti (2004) has shown that efficiency is always (that is, in our context, for any given γ) attainable with two-stage "generalized Groves" mechanisms where a final allocation is chosen at stage one, and where, subsequently, monetary transfers that depend on the realized ex-post utilities of all agents at that allocation are executed at stage two.[13] In particular, such mechanisms would require ex-post transfers across all existing partnerships, contingent on the previously realized surplus in each of these pairs. We think that using ex-post information (whether reported or verifiable) to this extent is somewhat unrealistic in the present environment. For example, group manipulations by partners should be an issue for any mechanism that imposes ex-post transfers across pairs. Our fixed-proportion sharing also needs ex-post information, but uses it in a much more limited way to* divide surplus within pairs. *In particular, there are no contingent payments between pairs or to/from a potential matchmaker after partnerships have formed.*

Our second main result deals with the special case where agents' attributes are one-dimensional. If $n = 1$, then Condition 2.1 implies that $y \mapsto (\partial_x v)(x, y)$ is either strictly increasing or strictly decreasing. Consequently, v either has strictly increasing differences or strictly decreasing differences in (x, y).[14] That is, v is either strictly supermodular or strictly submodular. This is the classical one-dimensional assortative/anti-assortative framework à la Becker (1973). We treat here the supermodular case. The submodular one is analogous.

In the one-dimensional supermodular case we find that the class of sharing rules that is compatible with efficient matching is larger, and strictly contains the class of constant rules obtained above.

Theorem 2.2. *Let $n = 1$, $I, J \geq 2$ and assume that v is strictly supermodular. Then, the efficient matching is implementable in ex-post equilibrium if and only if both γv and $(1 - \gamma)v$ are supermodular.*

We derive Theorem 2.2 by applying a characterization result due to Bergemann and Välimäki (2002). These authors have provided a necessary as well as a set of sufficient conditions for efficient ex-post implementation for one-dimensional types. The logic of our proof is as follows. We first verify that monotonicity in the sense of Definition 4 of Bergemann and Välimäki is satisfied for strictly supermodular match surplus. This is the

[13] The generalized Groves mechanism has the problem that it does not provide *strict* incentives for truthful reporting of ex-post utilities.

[14] See also Topkis (1998).

first part of their set of sufficient conditions (Proposition 3). Then, we show that their necessary condition (Proposition 1) implies that γv and $(1-\gamma)v$ must be supermodular. Finally, we show that the second part of the sufficient conditions is satisfied as well if γv and $(1-\gamma)v$ are supermodular.

2.4 Conclusion

We have introduced a novel two-sided matching model with a finite number of agents, two-sided incomplete information, interdependent values, and multi-dimensional attributes. We have shown that fixed-proportion sharing rules are the only ones conducive for efficiency in this setting. While our present result is agnostic about the preferred proportion, augmenting our model with, say, a particular ex-ante investment game will introduce new, additional forces that can be used to differentiate between various constant sharing rules.

2.5 Appendix for Chapter 2

We prepare the proof of Theorem 2.1 by a sequence of lemmas. The key step is Lemma 2.4 below. It will be very useful to introduce a cross-difference (two-cycle) linear operator F, which acts on functions $f : X \times Y \to \mathbb{R}$. The operator F_f has arguments $x^1 \in X^1 = X$, $x^2 \in X^2 = X$, $y^1 \in Y^1 = Y$ and $y^2 \in Y^2 = Y$, and it is defined as follows:[15]

$$F_f(x^1, x^2, y^1, y^2) := f(x^1, y^1) + f(x^2, y^2) - f(x^1, y^2) - f(x^2, y^1).$$

We also define the sets

$$A := \{(x^1, x^2, y^1, y^2) \in X \times X \times Y \times Y | F_v(x^1, x^2, y^1, y^2) = 0\},$$

and

$$A_0 := \{(x^1, x^2, y^1, y^2) \in A | \nabla F_v(x^1, x^2, y^1, y^2) \neq 0\},$$

[15]We choose superscripts here because x_1 is already reserved for the type of employer e_1, and so on.

where

$$\nabla F_v(x^1, x^2, y^1, y^2) = (\nabla_{X^1} F_v, \nabla_{X^2} F_v, \nabla_{Y^1} F_v, \nabla_{Y^2} F_v)(x^1, x^2, y^1, y^2).$$

Whenever $x_1 \neq x_2$ or $y_1 \neq y_2$, Condition 2.1 implies that $\nabla F_v(x_1, x_2, y_1, y_2) \neq 0$. This is repeatedly used below.

Lemma 2.1. *Let $n \in \mathbb{N}$, $I = J = 2$, and let Condition 2.1 be satisfied. If the efficient matching is ex-post implementable, then the following implications hold for all (x_1, x_2, y_1, y_2):*

$$F_v(x_1, x_2, y_1, y_2) \geq (\leq)\, 0 \Rightarrow F_{\gamma v}(x_1, x_2, y_1, y_2) \geq (\leq)\, 0, \tag{2.1}$$

$$F_v(x_1, x_2, y_1, y_2) \geq (\leq)\, 0 \Rightarrow F_{(1-\gamma)v}(x_1, x_2, y_1, y_2) \geq (\leq)\, 0. \tag{2.2}$$

Proof of Lemma 2.1. There are only two alternative matchings, $m_1 = ((e_1, w_1), (e_2, w_2))$ and $m_2 = ((e_1, w_2), (e_2, w_1))$. Since the efficient matching is ex-post implementable, the taxation principle for ex-post implementation implies that there must be "transfer" functions $t^{e_1}_{m_1}(x_2, y_1, y_2)$ and $t^{e_1}_{m_2}(x_2, y_1, y_2)$ for employer e_1 such that

$$\begin{aligned} F_v(x_1, x_2, y_1, y_2) \quad &> \quad (<)\, 0 \Rightarrow \\ (\gamma v)(x_1, y_1) + t^{e_1}_{m_1}(x_2, y_1, y_2) \quad &\geq \quad (\leq)\, (\gamma v)(x_1, y_2) + t^{e_1}_{m_2}(x_2, y_1, y_2). \end{aligned} \tag{2.3}$$

For $y_1 \neq y_2$, we have $(\nabla_{X^1} F_v)(x_2, x_2, y_1, y_2) = (\nabla_X v)(x_2, y_1) - (\nabla_X v)(x_2, y_2) \neq 0$ by Condition 2.1. Hence, in every neighborhood of $x_1 = x_2$, there are x_1' and x_1'' such that $F_v(x_1', x_2, y_1, y_2) > 0$ and $F_v(x_1'', x_2, y_1, y_2) < 0$. Since γv is continuous, relation (2.3) pins down the difference of transfers as:

$$t^{e_1}_{m_1}(x_2, y_1, y_2) - t^{e_1}_{m_2}(x_2, y_1, y_2) = (\gamma v)(x_2, y_2) - (\gamma v)(x_2, y_1).$$

Plugging this back into (2.3) yields for all (x_1, x_2, y_1, y_2) with $y_1 \neq y_2$:

$$F_v(x_1, x_2, y_1, y_2) > (<)\, 0 \Rightarrow F_{\gamma v}(x_1, x_2, y_1, y_2) \geq (\leq)\, 0. \tag{2.4}$$

As $F_v(x_1, x_2, y, y) = F_{\gamma v}(x_1, x_2, y, y) = 0$, relation (2.4) holds for all (x_1, x_2, y_1, y_2). However, every neighborhood of any $(x_1, x_2, y_1, y_2) \in A$ contains both points at which

F_v is strictly positive and points at which F_v is strictly negative. Whenever $x_1 \neq x_2$ or $y_1 \neq y_2$, this follows immediately from $\nabla F_v(x_1, x_2, y_1, y_2) \neq 0$. Otherwise, if $x_1 = x_2$ and $y_1 = y_2$, one may perturb x_2 by an arbitrarily small amount to some x_2' (staying in A since $y_1 = y_2$) and apply the argument to (x_1, x_2', y_1, y_2).

Using continuity of γv, (2.4) may thus be strengthened to (2.1). A completely analogous argument applies for worker w_1 and yields (2.2). □

To prove Theorem 2.1, we only need local versions of (2.1) and (2.2) at profiles where the efficient matching changes. These are available for general $I, J \geq 2$:

Lemma 2.2. *Let $n \in \mathbb{N}$, $I, J \geq 2$ and let Condition 2.1 be satisfied. If the efficient matching is ex-post implementable, then for all $(x_1, x_2, y_1, y_2) \in A$, there is an open neighborhood $U_{(x_1,x_2,y_1,y_2)} \subset X \times X \times Y \times Y$ of (x_1, x_2, y_1, y_2) such that for all $(x_1', x_2', y_1', y_2') \in U_{(x_1,x_2,y_1,y_2)}$:*

$$F_v(x_1', x_2', y_1', y_2') \geq (\leq)\, 0 \Rightarrow F_{\gamma v}(x_1', x_2', y_1', y_2') \geq (\leq)\, 0, \tag{2.5}$$

$$F_v(x_1', x_2', y_1', y_2') \geq (\leq)\, 0 \Rightarrow F_{(1-\gamma)v}(x_1', x_2', y_1', y_2') \geq (\leq)\, 0. \tag{2.6}$$

Proof of Lemma 2.2. Given $(x_1, x_2, y_1, y_2) \in A$, fix the types of all other employers and workers (x_i for $i \neq 1, 2$, y_j for $j \neq 1, 2$) such that there is an open neighborhood $U_{(x_1,x_2,y_1,y_2)}$ of (x_1, x_2, y_1, y_2) with the following property: for all $(x_1', x_2', y_1', y_2') \in U_{(x_1,x_2,y_1,y_2)}$, the efficient matching for the profile $(x_1', x_2', x_3, ..., x_I, y_1', y_2', y_3, ..., y_J)$ either matches e_1 to w_1 and e_2 to w_2, or e_1 to w_2 and e_2 to w_1 (depending on the sign of $F_v(x_1', x_2', y_1', y_2')$). From here on, the proof parallels the one of Lemma 2.1. □

Lemma 2.2 has the immediate consequence that *on* A_0, the gradients of F_v, $F_{\gamma v}$ and $F_{(1-\gamma)v}$ must all point in the same direction:

Lemma 2.3. *Let $n \in \mathbb{N}$, $I, J \geq 2$ and let Condition 2.1 be satisfied. If the efficient matching is ex-post implementable, then there is a unique function $\lambda : A_0 \to [0, 1]$ satisfying*

$$\nabla F_{\gamma v}(x_1, x_2, y_1, y_2) = \lambda(x_1, x_2, y_1, y_2) \nabla F_v(x_1, x_2, y_1, y_2) \tag{2.7}$$

for all $(x_1, x_2, y_1, y_2) \in A_0$.

Proof of Lemma 2.3. Since $\nabla F_v(x_1, x_2, y_1, y_2) \neq 0$ for all $(x_1, x_2, y_1, y_2) \in A_0$, (2.5) yields a unique $\lambda(x_1, x_2, y_1, y_2) \geq 0$ with

$$\nabla F_{\gamma v}(x_1, x_2, y_1, y_2) = \lambda(x_1, x_2, y_1, y_2) \nabla F_v(x_1, x_2, y_1, y_2).$$

Moreover, $\nabla F_{(1-\gamma)v}(x_1, x_2, y_1, y_2) = (1 - \lambda(x_1, x_2, y_1, y_2)) \nabla F_v(x_1, x_2, y_1, y_2)$ and (2.6) therefore implies $\lambda(x_1, x_2, y_1, y_2) \in [0, 1]$. □

The crucial step in the proof follows now. It shows that for $n \geq 2$ the function λ must be constant. This constant corresponds then to a particular fixed-proportion sharing rule.

Lemma 2.4. *Let $n \geq 2$, $I, J \geq 2$ and let Condition 2.1 be satisfied. Then the function λ from Lemma 2.3 must be constant: there is a $\lambda_0 \in [0, 1]$ such that $\lambda \equiv \lambda_0$.*

Proof of Lemma 2.4. Let us spell out the system of equations (2.7):

$$\begin{aligned}
(\nabla_X \gamma v)(x_1, y_1) - (\nabla_X \gamma v)(x_1, y_2) &= \lambda(x_1, x_2, y_1, y_2)((\nabla_X v)(x_1, y_1) - (\nabla_X v)(x_1, y_2)) \\
(\nabla_X \gamma v)(x_2, y_2) - (\nabla_X \gamma v)(x_2, y_1) &= \lambda(x_1, x_2, y_1, y_2)((\nabla_X v)(x_2, y_2) - (\nabla_X v)(x_2, y_1)) \\
(\nabla_Y \gamma v)(x_1, y_1) - (\nabla_Y \gamma v)(x_2, y_1) &= \lambda(x_1, x_2, y_1, y_2)((\nabla_Y v)(x_1, y_1) - (\nabla_Y v)(x_2, y_1)) \\
(\nabla_Y \gamma v)(x_2, y_2) - (\nabla_Y \gamma v)(x_1, y_2) &= \lambda(x_1, x_2, y_1, y_2)((\nabla_Y v)(x_2, y_2) - (\nabla_Y v)(x_1, y_2)).
\end{aligned} \tag{2.8}$$

Given any $(x_1, x_2, y_1, y_2) \in A_0$, one obtains the same system of equations at $(x_2, x_1, y_1, y_2) \in A_0$, albeit for $\lambda(x_2, x_1, y_1, y_2)$. Thus, the function λ is symmetric with respect to x_1 and x_2. Similarly, it is symmetric with respect to y_1 and y_2. Next, for given $x_1 \in X$ and $y_1 \neq y_2$, the vectors in the first equation of (2.8) (with $(\nabla_X v)(x_1, y_1) - (\nabla_X v)(x_1, y_2) \neq 0$ on the right hand side) do not depend on how (x_1, y_1, y_2) is completed by x_2 to yield a full profile that lies in A_0. Consequently, $\lambda(x_1, x_2, y_1, y_2) = \lambda(x_1, x_1, y_1, y_2)$ for all these possible choices.

We next show that for a given x_1, λ does in fact not depend on y_1 and y_2 as long as $y_1 \neq y_2$. To this end, start with any $x_1 \in X$ and $y_1 \neq y_2$. We will show that for all $y_2' \neq y_1$ it holds

$$\lambda(x_1, x_1, y_1, y_2) = \lambda(x_1, x_1, y_1, y_2'). \tag{2.9}$$

Then, by symmetry of λ, $\lambda(x_1, x_1, y_1, y_2) = \lambda(x_1, x_1, y_2', y_1)$, and repeating the argument will yield that λ is indeed independent of y_1 and y_2 as long as $y_1 \neq y_2$.

So, let us prove (2.9). Using the first equation of (2.8), we have:

$$\begin{aligned} & \lambda(x_1, x_1, y_1, y_2)((\nabla_X v)(x_1, y_1) - (\nabla_X v)(x_1, y_2)) \\ = \; & ((\nabla_X \gamma v)(x_1, y_1) - (\nabla_X \gamma v)(x_1, y_2')) + ((\nabla_X \gamma v)(x_1, y_2') - (\nabla_X \gamma v)(x_1, y_2)) \\ = \; & \lambda(x_1, x_1, y_1, y_2')((\nabla_X v)(x_1, y_1) - (\nabla_X v)(x_1, y_2')) \\ + \; & \lambda(x_1, x_1, y_2', y_2)((\nabla_X v)(x_1, y_2') - (\nabla_X v)(x_1, y_2)). \end{aligned}$$

It follows that

$$\begin{aligned} & (\lambda(x_1, x_1, y_1, y_2') - \lambda(x_1, x_1, y_1, y_2))\,((\nabla_X v)(x_1, y_1) - (\nabla_X v)(x_1, y_2')) \\ + \; & (\lambda(x_1, x_1, y_2', y_2) - \lambda(x_1, x_1, y_1, y_2))\,((\nabla_X v)(x_1, y_2') - (\nabla_X v)(x_1, y_2)) \\ = \; & 0. \end{aligned} \tag{2.10}$$

Two cases must now be distinguished.

Case 1: $(\nabla_X v)(x_1, y_1) - (\nabla_X v)(x_1, y_2')$ and $(\nabla_X v)(x_1, y_2') - (\nabla_X v)(x_1, y_2)$ are linearly independent. Then, it follows from (2.10) that $\lambda(x_1, x_1, y_1, y_2') = \lambda(x_1, x_1, y_1, y_2)$.

Case 2: $(\nabla_X v)(x_1, y_1) - (\nabla_X v)(x_1, y_2')$ and $(\nabla_X v)(x_1, y_2') - (\nabla_X v)(x_1, y_2)$ are linearly dependent. In this case, pick some $y_2'' \in Y$ such that $(\nabla_X v)(x_1, y_1) - (\nabla_X v)(x_1, y_2'')$ and $(\nabla_X v)(x_1, y_2'') - (\nabla_X v)(x_1, y_2)$ are linearly independent. This is always possible since $(\nabla_X v)(x_1, \cdot)$ maps open neighborhoods of y_1 one-to-one into $\mathbb{R}^n$, and since for $n \geq 2$, there is no one-to-one continuous mapping from an open set in $\mathbb{R}^n$ to the real line $\mathbb{R}$.[16]

From Case 1, we obtain $\lambda(x_1, x_1, y_1, y_2'') = \lambda(x_1, x_1, y_1, y_2)$. Since $(\nabla_X v)(x_1, y_1) - (\nabla_X v)(x_1, y_2')$ and $(\nabla_X v)(x_1, y_2') - (\nabla_X v)(x_1, y_2'')$ are also linearly independent, we then get $\lambda(x_1, x_1, y_1, y_2') = \lambda(x_1, x_1, y_1, y_2'')$, and hence (2.9) follows.

The third equation of (2.8) may be now used in an analogous way to show that for a given y_1, $\lambda(x_1, x_2, y_1, y_1)$ does not depend on x_1 and x_2, as long as $x_1 \neq x_2$.

The final ingredient is the following observation: for every $(x_1, x_1, y_1, y_2) \in A_0$, there is a $x_2 \neq x_1$ with $(x_1, x_2, y_1, y_2) \in A_0$. Indeed, $(\nabla_{X^2} F_v)(x_1, x_1, y_1, y_2) \neq 0$, so that the set of x_2 for which $(x_1, x_2, y_1, y_2) \in A_0$ is given locally (in a neighborhood of $x_2 = x_1$) by

[16] This is a special case of Brouwer's (1911) classical dimension preservation result: for $k < m$, there is no one-to-one, continuous function from a non-empty open set U of $\mathbb{R}^m$ into $\mathbb{R}^k$.

a differentiable manifold of dimension $n-1$. Since $n \geq 2$, this manifold must contain points other than x_1. A similar argument applies to $(x_1, x_2, y_1, y_1) \in A_0$.

To conclude the proof, we show that λ is constant on $\{(x_1, x_2, y_1, y_2) \in A_0 | x_1 \neq x_2 \text{ and } y_1 \neq y_2\}$. This set is non-empty by the previous observation (and we have already seen that $\lambda(x_1, x_2, y_1, y_2) = \lambda(x_1, x_1, y_1, y_2)$ and $\lambda(x_1, x_2, y_1, y_2) = \lambda(x_1, x_2, y_1, y_1)$, so that λ is constant on all of A_0 then). Given any $(x_1, x_2, y_1, y_2), (x_1', x_2', y_1', y_2') \in A_0$ with $x_1 \neq x_2$, $y_1 \neq y_2$, $x_1' \neq x_2'$ and $y_1' \neq y_2'$, we have:

$$\begin{aligned}\lambda(x_1, x_2, y_1, y_2) &= \lambda(x_1, x_1, y_1, y_2) = \lambda(x_1, x_1, y_1', y_2') \\ &= \lambda(x_1, x_2'', y_1', y_2') = \lambda(x_1, x_2'', y_1', y_1') \\ &= \lambda(x_1', x_2', y_1', y_1') = \lambda(x_1', x_2', y_1', y_2'),\end{aligned}$$

where $x_2'' \neq x_1$ is any feasible profile completion for (x_1, y_1', y_2'). $\square$

We are now finally ready to prove Theorem 2.1.

