Discrete Symmetries and their Stringy Origin

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Para “el dotor” y “la jiera”
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Contents

CHAPTER 1

Introduction

- Pero papá - le dijo Josep llorando - si Dios no existe, ¿Quién hizo el mundo?
- Tonto- dijo el obrero, cabizbajo, casi en secreto -. Tonto. Al mundo lo hicimos nosotros, los albañiles.

Eduardo Galeano, El libro de los abrazos.

After about fifty years since its proposal, the standard model (SM) continues to amaze us with its accurate description of the fundamental constituents of matter and their interactions at the microscopic level. The latest of its experimental triumphs was the discovery of the Higgs boson at the Large Hadron Collider (LHC) [1, 2]. This particle is the crucial artifact for the breakdown of the electroweak symmetry, from which all fundamental particles acquire their masses [3, 4].

In the standard model [5–7], the three families of quarks and leptons are interpreted as chiral fermions transforming as irreducible representations of the gauge symmetry $G_{\mathrm{SM}} = \mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$. The strong and electroweak interactions are described by the exchange of vector bosons: Eight gluons for quantum chromodynamics, the massive Z_μ and $W^\pm_\mu$ which are the carriers of the weak force, and the photon A_μ of quantum electrodynamics.

Despite of its remarkable features, there are many reasons to think there must be a more fundamental theory beyond the standard model. One of these reasons is that it does not include gravity. In contrast to all other forces, gravity is mediated by a spin two particle: The graviton. Theories of this type are very reluctant to a quantum description. This also poses significant challenges to the understanding of our universe, especially at its early stages, where we expect quantum gravity to play the main role. From cosmological observations we know that our universe is expanding in an accelerated way. This behavior can be accounted for by introducing constant term in Einstein's equations: The cosmological constant. This cosmological constant has an extremely small value ($10^{-122}\,M_{\rm Pl}^4$), so small that it can not be interpreted as the vacuum energy of any known quantum field theory. Even more paradoxical is perhaps the contribution of this very small number to the energy density of the universe, form the results of Planck collaboration we know that dark energy accounts for roughly 70% [8] of it. Furthermore, the measurement of the CMB spectrum as well as astrophysical observations of the rotation curves of galaxies and clusters, they all lead to the conclusion that there is a "dark" type of matter which contributes 25% of the universe fuel. This dark matter is not accounted for by the standard model. Similarly as in the case of dark matter, there are many other arguments for particles beyond the standard model. For instance, the experimental evidence in favor of neutrino oscillations [9]. In order for neutrinos to change flavor (i.e. oscillate), at least one of them must be massive. This is in contradiction to the SM. A possible solution to this problem is to introduce an extra particle: A right handed neutrino. The neutrino masses will then be generated via the so called seesaw mechanism: The bigger the mass of the right handed neutrino, the smaller the mass of the SM neutrino.

Another drawback of the SM is the so-called hierarchy problem. The standard model is a consistent quantum field theory, i.e. all of its quantum corrections can be kept well under control up to a certain cutoff scale Λ, at which new physics becomes relevant. However, when one examines loop corrections to the Higgs mass, one observes that it is pushed towards the cutoff by quadratic divergencies. Now we know that the Higgs mass is roughly $125\,\mathrm{GeV}$, so we have to ask what is the mechanism which stabilizes the electroweak scale? Another important question has to do with the following observation: In the standard model, the masses for the fields as well as the Fermi constant G_F are generated by a dynamical mechanism. In the SM we have about 20 constant parameters needed to adjust the fermion masses (Yukawa couplings), mixing angles, CP violation phases, etc. The question which remains is wether or not these constant parameters are also dynamically generated. One can take this maieutical approach even further, and ask why is the symmetry of the standard model precisely $\mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ and not any other? or why there are only three generations of quarks and leptons? or even beyond, why do we live in four dimensions? Obviously, a more fundamental theory is needed in order to address these questions.

Many brilliant proposals have been made to solve the problems just discussed, but all of them still await for experimental confirmation. One of the most relevant concerns the program of unification. Perhaps one of the great conceptual achievements of the SM was the unified description of the strong and electroweak forces in terms of Abelian and non-Abelian gauge symmetries, each of them with a corresponding coupling constant. It was soon realized that the renormalization group flow of the coupling constants drives them very close to each other at energy scales of about $10^{14} - 10^{15}\,\mathrm{GeV}$. If they happen to actually meet at some point, there is the possibility that at higher energies, all interactions are described in terms of a single Lie group. This is the underlying idea behind grand unification [10]. The simplest example of a grand unified theory is perhaps SU(5). In this theory the SM generations can be obtained from the anti-fundamental and two-index antisymmetric representations

$$
\begin{aligned}
\mathbf{10} &\to \underbrace{(\mathbf{3},\mathbf{2})}_{Q} + \underbrace{(\bar{\mathbf{3}},\mathbf{1})}_{\bar{u}} + \underbrace{(\mathbf{1},\mathbf{1})}_{\bar{e}}\,, \\
\bar{\mathbf{5}} &\to \underbrace{(\bar{\mathbf{3}},\mathbf{1})}_{\bar{d}} + \underbrace{(\mathbf{1},\mathbf{2})}_{L}\,.
\end{aligned}
\tag{1.1}
$$

One step further, at the level of SO(10) one finds that not only the interactions, but also a full family gets unified in the $\mathbf{16}$-plet

$$
\mathbf{16} \to Q + \bar{u} + \bar{e} + L + \bar{d} + \bar{\nu}_R\,, \tag{1.2}
$$

where the additional singlet $\bar{\nu}_R$ can be interpreted as a right handed neutrino. Among the positive features of GUT models, we have a prediction for the weak mixing angle θ_W as well as an explanation for the quantization of the electric charge. In particular, one of the problems of GUTs is the presence of additional particles in the spectrum, such as the lepto-quarks, which can mediate exceedingly fast proton decay. The is another problem which concerns the Higgs. In grand unified theories, the Higgs usually comes accompanied of a color triplet, which needs to be decoupled from the low energy due to the issues with proton decay mentioned above. However, in standard GUT models it turns very hard to lift the triplets while keeping the Higgs doublets light. This is the doublet-triplet splitting problem.

Precision measurements of the gauge coupling constants have made it less and less likely that they unify. However, if one assumes the presence of additional particles with intermediate massess, this introduces a kink in the running which might restore unification. This is actually the case in supersymmetric models. Supersymmetry (SUSY) is a symmetry without precedents in particle physics. It transforms fermions into bosons, and viceversa [11]. This is very appealing for the SM since the contribution of the supersymmetric particles cancels the quadratic divergence in the Higgs mass and thus, stabilizes the electroweak scale. Since no supersymmetry has been observed to date, it then follows that this symmetry needs to be broken at a higher scale. If we want SUSY to remain a solution for the hierarchy problem, the breaking scale must not be that far from the electroweak one. This keeps hope alive that the first signals for SUSY will be seen at the second run of the LHC.

In the minimal supersymmetric version of the standard model (MSSM), one simply introduces the superpartners of the SM model fields. The bosonic counterparts of quarks and leptons are called squarks and sleptons respectively. Similarly, one has the fermion companions of the gauge bosons: The gauginos. In the Higgs sector the situation is a bit more subtle as one sees that the usual Higgs which we denote by H_d does not suffice to generate all quark masses, thus we have to introduce another Higgs field H_u to give mass to the up-type quarks. The fermion companions of these fields are called the Higgsinos. Of course, doubling the spectrum can spoil the phenomenology of the models. In particular, some squark couplings can make the proton very unstable. We would need of additional symmetries in order to control all these dangerous operators. One of the simplest symmetries one can invoke is a $\mathbb{Z}_2$ matter parity [12]. This symmetry has a very important implication, namely it makes the lightest supersymmetric particle (LSP) stable. This LSP is one of the most favored candidates for dark matter [13].

Motivated by the possibility for gauge couplings to unify within the MSSM, one can contemplate the possibility of supersymmetric GUTs. Among the many challenges for these models, there are the usual problems with proton decay which in supersymmetric models get more severe due to the presence of additional fields. Similarly, in SUSY GUTs we would have to address another puzzle. Note that the Higgses H_u and H_d are vector-like with respect to all quantum numbers in the standard model, so that one may expect that the bilinear coupling $H_u H_d$ is induced at a higher scale. However, such a high scale coupling would remove the Higgses from the low energy. The question of why the coupling in the Higgs bilinear is small, is commonly referred to as the μ-problem.

Another very appealing proposal is the existence of extra dimensions. In this picture, gravity propagates over the whole internal space while the gauge symmetry is localized over a certain region. This could explain why gravity is so weak compared to all forces, as it gets diluted in comparison to the other interactions. One may assume that the gauge symmetry is that of the standard model but one can also not exclude that over certain subregions the symmetry enhances further, say to SU(5) or SO(10). Thus, one can think that the matter of the standard model is localized at these special subregions, while the Higgs fields are free to propagate over the entire space. This solves the doublet-triplet splitting problem because the Higgs(es) are not complete representations of the unified group. Thus they will not be accompanied by the dangerous triplets mediating proton decay. This picture in which complete and split representations coexist is commonly referred to as local grand unification [14].

So far, we have discussed SUSY as a global symmetry of the theory. By gauging supersymmetry one obtains a theory which is invariant under local Lorentz transformations. This gauged supersymmetry is called supergravity and it could be a first step towards a quantum theory of gravity. However, theories of this type turn out to be non renormalizable. The reason for many of the divergencies in quantum approaches to a theory of gravity are due to the point-like nature of particles. This is somehow intuitive as quantum gravity must certainly encode a description of a quantum spacetime. In string theory, these effects are accounted for by introducing the string scale as a fundamental scale of the theory, which in turn makes it finite in the ultraviolet. One of the early observations about string theory is that it exhibits a massless spin two field in the spectrum. In this sense gravity is automatically included in string theory [15]. Early observations also lead to the conclusion that in order to avoid vacuum instabilities, the string must be supersymmetric, i.e. a superstring. Similarly it was shown that string theory is only consistent in ten dimensions. There are five superstring theories: Two which include both closed and open oriented strings denoted as type IIA and type IIB, two including only closed strings which are known as Heterotic and exhibit the gauge symmetry of either $E_8 \times E_8$ (HE) or SO(32) (HO), and finally there is a theory of unoriented strings known as type I. Over the years, it has been shown that these theories enjoy of a very rich mathematical structure and that they are not independent from each other but related by an ever growing network of dualities [16]. This inspired an image in which all superstrings are nothing but limiting descriptions of an underlying eleven dimensional M-theory [**Witten:1995ex**].

With this very naïve picture in mind, we can already appreciate the value of superstring theories for particle phenomenology. They encompass all of the imaginable ingredients for physics beyond the standard model.

- They provide a consistent quantum description of gravity.
- They include supersymmetry by default.
- They allow for a picture of grand unification as

$$\mathrm{SU}(5) \subset \mathrm{SO}(10) \subset \mathrm{E}_8\,,\ \mathrm{SO}(32)\,. \tag{1.3}$$

- They are theories with extra dimensions, so that the picture of local GUTs is a possibility in these setups.

Nevertheless, considering a string theory as an underlying framework for particle physics makes face new challenges. We have to engineer a way to make contact with four dimensions and the gauge symmetry G_{SM}. This is possible if we assume that six spatial dimensions are compactified, i.e. they are rolled up to a very small space. This compactification will have dramatic implications for the particle content and the interactions that we observe in four dimensions. There have been many efforts towards finding compactifications which could resemble the physics within and beyond the standard model. This involves understanding which properties of the compactification are responsible for given features of the model. For example, it is desirable to have $\mathcal{N} = 1$ SUSY in the low energy in order to solve the hierarchy problem. In that case, the compactification space must be a Calabi-Yau manifold. These spaces are difficult to deal with and in most cases some physical information can be accessed only by means of topological quantities. Fortunately, there are some special compactifications which also lead to $\mathcal{N} = 1$, while allowing for a fully-fledged stringy description [17]. These spaces are called orbifolds and can be regarded as special limits of CY manifolds where the curvature gets concentrated at a finite amount of points. Orbifolds allow for an exceptional degree of computability. Due to this Berechenbarkeit, they have been used in almost any superstring theory for the purpose of model building [18, 19]. From those efforts we should remark that in the context of the HE string, many orbifold models have been found whose phenomenology reproduces most of the blueprints of the standard model [20–22]. This class of models is known as the heterotic mini-landscape. Furthermore it has been shown over the years that the orbifold equips the low energy effective field theory with a handful of discrete symmetries [23, 24]. As discussed already, these are of great value as they could serve to control the phenomenology of the models.

In this work we study the stringy origin of these discrete symmetries. This is done in the context of orbifold compactifications of the heterotic string. We devote special attention to R-symmetries, as they have been appreciated in SUSY GUTs as an alternative to solve the μ problem.

Another possibility to achieve discrete symmetries in the low energy is from the breaking of U(1) factors. We explore this possibility in the context of F-theory (the non-perturbative version of the type IIB string). Taking steps towards model building in a new class of F-theory vacua, we make use of additional U(1) s which are broken to matter parity and help us to control dangerous operators.

This work is organized as follows: In chapter 2 we discuss the basic features of ten dimensional heterotic string and its spectrum. Further we revisit the generalities of compactifications on CY manifolds. Then we discuss the orbifolded superstring and its corresponding particle spectrum. In chapter 3 we discuss the discrete symmetries exhibited by orbifold field theories. We use the correlators of the conformal field theory (CFT) to infer which Abelian and non-Abelian discrete symmetries are inherited from the geometry. We also observe that from some unbroken remnants of the Lorentz group in internal space we get discrete R-symmetries. Then we derive the charges for the fields under these R-symmetries. Finally we compute the anomalies for these and show that they are universal over a broad class of orbifolds. Chapter 4 is devoted to discuss the orbifold phenomenology. We start presenting the Zip-code of the mini-landscape: A preferred distribution in the orbifold geometry where (MS)SM fields are located. This configuration is favored by the discrete symmetries and ensures that the models constructed in this way exhibit a similar phenomenology as the SM. Further we introduce a new orbifold compactification based on the $\mathbb{Z}_2 \times \mathbb{Z}_4$ geometry. From a GUT approach we discuss alternatives for model building in this orbifold, and show that a similar Zip-code arises.

Having discussed orbifold models and their remarkable features, which are partially due to the presence of additional discrete symmetries. We turn to another corner of the string landscape, namely F-theory. We devote chapter 5 to a review of its properties and how it can be addressed via the perturbative type IIB as well as from dualities to heterotic and M-theory. In chapter 6 we discuss a class of F-theory models based on SU(5) grand unification, with extra U(1) symmetries. We take first steps towards model building in these class of models and show that the U(1) factors must be broken in order to generate the Yukawa couplings for the SM fields. We show that upon a suitable breakdown it is possible to retain the standard matter parity in this model. In the last chapter we present our conclusions and discuss avenues for future research.

List of publications

Parts of this work have been published in scientific journals

- D. K. Mayorga Pena, H. P. Nilles and P. -K. Oehlmann, *"A Zip-code for Quarks, Leptons and Higgs Bosons,"*
  JHEP **1212** (2012) 024 [arXiv:1209.6041 [hep-th]].

- N. G. Cabo Bizet, T. Kobayashi, D. K. Mayorga Pena, S. L. Parameswaran, M. Schmitz and I. Zavala, *"R-charge Conservation and More in Factorizable and Non-Factorizable Orbifolds,"*
  JHEP **1305** (2013) 076 [arXiv:1301.2322 [hep-th]].

- D. K. Mayorga Pena and P. -K. Oehlmann, *"Lessons from an Extended Heterotic Mini-Landscape,"*
  PoS Corfu **2012** (2013) 096 [arXiv:1305.0566 [hep-th]].

- N. G. C. Bizet, T. Kobayashi, D. K. M. Pena, S. L. Parameswaran, M. Schmitz and I. Zavala, *"Discrete R-symmetries and Anomaly Universality in Heterotic Orbifolds,"*
  JHEP **1402** (2014) 098 [arXiv:1308.5669 [hep-th]].

- S. Krippendorf, D. K. M. Pena, P. K. Oehlmann, F. Rühle, *"Rational F-theory GUTs without Exotics,"*
  arXiv:1401.5084 [hep-th].

CHAPTER 2

Heterotic orbifolds

Que integre las ciencias y las artes a la canasta familiar, de acuerdo con los designios de un gran poeta de nuestro tiempo que pidió no seguir amándolas por separado como a dos hermanas enemigas.

Gabriel García Márquez, Por un país al alcance de los niños.

We devote this chapter to revisiting the basic features of the heterotic string theory. After that, we discuss compactifications leading to an effective theory in four dimensions. We also present general arguments in favor of Calabi-Yau manifolds as promising compactification alternatives. A simpler version of these are orbifolds, which, despite of their simplicity, have shown to be rich grounds for particle phenomenology. In the remainder of this chapter we study the geometric properties of these spaces as well as the particle spectra of heterotic orbifold models.

2.1 The Heterotic String

In a closed string theory, the so-called left- and right-moving sectors decouple [15]. This fact is exploited to construct the heterotic string, by taking the left-moving sector to be spanned by 26 bosonic degrees of freedom $X_L^{M,I}$ $M = 0, \ldots, 9$, $I = 1, \ldots, 16$, while the right-moving one is spanned by those of the ten dimensional superstring X_R^M, ψ^M $M = 0, \ldots, 9$. This choice implies that the resulting theory possesses $(\mathcal{N}_L, \mathcal{N}_R) = (0, 1)$ world-sheet supersymmetries.

The first point to take care of in the heterotic theory is the obvious mismatch among the bosonic left- and right-moving degrees of freedom. One recognizes a physical dimension X^M to be composed by its left- and right-moving parts. This implies that the target space of the heterotic string is truly ten dimensional [25], and that the extra bosonic components X^I must be adequately compactified[1].

Before turnings to the world sheet (WS) action, let us briefly comment on some redefinitions. We start by taking (τ, σ) to be coordinates on the world sheet, then we Wick rotate the τ component and write $z = e^{-2\pi \mathrm{i}(\mathrm{i}\tau+\sigma)}$, $\bar{z} = e^{-2\pi \mathrm{i}(\mathrm{i}\tau-\sigma)}$. In these terms, the WS action reads;

$$S = \frac{1}{\pi} \int d^2 z \left(2\partial X_M \bar{\partial} X^M + i\psi_M \bar{\partial}\psi^M + 2\partial X_I \bar{\partial} X^I \right), \tag{2.1}$$

where we have defined $\partial = \partial_z$ and $\bar{\partial} = \partial_{\bar{z}}$. Similarly it can be shown that the equations of motion for X^μ allow us to write it in the following form

$$X^M(z, \bar{z}) = X_L^M(z) + X_R^M(\bar{z}) \,. \tag{2.2}$$

As discussed before, heterotic strings are subjected to closeness constraints:

$$X^M(e^{2\pi \mathrm{i}} z, e^{-2\pi \mathrm{i}} \bar{z}) = X^M(z, \bar{z}) \,, \tag{2.3}$$

from which follows that X^M must obey the following mode expansion

$$X^M(z, \bar{z}) = X_0^M + \frac{p_L^M}{2} \ln(z) + \frac{p_R^M}{2} \ln(\bar{z}) + \sum_{\substack{n\in\mathbb{Z} \\ n\neq 0}} \frac{1}{n} \left[\alpha_n^M z^n + \tilde{\alpha}_n^M \bar{z} e^n \right] \,. \tag{2.4}$$

The previous equation, together with the boundary condition (2.3) imply that left and right moving momenta must match: $p_L = p_R \equiv p/2$.

Let us now discuss the fermionic sector. It is well known that on the fermions we can impose either periodic (Ramond) or anti-periodic (Neveu-Schwarz) boundary conditions: $\psi^M(e^{2\pi \mathrm{i}} \bar{z}) = \pm \psi^M(\bar{z})$. The mode expansions in either case are given by

$$\psi^M(\bar{z}) = \begin{cases} \sum_{n\in\mathbb{Z}} d_n^M \bar{z}^n & \text{(R)} \\ \sum_{r\in\mathbb{Z}} b_{r+\frac{1}{2}}^M \bar{z}^{r+\frac{1}{2}} & \text{(NS)} \end{cases} \,. \tag{2.5}$$

[1] In the earliest formulation of the heterotic theory [26], the left movers were described in terms of 10 bosonic and 32 fermionic coordinates. Note that these degrees of freedom made a contribution to the conformal anomaly which coincides with that of the bosonic string, i.e. $10 + 32/2 = 26$. In this *fermionic description* the dimensionality of the heterotic theory becomes transparent.

The sixteen extra left movers can be compactified on a torus $\mathbb{T}^6 = \mathbb{R}^{16}/\Lambda_{16}$, with Λ_{16} a 16 dimensional lattice. Due to the compactification, strings can close up to windings, i.e.

$$X^I(z) = X^I(z) + 2\pi\Lambda^I\,, \quad I = 1, ..., 16\,, \tag{2.6}$$

with $\Lambda \in \Lambda_{16}$. These boundary conditions are consistent with the mode expansion

$$X^I(z) = X_0^I + \frac{p^I}{2}\ln(z) + \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{1}{n}\alpha_n^I z^n\,. \tag{2.7}$$

Note that the previous expansion makes the momenta and winding modes coincide, so that the lattice Λ_{16} must be self dual. Additionally, it was shown that the vacuum to vacuum amplitude for this theory is modular invariant only if Λ_{16} is also even [27]. The only Euclidean lattices with such properties are the root lattice of $\mathrm{E}_8 \times \mathrm{E}_8$ and that of $Spin(32)/\mathbb{Z}_2$. As the choice of the lattice defines the perturbative gauge theory, choosing the first one defines the heterotic $\mathrm{E}_8 \times \mathrm{E}_8$ theory (HE). Similarly the choice of $Spin(32)/\mathbb{Z}_2$, leads to the heterotic SO(32) string (HO). The compact coordinates X^I are often refered to as the gauge degrees of freedom.

Having discussed the mode expansions, we can now proceed to quantize the theory. In the canonical approach, we obtain the following (anti-) commutation relations between the modes present in eqs. (2.4), (2.5) and (2.7)

$$\left[\tilde{\alpha}_n^M, \tilde{\alpha}_m^N\right] = n\delta_{n+m}\eta^{MN}, \quad \left[\alpha_n^M, \alpha_m^N\right] = n\delta_{n+m}\eta^{MN}, \quad \left[\alpha_n^I, \alpha_m^J\right] = n\delta_{n+m}\delta^{IJ}, \tag{2.8}$$

$$\left\{b_r^M, b_s^N\right\} = \delta_{r+s}\eta^{MN}, \quad \left\{d_n^M, d_m^N\right\} = \delta_{n+m}\eta^{MN}\,.$$

The reality conditions imposed on both bosonic and fermionic coordinates together with the previous equations, make it possible to interpret these modes as creation ($n, r < 0$) and annihilation operators ($n, r > 0$). One construct the states $|\varphi\rangle$ as excitations of the vacuum. However, certain care has to be taken to remove redundant states which arise due to the super-conformal invariance of the theory. In order to do so, we consider the energy momentum tensor $T_{\alpha\beta}$ and the WS supersymmetry current T_F,

$$\begin{aligned} T &= T_{zz} = -\partial X^M \partial X_M - \partial X^I \partial X_I, \\ \bar{T} &= T_{\bar{z}\bar{z}} = -\bar{\partial} X^M \bar{\partial} X_M - \frac{i}{2}\psi^M \bar{\partial}\psi_M, \\ T_F &= \psi^M \bar{\partial} X_M\,. \end{aligned} \tag{2.9}$$

The mode expansions for these operators are

$$T = \sum_{n\in\mathbb{Z}} L_n z^n, \quad \bar{T} = \sum_{n\in\mathbb{Z}} \bar{L}_n \bar{z}^n\,, \tag{2.10}$$

and

$$T_F = \begin{cases} \sum_{n\in\mathbb{Z}} F_n \bar{z}^n & \text{(R)} \\ \sum_{r\in\mathbb{Z}+\frac{1}{2}} G_{r+\frac{1}{2}} \bar{z}^{(r+\frac{1}{2})} & \text{(NS)} \end{cases}\,. \tag{2.11}$$

The previous expressions define the generators of the super-Virasoro algebra L_n, $\bar{L}_n$, F_n and G_n. With the aid of those, we define the string-Hilbert space $\mathcal{H}$ to be the vector space of all string states $|\varphi\rangle$ satisfying

$$\begin{aligned} L_n\,|\varphi\rangle &= \bar{L}_n\,|\phi\rangle = 0, \quad \forall n > 0\,, \\ F_n\,|\varphi\rangle &= G_{r+\frac{1}{2}}\,|\varphi\rangle = 0, \quad n, r > 0\,, \\ (L_0 - a_L)\,|\varphi\rangle &= (\bar{L}_0 - a_R)\,|\varphi\rangle = 0\,. \end{aligned} \tag{2.12}$$

In the last equation, the coefficients $a_{L/R}$ have been introduced to account for normal ordering effects. In the left-moving sector one has $a_L = 1$, while in the right-moving sector a_R equals 0 (R) or $\frac{1}{2}$ (NS).

To construct the physical states, it is more convenient to choose a particular gauge in which the above super-Virasoro constraints are succinct. In the light cone gauge, one fixes the coordinates $X^{\pm} \sim (X^0 \pm X^1)$. In this way, only the contributions from the transverse modes X^M, ψ^M $M = 2, ..., 9$ and X^I are of physical relevance [25]. Of course, this gauge breaks the Lorentz group SO(1,9) to the transverse SO(8), corresponding to the little group under which massless states transform. Upon quantization in the light cone gauge we can compute the spectrum of the theory. The mass operator is given by

$$M^2 = M_L^2 + M_R^2, \quad M_L^2 = M_R^2\,. \tag{2.13}$$

The left moving mass operator is given by

$$\frac{M_L^2}{4} = \frac{p^I p_I}{2} + N_L - 1\,, \tag{2.14}$$

where

$$N_L = \sum_{n=1}^{\infty} (\alpha_{-n}^M \alpha_n^M + \alpha_{-n}^I \alpha_n^I)\,, \tag{2.15}$$

is the oscillator number from the left moving sector. For the right movers one has

$$\frac{M_R^2}{4} = \begin{cases} \displaystyle\sum_{m=0}^{\infty} m d_{-m}^M d_m^M + N_R & \text{(R)} \\ \displaystyle\sum_{r=\frac{1}{2}}^{\infty} r b_{-r}^M b_r^M + N_R - \frac{1}{2} & \text{(NS)} \end{cases} \tag{2.16}$$

with $N_R = \sum_{n=1}^{\infty} \bar{\alpha}_{-n}^i \bar{\alpha}_n^i$. The above equation can be written more compactly after bosonization. In this approach, the contribution of two WS fermions can be accounted by a (anti-) holomorphic boson. In the light cone gauge we have four anti-holomorphic bosonic fields H^i $i = 0, 1, ..., 3$. The anti-commutation relations for the operators allow us to write any contribution of d_n^M or b_r^M in the form $e^{\mathrm{i}q^i H^i}$, with q being a weight in the vector (R) or spinor lattice (NS) of SO(8). In terms of the *H-momentum* q, the mass operator for the left movers reads

$$\frac{M_R^2}{4} = \frac{q^2}{2} + N_R - \frac{1}{2}\,. \tag{2.17}$$

Having the mass relations (2.14) and (2.17) we can find the massless spectrum of the theory. To do so, we first consider families of operators from the left and right moving sector, which have $M_{L,R} = 0$. From the side of the left movers we have the following possibilities:

(i) The oscillators α_{-1}^M, $M = 2, ..., 9$ contribute $N_L = 1$. Note that these operators furnish a vector representation $\mathbf{8}_v$ of SO(8).