Proof of Theorem 2.1. **ii)$\Rightarrow$ i):** Consider the case $I \leq J$. As in the proof of Lemma 2.1, we make use of the "taxation principle" for ex-post implementation. For employer e_i, and matching $m \in \mathcal{M}$ define $t_m^{e_i}(x_{-i}, y_1, ..., y_J) := \lambda_0 \sum_{l \neq i} v(x_l, y_{m(l)}) - h(y_{m(i)})$. Then, $(\gamma v)(x_i, y_{m(i)}) + t_m^{e_i}(x_{-i}, y_1, ..., y_J) = \lambda_0 \sum_{l=1}^{I} v(x_l, y_{m(l)}) + g(x_i)$, so that it is optimal for e_i to select a matching that maximizes aggregate welfare. Note that strict incentives for truth-telling can be provided only if $\lambda_0 > 0$. For worker w_j, define

$$\begin{aligned} t_m^{w_j}(x_1, ..., x_I, y_{-j}) := \quad & (1-\lambda_0) \sum_{k \in m(\mathcal{I}), k \neq j} v(x_{m^{-1}(k)}, y_k) \\ & + g(x_{m^{-1}(j)}) \mathbf{1}_{j \in m(\mathcal{I})} - h(y_j) \mathbf{1}_{j \notin m(\mathcal{I})}. \end{aligned}$$

Here, $\mathbf{1}_{j \in m(\mathcal{I})} = 1$ if $j \in m(\mathcal{I})$, and $\mathbf{1}_{j \in m(\mathcal{I})} = 0$ otherwise. Note that if $I = J$, then $j \in m(\mathcal{I})$ for all possible matchings m, so that the final (y_j-dependent) term always vanishes. If $I < J$, then h is constant by assumption, and the transfer does not depend on y_j. It follows that if w_j is matched in m, his utility is $((1-\gamma)v)(x_{m^{-1}(j)}, y_j) + t_m^{w_j}(x_1, ..., x_I, y_{-j}) = (1-\lambda_0) \sum_{k \in m(\mathcal{I})} v(x_{m^{-1}(k)}, y_k) - h(y_j)$. Otherwise, his utility is just $t_m^{w_j}(x_1, ..., x_I, y_{-j}) = (1-\lambda_0) \sum_{k \in m(\mathcal{I})} v(x_{m^{-1}(k)}, y_k) - h(y_j)$. Hence, it is optimal for w_j to select a matching that maximizes aggregate welfare, and strict incentives for truth-telling can be provided only if $\lambda_0 < 1$. This proves i) for $I \leq J$. The proof for the case $I \geq J$ is completely analogous.

i) $\Rightarrow$ ii): By Lemma 2.4, there is a $\lambda_0 \in [0,1]$ such that for all $x \in X$, $y_1, y_2 \in Y$ with $y_1 \neq y_2$ it holds (the profile may be completed to lie in A_0, e.g. by $x' = x$):

$$(\nabla_X \gamma v)(x, y_1) - (\nabla_X \gamma v)(x, y_2) \;=\; \lambda_0((\nabla_X v)(x, y_1) - (\nabla_X v)(x, y_2)).$$

Integrating along any path from x_2 to x_1 (X is open and connected in $\mathbb{R}^n$, hence path-connected) yields $F_{\gamma v}(x_1, x_2, y_1, y_2) = \lambda_0 F_v(x_1, x_2, y_1, y_2)$. Hence, by linearity of the operator F, we obtain that $F_{(\gamma - \lambda_0)v} \equiv 0$. A function of two variables has vanishing cross differences if and only if it is additively separable, so that we can write $(\gamma v)(x, y) = \lambda_0 v(x, y) + g(x) + h(y)$. This concludes the proof for the case where $I = J$.

It remains to prove that h must be constant if $I < J$ (the proof that g must be constant when $I > J$ is analogous). Given $y_1 \in Y$, Condition 2.1 implies that $(\nabla_Y v)(\cdot, y_1)$ vanishes at most in one point. Pick then any $x_1 \in X$ with $(\nabla_Y v)(x_1, y_1) \neq 0$. Set $y_2 = y_1$ and complete the type profile for ($i \neq 1$, $j \neq 1, 2$) such that, for an open neighborhood U of (y_1, y_1), the efficient matching changes only with respect to the partner of e_1 : either w_1 is matched to e_1 and w_2 remains unmatched, or w_2 is matched to e_1 and w_1 remains unmatched. For $(y_1', y_2') \in U$, it follows that $v(x_1, y_1') - v(x_1, y_2') \geq (\leq)\ 0$ implies $((1-\gamma)v)(x_1, y_1') - ((1-\gamma)v)(x_1, y_2') \geq (\leq)\ 0$. Hence, there is a $\mu(x_1, y_1) \geq 0$ such that

$$(1 - \lambda_0)(\nabla_Y v)(x_1, y_1) - (\nabla_Y h)(y_1) = \mu(x_1, y_1)(\nabla_Y v)(x_1, y_1).$$

In other words, $(\nabla_Y h)(y_1)$ and $(\nabla_Y v)(x_1, y_1)$ are linearly dependent. Finally, let x_1 vary and note that, by Condition 2.1, the image of $(\nabla_Y v)(\cdot, y_1)$ cannot be concentrated on a line (recall footnote 16). Thus, we obtain that $(\nabla_Y h)(y_1) = 0$. Since y_1 was arbitrary and Y is connected, it follows that the function h must constant. $\square$

Proof of Theorem 2.2. Let $I \leq J$ (the proof for $I \geq J$ is analogous). Consider some $i \in \mathcal{I}$ and a given, fixed type profile for all other agents $(x_{-i}, y_1, ..., y_J)$. Given any such type profile, we re-order the workers and employers other than i such that $x^{(1)} \geq ... \geq x^{(I-1)}$ and $y^{(1)} \geq ... \geq y^{(J)}$.

We now verify the monotonicity condition identified by Bergemann and Välimäki.[17]

[17] We only verify it for type profiles for which all these inequalities are strict. When some types coincide, it is still straightforward to verify monotonicity but we do not spell out the more cumbersome case distinctions here.

This requires that the set of types of agent i for which a particular social alternative is efficient forms an interval. Let then m_k, $k = 1, ..., I$ denote the matching that matches $x^{(l)}$ to $y^{(l)}$ for $l = 1, ..., k-1$, x_i to $y^{(k)}$ and $x^{(l)}$ to $y^{(l+1)}$ for $l = k, ..., I-1$. Then, for $k = 2, ..., I-1$ it holds that the set

$$\{x_i \in X | u_{m_k}(x_1, ..., x_I, y_1, .., y_J) \geq u_m(x_1, ..., x_I, y_1, .., y_J) \ , \forall m \in \mathcal{M}\}$$

is simply $[x^{(k)}, x^{(k-1)}]$. For $k = I$ the set is $(\inf X, x^{(I-1)}]$, and for $k = 1$ it is $[x^{(1)}, \sup X)$. Monotonicity for workers j is verified in the same way.

Next, the necessary condition of Bergemann and Välimäki, spelled out for our matching model, requires that at all "switching points" $x_i = x^{(k-1)}$ where the efficient allocation changes, it also holds that

$$\frac{\partial}{\partial x_i}((\gamma v)(x_i, y^{(k-1)}) - (\gamma v)(x_i, y^{(k)})) \geq 0.$$

Given x_i and $y' > y$ we can always complete these to a full type profile such that x_i is a change point at which the efficient match for x_i switches from y to y'. Hence $\frac{\partial}{\partial x}((\gamma v)(x, y') - (\gamma v)(x, y)) \geq 0$ for all x and $y' > y$. So, γv must have increasing differences, i.e. it is supermodular. Since $\frac{\partial}{\partial x_i}((\gamma v)(x_i, y^{(k-1)}) - (\gamma v)(x_i, y^{(k)})) \geq 0$ is satisfied for all $x_i \in X$ (not just at switching points!), the second part of the sufficient conditions of Bergemann and Välimäki is satisfied. The argument for workers (yielding supermodularity of $(1-\gamma)v$) is analogous. This completes the proof. $\square$

Chapter 3

On the optimality of small research tournaments

I study two open problems with regard to the optimal number of participants in the research tournament model of Fullerton and McAfee (1999), in which firms are asymmetric with respect to their marginal effort cost. I derive a sharp bound for the possible cost inefficiency associated with a tournament of size 2*, if costs are common knowledge and the procurer can charge non-discriminatory entry fees. The analysis generally supports arranging a small tournament with the two most efficient firms, but it also identifies some notable exceptions. If costs are private information of ex-ante symmetric firms prior to the tournament, Fullerton and McAfee's contestant selection auction has to be used to select the most efficient candidates, and to raise money in advance. I discuss the procurer's problem of stimulating a given expected aggregate research effort at lowest expected total cost by choosing the optimal tournament size. A closed form solution is derived for the case where marginal costs are uniformly distributed on* $[0, \bar{c}]$*. The result strongly favors the smallest possible tournament with two participants.*

3.1 Introduction

Research tournaments, or contests, are widely used as mechanisms to procure innovations. They may mitigate many of the problems that plague traditional procurement contracts in this case, such as non-verifiable quality of the innovation, or difficulties to monitor and verify the efforts and costs of suppliers. Properly designed contests successfully foster competition between potential suppliers, while requiring relatively little information on the part of the procurer.[1] In an influential paper, Fullerton and McAfee (1999) (henceforth (FM)) studied how to design a fixed-prize research tournament when firms/suppliers are heterogenous (or *asymmetric*) with respect to their cost of exerting research effort, and when the research technology is *stochastic*. (FM) focused on how asymmetries affect the answers to two major design questions for the procurer: how many participants should be admitted to the tournament? How should contestants be selected from a pool of n candidates when costs are *private information* prior to the tournament? The purpose of this chapter is to address two open questions with regard to the optimal number of contestants. One problem concerns the complete information case, while the other, more fundamental one deals with the case of private information. Both problems arise naturally from the work of (FM), and my analysis builds on their model and results. I therefore summarize these before explaining the open problems and the results that I obtain.

(FM) provided a complete characterization of firms' equilibrium research efforts and expected profits in a fixed-prize tournament with prize P, when the stochastic research technology is of the "independent draws" type (leading to a standard Tullock success function, see also Baye and Hoppe, 2003), and when the heterogeneous effort costs of all m participants are common knowledge. Having more firms in the tournament increases aggregate research effort (which determines the distribution of the quality of the best innovation) but decreases equilibrium profits and hence firms' valuations for entering the tournament. Taking this tradeoff into account, the procurer's optimization problem involves stimulating given levels of aggregate research effort at the lowest possible total cost. For the case where the procurer can set *non-discriminatory entry fees*, (FM) derived a condition on the structure of heterogeneity which ensures that it is optimal to host a very small tournament with the two most efficient firms only.[2] These are selected by setting an appropriate entry fee.

Even if one has a clear idea what the optimal number of participants is, another key issue arises when effort costs are private information before the tournament: how can one make sure that the most efficient firms enter? (FM) advocated the use of an all-pay

[1] See for instance Che and Gale (2003) and Fullerton and McAfee (1999) for very readable accounts of some of the most appealing features of contests and tournaments. These papers also provide a number of interesting examples.

[2] The condition is sufficient even if duplication of fixed costs plays no role.

entry auction with a small interim prize (the “contestant selection auction”), in which firms bid for participation in the final tournament (with prize P and m participants). They analyzed a setup of incomplete information with ex-ante symmetric firms whose actual types, i.e. their marginal costs of conducting research, are drawn from a common and commonly known distribution. Firms’ types are private information prior to entry. However, in case a firm is admitted, it learns about the costs of its (small number of) competitors, so that the research tournament with asymmetric firms and complete information can be used as a building block for the analysis. (FM) identified conditions on the distribution of types under which standard discriminatory-price and uniform-price entry auctions do not possess an efficient symmetric equilibrium, i.e. an equilibrium that always selects the candidates with the lowest costs. In this sense, these auctions may fail as selection mechanisms. The authors nicely demonstrated that, in sharp contrast, the contestant selection auction is much more likely to have an efficient equilibrium. In particular, this is always the case when types are independent.

I now describe the two open problems. First, while the sufficient condition for optimality of $m = 2$ in the complete information model is not implausible, there are also many interesting cases in which it is violated. In particular, this often happens when firms’ costs are drawn from a common distribution as described above. I therefore drop the condition entirely. Instead, I derive a sharp bound for the loss that may possibly occur - compared to the optimal tournament size - if entry is restricted to only two firms. I also illustrate the meaningfulness of the numerical value of this bound. The analysis confirms that setting $m = 2$ is a good idea in many cases. However, it also provides an intuitive answer as to when admitting more contestants may be particularly beneficial. Roughly speaking, this is the case if a) there is a substantial but not too extreme cost asymmetry between the two most efficient firms and b) there are other firms which are (almost) as efficient as the second best firm.

Second and importantly, it is not clear why picking a tournament size of $m = 2$ should be a good idea, or even optimal, when firms’ costs are private information. In this case, setting optimal entry fees is infeasible: the contestant selection auction has to be used to select participants efficiently and to collect money before the tournament, in the form of all-pay bids. Both (FM)’s sufficient condition and the loss analysis carried out in Section 3.2.2 of this chapter rely on complete information, and in particular on the procurer’s ability to set optimal non-discriminatory entry fees. Therefore, I further investigate the incomplete information model that was used by (FM) to argue in favor of the contestant selection auction. I discuss the expression that must be analyzed for a designer's objective of "procuring" a given expected aggregate effort at lowest expected total cost and find a closed form solution for the important special case in which marginal costs are uniformly distributed on $[0, \bar{c}]$ (see Section 3.3 for additional motivation). This exercise turns out to be fairly involved but still analytically tractable (see the proof of Theorem 3.3). The results strongly support setting $m = 2$.

A comprehensive review of the literature on the optimal design of contests would go well beyond the scope of this chapter. Still, let me mention some related papers. These also provide further references and point out connections with several important strands of the economic literature, such as the literature on labor tournaments following Lazear and Rosen (1981), and the large body of work on all-pay auctions under either complete or incomplete information. Very closely related to (FM) is a seminal contribution for the case of symmetric firms by Taylor (1995), who also found that limiting the number of contestants is beneficial. Fullerton, Linster, McKee, and Slate (2002) and Schöttner (2008) compared fixed-prize tournaments to first-price auctions (firms bid a combination of quality and price after having conducted research) in two different models with stochastic research technology and symmetric firms. Che and Gale (2003) studied a model with *deterministic* technology, complete information and (possibly) asymmetric firms. They showed that inviting the two most efficient firms to participate in a first-price auction (handicapping the better firm by a maximum allowable price) is optimal within a much broader class of possible contests. In a model with a given number of ex-ante symmetric participants, deterministic technology and privately known cost types, Moldovanu and Sela (2001) demonstrated that it may be better to award multiple prizes rather than just a single one if the cost function is convex. However, a single prize is better for concave or linear costs (as is the case here).

The chapter is organized as follows. Section 3.2.1 describes (FM)'s complete information tournament model with asymmetric firms, introduces notation, and collects the known results that are needed later on. Section 3.2.2 studies the complete information model without the sufficient condition of (FM). In particular, the sharp bound mentioned above is developed. Section 3.3 contains the analysis and results for the incomplete information model, in which the contestant selection auction is used to determine participants. All proofs that are not in the main text may be found in Section 3.4.

3.2 Optimal tournament size under complete information

3.2.1 The model and some known results

In this section, I present the basic complete information tournament model, as well as the results from (FM) that will be used.[3] I adopt their notation so as to facilitate a comparative reading.

Risk-neutral firms $j = 1, ..., n$ ($n \geq 2$) have costs for making an effort $z_j \in \mathbb{R}_+$ when they take part in a simultaneous-move research tournament with m participants and prize $P > 0$. Costs are given by $\gamma + c_j z_j$ (where $\gamma \geq 0$ and $c_j > 0$) if $z_j > 0$, and by 0 if $z_j = 0$.[4] If firm j exerts effort z_j, it produces an innovation of random quality $x_j \in [0, \bar{x}]$, whose c.d.f. is given as $F^{z_j}(x_j)$. This simple stochastic research technology corresponds to a continuous version of modeling independent draws from the distribution F, which is assumed to be absolutely continuous with respect to Lebesgue measure. Firms' research activities are independent, so that F^Z is the distribution function of the winning quality, where $Z = \sum_j z_j$ is the aggregate effort. In this model, winning probabilities are determined by a standard Tullock success function. That is, the probability that firm j produces the highest quality and wins the tournament (and hence the prize P) is $\frac{z_j}{Z}$. Each others' costs are common knowledge among the m participants.

(FM) showed the following: if M denotes the set of contestants ($|M| \leq m$) who choose strictly positive effort in equilibrium, then for each $i \in M$, effort z_i and expected profit π_i before substracting γ are given by

$$z_i = \frac{P(|M| - 1)}{\sum_{j \in M} c_j} \left[1 - \frac{c_i(|M| - 1)}{\sum_{j \in M} c_j} \right],$$

$$\pi_i = P \left[1 - \frac{c_i(|M| - 1)}{\sum_{j \in M} c_j} \right]^2.$$

If $\gamma > 0$, the additional constraint $\pi_i \geq \gamma$ must be satisfied for all $i \in M$. (FM) showed that there is a unique set of lowest-cost contestants who choose strictly positive effort in equilibrium if $\gamma = 0$ (which is the case that I consider below). There may be some

[3] For more details, their paper should be consulted.

[4] The fixed cost $\gamma \geq 0$ is incurred only if $z_j > 0$.

ambiguity if $\gamma > 0$. However, even in that case one may essentially restrict attention to the equilibrium in which only (a γ-dependent number of) lowest-cost contestants actively participate.[5] For convenience, and following (FM), I label all n firms such that marginal costs are ordered: $c_1 \leq c_2 \leq ... \leq c_n$.

It is intuitive, and confirmed by the model, that high fixed costs are a strong reason for having a small number of participants. To enhance a clear analysis of the *consequences of asymmetries in marginal costs* for the optimal size of the tournament, I eliminate this effect and set $\gamma = 0$. Note that

$$Z = \sum_{i \in M} z_i = \frac{P(|M| - 1)}{\sum_{j \in M} c_j}, \tag{3.1}$$

and that it is profitable for firm m to be in a tournament with $1, ..., m-1$ if and only if

$$\frac{(m-1)c_m}{\sum_{j=1}^{m} c_j} < 1. \tag{3.2}$$

This is condition (5) in (FM). It is straightforward to see that the left hand side of (3.2) is strictly increasing in m as long as it is smaller than 1, and that if $2 \leq \bar{m} \leq n$ is the maximal number such that (3.2) is satisfied for all $m \leq \bar{m}$, then the condition is violated for all k with $n \geq k > \bar{m}$ (compare (FM)).

For a given P, varying the number of participants from 2 up to $\bar{m}$ goes along with increasing total effort. Indeed, let $m + 1 \leq \bar{m}$, so that (3.2) implies $c_{m+1} < \frac{\sum_{j=1}^{m+1} c_j}{m}$ which is also equivalent to $c_{m+1} < \frac{\sum_{j=1}^{m} c_j}{m-1}$. For the comparison of total efforts with $m+1$ and m contestants this yields

$$\frac{Pm}{\sum_{j=1}^{m+1} c_j} > \frac{Pm}{\frac{m}{m-1} \sum_{j=1}^{m} c_j} = \frac{P(m-1)}{\sum_{j=1}^{m} c_j}.$$

On the other hand, firms' equilibrium profits decrease with more participants, which diminishes the procurer's ability to collect money in advance, through entry fees or entry auctions, that may be used to partly finance the prize.

As (FM) noted, with complete information the procurer can charge a non-discriminatory

[5] See Lemma 1 in (FM).

entry fee of (slightly below)

$$E = P\left[1 - \frac{c_m(m-1)}{\sum_{j=1}^{m} c_j}\right]^2$$

to induce a tournament with exactly $m \leq \bar{m}$ low-cost participants. If he wants to motivate a target effort of Z, by (3.1) he must set $P = \frac{Z\sum_{j=1}^{m} c_j}{m-1}$. Hence, the total cost of "procuring" effort Z with $m \leq \bar{m}$ competitors is given by

$$\begin{aligned} TC_m = P - mE &= \frac{Z\sum_{j=1}^{m} c_j}{m-1}\left(1 - m\left[1 - \frac{c_m(m-1)}{\sum_{j=1}^{m} c_j}\right]^2\right) \\ &= Z\sum_{j=1}^{m} c_j\left(-1 + 2\Delta_m - \frac{m-1}{m}\Delta_m^2\right), \end{aligned} \tag{3.3}$$

where $\Delta_m = \frac{mc_m}{\sum_{j=1}^{m} c_j}$. In the symmetric case where all costs are equal to the same c, all TC_m are equal to Zc (just plug in $\Delta_m = 1$). So, the interesting questions in the non-discriminatory complete information framework are the following. How do asymmetries in marginal costs (which imply that some firms earn net profits) affect the TC_m? Do asymmetries generally favor any particular number of participants, do they at least tend to do so? A theorem of (FM) partly answers these questions.

Theorem 3.1. *(Theorem 3 of (FM)) If Δ_m is nondecreasing in m, then the total cost TC_m of stimulating a given aggregate effort level Z is minimized at $m = 2$.*

3.2.2 A sharp bound for general asymmetric cost structures

The sufficient condition of Theorem 3.1 requires that marginal costs are separated from each other by certain gaps. It has some appeal, but there are also many cases of interest in which it does not hold. For instance, the condition does not apply whenever one of the firms is a sufficiently close competitor for another one. Most importantly, the condition is violated for many realizations when firms' costs are randomly drawn from an ex-ante distribution (see Section 3.3). I do not make any assumptions about the structure of marginal costs (reflected by the ratios Δ_m) here. Consequently, $m = 2$ is not always optimal. However, it is possible to derive a sharp, worst case, lower bound for the ratio of the optimal TC_m over TC_2. This bound is quite close to 1 while, in contrast, having too many firms in the tournament can be very expensive for some cost structures (see Example 3.1 for an illustration). In this sense, hosting a tournament with the two most efficient firms is generally not a bad idea.[6] However, the analysis also yields the following insights as by-products. First, it is better to have more than two contestants whenever there exist close competitors for firm 2. Secondly, increasing the number of participants can be really profitable only for intermediate degrees of asymmetry between firms 1 and 2, and the effect is maximal for $\Delta_2 = \sqrt{2}$ (see Remark 3.1).

Theorem 3.2. *It holds*

$$\begin{aligned}\inf_{n\geq 2,\,0<c_1\leq\ldots\leq c_n,\,1\leq m<\bar{m}} \frac{TC_{m+1}}{TC_2} &= \inf_{\Delta_2\in[1,2)} \frac{1}{2\left[2\left(1-\sqrt{\frac{1}{2}}\right)-\left(\sqrt{\frac{1}{2}\Delta_2}-\sqrt{\frac{1}{\Delta_2}}\right)^2\right]} \\ &= \frac{1}{4\left(1-\sqrt{\frac{1}{2}}\right)} \approx 0.85. \end{aligned} \tag{3.4}$$

To prove Theorem 3.2, I first rewrite formula (3.3) in a form that is more suitable for examining the ratios $\frac{TC_{m+1}}{TC_2}$, $\bar{m} > m \geq 1$.

$$\begin{aligned} TC_{m+1} &= Z\sum_{j=1}^{m+1} c_j\left(-1+2\Delta_{m+1}-\frac{m}{m+1}\,\Delta_{m+1}^2\right) \\ &= Z\frac{(m+1)c_{m+1}}{\Delta_{m+1}}\left[2\left(1-\sqrt{\frac{m}{m+1}}\right)\Delta_{m+1}-\left(\sqrt{\frac{m}{m+1}}\Delta_{m+1}-1\right)^2\right] \\ &= Z(m+1)c_{m+1}\left[2\left(1-\sqrt{\frac{m}{m+1}}\right)-\left(\sqrt{\frac{m}{m+1}\Delta_{m+1}}-\sqrt{\frac{1}{\Delta_{m+1}}}\right)^2\right]. \end{aligned}$$

This yields for $\bar{m} > m \geq 1$,

$$\begin{aligned} \frac{TC_{m+1}}{TC_2} &= \frac{(m+1)c_{m+1}}{2c_2}\,\frac{\left[2\left(1-\sqrt{\frac{m}{m+1}}\right)-\left(\sqrt{\frac{m}{m+1}\Delta_{m+1}}-\sqrt{\frac{1}{\Delta_{m+1}}}\right)^2\right]}{\left[2\left(1-\sqrt{\frac{1}{2}}\right)-\left(\sqrt{\frac{1}{2}\Delta_2}-\sqrt{\frac{1}{\Delta_2}}\right)^2\right]} \\ &= \left(\prod_{j=2}^{m}\alpha_j\right)\frac{(m+1)\left[2\left(1-\sqrt{\frac{m}{m+1}}\right)-\left(\sqrt{\frac{m}{m+1}\Delta_{m+1}}-\sqrt{\frac{1}{\Delta_{m+1}}}\right)^2\right]}{2\left[2\left(1-\sqrt{\frac{1}{2}}\right)-\left(\sqrt{\frac{1}{2}\Delta_2}-\sqrt{\frac{1}{\Delta_2}}\right)^2\right]}. \end{aligned} \tag{3.5}$$

[6] The argument is reinforced by the observations that the bound is developed for the case $\gamma = 0$, and that there may be important additional costs related to conducting the tournament (evaluation, etc.) that lie beyond the framework of the model.

Here $\alpha_j := \frac{c_{j+1}}{c_j}$, and an empty product is equal to 1 by convention. By the ordering of costs, $\Delta_{m+1} \geq 1$. On the other hand, $\frac{m}{m+1}\Delta_{m+1} < 1$ is necessary for firm $m+1$ to enter and to exert positive effort (i.e., for $m < \bar{m}$). The following lemma establishes a straightforward recursive formula for Δ_m, as well as a necessary and sufficient condition for the implication ($m \leq \bar{m} \Rightarrow m+1 \leq \bar{m}$) in terms of the sequence of marginal costs.

Lemma 3.1. *i) It holds*

$$\Delta_{m+1} = \frac{(m+1)c_{m+1}}{mc_m + \Delta_m\, c_{m+1}}\, \Delta_m.$$

ii) If $\frac{m-1}{m}\,\Delta_m < 1$, then

$$\frac{m}{m+1}\,\Delta_{m+1} < 1 \;\Leftrightarrow\; c_{m+1} < \left(\frac{m-1}{m}\,\Delta_m\right)^{-1} c_m.$$

Consequently, each structure of marginal costs that may lead to a situation in which $m+1$ firms participate in equilibrium is characterized by a sequence of α_j satisfying

$$\alpha_j \in \left[1, \left(\frac{j-1}{j}\,\Delta_j\right)^{-1}\right) \text{ for } j = 2, ..., m.$$

This is all information that is available a priori for studying how small the expression (3.5) may get for arbitrary sequences of marginal costs (that do not satisfy an additional monotonicity property like in Theorem 3.1). At first glance, it thus seems rather difficult to find a sharp lower bound for the ratio (3.5). The next observation drastically simplifies this task.

Lemma 3.2. *Consider any $2 \leq m \leq \bar{m}$. If it holds in addition that $c_{m+1} = c_m$, then $TC_{m+1} \leq TC_m$, i.e. it is weakly better to have a tournament with $m+1$ rather than m participants. The inequality is strict unless $\Delta_m = 1$.*

Lemma 3.2 enables the following proof of Theorem 3.2.

Proof of Theorem 3.2. Lemma 3.2 shows that for any situation in which a tournament with $m > 2$ firms is better than one with 2, one may construct a case in which total costs are even lower with $m+1$ firms (by adding another firm whose marginal cost equals that of firm m).

Returning to the expression for the ratio of total costs (3.5), $\frac{m}{m+1}\Delta_{m+1} \in \left[\frac{m}{m+1}, 1\right)$ and $\Delta_{m+1}^{-1} \in \left(\frac{m}{m+1}, 1\right]$ imply

$$\left(\sqrt{\frac{m}{m+1}\Delta_{m+1}} - \sqrt{\frac{1}{\Delta_{m+1}}}\right)^2 = O\left(\frac{1}{m^2}\right).$$

This follows from $\sqrt{x} = 1 + \frac{1}{2}(x-1) + O((x-1)^2)$ as $x \to 1$, which furthermore yields

$$1 - \sqrt{\frac{m}{m+1}} = \frac{1}{2(m+1)} + O\left(\frac{1}{m^2}\right).$$

Hence, asymptotically only the latter term determines the numerator in (3.5). For a given $\Delta_2 \in [1,2)$, one may thus reach convergence to the infimum of (3.5) over all admissible cost structures by setting $\alpha_j = 1$ for all $j \geq 2$. The infimum is given by

$$\lim_{m\to\infty} \frac{2(m+1)\left(1-\sqrt{\frac{m}{m+1}}\right)}{2\left[2\left(1-\sqrt{\frac{1}{2}}\right) - \left(\sqrt{\frac{1}{2}\Delta_2} - \sqrt{\frac{1}{\Delta_2}}\right)^2\right]} = \frac{1}{2\left[2\left(1-\sqrt{\frac{1}{2}}\right) - \left(\sqrt{\frac{1}{2}\Delta_2} - \sqrt{\frac{1}{\Delta_2}}\right)^2\right]}.$$

This expression is minimized at $\Delta_2 = \sqrt{2}$ with value $\frac{1}{4\left(1-\sqrt{\frac{1}{2}}\right)}$. □

Remark 3.1. *As indicated above, the proof yields some additional insights: whether it can be really profitable to allow entry of more than two firms depends crucially on the the asymmetry between the two strongest firms. In the extremal cases of equal strength ($\Delta_2 = 1$) and drastic superiority of firm 1 ($\Delta_2 \to 2$), $m = 2$ is always optimal. Indeed, in these cases the denominator of the ratio* (3.5) *is equal to* 1. *The closer the asymmetry is to the geometric mean of these extremes, the more profitable it may be to allow for more contestants to increase competition, but only if there are other firms which are (almost) as strong as firm 2.*

To conclude this section, I construct an ad hoc example which shows that the cost inefficiency from having too many firms in the tournament may be significantly higher than the one associated with the worst case bound of Theorem 3.2.

Example 3.1. *Consider $\Delta_j = 1 + \frac{j-2}{m-1}\frac{1}{m}$ for $j = 2, ..., m+1$, and for $m = 5$.*[7] *Then $\frac{TC_6}{TC_2} \approx 1.38$.*

[7]For a fully rigorous argument, Δ_{m+1} has to be slightly below $\frac{m+1}{m}$.

3.3 Incomplete information, (FM)'s contestant selection auction, and optimal tournament size

In this section, I study the optimal number of participants for the model of *incomplete information* with ex-ante symmetric firms that was used by (FM) to argue in favor of the contestant selection auction. The crucial observation, which seems to have gone unnoticed in (FM), is that a separate and more involved analysis is needed for that purpose. Both Theorem 3.1 and the results of Section 3.2.2 have been obtained assuming complete information and the use of non-discriminatory entry fees. Therefore, these results are not applicable (even though they are useful to guide intuition).

The type of each firm, i.e. its marginal cost, is drawn ex-ante from a common and commonly known distribution H with support $[\underline{c}, \bar{c}] \subset \mathbb{R}_+$ and density h. Types are private information prior to the tournament. However, they become common knowledge among the selected contestants before effort choices are made, so that efforts and profits can be computed as in Section 3.2.1. (FM) showed that the contestant selection auction, an all-pay entry auction with a small interim prize for entry, performs well in selecting the most efficient contestants under incomplete information. In contrast, entry fees, uniform-price auctions and discriminatory-price auctions may perform very poorly (see also Section 3.1). In particular, for independent types the contestant selection auction always has a symmetric equilibrium with a bidding function that is strictly decreasing in marginal cost (Theorem 5 of (FM)). Moreover, for every P and m, the expected total cost of conducting the tournament, which takes the revenue from the pre-tournament auction into account, is then formally equivalent to the expression that would arise for an efficient uniform-price auction (Theorem 6 of (FM)). To be precise about the latter point, let $h_{n-m,n}$ denote the density of the $m+1$st - lowest out of n independent draws from H. Moreover, let $\psi(c, c)$ denote the conditional expected profit of a firm of type c which enters the tournament as the weakest one of m contestants, and which happens to have the same cost as the mth-strongest out of all remaining $n-1$ candidates (the "marginal" firm).[8] Lemma 3.3 below follows easily from Theorem 6 of (FM).[9]

[8] $\psi(c, c)$ is the usual candidate for equilibrium bidding in a uniform-price auction.

[9] For completeness, a proof is included in Section 3.4. The reason is that (FM) presented their results in a slightly different model, in which firms/agents are characterized by an ability parameter w. w and c are related by $w = \frac{1}{c}$.

Lemma 3.3. *With independent types, the expected total cost of conducting a tournament with prize P, when the m participants are determined by means of (FM)'s contestant selection auction, is*

$$P - m \int_{\underline{c}}^{\bar{c}} \psi(c,c)\, h_{n-m,n}(c)\, dc. \tag{3.6}$$

For the rest of this chapter, it will be convenient to *drop the assumption* that $c_1,...,c_{m-1}$ are ordered. Spelling out $\psi(c,c)$ explicitly for $\gamma = 0$, one finds (compare (FM), where the "max" was forgotten):

$$\psi(c,c) = \int_{\underline{c}}^{c} ... \int_{\underline{c}}^{c} P \max\left(1 - \frac{c(m-1)}{c + \sum_{j=1}^{m-1} c_j}, 0\right)^2 \prod_{j=1}^{m-1} \frac{h(c_j)}{H(c)}\, dc_1...dc_{m-1}. \tag{3.7}$$

Absent further (and necessarily special) assumptions about the procurer's utility as a function of i) the quality of the winning innovation, and ii) monetary costs, the most natural and interesting optimization problem is the following, which is analogous to the one in the complete information case: "procure" a given *expected* effort $\bar{Z}$ at the lowest possible expected total cost. Remember from (3.1) that the relationship between prize P, total equilibrium effort $Z = \sum_{j \in M} z_j$ (where M is the set of firms that make strictly positive effort) and marginal costs is $Z = \frac{P(|M|-1)}{\sum_{j \in M} c_j}$. Thus, to target an expected effort of $\bar{Z}$ with m participants, the procurer must set

$$P = \frac{\bar{Z}}{E_n \left[\frac{|M|-1}{\sum_{j \in M} c_j} \,\middle|\, |M| \le m \right]}.$$

The conditional expectation in the denominator is the expected total effort made by the m lowest cost firms (which are selected by the all-pay auction) in a tournament with prize 1. The subscript n indicates the dependence of this expression on the total number of firms. Plugging $\psi(c,c)$ and the relationship for P into (3.6), one obtains an expression for the expected total cost of procuring $\bar{Z}$ from m contestants.

$$ETC^{\bar{Z}}_{m,n} := \frac{\bar{Z}}{E_n\left[\frac{|M|-1}{\sum_{j\in M} c_j} \,\middle|\, |M| \leq m\right]} \times$$
$$\int_{\underline{c}}^{\bar{c}} \int_{[\underline{c},c]^{m-1}} \left[1 - m\max\left(1 - \frac{c(m-1)}{c + \sum_{j=1}^{m-1} c_j}, 0\right)^2\right] \prod_{j=1}^{m-1} \frac{h(c_j)}{H(c)}\, dc_1 ... dc_{m-1}\, h_{n-m,n}(c)\, dc. \tag{3.8}$$

While this looks similar to the first line of (3.3), formula (3.8) is not just an average of the TC_m. Rather, the expectations of the cost of conducting a tournament with prize $P = 1$ and of the corresponding total effort are taken separately.

It seems very hard to find general conditions that relate properties of H to the optimal m, for each given n. In the remainder of this chapter, I solve the important special case $H \sim U[0, \bar{c}]$ explicitly (without further loss of generality, $\bar{c} = 1$). *The results show that the smallest possible tournament, i.e. $m = 2$, is by far the most cost efficient one.* Let me give some motivation first. $\underline{c} = 0$ is important to make the problem interesting. Indeed, the assumptions of incomplete information prior to the tournament and complete information after entry seem reasonable only if n is quite large. On the other hand, if $\underline{c} > 0$, then for given H and large n, there are no significant asymmetries between the lowest-cost firms. Without fixed costs, it then does not really matter whether e.g. $m = 2$ or $m = 3$: if $\underline{c} > 0$, it holds that

$$\lim_{n\to\infty} ETC^{\bar{Z}}_{m,n} = \frac{\bar{Z}\, m\underline{c}}{m-1}\left[1 - m\left(1 - \frac{m-1}{m}\right)^2\right] = \bar{Z}\underline{c}.$$

In contrast, $\underline{c} = 0$ may induce an interesting problem where asymmetries between the few best candidates remain important even though the total number of firms is large. The case $H \sim U[0,1]$ has nice additional homogeneity properties. In particular, certain conditional expectations of expressions that depend on marginal costs relative to each other only, such as $\frac{c(m-1)}{c+\sum_{j=1}^{m-1} c_j}$, do not depend on n. This is very helpful for computations.

In Theorem 3.3 below, I do the main step towards solving the optimization problem by deriving a closed form solution for $ETC^{\bar{Z}}_{m,n}$.

Theorem 3.3. *Let $H \sim U[0,1]$. Then for any $n \geq 2$ and $2 \leq m \leq n$, the expected total*

m	κ_m	$\ln 2 + \sum_{j=3}^{m} \frac{\beta_j}{(j-1)!}$	$\frac{n}{\bar{Z}}\, ETC_{m,n}^{\bar{Z}}$
2	0.113706	0.693147	1.11461
3	0.012558	0.716395	1.34329
4	0.001373	0.718123	1.38487
5	0.000142	0.718268	1.39125
6	0.000014	0.718281	1.39210

Table 3.1: Expected money raised per contestant for $P = 1$, normalized aggregate expected effort for $P = 1$, and normalized expected total cost, for $m = 2, ..., 6$.

cost for stimulating expected effort $\bar{Z}$ in a tournament with m participants is

$$ETC_{m,n}^{\bar{Z}} = \frac{\bar{Z}}{n} \frac{1 - m\kappa_m}{\ln 2 + \sum_{j=3}^{m} \frac{\beta_j}{(j-1)!}},$$

where $\kappa_2 = \frac{3}{2} - 2\ln 2$, and for $m \geq 3$

$$\begin{aligned}\kappa_m = & \frac{1}{(m-2)!}\left[(\ln(m-1) - \ln m)\left((m-2)m^{m-3} - 2(m-1)m^{m-2} + m^m\right)\right] \\ & + \frac{1}{(m-2)!}\left[\sum_{j=0}^{m-4}\left(\frac{m-2}{m-3-j} - \frac{2(m-1)}{m-2-j} + \frac{m}{m-1-j}\right) m^j\right] \\ & + \frac{1}{(m-2)!}\left[\left(2 - \frac{3}{2}m\right) m^{m-3} + m^{m-1}\right],\end{aligned}$$

and

$$\begin{aligned}\beta_m = & (m-1)\left[-m^{m-2}(\ln(m-1) - \ln m) - \sum_{j=0}^{m-3} \frac{m^j}{m-2-j}\right] \\ & +(m-2)\left[(m-1)^{m-2}(\ln(m-2) - \ln(m-1)) + \sum_{j=0}^{m-3} \frac{(m-1)^j}{m-2-j}\right].\end{aligned}$$

Theorem 3.3 settles the question about the optimal number of contestants for the case $H \sim U[0,1]$. It reveals that two contestants are optimal because the decrease in expected money raised by the entry auction that goes along with increasing tournament size strongly dominates the positive effect of increasing aggregate effort. This is shown in Table 3.1 for $m = 2, ..., 6$. Note in particular the big jump in expected costs from $m = 2$ to $m = 3$.