(ii) The Lorentz scalars α_{-1}^I, and $e^{\mathrm{i}p_I X^I}$ with $p^2 = 2$. The former relation is only satisfied by the 480 roots of SO(32) or $\mathrm{E}_8 \times \mathrm{E}_8$ depending on the choice of Λ_{16}. In this way the sixteen oscillators $\tilde{\alpha}_{-1}^I$ are interpreted as the Cartan generators.

For the right moving sector, eq. (2.16) implies that the only operators leading to a massless state are characterized by $N_R = 0$ and $q^2 = 1$. This condition is only satisfied by the following

H-momenta[2]

(i) $q^{(1)} = \big(\underline{\pm 1, 0, 0, 0}\big)$, i.e. the weights of the vector representation $\mathbf{8}_v$ of SO(8).

(ii) $q^{(1/2)} = \big(\big[\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\big]\big)$, which give the weights of the spinor representation $\mathbf{8}_s$ of SO(8).

After tensoring these operators we finally obtain the massless spectrum of the heterotic theory:

(i) The states $e^{\mathrm{i}q^{(1)}\cdot H} e^{\mathrm{i}p_I X^I}\left|0\right\rangle$ and $e^{\mathrm{i}q^{(1/2)}\cdot H}\alpha^I_{-1}\left|0\right\rangle$ are vectors of SO(8), and transform in the adjoint representation $(\mathbf{248}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{248})$ of $E_8 \times E_8$ or $\mathbf{496}$ of SO(32). Hence, they are interpreted as the gauge bosons A^a_M of the theory. The index a runs over the adjoint of the Lie group. Similarly, using the operator $e^{\mathrm{i}q^{(1/2)}\cdot H}$ instead of $e^{\mathrm{i}q^{(1)}\cdot H}$ we obtain the gauginos χ^a.

(ii) The states $e^{\mathrm{i}q^{(1)}\cdot H}\alpha^i_{-1}\left|0\right\rangle$ and $e^{\mathrm{i}q^{(1/2)}\cdot H}\alpha^i_{-1}\left|0\right\rangle$ yield the $\mathcal{N}=1$ supergravity multiplet. The states with bosonic H-momentum decompose into the following irreducible representations:

$$\mathbf{8}_v \otimes \mathbf{8}_v = \mathbf{1} + \mathbf{28} + \mathbf{35}_v, \tag{2.18}$$

corresponding to the graviton g_{MN} ($\mathbf{35}_v$), the dilation ϕ ($\mathbf{1}$), and the antisymmetric two form B_{MN} ($\mathbf{28}$). For the remaining 64 fermionic states one has the decomposition[3]

$$\mathbf{8}_s \otimes \mathbf{8}_v = \mathbf{8}_c + \mathbf{56}_c. \tag{2.19}$$

These irreducible representations are interpreted as the dilatino λ ($\mathbf{8}_c$) and the gravitino ψ_M ($\mathbf{56}_c$).

2.2 Compactifications to four dimensions

In the low energy regime massive string excitations remain uninteresting since their masses are of the order of the string scale $M_s \sim 10^{17}$ GeV [28, 29]. In this sense, the massless spectrum suffices to write an effective Lagrangian for $\mathcal{N}=1$ supergravity[4] coupled to Yang Mills theory (YM) [16]

$$S_{\mathrm{HE}} = \frac{1}{2}\int d^{10}x\sqrt{-g}\left[R - |d\phi|^2 - \frac{3}{8\phi^2}|H_3|^2 - \frac{g^2}{4\phi}\mathrm{tr}(|F|^2) + \ldots\right], \tag{2.20}$$

[2] Here $\big(\underline{\pm 1, 0, 0, 0}\big)$ means all possible permutations and $\big(\big[\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\big]\big)$ describe all the combinations with an even number of minus signs.

[3] The subindex c denotes the cospinor representation of SO(8).

[4] For a more detailed discussion of ten dimensional supergravities, specially those of the type II, the reader is referred to section 5.1.

where g is the YM coupling strength and $H_3 = dB - \omega_3$, with ω_3 being the Chern-Simons three form

$$\omega_3 = A^a \wedge F^a - \frac{1}{3} g f_{abc} A^a \wedge A^b \wedge A^c \,, \tag{2.21}$$

From now on we concentrate on the $\mathrm{E}_8 \times \mathrm{E}_8$ theory, so that the indices a run over the adjoint $(\mathbf{248}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{248})$.

Since our intention is to make contact with particle physics. We are interested in a certain regime of the heterotic theory in which only four space-time dimensions are manifest. This regime can be attained if one confines six spatial dimensions to a certain compact space. For that purpose, let the 10D space $\mathcal{M}_{10}$ decompose as a product of Minkowski space-time $\mathcal{M}_{3,1}$ and a compact Riemannian manifold $\mathcal{M}_6$

$$\mathcal{M}_{10} = \mathcal{M}_{3,1} \times \mathcal{M}_6 \,. \tag{2.22}$$

One can then assume that the volume of $\mathcal{M}_6$ is small enough, so that the extra dimensions remain unobserved by state-of-the-art experiments. The decomposition of $\mathcal{M}_{10}$ as a direct product ensures that the Lorentz symmetry SO(1,3) $\subset$ SO(1,9) remains unbroken. The simplest candidate for $\mathcal{M}_6$ is a six dimensional torus $\mathbb{T}^6$. In this case, the Lorentz group exhibits the minimal breaking

$$\mathrm{SO(9,1)} \to \mathrm{SO(3,1)} \times \mathrm{SO(6)} \cong \mathrm{SO(3,1)} \times \mathrm{SU(4)} \,. \tag{2.23}$$

For the transverse SO(8) this gets translated into

$$\mathrm{SO(8)} \to \mathrm{SO(2)} \times \mathrm{SO(6)} \cong \mathrm{U(1)} \times \mathrm{SU(4)}, \tag{2.24}$$

in which the U(1) $\subset$ SO(1,3) is associated with the helicity. In four dimensions, the SU(4) factor is seen can only be seen as an internal symmetry treating bosons and fermions in a different manner, i.e. an R-symmetry. To compute the spectrum of the heterotic theory on this particular background, let us first note that the winding numbers in the compact coordinates do not contribute further massless states. Thus, the massless states can be simply read off from decomposing the representations found in the previous section according to the breaking (2.24). For instance, in the case of the gravitino one obtains

$$\mathbf{56}_c \to \mathbf{4}_{3/2} \oplus \overline{\mathbf{4}}_{-3/2} \oplus \mathbf{4}_{1/2} \oplus \overline{\mathbf{4}}_{-1/2} \oplus \overline{\mathbf{20}}_{1/2} \oplus \mathbf{20}_{-1/2} \,. \tag{2.25}$$

Note that from the previous decomposition one obtains one fundamental of SU(4) carrying helicity $3/2$. This observation, in the light of the four dimensional theory implies that we have four gravitini. In this way we managed to obtain a four dimensional theory with $\mathcal{N} = 4$ SUSY. We have then achieved one of our goals, but we are still far from a realistic description because theories with more than one supersymmetry do not allow for a chiral spectrum. We ignore the possibility of having no surviving supersymmetries in the four dimensional theory, and focus from now (and over the whole of this work) on compactifications with $\mathcal{N}=1$ SUSY.

The remainder of this section is devoted to work out the properties of the manifold $\mathcal{M}_6$ that are necessary to keep one supersymmetry unbroken. This amounts to find a SUSY transformation leaving all backgrounds fields invariant. This statement holds true for the bosonic fields (at least at the classical level), so that one only has to require the variation of the fermionic fields to be zero. These variations are

$$\begin{aligned}\delta_\epsilon\psi_M &= D_M\epsilon + \frac{1}{32\phi}\left(\Gamma_M^{NPQ} - 9\delta_M^N\Gamma^{PQ}\right)H_{NPQ}\epsilon + \dots\,,\\ \delta_\epsilon\lambda &= -\frac{1}{\sqrt{2\phi}}\Gamma^M\partial_M\Phi\epsilon + \frac{g}{8\sqrt{2\phi}}\Gamma^{MNP}H_{MNP}\epsilon + \dots\\ \delta_\epsilon\chi^a &= -\frac{1}{4g\sqrt{\phi}}F^a_{MN}\Gamma^{MN}\epsilon\,,\quad M,N=0,...,9\,,\end{aligned} \tag{2.26}$$

In searching for a solution for this system of equations we have to specify the metric and the dilaton backgrounds, as well as the three form potential H_3. Recall that the former is subject to a Bianchi identity, which can be derived from the supergravity action (2.20), and reads $dH_3 = -\mathrm{tr}(F\wedge F)$. Due to anomaly considerations [30], the Bianchi identity gets corrected in string theory, taking the following form

$$dH_3 = \mathrm{tr}(R\wedge R) - \mathrm{tr}(F\wedge F) \tag{2.27}$$

We can write the YM field strength F in terms of a holomorphic vector bundle $V = V_1\times V_2$. In this way, we recognize the terms on the right hand side of the previous equation, as the second Chern characters of the tangent bundle $T\mathcal{M}_6$ and the gauge bundle V. We do not attempt a comprehensive study of these objects, for which we refer to e.g. [31, 32].

Back to eqs. (2.26), we can study the simplest case in which $H_3=0$. This in turn implies

$$\mathrm{tr}(R\wedge R) = \mathrm{tr}(F\wedge F)\,. \tag{2.28}$$

Thus, we observe that the vanishing of the gravitino variation, implies $D_M\epsilon = 0$. We also observe that for a constant dilaton profile, the dilatino variation is zero. We can now take advantage of the fact that $\mathcal{M}_{10}$ is a product manifold. Let μ,ν label coordinates in Minkowski space, and

$m, n = 4, ..., 9$ label coordinates in $\mathcal{M}_6$. Taking the spinor $\epsilon \in \mathbf{8}_s$ to decompose according to the breaking (2.24), i.e. $\epsilon = \epsilon_{1/2} + \bar{\epsilon}_{-1/2}$, with $\epsilon_{1/2}$ ($\bar{\epsilon}_{-1/2}$) transforming (anti-) fundamental of SU(4), and the subindices $1/2$, $-1/2$ denoting the corresponding chiralities. Since ϵ is covariantly constant, we obtain the following Bianchi identity for the internal space

$$[D_m, D_n]\epsilon = \frac{1}{4} R_{mnpq}\Gamma^{pq}\epsilon_{1/2} = 0\,. \tag{2.29}$$

Since the Γ^{pq} are precisely the generators of the Lie algebra SU(4), we find a non trivial solution for $\epsilon_{1/2}$ only if the combinations $R_{mnpq}\Gamma^{pq}$ are restricted to a subgroup of the full SU(4). In particular if $R_{mnpq}\Gamma^{pq}$ run over the Lie algebra of SU(3), the spinor $\epsilon_{1/2}$ decomposes according to the branching rule $\mathbf{4} = \mathbf{3} \oplus \mathbf{1}$. This leaves us with a single covariantly constant spinor in 4D. Furthermore, the condition (2.29) also implies

$$R_{mn}\Gamma^{n}\epsilon = 0\,. \tag{2.30}$$

The previous observations (in the case $H_3 = 0$) imply that a surviving supersymmetry in four dimensions is achieved by compactification on a Ricci flat manifold $\mathcal{M}_6$, with SU(3) holonomy (i.e. the structure group of the tangent bundle $T\mathcal{M}_6$). Due to that Calabi-Yau manifolds seem to be the immediate choice for $\mathcal{M}_6$, as they allow for a Ricci flat metric with SU(3) holonomy [33].

So far we have not discussed the restrictions on the YM field strength leading to vanishing variations of the gaugino. Recall that the spin connection needs to be embedded into the background Yang Mills connection A^a (see eq. (2.21)). The simplest possible choice, known as the standard embedding takes the gauge connection to have SU(3) as its structure group. This will in turn break one of the E_8 factors to E_6 (the commutant of SU(3) in E_8), leaving an unbroken $E_6 \times E_8$ gauge symmetry in the 4D theory. More in general, if we solve eq. (2.28) by considering a holomorphic vector bundle[5] $V = V_1 \times V_2$, with V_1 and V_2 having structure groups H_1 and H_2 inside each of the E_8 factors. In this case, $\mathcal{M}_6$ will not have a Ricci flat metric, but will still be Calabi-Yau, as it has a vanishing first Chern class $c_1(T\mathcal{M}_6) = 0$ [34]. Analogously as in the standard embedding, the resulting gauge group of the four dimensional theory will be $G_1 \times G_2$, with G_1 and G_2 being the comutants of H_1 and H_2 in E_8, respectively.

The departure from the standard embedding defines an avenue for heterotic model building. Depending on the choice of the structure bundle, it is possible to obtain, for instance, an SU(5) or an SO(10) symmetry, which are well motivated candidates for a grand unified theory. For a

[5] Consistency with $\mathcal{N} = 1$ SUSY in 4D, also requires the bundle V to be poly-stable [31].

detailed account on model building efforts in this context, the reader is referred to [35–37].

One could naïvely imagine that the Calabi-Yau manifold can be deformed in such a way, that all the curvature gets confined to a finite number of points. At first one might worry about these singularities, but it turns out that as strings are extended objects, one can write a consistent string theory on those backgrounds [17, 38]. These singular limits correspond to a special class of orbifolds, whose discrete holonomy is a subgroup of SU(3) [17]. Despite of their simplicity, heterotic orbifold compactifications are versatile enough as to allow for semi-realistic four dimensional theories. The features of these compactifications are revisited in the forthcoming sections.

2.2.1 Orbifolds

A toroidal orbifold results from the quotient of a torus by one of its isometries. Of particular interest for us are those six-dimensional orbifolds which could serve as compactification spaces for the heterotic theory, leaving one unbroken supersymmetry. To start with, let us complexify the coordinates of the would-be internal space

$$Z^i = X^{2i+2} + \mathrm{i}X^{2i+3}\,, \quad \bar{Z}^i = (Z^i)^*\,, \quad i = 1,2,3, \tag{2.31}$$

Given a six dimensional lattice Γ and a finite isometry group $\mathrm{P} \subset \mathrm{Aut}(\Gamma)$, one can construct a six dimensional orbifold

$$\mathbb{O}^6 = \frac{\mathbb{C}^3}{\mathrm{P} \ltimes \Gamma} = \frac{\mathbb{T}^6}{\mathrm{P}}\,. \tag{2.32}$$

The isometry group P is usually referred to as the point group. Additionally, we can define the space group

$$\mathrm{S} = \mathrm{P} \ltimes \Gamma = \{(\theta,\lambda)\,|\,\theta \in \mathrm{P}\,, \lambda \in \Gamma\}\,. \tag{2.33}$$

From now on we restrict our discussion to the Abelian point groups, for which P is given as a direct product of cyclic groups $\mathbb{Z}_N$. As all elements of P can be brought into a diagonal form, we assume, without loss of generality, that this happens precisely for the complex basis given in eq. (2.31). Under this assumption, the action of a given element $\theta \in \mathrm{P}$ is given by:

$$\theta \,:\, Z \mapsto \mathrm{diag}(e^{2\pi \mathrm{i} v^1}, e^{2\pi \mathrm{i} v^2}, e^{2\pi \mathrm{i} v^3}) Z\,, \tag{2.34}$$

with some the coefficients v^1, v^2 and v^3. Note that, generically, $\theta \in \mathrm{U(3)}$. As P coincides with the discrete holonomy of $\mathbb{O}^6$, in order to have $\mathcal{N} = 1$ SUSY we must restrict to Abelian point

P	v_N
$\mathbb{Z}_3$	$\frac{1}{3}(1,1,-2)$
$\mathbb{Z}_4$	$\frac{1}{4}(1,1,-2)$
$\mathbb{Z}_{6-\mathrm{I}}$	$\frac{1}{6}(1,1,-2)$
$\mathbb{Z}_{6-\mathrm{II}}$	$\frac{1}{6}(1,2,-3)$
$\mathbb{Z}_7$	$\frac{1}{7}(1,2,-3)$
$\mathbb{Z}_{8-\mathrm{I}}$	$\frac{1}{8}(1,2,-3)$
$\mathbb{Z}_{8-\mathrm{II}}$	$\frac{1}{8}(1,3,-4)$
$\mathbb{Z}_{12-\mathrm{I}}$	$\frac{1}{12}(1,4,-5)$
$\mathbb{Z}_{12-\mathrm{II}}$	$\frac{1}{12}(1,4,-5)$

(a)

P	v_N	v_M
$\mathbb{Z}_2\times\mathbb{Z}_2$	$\frac{1}{2}(1,0,-1)$	$\frac{1}{2}(0,1,-1)$
$\mathbb{Z}_3\times\mathbb{Z}_3$	$\frac{1}{3}(1,0,-1)$	$\frac{1}{3}(0,1,-1)$
$\mathbb{Z}_2\times\mathbb{Z}_4$	$\frac{1}{2}(1,0,-1)$	$\frac{1}{4}(0,1,-1)$
$\mathbb{Z}_4\times\mathbb{Z}_4$	$\frac{1}{4}(1,0,-1)$	$\frac{1}{4}(0,1,-1)$
$\mathbb{Z}_2\times\mathbb{Z}_{6-\mathrm{I}}$	$\frac{1}{2}(1,0,-1)$	$\frac{1}{6}(0,1,-1)$
$\mathbb{Z}_2\times\mathbb{Z}_{6-\mathrm{II}}$	$\frac{1}{2}(1,0,-1)$	$\frac{1}{6}(1,1,-2)$
$\mathbb{Z}_3\times\mathbb{Z}_6$	$\frac{1}{3}(1,0,-1)$	$\frac{1}{3}(0,1,-1)$
$\mathbb{Z}_6\times\mathbb{Z}_6$	$\frac{1}{6}(1,0,-1)$	$\frac{1}{6}(0,1,-1)$

(b)

Table 2.1: Abelian point groups P containing (a) one and (b) two cyclic factors. These are the only alternatives consistent with $\mathcal{N}=1$ supersymmetry [28].

groups in the Cartan of SU(3). This implies the condition

$$v^1+v^2+v^3=0 \bmod 1\,, \tag{2.35}$$

so that P is composed of at most two cyclic factors.

In the simplest case P $=\mathbb{Z}_N$ one has only one generator θ, associated to a *twist vector*[6]

$$v_N \equiv (0,v^1,v^2,v^3)\,. \tag{2.36}$$

The cyclicity condition ($\theta^N=1$) implies that the twists v_N^i are all of order N i.e. $Nv_N^i = 0 \bmod 1$, for $i=1,2,3$. In the case of $P=\mathbb{Z}_N\times\mathbb{Z}_M$ one has two generators θ and ω related to the twists vectors v_N and v_M of order N and M respectively. Each of these twist vectors has to satisfy eq. (2.35). The requirement of P being an isometry of the lattice, together with eq. (2.35) restrict the periods N and M to take only few values. The complete classification of Abelian point groups P $\subset$ SU(3) is given in table 2.1.

In the complex basis of $\mathbb{C}^3$, where the elements of the point group are diagonal 3×3 matrices, the underlying lattice can be aligned in various ways inside $\mathbb{C}^3$. An orbifold $\mathbb{O}^6$ is called *fac-*

[6] The first entry is introduced to indicate that there is no rotation along the generator of the transverse component of the Lorentz group SO(1,3).

torizable if its compactification lattice Γ_6 can be continuously deformed[7] to a direct product of three sublattices, each of which lies completely in one complex plane. If that is not the case, each vector of the basis has to be specified by three complex coordinates which are in general non-zero.

2.3 Strings on orbifolds

The orbifold geometry provides two different classes of boundary conditions for closed strings. The boundary conditions (2.3), provide the conformal blocks needed to construct the physical vertices of the *untwisted sector*. Among those which are of relevance for massless states we have the identity operator, the left moving oscillators α^i_{-1}, α^{i*}_{-1}, the exponents $e^{\mathrm{i}p^I X^I}$ with $p \in \Gamma_{16}$ and the fermionic fields $e^{\mathrm{i}q^a \cdot H}$. The second class of boundary conditions allows the string to close up to the combined action of point group elements and lattice vectors [39]

$$Z^i(e^{2\pi\mathrm{i}}z, e^{-2\pi\mathrm{i}}\bar{z}) = (gZ)^i(z,\bar{z}), \quad i = 1,2,3\,, \tag{2.37}$$

for a non-trivial $g \in \mathrm{S}$ which is called the *constructing element* of the string. For concreteness we consider the case of a $\mathbb{Z}_N$ orbifold[8]. The twisted boundary conditions associated to a generic constructing element $g = (\theta^k, \lambda)$, are more explicitly given by

$$Z^i(e^{2\pi\mathrm{i}}z, e^{-2\pi\mathrm{i}}\bar{z}) = e^{2\pi i v^i_g} Z^i(z,\bar{z}) + \lambda^i\,, \tag{2.38}$$

where the we have introduced the *local twist* $v_g \equiv k v_N$. All twisted strings closing up to a point group element θ^k, are said to belong to the k^{th} *twisted sector*[9] T_k. Notice that strings closed by g and hgh^{-1} (for some $h \in \mathrm{S}$) are physically equivalent, that is, twisted states are associated with conjugacy classes

$$[g] = \left\{hgh^{-1} | h \in S\right\} \tag{2.39}$$

and not to individual space group elements.

A mode expansion consistent with equation (2.38) has the center of mass fixed at $Z_0 = (1 -$

[7] Note that the only possible deformations permitted for a certain lattice are those which commute with the point group P.

[8] In the case of $\mathbb{Z}_N \times \mathbb{Z}_M$ the corresponding constructing element will be of the form $g = (\theta^{k_1}\omega^{k_2}, \lambda)$ and the local twist $v_g \equiv k_1 v_N + k_2 v_N$.

[9] Similarly for $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds, the twisted sector $T_{(k,l)}$ is associated to the element $\theta^k\omega^l \in \mathrm{P}$

$\theta^k)^{-1}\lambda$, and has the form

$$Z^i(z,\bar{z}) = Z_0^i + \frac{i}{2}\sum_{n\in\mathbb{Z}}\left(\frac{\alpha^i_{n-w^i}}{n-w^i}z^{(n-w^i)} + \frac{\alpha^i_{n+w^i}}{n+w^i}\bar{z}^{(n+w^i)}\right), \tag{2.40}$$

with $w^i = v_g^i \bmod 1$ $(0 \leq w^i < 1)$. The previous expansion provides a set of *twisted bosonic oscillators* $\tilde{\alpha}^i_{n-w^i}$, $\alpha^i_{n-w^i}$ for the right- and left-moving parts of the string. The conjugate version to (2.40) reads

$$\bar{Z}^i(z,\bar{z}) = \bar{Z}_0^i + \frac{i}{2}\sum_{n\in\mathbb{Z}}\left(\frac{\alpha^{i*}_{n+w^i}}{n+w^i}z^{(n+w^i)} + \frac{\tilde{\alpha}^{i*}_{n-w^i}}{n-w^i}\bar{z}^{(n-w^i)}\right). \tag{2.41}$$

The above relations are used to quantize the theory in the light cone gauge. As a result one obtains that the commutation relations for the oscillators are all trivial except for:

$$\left[\alpha^i_{n-w^i}, \alpha^{j*}_{-m+w^i}\right] = (n+w^i)\delta^{ij}\delta_{nm}, \quad \left[\tilde{\alpha}^i_{n+w^i}, \tilde{\alpha}^{j*}_{-m-w^i}\right] = (n-w^i)\delta^{ij}\delta_{nm}. \tag{2.42}$$

The real fermion fields ψ_M, $M = 4,\ldots,9$, can be compactified identically to the bosonic coordinates (see eq. (2.31)). The twisted boundary conditions for these fields are similar to those of the standard R and NS sectors, but weighted by the phases introduced by the local twist. These phases shift the weight of the fermionic operators, and as a consequence, the bosonized version of the twisted fermions is of the form $e^{\mathrm{i}q^{(a)}_{sh}\cdot H}$. The shifted H-momentum is $q^{(a)}_{sh} = q^{(a)} + v_g$, with $q^{(a)}$, either in the vector $(a = 1)$ or spinor lattice $(a = 1/2)$ of SO(8).

2.3.1 Gauge embeddings

As observed in section 5.5, there is a Bianchi identity correlating the background geometry and the YM field strength background. This fact, translated to the orbifold implies that the space group must act non-trivially on the gauge coordinates. Otherwise, it is not possible for the theory to have a modular invariant partition function. In this spirit, let us consider the embedding

$$\mathrm{S} \hookrightarrow G \subset \mathrm{Aut}(\mathrm{E}_8 \times \mathrm{E}_8) \tag{2.43}$$

$$g \mapsto G_g\,. \tag{2.44}$$

For the constructing element g, the boundary condition on the gauge coordinates reads

$$X^I(z) = (G_g X)^I(z) + \pi\Lambda^I, \tag{2.45}$$

with Λ as in eq. (2.7). In order to proceed further, we can take G as a subgroup of the inner automorphisms of $E_8 \times E_8$. In that case the action of G_g can be realized as a shift [40]

$$(G_g X)^I(z) = X^I(z) + \pi V_g^I\,, \tag{2.46}$$

where V_g is some 16 dimensional vector. Consistency of the embedding implies that V_g respects the product structure of S up to lattice identifications. Comparing the previous equation with the untwisted case eq. (2.7), we see that the embedding simply shifts the gauge momentum, i.e. $p_{sh} = p + V_g$. Analogously to section 5.5, G will break the $E_8 \times E_8$ symmetry to a smaller subgroup. Note that the boundary conditions (2.45) simply shift the gauge momentum but do not modify the mode expansions. Since the oscillators α^I_{-1} also appear in the twisted mode expansions and, as they are identified with the Cartan generators (see sect. 2), we can anticipate that embeddings of the form (2.46) lead to a breaking of the $E_8 \times E_8$ gauge symmetry which preserves its rank[10]. For the Abelian orbifolds consistent with $\mathcal{N} = 1$ SUSY in 4D, the translational embeddings can be described by

$$\mathbb{Z}_N : \qquad (\theta^k, n_\alpha e_\alpha) \quad \mapsto kV_N + n_\alpha W_\alpha\,, \tag{2.47}$$

$$\mathbb{Z}_N \times \mathbb{Z}_M : \quad (\theta^{k_1}\omega^{k_2}, n_\alpha e_\alpha) \quad \mapsto k_1 V_N + k_2 V_M + n_\alpha W_\alpha\,, \tag{2.48}$$

with the vectors e_α, $\alpha = 1, ..., 6$ spanning a basis for the six dimensional lattice Γ. The simplest case in which one just embeds the point group ($W_\alpha = 0$) is in agreement with modular invariance, provided a suitable choice of the embedding vector(s) V_N (and V_M). The further embedding of the six dimensional lattice as *discrete Wilson lines* W_α is a freedom one has to further break the gauge group, or to reduce certain matter representations [42].