3.4 Appendix for Chapter 3

Proof of Lemma 3.1. To establish i), note that

$$\frac{1}{\Delta_{m+1}} = \frac{\sum_{j=1}^{m+1} c_j}{(m+1)c_{m+1}} = \frac{1}{\Delta_m}\frac{mc_m}{(m+1)c_{m+1}} + \frac{1}{m+1} = \frac{mc_m + \Delta_m\, c_{m+1}}{(m+1)c_{m+1}\Delta_m}.$$

ii) By i), we have

$$\begin{aligned}\frac{m}{m+1}\,\Delta_{m+1} < 1 \quad &\Leftrightarrow \quad mc_{m+1}\Delta_m < mc_m + \Delta_m\, c_{m+1} \Leftrightarrow (m-1)c_{m+1}\Delta_m < mc_m \\ &\Leftrightarrow \quad c_{m+1} < \frac{m}{m-1}\Delta_m^{-1}\, c_m.\end{aligned}$$

□

Proof of Lemma 3.2. From the technical appendix of (FM) it follows that

$$\begin{aligned} TC_{m+1} - TC_m \quad &= \quad Z\sum_{j=1}^{m+1} c_j \left[(\Delta_{m+1} - \Delta_m)\left(2 - \frac{m}{m+1}\Delta_{m+1} - \frac{m-1}{m}\Delta_m\right)\right] \\ &+ \quad Z\sum_{j=1}^{m+1} c_j \left[\frac{\Delta_{m+1}}{m+1}(\Delta_m - 1)\left(1 - \frac{m-1}{m}\Delta_m\right)\right].\end{aligned}$$

The factor multiplied with $Z\sum_{j=1}^{m+1} c_j$ has been split into two additive parts merely due to limitations of space. I examine this factor for the case $c_{m+1} = c_m$, which by

Lemma 3.1 implies $\Delta_{m+1} = \frac{(m+1)\Delta_m}{m+\Delta_m}$. This yields

$$\begin{aligned}
&(\Delta_{m+1} - \Delta_m)\left(2 - \frac{m}{m+1}\Delta_{m+1} - \frac{m-1}{m}\Delta_m\right) + \frac{\Delta_{m+1}}{m+1}(\Delta_m - 1)\left(1 - \frac{m-1}{m}\Delta_m\right) \\
=& \left(\frac{(m+1)\Delta_m}{m+\Delta_m} - \Delta_m\right)\left(2 - \frac{m\Delta_m}{m+\Delta_m} - \frac{m-1}{m}\Delta_m\right) \\
&+ \frac{\Delta_m}{m+\Delta_m}(\Delta_m - 1)\left(1 - \frac{m-1}{m}\Delta_m\right) \\
=& \frac{\Delta_m(1-\Delta_m)}{m+\Delta_m}\,\frac{2(m+\Delta_m) - m\Delta_m - \frac{m-1}{m}\Delta_m(m+\Delta_m)}{m+\Delta_m} \\
&+ \frac{(\Delta_m - 1)\left(\Delta_m - \frac{m-1}{m}\Delta_m^2\right)}{m+\Delta_m} \\
=& \frac{\Delta_m - 1}{(m+\Delta_m)^2}\left[(m+\Delta_m)\left(\Delta_m - \frac{m-1}{m}\Delta_m^2\right) - \Delta_m\left(2m + (3-2m)\Delta_m - \frac{m-1}{m}\Delta_m^2\right)\right] \\
=& \frac{\Delta_m - 1}{(m+\Delta_m)^2}\left[m\Delta_m + \Delta_m^2 - (m-1)\Delta_m^2 - 2m\Delta_m - (3-2m)\Delta_m^2\right] \\
=& \frac{\Delta_m - 1}{(m+\Delta_m)^2}\left[(m-1)\Delta_m^2 - m\Delta_m\right] \\
=& \frac{\Delta_m(\Delta_m - 1)}{(m+\Delta_m)^2}\left[(m-1)\Delta_m - m\right]
\end{aligned}$$

Note that $(m-1)\Delta_m - m < 0$ by assumption. The other factor is ≥ 0, and equality holds only in the boundary case $\Delta_m = 1$. This proves the claim. $\square$

Calculations for Example 3.1. Note that $\Delta_2 = 1$ and $\Delta_{m+1} = \frac{m+1}{m}$. (3.5) implies

$$\frac{TC_6}{TC_2} = \left(\prod_{j=2}^{5} \alpha_j\right) \frac{6\left(1 - \sqrt{\frac{5}{6}}\right)\left(1 + \sqrt{\frac{5}{6}}\right)}{2\left(1 - \sqrt{\frac{1}{2}}\right)\left(1 + \sqrt{\frac{1}{2}}\right)} = \prod_{j=2}^{5} \alpha_j.$$

From Lemma 3.1,

$$\Delta_{j+1} = \frac{(j+1)c_{j+1}}{jc_j + \Delta_j\, c_{j+1}}\,\Delta_j = \frac{\Delta_j}{\frac{j}{(j+1)\alpha_j} + \frac{\Delta_j}{j+1}},$$

which yields after a few steps

$$\alpha_j = \frac{j\,\Delta_{j+1}}{\Delta_j(j+1-\Delta_{j+1})}.$$

For the current example, this implies $\alpha_2 = \frac{2(1+\frac{1}{20})}{3-(1+\frac{1}{20})} = \frac{14}{13}$. Similarly, $\alpha_3 = \frac{220}{203}$, $\alpha_4 = \frac{920}{847}$, $\alpha_5 = \frac{25}{23}$ and thus

$$\frac{TC_6}{TC_2} = \prod_{j=2}^{5} \alpha_j \approx 1.38.$$

□

Proof of Lemma 3.3. In (FM), $[\underline{w}, \bar{w}]$ (or $[\underline{w}, +\infty)$) denotes the support of the distribution of an ability type, with c.d.f. $\widetilde{H}$ and density $\widetilde{h}$. $\widetilde{h}_{m+1,n}$ is the density of the $m+1$st-highest draw out of n independent draws from $\widetilde{H}$, and $\widetilde{\psi}(w,w)$ is the analog of $\psi(c,c)$. According to Theorem 6 of (FM) then, with independent types, the expected total cost of conducting a tournament with m contestants when these are selected by the contestant selection auction is

$$P - m \int_{\underline{w}}^{\bar{w}} \widetilde{\psi}(w,w)\, \widetilde{h}_{m+1,n}(w) dw. \tag{3.9}$$

Set $\bar{w} := \frac{1}{\underline{c}}$, $\underline{w} := \frac{1}{\bar{c}}$, and $w = \frac{1}{c}$ in between. Define $\widetilde{\psi}(w,w)$ via $\widetilde{\psi}(w,w) := \psi(\frac{1}{w}, \frac{1}{w})$. Since $dc = -\frac{1}{w^2}\, dw$, and since increasing c means decreasing w, it holds that $\widetilde{h}(w) = -h(c)\, \frac{dc}{dw} = \frac{1}{w^2}\, h(\frac{1}{w})$. Moreover, $\widetilde{H}(w) = 1 - H\left(\frac{1}{w}\right)$.

From the general formula for order statistics, it follows that

$$\begin{aligned}
\widetilde{h}_{m+1,n}(w) &= \frac{n!}{(n-m-1)!\, m!}\, \widetilde{H}(w)^{n-m-1} \left(1 - \widetilde{H}(w)\right)^m \widetilde{h}(w) \\
&= \frac{n!}{m!\,(n-m-1)!}\, H\left(\frac{1}{w}\right)^m \left(1 - H\left(\frac{1}{w}\right)\right)^{n-m-1} \frac{h\left(\frac{1}{w}\right)}{w^2} \\
&= \frac{h_{n-m,n}(\frac{1}{w})}{w^2}.
\end{aligned}$$

Hence, expression (3.9) becomes

$$P - m \int_{\underline{w}}^{\bar{w}} \psi\left(\frac{1}{w}, \frac{1}{w}\right) \frac{h_{n-m,n}\left(\frac{1}{w}\right)}{w^2}\, dw = P - m \int_{\underline{c}}^{\bar{c}} \psi(c,c)\, h_{n-m,n}(c)\, dc.$$

□

Proof of Theorem 3.3. Observe that $\frac{h(c_j)}{H(c)} = \frac{1}{c}$. Let $\mathrm{vol}_k(A)$ denote the k-dimensional Hausdorff measure of set A. Making use of the Coarea formula for the mapping $(c_1, ... c_{m-1}) \mapsto \sum_{j=1}^{m-1} c_j$, I find for the crucial term in the second factor of $ETC^{\bar{Z}}_{m,n}$

in (3.8), for $m \geq 3$:

$$
\begin{aligned}
&\int_{[0,c]^{m-1}} \max\left(1 - \frac{c(m-1)}{c + \sum_{j=1}^{m-1} c_j}, 0\right)^2 \frac{1}{c^{m-1}}\, dc_1 ... dc_{m-1} \\
&= \frac{1}{c^{m-1}} \int_{(m-2)c}^{(m-1)c} \left(1 - \frac{(m-1)c}{c+a}\right)^2 \mathrm{vol}_{m-2}\left(\left\{\sum_{j=1}^{m-1} c_j = a\right\} \cap [0,c]^{m-1}\right) \sqrt{\frac{1}{m-1}}\, da \\
&= \frac{1}{c^{m-1}} \int_0^c \left(1 - \frac{(m-1)c}{mc-a}\right)^2 \\
&\quad \mathrm{vol}_{m-2}\left(\left\{\sum_{j=1}^{m-1} c_j = (m-1)c - a\right\} \cap [0,c]^{m-1}\right) \sqrt{\frac{1}{m-1}}\, da \\
&= \frac{1}{c^{m-1}} \int_0^c \left(\frac{c-a}{mc-a}\right)^2 \mathrm{vol}_{m-2}\left(\left\{\sum_{j=1}^{m-1} c_j = a\right\} \cap [0,c]^{m-1}\right) \sqrt{\frac{1}{m-1}}\, da \\
&= \frac{1}{c^{m-1}} \int_0^c \left(\frac{c-a}{mc-a}\right)^2 \frac{\sqrt{m-1}}{(m-2)!\sqrt{2^{m-2}}} (\sqrt{2}a)^{m-2} \sqrt{\frac{1}{m-1}}\, da \\
&= \frac{1}{c^{m-1}(m-2)!} \int_0^c \left(\frac{c^2 a^{m-2} - 2ca^{m-1} + a^m}{(mc-a)^2}\right) da. \qquad (3.10)
\end{aligned}
$$

For the step from line 3 to 4, symmetry with respect to the corners of $[0,c]^{m-1}$ was invoked. In the second but last step, I plugged in a well known formula for the volume of a simplex of unit side length.

Expression (3.10) shows that a formula for integrals of the form $\int_0^c \frac{a^k}{(b-a)^2}\, da$ for $k \geq 1$ and $b > c$ is needed. Integrating by parts once,

$$\int_0^c \frac{a^k}{(b-a)^2}\, da \;=\; \frac{c^k}{b-c} \;-\; k \int_0^c \frac{a^{k-1}}{b-a}\, da.$$

Noting that $a^{k-1} = a^{k-1} - b^{k-1} + b^{k-1} = (a-b)\left(\sum_{j=0}^{k-2} a^{k-2-j}\, b^j\right) + b^{k-1}$ (with the convention that sums running from $j = 0$ to -1 are zero), it is straightforward to establish

$$\int_0^c \frac{a^k}{(b-a)^2}\, da \;=\; \frac{c^k}{b-c} \;+\; kb^{k-1}(\ln(b-c) - \ln b) \;+\; \sum_{j=0}^{k-2} k\, b^j \frac{c^{k-1-j}}{k-1-j}.$$

After plugging in and collecting terms, which I omit here for the sake of brevity, equation (3.10) turns into:

$$\begin{aligned}
&\int_{[0,c]^{m-1}} \max\left(1-\frac{c(m-1)}{c+\sum_{j=1}^{m-1} c_j},0\right)^2 \frac{1}{c^{m-1}}\, dc_1...dc_{m-1} \\
&= \frac{1}{(m-2)!}\left[(\ln(m-1)-\ln m)\left((m-2)m^{m-3}-2(m-1)m^{m-2}+m^m\right)\right] \\
&+ \frac{1}{(m-2)!}\left[\sum_{j=0}^{m-4}\left(\frac{m-2}{m-3-j}-\frac{2(m-1)}{m-2-j}+\frac{m}{m-1-j}\right)m^j\right] \\
&+ \frac{1}{(m-2)!}\left[\left(2-\frac{3}{2}m\right)m^{m-3}+m^{m-1}\right] = \kappa_m.
\end{aligned}$$

This expression does not depend on c or n, so that κ_m is also the expected money raised per contestant in the entry auction (for $P=1$), irrespectively of n. A separate calculation is necessary for κ_2.

$$\begin{aligned}
\kappa_2 &= \int_0^c \left(1-\frac{c}{c+c_1}\right)^2 \frac{1}{c}\, dc_1 = \frac{1}{c}\int_0^c \left(\frac{c_1}{c+c_1}\right)^2 dc_1 \\
&= \frac{1}{c}\left[-\frac{c}{2}+\int_0^c \frac{2c_1}{c+c_1} dc_1\right] = -\frac{1}{2}+\frac{2}{c}\left[c\ln(2c)-\int_0^c \ln(c+c_1)\, dc_1\right] \\
&= -\frac{1}{2}+\frac{2}{c}\left[c\ln(2c)-2c\ln(2c)+c+c\ln c\right] \\
&= \frac{3}{2}+2(\ln c-\ln(2c)) = \frac{3}{2}-2\ln 2.
\end{aligned}$$

Concerning the terms $\gamma_{m,n} := E_n\left[\frac{|M|-1}{\sum_{j\in M} c_j}\middle| |M|\le m\right]$, a separate calculation is again needed for $m=2$.

$$\begin{aligned}
\gamma_{2,n} = E_n\left[\frac{|M|-1}{\sum_{j\in M} c_j}\middle| |M|\le 2\right] &= \int_0^1\int_0^c \frac{1}{c+c_1}\frac{1}{c}\, dc_1\, h_{n-1,n}(c)\, dc \\
&= \int_0^1\int_0^c \frac{1}{c+c_1}\, dc_1\, n(n-1)(1-c)^{n-2}\, dc \\
&= (\ln 2)\, n(n-1)\int_0^1 (1-c)^{n-2}\, dc = n\ln 2.
\end{aligned}$$

While it is practically impossible to compute the terms for higher m directly, they may be computed recursively. Indeed, allowing for an additional contestant has an effect on aggregate equilibrium effort only in those cases where the cost of the new firm is low enough in the sense of inequality (3.2). Also, in all these relevant cases, if the new firm is not allowed to participate, then all the lower cost firms make positive effort because of the monotonicity of the right hand side of (3.2) that was mentioned in Section 3.2.1. I use the equivalent formulation of (3.2) which was also mentioned in Section 3.2.1 to describe the domain of integration. Consequently, for $m \geq 3$,

$$\begin{aligned}
&\gamma_{m,n} - \gamma_{m-1,n} \\
&= \int_0^1 \frac{h_{n-m+1,n}(c)}{c^{m-1}} \int_{[0,c]^{m-1}} \left(\frac{m-1}{c + \sum_{j=1}^{m-1} c_j} - \frac{m-2}{\sum_{j=1}^{m-1} c_j} \right) I_{\{(m-2)c < \sum_{j=1}^{m-1} c_j\}} \, dc_1 ... dc_{m-1} \, dc.
\end{aligned}$$

The inner integral may again be computed by using the Coarea formula and the little symmetry trick:

$$\begin{aligned}
&\int_{[0,c]^{m-1}} \left(\frac{m-1}{c + \sum_{j=1}^{m-1} c_j} - \frac{m-2}{\sum_{j=1}^{m-1} c_j} \right) I_{\{(m-2)c < \sum_{j=1}^{m-1} c_j\}} \, dc_1 ... dc_{m-1} \\
&= \int_{(m-2)c}^{(m-1)c} \left(\frac{m-1}{c+a} - \frac{m-2}{a} \right) \mathrm{vol}_{m-2} \left(\left\{ \sum_{j=1}^{m-1} c_j = a \right\} \cap [0,c]^{m-1} \right) \sqrt{\frac{1}{m-1}} \, da \\
&= \int_0^c \left(\frac{m-1}{mc-a} - \frac{m-2}{(m-1)c-a} \right) \mathrm{vol}_{m-2} \left(\left\{ \sum_{j=1}^{m-1} c_j = a \right\} \cap [0,c]^{m-1} \right) \sqrt{\frac{1}{m-1}} \, da \\
&= \int_0^c \left(\frac{m-1}{mc-a} - \frac{m-2}{(m-1)c-a} \right) \frac{\sqrt{m-1}}{(m-2)!} a^{m-2} \sqrt{\frac{1}{m-1}} \, da \\
&= \frac{1}{(m-2)!} \int_0^c \left((m-1) \frac{a^{m-2}}{mc-a} - (m-2) \frac{a^{m-2}}{(m-1)c-a} \right) da.
\end{aligned}$$

Now,

$$\begin{aligned}
&\int_0^c \left((m-1) \frac{a^{m-2}}{mc-a} - (m-2) \frac{a^{m-2}}{(m-1)c-a} \right) da \\
&= (m-1) \left[-(mc)^{m-2} (\ln(m-1) - \ln m) - \sum_{j=0}^{m-3} \frac{m^j}{m-2-j} c^{m-2} \right] \\
&+ (m-2) \left[((m-1)c)^{m-2} (\ln(m-2) - \ln(m-1)) + \sum_{j=0}^{m-3} \frac{(m-1)^j}{m-2-j} c^{m-2} \right] \\
&= c^{m-2} \beta_m.
\end{aligned}$$

Hence,

$$
\begin{aligned}
\gamma_{m,n} - \gamma_{m-1,n} &= \int_0^1 \frac{h_{n-m+1,n}(c)}{c^{m-1}} \frac{1}{(m-2)!} c^{m-2}\, \beta_m \, dc \\
&= \beta_m \frac{1}{(m-2)!} \int_0^1 \frac{n!}{(n-m)!\,(m-1)!} (1-c)^{n-m}\, c^{m-2}\, dc \\
&= \beta_m \frac{1}{(m-2)!} \frac{n!}{(n-m)!\,(m-1)!} \frac{(m-2)!(n-m)!}{(n-1)!} \\
&= \frac{n}{(m-1)!}\, \beta_m .
\end{aligned}
$$

This shows

$$
\gamma_{m,n} = n \left(\ln 2 + \sum_{j=3}^{m} \frac{\beta_j}{(j-1)!} \right).
$$

Q.E.D. □

Chapter 4

Revenue maximization in the dynamic knapsack problem

We analyze maximization of revenue in the dynamic and stochastic knapsack problem where a given capacity needs to be allocated by a given deadline to sequentially arriving agents. Each agent is described by a two-dimensional type that reflects his capacity requirement and his willingness to pay per unit of capacity. Types are private information. We first characterize implementable policies. Then we solve the revenue maximization problem for the special case where there is private information about per-unit values, but capacity needs are observable. After that we derive two sets of additional conditions on the joint distribution of values and weights under which the revenue maximizing policy for the case with observable weights is implementable, and thus optimal also for the case with two-dimensional private information. In particular, we investigate the role of concave continuation revenues for implementation. We also construct a simple policy for which per-unit prices vary with requested weight but not with time, and we prove that it is asymptotically revenue maximizing when available capacity and time to the deadline both go to infinity. This highlights the importance of nonlinear as opposed to dynamic pricing.

4.1 Introduction

The knapsack problem is a classical combinatorial optimization problem with numerous practical applications: several objects with given, known capacity needs (or weights) and given, known values must be packed into a "knapsack" of given capacity to maximize the total value of the included objects. In the dynamic and stochastic version (see Ross and Tsang, 1989), objects sequentially arrive over time and their weight-value combination is stochastic but becomes known to the designer at arrival times. Objects cannot be recalled later, so it must be decided upon arrival whether an object is included or not. Several applications that come to mind are logistic decisions in the freight transportation industry, the allocation of fixed capacities in the travel and leisure industries (e.g., airlines, trains, hotels, rental cars), the allocation of fixed equipment or personnel in a given period of time, the allocation of fixed budgets to investment opportunities that appear sequentially, and the allocation of dated advertising space on web portals.

In the present study, we add incomplete information to the dynamic and stochastic setting. In this way, we obtain a dynamic monopolistic screening problem: there is a finite number of periods, and at each period a request for capacity arrives from an agent who is impatient and privately informed about both his valuation per unit of capacity and the needed capacity.[1] Each agent derives positive utility if he gets the needed capacity (or more), and zero utility otherwise. The designer accepts or rejects the requests so as to maximize the revenue obtained from the allocation.

The dynamic and stochastic knapsack problem with complete information about values and requests was analyzed by Papastavrou, Rajagopalan and Kleywegt (1996) and by Kleywegt and Papastavrou (2001). These authors characterized optimal policies in terms of weight-dependent value thresholds. Kincaid and Darling (1963) and Gallego and Van Ryzin (1994) examined a setting that can be reinterpreted as having (one-dimensional) incomplete information about values, but in their frameworks all requests have the same known weight.[2] In particular, Gallego and Van Ryzin showed that optimal expected revenue is concave in capacity in the case of equal weights. Kleywegt and Papastavrou gave examples showing that total value is not necessarily globally concave in capacity if the weight requests are heterogeneous, and they provided a sufficient condition for this structural property to hold. Gallego and Van Ryzin also showed that the optimal policy, which exhibits complicated time dynamics, can often be replaced by a simple time-independent policy without much loss: the simple policy performs asymptotically optimal as the number of periods and the units to be sold go to infinity. Finally, Gershkov

[1] Our results are easily extended to the setting where arrivals are stochastic and/or time is continuous.

[2] We refer the reader to the book by Talluri and Van Ryzin (2004) for references to the large literature on revenue (or yield) management that adopts variations on these models.

and Moldovanu (2009) generalized the Gallego-Van Ryzin model to incorporate objects with the same weight but with several qualities that are equally ranked by all agents, independently of their types (which are also one-dimensional).

Problems of multi-dimensional mechanism design are usually complex and difficult to solve: the main problem is that incentive compatibility - which in the one-dimensional case often reduces to a monotonicity constraint - imposes, besides a monotonicity requirement, an integrability constraint that is not easily included in maximization problems (see e.g. Rochet, 1985; Armstrong, 1996; Jehiel, Moldovanu and Stacchetti, 1999; and the survey of Rochet and Stole, 2003). Our implementation problem is special though because utilities have a special form and useful deviations in the weight dimension can only be one-sided (upward). This feature allows a less cumbersome characterization of implementable policies that can be embedded in the dynamic analysis under certain conditions on the joint distribution of values and weights of the arriving agents. Other multi-dimensional mechanism design problems with restricted deviations in one or more dimensions were studied for example by Blackorby and Szalay (2007), Che and Gale (2000), Iyengar and Kumar (2008), Kittsteiner and Moldovanu (2005), and Pai and Vohra (2009).

4.1.1 Outline and preview of results

We first characterize implementable policies, as explained above. Then, we solve the revenue maximization problem for the case where there is private information about per-unit values, but weights are observable. We will sometimes refer to this as the relaxed problem. Under a standard monotonicity assumption on virtual values, this is the virtual value analog of the problem solved by Papastavrou, Rajagopalan and Kleywegt (1996). The resulting optimal policy is Markovian and deterministic, and it has a threshold property with respect to virtual values. It is important to emphasize that this policy need not be implementable for the case where both values and weights are unobservable, unless additional conditions are imposed. Our main results in Section 4.4 are therefore concerned with the implementability of the relaxed optimal solution: we derive two different sets of additional conditions on the joint distribution of values and weights under which the revenue maximizing policy for the case with observable weights is implementable, and thus optimal also for the case with two-dimensional private information. The first set of conditions - which is satisfied in a variety of intuitive settings

- features a hazard rate ordering that expresses a form of positive correlation between weights and values. It ensures that the incentive constraint in the capacity dimension is never binding. Related conditions can be found in previous work on multi-dimensional mechanism design with restricted deviations mentioned above, e.g. in the papers of Pai and Vohra (2009), Iyengar and Kumar (2008), or Blackorby and Szalay (2007). More interestingly, we also draw a connection between incentive compatibility and the structural property of concavity of revenue in capacity. Concavity of optimal expected revenue in the relaxed problem creates a tendency to set higher virtual value thresholds for higher capacity requests. It is then less attractive for agents to overstate their capacity needs, which facilitates the implementation of the relaxed solution by relaxing the incentive constraints. We quantify this relation in our second set of additional conditions: concavity of revenue combined with a (substantial) weakening of the hazard rate order imply implementability of the relaxed solution. For completeness, we also briefly translate to our model the sufficient condition for concavity of revenue due to Papastavrou, Rajagopalan and Kleywegt so as to obtain a condition on the model's primitives.

The last part of this chapter contains another main result. We construct - for general distributions of weights and values - a time-independent, nonlinear price schedule which is asymptotically revenue maximizing when the available capacity and the time to the deadline both go to infinity, and when weights are observable. This extends an asymptotic result by Gallego and van Ryzin (1994) (for a detailed discussion, see Section 4.5) and suggests that complicated dynamic pricing may not be that important for revenue maximization if the distribution of agents' types is known. Our result emphasizes though that nonlinear pricing remains asymptotically important in dynamic settings. As a nice link to the first part of the paper, the constructed price schedule turns out to be implementable for the case with two-dimensional private information if the weakened hazard rate condition employed in our discussion of concavity is satisfied. Since prices are time-independent, the policy is also immune to strategic buyer arrivals (which we do not model explicitly here). We also point out that a policy that varies with time but not with requested weight (whose asymptotic optimality in the complete information case was established by Lin, Lu and Yao, 2008) is usually not optimal under incomplete information.

The chapter is organized as follows. In Section 4.2 we present the dynamic model and the informational assumptions about values and weights. In Section 4.3 we characterize incentive compatible allocation policies. In Section 4.4 we focus on dynamic revenue maximization. We first characterize the revenue maximizing policy for the relaxed problem. We then offer two results that exhibit conditions under which the above policy is incentive compatible, and thus optimal also for the case where both values and weights are private information. Section 4.5 contains the asymptotic analysis. All proofs are in the Appendix.

4.2 The model

The designer has a "knapsack" of given capacity $C \in \mathbb{R}$ that he wants to allocate in a revenue-maximizing way to several agents in at most $T < \infty$ periods. In each period, an impatient agent arrives with a demand for capacity characterized by a weight (or quantity request) w, and by a per-unit value v.[3] While the realization of the random vector (w, v) is private information to the arriving agent, its distribution is assumed to be common knowledge and given by the joint cumulative distribution function $F(w, v)$, with continuously differentiable density $f(w, v) > 0$, defined on $[0, \infty)^2$. w, v and wv all have finite expected value. Demands are independent across different periods.[4]

In each period, the designer decides on a capacity to be allocated to the arriving agent (possibly none) and on a monetary payment. Type (w, v)'s utility is given by $wv - p$ if at price p he is allocated a capacity $w' \geq w$ and by $-p$ if he is assigned an insufficient capacity $w' < w$. Each agent observes the remaining capacity of the designer.[5] Finally, we assume that for all w, the conditional virtual value functions $\hat{v}(v, w) := v - \frac{1-F(v|w)}{f(v|w)}$ are unbounded as a function of v and strictly monotone increasing with $\frac{\partial}{\partial v}\hat{v}(v, w) > 0$ for all (w, v).

[3] It is an easy extension to assume that the arrival probability per period is given by $p < 1$.

[4] As pointed out by a referee, the results of Sections 4.3 and 4.4 apply also, with the obvious modifications, if types in different periods are independent but not necessarily drawn from identical distributions.

[5] Alternatively, we can assume that each agent observes the entire history of the previous allocations. These assumptions are innocuous in the following sense: when we analyze revenue maximization in Section 4.4, we first solve for the optimal policy in the relaxed problem with observable weight types w. We then provide conditions for when this relaxed solution is implementable. Since in the case of observable weight requests, the designer cannot gain by hiding the available capacity, he cannot increase expected revenue by hiding the remaining capacity in the original problem.

4.3 Incentive compatible policies

To characterize the revenue maximizing scheme, we may restrict attention, without loss of generality, to direct mechanisms where every agent, upon arrival, reports a type (w, v) and the mechanism then specifies an allocation and a payment. In this section, we characterize incentive compatibility for a class of allocation policies that necessarily contains the revenue-maximizing one. The schemes we develop also have an obvious and immediate interpretation as indirect mechanisms, where the designer sets menus of per-unit prices that depend on time and on the remaining capacity.

An allocation rule is called *deterministic* and *Markovian* if, at any period $t = 1, ..., T$ and for any possible type of agent arriving at t, it uses a non-random allocation rule that depends only on the arrival time t, on the declared type of the arriving agent, and on capacity that is still available, denoted by c. The restriction to these policies is innocuous as shown in Section 4.4.

For the purpose of revenue maximization, we can assume without loss of generality that a deterministic Markovian allocation rule for time t with remaining capacity c has the form $\alpha_t^c : [0, +\infty)^2 \to \{1, 0\}$ where 1 (0) means that the reported capacity demand w is satisfied (not satisfied). Indeed, it never makes sense to allocate an insufficient quantity $0 < w' < w$ because individually rational agents are not willing to pay for this. Alternatively, allocating more capacity than the reported demand is useless as well: such allocations do not further increase agents' utility while they may decrease continuation revenues for the designer. Finally, note that all feasible allocation rules α_t^c must satisfy $\alpha_t^c(w, v) = 0$ for all $w > c$. $q_t^c : [0, +\infty)^2 \to \mathbb{R}$ will denote a payment rule associated with such a α_t^c.

Proposition 4.1. *A feasible, deterministic, Markovian allocation rule $\{\alpha_t^c\}_{t,c}$ is implementable if and only if for every t and every c it satisfies the following two conditions.*[6]

i) For all $(w, v), v' \geq v :\ \alpha_t^c(w, v) = 1 \Rightarrow \alpha_t^c(w, v') = 1$.

ii) The function $wp_t^c(w)$ is non-decreasing in w, where $p_t^c(w) = \inf\{v| \, \alpha_t^c(w, v) = 1\}$.[7]

[6] Here, we use "implementable" in the standard sense from the mechanism design literature. An allocation rule is implementable if we can associate to it a payment rule such that any agent finds it optimal to truthfully reveal her type when faced with the combined allocation-payment scheme.

[7] We set $p_t^c(w) = \infty$ if the set $\{v| \, \alpha_t^c(w, v) = 1\}$ is empty.

When the above two conditions are satisfied, the allocation rule $\{\alpha_t^c\}_{t,c}$, together with the payment rule

$$q_t^c(w,v) = \begin{cases} wp_t^c(w) & \text{if } \alpha_t^c(w,v) = 1 \\ 0 & \text{if } \alpha_t^c(w,v) = 0 \end{cases}$$

constitute an incentive compatible policy.

The threshold property embodied in condition i) of Proposition 4.1 is standard and is a natural feature of welfare maximizing rules under complete information. When there is incomplete information in the value dimension, this condition imposes limitations on the payments that can be extracted in equilibrium. Condition ii) is new: it reflects the limitations imposed in our model by the incomplete information in the weight dimension. We note that the above simple result is based on a combination of three main factors. (1) Due to our special utility function and to the pursued goal of revenue maximization, it is sufficient to consider only policies that allocate either the demanded weight to the agent or nothing. (2) The monotonicity requirement behind incentive compatibility boils down to the above simple conditions. (3) The integrability condition is automatically satisfied by all monotone allocation rules in the considered class. In general, one has to consider more allocation functions, more implications of monotonicity, and potentially an integrability constraint.

4.4 Dynamic revenue maximization

We first demonstrate how the dynamic revenue maximization problem can be solved if w is observable. This is, essentially, the dynamic programming problem analyzed by Papastavrou, Rajagopalan and Kleywegt (1996), translated from values to virtual values. Nevertheless, the logic of the derivation is somewhat involved, so we detail it below.

1. Without loss of generality, we can restrict attention to Markovian policies. The optimality of Markovian, possibly randomized, policies is standard for all models where, as is the case here, the per-period rewards and transition probabilities are history-independent.[8]

[8] See for example Theorem 11.1.1 in Puterman (2005) which shows that, for any history-dependent policy, there is a Markovian, possibly randomized, policy with the same payoff.

2. If there is incomplete information about v, but complete information about the weight requirement w, then Markovian, deterministic and implementable policies α_t^c are characterized for each t and c by the threshold property of condition i) in Proposition 4.1.

3. Naturally, in the given revenue maximization problem with complete information about w, we need to restrict attention to interim individually rational policies where no agent ever pays more than the utility obtained from her actual capacity allocation. It is easy to see that, for any Markov, deterministic and implementable allocation rule α_t^c, the maximal, individually rational payment function which supports it is the one given in Proposition 4.1. Otherwise, the designer pays some positive subsidy to the agent, and this cannot be revenue-maximizing.

4. At each period t, and for each remaining capacity c, the designer's problem under complete information about w is equivalent to a simpler, one-dimensional static problem where a known capacity needs to be allocated to the arriving agent, and where the seller has a salvage value for each remaining capacity: the salvage values in the static problem correspond to the continuation revenues in the dynamic version. Analogous to the analysis of Myerson (1981), each static revenue maximization problem has a monotone (in the sense of condition i) in Proposition 4.1), non-randomized solution as long as for any weight w, the agent's conditional virtual valuation $v - \frac{1-F(v|w)}{f(v|w)}$ is increasing in v.[9] If thresholds/per-unit prices are set at $p_t^c(w)$ ($p_t^c(w) = +\infty$ for $w > c$ reflects the fact that such demands are always rejected) in period $t \leq T$ (so $T+1-t$ periods, including the current one, remain until the deadline) with remaining capacity c and if the optimal Markovian policy is followed from time $t+1$ onward, then the expected revenue $R(c, T+1-t)$ can be written as

$$\begin{aligned} R(c,T+1-t) \;=\; & \int_0^c wp_t^c(w)(1-F(p_t^c(w)|w))\bar{f}_w(w)\,dw \\ + \;& \int_0^\infty \left[(1-F(p_t^c(w)|w))R^*(c-w,T-t) + F(p_t^c(w)|w)R^*(c,T-t)\right]\bar{f}_w(w)\,dw, \end{aligned}$$

[9]Note that the optimal policy continues to be deterministic even if virtual valuations are not monotonic. This follows by an argument that is similar to the one given by Riley and Zeckhauser (1983). We nevertheless keep the monotonicity assumption for simplicity, and because we need related conditions for some of the subsequent results.

where $\bar{f}_w$ denotes the marginal density in w, and where R^* denotes optimal expected revenues, with $R^*(c,0) = 0$ for all c. The revenue maximizing unit prices will be called $p_t^c(w)$ from now on. For $w \leq c$, they are determined by the first-order conditions

$$w\left(p_t^c(w) - \frac{1 - F(p_t^c(w)|w)}{f(p_t^c(w)|w)}\right) = R^*(c, T-t) - R^*(c-w, T-t).^{10}$$

5. By backward induction, and by the above reasoning, the seller has a Markov, non-randomized optimal policy in the dynamic problem with complete information about w. Note also that, by a simple duplication argument, $R^*(c, T+1-t)$ must be monotone non-decreasing in c.

If the above solution to the relaxed problem satisfies the incentive compatibility constraint in the weight dimension, i.e. if $wp_t^c(w)$ happens to be monotone as required by condition ii) of Proposition 4.1, then the associated allocation where $\alpha_t^c(w,v) = 1$ if and only if $v \geq p_t^c(w)$ is also implementable in the original problem with incomplete information about both v and w. It then constitutes the revenue maximizing scheme that we are after. The next example illustrates that condition ii) of Proposition 4.1 can be binding.

Example 4.1. *Assume that $T = 1$. The distribution of the agents' types is given by the following stochastic process. First, the weight request w is realized according to an exponential distribution with parameter λ. Next, the per-unit value of the agent is sampled from the following distribution*

$$F(v|w) = \begin{cases} 1 - e^{-\overline{\lambda} v} & \textit{if} \quad w > w^* \\ 1 - e^{-\underline{\lambda} v} & \textit{if} \quad w \leq w^* \end{cases}$$

where $\overline{\lambda} > \underline{\lambda}$ and $w^* \in (0,c)$.

In this case, for an observable weight request, the seller charges the take-it-or-leave-it offer of $\frac{1}{\underline{\lambda}}$ ($\frac{1}{\overline{\lambda}}$) per unit if the weight request is smaller than or equal to (larger than) w^*.

[10] Note that by our assumption of unbounded conditional virtual values, which is a mild assumption for distributions with unbounded support, these first-order conditions always admit a solution and must be satisfied at the optimum.

This implies that

$$wp_t^c(w) = \begin{cases} \frac{w}{\overline{\lambda}} & \text{if} \quad c \geq w > w^* \\ \frac{w}{\underline{\lambda}} & \text{if} \quad w \leq w^* \end{cases}$$

and, therefore, $wp_t^c(w)$ is not monotone.

4.4.1 The hazard rate stochastic ordering

A simple sufficient condition that guarantees implementability of the relaxed solution is a particular stochastic ordering of the conditional distributions of per-unit values: the conditional distribution given a higher weight should be (weakly) statistically higher in the hazard rate order than the conditional distribution given a lower weight. This is similar to conditions found in static frameworks by Pai and Vohra (2009), Iyengar and Kumar (2008), or Blackorby and Szalay (2007).

Theorem 4.1. *For each c, t, and w, let $p_t^c(w)$ denote the solution to the problem of maximizing expected revenue under complete information about w, determined recursively by the Bellman equation*

$$w\left(p_t^c(w) - \frac{1 - F(p_t^c(w)|w)}{f(p_t^c(w)|w)}\right) = R^*(c, T-t) - R^*(c-w, T-t). \tag{4.1}$$

Assume that the following conditions hold.

i) For any w, the conditional hazard rate $\frac{f(v|w)}{1-F(v|w)}$ is non-decreasing in v.[11]

ii) For any $w' \geq w$, and for any v, $\frac{f(v|w)}{1-F(v|w)} \geq \frac{f(v|w')}{1-F(v|w')}$.

Then, $wp_t^c(w)$ is non-decreasing in w, and, consequently, the underlying allocation where $\alpha_t^c(w, v) = 1$ if and only if $v \geq p_t^c(w)$ is implementable. In particular, (4.1) *characterizes the revenue maximizing scheme under incomplete information about both values and weights.*

An important special case for which the conditions of Theorem 4.1 hold is where the distribution of per-unit values is independent of the distribution of weights and has an increasing hazard rate.

[11] Note that this condition already implies the needed monotonicity in v of the conditional virtual value for all w.

4.4.2 The role of concavity

A major result for the case where capacity comes in discrete units, and where all weights are equal is that optimal expected revenue is concave in capacity.[12] This is a very intuitive property since it says that additional capacity is more valuable to the designer when capacity itself is scarce. Due to the more complicated combinatorial nature of the knapsack problem with heterogeneous weights, concavity need not generally hold (see Papastavrou, Rajagopalan and Kleywegt, 1996, for examples where concavity of expected welfare in the framework with complete information fails). When concavity does hold, the optimal per-unit virtual value thresholds for the relaxed problem increase with weight, which facilitates implementation for the case of two-dimensional private information.

Our main result in this subsection identifies a condition on the distribution of types that, together with concavity of the expected revenue in the remaining capacity, ensures that for each t and c, $wp_t^c(w)$ is increasing.

Theorem 4.2. *Assume the following conditions.*

i) The expected revenue $R^*(c, T+1-t)$ is a concave function of c for all times t.

ii) For any $w \leq w'$, $v - \frac{1-F(v|w)}{f(v|w)} \geq \frac{vw}{w'} - \frac{1-F(\frac{vw}{w'}|w')}{f(\frac{vw}{w'}|w')}$.

For each c, t, and w, let $p_t^c(w)$ denote the solution to the problem of maximizing expected revenue under complete information about w, determined recursively by (4.1). *Then $wp_t^c(w)$ is non-decreasing in w, and hence the underlying allocation where $\alpha_t^c(w,v) = 1$ if and only if $v \geq p_t^c(w)$ is implementable. In particular,* (4.1) *characterizes the revenue maximizing scheme under incomplete information about both values and weights.*

Remark 4.1. *The sufficient conditions for implementability used in Theorem 4.1 are, taken together, stronger than condition ii) in Theorem 4.2. To see this, assume that for any w, the conditional hazard rate $\frac{f(v|w)}{1-F(v|w)}$ is increasing in v, and that for any $w' \geq w$ and for all v, $\frac{f(v|w)}{1-F(v|w)} \geq \frac{f(v|w')}{1-F(v|w')}$. This yields:*

$$v - \frac{1-F(v|w)}{f(v|w)} \geq \frac{vw}{w'} - \frac{1-F(\frac{vw}{w'}|w)}{f(\frac{vw}{w'}|w)} \geq \frac{vw}{w'} - \frac{1-F(\frac{vw}{w'}|w')}{f(\frac{vw}{w'}|w')},$$

[12] See Gallego and van Ryzin (1994) for a continuous time framework with Poisson arrivals and Bitran and Mondschein (1997) for a discrete time setting.

where the first inequality follows by the monotonicity of the hazard rate, and the second follows by the stochastic order assumption. Note also that condition ii) of Theorem 4.2 will play an important role for implementability of the asymptotically optimal policy that we construct in Section 4.5.