Let us now consider the actual conditions on the embedding. Specifically, we discuss these conditions for $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds, but they carry straightforwardly over to those of the $\mathbb{Z}_N$-type. First, since θ^N and ω^M equal the identity element one has to guarantee that their action is trivial on the gauge coordinates. Consistency of the embedding implies that NV_N and MV_M must belong to the $E_8 \times E_8$ lattice. Similarly for each basis vector $e_\alpha \in \Gamma$ one has to find the smallest N_α which yields

$$\sum_{k=1}^{N_\alpha - 1} \theta^k e_\alpha = 0\,, \tag{2.49}$$

[10] The rank reducing breakings can be achieved if one allows G to contain outer automorphisms of $E_8 \times E_8$, for orbifold models of this type see e.g. [41].

for any $\vartheta \in$ P. The number N_α is known as the *order of the Wilson line*. In analogy with the $V_{M,N}$, $N_\alpha W_\alpha$ must also be a vector of the gauge lattice. Furthermore, if $\vartheta e_\alpha = m_{\alpha\beta} e_\beta$, then W_α and $m_{\alpha\beta} W_\beta$ are equal up to an $\mathrm{E}_8 \times \mathrm{E}_8$ lattice vector. This *equivalence* of the Wilson lines is denoted as $W_\alpha \simeq m_{\alpha\beta} W_\beta$.

In addition to the previous requirements, there are additional conditions coming from modular invariance of the partition function [43]

$$N(V_N^2 - v_N^2) = 0 \bmod 2, \tag{2.50}$$

$$M(V_M^2 - v_M^2) = 0 \bmod 2, \tag{2.51}$$

$$\gcd(N, M)(V_N \cdot V_M - v_N \cdot v_M) = 0 \bmod 2, \tag{2.52}$$

$$N_\alpha(W_\alpha \cdot V_i) = 0 \bmod 2, \tag{2.53}$$

$$\gcd(N_\alpha, N_\beta)(W_\alpha \cdot W_\alpha) = 0 \bmod 2. \tag{2.54}$$

The simplest possible embedding one can imagine for the $\mathbb{Z}_N$ orbifold is one in which the shift V_N takes the form

$$V_N = (v^1, v^2, v^3, 0, 0, 0, 0, 0) \oplus (0, 0, 0, 0, 0, 0, 0, 0)\,, \quad W_\alpha = 0\,. \tag{2.55}$$

In this case the one of the E_8 factors gets broken to E_6, times additional U(1) factors. The embedding (2.55) is the orbifold analogue of the standard embedding we discussed in the context of smooth CYs.

2.3.2 Twisted vacua and space group invariant states

In analogy with the untwisted sector, we expect that the oscillators $\alpha^i_{-w^i}$, $\alpha^{i*}_{-\bar{w}^i}$ ($\bar{w}^i \equiv -w^i \bmod 1$, $0 < \bar{w}^i \leq 1$), play a crucial role in the construction of twisted states. In the conformal field theory, these oscillators are analogous to the primaries ∂Z^i and $\partial \bar{Z}^i$, respectively. For those, we note that the twisted boundary condition (2.38) gets translated into

$$\begin{aligned} \partial Z^i(e^{2\pi \mathrm{i}} z, e^{-2\pi \mathrm{i}} \bar{z}) &= e^{2\pi i v_g^i} \partial Z^i(z, \bar{z})\,, \\ \partial \bar{Z}^i(e^{2\pi \mathrm{i}} z, e^{-2\pi \mathrm{i}} \bar{z}) &= e^{-2\pi i v_g^i} \partial \bar{Z}^i(z, \bar{z})\,, \end{aligned} \tag{2.56}$$

implying that in the vicinity of $z = 0$, the fields ∂Z^i and $\partial \bar{Z}^i$ have a branch cut. The standard way to deal with these branch singularities is to introduce *twist fields* $\sigma(z, \bar{z})$ [44, 45] with the following operator product expansions (OPEs)

$$\begin{aligned}\partial Z^i(z,\bar z)\sigma(w,\bar w) &= (z-w)^{-w^i}\tau+\dots\,,\\ \partial\bar Z^i(z,\bar z)\sigma(w,\bar w) &= (z-w)^{-\bar w^i}\tilde\tau+\dots\,,\end{aligned} \tag{2.57}$$

with τ, $\tilde\tau$ being excited twist fields, which are of no relevance for the massless spectrum. One can think of the twist field σ as the field that creates a twisted vacuum $|\sigma\rangle = \sigma(0,0)\,|0\rangle$ from the untwisted one. In a give twisted sector one has one (or more) twist field per conjugacy classes, so that the twisted vacua are degenerate. The conformal weights of σ are given by

$$\Delta_\sigma = \bar\Delta_\sigma = \frac{1}{2}\sum_{i=1}^{3} w^i(1-w^i)\,, \tag{2.58}$$

corresponding to the vacuum energy of $|\sigma\rangle$.

Having introduced the twist fields we are to face another problem. Namely the OPEs (2.57) evidence the fact that ∂Z^i, $\partial\bar Z^i$ and σ are not mutually local. Locality is restored by requiring the spectrum to be invariant under the full space group S. This requirement is precisely what leads to an $\mathcal{N}=1$ supersymmetric spectrum.

In order to describe the transformation of the states under S, we first consider the transformation of the individual conformal blocks. For the untwisted sector one has the oscillators α^μ_{-1} ($\mu=2,3$) which generate the vector representation of the transverse Lorentz group in 4D. Similarly, one has the Cartan generators α^I_{-1}. Both of these operators are invariant under the space group. The transformation for the remaining oscillators under a generic element $h\in$ S, is given by

$$\begin{aligned}\alpha^i_{-w^i} &\xrightarrow{h} e^{+2\pi\mathrm{i}v^i_h}\,\alpha^i_{-w^i}\,,\\ \alpha^{i*}_{-\bar w^i} &\xrightarrow{h} e^{-2\pi\mathrm{i}v^i_h}\alpha^{i*}_{-\bar w^i}\,.\end{aligned} \tag{2.59}$$

From eq. (2.46) we obtain the transformation for the momentum modes in gauge space

$$e^{\mathrm{i}p^I_{sh}X_I} \xrightarrow{h} e^{\mathrm{i}p^I_{sh}(G_hX)_I} = e^{2\pi\mathrm{i}p_{sh}\cdot V_h}e^{\mathrm{i}p^I_{sh}X_I}\,. \tag{2.60}$$

The H-momenta exhibit a similar transformation behavior [46]:

$$e^{\mathrm{i}q_{sh}(a)\cdot H} \xrightarrow{h} e^{-2\pi\mathrm{i}q_{sh}(a)\cdot v_h}e^{\mathrm{i}q_{sh}(a)\cdot H}\,. \tag{2.61}$$

In order to see how h acts on the twist fields it is convenient to decompose them into a sum of auxiliary twists σ_g, one for each element in the conjugacy class $[g]$

$$\sigma \sim \sum_{g' \in [g]} e^{2\pi \mathrm{i} \tilde{\gamma}(g')} \sigma_{g'} \,, \tag{2.62}$$

where the phases $\tilde{\gamma}(g')$ are yet to be determined. For the auxiliary twists one has the following transformation behavior

$$\sigma_g \xrightarrow{h} e^{2\pi \mathrm{i} \Phi(g,h)} \sigma_{hgh^{-1}} \,, \tag{2.63}$$

with the *vacuum phase* $\Phi(g,h) = -\frac{1}{2}\left(V_g \cdot V_h - v_g \cdot v_h\right)$ [43]. Note that $\Phi(g',h)$ is the same for all $g' \in [g]$. The resulting transformation for the twist field σ reads

$$\sigma \xrightarrow{h} e^{2\pi \mathrm{i}[\gamma_h + \Phi(g,h)]} \sigma \,, \tag{2.64}$$

where we have defined

$$\gamma_h = \left[\tilde{\gamma}(g') - \tilde{\gamma}(hg'h^{-1})\right] \bmod 1, \text{ for some } g' \in [g] \,, \tag{2.65}$$

together with the condition

$$-\tilde{\gamma}(hg'h^{-1}) + \tilde{\gamma}(g') = -\tilde{\gamma}(hgh^{-1}) + \tilde{\gamma}(g) \bmod 1 \,, \tag{2.66}$$

for any pair of elements $g', g \in [g]$.

Having discussed the transformation of the operators, we can now consider a generic twisted state

$$|\varphi\rangle = \left(\prod_{i=1}^{3} (\alpha^i_{-w^i})^{\mathcal{N}^i_L} (\alpha^i_{-\bar{w}^i})^{\bar{\mathcal{N}}^i_L}\right) e^{\mathrm{i} q^{(a)}_{sh} \cdot H} e^{\mathrm{i} p_{sh} \cdot X} \sigma \,|0\rangle \,, \tag{2.67}$$

subject to the masslessness condition [39]

$$\frac{p^2_{sh}}{2} + N_L - 1 + \Delta_\sigma = \frac{(q^{(a)}_{sh})^2}{2} - \frac{1}{2} + \bar{\Delta}_\sigma = 0 \,, \tag{2.68}$$

where, for the specific case of $|\varphi\rangle$, the left moving oscillator number N_L takes the form

$$N_L = w^i \mathcal{N}^i_L + \bar{w}^i \bar{\mathcal{N}}^i_L \,. \tag{2.69}$$

Combining equations (2.59), (2.60), (2.61) and (2.64), we can finally obtain the transformation behavior of $|\varphi\rangle$ under S,

$$|\varphi\rangle \xrightarrow{h} \exp\{2\pi\mathrm{i}[p_{sh}\cdot V_h - v_h^i(q_{sh}^{(a)\,i} - \mathcal{N}_L^i + \bar{\mathcal{N}}_L^i) + \gamma_h + \Phi(g,h)]\}\ |\varphi\rangle\ . \tag{2.70}$$

In order for $|\varphi\rangle$ to be part of the physical spectrum the phase it picks when transformed by any h in S must be trivial. We can distinguish two possibilities:

(i) If there exists an element $g \in [g]$ which commutes with a certain h, from eq. (2.65) it follows that $\gamma_h = 0 \bmod 1$. In this case eq. (2.70) becomes a projection condition

$$p_{sh}\cdot V_h - v_h^i(q_{sh}^{(a)\,i} - \mathcal{N}_L^i + \bar{\mathcal{N}}_L^i) + \Phi(g,h) = 0 \bmod 1\,, \tag{2.71}$$

which are the so-called *orbifold projection conditions* and are responsible for $\mathcal{N}=1$ SUSY.

(ii) In all other cases, the transformation (2.70) is made trivial after fixing the gamma-phases, i.e.

$$\gamma_h = -p_{sh}\cdot V_h + v_h^i(q_{sh}^{(a)\,i} - \mathcal{N}_L^i + \bar{\mathcal{N}}_L^i) - \Phi(g,h) = \tilde{\gamma}(g) - \tilde{\gamma}(hgh^{-1}) \bmod 1\,. \tag{2.72}$$

In this way, space group invariance fixes all $\tilde{\gamma}(g')$ in (2.62) except for one, which can be reabsorbed as an overall phase in σ. Since the phases in (2.72) are fixed differently for each physical state $|\varphi\rangle$, there will be, in general, more than one twist field σ per conjugacy class.

2.3.3 Massless Spectrum

Given the many alternatives for compactification and choices for the gauge embedding, there is a bestiary of orbifold spectra. In this section we do not aim at explicit constructions, but to sketch how the spectra are computed, and to describe their general features. The gauge group, particles and interactions in a given orbifold can be computed with the aid of the C++ orbifolder [47].

To see how $\mathcal{N}=1$ SUSY is achieved in orbifold models, let us consider the H-momenta $q^{(a)}$ introduced in section 2. Being weights in the representations $\mathbf{8}_v$ and $\mathbf{8}_s$ of SO(8), under the breaking (2.24), they decompose as

$$\begin{aligned}\mathbf{8}_v &\to \mathbf{1}_1 \oplus \mathbf{6}_0 \otimes \mathbf{1}_{-1},\\ \mathbf{8}_s &\to \mathbf{4}_{1/2} \oplus \overline{\mathbf{4}}_{-1/2}\,.\end{aligned} \tag{2.73}$$

From the 4D perspective $\mathbf{1}_1$ and $\mathbf{4}_{1/2}$ are interpreted as a vector boson and four fermions with positive helicity, respectively. The weights $\mathbf{1}_{-1}$ and $\overline{\mathbf{4}}_{-1/2}$ correspond to the CPT conjugates of the former ones. Similarly, out of $\mathbf{6}_0$ one gets three complex scalars plus conjugates. From the weights of these representations we see that the only $q^{(a)}$ for which $e^{\mathrm{i}q^{(a)}\cdot H}$ is space group invariant are

$$q_{\pm}^{(1)} = (1,0,0,0) \quad \text{and} \quad q_{\pm}^{(1)} = (\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2})\,. \tag{2.74}$$

We associate these weights to an $\mathcal{N}=1$ vector multiplet in 4D. Regarding the scalars in $\mathbf{6}_0$ and the remaining fermions from $\overline{\mathbf{4}}_{-1/2}$, the projection (2.61) induces a natural pairing among them

$$q^{(1)} = q^{(1/2)} + (\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2})\,, \tag{2.75}$$

from which we identify the SUSY generator with the cospinor weight $q_{\mathrm{SUSY}} = (\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$. As a consequence of the previous arguments it follows that any of the combinations

$$\begin{aligned} q_1^{(1)} &= (0,1,0,0)\,, & q_1^{(1/2)} &= (-\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2})\,,\\ q_2^{(1)} &= (0,0,1,0)\,, & q_2^{(1/2)} &= (-\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2})\,,\\ q_3^{(1)} &= (0,0,0,1)\,, & q_3^{(1/2)} &= (-\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2})\,, \end{aligned} \tag{2.76}$$

can be used to form a left chiral superfield. For the case of $\mathbb{Z}_N$ orbifolds, the shifted H-momenta leading to massless states are summarized in table 2.2, there we only include the bosonic weights, the fermionic counterparts can be computed with the aid of eq. (2.75). To conlude let us revisit the

	T_1	T_2	T_3	T_4	T_5	T_6
$\mathbb{Z}_3$	$\frac{1}{3}(0,1,1,1)$					
$\mathbb{Z}_4$	$\frac{1}{4}(0,1,1,2)$	$\frac{1}{2}(0,1,1,0)$				
$\mathbb{Z}_{6-\mathrm{I}}$	$\frac{1}{6}(0,1,1,4)$	$\frac{1}{3}(0,1,1,1)$	$\frac{1}{2}(0,1,1,0)$			
$\mathbb{Z}_{6-\mathrm{II}}$	$\frac{1}{6}(0,1,2,3)$	$\frac{1}{3}(0,1,2,0)$	$\frac{1}{2}(0,1,0,1)$	$\frac{1}{3}(0,2,1,0)$		
$\mathbb{Z}_7$	$\frac{1}{7}(0,1,2,4)$	$\frac{1}{7}(0,2,4,1)$		$\frac{1}{7}(0,4,1,2)$		
$\mathbb{Z}_{8-\mathrm{I}}$	$\frac{1}{8}(0,2,1,5)$	$\frac{1}{4}(0,2,1,1)$		$\frac{1}{2}(0,0,1,1)$	$\frac{1}{8}(0,2,5,1)$	
$\mathbb{Z}_{8-\mathrm{II}}$	$\frac{1}{8}(0,1,3,4)$	$\frac{1}{4}(0,1,3,0)$	$\frac{1}{8}(0,3,1,4)$	$\frac{1}{2}(0,1,1,0)$		$\frac{1}{4}(0,3,1,0)$
$\mathbb{Z}_{12-\mathrm{I}}$	$\frac{1}{12}(0,4,1,7)$	$\frac{1}{6}(0,4,1,1)$	$\frac{1}{4}(0,0,1,3)$ $\frac{1}{4}(0,0,3,1)^\clubsuit$	$\frac{1}{3}(0,1,1,1)$	$\frac{1}{12}(0,4,7,1)^\clubsuit$	$\frac{1}{2}(0,0,1,1)$
$\mathbb{Z}_{12-\mathrm{II}}$	$\frac{1}{12}(0,1,5,6)$	$\frac{1}{6}(0,1,5,0)$ $\frac{1}{6}(0,5,1,0)^\clubsuit$	$\frac{1}{4}(0,1,1,2)$	$\frac{1}{3}(0,1,2,0)$ $\frac{1}{3}(0,2,1,0)^\clubsuit$	$\frac{1}{12}(0,5,1,6)$	$\frac{1}{2}(0,1,1,0)$

Table 2.2: Shifted H-momenta of negative helicity for $\mathbb{Z}_N$ orbifold models [39]. The weights marked with ♣ belong to the inverse twisted sector of that where they are shown.

content of the untwisted sector. As we pointed out already, the untwisted spectrum is composed of the states from the ten dimensional theory which survive the orbifold projections.

(i) The states of the form $e^{\mathrm{i}q_+^{(a)}\cdot H}\alpha_{-1}^{\mu}\left|0\right\rangle$, $a=1,1/2$, $\mu=2,3$ together with their CPT conjugates, give rise to the supergravity multiplet and a chiral multiplet in four dimensions. The decomposition of the bosonic part leads to a real spin 2 particle which we identify as the 4D graviton and a complex scalar corresponding to the axion-dilaton field. Similarly, the fermionic part decomposes into the gravitino and the dilatino.

(ii) We denote the internal components of the 10D SUGRA multiplet which are not projected out by

$$T_{i\bar{j}} = e^{\mathrm{i}q_i^{(a)}\cdot H}\alpha_{-1}^{i}\left|0\right\rangle \quad U_{ij} = e^{\mathrm{i}q_i^{(a)}\cdot H}\alpha_{-1}^{i*}\left|0\right\rangle , \tag{2.77}$$

with $q_i^{(a)}$ as given in eq. (2.76). These *moduli* fields are related to geometric variations of the internal manifold. $U_{i\bar{j}}$ are the Kähler moduli, whereas U_{ij} account for complex structure deformations. Note that we always have three Kähler moduli $U_{i\bar{i}}$ in the spectrum. The presence of further moduli depends on the explicit orbifold geometry [48].

(iii) The states $e^{\mathrm{i}q_+^{(a)}\cdot H}e^{\mathrm{i}p_I X^I}\left|0\right\rangle$, satisfying $p\in\Gamma_{16}$, $p^2=2$ and

$$p\cdot V_N = 0 \bmod 1, \quad p\cdot V_M = 0 \bmod 1, \quad p\cdot W_\alpha = 0 \bmod 1 \quad \forall W_\alpha, \tag{2.78}$$

correspond to the non-Abelian gauge bosons of the four dimensional theory. As the embedding is is rank preserving, the Cartan elements $e^{\mathrm{i}q_+^{(a)}\cdot H}\tilde{\alpha}_{-1}^{I}\left|0\right\rangle$ are all present in the spectrum.

(iv) One may also find some leftover fields $e^{\mathrm{i}q_i^{(a)}\cdot H}e^{\mathrm{i}p_I X^I}\left|0\right\rangle$, which satisfy

$$p\cdot V_1 - v_1^i = 0 \bmod 1, \quad p\cdot V_2 - v_2^i = 0 \bmod 1, \quad p\cdot W_\alpha = 0 \bmod 1 \quad \forall W_\alpha. \tag{2.79}$$

Such states are chiral superfields of the four dimensional theory. The gauge momentum p specifies the charges of the field under the diverse gauge factors. A generic blueprint of these models is the presence of an anomalous U(1) symmetry [46]. The anomalies associated to this U(1) will be cancelled by shifts in the imaginary part of the axio-dilaton field previously discussed, along the lines of the Green-Schwarz mechanism. However, the large Fayet Iliopoulos term associated to this U(1) factor, will force some chiral fields in the spectrum to acquire a vacuum expectation value (VEV). This allows for a geometrical interpretation: the VEV fields can be related to additional Kähler moduli, in a geometry where some of the fixed points are smoothened out [32].

In the twisted sectors we will only find chiral superfields. In order to construct the full twisted spectrum, one first has to look at all inequivalent fixed points and their corresponding generators (i.e. constructing elements). Having the H-momenta as in table 2.2, one considers combinations of oscillators $\alpha^{i}_{-w^i}$, $\alpha^{i*}_{-\bar{w}^i}$ such that $N_L - 1 + \Delta_\sigma < 0$. For each of those, and for each constructing element, one searches for shifted momenta p_{sh} which make the oscillator combination to be in agreement with eq. (2.68). The combination of oscillators, and gauge momentum we found corresponds to a massless state. Next one has to look for the projectors associated to the constructing element, and if the state under consideration survives, then it is part of the twisted spectrum.

For the twisted sectors one observes that for each of the orbifold fixed points (conjugacy classes) there will be a given set of projection conditions. By setting these local projections to act on the original $E_8 \times E_8$ generators, we see that at the fixed points, the gauge group is generically enhanced, with the matter living there forming complete representations of the enhanced gauge group. Thus, over the orbifold, there is a non trivial topography of gauge groups. The four dimensional gauge symmetry results from the intersection of all of these gauge factors [49].

CHAPTER 3

Discrete symmetries in orbifold models

Tout art, comme toute science, est un moyen de communication entre les hommes. Il est évident que l'efficacité et l'intensité de la communication diminuent et tendent à s'annuler dès l'instant qu'un doute s'installe sur la vérité de ce qui est dit, sur la sincérité de ce qui est exprimé (imagine-t-on, par exemple, une science au second degré?).

Michel Houellebecq, Approches du désarroi.

In order to obtain an effective field theory for orbifold models, we have to construct the interaction terms. These can be accessed via the conformal field theory (CFT). Among the many interaction terms, those belonging to the superpotential are of outmost relevance for particle phenomenology. They are also easier to track in the CFT since they are purely holomorphic. In this chapter we introduce the techniques required to deal with these operators. Nevertheless, we will not aim to compute them exactly. Instead, we will discuss the conditions needed for some couplings to vanish. In many cases, the vanishing of a certain class of operators can be related to the presence of discrete symmetries in the effective field theory. As we will observe, many of these symmetries have a clear interpretation as symmetries of the orbifold geometry. Among the various symmetries considered, we devote special attention to R-symmetries, which are discrete remnants of the Lorentz group in internal space that happen to survive the orbifolding process. We provide a prescription to compute the R-charges of the fields, and show that these are anomaly universal. The results of this section are used to compute the discrete symmetries exhibited by the $\mathbb{Z}_2 \times \mathbb{Z}_4$ orbifold geometry (see section 4.2.2).

3.1 Physical Vertices and Correlation Functions

For the purpose of computing amplitudes, it is more convenient to think of the physical state $|\varphi\rangle$ (see eq. (2.67)) in terms of the vertex operator which creates it, namely

$$V_{-a} = e^{-a\phi} \left(\prod_{i=1}^{3} (\partial Z^i)^{\mathcal{N}_L^i} (\partial \bar{Z}^i)^{\bar{\mathcal{N}}_L^i} \right) e^{\mathrm{i} q_{sh}^{(a)} \cdot H} e^{\mathrm{i} p_{sh} \cdot X} \sigma \,. \tag{3.1}$$

We have already developed an intuition for the conformal fields appearing in the previous equation. The only new piece is the scalar field φ which is part of the superconformal ghost system [45]. The subscript $-a$ denotes the conformal ghost charge. Note that a also labels the H-momentum, so that V_{-1} is associated with a boson, and $V_{-1/2}$ with a fermion. Note that untwisted fermionic and bosonic vertex operators have the same form as (3.1) but with all momenta unshifted and with the twist field replaced by the identity operator. Furthermore it is important to recall that the vertices are expressed in the zero 4D momentum limit which is enough for the matters we are interested in. Also the cocycle factors [50] arising from consistency with fermionic anti-commutation properties as well as the correct normalization factors [51] have been omitted for simplicity.

Since the prescription (2.75) helps us to pair up a bosonic weight with its SUSY companion, we can also pair up vertex operators, which will be interpreted as components (ϕ, ψ) of a left chiral superfield Φ. We are interested in a generic L-point coupling Φ^{L}. The presence of this coupling in the superpotential is related to a non-vanishing correlator of the form $\psi\psi\phi^{\mathrm{L}-2}$. In particular, we want to study this correlator at tree level [52]. There are off course loop contributions, but these will be exponentially suppressed by the dilaton field, and hence they are part of the Kähler potential. Higher order contributions to the Φ^{L} coupling will only be allowed in the superpotential after moduli stabilization [53, 54]. The reason for which we ignore these contributions is the following: We are not really interested in the value of the correlators but wether or not they are zero. If the coupling vanishes, this hints at the possibility that the coupling is forbidden by a given symmetry. Hence, if a forbidden coupling gets induced at loop level, this implies that the symmetry of our interest gets broken in the moduli stabilization process.

An L-point correlator at tree level corresponds to the emission of L string states in the background of the sphere. In the correlator, the emission vertices have to be accommodated in the correlator in such a way the background charge of two on the sphere is cancelled. This makes it necessary to write some bosonic operators in a different ghost picture

$$V_0 = e^{\phi} T_F V_{-1}\,, \tag{3.2}$$

where T_F is the world sheet supersymmetry current introduced in section 2.1. In complexified coordinates this current takes the form

$$T_F = \bar{\partial} Z^i \bar{\psi}^i + \bar{\partial} \bar{Z}^i \psi^i\,, \tag{3.3}$$

with $\psi^j = \exp\{-\mathrm{i} q_j \cdot H\}$ and $(q_j)^i = \delta^i_j$. This picture changing operation allows some bosonic vertices to have zero conformal charge at the price of introducing the right moving oscillators $\bar{\partial} Z^j$ and $\bar{\partial}\bar{Z}^j$. The tree level correlation function can then be expressed as

$$\mathcal{F} = \left\langle V_{-1/2}(z_1, \bar{z}_1) V_{-1/2}(z_2, \bar{z}_2) V_{-1}(z_3, \bar{z}_3) V_0(z_4, \bar{z}_4) \ldots V_0(z_L, \bar{z}_L) \right\rangle\,, \tag{3.4}$$

where each $V_\alpha = V(z_\alpha, \bar{z}_\alpha)$ (independently of the conformal charge) represents a certain physical state from the massless spectrum. The correlator $\mathcal{F}$ is now the object of our interest. As already mentioned, we will not compute it explicitly. Instead, we will study the properties of the conformal blocks in $\mathcal{F}$ which make it vanish. In the following we will focus on $\mathbb{Z}_N$ orbifolds for concreteness, but the results can be straightforwardly extended to those of the $\mathbb{Z}_N \times \mathbb{Z}_M$ type.

3.1.1 Gauge invariance

In the correlator $\mathcal{F}$ we must integrate over the gauge coordinates. Since the orbifold CFT is non-interacting, the fields $e^{\mathrm{i}p_{sh}\cdot X}$ are uncorrelated to any of the other pieces composing the vertex (3.1). Thus we can factorize them and integrate out the gauge coordinates X^I. After integration one gets [51]

$$\langle \prod_{\alpha=1}^{\mathrm{L}} e^{\mathrm{i}p^I_{sh\ \alpha} X_I(z_\alpha)} \rangle = \delta^{16}\left(\sum_{\alpha=1}^{\mathrm{L}} p_{sh\ \alpha}\right) \prod_{\substack{\alpha,\beta=1\\ \alpha<\beta}}^{\mathrm{L}} (z_\alpha - z_\beta)^{\frac{p_\alpha \cdot p_\beta}{2}}\,, \tag{3.5}$$

from which it follows that $\mathcal{F}$ is non-zero only if

$$\sum_{\alpha=1}^{\mathrm{L}} p^I_{sh\ \alpha} = 0\,. \tag{3.6}$$

Recall that p_{sh} gives the charges of the fields under gauge transformations. Hence eq. (3.6) reproduces the requirement of invariance under gauge transformations for each of the couplings present in the superpotential.