We next modify a result of Papastavrou, Rajagopalan and Kleywegt (1996) so as to identify conditions on the joint distribution $F(w, v)$ that imply concavity of expected revenue with respect to c for all periods, as required by Theorem 4.2.[13] It is convenient to introduce the joint distribution of weight and total valuation $u = vw$, which we denote by $G(w, u)$ with density $g(w, u)$. By means of a transformation of variables, the densities f and g are related by $wg(w, wv) = f(w, v)$. In particular, marginal densities in w coincide, i.e.

$$\bar{f}_w(w) = \int_0^\infty f(w, v)\, dv = \int_0^\infty g(w, u)\, du = \bar{g}_w(w).$$

Under our assumptions, the virtual total value is increasing in u with strictly positive derivative for any given w. This follows from the identity

$$\hat{u}(u, w) := u - \frac{1 - G(u|w)}{g(u|w)} = wv - \frac{1 - F(v|w)}{f(v|w)/w} = w\hat{v}(v, w).$$

We write $\hat{u}^{-1}(\hat{u}, w)$ for the inverse of $\hat{u}(u, w)$ with respect to u and define a distribution $\hat{G}(\hat{u}, w)$ by both $\hat{G}(\hat{u}|w) := G(\hat{u}^{-1}(\hat{u}, w)|w)$ for all w and $\bar{\hat{g}}_w(w) := \bar{g}_w(w)$. On the level of $\hat{v}$, this corresponds to $\hat{F}(\hat{v}|w) = F(\hat{v}^{-1}(\hat{v}, w)|w)$ and $\bar{\hat{f}}_w(w) = \bar{f}_w(w)$.

Theorem 4.3. *Assume that the conditional distribution $\hat{G}(w|\hat{u})$ is concave in w for all $\hat{u}$, that both $\hat{g}(w|\hat{u})$ and $\frac{d}{dw}\hat{g}(w|\hat{u})$ are bounded, and that the total virtual value $\hat{u}$ has a finite mean. Then, in the revenue maximization problem where the designer has complete information about w, the expected revenue $R^*(c, T + 1 - t)$ is concave as a function of c for all times t.*

Example 4.2. *A simple setting where the conditions of Theorem 4.2 are satisfied while those of Theorem 4.1 are violated is obtained as follows. Assume that $G(w, u)$ is such*

[13] In the Appendix we also provide an elementary proof of the result of Papastavrou, Rajagopalan and Kleywegt, since a proof is neither contained in the above-mentioned paper nor in the related one by Kleywegt and Papastavrou (2001). Moreover, we were unable to find a general result from finite horizon stochastic dynamic programming that ensures concavity of expected value in the state variable c, which is only a part of the relevant state description.

that u and w are independent, the hazard rate $\frac{g_u(u)}{1-G_u(u)}$ is non-decreasing, and G_w is concave.[14] *Then condition i) of Theorem 4.2 is satisfied according to Theorem 4.3 because the $\hat{G}(w|\hat{u})$ are concave. Consider then $w < w'$. By independence of u and w, we have $w'\hat{v}(\frac{vw}{w'}, w') = \hat{u}(vw, w') = \hat{u}(vw, w) = w\hat{v}(v, w)$ and hence $\hat{v}(v, w) = \frac{w'}{w}\hat{v}(\frac{vw}{w'}, w')$. As $\frac{w'}{w} > 1$, this implies condition ii) of Theorem 4.2 in the relevant domain where virtual values are non-negative. However, as we show now, condition ii) of Theorem 4.1, i.e. the hazard rate ordering, is violated. Indeed, the equation we have just derived implies also that $\frac{f(v|w)}{1-F(v|w)} = \frac{w}{w'}\frac{f(\frac{vw}{w'}|w')}{1-F(\frac{vw}{w'}|w')}$. But the conditional hazard rates of F are non-decreasing (because G_u has non-decreasing hazard rate) and $\frac{w}{w'} < 1$, so that $\frac{f(v|w)}{1-F(v|w)} = \frac{w}{w'}\frac{f(\frac{vw}{w'}|w')}{1-F(\frac{vw}{w'}|w')} < \frac{f(v|w')}{1-F(v|w')}$, which contradicts the hazard rate ordering of Theorem 4.1.*

4.5 Asymptotically optimal and time-independent pricing

The optimal policy identified above requires price adjustments in every period and for any quantity request w. These dynamics are arguably too complicated to be applied in practice. Gallego and van Ryzin (1994) used an asymptotic argument to show that the theoretical gain from optimal dynamic pricing compared to a suitably chosen, time-independent policy is usually small in the setting with unit demands. Our main theorem in this section extends their result to the dynamic knapsack problem with general distribution of types. We construct a static nonlinear price schedule that uses the existing correlations between w and v, and we show that it is asymptotically optimal if both capacity and time horizon go to infinity.

While the basic strategy of the proof follows the suggestion made by Gallego and van Ryzin, there are several major differences. In fact, in Section 5 of their paper these authors also considered the case of heterogeneous capacity demands. However, they assumed that weights and values are independent and, most importantly, their optimality benchmark does not even allow per-unit prices to depend on weight requests. But, as we saw above, such weight dependency is a general property of the dynamically optimal solution, even if w and v are independent. We therefore take our solution of the relaxed problem as the optimality benchmark, and we also consider general type distributions F.

[14] We also assume that the other mild technical conditions of Theorem 4.3 are satisfied.

As above, we start by focusing on the case of observable weights. We then show that condition ii) of Theorem 4.2 is a sufficient condition for implementability for the case with two-dimensional private information.

Like Gallego and van Ryzin, we first solve a simpler, suitably chosen deterministic maximization problem. The revenue obtained in the solution to that problem provides an upper bound for the optimal expected revenue of the stochastic problem, and the solution itself suggests the use of per-unit prices that depend on weight requests, but that are constant in time. We next show that the derived policy is asymptotically optimal also in the original stochastic problem where both capacity and time go to infinity: the ratio of expected revenue from following the considered policy over expected revenue from the optimal Markovian policy converges to one. Moreover, there are various ways to quantify this ratio for moderately large capacities and time horizons.

Let us first recall some assumptions and introduce further notation. The marginal density $\bar{f}_w(w)$ and the conditional densities $f(v|w)$ pin down the distribution of (independent) arriving types $(w_t, v_t)_{t=1}^T$. Given w, the demanded per-unit price p and the probability λ^w of a request being accepted are related by $\lambda^w(p) = 1 - F(p|w)$. Let $p^w(\lambda)$ be the inverse of λ, and note that this is well defined on $(0, 1]$. Because of monotonicity of conditional virtual values, the instantaneous (expected) per-unit revenue functions $r^w(\lambda) := \lambda\, p^w(\lambda)$ are strictly concave, and each one attains a unique interior maximum. Indeed, $p^w(\lambda) = F(\cdot|w)^{-1}(1-\lambda)$ and hence

$$\begin{aligned}
\frac{d}{d\lambda} r^w(\lambda) &= p^w(\lambda) - \lambda \frac{1}{f(p^w(\lambda)|w)} = p^w(\lambda) - \frac{1 - F(p^w(\lambda)|w)}{f(p^w(\lambda)|w)} = \hat{v}(p^w(\lambda), w), \\
\frac{d^2}{d\lambda^2} r^w(\lambda) &= -\left(\frac{\partial}{\partial v}\hat{v}\right)(p^w(\lambda), w) \frac{1}{f(p^w(\lambda)|w)} < 0.
\end{aligned}$$

Consequently, r^w is strictly concave, strictly increasing up to the $\lambda^{w,*}$ that satisfies $\hat{v}(p^w(\lambda^{w,*}), w) = 0$, and strictly decreasing from there on.

4.5.1 The deterministic problem

We now formulate an auxiliary deterministic problem that closely resembles the relaxed stochastic problem. Let $\mathrm{Cap} : (0, \infty) \to (0, \infty)$, $w \mapsto \mathrm{Cap}(w)$ be a measurable function.

Consider the problem:

$$\max_{\mathrm{Cap}(\cdot)} \int_0^\infty \max_{(\lambda_t^w)_{t=1,\dots,T}} \left(\sum_{t=1}^T r^w(\lambda_t^w) \right) w\bar{f}_w(w)\, dw, \tag{4.2}$$

subject to

$$\sum_{t=1}^T \lambda_t^w\, w\bar{f}_w(w) \le \mathrm{Cap}(w) \text{ a.s.} \quad \text{and} \quad \int_0^\infty \mathrm{Cap}(w)\, dw \;\le\; C. \tag{4.3}$$

In words, we analyze a problem where the following statements are true.

1. The capacity C needs to be divided into capacities $\mathrm{Cap}(w)$, one for each w.
2. In each w subproblem, a deterministic quantity request of $w\bar{f}_w(w)$ arrives in each period, and λ_t^w determines a share (not a probability!) of this request that is accepted and sold at per-unit price $p^w(\lambda_t^w)$.
3. In each subproblem, the allocated capacity over time cannot exceed $\mathrm{Cap}(w)$, and total allocated capacity in all subproblems $\int_0^\infty \mathrm{Cap}(w)\, dw$ cannot exceed C.
4. The designer's goal is to maximize total revenue. We call the revenue at the solution $R^d(C,T)$.

As r^w is strictly concave and increasing up to $\lambda^{w,*}$, it is straightforward to verify that, given a choice $\mathrm{Cap}(w)$, the solution to the w subproblem

$$\max_{(\lambda_t^w)_{t=1,\dots,T}} \left(\sum_{t=1}^T r^w(\lambda_t^w) \right) w\bar{f}_w(w) \;\text{ such that }\; \sum_{t=1}^T \lambda_t^w\, w\bar{f}_w(w) \le \mathrm{Cap}(w)$$

is given by

$$\lambda_t^w \equiv \lambda^{w,d} := \begin{cases} \lambda^{w,*} & \text{if } \lambda^{w,*} \le \frac{\mathrm{Cap}(w)}{Tw\bar{f}_w(w)} \\ \frac{\mathrm{Cap}(w)}{Tw\bar{f}_w(w)} & \text{else.} \end{cases} \tag{4.4}$$

Accordingly, the revenue in the w subproblem is $r^w(\lambda^{w,d})Tw\bar{f}_w(w)$.

Proposition 4.2. *The solution to the deterministic problem given by* (4.2) *and* (4.3) *is characterized by* *i)* $\hat{v}(p^w(\lambda^{w,d}), w) = \beta(C,T) = \text{const}$,

ii) $\lambda_t^w = \lambda^{w,d} = \frac{\text{Cap}(w)}{Tw\bar{f}_w(w)}$,

iii) $\int_0^\infty \text{Cap}(w)\,dw \;=\; \min(C, T\int_0^\infty \lambda^{w,*} w\bar{f}_w(w)\,dw)$.

To get an intuition for the above result, observe that the marginal increase of the optimal revenue for the w subproblem from marginally increasing $\text{Cap}(w)$ is

$$\left(\frac{d}{d\lambda} r^w\right)\left(\frac{\text{Cap}(w)}{Tw\bar{f}_w(w)}\right) \;=\; \hat{v}(p^w(\lambda^{w,d}), w) \;\text{ if }\; \lambda^{w,*} \;>\; \frac{\text{Cap}(w)}{Tw\bar{f}_w(w)},$$

and 0 otherwise. Proposition 4.2 says that, optimally, the capacity should be split in such a way that the marginal revenue from increasing $\text{Cap}(w)$ is the same for all w. Actually solving the problem amounts to the simple static exercise of determining the constant $\beta(C,T)$ in accordance with the integral feasibility constraint.

The above construction is justified by the following two-step argument: on the one hand, we show in Theorem 4.4 below that the optimal revenue in the deterministic problem, $R^d(C,T)$, bounds from above the optimal revenue in the original stochastic case. On the other hand, as we show in Section 4.5.2, the optimal solution of the deterministic problem serves to define a simple time-independent policy that in the stochastic problem captures revenues $R^{TI}(C,T)$ such that $\frac{R^{TI}(C,T)}{R^d(C,T)}$ converges to 1 as C and T go to infinity. Combining these two points yields the kind of asymptotic optimality result we want to establish.

Since we assume here that weights are observable, a Markovian policy α for the original stochastic problem is characterized by the acceptance probabilities $\lambda_t^{w_t}[c_t]$ contingent on current time t, remaining capacity c_t, and weight request w_t. Expected revenue from policy α at the beginning of period t (i.e. when there are $(T-t+1)$ periods left) with remaining capacity c_t is given by

$$R_\alpha(c_t, T-t+1) = E_\alpha\left[\sum_{s=t}^{T} w_s\, p^{w_s}(\lambda_s^{w_s}[c_s])\, I_{\{v_s \geq p^{w_s}(\lambda_s^{w_s}[c_s])\}}\right]$$

such that

$$\sum_{s=t}^{T} w_s\, I_{\{v_s \geq p^{w_s}(\lambda_s^{w_s}[c_s])\}} \leq c_t.$$

Here, the constraint must hold almost surely when following α. As before, we write $R^*(c_t, T-t+1)$ for the optimal revenue, i.e. the supremum of expected revenues taken over all feasible Markovian policies α.

Theorem 4.4. *For any capacity C and deadline T, it holds that $R^*(C,T) \leq R^d(C,T)$.*

4.5.2 A simple policy for the stochastic problem

Having established the upper bound of Theorem 4.4, we proceed with the second part of our two-step argument outlined in the preceding section. We use the optimal solution of the deterministic problem to define a w-contingent yet time-independent policy α_{TI} for the stochastic case as follows.

1. Given C and T, solve the deterministic problem to obtain $\beta(C,T)$, $\lambda^{w,d}$ and thus $p^{w,d} := p^w(\lambda^{w,d}) = \hat{v}^{-1}(\beta(C,T), w)$.
2. In the stochastic problem charge these weight-contingent prices $p^{w,d}$ for the entire time horizon, provided that the quantity request does not exceed the remaining capacity. Else, charge a price equal to $+\infty$ (i.e., reject the request).

Note that under condition ii) of Theorem 4.2, the time-independent policy α_{TI} *is also implementable if weights are not observable*! Indeed, setting all virtual valuation thresholds equal to a constant is like setting them optimally for linear and hence concave salvage values.

We now determine how well the time-independent policy constructed above performs compared to the optimal Markovian policy. Recall that we do this by comparing its expected revenue, $R^{TI}(C,T)$, with the optimal revenue in the deterministic problem, $R^d(C,T)$, which, as we know by Theorem 4.4, provides an upper bound for the optimal revenue in the stochastic problem, $R^*(C,T)$.

Theorem 4.5. *i) For any joint distribution of values and weights*

$$\lim_{C,T\to\infty, \frac{C}{T}=\text{const}} \frac{R^{TI}(C,T)}{R^d(C,T)} = 1.$$

ii) Assume that w and v are independent. Then

$$\frac{R^{TI}(C,T)}{R^d(C,T)} \geq \left(1 - \frac{\sqrt{E[w^2]/E[w]}}{2\sqrt{\min(C, \lambda^* E[w]T)}}\right).$$

In particular, $\lim_{\min(C,T)\to\infty} \frac{R^{TI}(C,T)}{R^d(C,T)} = 1.$

We have chosen to focus on these two general limit results, but various other quantitative results could be proven by similar techniques at the expense of slightly more technical effort and possibly some further assumptions on the distribution F. This should be clear from the proof in the Appendix. Since the policy α_{TI} is stationary, it does not generate incentives to postpone arrivals even in a more complex model where buyers are patient and can choose their arrival time.

Remark 4.2. *In a complete information knapsack model, Lin, Lu and Yao (2008) studied policies that start by accepting only high value requests and then switch over to accepting also lower values as time goes by. They established asymptotic optimality of such policies (with carefully chosen switch-over times) as available capacity and time go to infinity. In other words, their prices are time-dependent but do not condition on the weight request. It is easy to show that, in our incomplete information model, such policies are in general suboptimal. Consider first a one-period example where the seller has capacity 2, and where the arriving agent has either a weight request of 1 or 2 (equally likely). If the weight request is 1(2), the agent's per-unit value distributes uniformly between 0 and 1 (between 1 and 2). The optimal mechanism in this case is as follows: if the buyer requests one unit, the seller sells it for a price of 0.5, and if the buyer requests two units, the seller sells each unit at a price of 1. Note that this policy is implementable since the requested per-unit price is monotonically increasing in the weight request. The expected revenue is* $\frac{9}{8}$*. If, however, the seller is forced to sell all units at the same per-unit price without conditioning on the weight request, he will charge the price of 1 for each unit, yielding an expected revenue of 1, and thus loose* $\frac{1}{8}$ *versus the optimal policy. Now replicate this problem so that there are T periods and capacity C=2T. Then, the expected revenue from the optimal mechanism is* $\frac{9}{8}T$*, while the expected revenue from the constrained mechanism is only T. Obviously, the constrained mechanism is not asymptotically optimal.*

4.6 Appendix for Chapter 4

Proof of Proposition 4.1. $\Longrightarrow$. So assume that conditions i) and ii) are satisfied and define for any t, c,

$$q_t^c(w,v) = \begin{cases} wp_t^c(w) & \text{if } \alpha_t^c(w,v) = 1 \\ 0 & \text{if } \alpha_t^c(w,v) = 0. \end{cases}$$

Consider then an arrival of type (w, v) in period t with remaining capacity c. There are two cases.

a) $\alpha_t^c(w,v) = 1$. In particular, $v \geq p_t^c(w)$. Then, truth-telling yields utility $w(v - p_t^c(w)) \geq 0$. Assume that the agent reports instead $(\widehat{w}, \widehat{v})$. If $\alpha_t^c(\widehat{w}, \widehat{v}) = 0$, then the agent's utility is zero and the deviation is not profitable. Assume then that $\alpha_t^c(\widehat{w}, \widehat{v}) = 1$. By the form of the utility function, a report of $\widehat{w} < w$ is never profitable. But, for $\widehat{w} \geq w$, the agent's utility is $wv - \widehat{w}p_t^c(\widehat{w}) \leq w(v - p_t^c(w))$, where we used condition ii). Therefore, such a deviation is also not profitable.

b) $\alpha_t^c(w,v) = 0$. In particular, $v \leq p_t^c(w)$. Truth-telling yields here utility of zero. Assume that the agent reports instead $(\widehat{w}, \widehat{v})$. If $\alpha_t^c(\widehat{w}, \widehat{v}) = 0$, then the agent's utility remains zero, and the deviation is not profitable. Assume then that $\alpha_t^c(\widehat{w}, \widehat{v}) = 1$. By the form of the utility function, a report of $\widehat{w} < w$ is never profitable. Thus, consider the case where $\widehat{w} \geq w$. In this case, the agent's utility is $wv - \widehat{w}p_t^c(\widehat{w}) \leq w(v - p_t^c(w)) \leq 0$, where we used condition ii). Therefore, such a deviation is also not profitable.

$\Longleftarrow$. Consider now an implementable, deterministic and Markovian allocation policy $\{\alpha_t^c\}_{t,c}$. Assume first, by contradiction, that condition i) is not satisfied. Then, there exist (w, v) and (w, v') such that $v' > v$, $\alpha_t^c(w,v) = 1$ and $\alpha_t^c(w,v') = 0$. We obtain the chain of inequalities $wv' - q_t^c(w,v) > wv - q_t^c(w,v) \geq -q_t^c(w,v')$ where the second inequality follows by incentive compatibility for type (w, v). This shows that a deviation to a report (w, v) is profitable for type (w, v'), a contradiction to implementability. Therefore, condition i) must hold.

In particular, note that for any two types who have the same weight request, (w, v) and (w, v'), if both are accepted, i.e. $\alpha_t^c(w,v) = \alpha_t^c(w,v') = 1$, the payment must be the same (otherwise the type which needs to make the higher payment would deviate and report the other type). Denote this payment by $r_t^c(w)$. Note also that any two types (w, v) and (w', v') such that $\alpha_t^c(w,v) = \alpha_t^c(w',v') = 0$ must also make the same payment (otherwise the type that needs to make the higher payment would deviate and report the other type) and denote this payment by s.