3.1.2 Discrete symmetries from the space group

Among the pieces which appear in the correlation function (3.4), there is a product of twist fields. The expectation value for this product of fields vanishes if it does not preserve the vacuum, i.e.

$$\mathbb{1} \in \prod_{\alpha=1}^{L} \sigma_\alpha \,. \tag{3.7}$$

This defines the *space group selection rule*. From the OPEs for the twists one can observe that the above relation holds only if it is possible to find a combination $g_1' \in [g_1]$, $g_2' \in [g_2]$,...,$g_L' \in [g_L]$ such that

$$g_1' \cdot g_2' \cdot ..., \cdot g_L' = (\mathbb{1}, 0)\,. \tag{3.8}$$

Given the structure of the space group, we are allowed to split the last condition into its twist and lattice part. It implies that the product of twists must be equal to the identity, i.e.

$$\sum_{\alpha=1}^{L} k_\alpha = 0 \bmod N\,, \tag{3.9}$$

where we have written $g_\alpha = (\theta^{k_\alpha}, \lambda_\alpha)$. The previous relation is known as the *point group selection rule*, and can be seen as a discrete $\mathbb{Z}_N$ symmetry from the perspective of the effective field theory. For $\mathbb{Z}_N \times \mathbb{Z}_M$ this selection rule provides two discrete symmetries, one per each factor. Equation (3.7) also implies that there exists a set of lattice vectors $\tau_\alpha \in \Gamma_6$ and some numbers $j_\alpha \in \mathbb{Z}$ which fulfill

$$\sum_{\alpha=1}^{\mathrm{L}} \left(\prod_{\beta=\alpha+1}^{\mathrm{L}} \theta^{k_\beta} \right) [\theta^{j_\alpha}\lambda_\alpha + (\mathbb{1} - \theta^{k_\alpha})\tau_\alpha] = 0\,. \tag{3.10}$$

This expression can be rewritten in terms of the fixed points $Z_\alpha = (\mathbb{1} - \theta^{k_\alpha})\lambda_\alpha$ as

$$\sum_{\alpha=1}^{L} \left(\prod_{\beta=\alpha+1}^{L} \theta^{k_\beta} \right) (\mathbb{1} - \theta^{k_\alpha})[\theta^{j_\alpha} Z_\alpha + \tau_\alpha] = 0\,. \tag{3.11}$$

This *fixed point selectivity* determines the configuration of fixed points in which fields can sit in order to give a non vanishing coupling. In general, the previous relation also induces some Abelian discrete symmetries. Their explicit form depends highly on the choice for the compactification lattice Γ.

3.1.3 Non-Abelian family symmetries

In the absence of Wilson lines we could have multiple fixed points nesting identical physical states. In many cases, these fixed points happen to be indistinguishable in the orbifold geometry, and thanks to this, the CFT becomes invariant under certain permutation symmetries. Similarly as with the point group selectivity, permutation symmetries evade a generic description as they are very dependent on the compactification lattice. In order to get an impression of them, here we illustrate how they arise in very simple examples, which have been discussed in ref [24]. First, consider the orbicircle $S^1/\mathbb{Z}_2$. It contains two identical fixed points, so that the orbifold exhibits an S_2 permutation symmetry. Let us denote by V_α and $V_{\alpha'}$ identical vertices creating states at each of the fixed points. Thus, under the action of the permutation generator S, the vertices transform according to

$$\begin{pmatrix} V_\alpha \\ V_{\alpha'} \end{pmatrix} \xrightarrow{S} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} V_\alpha \\ V_{\alpha'} \end{pmatrix} , \tag{3.12}$$

Furthermore, if one applies the point group selectivity (3.11) in this case, one finds that it contributes a $\mathbb{Z}_2$ symmetry under which V has charge 0 and V' has charge 1 (the choice of the invariant state has no relevance). If we denote by T the generator of this symmetry, we find

$$\begin{pmatrix} V_\alpha \\ V_{\alpha'} \end{pmatrix} \xrightarrow{T} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} V_\alpha \\ V_{\alpha'} \end{pmatrix} . \tag{3.13}$$

From the action of T and S we can readily infer that together they generate a D_4 symmetry under which V_α and $V_{\alpha'}$ transform in the doublet representation D. This situation is common in orbifold compactifications: In the absence of a mechanism which distinguishes among degenerate fixed points, for example a Wilson line, the contribution of the fixed point selectivity rule will usually enhance to a non-Abelian discrete symmetry. In particular, the physical states will merge into irreducible representations of the non-Abelian group. For the case of D_4 we have five irreducible representations, the doublet D and four one dimensional representations A_1, A_2, A_3 and A_4. The action of S and T on them is more or less intuitive: A_1 is the invariant singlet, A_2 picks a -1 under T, A_3 picks a -1 under S and A_4 transforms with -1 under both.

One step further one can consider the orbifold $\mathbb{T}^2/\mathbb{Z}_2$, which can be treated as a product of orbicircles while keeping in mind that the $\mathbb{Z}_2$ identifications must act simultaneously on both of them. This implies that the family symmetry in this orbifold is $D_4 \times D_4/\mathbb{Z}_2$. A similar procedure can be used to infer the family symmetries in a given six dimensional orbifold. For a particular example, the reader is referred to section 4.2.2, where we discuss the non-Abelian discrete symmetries in the $\mathbb{Z}_2 \times \mathbb{Z}_4$ orbifold.

3.1.4 H-momentum conservation

Back to the correlator (3.4), one can further integrate out the bosonized fermions H^i. In doing so, we will find an expression analogous to (3.6). Recall that in the correlator, $\mathrm{L}-3$ vertices have been set to the 0 picture (see eq. (3.2)) using the operator T_F which, in turn, depends on the fields H^i. More precisely, T_F is writen as a sum over the complex coordinates in compact space. After expanding such terms in the correlation function (3.4) and integrating the H^i, one can observe that for the coupling not to vanish, at least one of the terms in the expansion has to fulfill

$$q_{sh\ 1}^{(1/2)\ i} + q_{sh\ 2}^{(1/2)\ i} + q_{sh\ 3}^{(1)\ i} + \sum_{\alpha=4}^{L} q_{sh\ \alpha}^{(1)\ i} + (\mathcal{N}_R^i - \bar{\mathcal{N}}_R^i) = 0, \tag{3.14}$$

where $\mathcal{N}_R^i$ and $\bar{\mathcal{N}}_R^i$ count how many times the term -either $\bar{\partial} Z^j \bar{\psi}^j$ or its conjugate- is present in the piece under consideration. As can be seen from the structure of T_F, these numbers ($\bar{\mathcal{N}}_R^i$ and $\mathcal{N}_R^i$) coincide with the amount of right moving oscillators involved. In contrast to $\mathcal{N}_L^i$ and $\bar{\mathcal{N}}_L^i$, the right moving oscillator numbers do not have a counterpart as quantum numbers for the physical states. Because of that, both $\mathcal{N}_R^i$ and $\bar{\mathcal{N}}_R^i$ have to be regarded as a property of the correlation function itself.

In any orbifold, the three internal weights $q_{sh}^{(1/2)\ i}$ $(i = 1, 2, 3)$ for a massless left chiral state add up to $1/2$. Summing over (3.14) one finds

$$-(\mathrm{L}-3) + \sum_i (\mathcal{N}_R^i - \bar{\mathcal{N}}_R^i) = 0\,. \tag{3.15}$$

from which we infer that necessarily $\bar{\mathcal{N}}_R^i = 0$, so that the *H-momentum conservation* rule can be written as

$$\sum_{\alpha=1}^{L} q_{sh\ \alpha}^{(1)\ i} = -1 - \mathcal{N}_R^i\,, \tag{3.16}$$

One can use table 2.2 to see that all entries $q_{sh\ \alpha}^{(1)\ i}$ are negative or zero. By virtue of the space group selection rule, we also see that in any allowed coupling they add to an integer. From this it follows that eq. (3.16) is generically satisfied, except for some seldom couplings where the

$q_{sh\ \alpha}^{(1)\ i}$ add up to zero. This happens for example in the $\mathbb{Z}_{6-\mathrm{II}}$ orbifold: A tree point coupling involving only states from the θ^4 sector is allowed by the point group, but the H-momenta in the third complex plane add up to zero[1]. Even though we have identified some couplings which are zero, the condition which set them to vanish is not a property of the states but of the coupling itself. In contrast to the previous cases, the zeroes of H-momentum are not protected by any symmetry. Hence we expect them to be generated after moduli stabilization, or away from the orbifold point. Despite of that, H-momentum conservation remains as a valuable condition since it encodes, in a certain sense, the Lorentz invariance condition on internal space, or at least its right moving part.

3.2 Discrete R-symmetries from orbifold isometries

At this point we have integrated out many of the conformal fields present in the correlation function. Assuming that it is invariant under all selection rules we have discussed so far, $\mathcal{F}$ takes the form

$$\mathcal{F} \sim \left\langle \prod_{\alpha=1}^{\mathrm{L}} \left(\prod_{i=1}^{3} (\partial Z^i)^{\mathcal{N}_L^i{}_\alpha} (\partial \bar{Z}^i)^{\bar{\mathcal{N}}_L^i{}_\alpha} (\bar{\partial} Z^i)^{\mathcal{N}_R^i} \right) \sigma_\alpha \right\rangle . \tag{3.17}$$

Note that we have not integrated out the twisted fields due to the OPEs (2.57). For a thorough discussion of how to deal with this object see e.g. [45, 51]. For our purposes it suffices the observation that the oscillators present in eq. as well as the twist fields encode valuable geometrical information. On the one hand, the fields ∂Z^i, $\bar{\partial} Z^i$ and, $\partial \bar{Z}^i$ must exhibit the symmetries of the internal space. On the other hand, the twists σ_α are related to the fixed points in the orbifold. Thus, we expect the correlator $\mathcal{F}$ to be invariant under isometries of the orbifold space[2].

We focus on orbifold isometries which are also be isometries of the lattice Γ. Thus, we proceed to dissect the automorphism group of Γ ($\mathrm{Aut}(\Gamma)$) into smaller subgroups, depending on what are their effects on the point and space groups. Any symmetry which could be of relevance for the orbifold must have a closed action on P. The set of symmetries fulfilling this property form the following group

$$A = \left\{ \varrho \in \mathrm{Aut}(\Gamma) \mid \varrho \vartheta \varrho^{-1} \in \mathrm{P},\ \forall \vartheta \in \mathrm{P} \right\} . \tag{3.18}$$

[1] A similar situation occurs in $\mathbb{Z}_{8-\mathrm{II}}$. A coupling of the form $\theta^6\theta^6\theta^4$ is also forbidden by H-momentum.

[2] This can be explicitly seen when computing the correlator. There one observes that the orbifold isometries are symmetries of the instanton solutions for ∂Z^i and $\bar{\partial} Z^i$.

Clearly any vector in the compact dimensions will transform under the elements of Aut(Γ_6). Consequently the action of any $\varrho \in A$ on a space group element $h = (\theta^k, \lambda) \in S$ is given by

$$\varrho(h) = (\varrho\theta^k\varrho^{-1}, \varrho\lambda)\,. \tag{3.19}$$

Note that any transformation $\varrho \in A$ will preserve the network of identifications, i.e. given two elements $g_1, g_2 \in$ S which belong to the same conjugacy class, then $\varrho(g_1) \sim \varrho(g_2)$. Since the twists σ_α span over complete conjugacy classes, we see that the action of $\varrho \in A$ can be set to map among twist fields. We further define the subgroups

$$\begin{aligned} B &= \{\varrho \in A \mid [\varrho, \theta] = 0\}\ , \\ C &= \{\varrho \in B \mid \varrho(h) \in [h]\,,\ \forall h \in \mathrm{S}\}\ , \\ D &= \{\varrho \in C \mid \det(\varrho) = 1\}\ . \end{aligned} \tag{3.20}$$

In other words B is the subgroup of symmetries which map between conjugacy classes of the same twisted sector. C is defined as the subgroup of automorphisms which preserve all the conjugacy classes of the space group and D contains all elements in C which belong to SO(6). It is easy to show that these subgroups fulfill the following normalcy chain

$$D \lhd C \lhd B \lhd A\,, \tag{3.21}$$

and hence it makes sense to define the corresponding quotient groups $E = C/D$, $F = B/C$ and $G = A/B$. Therefore, the elements in E will be reflections leaving the conjugacy classes invariant. The elements in F will exchange between conjugacy classes, and the elements in G trade different twisted sectors.

From the previous discussion we see that in the conformal field theory, the groups F and G exchange between correlators, whereas elements in D must leave them invariant. The symmetries in E are outside of the internal Lorentz group SU(4), these interchange the oscillators ∂Z^i and $\partial\bar{Z}^i$, and thus they do not leave the vertices invariant. From now on we will focus on elements in D as they allow for the simplest interpretation. Consider an element $\varrho \in D$. Since it commutes with the point group, it can be generally written in a block diagonal form

$$\varrho = \mathrm{diag}(e^{2\pi\mathrm{i}\xi^1}, e^{2\pi\mathrm{i}\xi^2}, e^{2\pi\mathrm{i}\xi^3})\,. \tag{3.22}$$

By definition, given a $g \in$ S, $\varrho(g)$ is conjugate to g, and hence there exists a space group element h_g such that

$$\varrho(g) = h_g g h_g^{-1}\,. \tag{3.23}$$

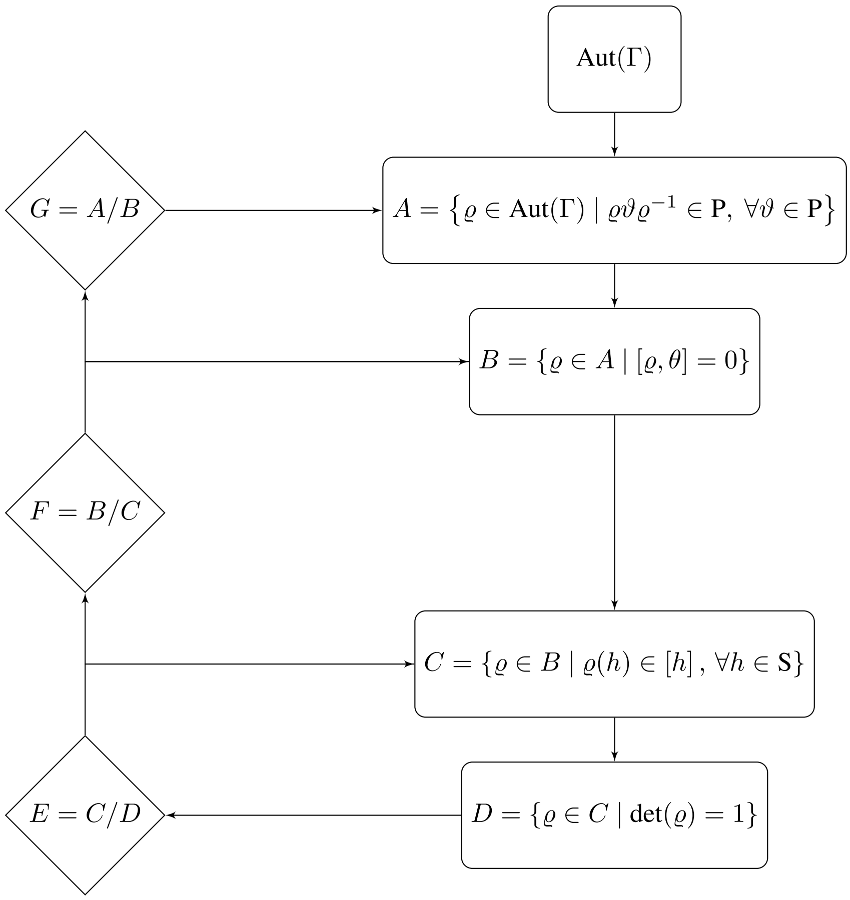


Figure 3.1: The automorphism group for the six dimensional lattice allows for a decomposition into the subgroups A, B, C and D. Provided the normalcy relations between them, one can construct the quotients E, F and G which allow for a simpler interpretation.

Writing $g = (\theta^k, \lambda)$ and $\varrho(g) = (\theta^k, \varrho\lambda)$, $h_g = (\theta^l, \mu)$ can be determined by finding a solution to the equation

$$\mu = (1 - \theta^k)^{-1}(\varrho - \theta^l)\lambda\,. \tag{3.24}$$

In analogy to (2.63), the most general transformation behavior for the auxiliary twist fields under ϱ is given by

$$\sigma_g \xrightarrow{\varrho} e^{2\pi\mathrm{i}\Phi_\varrho(g)}\sigma_{\varrho(g)}\,, \tag{3.25}$$

which, for the twist fields as written in eq. (2.62), implies

$$\sigma \xrightarrow{\varrho} \sum_{g' \in [g]} e^{2\pi\mathrm{i}[-\tilde{\gamma}(\varrho(g')) + \tilde{\gamma}(g') + \Phi_\varrho(g')]} e^{2\pi\mathrm{i}\tilde{\gamma}(\varrho(g'))}\sigma_{\varrho(g')}\,. \tag{3.26}$$

Since ϱ preserves conjugacy classes, the vertex operators have to be invariant up to phases. This means that we have to require the structure of σ to be preserved, i.e.

$$\tilde{\gamma}(g) - \tilde{\gamma}(\varrho(g)) + \Phi_\varrho(g) = \tilde{\gamma}(hgh^{-1}) - \tilde{\gamma}(\varrho(hgh^{-1})) + \Phi_\varrho(hgh^{-1}) \bmod 1\,. \tag{3.27}$$

Using $\varrho(hgh^{-1}) = \varrho(h)\varrho(g)\varrho(h)^{-1}$ together with the definition of the gamma phases (see eq. (2.65)), we obtain

$$\Phi_\varrho(hgh^{-1}) = \Phi_\varrho(g) + \gamma_h - \gamma_{\varrho(h)} \bmod 1. \tag{3.28}$$

Note that this equation implies that once we know the phase $\Phi_\varrho(g)$ for a given representative $g \in [g]$, the phases for all other elements of the conjugacy class are automatically fixed. Note also that the phase $\Phi_\varrho(g)$ is an intrinsic property of the twist σ_g, whereas the gamma phases depend, via (2.72), on the quantum numbers of the corresponding state. Therefore, if (3.28) is to be fulfilled for all physical states, the vacuum-phases and gamma-phases must independently fulfill

$$\begin{aligned} \Phi_\varrho(hgh^{-1}) - \Phi_\varrho(g) &= 0 \bmod 1\,, \\ \gamma_h - \gamma_{\varrho(h)} &= 0 \bmod 1\,, \end{aligned} \tag{3.29}$$

for all space group elements $g, h \in \mathrm{S}$. Now, plugging (3.23) into (3.27) and using (3.29) we find

$$\gamma_{h_g} = \gamma_{h_{g'}} \bmod 1\,, \tag{3.30}$$

for all $g' \in [g]$. This permits the transformation of the σ twist to be recast to the desired form

$$\sigma \xrightarrow{\varrho} e^{2\pi\mathrm{i}[\gamma_{h_g} + \Phi_\varrho(g)]}\sigma\,. \tag{3.31}$$

Note that we have left the phases $\Phi_\varrho(g)$ undetermined. In fact, one can show that the space group selection rule together with the OPEs for the twist fields imply [3]

$$\sum_{\alpha=1}^{\mathrm{L}} \Phi_\varrho(g_\alpha) = 0 \bmod 1\,, \tag{3.32}$$

so that, the transformation of the twist field σ is fixed by the gamma phase γ_{h_g}. Having fixed this transformation we consider now the oscillators. In analogy with eq. (2.59) we write

$$\partial Z^i \xrightarrow{\varrho} e^{2\pi\mathrm{i}\xi^i}\partial Z^i\,, \quad \partial \bar{Z}^i \xrightarrow{\varrho} e^{2\pi\mathrm{i}\xi^i}\partial \bar{Z}^i\,, \quad \bar{\partial} Z^i \xrightarrow{\varrho} e^{2\pi\mathrm{i}\xi^i}\bar{\partial} Z^i\,. \tag{3.33}$$

[3] Using the space group selection rule, the leading term in the OPE of all auxiliary twist fields involved in the coupling is proportional to the identity, which transforms trivially under ϱ.

Finally, the transformation behavior of the correlator (3.2) under ϱ can be obtained. It follows that $\mathcal{F}$ is only invariant in the case

$$\sum_i \xi^i \left(\sum_{\alpha=1}^{\mathrm{L}} \left(\mathcal{N}^i_{L\,\alpha} - \bar{\mathcal{N}}^i_{L\,\alpha} \right) + \mathcal{N}^i_R \right) + \sum_{\alpha=1}^{\mathrm{L}} \gamma_{h_{g_\alpha}} = 0 \bmod 1\,. \tag{3.34}$$

Note again that the non-physical quantity $\mathcal{N}^i_R$ appears. We can get rid of this term thanks to the H-momentum conservation condition (3.16). The final result reads[4]

$$\sum_{\alpha=1}^{\mathrm{L}} \left(\sum_{i=1}^{3} \xi^i \left[q^{(1)\,i}_{sh\ \alpha} - \mathcal{N}^i_{L\,\alpha} + \bar{\mathcal{N}}^i_{L\,\alpha} \right] - \gamma_{h_{g_\alpha}} \right) = - \sum_{i=1}^{3} \xi^i \bmod 1\,. \tag{3.35}$$

In the case $\sum_i \xi^i \neq 0 \bmod 1$, this condition looks precisely like the requirement of invariance under an R-symmetry. In that case, take M to be the smallest integer such that

$$R \equiv -M \sum_i \xi^i \tag{3.36}$$

is an integer too. Then eq. (3.35) takes the more familiar form

$$\sum_{\alpha=1}^{\mathrm{L}} r_\alpha = R \bmod M\,, \qquad \text{with} \qquad r_\alpha = \sum_{i=1}^{3} M\xi^i \left[q^{(1)\,i}_{sh\ \alpha} - \mathcal{N}^i_{L\,\alpha} + \bar{\mathcal{N}}^i_{L\,\alpha} \right] - M\gamma_{h_{g_\alpha}}. \tag{3.37}$$

Thus, from the invariance of the correlator $\mathcal{F}$ under the action of any $\varrho \in D$, we have derived a quantity that can be readily interpreted as a $\mathbb{Z}^R_M$ discrete symmetry. The quantity R in our notation, denotes the charge of the superpotential[5], and r_α correspond to the charges of the chiral superfields. In table 3.1 we provide an account of the R-symmetry generators expected for different types of $\mathbb{Z}_N$ orbifolds. Note that the R-symmetries are very sensitive to the compactification lattice. Thus, our results only apply for the orbifold models with the specific compactification lattice given in table 3.1. As a final remark, notice that the R-charges for the axio-dilaton, as well as the U and T moduli are zero. This implies that the R-symmetries could survive after moduli stabilization. However, we also discussed in section 2.3.3, that generically some twisted fields acquire VEVs as needed to cancel the FI term. Since these twisted fields carry non trivial charges, the R-symmetries are broken by the VEV configurations. However, for phenomenological reasons R-symmetries are desired in the low energy. Hence, one has to consider very special vacua where at least one R-symmetry survives. We will elaborate more on these matters

[4] This result was derived also in ref. [55] for the specific case of the $\mathbb{Z}_{6-\mathrm{II}}$

[5] For a comprehensive summary of R-charge conventions we refer to [56].

orbifold	lattice	twist	ϱ			R	M
			ξ^1	ξ^2	ξ^3		
$\mathbb{Z}_4$	SO(4)^2 × SU(2)^2	$(0, \frac{1}{4}, \frac{1}{4}, -\frac{2}{4})$	$1/4$	$1/4$	0	-1	2
			$1/2$	0	0	-1	2
			0	0	$-1/2$	$+1$	2
$\mathbb{Z}_4$	SU(4)^2	$(0, \frac{1}{4}, \frac{1}{4}, -\frac{2}{4})$	$1/2$	0	0	-1	2
			0	$1/2$	0	-1	2
$\mathbb{Z}_{6-\mathrm{I}}$	$G_2 \times G_2 \times$ SU(3)	$(0, \frac{1}{6}, \frac{1}{6}, -\frac{2}{6})$	$1/6$	$1/6$	0	-1	3
			0	0	$-1/3$	$+1$	3
$\mathbb{Z}_{6-\mathrm{II}}$	$G_2 \times$ SU(3) × SU(2)^2	$(0, \frac{1}{6}, \frac{2}{6}, -\frac{3}{6})$	$1/6$	0	0	-1	6
			0	$1/3$	0	-1	3
			0	0	$-1/2$	$+1$	2
$\mathbb{Z}_{8-\mathrm{I}}$	SO(9) × SO(5)	$(0, \frac{1}{8}, -\frac{3}{8}, \frac{2}{8})$	$1/4$	$-3/4$	0	$+1$	2
			0	0	$1/2$	-1	2
$\mathbb{Z}_{8-\mathrm{II}}$	SO(8) × SO(4)	$(0, \frac{1}{8}, \frac{3}{8}, -\frac{4}{8})$	$1/8$	$3/8$	0	-1	2
			0	0	$-1/2$	$+1$	2
$\mathbb{Z}_{12-\mathrm{I}}$	SU(3) × F_4	$(0, \frac{4}{12}, \frac{1}{12}, -\frac{5}{12})$	$1/3$	0	0	-1	3
			0	$1/12$	$-5/12$	$+1$	3
$\mathbb{Z}_{12-\mathrm{II}}$	$F_4 \times$ SO(4)	$(0, \frac{1}{12}, \frac{5}{12}, -\frac{6}{12})$	$1/12$	$5/12$	0	-1	2
			0	0	$-1/2$	$+1$	2

Table 3.1: Summary of point groups studied with their corresponding lattices and orbifold isometries. The charge of the superpotential R and the order of the symmetry M are also given.

when discussing the $\mathbb{Z}_2 \times \mathbb{Z}_4$ orbifold (see sect. 4.4.1).