Assume now, by contradiction, that condition ii) does not hold. Then there exist w and w' such that $w' > w$ but $w'p_t^c(w') < wp_t^c(w)$. In particular, $w'p_t^c(w') < \infty$, and therefore $p_t^c(w') < \infty$.

Assume first that $p_t^c(w) < \infty$. We have $w' p_t^c(w') - r_t^c(w') = w p_t^c(w) - r_t^c(w) = -s$ because, by incentive compatibility, both types $(w, p_t^c(w))$ and $(w', p_t^c(w'))$ must be indifferent between getting their request and not getting it. Since by assumption $w' p_t^c(w') < w p_t^c(w)$, we obtain that $r_t^c(w') < r_t^c(w)$. Consider now a type (w, v) for which $v > p_t^c(w)$. By reporting truthfully, this type gets utility $wv - r_t^c(w)$, while by deviating to (w', v) he gets utility $wv - r_t^c(w') > wv - r_t^c(w)$, a contradiction to incentive compatibility.

Assume now that $p_t^c(w)$ is infinite, and therefore $w p_t^c(w)$ is infinite. Consider a type (w', v) where $v > p_t^c(w')$. The utility of this type is $w'v - r_t^c(w') > w' p_t^c(w') - r_t^c(w') = -s$. In particular, $r_t^c(w')$ must be finite. By reporting truthfully, a type (w, v) gets utility $-s$, while by deviating to a report of (w', v) he gets $wv - r_t^c(w')$. For v large enough, we obtain $wv - r_t^c(w') > -s$, a contradiction to implementability.

Thus, condition ii) must hold and, in particular, the payment $r_t^c(w)$ is monotonic in w. □

Proof of Theorem 4.1. Let $w < w'$. We need to show that $w p_t^c(w) - w' p_t^c(w') \leq 0$. If $p_t^c(w) \leq p_t^c(w')$, the result is clear. Assume then that $p_t^c(w) > p_t^c(w')$. We obtain the following chain of inequalities:

$$
\begin{aligned}
& w\left(\frac{1 - F(p_t^c(w)|w)}{f(p_t^c(w)|w)}\right) - w'\left(\frac{1 - F(p_t^c(w')|w')}{f(p_t^c(w')|w')}\right) \\
\leq \;& w'\left(\frac{1 - F(p_t^c(w)|w)}{f(p_t^c(w)|w)} - \frac{1 - F(p_t^c(w')|w')}{f(p_t^c(w')|w')}\right) \\
\leq \;& w'\left(\frac{1 - F(p_t^c(w')|w)}{f(p_t^c(w')|w)} - \frac{1 - F(p_t^c(w')|w')}{f(p_t^c(w')|w')}\right) \leq 0,
\end{aligned}
$$

where the second inequality follows by the monotonicity of the hazard rate and the third follows by the hazard rate ordering condition.

Since $R^*(c - w, T - t)$ is monotonically decreasing in w, we obtain that

$$
\begin{aligned}
w\left(p_t^c(w) - \frac{1 - F(p_t^c(w)|w)}{f(p_t^c(w)|w)}\right) &\leq w'\left(p_t^c(w') - \frac{1 - F(p_t^c(w')|w')}{f(p_t^c(w')|w')}\right) \Leftrightarrow \\
w p_t^c(w) - w' p_t^c(w') &\leq w\left(\frac{1 - F(p_t^c(w)|w)}{f(p_t^c(w)|w)}\right) - w'\left(\frac{1 - F(p_t^c(w')|w')}{f(p_t^c(w')|w')}\right) \leq 0,
\end{aligned}
$$

where the last inequality follows by the derivation above. Hence $w p_t^c(w) - w' p_t^c(w') \leq 0$

as desired. □

Proof of Theorem 4.2. For any concave function ϕ, and for any $x < y < z$ in its domain, the well known "Three Chord Lemma" asserts that

$$\frac{\phi(y)-\phi(x)}{y-x} \geq \frac{\phi(z)-\phi(x)}{z-x} \geq \frac{\phi(z)-\phi(y)}{z-y}.$$

Consider then $w < w'$ and let $x = c - w' < y = c - w < z = c$. For the case of a concave revenue, the lemma then yields

$$\begin{aligned}\frac{R^*(c-w,T-t)-R^*(c-w',T-t)}{w'-w} \quad &\geq \quad \frac{R^*(c,T-t)-R^*(c-w',T-t)}{w'} \\ &\geq \quad \frac{R^*(c,T-t)-R^*(c-w,T-t)}{w}.\end{aligned}$$

We obtain in particular

$$\begin{aligned} p_t^c(w') - \frac{1-F(p_t^c(w')|w')}{f(p_t^c(w')|w')} &= \frac{R^*(c,T-t)-R^*(c-w',T-t)}{w'} \\ \geq \quad \frac{R^*(c,T-t)-R^*(c-w,T-t)}{w} &= p_t^c(w) - \frac{1-F(p_t^c(w)|w)}{f(p_t^c(w)|w)},\end{aligned}$$

which yields

$$p_t^c(w') - \frac{1-F(p_t^c(w')|w')}{f(p_t^c(w')|w')} \geq p_t^c(w) - \frac{1-F(p_t^c(w)|w)}{f(p_t^c(w)|w)} \geq \frac{w}{w'}p_t^c(w) - \frac{1-F(\frac{w}{w'}p_t^c(w)|w')}{f(\frac{w}{w'}p_t^c(w)|w')}$$

where the last inequality follows by the condition in the statement of the Theorem. Since virtual values are increasing, this yields $p_t^c(w') \geq \frac{w}{w'}p_t^c(w) \Leftrightarrow w'p_t^c(w') \geq wp_t^c(w)$ as desired. □

For the proof of Theorem 4.3, we first need a lemma on maximization of expected welfare under complete information. The result appears (without proof) in Papastavrou, Rajagopalan and Kleywegt (1996).

Lemma 4.1. *Assume that the total value u has finite mean, and that both $g(w|u)$ and $\frac{d}{dw}g(w|u)$ are bounded and continuous. Consider the allocation policy that maximizes expected welfare under complete information (i.e., upon arrival the agent's type is revealed to the designer). If $G(w|u)$ is concave in w for all u, then the optimal expected welfare,* *denoted U_t^c, is twice continuously differentiable and concave in the remaining capacity c for all periods $t \leq T$.*

Proof of Lemma 4.1. Note that, for notational convenience throughout this proof, we index optimal expected welfare by the current time t and not by periods remaining to deadline. By standard arguments, the optimal policy for this unconstrained dynamic optimization problem is deterministic and Markovian, and U_t^c is non-decreasing in remaining capacity c by a simple strategy duplication argument. Moreover, the optimal policy can be characterized by weight thresholds $w_t^c(u) \leq c$: If c remains at time t and a request whose acceptance would generate value u arrives, then it is accepted if and only if $w \leq w_t^c(u)$. If $U_{t+1}^c \geq u$, then the weight threshold must satisfy the indifference condition

$$u = U_{t+1}^c - U_{t+1}^{c-w_t^c(u)}. \tag{4.5}$$

Otherwise, we have $w_t^c(u) = c$.
We now prove the lemma by backward induction. At time $t = T$, i.e. in the deadline period, it holds that

$$U_T^c = \int_0^\infty G(c|u)u\,\bar{g}_u(u)\,du.$$

This is concave in c because all $G(c|u)$ are concave by assumption, because $u\,\bar{g}_u(u)$ is positive, and because the distribution of u has a finite mean. Since both $g(w|u)$ and $\frac{d}{dw}g(w|u)$ are bounded and continuous, U_T^c is also twice continuously differentiable.
Suppose now that the lemma has been proven down to time $t+1$. The optimal expected welfare at t provided that capacity c remains may be written as

$$U_t^c = \int_0^\infty \left[uG(w_t^c(u)|u) + \int_0^{w_t^c(u)} U_{t+1}^{c-w} g(w|u)\,dw + (1 - G(w_t^c(u)|u))U_{t+1}^c\right]\bar{g}_u(u)\,du. \tag{4.6}$$

We proceed to show concavity with respect to c of the term in brackets, for all u. This in turn implies concavity of U_t^c and hence, with a short additional argument for differentiability, is sufficient to conclude the induction step. We distinguish the cases $u > U_{t+1}^c$ for which the indifference condition (4.5) does not hold, and $u \leq U_{t+1}^c$ for which it does. For both cases, we demonstrate that the second derivative (one-sided if necessary) of the bracket term with respect to c is non-positive, and thus establish global concavity.

Case 1: $u > U_{t+1}^c$. The bracket term becomes $uG(c|u) + \int_0^c U_{t+1}^{c-w} g(w|u)\,dw + (1 -$

$G(c|u))U^c_{t+1}$. By continuity of U^c_{t+1}, this representation also holds in a small interval around c. We find

$$\begin{aligned}
&\frac{d}{dc}\left[uG(c|u) + \int_0^c U^{c-w}_{t+1} g(w|u)\,dw + (1-G(c|u))U^c_{t+1}\right] \\
&= ug(c|u) + \int_0^c \frac{d}{dc} U^{c-w}_{t+1} g(w|u)\,dw + U^0_{t+1}\, g(c|u) \\
&\quad -g(c|u)\, U^c_{t+1} + (1-G(c|u))\frac{d}{dc}U^c_{t+1} \\
&= (u - U^c_{t+1})g(c|u) + \int_0^c \frac{d}{dc} U^{c-w}_{t+1} g(w|u)\,dw + (1-G(c|u))\frac{d}{dc}U^c_{t+1}
\end{aligned}$$

and

$$\begin{aligned}
&\frac{d^2}{dc^2}\left[uG(c|u) + \int_0^c U^{c-w}_{t+1} g(w|u)\,dw + (1-G(c|u))U^c_{t+1}\right] \\
&= (u - U^c_{t+1})g'(c|u) - g(c|u)\frac{d}{dc}U^c_{t+1} + \int_0^c \frac{d^2}{dc^2} U^{c-w}_{t+1} g(w|u)\,dw \\
&+ \frac{d}{dw}U^w_{t+1}\Big|_{w=0} g(c|u) - g(c|u)\frac{d}{dc}U^c_{t+1} + (1-G(c|u))\frac{d^2}{dc^2}U^c_{t+1}.
\end{aligned} \tag{4.7}$$

The last term is non-positive by the concavity of U^c_{t+1}, the first term is non-positive because $u > U^c_{t+1}$ and because $G(c|u)$ has a non-increasing density by assumption. In addition, $g(c|u)\frac{d}{dc}U^c_{t+1}$ is non-negative, and hence (4.7) is bounded from above by

$$\int_0^c \frac{d^2}{dc^2} U^{c-w}_{t+1} g(w|u)\,dw + g(c|u)\left(\frac{d}{dw}U^w_{t+1}\Big|_{w=0} - \frac{d}{dc}U^c_{t+1}\right).$$

But $\int_0^c \frac{d^2}{dc^2} U^{c-w}_{t+1} g(w|u)\,dw$ may be bounded from above by $g(c|u)\int_0^c \frac{d^2}{dc^2} U^{c-w}_{t+1}\,dw$ because of the decreasing density and because $\frac{d^2}{dc^2}U^{c-w}_{t+1} \le 0$. Thus,

$$\begin{aligned}
&\frac{d^2}{dc^2}\left[uG(c|u) + \int_0^c U^{c-w}_{t+1} g(w|u)\,dw + (1-G(c|u))U^c_{t+1}\right] \\
&\le g(c|u)\left[\int_0^c \frac{d^2}{dc^2} U^{c-w}_{t+1}\,dw + \frac{d}{dw}U^w_{t+1}\Big|_{w=0} - \frac{d}{dc}U^c_{t+1}\right] \\
&= g(c|u)\left[\int_0^c \frac{d^2}{dw^2} U^{c-w}_{t+1}\,dw + \frac{d}{dw}U^w_{t+1}\Big|_{w=0} - \frac{d}{dc}U^c_{t+1}\right] = 0.
\end{aligned} \tag{4.8}$$

Case 2: $u \le U^c_{t+1}$. Here $u = U^c_{t+1} - U^{c-w^c_t(u)}_{t+1}$. Consequently, the bracket term in

(4.6) becomes

$$U_{t+1}^{c} - U_{t+1}^{c-w_t^c(u)} G(w_t^c(u)|u) + \int_0^{w_t^c(u)} U_{t+1}^{c-w} g(w|u)\,dw. \tag{4.9}$$

Before computing its first and second derivatives, we differentiate the identity $u = U_{t+1}^{c} - U_{t+1}^{c-w_t^c(u)}$ to obtain an expression for $\frac{d}{dc} w_t^c(u)$ (derivative from the right if $u = U_{t+1}^c$):

$$0 = \frac{d}{dc} U_{t+1}^{c} - \frac{d}{dw} U_{t+1}^{w}\Big|_{w=c-w_t^c(u)} \left(1 - \frac{d}{dc} w_t^c(u)\right).$$

Since indeed $\frac{d}{dw} U_{t+1}^{w} > 0$ in our setup with strictly positive densities, this implies

$$\frac{d}{dc} w_t^c(u) = \frac{\frac{d}{dw} U_{t+1}^{w}\Big|_{w=c-w_t^c(u)} - \frac{d}{dc} U_{t+1}^{c}}{\frac{d}{dw} U_{t+1}^{w}\Big|_{w=c-w_t^c(u)}}. \tag{4.10}$$

By concavity of U_{t+1}^c, its derivative is non-increasing and hence the identity (4.10) yields in particular $\frac{d}{dc} w_t^c(u) \geq 0$. We now compute the derivatives of (4.9):

$$\begin{aligned}
&\frac{d}{dc}\left[U_{t+1}^{c} - U_{t+1}^{c-w_t^c(u)} G(w_t^c(u)|u) + \int_0^{w_t^c(u)} U_{t+1}^{c-w} g(w|u)\,dw\right] \\
&= \frac{d}{dc} U_{t+1}^{c} - \frac{d}{dw} U_{t+1}^{w}\Big|_{w=c-w_t^c(u)} \left(1 - \frac{d}{dc} w_t^c(u)\right) G(w_t^c(u)|u) \\
&\quad - U_{t+1}^{c-w_t^c(u)} g(w_t^c(u)|u) \frac{d}{dc} w_t^c(u) \\
&\quad + U_{t+1}^{c-w_t^c(u)} g(w_t^c(u)|u) \frac{d}{dc} w_t^c(u) + \int_0^{w_t^c(u)} \frac{d}{dc} U_{t+1}^{c-w} g(w|u)\,dw \\
&\overset{(4.10)}{=} \frac{d}{dc} U_{t+1}^{c} - \frac{d}{dc} U_{t+1}^{c}\, G(w_t^c(u)|u) + \int_0^{w_t^c(u)} \frac{d}{dc} U_{t+1}^{c-w} g(w|u)\,dw \\
&= \frac{d}{dc} U_{t+1}^{c} (1 - G(w_t^c(u)|u)) + \int_0^{w_t^c(u)} \frac{d}{dc} U_{t+1}^{c-w} g(w|u)\,dw.
\end{aligned}$$

Thus,

$$
\begin{aligned}
&\frac{d^2}{dc^2}\left[U_{t+1}^{c} - U_{t+1}^{c-w_t^c(u)} G(w_t^c(u)|u) + \int_0^{w_t^c(u)} U_{t+1}^{c-w} g(w|u)\,dw\right] \\
&= \frac{d^2}{dc^2}U_{t+1}^{c}(1 - G(w_t^c(u)|u)) - \frac{d}{dc}U_{t+1}^{c}\, g(w_t^c(u)|u)\frac{d}{dc}w_t^c(u) \\
&\quad + \frac{d}{dw}U_{t+1}^{w}\Big|_{w=c-w_t^c(u)} g(w_t^c(u)|u)\frac{d}{dc}w_t^c(u) + \int_0^{w_t^c(u)} \frac{d^2}{dc^2}U_{t+1}^{c-w} g(w|u)\,dw \\
&\leq g(w_t^c(u)|u)\frac{d}{dc}w_t^c(u)\left(\frac{d}{dw}U_{t+1}^{w}\Big|_{w=c-w_t^c(u)} - \frac{d}{dc}U_{t+1}^{c}\right) + \int_0^{w_t^c(u)} \frac{d^2}{dw^2}U_{t+1}^{c-w} g(w|u)\,dw.
\end{aligned}
$$

For the final inequality we used concavity of U_{t+1}^c, as well as $\frac{d^2}{dc^2}U_{t+1}^{c-w} = \frac{d^2}{dw^2}U_{t+1}^{c-w}$. Noting that (4.10) implies that $\frac{d}{dc}w_t^c(u) \leq 1$ and once more using concavity of U_{t+1}^c, we may bound the first term from above. Since $g(w|u)$ is non-increasing in w, we can also bound the second term to obtain

$$
\begin{aligned}
&\frac{d^2}{dc^2}\left[U_{t+1}^{c} - U_{t+1}^{c-w_t^c(u)} G(w_t^c(u)|u) + \int_0^{w_t^c(u)} U_{t+1}^{c-w} g(w|u)\,dw\right] \\
&\leq g(w_t^c(u)|u)\left(\frac{d}{dw}U_{t+1}^{w}\Big|_{w=c-w_t^c(u)} - \frac{d}{dc}U_{t+1}^{c} + \int_0^{w_t^c(u)} \frac{d^2}{dw^2}U_{t+1}^{c-w}\,dw\right) = 0.
\end{aligned}
\tag{4.11}
$$

Taken together, (4.8) and (4.11) establish concavity of the integrand in (4.6) with respect to c. This implies that U_t^c is concave. Having a second look at the computations just performed reveals that the integrand in (4.6) has a kink in the second derivative at $u = U_{t+1}^c$. However, this event has measure zero for any given c, so that we also get that U_t^c is twice continuously differentiable. This completes the induction step. □

Proof of Theorem 4.3. The main idea of the proof is to translate the problem of maximizing revenue when w is observable into the problem of maximizing welfare with respect to virtual values (rather than values) and to use Lemma 4.1 then.

To begin with, note that there is a dual way to describe the policy that maximizes expected welfare under complete information. In the proof of Lemma 4.1, we characterized it by optimal weight thresholds $w_t^c(u)$. Alternatively, given any requested quantity $w \leq c$, we may set a valuation per unit threshold $v_t^c(w)$. Requests above this valuation are accepted, those below are not. Such optimal thresholds are characterized by the Bellman-type condition

$$wv_t^c(w) = U_{t+1}^c - U_{t+1}^{c-w}.$$

Thus, using $v_t^c(w) = +\infty$ for $w > c$, one way to write optimal expected welfare under complete information is

$$\begin{aligned} U_t^c &= \int_0^c w \int_{v_t^c(w)}^\infty v f(v|w)\, dv\, \bar{f}_w(w)\, dw \\ &\quad + \int_0^\infty \Big[(1 - F(v_t^c(w)|w))U_{t+1}^{c-w} + F(v_t^c(w)|w)U_{t+1}^c\Big]\, \bar{f}_w(w)\, dw. \end{aligned} \tag{4.12}$$

In contrast, the optimal expected revenue with complete information about w but incomplete information about v satisfies

$$\begin{aligned} R^*(c, T+1-t) &= \int_0^c w\, p_t^c(w)\, (1 - F(p_t^c(w)|w))\, \bar{f}_w(w)\, dw \qquad (4.13) \\ + & \int_0^\infty [(1 - F(p_t^c(w)|w))R^*(c-w, T-t) \;+\; F(p_t^c(w)|w)R^*(c, T-t)]\; \bar{f}_w(w)\, dw, \end{aligned}$$

where $p_t^c(w)$ are the per-unit prices from (4.1). We rephrase this in terms of $\hat{F}$, whose definition required monotonicity of virtual values. Setting $\hat{v}_t^c(w) := \hat{v}(p_t^c(w), w)$ we have on the one hand

$$F(p_t^c(w)|w) = \hat{F}(\hat{v}_t^c(w)|w).$$

On the other hand,

$$\begin{aligned} p_t^c(w)\, (1 - F(p_t^c(w)|w)) &= \int_{p_t^c(w)}^\infty [v\, f(v|w) - (1 - F(v|w))]\; dv \\ &= \int_{p_t^c(w)}^\infty \hat{v}(v, w)\, \hat{f}(\hat{v}(v, w)|w) \frac{d}{dv}\hat{v}(v, w)\, dv \\ &= \int_{\hat{v}_t^c(w)}^\infty \hat{v}\, \hat{f}(\hat{v}|w)\, d\hat{v}. \end{aligned}$$

Plugging this and the identities for the marginal densities in w into (4.13), we obtain:

$$\begin{aligned} R^*(c, T+1-t) \;&=\; \int_0^c w \int_{\hat{v}_t^c(w)}^\infty \hat{v} \hat{f}(\hat{v}|w)\, d\hat{v}\, \bar{\hat{f}}_w(w)\, dw \\ + \; & \int_0^\infty \Big[(1 - \hat{F}(\hat{v}_t^c(w)|w))R^*(c-w, T-t) \;+\; \hat{F}(\hat{v}_t^c(w)|w)R^*(c, T-t)\Big]\; \bar{\hat{f}}_w(w)\, dw. \end{aligned}$$

Comparing this with (4.12), it follows that maximizing expected revenue when w is observable is equivalent to maximizing expected welfare with respect to the distribution of weight and conditional virtual valuation (note the identical zero boundary values at $T+1$). Invoking Lemma 4.1 applied to $\hat{G}$, we see that $R^*(c, T+1-t)$ is concave with respect to c for all t (note that the fact that the support of virtual valuations contains also negative numbers does not matter for the argument of Lemma 4.1). $\square$

Proof of Proposition 4.2. The proposition is an immediate consequence of the characterization (4.4) of optimal solutions for the w subproblems given $\mathrm{Cap}(w)$, and of a straightforward variational argument ensuring that marginal revenues from marginal increase of $\mathrm{Cap}(w)$ must be constant almost surely in w. $\square$

Proof of Theorem 4.4. We need to distinguish two cases.