3.2.1 Further R-symmetry candidates

The symmetries from the group F and G are more difficult to track in the CFT. While the group G is usually trivial, there are some simple examples of F which allow for a simpler interpretation. An element $\zeta \in F$ has a well defined action on the CFT, only if the fixed points which get mapped to each other under ζ allocate identical representations. The first observation about the any $\zeta \in F$ is that for all cases considered, the elements in F can be written in a block diagonal form as

$$\zeta = \mathrm{diag}(e^{2\pi \mathrm{i}\eta^1}, e^{2\pi \mathrm{i}\eta^2}, e^{2\pi \mathrm{i}\eta^3}). \tag{3.38}$$

In some sense, the symmetries from F remind us of the permutation symmetries discussed in section 3.1.3. However, for the cases we are considering, due to eq. (3.38), the group F is always Abelian. For those vertex operators which are eigenstates of ζ, the charges are identical to those in (3.37):

$$r_\alpha = \sum_{i=1}^{3} M\eta^i \left[q_{sh\ \alpha}^{(a)\ i} - \mathcal{N}_{L\ \alpha}^i + \bar{\mathcal{N}}_{L\ \alpha}^i\right] - M\gamma_{h_{g\alpha}}. \tag{3.39}$$

Now let us contemplate vertices from non-invariant fixed points. For simplicity, let us assume that the role of ζ is simply to exchange among some conjugacy classes, i.e.

$$[g] \overset{\zeta}{\leftrightarrow} [g'], \quad g \nsim g'. \tag{3.40}$$

This implies that a vertex V from $[g]$ gets mapped to its counterpart V', where V and V' share the same quantum numbers[6]. Writing the twist fields involved in constructing V and V' as

$$\begin{aligned} \sigma &\sim \sum_{g\in[g]} e^{2\pi\mathrm{i}\tilde{\gamma}(g)}\sigma_g\,, \\ \sigma' &\sim \sum_{g'\in[g']} e^{2\pi\mathrm{i}\tilde{\gamma}'(g')}\sigma_{g'}\,, \end{aligned}$$

their transformation under ζ is given by[7]

$$\begin{aligned} \sigma &\xrightarrow{\zeta} \exp\{2\pi\mathrm{i}[\tilde{\gamma}(g) - \tilde{\gamma}'(\zeta(g)) + \Phi_\zeta(g)]\}\sigma'\,, \\ \sigma' &\xrightarrow{\zeta} \exp\{2\pi\mathrm{i}[\tilde{\gamma}'(g') - \tilde{\gamma}(\zeta(g')) + \Phi_\zeta(g')]\}\sigma\,, \end{aligned} \tag{3.41}$$

and and hence, the transformation of V and V' reads

$$\begin{aligned} V &\xrightarrow{\zeta} \exp\{2\pi\mathrm{i}[-\eta^i(q_{sh}^{(a)\ i} - \mathcal{N}_L^i + \bar{\mathcal{N}}_L^i) + \tilde{\gamma}(g) - \tilde{\gamma}'(\zeta(g)) + \Phi_\zeta(g)]\}V'\,, \\ V' &\xrightarrow{\zeta} \exp\{2\pi\mathrm{i}[-\eta^i(q_{sh}^{(a)\ i} - \mathcal{N}_L^i + \bar{\mathcal{N}}_L^i) + \tilde{\gamma}'(g') - \tilde{\gamma}(\zeta(g')) + \Phi_\zeta(g')]\}V\,, \end{aligned} \tag{3.42}$$

[6] Note that although V and V' are associated with different conjugacy classes, one can show that if a coupling $\langle V_1 \ldots V_{\mathrm{L}-1}V\rangle$ is allowed by all selection rules, then so is $\langle V_1 \ldots V_{\mathrm{L}-1}V'\rangle$.

[7] In general one can allow for vacuum phases for the twist fields under this transformation. However they turn out to be irrelevant for our discussion in the same way as we observed for the group D.

respectively. Recall that $q_{sh}^{(a)\ i}$, $\mathcal{N}_L^i$ and $\bar{\mathcal{N}}_L^i$ are the same for both V and V'. Note that, a priori, the transformation phases in (3.41), (3.42) cannot be related to physical gamma-phases since g and $\zeta(g)$ belong to different conjugacy classes. Since V and V' differ only in their conjugacy classes and carry identical quantum numbers, one can write them consistently in a basis of eigenstates of ζ

$$V^{(s)} = V + e^{2\pi \mathrm{i}(\delta+s)} V' \,, \quad s = 0\,, \tfrac{1}{2}\,, \tag{3.43}$$

in which δ is a phase fixed so that the operators $V^{(s)}$ are invariant under ζ up to a phase. Using equations (3.42) and (3.43), we can fix δ and write the transformation behavior of $V^{(s)}$ under ζ as

$$V^{(s)} \xrightarrow{\zeta} \exp\left\{2\pi \mathrm{i}\left[-\eta^i(q_{sh}^{(a)\ i} - \mathcal{N}_L^i + \bar{\mathcal{N}}_L^i) + \tfrac{1}{2}\left(\gamma_{h_g} + \gamma'_{h_{g'}}\right) + s\right]\right\} V^{(s)} \,. \tag{3.44}$$

The space group elements h_g and $h_{g'}$ satisfy

$$\zeta(g') = h_g g h_g^{-1} \,, \quad \zeta(g) = h_{g'} g' h_{g'}^{-1} \,, \tag{3.45}$$

for any combination of representatives g and g'. Their corresponding gamma phases are given by

$$\gamma_{h_g} = \tilde{\gamma}(g) - \tilde{\gamma}(h_g g h_g^{-1}) \,, \quad \gamma'_{h_{g'}} = \tilde{\gamma}'(g') - \tilde{\gamma}'(h_{g'} g' h_{g'}^{-1}) \,. \tag{3.46}$$

From the transformation property of the $V^{(s)}$ we can now read off their corresponding R-charges

$$r^{(s)} = M \sum_{i=1}^{3} \eta^i(q_{sh}^{(a)\ i} - \mathcal{N}_L^i + \bar{\mathcal{N}}_L^i) - \tfrac{1}{2} M \left(\gamma_{h_g} + \gamma'_{h_{g'}}\right) - Ms \,, \tag{3.47}$$

where M is the smallest integer such that

$$R \equiv -M \sum_{i=1}^{3} \eta^i \in \mathbb{Z} \,. \tag{3.48}$$

From the previous results we see that the low energy effective field theory is more conveniently described in terms of ζ eigenstates. In this basis the corresponding R-symmetry for the correlators among those fields is given by

$$\sum_{\alpha=1}^{\mathrm{L}} r_\alpha = R \bmod M \,. \tag{3.49}$$

The very simple result we just obtained for the elements in F has remarkable implications. It tells us that not only elements from D are responsible for the R-symmetries in the effective theory. In contrast to those emerging from D, these novel R-symmetries can be broken by Wilson line configurations that destroy the state degeneracy at the non-invariant fixed points. Note that in our derivation we assumed that ζ at most *interchanges* pairs of conjugacy classes, but in principle more intricate transformation patterns can emerge, particularly in the case of non-factorizable orbifolds. We expect that in those cases, the charges can be computed in a similar fashion.

3.3 Universal discrete anomalies

The ten dimensional heterotic theory contains the ingredients necessary for the Green-Schwarz mechanism to cancel all of its anomalies [30]. We expect the same to hold after compactification. With regards to the continuous symmetries, it was already pointed out that orbifold models exhibit at most one anomalous U(1) symmetry. This anomaly is cancelled by a shift in the universal axion we found when considering the massless spectrum of the orbifold.

From the field theory perspective one can see that there are also anomalies related to the discrete symmetries [57, 58]. Those discrete symmetries we have considered along this chapter are well motivated from the CFT point of view, and hence we expect that if they are anomalous, there is a shift of the universal axion which fixes the problem. In particular, since we have a single axion, its shift must be able to cancel all anomalies of a given discrete symmetry. This condition is known as anomaly universality [23].

Consider the simplest case of a $\mathbb{Z}_M$ symmetry (like those one gets from the space group selection rule). In the field theory, the fermionic measure in the path integral transforms under the $\mathbb{Z}_M$ symmetry according to

$$\begin{aligned}\mathcal{D}\psi\mathcal{D}\overline{\psi} \to \mathcal{D}\psi\mathcal{D}\overline{\psi}\exp\Bigg[-2\pi\mathrm{i}\,\frac{1}{M}\Bigg(\sum_a A_{G_a^2-\mathbb{Z}_M}\cdot\frac{1}{16\pi^2}\int \mathrm{tr}\{F_a\wedge F_a\} \\ +A_{\text{grav.}^2-\mathbb{Z}_M}\cdot\frac{1}{284\pi^2}\int \mathrm{tr}\{R\wedge R\}\Bigg)\Bigg]\,, \qquad (3.50)\end{aligned}$$

as can be seen from applying Fujikawa's method [59], where a runs over all gauge factors. The Pontryagin indices

$$\frac{\ell(\mathbf{N}_a)}{16\pi^2}\int \mathrm{tr}\{\mathcal{F}_a\wedge\mathcal{F}_a\} \quad \text{and} \quad \frac{1}{2}\,\frac{1}{284\pi^2}\int \mathrm{tr}\{\mathcal{R}\wedge\mathcal{R}\} \qquad (3.51)$$

are integer valued [60, 61], with $\ell(\mathbf{N}_a)$ being the Dynkin index of the fundamental representation. In general anomalies will be universal up to Kǎc-moody labels, which for non-Abelian gauge bosons and gravitons are equal to 1. Since we are interested in the universality of the discrete anomalies, we do not consider those involving U(1) gauge symmetries in the sequel. The anomaly coefficients of our interest are

$$A_{G_a^2-\mathbb{Z}_M} = \sum_\alpha q_\alpha \ell(\mathbf{R}_a^\alpha)\,, \quad A_{\text{grav.}^2-\mathbb{Z}_M} = \sum_\alpha q_\alpha \text{dim}\{\mathbf{R}^\alpha\}\,, \tag{3.52}$$

in which we sum over all chiral representations $\mathbf{R}^\alpha$, with q_α being their corresponding $\mathbb{Z}_M$ charges. It can be shown that the path integral obeys a similar transformation for the case of R- and non-Abelian discrete symmetries, with certain redefinitions in the anomaly coefficients. Let us consider non-Abelian symmetries at first. Given a non-Abelian discrete group G, and an element $\rho \in G$, after transforming the path integral with ρ one gets the following coefficients

$$A_{G_a^2-\rho} = \sum_\alpha \delta_\rho(\mathbf{D}_\alpha)\ell(\mathbf{R}_a^\alpha)\,, \tag{3.53}$$

$$A_{\text{grav.}^2-\rho} = \sum_\alpha \delta_\rho(\mathbf{D}_\alpha)\text{dim}\{\mathbf{R}^\alpha\}\,, \tag{3.54}$$

in which $\mathbf{D}_\alpha$ is the representation of G under which $\mathbf{R}^\alpha$ transforms. In this case the coefficient M appearing in (3.50) is the smallest integer such that $\rho^M = \mathbb{1}$. The "charges" $\delta_\rho(\mathbf{D}_\alpha)$ are computed as [62]

$$\delta_\rho(\mathbf{D}_\alpha) = \frac{M}{2\pi \mathrm{i}} \ln\det(\rho(\mathbf{D}_\alpha))\,. \tag{3.55}$$

Similarly, in the case of a $\mathbb{Z}_M^R$ symmetry, the anomaly coefficients read[8] [23, 57, 58]

$$A_{G_a^2-\mathbb{Z}_M^R} = C_2(G_a)\frac{R}{2} + \sum_\alpha \left(r_\alpha - \frac{R}{2}\right)\ell(\mathbf{R}_a^\alpha)\,, \tag{3.56}$$

$$\begin{aligned} A_{\text{grav.}^2-\mathbb{Z}_M^R} = {} & \left(-21 - 1 - N_T - N_U + \sum_a \text{dim}\{\text{adj}(G_a)\}\right)\frac{R}{2} \\ & + \sum_\alpha \left(r_\alpha - \frac{R}{2}\right)\cdot \text{dim}\{\mathbf{R}^\alpha\}\,, \end{aligned} \tag{3.57}$$

[8] Recall that gauginos and matter fermions both contribute to the anomaly. The charge of the fermions can be inferred from the piece $\theta\psi \subset \Phi$: if the charge of the multiplet Φ is denoted by r, then the charge of the fermion is $r - R/2$. Analogously, the gauginos appear in the vector multiplet in the form $\overline{\theta\theta}\theta\lambda$, so that their charge is $R/2$.

with $C_2(G_a)$ being the quadratic Casimir of G_a, i.e. the Dynkin index of the adjoint representation. In eq. (3.57), the contributions of -21 and -1 correspond to the gravitino and dilatino respectively, N_T and N_U are the number of T- and U-modulini and a runs over all gauge factors (including U(1) symmetries).

The discrete non-R symmetries have been previously considered in ref. [23]. There it has been shown that all of them satisfy universality relations. For the R-symmetries, the story is a bit more subtle because the gamma phase contribution (3.35) was found only recently. If our definition of the R-charges is to lead universal anomalies, the following conditions must hold

$$A_{G_a^2-\mathbb{Z}_M^R} \bmod M\ell(\mathbf{N}_a) = A_{G_b^2-\mathbb{Z}_M^R} \bmod M\ell(\mathbf{N}_b)\,, \tag{3.58}$$

$$A_{G_a^2-\mathbb{Z}_M^R} \bmod M\ell(\mathbf{N}_a) = \frac{1}{24}\left(A_{\text{grav.}^2-\mathbb{Z}_M^R} \bmod \frac{M}{2}\right)\,, \tag{3.59}$$

for any two gauge factors $G_{a,b}$.

To check the universality of the anomalies we focus on the orbifolds presented in table 3.1 with their corresponding isometries. We used the C++ orbifolder [47] to compute the spectrum and the corresponding anomalies for all of the embeddings classified in [63, 64] *in the absence of Wilson lines*, employing the R-charge assignment given in eq. (3.37). In all models the R-anomalies satisfy universality conditions. Furthermore we considered models *with Wilson lines*. For each of the allowed shift embeddings we randomly generated 10 000 Wilson line configurations and found that in these cases the R-charges computed from eq. (3.37) show universality relations for all orbifolds studied. This is an overwhelming result and a strong hint that the R-charges derived here are correct.

3.3.1 An explicit example

To illustrate our results, we consider an example based on the $\mathbb{Z}_4$ orbifold on the lattice of SO(4)^2 $\times$ SU(2)^2, with the twist as given in table 3.1. One easily sees that a basis of generators for the group D is given by

$$\begin{aligned}\varrho_1 = \mathrm{diag}(e^{2\pi \mathrm{i}\frac{1}{4}}, e^{2\pi \mathrm{i}\frac{1}{4}}, 1)\,, \quad \varrho_2 = (\theta_1)^2 = \mathrm{diag}(e^{2\pi \mathrm{i}\frac{1}{2}}, 1, 1)\,,\\ \varrho_3 = \mathrm{diag}(1, 1, e^{-2\pi \mathrm{i}\frac{1}{4}})\,.\end{aligned} \tag{3.60}$$

In order to compute the charges we need to compute the space group elements h_g for all the conjugacy classes. The computation of these is discussed in appendix A. For the $\mathbb{Z}_4$ orbifold example, we take the following shift embedding, which is in agreement with eqs. (2.50)-(2.54)

$$\begin{aligned} V &= \left(-1,-\tfrac{3}{4},0,0,0,0,0,\tfrac{1}{4},0,0,0,0,0,0,0,\tfrac{1}{2}\right)\,, \\ W_1 = W_2 &= \left(\tfrac{7}{4},\tfrac{1}{4},-\tfrac{3}{4},-\tfrac{1}{4},\tfrac{1}{4},\tfrac{1}{4},\tfrac{5}{4},\tfrac{1}{4},-\tfrac{7}{4},-\tfrac{1}{4},-\tfrac{1}{4},-\tfrac{1}{4},-\tfrac{1}{4},-\tfrac{1}{4},\tfrac{1}{4},\tfrac{7}{4}\right)\,, \\ W_3 = W_4 &= \left(-\tfrac{1}{2},-\tfrac{3}{2},-\tfrac{3}{2},1,-\tfrac{3}{2},\tfrac{3}{2},\tfrac{3}{2},1,-\tfrac{1}{4},-\tfrac{7}{4},-\tfrac{5}{4},-\tfrac{1}{4},\tfrac{1}{4},\tfrac{1}{4},\tfrac{3}{4},-\tfrac{7}{4}\right)\,, \\ W_5 &= \left(0,-\tfrac{1}{2},\tfrac{3}{2},-\tfrac{3}{2},\tfrac{3}{2},\tfrac{3}{2},1,\tfrac{3}{2},1,-2,0,1,\tfrac{1}{2},2,1,-\tfrac{3}{2}\right)\,, \\ W_6 &= 0\,. \end{aligned} \tag{3.61}$$

Recall the identifications for the Wilson lines: $W_1 \sim W_2$, $W_3 \sim W_4$ (see fig. 3.2). This

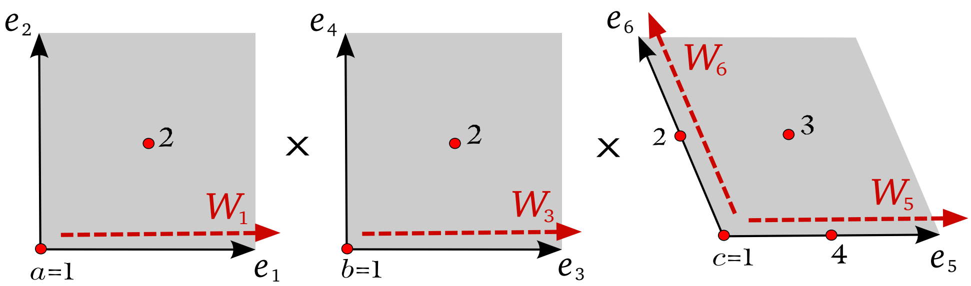


Figure 3.2: Wilson line configuration for the $\mathbb{Z}_4$ orbifold studied in the text.

embedding leaves the following gauge symmetry unbroken

$$\mathrm{SU}(4)_1 \times \mathrm{SU}(2)_1 \times \mathrm{SU}(2)_2 \times \mathrm{SU}(4)_2 \times \mathrm{SU}(2)_3 \times \mathrm{U}(1)^7 \subset \mathrm{E}_8 \times \mathrm{E}_8\,, \tag{3.62}$$

where the subindices have been introduced to distinguish between identical gauge factors. The anomaly coefficients obtained for this specific orbifold model are

$$\begin{aligned} A_{\text{grav.}^2-\varrho_1} &= -76\,, & A_{\text{grav.}^2-\varrho_2} &= 94\,, & A_{\text{grav.}^2-\varrho_3} &= 84\,, \\ A_{SU(4)_1^2-\varrho_1} &= -3\,, & A_{SU(4)_1^2-\varrho_2} &= 3\,, & A_{SU(4)_1^2-\varrho_3} &= -1\,, \\ A_{SU(2)_1^2-\varrho_1} &= -5\,, & A_{SU(2)_1^2-\varrho_2} &= 1\,, & A_{SU(2)_1^2-\varrho_3} &= 5\,, \\ A_{SU(2)_2^2-\varrho_1} &= -11\,, & A_{SU(2)_2^2-\varrho_2} &= 6\,, & A_{SU(2)_2^2-\varrho_3} &= 5\,, \\ A_{SU(4)_2^2-\varrho_1} &= -3\,, & A_{SU(4)_2^2-\varrho_2} &= -1\,, & A_{SU(4)_2^2-\varrho_3} &= -1\,, \\ A_{SU(2)_3^2-\varrho_1} &= -11\,, & A_{SU(2)_3^2-\varrho_2} &= 3\,, & A_{SU(2)_3^2-\varrho_3} &= 5\,. \end{aligned} \tag{3.63}$$

One can straightforwardly check that all of these values satisfy the universality conditions (3.58) and (3.59). This model also serves to discuss the effects of the new R-symmetries emerging from F. Note that

$$\zeta = \mathrm{diag}(e^{2\pi \mathrm{i}\frac{1}{4}}, 1, 1) \in F\,, \tag{3.64}$$

interchanges the fixed points

$$Z_g = \frac{e_2 + e_3}{2} \overset{\zeta}{\leftrightarrow} Z_{g'} = \frac{e_2 + e_4}{2}, \tag{3.65}$$

which are generated by space group elements $g = (\theta^2, e_2 + e_3)$ and $g' = (\theta^2, e_2 + e_4)$ belonging to different conjugacy classes. This is illustrated in figure 3.3. Note that in our example we have

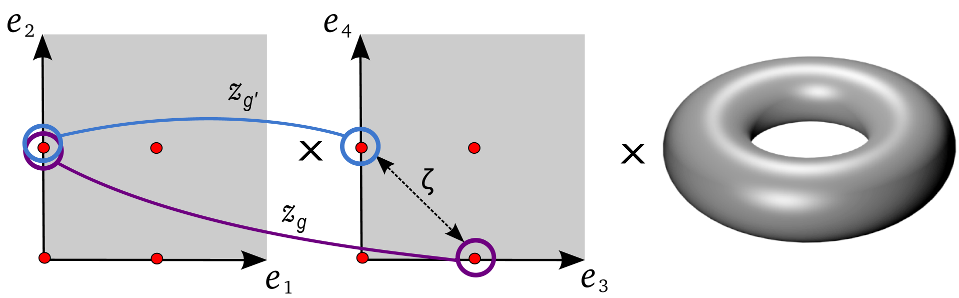


Figure 3.3: Representation of the θ_1 action on the T_2 sector fixed points of the $\mathbb{Z}_4$ orbifold studied.

chosen $W_1 = W_2$, $W_3 = W_4$, so that the transformation ζ respects the Wilson line structure. As an example consider the states specified by the following quantum numbers

$$p_{sh} = \left(-\tfrac{3}{4}, \tfrac{1}{4}, -\tfrac{1}{4}, -\tfrac{1}{4}, -\tfrac{1}{4}, -\tfrac{1}{4}, -\tfrac{1}{4}, -\tfrac{1}{4}, 0, 0, -\tfrac{1}{2}, \tfrac{1}{2}, 0, 0, 0, 0\right), \tag{3.66}$$
$$q_{sh} = \left(0, -\tfrac{1}{2}, -\tfrac{1}{2}, 0\right), \tag{3.67}$$

with no left-moving oscillators. The p_{sh} presented is the highest weight of the representation $(\mathbf{1}, \mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{2})$ with all U(1) charges equal to zero. Two identical copies of this state live at the fixed points under consideration. The elements h_g and $h_{g'}$ needed to compute the R-charges are given by

$$h_g = (\theta, e_3) \quad \text{and} \quad h_{g'} = (\theta, 0), \tag{3.68}$$

and the corresponding gamma-phases are $\gamma(g) = \gamma(g') = 3/4$. With this information we can compute the R-charges for the eigenstates of ζ to be

$$r^{(s)} = -7/2 - 4s. \tag{3.69}$$

We also computed the anomaly coefficients for the R-symmetry ζ, with a scan of over 100.000 randomly generated models. In all cases the anomalies turned out to be universal. Note that, in our example, $\zeta^2 = \varrho_2 \in D$. This implies that the R-charges under ϱ_2 are twice those under ζ up to multiples of 2. This implies that one can safely take ζ and ϱ_1 as a basis for all R-symmetries in the $\mathbb{Z}_4$ orbifold.

CHAPTER 4

An extension of the heterotic mini-landscape

The cradle rocks above an abyss, and common sense tells us that our existence is but a brief crack of light between two eternities of darkness.

Vladimir Nabokov, Speak, Memory: A Memoir.

Having discussed some of the formal tools, we are now in shape to discuss the phenomenology of orbifold models. We are interested in orbifold compactifications which could help us make contact with particle physics, i.e. we aim to achieve the gauge group of the standard model G_{SM} (times additional factors), three families of quarks and leptons, a Higgs sector, etc. We also expect the interactions among these fields to be in a good shape. Here we present the heterotic mini-landscape: A class of models based on the $\mathbb{Z}_{6-\mathrm{II}}$ obifold. The low energy version of these models resemble the properties of the MSSM to a surprising degree. Motivated by these observations, we construct a new class of orbifold models based on the $\mathbb{Z}_2 \times \mathbb{Z}_4$ geometry. We construct the allowed gauge embeddings and discuss the discrete symmetries in this specific case. We also take first steps towards model building based on SO(10) (or E_6) local grand unification. There we show that the features of our models are very similar to those from the mini-landscape.

4.1 The particle Zip-code in mini-landscape models

The $\mathbb{Z}_{6-\mathrm{II}}$ orbifold of our interest is based on the lattice $G_2 \times \mathrm{SU}(3) \times \mathrm{SU}(2)^2$. This lattice exhibits the point group group symmetry associated to the following twist vector

$$v = \frac{1}{6}(0, 1, 2, -3)\,. \tag{4.1}$$

In the four dimensional theory we are aiming to achieve the gauge group of the SM (times some extra factors). From the orbifold point of view, such a gauge group originates as the intersection of all gauge factors which are characteristic to the various orbifold locations, in particular, there are some fixed points/tori where the gauge symmetry is enhanced to e.g. $SO(10)$ or even E_6. Since localized fields transform as complete representations of the local gauge group, it then follows that orbifold compactifications allow for the coexistence of complete and split multiplets of an underlying GUT.

From the previous arguments it follows that in a local SO(10) (E_6) GUT it is possible to have the three generations of the standard model arising from complete $\mathbf{16}$-plets (or $\mathbf{27}$-plets) at points where the gauge symmetry enhances to SO(10) (or E_6). This observation motivates the following search strategy: One can start with a shift embedding which leaves and SO(10) or an E_6 factor unbroken in the bulk. Then one considers Wilson line configurations which break the GUT factor down to G_{SM}. One will generically find points on the orbifold where the original SO(10) (E_6) remains unbroken. If these points contain a $\mathbf{16}$-plet, the complete multiplet will survive. Then some matter generations in the standard model might very well be complete multiplets of SO(10) while others might arise as a patchwork of split multiples emerging from different orbifold locations.

The starting point of the mini-landscape were the following SO(10) shifts [65]

$$V_1^{\mathrm{SO(10)}} = \left(\frac{1}{3}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0, 0\right) \left(\frac{1}{3}, 0, 0, 0, 0, 0, 0, 0\right) , \tag{4.2}$$

$$V_2^{\mathrm{SO(10)}} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, 0, 0, 0\right) \left(\frac{1}{6}, \frac{1}{6}, 0, 0, 0, 0, 0, 0\right) , \tag{4.3}$$

together with the following

$$V_1^{\mathrm{E}_6} = \left(\frac{1}{2}, \frac{1}{3}, \frac{1}{6}, 0, 0, 0, 0, 0\right) (0, 0, 0, 0, 0, 0, 0, 0) , \tag{4.4}$$

$$V_2^{\mathrm{E}_6} = \left(\frac{2}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, 0, 0, 0\right) \left(\frac{1}{6}, \frac{1}{6}, , 0, 0, 0, 0, 0, 0\right) , \tag{4.5}$$

leading to an E_6 factor. Now we can consider the Wilson lines. The orbifold under consideration allows for three Wilson lines: One of order three which we denote as W_3, related to the vectors in the second complex plane, and two of order two W_4 and W_5 in the third. It is observed that if only W_4 and W_5 are on, there will be three fixed points in the T_1 (T_5) sector, where the GUT factor remains unbroken. If these fixed points nest $\mathbf{16}$-plets, then we have a complete family structure. Moreover, this approach turns to be unsuccessful because two Wilson lines of order two do not suffice to break the SO(10) factor down to the standard model gauge group [20].

Another alternative is to take one Wilson line of order three and one of order two, so that instead of three, one has two fixed points in the first (fifth) twisted sector where the states furnish complete representations of the unified group. The third family gets completed by pieces coming from the untwisted and twisted sectors. This strategy was more successful and out of all possible models about one percent of them were found to have the SM spectrum plus vector like exotics, in addition to a non-anomalous $U(1)_Y$ with the standard SU(5) normalization.

Among the plausible states which will serve to complete the third family, we can distinguish between three cases

- fields which are free to propagate in the 10-dimensional bulk
- fields sitting at fixed points in the extra dimensions (representing "3-branes")
- fields which can only propagate along fixed tori in compact space (representing "5-branes")

This distinction is relevant for our discussion due to the observation that the geography of the fields in the extra dimensions plays a crucial role in the low energy effective field theory. This is not entirely surprising as we have seen already that the charges of the fields under the discrete symmetries are related to their location in internal space. A similar situation occurs with the supersymmetries: fields in the bulk feel remnants of $\mathcal{N} = 4$ SUSY, in contrast to those sitting at fixed tori and fixed points which experience remnants of $\mathcal{N} = 2$ and $\mathcal{N} = 1$, respectively. Arguments of this type motivate the idea that from imposing phenomenological constraints, a particular distribution of the SM fields in the extra dimensions might emerge, this is what we call the Zip-code of the mini-landscape. Bearing this in mind, we dedicate the remainder of this section to analyze the preferred locations for the SM fields observed in the mini-landscape as well as the physical implications of this peculiar distribution.

4.1.1 The Higgs system

The minimal supersymmetric version of the SM contains one pair of Higgs superfields H_u and H_d. With regards to this vector-like pair, two important questions which raise from the phenomenological side need to be addressed: The first one is the doublet triplet splitting problem. As stressed before, local GUTs allow for certain fields to come in incomplete representations of the gauge symmetry, provided these fields originate from a certain region which does not experience the gauge enhancement. We expect the Higgses to live in those regions, so that the dangerous triplets which could accompany them, are definitely projected out of the spectrum.

The second situation which needs to be addressed here is the so called μ-problem. Generally in string constructions one gets more than one pair of Higgses, so that one has to face the question of why all except one of them become heavy. For a large class of mini–landscape models we find that this problem is solved in a miraculous way [65, 66]: In these models one pair of Higgs doublets lives at the untwisted sector. Such Higgses happen to be vector-like under all possible symmetries, and this is of great interest since any R-symmetry in the model will thus forbid a coupling of the form $\mu H_u H_d$ to be present in the superpotential $(\mathcal{W})$ [65–67]. The μ term can be generated after SUSY breakdown. In particular, gravity mediation mechanisms relate the μ term directly to the gravitino mass $m_{3/2}$, as observed earlier in the context of field theoretic models [68]. R-symmetries are crucial for this mechanism to work, and as observed in section 3.2, they arise naturally in orbifold models.

The first lesson from the mini–landscape is then that *the Higgs pair H_u and H_d should live in the bulk (untwisted sector).* They correspond to gauge fields in extra dimensions, compatible with the picture of gauge-Higgs unification and represent continuous Wilson lines as discussed in ref. [69–71].

4.1.2 The top-quark

The mass of the top quark is of the order of the electroweak scale, one thus expects the gauge and the top-Yukawa couplings to be of the same order, exhibiting gauge-top unification [72]. From the stringy point of view, these couplings are given directly by the string coupling and we expect the top-quark Yukawa coupling at the trilinear level in the superpotential. Given the fact that H_u is a field in the untwisted sector there remain only few allowed alternatives for the location of the top quark.

In the mini–landscape we find that, in most of the models, the left- and right-handed top-multiplets belong to the untwisted sector (bulk) and this is a guarantee for a sufficiently large Yukawa coupling. The location of the other members of the third family is rather model dependent, they are distributed over various sectors. Very often the top-quark Yukawa coupling is the only trilinear Yukawa coupling one finds in the model.

This is the second lesson from the mini–landscape: *left- and right-handed top-quark multiplets* *should be bulk fields*.

4.1.3 The first two families of quarks and leptons

The Mini-Landscape models provide a grand unified picture with families in the 16-dimensional spinor representation of SO(10) (in the case of E_6 enhancements, the families are expected to descend from the $\mathbf{27}$-plet). The analysis shows that three family models can only be achieved with at most two families as complete SO(10) representations [21]. Since the top-family will originate as a patchwork, the localized $\mathbf{16}$-plets correspond to the two light families, these live at fixed points of the orbifold twist. Due to their location, no trilinear coupling is allowed for these fields and for this reason quark-and lepton masses are suppressed. It is also observed that these families transform as a doublet under a D_4 family symmetry [73, 74] inherited from the geometry. This symmetry can be used to avoid the problem of flavor changing neutral currents. Thus we have a third lesson: *the first two families are located at fixed points where the gauge symmetry is enhanced and enjoy of a non trivial transformation under a family symmetry*.

4.1.4 The pattern of supersymmetry breakdown

The field configuration discussed in sects. 4.1.1 to 4.1.3 has many implications for the low energy, some of which have been already stressed. So far we have not discussed the breakdown of supersymmetry which is also sensitive to the SM Zip-code. This discussion is more involved since it has to address the question of moduli stabilization. Supersymmetry can be broken by gaugino condensation in the hidden sector G_{Hidden} [75, 76]. The hidden groups one obtains in Mini-Landscape models are consistent with a gravitino mass in the (multi) TeV-range [77] (provided the dilaton is fixed at a realistic grand unified gauge coupling). If moduli stabilization proceeds along the lines of [53, 54], we would then keep a run-away dilaton and a positive vacuum energy. It can be shown that by adjusting the vacuum energy with a matter superpotential (downlifting the vacuum energy) one can fix the dilaton as well. The resulting picture [78] is reminiscent of a scheme known as mirage mediation [79–81], at least for gaugino masses and A-parameters: As bulk fields, gauginos experience remnants of $\mathcal{N} = 4$ SUSY and due to that, their masses are suppressed by a factor of $\log(M_{\text{Planck}}/m_{3/2})$ [82]. Scalar masses are more model dependent and could be as large as the gravitino mass $m_{3/2}$ [83]. However, since Higgses and top quark are also bulk fields, the same suppression factor is expected for the mass of Higgsinos and the stop. Thus, *one expects $m_{3/2}$ and scalar masses of the first two families in the multi-TeV range while stops and Higgsinos are in the TeV range.* This is a result of the location of fields in the extra dimensions and provides the fourth lesson of the Mini–Landscape [84].

4.2 The $\mathbb{Z}_2 \times \mathbb{Z}_4$ orbifold

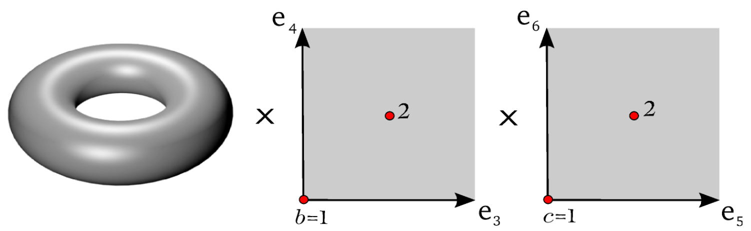
The $\mathbb{Z}_2 \times \mathbb{Z}_4$ orbifold of our interest is based on the factorizable lattice spanned by the roots of SU(2)^2 $\times$ SO(4) $\times$ SO(4). Out of the resulting torus, we mod out the point group generated by the following twist vectors

$$v_2 = (0, \tfrac{1}{2}, -\tfrac{1}{2}, 0)\,, \quad v_4 = (0, 0, \tfrac{1}{4}, -\tfrac{1}{4})\,, \tag{4.6}$$

which clearly correspond to isometries of the lattice and comply with the conditions for $\mathcal{N} = 1$ SUSY in 4D. The v_2 twist indicates that the $\mathbb{Z}_2$ generator acts as a simultaneous reflection on the first two planes, while leaving the third invariant. Similarly, the twist v_4 implies that the $\mathbb{Z}_4$ generator rotates the second and third planes by $\pi/2$ counter and clockwise, respectively.

Here we start our discussion of the geometry of the orbifold, but many stringy properties of the fixed point/tori are left to be discussed in the upcoming sections, where the appropriate machinery is constructed. Lets denote by θ and ω the generators of the $\mathbb{Z}_2$ and $\mathbb{Z}_4$ factors. The fixed points/tori of $\theta^{k_1}\omega^{k_2}$ are said to belong to the twisted sector $T_{(k_1,k_2)}$. Each particular sector is studied separately in tables 4.1 to 4.5. For the sectors $T_{(0,1)}$, $T_{(0,2)}$ and $T_{(0,3)}$ one finds fixed tori due to the trivial action of ω on the first complex plane. A similar situation will happen for the case of $T_{(1,0)}$ and $T_{(1,2)}$ where, respectively the third and second planes are invariant. The only sectors where one finds fixed points are $T_{(1,1)}$ and $T_{(1,3)}$. For each twisted sector, the fixed points/tori on the fundamental domain are a maximal set of representatives for the conjugacy classes of the space group (which act non freely on $\mathbb{C}^3$). For model building purposes it is crucial to find the generating elements of each inequivalent fixed point, in order to determine the right gauge shift and its corresponding set of commuting elements. The fixed points have been labeled with indices a, b and c denoting the location on each complex plane. The twisted sectors $T_{(0,1)}$ and $T_{(0,3)}$ as well as $T_{(1,1)}$ and $T_{(1,3)}$ are the inverse of each other and have the same fixed points/tori but the generating elements differ depending on which sector one is considering (see tables 4.1 and 4.4). The remaining sectors are self inverse and are generated by point group elements of order two, so that in general, one has further identifications induced by the $\mathbb{Z}_4$ generator. Similar as in the $\mathbb{Z}_{6-\mathrm{II}}$ case, those fixed tori which do not have a unique representative within the fundamental domain we call *special*. One can show that there is no element of the form $(\theta, \lambda) \in S$ which commutes with the generating elements of those special fixed tori. This feature will again lead to an enhancement of the local gauge symmetry, but in contrast to the $\mathbb{Z}_{6-\mathrm{II}}$ case, the presence of the additional $\mathbb{Z}_2$ twist forbids the local enhancement of the supersymmetries.

Note that in the sectors $T_{(1,0)}$ and $T_{(1,2)}$ the identifications under the $\mathbb{Z}_4$ generator will lead to only three inequivalent fixed points in the $SO(4)$ plane. This is the same amount of fixed points one finds for the $SU(3)$ plane of the $\mathbb{Z}_{6-\text{II}}$ orbifold.



$T_{(0,1)}$

b \ c	1	2
1	0	e_6
2	e_3	$e_3 + e_6$

$T_{(0,3)}$

b \ c	1	2
1	0	e_5
2	e_4	$e_4 + e_5$

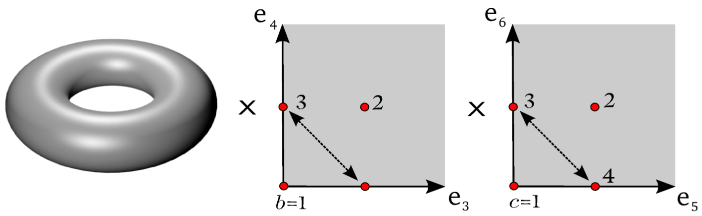
Figure 4.1: Fixed tori of the $T_{(0,1)}$ and $T_{(0,3)}$ sectors. The tables above help to deduce the generating element of each fixed torus. Consider a fixed torus located at the position b and c in the last two planes, such fixed torus is generated by a space group element (ω, λ_{bc}) if the fixed torus belongs to the $T_{(1,0)}$ sector, or $(\omega^3, \lambda'_{bc})$ for $T_{(0,3)}$. The lattice vectors λ_{bc} and λ'_{bc} can be found in the bc-th entry of the left and right tables below the picture.

4.2.1 Gauge embeddings

As a first step towards model building in $\mathbb{Z}_2 \times \mathbb{Z}_4$ we are interested in the gauge groups that are allowed by this geometry. Our intention is to classify all embeddings V_2, V_4, W_α consistent with the modular invariance conditions (2.50)-(2.52). There are also constraints due to the consistency of the embedding with the space group. These fix the order of the shifts V_2, V_4 to be 2 and 4, i.e. the vectors $2V_2$ and $4V_4$ must belong to the $E_8 \times E_8$ lattice. For the Wilson lines W_α one has the following situation:

(i) In the first complex plane one has two Wilson lines W_1 and W_2, because the point group action does not relate the lattice vectors e_1 and e_2. These lines are of order two, since

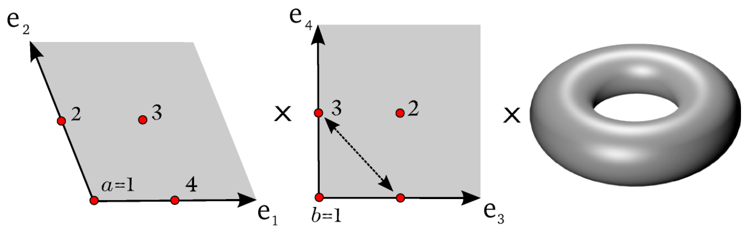
$$\theta e_\alpha + e_\alpha = 0\,, \quad \alpha = 1, 2\,.$$



$T_{(0,2)}$

b \ c	1	2	3	4
1	0	e_5+e_6	e_6	
2	e_3+e_4	$e_3+e_4+e_5+e_6$	$e_3+e_4+e_5+e_6$	
3	e_4	$e_4+e_5+e_6$	e_4+e_6	e_4+e_5

Figure 4.2: Fixed tori of $T_{(0,2)}$. The arrows in the last two planes have to be understood as identifications acting simultaneously such that they reproduce the effects of the $\mathbb{Z}_4$ generator of the point group. Similarly as in table 4.1, the generating elements can be found below the picture. Those entries which are left blank do not correspond to additional inequivalences. Shaded cells have been put to denote special fixed tori.



$T_{(1,0)}$

a \ b	1	2	3
1	0	e_3+e_4	e_4
2	e_2	$e_2+e_3+e_4$	e_2+e_4
3	e_1+e_2	$e_1+e_2+e_3+e_4$	$e_1+e_2+e_4$
4	e_1	$e_1+e_3+e_4$	e_1+e_4

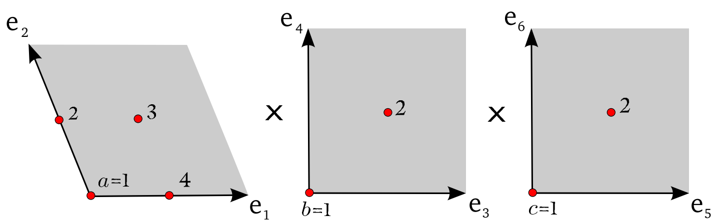
Figure 4.3: Fixed tori of the $T_{(1,0)}$ sector. This sector is associated to the $\mathbb{Z}_2$ generator of the point group. For this reason, the fixed tori are identified only up to rotations by $\pi/2$ on the second plane. The special fixed tori are associated to the shaded cells in the table.

(ii) In the second and third planes we have the identifications:

$$\omega e_3 = e_4\,, \quad \omega^3 e_5 = e_6\,, \tag{4.7}$$

so that we only have one inequivalent Wilson line per complex plane[1]. These ones we will

[1] This equivalence is up to lattice vectors from the gauge lattice, which we omit for simplicity.



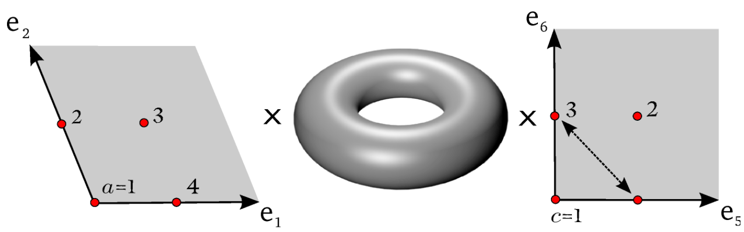
$T_{(1,1)}$

a \ bc	11	12	21	22
1	0	e_6	e_4	$e_4 + e_6$
2	e_2	$e_2 + e_6$	$e_2 + e_4$	$e_2 + e_4 + e_6$
3	$e_1 + e_2$	$e_1 + e_2 + e_6$	$e_1 + e_2 + e_4$	$e_1 + e_2 + e_4 + e_6$
4	e_1	$e_1 + e_6$	$e_1 + e_4$	$e_1 + e_4 + e_6$

$T_{(1,3)}$

a \ bc	11	12	21	22
1	0	e_5	e_3	$e_3 + e_5$
2	e_2	$e_2 + e_5$	$e_2 + e_3$	$e_2 + e_3 + e_5$
3	$e_1 + e_2$	$e_1 + e_2 + e_5$	$e_1 + e_2 + e_3$	$e_1 + e_2 + e_3 + e_5$
4	e_1	$e_1 + e_5$	$e_1 + e_3$	$e_1 + e_3 + e_5$

Figure 4.4: Fixed points of the sectors $T_{(1,1)}$ and $T_{(1,3)}$.



$T_{(1,2)}$

a \ c	1	2	3
1	0	$e_5 + e_6$	e_6
2	e_2	$e_2 + e_5 + e_6$	$e_2 + e_6$
3	$e_1 + e_2$	$e_1 + e_2 + e_5 + e_6$	$e_1 + e_2 + e_6$
4	e_1	$e_1 + e_5 + e_6$	$e_1 + e_6$

Figure 4.5: Fixed tori of $T_{(1,2)}$. The picture is very similar to that one observes in $T_{(1,0)}$

denote by W_3 and W_4. Due to the conditions

$$\omega^2 e_3 + e_3 = 0\,, \qquad \text{and} \qquad \omega^2 e_5 + e_5 = 0\,, \tag{4.8}$$

these Wilson lines are of order two as well.

In contrast to the Wilson line backgrounds, the shift vectors V_2 and V_4 must be non-trivial. Since we are interested in obtaining all inequivalent models, the simplest alternative is to assume all WLs are off and search for consistent pairs (V_2, V_4). Crucial for our exploration is that if two embeddings differ by lattice automorphisms or special combinations of lattice vectors, the models they lead to are identical. This implies that we can zoom in a particular region of the $\mathrm{E}_8 \times \mathrm{E}_8$ lattice, and from this region we can find all possible models. Since the lattice is the direct sum of two E_8 root lattices, we can consider two eight dimensional vectors and study them independently. For V_4 we took the ten inequivalent shifts found for the $\mathbb{Z}_4$ orbifold [63], with these vectors at hand we explored the corresponding $\mathbb{Z}_2$ shifts consistent with each of the V_4's.

It can be shown that the gauge group of a certain embedding does not change if one adds lattice vectors to the shifts. As for now we are only interested in the diversity of gauge groups allowed by the $\mathbb{Z}_2 \times \mathbb{Z}_4$ geometry, we take any given pair of vectors as equivalent if they differ by a lattice translation. As the V_2 shifts must be half of a lattice vector, it suffices to consider all half lattice vectors of E_8 whose norm is smaller than two. After this we are left with 75 combinations for a single E_8. These combinations are then paired together to form sixteen dimensional vectors. Only those pairings which agree with modular invariance are useful for model building. Allowed embeddings must satisfy

$$2\left(V_2^2 - \frac{1}{2}\right) = 0 \bmod 2\,, \tag{4.9}$$

$$4\left(V_4^2 - \frac{1}{8}\right) = 0 \bmod 2\,, \tag{4.10}$$

in addition to a mixed constraint[2]

$$4\left(V_2 \cdot V_4 + \frac{1}{8}\right) = 0 \bmod 1\,. \tag{4.11}$$

We found 144 combinations which survive these conditions and the gauge groups for these models were computed. Among the relevant features of the gauge structures we found, it is worth to highlight the large number of embeddings in contrast to $\mathbb{Z}_{6-\mathrm{II}}$ where only 61 are possible. In our model, 35 embeddings contain an SO(10) factor, we also have 26 E_6 models and 25 containing a

[2] Our construction misses the effects of the lattice vectors, hence the modular invariace condition which mixes between V_2 and V_4 [43] gets milder. Embeddings which can be made modular invariant necessarily have to satisfy eq. 4.11

SU(5) factor. Recall that in the $\mathbb{Z}_{6-\text{II}}$ orbifold one has 14 SO(10) models, 16 with E_6 and 4 with SU(5).

4.2.2 Discrete symmetries

In this section we discuss the discrete symmetries of the $\mathbb{Z}_2 \times \mathbb{Z}_4$ orbifold. If we consider a generic L-point coupling involving fields Φ_α from the sectors $T_{(m_\alpha, n_\alpha)}$, the point group selection rule implies

$$\sum_{\alpha=1}^{L} m_\alpha = 0 \bmod 2\,, \tag{4.12}$$

$$\sum_{\alpha=1}^{L} n_\alpha = 0 \bmod 4\,, \tag{4.13}$$

such that the superpotential possesses a $\mathbb{Z}_2$ and a $\mathbb{Z}_4$ discrete symmetry, where the charges of each state permit to determine which twisted sector it belongs to. One finds that the lattice part of equation (3.8), introduces additional discrete symmetries. In order to infer the discrete symmetries arising from the fixed point selectivity (see eq. (3.11)), we consider the problem planewise: The first plane contributes two $\mathbb{Z}_2$ symmetries because the vectors e_1 and e_2 are not further identified under the point group. For the remaining planes one finds that each contributes a $\mathbb{Z}_2$ discrete symmetry where the charges of the fields depend on their corresponding location (see table 4.1).

	$a=1$	$a=2$	$a=3$	$a=4$	$b=1,2$	$b=3$	$c=1,2$	$c=3,4$
$\mathbb{Z}_2^1$	0	1	1	0				
$\mathbb{Z}_2^{1'}$	0	0	1	1				
$\mathbb{Z}_2^2$					0	1		
$\mathbb{Z}_2^3$							0	1

Table 4.1: Symmetries resulting from the lattice parts of the space group selection rule. $\mathbb{Z}_2^1$ and $\mathbb{Z}_2^{1'}$ result from the first complex plane, $\mathbb{Z}_2^2$ and $\mathbb{Z}_2^3$ from the second and third, respectively. The indices a, b, c label the fixed points in each complex plane according to the notation adopted in figures 4.1 to 4.5.

The discrete symmetries we have just presented describe a manner to track the location of a certain field on the orbifold. In the absence of Wilson lines, these discrete symmetries will enhance to a non-Abelian factor. To see which are the corresponding non-Abelian symmetries we start with the buiding blocks discussed in section 3.1.3. Let us consider the first plane, since only a $\mathbb{Z}_2$ identification acts there, it can be regarded as the direct product of two orbicircles $S^1/\mathbb{Z}_2$, as long as one keeps in mind that the $\mathbb{Z}_2$ identification should act simultaneously on both of them. On each orbicircle, one finds that there is a freedom to exchange between the fixed points i.e. a S_2 symmetry, whose multiplicative closure with the $\mathbb{Z}_2 \times \mathbb{Z}_2$ space group selection rule leads to a D_4 symmetry. The resulting symmetry on that plane, ends up being the direct product of two D_4 factors, divided out by the simultaneous $\mathbb{Z}_2$ orbifold action we have mentioned previously. Under this flavor symmetry, untwisted states transform in the trivial representation (A_1, A_1) where A_1 denotes the invariant singlet under D_4, while twisted states sitting at the four fixed points furnish a four dimensional representation (D, D), with D being the D_4 doublet[3].

For the remaining planes consider a $\mathbb{T}^2/\mathbb{Z}_4$, in the first twisted sector one finds two fixed points along the diagonal in the fundamental domain (see e.g. the second plane in figure 4.1), for which one finds again a S_2, which combined to the $\mathbb{Z}_4 \times \mathbb{Z}_2$ space group symmetries, leading to $(D_4 \times \mathbb{Z}_4)/\mathbb{Z}_2$. When it comes to the representations one can easily see that states from the first twisted sector come in doublet representations D and the fixed points of the second twisted sector allow us to realize all one dimensional representations of the D_4. Consider for instance the fixed points in the third plane of figure 4.2, and take $|\varphi_1\rangle$ and $|\varphi_2\rangle$ as identical states sitting at $c = 1$ and 2 (Note that these states are invariant under the $\mathbb{Z}_4$ lattice automorphism). The symmetric and antisymmetric combinations of these states will transform as A_1 and A_2 under D_4. The remaining fixed points $c = 3, 4$ get identified under the $\mathbb{Z}_4$ rotation. One can take then even and odd combinations under such a rotation, and this gives rise to the representations A_3 and A_4 of D_4. The third twisted sector is the inverse of the first one, so that states there transform also as doublets. The charge of a given state under the $\mathbb{Z}_4$ is just the order of the twisted sector it belongs to.

In order to obtain the complete symmetry group for the six dimensional orbifold we have to take the direct product of the previous factors and mod out the point group identifications. Finally one obtains

$$G_{\text{Flavor}} = \frac{\left(\frac{D_4 \times D_4}{\mathbb{Z}_2}\right) \times \left(\frac{D_4 \times \mathbb{Z}_4}{\mathbb{Z}_2}\right) \times \left(\frac{D_4 \times \mathbb{Z}_4}{\mathbb{Z}_2}\right)}{\mathbb{Z}_2 \times \mathbb{Z}_4} = \frac{D_4^4 \times \mathbb{Z}_4}{\mathbb{Z}_2^4}. \tag{4.14}$$

This can be used to see how different twisted sectors transform under this flavor symmetry:

- The bulk states are all flavor singlets.

[3] The notation we use to denote the D_4 representation is the same as in ref. [74].

- For $T_{(0,1)}$ and $T_{(1,3)}$, the four fixed points form states transforming as $(A_1, A_1, D, D)_1$ and $(A_1, A_1, D, D)_3$, respectively. This notation indicates just that the states transform as doublets under the latter two D_4 factors in eq. (4.14), while the subindices are the charges under the $\mathbb{Z}_4$.

- For $T_{(0,2)}$, the states are of the form $(A_1, A_1, A_i, A_j)_2$ with $i, j = 1, 2, 3, 4$. The states with $i, j = 1, 2$ correspond to the ordinary fixed tori. For the special ones one has six alternatives. Depending on how the other pieces in the physical state transform under the $\mathbb{Z}_4$ generator of the point group, one has to choose between the invariant (even) combinations

$$\begin{array}{lll} (A_1, A_1, A_1, A_3)_2, & (A_1, A_1, A_2, A_3)_2, & (A_1, A_1, A_3, A_3)_2, \\ (A_1, A_1, A_3, A_1)_2, & (A_1, A_1, A_3, A_2)_2, & (A_1, A_1, A_4, A_4)_2, \end{array}$$

  or the odd ones

$$\begin{array}{lll} (A_1, A_1, A_1, A_4)_2, & (A_1, A_1, A_2, A_4)_2, & (A_1, A_1, A_3, A_4)_2, \\ (A_1, A_1, A_4, A_1)_2, & (A_1, A_1, A_4, A_2)_2, & (A_1, A_1, A_4, A_3)_2, \end{array}$$

  in order to build a state which is space group invariant[4]

- For $T_{(1,0)}$, the states sitting at ordinary tori are of the form $(D, D, A_i, A_1)_0$ $i = 1, 2$. At the special ones we have either $(D, D, A_3, A_1)_0$ for even or $(D, D, A_4, A_1)_0$ for odd states.

- A similar situation occurs in the $T_{(1,2)}$, where ordinary tori transform in the representation $(D, D, A_1, A_i)_0$ $i = 1, 2$. For the special singularities one has either $(D, D, A_1, A_3)_0$ for even or $(D, D, A_1, A_4)_0$ for odd states.

- States in $T_{(1,1)}$ or $T_{(1,3)}$ transform as $(D, D, D, D)_1$ or $(D, D, D, D)_3$, respectively.

The breakdown of the flavor group induced by the Wilson lines occurs blockwise: note that each of the Wilson lines W_1 or W_2 is associated to each of the orbicircles in the first plane. The effect of such Wilson lines is to break the permutation symmetry. This reduces D_4 to its Abelian part, i.e. the space group selection rule.

For the case of the Wilson lines W_3 and W_4, the flavor group on the $\mathbb{T}^2/\mathbb{Z}_4$ block gets broken according to

$$(D_4 \times \mathbb{Z}_4)/\mathbb{Z}_2 = S_2 \ltimes (\mathbb{Z}_4^p \times \mathbb{Z}_2^l) \to (\mathbb{Z}_4 \times \mathbb{Z}_2)\,. \tag{4.15}$$

[4] Note that it depends on the other quantum numbers $(p_{sh}, q_{sh}, \mathcal{N}_L^i, \bar{\mathcal{N}}_L^i)$ which combination, either the symmetric or the antisymmetric, is taken to construct the physical state

were $\mathbb{Z}_4^p$ and $\mathbb{Z}_2^l$ are discrete factors related to the point group and the lattice, respectively. With regards to the R-symmetries, we see that a rotation of $\pi/2$ in either the second or the third planes will give us an element in F, thus we take the one in the second plane as the generator, in addition to that, the rotations of π on the first or the second planes will be elements in the group D. The R-charges for the fields will be computed accordingly as in the previous chapter.