Case 1: Assume that $C > T\int_0^\infty \lambda^{w,*} w \bar{f}_w(w)\,dw$. In this case, $\beta(C,T) = 0$ and $R^d(C,T) = T\int_0^\infty r^w(\lambda^{w,*}) w \bar{f}_w(w)\,dw$. We also know that $R^*(C,T) \le R^*(+\infty,T)$, where $R^*(+\infty,T)$ denotes the optimal expected revenue from a stochastic problem without any capacity constraint. But for such a problem, the optimal Markovian policy maximizes at each period the instantaneous expected revenue upon observing w_t, $w_t\, r^{w_t}(\lambda)$. That is, the optimal policy sets $\lambda_t^{w_t}[+\infty] = \lambda^{w,*}$. Thus,

$$R^*(C,T) \le R^*(+\infty,T) = T\int_0^\infty w r^w(\lambda^{w,*}) \bar{f}_w(w)\,dw = R^d(C,T).$$

Case 2: Assume now that $C \le T\int_0^\infty \lambda^{w,*} w \bar{f}_w(w)\,dw$. For $\mu \ge 0$, consider the unconstrained maximization problem

$$\max_{\mathrm{Cap}(\cdot)} \left[\int_0^\infty r^w\left(\frac{\mathrm{Cap}(w)}{Tw\bar{f}_w(w)}\right) Tw\bar{f}_w(w)\,dw \;+\; \mu\left(C - \int_0^\infty \mathrm{Cap}(w)\,dw\right)\right].$$

The Euler-Lagrange equation is $\left(\frac{d}{d\lambda} r^w\right)\left(\frac{\mathrm{Cap}(w)}{Tw\bar{f}_w(w)}\right) = \mu$. Hence, if we write $R^d(C,T,\mu)$ for the optimal value of the above problem and if we let $\mu = \beta(C,T)$, where $\beta(C,T)$ is the constant from Proposition 4.2, then the solution equals the one of the constrained deterministic problem. In particular $\int_0^\infty \mathrm{Cap}(w)\,dw = C$, and $R^d(C,T,\beta(C,T)) = R^d(C,T)$.

Recall that for the stochastic problem and for any Markovian policy α we have

$$R_\alpha(C,T) = E_\alpha\left[\sum_{t=1}^{T} w_t\, p^{w_t}(\lambda_t^{w_t}[c_t])\, I_{\{v_t \ge p^{w_t}(\lambda_t^{w_t}[c_t])\}}\right],$$

and define

$$R_\alpha(C,T,\beta(C,T)) = R_\alpha(C,T) + \beta(C,T)\left(C - E_\alpha\left[\sum_{t=1}^{T} w_t\, I_{\{v_t \geq p^{w_t}(\lambda_t^{w_t}[c_t])\}}\right]\right).$$

Since for any policy α that is admissible in the original problem, it holds that

$$\sum_{t=1}^{T} w_t\, I_{\{v_t \geq p^{w_t}(\lambda_t^{w_t}[c_t])\}} \leq C \quad \text{a.s.},$$

we have $R_\alpha(C,T) \leq R_\alpha(C,T,\beta(C,T))$. We show below that, for arbitrary α (which may or may not satisfy the capacity constraint), it holds that

$$R_\alpha(C,T,\beta(C,T)) \leq R^d(C,T,\beta(C,T)). \tag{4.14}$$

This yields, for any α that is admissible in the original problem,

$$R_\alpha(C,T) \leq R_\alpha(C,T,\beta(C,T)) \leq R^d(C,T,\beta(C,T)) = R^d(C,T).$$

Taking the supremum over α then concludes the proof for the second case.

It remains to prove (4.14). The argument uses the filtration $\{\mathcal{F}_t\}_{t=1}^T$ of σ-algebras that contain information prior to time t (in particular the value of c_t) and, in addition, the currently observed w_t.

$$
\begin{aligned}
R_\alpha(C,T,\beta(C,T)) &= E_\alpha\left[\sum_{t=1}^{T} w_t\left(p^{w_t}(\lambda_t^{w_t}[c_t]) - \beta(C,T)\right) I_{\{v_t\geq p^{w_t}(\lambda_t^{w_t}[c_t])\}}\right] + \beta(C,T)C \\
&= E_\alpha\left[\sum_{t=1}^{T} E_\alpha\left[w_t\left(p^{w_t}(\lambda_t^{w_t}[c_t]) - \beta(C,T)\right) I_{\{v_t\geq p^{w_t}(\lambda_t^{w_t}[c_t])\}}|\mathcal{F}_t\right]\right] + \beta(C,T)C \\
&= E_\alpha\left[\sum_{t=1}^{T} w_t\left(p^{w_t}(\lambda_t^{w_t}[c_t]) - \beta(C,T)\right) E_\alpha\left[I_{\{v_t\geq p^{w_t}(\lambda_t^{w_t}[c_t])\}}|\mathcal{F}_t\right]\right] + \beta(C,T)C \\
&= E_\alpha\left[\sum_{t=1}^{T} w_t\left(r^{w_t}(\lambda_t^{w_t}[c_t]) - \beta(C,T)\lambda_t^{w_t}[c_t]\right)\right] + \beta(C,T)C \\
&\leq E_\alpha\left[\sum_{t=1}^{T} w_t\left(r^{w_t}(\lambda^{w_t,d}) - \beta(C,T)\lambda^{w_t,d}\right)\right] + \beta(C,T)C \\
&= E_{(w_t)_{t=1}^T}\left[\sum_{t=1}^{T} w_t\left(r^{w_t}(\lambda^{w_t,d}) - \beta(C,T)\lambda^{w_t,d}\right)\right] + \beta(C,T)C \\
&= T\int_0^\infty (r^w(\lambda^{w,d}) - \beta(C,T)\lambda^{w,d}) w\bar{f}_w(w)\,dw + \beta(C,T)C = R^d(C,T,\beta(C,T)).
\end{aligned}
$$

For the inequality, we have used that $\lambda^{w,d}$ maximizes $r^w(\lambda) - \beta(C,T)\lambda$. $\square$

For the proof of Theorem 4.5, we first need a lemma.

Lemma 4.2. *Let $R^{TI}(C,T)$ be the expected revenue obtained form the stationary policy α_{TI}. Let $(\widetilde{w}_t,\widetilde{v}_t)_{t=1}^T$ be an independent copy of the process $(w_t,v_t)_{t=1}^T$. Then*

i)

$$
R^{TI}(C,T) = E_{(w_t)_{t=1}^T}\left[\sum_{t=1}^{T} r^{w_t}(\lambda^{w_t,d})\, w_t\left(1 - P\left[\sum_{s=1}^{t-1}\widetilde{w}_s I_{\{\widetilde{v}_s\geq p^{\widetilde{w}_s,d}\}} > C - w_t\right]\right)\right],
$$

ii)

$$
\frac{R^{TI}(C,T)}{R^d(C,T)} \geq 1 - \frac{\sum_{t=1}^T \int_0^\infty r^w(\lambda^{w,d}) w\left(\min\left(1, \frac{(t-1)\sigma_d^2}{((T-t+1)\mu_d - w)^2}\right) I^1 + I^2\right)\bar{f}_w(w)\,dw}{T\int_0^\infty r^w(\lambda^{w,d}) w\bar{f}_w(w)\,dw}.
$$

where $\mu_d := \frac{\min(C, T\int_0^\infty \lambda^{w,*} w\bar{f}_w(w)\,dw)}{T}$, $I^1 := I_{\{w\leq(T-t+1)\mu_d\}}$, $I^2 := I_{\{w>(T-t+1)\mu_d\}}$, and $\sigma_d^2 := E[w^2 I_{\{v\geq p^{w,d}\}}] - \mu_d^2 = \int_0^\infty w^2\lambda^{w,d}\bar{f}_w(w)\,dw - \mu_d^2$.

Proof of Lemma 4.2. $R^{TI}(C,T)$ may be written as

$$\begin{aligned} R^{TI}(C,T) &= E_{(w_t,v_t)_{t=1}^T}\left[\sum_{t=1}^T p^{w_t,d}\, w_t\, I_{\{v_t \geq p^{w_t,d}\}}\, I_{\left\{\sum_{s=1}^{t-1} w_s I_{\{v_s \geq p^{w_s,d}\}} \leq C - w_t\right\}}\right] \\ &= E_{(w_t)_{t=1}^T}\left[\sum_{t=1}^T r^{w_t}(\lambda^{w_t,d})\, w_t\right] \\ &\quad - E_{(w_t,v_t)_{t=1}^T}\left[\sum_{t=1}^T p^{w_t,d}\, w_t\, I_{\{v_t \geq p^{w_t,d}\}}\, I_{\left\{\sum_{s=1}^{t-1} w_s I_{\{v_s \geq p^{w_s,d}\}} > C - w_t\right\}}\right]. \end{aligned}$$

To simplify the second term, we use the fact that v_t and $(w_s, v_s)_{s=1}^{t-1}$ are independent conditional on w_t.

$$\begin{aligned} &E_{(w_t,v_t)_{t=1}^T}\left[\sum_{t=1}^T p^{w_t,d}\, w_t\, I_{\{v_t \geq p^{w_t,d}\}}\, I_{\left\{\sum_{s=1}^{t-1} w_s\, I_{\{v_s \geq p^{w_s,d}\}} > C - w_t\right\}}\right] \\ &= E_{(w_t,v_t)_{t=1}^T}\left[\sum_{t=1}^T E\left[p^{w_t,d}\, w_t\, I_{\{v_t \geq p^{w_t,d}\}}\, I_{\left\{\sum_{s=1}^{t-1} w_s\, I_{\{v_s \geq p^{w_s,d}\}} > C - w_t\right\}} \mid w_t\right]\right] \\ &= E_{(w_t,v_t)_{t=1}^T}\left[\sum_{t=1}^T p^{w_t,d}\, w_t\, E\left[I_{\{v_t \geq p^{w_t,d}\}} | w_t\right]\, E\left[I_{\left\{\sum_{s=1}^{t-1} w_s\, I_{\{v_s \geq p^{w_s,d}\}} > C - w_t\right\}} \mid w_t\right]\right] \\ &= E_{(w_t,v_t)_{t=1}^T}\left[\sum_{t=1}^T p^{w_t,d}\, w_t\, \lambda^{w_t,d}\, P\left[\sum_{s=1}^{t-1} \widetilde{w}_s\, I_{\{\widetilde{v}_s \geq p^{\widetilde{w}_s,d}\}} > C - w_t\right]\right] \\ &= E_{(w_t)_{t=1}^T}\left[\sum_{t=1}^T r^{w_t}(\lambda^{w_t,d})\, w_t\, P\left[\sum_{s=1}^{t-1} \widetilde{w}_s\, I_{\{\widetilde{v}_s \geq p^{\widetilde{w}_s,d}\}} > C - w_t\right]\right]. \end{aligned}$$

This establishes i). Recall that $R^d(C,T) = T\int_0^\infty r^w(\lambda^{w,d}) w \bar{f}_w(w)\, dw$ then. Observe furthermore that $\lambda^{w,d}$ depends on C and T only through the ratio $\frac{C^{\text{eff}}}{T}$, where $C^{\text{eff}} = \min(C, T\int_0^\infty \lambda^{w,*} w \bar{f}_w(w)\, dw)$, via $E[wI_{\{v \geq p^{w,d}\}}] = \int_0^\infty w\lambda^{w,d}\bar{f}_w(w)\, dw = \frac{C^{\text{eff}}}{T} = \mu_d$. Observe first that

$$\begin{aligned} P\left[\sum_{s=1}^{t-1} \widetilde{w}_s\, I_{\{\widetilde{v}_s \geq p^{\widetilde{w}_s,d}\}} > C - w_t\right] &\leq P\left[\sum_{s=1}^{t-1} \widetilde{w}_s\, I_{\{\widetilde{v}_s \geq p^{\widetilde{w}_s,d}\}} > T\mu_d - w_t\right] \\ = P\left[\sum_{s=1}^{t-1} \widetilde{w}_s\, I_{\{\widetilde{v}_s \geq p^{\widetilde{w}_s,d}\}} - (t-1)\mu_d > (T - t + 1)\mu_d - w_t\right]&. \end{aligned}$$

We trivially bound the last expression by 1 if $(T - t + 1)\mu_d - w_t \leq 0$, and otherwise use Chebychev's inequality to deduce

$$
\begin{aligned}
& P\left[\sum_{s=1}^{t-1} \widetilde{w}_s\, I_{\{\widetilde{v_s} \geq p^{\widetilde{w}_s,d}\}} - (t-1)\mu_d > (T-t+1)\mu_d - w_t\right] \\
\leq \;& P\left[\left(\sum_{s=1}^{t-1} \widetilde{w}_s\, I_{\{\widetilde{v_s} \geq p^{\widetilde{w}_s,d}\}} - (t-1)\mu_d\right)^2 > ((T-t+1)\mu_d - w_t)^2\right] \\
\leq \;& \frac{E\left[\left(\sum_{s=1}^{t-1} \widetilde{w}_s\, I_{\{\widetilde{v_s} \geq p^{\widetilde{w}_s,d}\}} - (t-1)\mu_d\right)^2\right]}{((T-t+1)\mu_d - w_t)^2} = \frac{(t-1)\sigma_d^2}{((T-t+1)\mu_d - w_t)^2}.
\end{aligned}
$$

As we are bounding a probability, we can replace this estimate by the trivial bound 1 again whenever this is better, i.e. if w_t is smaller than but close to $(T-t+1)\mu_d$. Thus,

$$
\begin{aligned}
& E_{(w_t,v_t)_{t=1}^T}\left[\sum_{t=1}^{T} p^{w_t,d}\, w_t\, I_{\{v_t \geq p^{w_t,d}\}}\, I_{\{\sum_{s=1}^{t-1} w_s\, I_{\{v_s \geq p^{w_s,d}\}} > C - w_t\}}\right] \\
\leq \;& \sum_{t=1}^{T} \int_0^\infty r^w(\lambda^{w,d})\, w \left(\min\left(1, \frac{(t-1)\sigma_d^2}{((T-t+1)\mu_d - w)^2}\right) I^1 + I^2\right) \bar{f}_w(w)\, dw.
\end{aligned}
$$

Finally, dividing by $R^d(C,T)$ yields ii). □

Proof of Theorem 4.5. We first prove i). I^1 and I^2 are defined as in Lemma 4.2, and the starting point for the proof is estimate ii) from that lemma.

Note that $r^w(\lambda^{w,d}) w \bar{f}_w(w)$ is an integrable upper bound for

$$
r^w(\lambda^{w,d})\, w \left(\min\left(1, \frac{(t-1)\sigma_d^2}{((T-t+1)\mu_d - w)^2}\right) I^1 + I^2\right) \bar{f}_w(w).
$$

Moreover, for fixed w, for arbitrary $\eta \in (0,1)$ and for $t \leq \eta T$ we have $w < (1-\eta)T\mu_d$ eventually as $T, C \to \infty$, $\frac{C}{T} = \text{const}$. Moreover,

$$
\frac{(t-1)\sigma_d^2}{((T-t+1)\mu_d - w)^2} \leq \frac{\eta T \sigma_d^2}{((1-\eta)T\mu_d - w)^2} \to 0, \text{ as } T \to \infty.
$$

The Dominated Convergence Theorem implies then that

$$
\int_0^\infty r^w(\lambda^{w,d})\, w \left(\min\left(1, \frac{(t-1)\sigma_d^2}{((T-t+1)\mu_d - w)^2}\right) I^1 + I^2\right) \bar{f}_w(w)\, dw \to 0,
$$

in the considered limit, for arbitrary $\eta \in (0,1)$ and for $t \leq \eta T$. Consequently, also the term that is subtracted in estimate ii) of Lemma 4.2 converges to zero. This proves i).

We now prove ii). A straightforward application of the proof by Gallego and Van Ryzin is possible for this last part. For completeness, we spell it out. If w and v are independent, all the $\lambda^{w,d}$ for different w coincide, as do the $\lambda^{w,*}$. Call them λ^d and λ^*, respectively. We then have

$$\begin{aligned} R^{TI}(C,T) &= p(\lambda^d) E\left[\min\left(C, \sum_{t=1}^{T} w_t\, I_{\{v_t \geq p(\lambda^d)\}}\right)\right] \\ &= p(\lambda^d) E\left[\sum_{t=1}^{T} w_t\, I_{\{v_t \geq p(\lambda^d)\}} - \left(\sum_{t=1}^{T} w_t\, I_{\{v_t \geq p(\lambda^d)\}} - C\right)^+\right]. \end{aligned}$$

We use now the following estimate for $E[(X-k)^+]$, where X is a random variable with mean m and variance σ^2 and where k is a constant:

$$E[(X-k)^+] \leq \frac{\sqrt{\sigma^2 + (k-m)^2} - (k-m)}{2}.$$

Note that by independence

$$\begin{aligned} E\left[\sum_{t=1}^{T} w_t\, I_{\{v_t \geq p(\lambda^d)\}}\right] &= E[w] T \lambda^d, \\ Var\left[\sum_{t=1}^{T} w_t\, I_{\{v_t \geq p(\lambda^d)\}}\right] &= T\left(E[(w\, I_{\{v \geq p(\lambda^d)\}})^2] - E[w]^2 (\lambda^d)^2\right) \\ &= T\left(E[w^2]\lambda^d - E[w]^2(\lambda^d)^2\right). \end{aligned}$$

If $\lambda^* T E[w] > C$ and hence if $\lambda^d = \frac{C}{TE[w]}$ this yields

$$R^{TI}(C,T) \geq R^d(C,T) - p(\lambda^d)\frac{\sqrt{TE[w^2]\lambda^d}}{2} = R^d(C,T)\left(1 - \frac{\sqrt{E[w^2]/E[w]}}{2\sqrt{C}}\right).$$

If $\lambda^* T E[w] \leq C$ and hence if $\lambda^d = \lambda^*$, then $C \geq E\left[\sum_{t=1}^{T} w_t\, I_{\{v_t \geq p(\lambda^d)\}}\right]$, so that

$E\left[\left(\sum_{t=1}^{T} w_t \, I_{\{v_t \geq p(\lambda^d)\}} - C\right)^+\right] \leq \frac{\sqrt{\sigma^2}}{2}$. Thus,

$$R^{TI}(C,T) \geq R^d(C,T) - p(\lambda^*)\frac{\sqrt{\lambda^* T E(w^2)}}{2} = R^d(C,T)\left(1 - \frac{\sqrt{E[w^2]/E[w]}}{2\sqrt{\lambda^* E(w) T}}\right).$$

Hence, we can conclude that

$$\frac{R^{TI}(C,T)}{R^d(C,T)} \geq \left(1 - \frac{\sqrt{E[w^2]/E[w]}}{2\sqrt{\min(C, T\lambda^* E[w]))}}\right).$$

□

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