4.3 Searching for realistic models

To judge the fertility of the shift embeddings discussed in the previous section, we can compute the spectrum at the GUT level and then analyze the possible effects of the Wilson lines. With this information one can implement a strategy to retain three families in the low energy.

At the GUT level, the multiplicities for the twisted states from $T_{(0,1)}$, $T_{(0,3)}$ and $T_{(1,3)}$ are equal to the number of fixed points/tori in each of the sectors. However, the sectors $T_{(1,0)}$, $T_{(0,2)}$ and $T_{(1,2)}$ contain some *special* fixed points which get identified to each other by the $\mathbb{Z}_4$ generator. These points enjoy an enhancement of the gauge symmetry, fields at those locations will furnish complete representations of the enhanced group. In those sectors, the fields sitting at normal fixed points will also be present at the special ones, but at those special locations some additional states might appear.

When it comes to the effects of the Wilson lines, there are some differences in the way they act on matter fields. In the untwisted sector, for instance, the states are sensitive to all the Wilson lines. As expected for fields which propagate in the bulk, no new states are introduced and out of those found at the GUT level only those which survive the WL projections will be part of the physical spectrum.

The Wilson lines act differently on the fixed points and hence split the multiplicities for the states in a given twisted sector. Thus, the Wilson lines do not only serve for the breakdown of the GUT factor to G_{SM}, but must also allow for a distribution in which the SM matter comes in three copies. Depending on the manner the WL configuration acts on the states sitting there, the singularities fit in the following classification:

Config.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
W_1		✓				✓	✓	✓				✓	✓	✓		✓
W_2			✓			✓			✓	✓		✓	✓		✓	✓
W_3				✓			✓		✓		✓	✓		✓	✓	✓
W_5					✓			✓		✓	✓		✓	✓	✓	✓

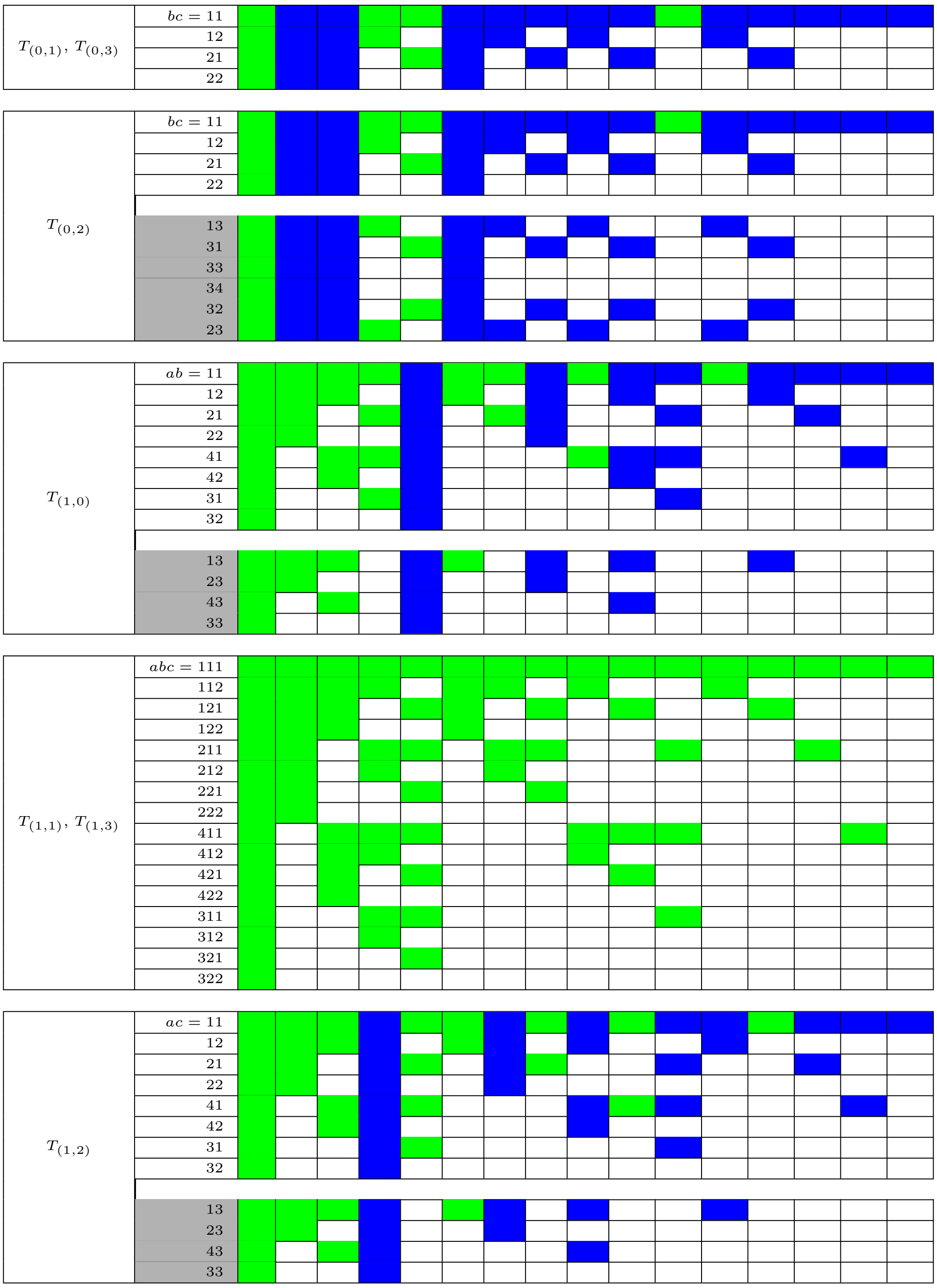


Figure 4.6: Protected (green) and split (blue) fixed points under different Wilson line configurations. The matter representations we found in the absence of Wilson lines will completely survive when sitting at a protected fixed point/torus. At split singularities they will decompose according to the local gauge group, some of these pieces will be projected out by the Wilson lines.

(i) At some fixed points/tori, the Wilson lines act trivially and hence, the matter sitting at those fixed points is the same as that one had at the GUT level. These singularities are called *protected* and they are the only locations in the orbifold where the twisted states furnish complete representations under the GUT group.

(ii) At the *split* singularities, the states are the same as those at the GUT level, but in this case the Wilson lines act as projectors, so that only some pieces of the GUT multiplets will survive. This is very similar to what occurs in the untwisted sector.

(iii) When the Wilson lines enter the mass equations for the states sitting at a certain singularity, the spectrum at this location is entirely disconnected from that at the GUT level. Such singularities we call *unshielded* because the matter sitting there is extremely sensitive to the specific choice of the WLs.

It is clear that having a spectrum at the GUT level only helps to make statements about protected and split singularities. Nevertheless, by insisting on a picture in which (some) families are complete GUT multiplets one can keep track of the splitting of the degeneracies to find out what are the relevant multiplicities under a given configuration. The results of our analysis are depicted in fig. 4.6. As pointed out in section 4.2 one can have four Wilson lines of order two. The combinatorics leads to sixteen configurations, each corresponding to a column. For each configuration we computed the embedding of the generators and their centralizers. Then we look at all those fixed points sharing the same corrections to the mass equation and the same projectors. Each fixed point/tori corresponds to a box in the table. The color code is the following: The green boxes represent protected fixed points, blue ones are split and the remainder are unshielded. To explain how to interpret our results consider for instance the configuration number 2, in which only W_1 is non trivial. Note that the first complex plane is a fixed torus of $T_{(0,1)}$ and $T_{(0,2)}$. This means that the constructing elements for the fixed tori at those sectors are independent of e_1, so that the mass equation at this sectors does not suffer from any modification. However, the presence of the twisted torus implies that any constructing element commutes with $(\mathbb{1}, e_1)$ so that W_1 projects out states. This is the reason why all fixed points from the sectors mentioned above appear blue in the second column. The remaining sectors contain some fixed points/tori which are only protected if they are located along the vertical axis of the first plane. For instance, this can be observed in table 4.3 where the $T_{(1,0)}$ sector has four ordinary and two special fixed points protected and thus highlighted in green.

4.4 Promising Candidates

For the untwisted sector, as well as protected and split singularities, we can use the spectrum at the GUT level to infer what kind of matter is likely to appear at any of these locations. We must remark that the matter states living at unshielded singularities can only be found after specifying the Wilson lines. To avoid an exhausting search for suitable Wilson lines we assume that the relevant fields of the MSSM do not sit at unshielded locations. If we want for the SO(10) models that the families are complete GUT representations, they must then arise from protected fixed points. If we consider E_6 models both split and protected singularities are favored to allocate a whole family, provided the existence of a Wilson line configuration which locally breaks the gauge group to SO(10), while leaving the $\mathbf{16} \subset \mathbf{27}$ untouched. In all other cases the families will arise as a patchwork of states originating from various locations in the orbifold.

In addition to the three families of the SM, we have to ensure that the interactions of the standard model are reproduced in an accurate manner. In particular, the presence of a heavy top in a given model can be checked before dealing with VEV configurations, particularly, if we assume none of the pieces involved in the top coupling comes from an unshielded singularity, the trilinear couplings must also be present at the GUT level. An operator of the form $\mathbf{16}\cdot\mathbf{16}\cdot\mathbf{10}$ (or $(\mathbf{27})^3$) allowed in an SO(10) (or E_6) model, will induce the desired $\overline{U}QH_u$, if the relevant pieces survive the Wilson line projections.

One can thus see that, among all possible sectors for the fields to originate from, the point group symmetry will leave us only with the following alternatives for a trilinear coupling[5]

1. UUU ,
2. $T_{(0,2)}T_{(0,2)}U$,
3. $T_{(1,0)}T_{(1,0)}U$,
4. $T_{(1,2)}T_{(1,2)}U$,
5. $T_{(0,1)}T_{(0,3)}U$,
6. $T_{(0,2)}T_{(1,2)}T_{(1,0)}$,
7. $T_{(1,3)}T_{(1,3)}T_{(0,2)}$,
8. $T_{(1,3)}T_{(0,3)}T_{(1,2)}$,
9. $T_{(1,3)}T_{(0,1)}T_{(1,0)}$.

Now we use the spectra to determine which of the previous couplings is supported by any of the models. We consider the spectrum in combination with the schematic action of the Wilson lines (see figure 4.6), with this information we intend to determine which models support any of the couplings depicted above. For conciseness, and since both SO(10) and E_6 models feature similar properties, we present our findings for SO(10) and defer the discussion about E_6 to the appendix B. Among all SO(10) embeddings only one was found to allow for three complete $\mathbf{16}$-plets. In

[5] In general, the point group itself gives more possibilities for a trilinear coupling, however all those involving $T_{(1,1)}$ are not viable since this sector does not contain any left-chiral state. One the other hand, R charge conservation forbids couplings such as $T_{(0,1)}T_{(0,1)}T_{(0,3)}$ which share a common fixed torus [85].

that model, all of the $\mathbf{16}$-pets belong to the $T_{(1,2)}$ sector, so that the top Yukawa must arise of the from $T_{(1,2)}T_{(1,2)}U$. The untwisted sector contains two $\mathbf{10}$-plets; unfortunately, none of them permit a gauge invariant coupling of the form $\mathbf{16}\cdot\mathbf{16}\cdot\mathbf{10}$. Hence, *for the $\mathbb{Z}_2\times\mathbb{Z}_4$ orbifold we constructed there is no SO(10) model in which the three families arise as complete GUT representations*

Configurations with only two complete families are more commonly found. These families usually sit at the same twisted sector and transform among each other due to an underlying D_4 flavor symmetry, which, as pointed out previously, is a consequence of a Wilson line being off. By looking at the spectra of these models one can search for allowed operators of the form $\mathbf{16}\cdot\mathbf{16}\cdot\mathbf{10}$ which could involve twisted fields. Surprisingly, couplings of this kind are not found, and hence the only alternative which is left for a trilinear coupling at the GUT level is UUU. Three embeddings were found to contain an SO(10) factor, with two complete families and a purely untwisted trilinear interaction; the remainder of this section is devoted to discuss them in detail.

The first promising embedding is realized by the vectors

$$\begin{aligned} V_2^{\text{SO(10)},1} &= (\tfrac{1}{2},\tfrac{1}{2},1,0,0,0,0,0)(1,1,1,0,0,0,0,0)\,,\\ V_4^{\text{SO(10)}1} &= (\tfrac{1}{2},\tfrac{1}{4},\tfrac{1}{4},0,0,0,0,0)(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},0,0,0,0,0)\,, \end{aligned}\tag{4.16}$$

which lead to the gauge group $[\text{SO}(10)\times\text{U}(1)^3]\times[\text{SO}(10)\times\text{SU}(4)]$ (the squared brackets are set to distinguish between the original E_8 factors). The first SO(10) factor is the relevant one and the twisted $\mathbf{16}$-plets appear at $T_{(1,2)}$. In order to achieve only two protected fixed tori in this sector one must have $W_3\neq 0$, $W_5=0$ and either $W_1=0$ or $W_2=0$ but not both.

The second embedding corresponds to

$$\begin{aligned} V_2^{\text{SO(10)},2} &= (2,0,0,0,0,-1,0,0)(\tfrac{3}{2},0,-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},0,\tfrac{1}{2})\,,\\ V_4^{\text{SO(10)},2} &= (1,0,0,0,0,0,0,0)(\tfrac{3}{4},\tfrac{1}{4},0,0,0,0,0,0)\,, \end{aligned}\tag{4.17}$$

its gauge group is $[\text{SO}(14)\times\text{U}(1)]\times[\text{SO}(10)\times\text{U}(1)^3]$, while the third one is generated by the shifts

$$\begin{aligned} V_2^{\text{SO(10)},3} &= (1,-\tfrac{1}{2},0,0,0,-\tfrac{1}{2},0,0)(\tfrac{5}{4},-\tfrac{1}{4},\tfrac{3}{4},\tfrac{3}{4},\tfrac{3}{4},\tfrac{3}{4},-\tfrac{1}{4},\tfrac{1}{4})\,,\\ V_4^{\text{SO(10)},3} &= (\tfrac{1}{2},0,0,0,0,0,0,0)(\tfrac{5}{4},-\tfrac{1}{4},-\tfrac{1}{4},-\tfrac{1}{4},-\tfrac{1}{4},-\tfrac{1}{4},\tfrac{1}{2},-\tfrac{1}{2})\,, \end{aligned}\tag{4.18}$$

leading to the gauge symmetry [SO(10) $\times$ SU(2)^2 $\times$ U(1)] $\times$ [SU(8) $\times$ U(1)]. For the last configurations, three Wilson lines are need in order to achieve two protected $\mathbf{16}$-plets in the $T_{(1,3)}$ sector. The Wilson line which remains off can be either W_1 or W_2.

As already pointed out, the presence of a trilinear Yukawa is somehow a necessary requirement. In models with two complete $SO(10)$ families the desired Yukawa can be achieved if the up-type Higgs and left- and right- handed components of the top quark are untwisted fields. However, there is a little drawback to overcome: we need to find a Wilson line configuration which ensures the coupling $\overline{U}QH_u$ ($\subset \mathbf{16}\cdot\mathbf{16}\cdot\mathbf{10}$) survives the projections. We have developed a search strategy which favors certain embeddings depending on their potential features, but we can not forget that these features can be spoiled by the Wilson lines. We have then arrived at a stage where we need of concrete WLs backgrounds to prove that our search strategy for realistic models indeed works.

4.4.1 A benchmark model

The embeddings previously discussed fit the Mini-Landscape Zip-code, at least at the GUT level. If such a picture can be retained after switching on the Wilson lines, one would expect the phenomenology of these models to develop along the lines of section 4.1. Here we explore one model and present a WL configuration which partially agrees with our expectations. We took the model defined by the vectors in eq. (4.18), whose complete spectrum at the GUT level is presented in table 4.2. At this stage one can see that the untwisted sector provides the following

U	1 $(\mathbf{16},\mathbf{2},\mathbf{1},\mathbf{1})_{0,-1}$ 1 $(\mathbf{16},\mathbf{1},\mathbf{2},\mathbf{1})_{0,1}$ 1 $(\mathbf{10},\mathbf{2},\mathbf{2},\mathbf{1})_{0,0}$ 1 $(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})_{-12,0}$ 1 $(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})_{12,0}$ 1 $(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{28})_{6,0}$ 1 $(\mathbf{1},\mathbf{1},\mathbf{1},\overline{\mathbf{28}})_{6,0}$

$T_{(0,1)}$	4 $(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{8})_{6,1}$ 4 $(\mathbf{1},\mathbf{1},\mathbf{1},\overline{\mathbf{8}})_{0,1}$
$T_{(0,2)}$	10 $(\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1})_{6,0}$ 10 $(\mathbf{10},\mathbf{1},\mathbf{1},\mathbf{1})_{-6,0}$ 6 $(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})_{-6,-2}$ 6 $(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})_{-6,2}$

$T_{(0,3)}$	4 $(\mathbf{1},\mathbf{1},\mathbf{1},\overline{\mathbf{8}})_{6,-1}$ 4 $(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{8})_{0,-1}$
$T_{(1,0)}$	4 $(\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{8})_{-3,0}$
$T_{(1,1)}$	-
$T_{(1,2)}$	4 $(\mathbf{1},\mathbf{2},\mathbf{1},\overline{\mathbf{8}})_{-3,0}$
$T_{(1,3)}$	16$(\mathbf{16},\mathbf{1},\mathbf{1},\mathbf{1})_{3,0}$ 16 $(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})_{3,1}$ 16 $(\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1})_{3,-1}$

Table 4.2: Matter spectrum corresponding to the shift embedding given in eq. (4.18). For each state, the bold numbers label its corresponding representation under $SO(10)\times SU(2)\times SU(2)\times SU(8)$ respectively, whereas the subindices label the charges under the $U(1)$ symmetries.

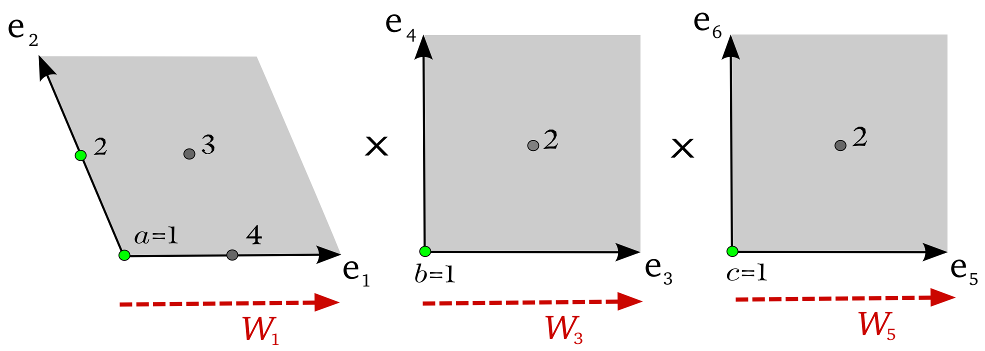


Figure 4.7: The fixed points of the $T_{(1,3)}$ sector. Note that for the model under consideration and in the case $W_{1,3,5} \neq 0$, the **16**-plets living at the fixed points $a = 1, 2$ ($b = 1, c = 1$) are not affected by the WL projections.

coupling

$$(\mathbf{16},\mathbf{2},\mathbf{1},\mathbf{1})_{0,-1} \cdot (\mathbf{16},\mathbf{1},\mathbf{2},\mathbf{1})_{0,1} \cdot (\mathbf{10},\mathbf{2},\mathbf{2},\mathbf{1})_{0,0}\,,$$

which is actually allowed by all CFT selection rules, this is the kind of coupling we expect to originate the top-Yukawa. We can also observe that in the $T_{(1,3)}$ sector we get 16 identical copies of the representation $(\mathbf{16},\mathbf{1},\mathbf{1},\mathbf{1})_{3,0}$, one sitting at each of the fixed points. Out of these, two will remain intact after the Wilson line configuration is set on. Aided by the C++ Orbifolder we searched for alternatives for W_1, W_3 and W_5 (see fig. 4.7). The Wilson line configuration is expected to break the SO(10) factor down to the desired G_{SM} with a non anomalous hypercharge generator $\mathrm{U}(1)_Y \subset \mathrm{SU}(5)$, while allowing for three net generations plus vector-like exotics. As an outcome of this exploration, roughly 200 inequivalent choices for the Wilson line configuration were found to comply with our requirements. A similar amount of models is found for the two other shift embeddings presented before. To illustrate our findings let us study the effects of the following choice for the Wilson lines

$$\begin{aligned}
W_1 &= (-\tfrac{1}{2},\tfrac{1}{2},-\tfrac{3}{2},-\tfrac{1}{2},0,-1,-1,2)(-\tfrac{3}{4},-\tfrac{7}{4},-\tfrac{1}{4},-\tfrac{1}{4},-\tfrac{1}{4},\tfrac{7}{4},\tfrac{3}{4},\tfrac{3}{4})\,,\\
W_3 &= (-1,\tfrac{3}{2},-\tfrac{1}{2},\tfrac{1}{2},0,\tfrac{1}{2},0,2)(-\tfrac{1}{2},1,-2,0,\tfrac{3}{2},-1,-\tfrac{3}{2},-\tfrac{3}{2})\,,\\
W_4 &= (-\tfrac{5}{4},\tfrac{5}{4},\tfrac{1}{4},-\tfrac{1}{4},\tfrac{3}{4},-\tfrac{1}{4},\tfrac{5}{4},\tfrac{9}{4})(0,1,1,2,-1,-\tfrac{1}{2},2,\tfrac{3}{2})\,.
\end{aligned}$$

This breaks the gauge group to $G_{\mathrm{SM}} \times \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)^9$, where the additional $\mathrm{SU}(3) \times \mathrm{SU}(2)$ is a subgroup from the SU(8) in the second E_8. As expected one of the U(1) s is anomalous, but the hypercharge generator is orthogonal to it. The splitting for the relevant untwisted fields at the

GUT level, in terms of G_{SM} is given by

$$\begin{aligned}
(\mathbf{10},\mathbf{2},\mathbf{2},\mathbf{1})_{0,0} &\to \overbrace{(\mathbf{1},\mathbf{2})_{-1/2,\cdots}}^{H_u} + \overbrace{(\mathbf{1},\mathbf{2})_{1/2,\cdots}}^{H_d}\,,\\
(\mathbf{16},\mathbf{2},\mathbf{1},\mathbf{1})_{0,-1} &\to (\mathbf{1},\mathbf{1})_{-1,\cdots} + \underbrace{(\overline{\mathbf{3}},\mathbf{1})_{1/3,\cdots}}_{U}\,,\\
(\mathbf{16},\mathbf{1},\mathbf{2},\mathbf{1})_{0,1} &\to \underbrace{(\mathbf{3},\mathbf{2})_{-1/6,\cdots}}_{Q}\,,
\end{aligned} \tag{4.19}$$

where the dots stand for additional U(1) charges different from the hypercharge. The previous splitting implies that the trilinear couplings $\overline{U}HQ$ is retained. The spectrum after the Wilson line breaking is presented in table 4.3. There one observes that in addition to those we get from the protected $\mathbf{16}$-plets in $T_{(1,3)}$, the are additional triplets coming from other locations. Note however that there are exactly three $(\mathbf{3},\mathbf{2})_{\frac{1}{6}}$, $(\overline{\mathbf{3}},\mathbf{1})_{-\frac{2}{3}}$ and $(\mathbf{1},\mathbf{1})_{-1}$. The spectrum contains four fields of the form $(\mathbf{1},\mathbf{2})_{-\frac{1}{2}}$ (including the Higgs H_u introduced in eq. (4.19)), hence we have only one Higgs pair for this model. When it comes to the down-type quarks we find 6 extra $(\overline{\mathbf{3}},\mathbf{1})_{-\frac{1}{3}}$ together with their complex conjugates. Some additional exotics are observed, those states come in vector-like pairs with respect to the SM and we expect them to decouple after assigning VEVs to some of the SM singlets n_i, s_i, $\tilde{v}_i$ and $\overline{s}_i$. We must insist that the choice of Wilson lines presented here serves merely to illustrate that it is possible to find Wilson lines leading to a three family model. However, one can see that this model can not follow the pattern depicted in section 4.1.4. To see this note that states like χ_i or m_i lack of a conjugate counterpart, so that they can only be decoupled from the spectrum in the case the VEV configuration breaks all non-Abelian factors in the hidden group, leaving no chances for gaugino condensation to occur.

4.4.2 Prospects: VEVs, Light Higgses and Decoupled Exotics

In order to retain a reasonable phenomenology we have to give VEVs to some singlet fields. This VEV configuration is also necessary to cancel the FI term induced by the anomalous $U(1)$ as well as to remove dangerous exotics from the low energy spectrum by means of the so called Froggatt-Nielsen mechanism [158]. Depending on the VEV fields, the $U(1)$s as well as discrete R and non-R symmetries will be generically broken in such a way that certain discrete combinations survive. Those combinations are of particular interest for us, since they can be used to control some dangerous operators and in particular, can be used to set the μ term to vanish in the supersymmetric vacuum.

In principle, the choice of the VEVs is not entirely arbitrary since one has to ensure F and

#	Rep.	label	#	Rep.	label
3	$(\overline{\mathbf{3}},\mathbf{1},\mathbf{1},\mathbf{1})_{\frac{2}{3}}$	$\overline{u}$	69	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})_0$	n
3	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})_{-1}$	$\overline{e}$	32	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})_{-\frac{1}{2}}$	r
3	$(\mathbf{3},\mathbf{2},\mathbf{1},\mathbf{1})_{-\frac{1}{6}}$	q	4	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})_{-\frac{1}{2}}$	b
4	$(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})_{\frac{1}{2}}$	l	30	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})_{\frac{1}{2}}$	$\overline{r}$
1	$(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})_{-\frac{1}{2}}$	$\overline{l}$	4	$(\mathbf{1},\mathbf{1},\overline{\mathbf{3}},\mathbf{1})_0$	s
9	$(\overline{\mathbf{3}},\mathbf{1},\mathbf{1},\mathbf{1})_{-\frac{1}{3}}$	$\overline{d}$	10	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})_0$	$\tilde{v}$
6	$(\mathbf{3},\mathbf{1},\mathbf{1},\mathbf{1})_{\frac{1}{3}}$	d	8	$(\mathbf{1},\mathbf{1},\mathbf{3},\mathbf{1})_0$	$\overline{s}$
6	$(\mathbf{3},\mathbf{1},\mathbf{1},\mathbf{1})_{-\frac{1}{6}}$	f	2	$(\mathbf{1},\mathbf{1},\overline{\mathbf{3}},\mathbf{1})_{\frac{1}{2}}$	χ
8	$(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})_0$	v	5	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})_{\frac{1}{2}}$	$\overline{b}$
1	$(\mathbf{3},\mathbf{1},\mathbf{1},\mathbf{2})_{-\frac{1}{6}}$	m	2	$(\mathbf{1},\mathbf{1},\overline{\mathbf{3}},\mathbf{1})_{-\frac{1}{2}}$	$\tilde{\chi}$
8	$(\overline{\mathbf{3}},\mathbf{1},\mathbf{1},\mathbf{1})_{\frac{1}{6}}$	$\overline{f}$			

Table 4.3: Matter spectrum obtained after switching on the WLs, the numbers in parenthesis label its corresponding representation under $SU(3)_C \times SU(2)_L \times SU(3) \times SU(2)$. The subindex labels the hypercharge.

D flatness of the new vacuum, however, given the large amount of VEVs and their generic size, it is assumed that such flat directions exist. Our main goal is to ensure the absence of the μ term in the superpotential, the common consensus is that this happens due to an R-symmetry. In the context of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold [86], some surviving R-symmetries in the low energy have been engineered [53]. Here we try the simplest alternatives for an R-symmetry, which turn out to have significant drawbacks, but we expect that in the more comprehensive exploration such drawbacks can be avoided.

Following the discussion from sections 3.2 and 3.2.1 one can deduce that the $\mathbb{Z}_2 \times \mathbb{Z}_4$geometry provides a $\mathbb{Z}_2$ R-symmetry in the first plane and two $\mathbb{Z}_4$ R-symmetries in the last two planes. Given an L point coupling $\Phi_1 \cdots \Phi_L$, it is only allowed if it fulfills the conditions

$$\sum_{\alpha=1}^{L} r_\alpha^1 = -1 \bmod 2\,, \quad \sum_{\alpha=1}^{L} r_\alpha^{2,3} = -1 \bmod 4\,,$$

in which r_α^i denotes the R-charge of the α-th field and it is given by

$$r_\alpha^i = q_\alpha^{(1)\,i} - \mathcal{N}_{N\,\alpha}^i + \bar{\mathcal{N}}_{N\,\alpha}^i\,,$$

where the weight $q^{(1)\,i}$ encodes the information about the fermionic excitations (see table 4.4) and $\mathcal{N}_L^i$ ($\bar{\mathcal{N}}_L^i$) counts the number of (anti-) holomorphic bosonic oscillators used to construct the physical state. With this information in mind we can reconsider the GUT bilinear (see eq. (4.19))

$$(\mathbf{10},\mathbf{2},\mathbf{2},\mathbf{1})_{0,0}\cdot(\mathbf{10},\mathbf{2},\mathbf{2},\mathbf{1})_{0,0}\supset H_uH_d\,.$$

This term is neutral under all selection rules, so that the coupling μH_uH_d is only absent in the superpotential if there is a leftover R-symmetry after the VEV configuration is chosen. From table 4.4 we can see that the R-charges are in close relation with the twisted sector. In particular note that by taking only VEV fields from $T_{(0,1)}$, $T_{(0,2)}$, $T_{(0,3)}$ and untwisted fields with $q=(0,0,-1,0)$ or $q=(0,0,0,-1)$, the $\mathbb{Z}_2$ R-symmetry from the first torus survives. However this does not work out because after we give VEVs to all possible singlets in the mentioned sectors, two $U(1)$s remain unbroken, and some SM fields are charged under them.

U	$(0,-1,0,0)$ $(0,0,-1,0)$ $(0,0,0,-1)$
$T_{(0,1)}$	$-\frac{1}{4}(0,0,1,3)$
$T_{(0,2)}$	$-\frac{1}{2}(0,0,1,1)$

$T_{(0,3)}$	$-\frac{1}{4}(0,0,3,1)$
$T_{(1,0)}$	$-\frac{1}{2}(0,1,0,1)$
$T_{(1,1)}$	-
$T_{(1,2)}$	$-\frac{1}{2}(0,1,1,0)$
$T_{(1,3)}$	$-\frac{1}{4}(0,2,1,1)$

Table 4.4: H-momenta q of negative helicity for the various sectors of $\mathbb{Z}_2\times\mathbb{Z}_4$.

In table 4.4 we give the H-momenta for all the twisted sectors. There one can see that, as expected all H-momentum entries add up to -1. Hence if one takes the sum of the of the three R-symmetries, one obtains a $\mathbb{Z}_2$ R-symmetry of the form

$$\sum_{\alpha=1}^{L}\left(\sum_{i=1}^{3}r_\alpha^i\right)=-1\bmod 2 \tag{4.20}$$

under this $\mathbb{Z}_2$ symmetry the contribution of the oscillators and gama phases in many states is such that their R-charge is of the form $0\bmod 2$. These states suffice to break all U(1) factors. However, they are not sufficient to lift all exotic fields, some of which are charges under the standard model gauge group.

We have then argued that the presence of a surviving R-symmetry in the low energy is a desirable feature which could alleviate the μ problem. However, as pointed out already, this symmetry is only a suitable alternative if the following two conditions are satisfied

- There are enough singlet fields with R-charge zero, so that their corresponding VEVs suffice the breaking of all extra $U(1)$s.
- The R-charge assignment for the fields allows for a mass term for all exotics in the model. Though this constraint is very hard to satisfy, one can look for situations in which the surviving R-symmetry is a mixture of any of the original ones with some non-R factors. There one expects more intricate charge assignments and hence more mass terms to be allowed, in contrast to the previous example.

CHAPTER 5

F-Theory Compactifications

Esse gancho que tens no braço não o inventaste tu, foi preciso que alguém tivesse a necessidade e a idéia, que sem aquela esta não ocorre, juntasse o couro e o ferro, e tamém estes navios que vês no rio, houve um tempo em que não tiveram velas, e outro tempo foi da invenção dos remos, outro o do leme, e, assim como o homem, bicho da terra, se faz marinheiro por necessidade, por necessidade se fará voador.

José Saramago, Memorial do Convento

In this chapter we contemplate another very promising framework for particle model building: F-theory. It corresponds to the non-perturbative description of the type-IIB superstring theory. As such, it permits to have localized gauge degrees of freedom, thus allowing for a local approach to particle model building.

Given the large duality web connecting the many string theories, it is not a surprise that F-theory is a close cousin of the heterotic theory. In particular, and in contrast to the (perturbative) type II theories, F-theory allows to realize exceptional gauge groups, which is one of the crucial features behind the overwhelming success of the heterotic brane world.

Here we account for a short introduction to the topic: We discuss the basic setup by briefly reviewing some features of the type IIB superstring. Then we review the dualities between F-theory, M-theory, and the heterotic string, and from these, we discuss how gauge symmetries, chiral matter and interactions arise. Even though this review is oriented in a phenomenological direction, for which reason we devote especial attention to fiber degenerations of the SU(5) type, the discussion of the explicit model building efforts and their outcome are deferred to the forthcoming chapter.

5.1 The type IIB superstring

In order to discuss some of the properties of the type IIB superstring we can use the Green-Schwarz formulation [87] in which the space time supersymmetry is explicit. For that purpose we start by reviewing the 10D, $\mathcal{N} = 1, 2$ supercurrents [16]. Firstly recall that the minimal spinor in ten dimensions is both chiral and Majorana. A Majorana spinor Q satisfies the constraint $\bar{Q} = Q^T C$, where $\bar{Q}$ is the Dirac conjugate and C is the charge conjugation operator. In the Majorana representation all Gamma matrices are real and $C = \Gamma_0$, so that the Majorana condition becomes a reality condition on the spinor Q. In ten dimensions a Majorana spinor has $2^{10/2} = 32$ real components. One can further halve the dimensionalty by projecting onto eigenstates of Γ_{11}

$$\Gamma_{11} Q^{\pm} = \pm Q^{\pm} \,, \tag{5.1}$$

with Γ_{11} being the product of all Gamma matrices. We can then show that the following operators

$$\mathcal{P}_{\pm} = \frac{\mathbb{1} \pm \Gamma_{11}}{2} \tag{5.2}$$

project into definite chirality spinors.

The $\mathcal{N} = 1$ superalgebra is then defined according to

$$\{Q^+_\alpha, Q^+_\beta\} = (C\Gamma^M \mathcal{P}_+)_{\alpha\beta} P_M \,. \tag{5.3}$$

With regards to the $\mathcal{N} = 2$ algebras one can choose the two generators to have opposite or identical chiralities. Depending on that choice one gets the type IIA (unchiral) or the type IIB (chiral) superalgebras. For the case of our interest, namely type IIB, the superalgebra reads

$$\{Q^{+\,A}_\alpha, Q^{+\,B}_\beta\} = \delta^{AB} (C\Gamma^M \mathcal{P}_+)_{\alpha\beta} P_M \,, \tag{5.4}$$

where the spinors $Q^{+\,A}$, ($A = 1, 2$) furnish an SO(2) doublet. The supertranslation group can be obtained upon exponentiation of a generic algebra element

$$X^M P_M + \bar{\theta}^{\,A}_{+} Q^{+\,A} \,, \tag{5.5}$$

where X^M are the ten dimensional bosonic coordinates and $\bar{\theta}^A_+$ are both antichiral Majorana spinors with Grassmann entries. Having introduced the superspace coordinates, we can take these fields as maps from the worldsheet, and use them to write down the worlsheet action. To do so, we first need to find super-Poincarè invariant forms. To start with, there is the 1-form $\Pi^M = \Pi_i^M d\sigma^i$, with σ^i $(i = 0, 1)$ being the WS coordinates and Π_i^M given by

$$\Pi_i^M = \partial_i X^M - \mathrm{i}\delta^{AB}\bar{\theta}_+^A \Gamma^M \partial_i \theta_+^B \,. \tag{5.6}$$

One can use these ten dimensional vectors to construct pullbacks of the metric to the world sheet: $g_{ij} = \Pi_i^M \eta_{MN} \Pi_j^N$. Next, one identifies the following invariant 3-form

$$h_3 = \Pi^M (d\bar{\theta}_+^1 \Gamma_M d\theta_+^1 - d\bar{\theta}_+^2 \Gamma_M d\theta_+^2) \,, \tag{5.7}$$

this form happens to be exact in the Minkowski background under consideration, i.e. $h_3 = dB_2$. The WS action then reads

$$S = -\int d\sigma \sqrt{-\det{(g)}} + \int B_2 \tag{5.8}$$

in which we have set the string tension to one. The first term is simply the Nambu-Goto action, and the second is the Wess-Zumino term, which makes the the action κ-symmetric, hence allowing the world–sheet supersymmetry to be linearly realized. In the above analysis we found a 2-form potential B_2 relevant to a 1-brane (i.e. the string). In that sense, the Wess-Zumino can be thought of as the minimal coupling of the string to B_2. It is also worth remarking that the previous equation can be extended to any Supergravity background beyond Minkowski space. In that case, B_2 becomes a combination of the superspace two form and the two index antisymmetric tensor of the corresponding background. In addition to that, the action receives additional contributions from the dilaton field. The bosonic part of the action in a more general background then reads

$$S = \int d\sigma \left\{ \sqrt{-\det{(\gamma)}} (\gamma^{ij} g_{ij} + \phi R^{(2)}) + \epsilon^{ij} B_{ij} \right\} \,. \tag{5.9}$$

Note that in the case of ϕ being constant, the new piece in the action involving the WS Ricci scalar $R^{(2)}$ becomes $\phi\chi$, where χ is the world sheet Euler number. From this we see that a perturbative expansion in $g_s = e^{\phi}$ suits the (closed) superstring theory. This is because scattering amplitudes at higher genus will be suppressed by a factor $e^{-2\phi(1-g)}$ (with g being the genus), due to the term e^{-S} present in the Euclidean path integral.

The massless fields of the theory can be read off from the content of a light-cone superfield $\phi(X^M, d\theta_+^A)$. The expansion of for this field reads

$$\begin{aligned} \phi(X^M, \theta_+^A) =& \phi + \mathrm{i}\bar{\theta}_+^A \lambda_-^A + \epsilon^{AB} \bar{\theta}_+^A \Gamma^M \theta_+^B L_M + \mathrm{i}\bar{\theta}_+^A \Gamma^{MNP} \theta_+^B H_{MNP}^{IJ} \\ &+ \epsilon^{AB} \bar{\theta}_+^A \Gamma^{MNPQL} \theta_+^B M_{MNPQL} + \dots \,. \end{aligned} \tag{5.10}$$

The restrictions on the above fields are such that under supertranslations the algebra closes on-shell [88–90]. In the first place it is required that the tensor H^{AB} is traceless, i.e.

$$H^{AB} = \begin{pmatrix} H & H' \\ H' & -H \end{pmatrix} . \tag{5.11}$$

In addition to that we have various Bianchi identities for the fields, namely $L = dC_0$ for a pseudoscalar 0-form. One also gets $H = dB_2$, $H' = dC_2$ and $M = dC_4$ for the self dual 5-form field M. Let us recall that the self duality of M is not enforced by superspace transformations, but a consequence of the chirality of the theory. So far we have not discussed the sector to which the above mentioned bosons belong, either RR or NSNS. In the superfield description the are distinguished depending on whether the field couples to a boson (NSNS) or to a fermion bilinear (RR). The bosonic NSNS sector is composed of the metric g (the Riemann tensor appears at θ^4 level in $\phi(X^\mu, \theta_+^I)$), the two index antisymmetric tensor B_2 and the dilaton ϕ. These fields contribute the following part of the action

$$S_{10} \supset \int d^{10}x \sqrt{-g} e^{-2\phi} \left[R + |d\phi|^2 - \frac{1}{3}|dB_2|^2 \right] . \tag{5.12}$$

Note the presence of the factor $e^{-2\phi}$ expected from closed string amplitudes in the background of the sphere. The RR fields of the IIB theory (C_0, C_2, C_4) are Abelian $2p$-forms which couple minimally to the WS via $2p + 1$-form field strengths

$$S_{10} \supset -\frac{1}{2} \sum_{p=1}^{4} \int d^{10}x \sqrt{-g} \frac{1}{(2p+1)!} |F_{2p+1}|^2 . \tag{5.13}$$

The term $e^{-2\phi}$ is absent because RR couplings to the fermions introduce branch cuts on the sphere, so that the argument following from the NSNS sector does not apply in this case [91]. Equation (5.13) has been written in the so-called democratic formulation (see e.g. [92]) in which the action is considered together with a set of duality relations defining the extra fields F_7 and F_9. Such duality relations read

$$F_5 = *F_5 , \quad F_7 = - * F_3 , \quad F_9 = *F_1 , \tag{5.14}$$

where the field strengths are given by

$$\begin{aligned} F_1 &= dC_0\,, \\ F_3 &= dC_2 - C_0 \wedge B_2\,, \\ F_5 &= dC_4 - dB_2 \wedge C_2 - dC_2 \wedge B_2\,. \end{aligned} \tag{5.15}$$

Note that the previous $2p+1$-forms mix lower order RR fields with the NSNS field B_2, but note that in the limit $B_2 = 0$ one recovers the usual relations $F_{2p+1} = dC_{2p}$. This democratic formulation suits better to deal with D-branes [93, 94]. The reason is that the Dp-branes source the p-form RR potentials. The Wess-Zumino action for the Dp-brane includes RR forms of all orders making the approach to D7- and D9-branes more intuitive.

5.2 $SL(2,\mathbb{Z})$ invariance and 7-brane monodromies

A very striking feature of the type IIB supergravity action described before is that it exhibits a global $SL(2,\mathbb{R})$ symmetry. This can be seen more easily by transforming to the Einstein frame metric

$$g^E_{MN} = e^{-\phi/2} g_{MN}\,. \tag{5.16}$$

In this frame the action reads

$$S^E_{10} \supset \int d^{10}x \sqrt{-g^E}\left[R^E + \frac{|d\tau|^2}{2(\mathrm{Im}\,\tau)^2} - \frac{|G_3|^2}{2\mathrm{Im}\,\tau}\right] + \ldots\,, \tag{5.17}$$

in which we have employed the redefinitions

$$\tau = C_0 + \mathrm{i}e^{-\phi}\,, \quad G_3 = F_3 - \tau dB_2\,. \tag{5.18}$$

The complex field τ is known as the axio-dilaton, as it combines the real NSNS scalar ϕ (the dilaton) and the RR pseudoscalar C_0. Now we can see that under the transformation

$$\tau \to \frac{a\tau + b}{c\tau + d}\,, \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})\,, \tag{5.19}$$

the second term in eq. (5.17) remains invariant. The same occurs for the third term, provided the $SL(2,\mathbb{R})$ transformation mixes the NSNS and RR 2-forms given by

$$\begin{pmatrix} C_2 \\ B_2 \end{pmatrix} \to M \begin{pmatrix} C_2 \\ B_2 \end{pmatrix} . \tag{5.20}$$

In fact, it can be shown that this symmetry extends to the full type IIB supergravity theory, including its fermionic sector. Note in particular that for a constant C_0, the transformation

$$M = \begin{pmatrix} 0 & 1 \\ -1 & C_0 \end{pmatrix} \tag{5.21}$$

maps $\phi \to -\phi$, i.e. $g_s \to g_s^{-1}$. Even though, we have not discussed whether or not the symmetry is protected from quantum effects, the previous result is very interesting as it somehow hints to the fact that the strong coupling limit of the type IIB superstring theory is again a type IIB superstring [95]. Due to the menagerie of effects which keep the superstring theory finite, the $SL(2,\mathbb{R})$ symmetry of the supergravity limit has scarce chances to survive at the quantum level. In particular, it is shown that D(-1) instanton effects break $SL(2,\mathbb{R})$ to its discrete subgroup $SL(2,\mathbb{Z})$ and that this surviving symmetry is in fact preserved at the quantum level. Note also that, under a generic transformation (5.19), the string coupling transforms according to

$$g_s \to \frac{g_s}{(cC_0 + d)^2 + c^2 g_s^2} , \tag{5.22}$$

such that the $SL(2,\mathbb{Z})$ suffices to exchange between strong and weak coupling limits. The fact that the $SL(2,\mathbb{Z})$ is a manifest symmetry of the type IIB theory, implies that the two form fields B_2 and C_2 are mapped to linear integer combinations. The coefficients are integer and not simply real is a consequence of the fact that many non pertubative objects of the theory carry topological charges under B_2 and C_2. These charges are integer quantized and this is somehow consistent with the $SL(2,\mathbb{Z})$ transformation for those. To see this in more detail, let us recall the WS action (5.9) from which we saw that the fundamental superstring carries one unit of charge under the NSNS B_2 field. As $SL(2,\mathbb{Z})$ mixes among B_2 and C_2, it thus follows that there must exist an analogous object which is electrically charged under C_2. This object is known as a D1 string. More in general, the $SL(2,\mathbb{Z})$ invariance predicts the existence of arbitrary (p,q) strings, which in the perturbative type IIB theory can only be reached as solitonic, non-perturbative objects [96].

As a short digression, let us briefly discuss some specific features of the brane zoo in type IIB. As already suggested the $p+1$-form RR potentials can be thought of as generalization of the electromagnetic potential, sourced by the D-branes. Brane configurations allow for a perturbative description if their backreaction on the geometry is negligible, at least asymptotically away from the branes. A Dp-brane can be regarded as a point source in the transverse $9-p$ directions. For the case of a single Dp-brane, the $p+1$-form potential corresponds to a solution to the Poisson equation in the transverse direction. For $p<7$ the solutions to the supergavity action (5.17), for the dilaton profile and C_{p+1} read [97]

$$e^{2\phi} = e^{2\phi_0} H_p^{(3-p)/2}\,, \quad C_{p+1} = e^{-\phi}(H_p^{-1}+1)dx^0 \wedge dx^1 \wedge \ldots \wedge dx^p\,, \tag{5.23}$$

with $H_p = 1 - Qr^{(p-7)}$, $r = \sum_{i>p}(x^i)^2$, and Q being the soliton charge, which can be computed by integrating the RR flux over an $8-p$ sphere at transverse infinity, i.e.

$$Q = \int_{S^{8-p}} d * F_{p+2}\,. \tag{5.24}$$

Note from eq. (5.23) that the dilaton asymptotes to ϕ_0 away from the brane, so that e^{ϕ_0} is taken as the effective value of the string coupling relevant for the large volume limit. Note also that for a D3-brane, in particular, the dilaton profile is constant over the whole space.

As already pointed out, the previous arguments do not apply to the D7-brane. In the two dimensional space transverse to the D7-brane, the solutions to the Poisson equation scale logarithmically with the distance to the source. For a single D7-brane ($Q=1$) located at z_0 in the transverse dimension, eq. (5.24) translates into

$$1 = \int_{S^2} d * F_9 = \oint_{S^1} F_1 = \oint_{S^1} dC_0\,, \tag{5.25}$$

which implies that when encircling the D7-brane, the C_0 form experiences the monodromy $C_0 \to C_0 + 1$ [98]. Thus, after compactifying the transverse coordinates $z = x^9 + \mathrm{i}x^{10}$, we observe that the axio-dilaton field admits the expansion[1]

$$\tau(z) = \frac{1}{2\pi \mathrm{i}} \ln\left(\frac{z-z_0}{\lambda_0}\right) + \ldots\,, \tag{5.26}$$

with some constant coefficient λ_0. The above result implies a very dramatic behavior for g_s. In particular, note that in the limit $|z-z_0| \to |\lambda_0|$, the string coupling becomes divergent. The perturbative description holds in the vicinity of the D7-brane ($|z-z_0| \ll |\lambda_0|$) where one

[1] We are interested in supergravity solutions which preserve half of the supersymmetries. In these situations τ must be a holomorphic function of z [99].

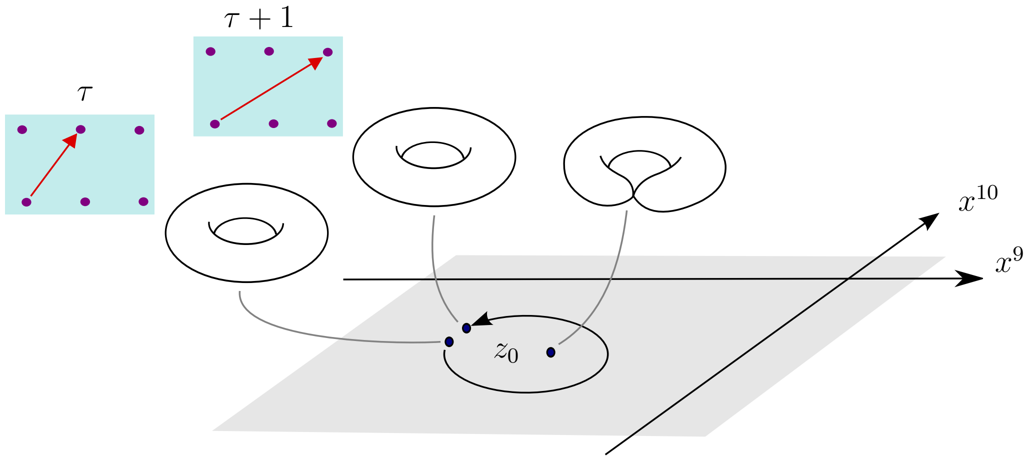


Figure 5.1: At the position z_0 the axio-dilaton τ diverges, hinting at the presence of a 7-brane. When carrying τ around z_0 it becomes $\tau + 1$. This is consistent with the $SL(2,\mathbb{Z})$ symmetry of the torus.

confirms that the supergravity solutions are approximately flat. In the limit $|z - z_0| \to \infty$, space becomes asymptotically flat, but due to the D7-brane it exhibits a deficit angle. Due to this and given the dramatic behavior of the string coupling in the presence of the D7-brane, a large volume approach, analogous to that discussed for the other type of branes, seems inadequate.

To conclude our account of the peculiar features of a D7-brane, let us comment on their interplay with the $SL(2,\mathbb{Z})$ symmetry of the type IIB theory. Note firstly that upon encircling the D7-brane, the axio-dilaton undergoes a monodromy

$$\tau(e^{2\pi \mathrm{i}}(z - z_0)) = \tau(z - z_0) + 1\,, \tag{5.27}$$

which can be reproduced by acting on τ with the following $SL(2,\mathbb{Z})$ transformation

$$A = M_{1,0} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\,. \tag{5.28}$$

As it was pointed out already, $SL(2,\mathbb{Z})$ can be used to build a (p,q)-string out of a fundamental one, i.e.

$$\begin{pmatrix} p \\ q \end{pmatrix} = g_{p,q} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} p & r \\ q & s \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\,. \tag{5.29}$$

In analogy with fundamental strings which end on D-branes, (p,q)-strings should end on $[p,q]$-branes. The corresponding monodromy of the axio-dilaton in the presence of such objects can be inferred from the previous equations

$$M_{p,q} = g_{p,q}\, M_{1,0}\, g_{p,q}^{-1} = \begin{pmatrix} 1-pq & p^2 \\ -q^2 & 1+pq \end{pmatrix}\,, \tag{5.30}$$

whose only eigenvector is a (p,q)-string itself. The relevant feature of the previous relation is that any $[p,q]$-brane can be brought into a D7-brane by means of an $SL(2,\mathbb{Z})$ action, so that, at least locally, the geometry in the vicinity of a $[p,q]$-brane can not be distinguished from that backreacted by an ordinary D7-brane. Instead, when one has different $[p,q]$-branes sitting on top of each other, the $SL(2,\mathbb{Z})$ action does not suffice to bring them all into a stack of $[1,0]$-branes. Under these circumstances, we expect new phenomena to appear which, in general, can not be dealt with using perturbative techniques.

Consistent D-brane configurations must be free of tadpoles. As D7-branes carry one unit of charge under B_2, in a consistent theory, the presence of more exotic objects is unavoidable. A very handy object which helps to construct consistent models while allowing for a perturbative description is the orientifold. Note that if we put a $[3,-1]$- and a $[1,-1]$-brane [100], the net monodromy of the resulting object is given by:

$$M_O = BC = \begin{pmatrix} -1 & 4 \\ 0 & -1 \end{pmatrix}\,, \tag{5.31}$$

where we have defined $B = M_{3,-1}$ and $C = M_{1,-1}$. Note that M_O acts on the fundamental string by orientation reversal. For this reason we identify this special combination of $[p,q]$-branes with a perturbative orientifold plane O7. Note from the above equation that when surrounding the orientifold, the axio-dilaton experiences the shift

$$\tau \to \tau - 4\,, \tag{5.32}$$

implying that orientifolds carry minus four units of charge under B_2. We can now construct consistent compactifications combining D7-branes and O7 planes, as four D7-branes together with an O7 plane account for a null overall charge, cancelling the global tadpole. In order for the tadpoles to cancel locally the branes and the orientifold must be coincidental. If this is the case, the net monodromy for the axio-dilaton is zero, so that its profile becomes constant along the tranverse directions. The resulting gauge symmetry associated to this configuration is SO(8), as can be confirmed from the usual Chan-Paton counting. Recall that a stack of N coincident branes leads to a $\mathrm{U}(N)$ gauge symmetry [101]. In general, $\mathrm{U}(N)$, $\mathrm{SO}(N)$, and $Sp(N)$ groups can be realized by combining branes and orientifolds [92]. In contrast to these, exceptional groups only make sense in a non-perturbative description.

5.3 Evidence for F-theory

The basic observation leading to F-theory has to do with the fact that, in the context of elliptic curves, the $SL(2,\mathbb{Z})$ symmetry appears as the modular symmetry of the torus. In two dimensions, the torus results from dividing the complex plane by a two dimensional lattice, i.e. $\mathbb{T}^2 = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ such that $\mathfrak{Im}(\tau) \neq 0$. Note that 1 and τ can be taken as lattice generators, in which case τ can be directly read as the complex structure of the tours. The lattice $(\mathbb{Z}+\tau\mathbb{Z})$ is identical to $(\mathbb{Z}+\tau'\mathbb{Z})$, provided τ and τ' are related by an $SL(2,\mathbb{Z})$ transformation.

The previous observation permits us to think of the axio-dilaton field as the complex structure of a genus one curve. The two extra dimensions spanned by the elliptic curve are non-physical but regarded as a bookkeeping device to track the behavior of the axio-dilaton in the transverse directions to the D7-brane. As the axio-dilaton takes a particular value at each point in ten-dimensional space time, the appropriate geometrical description is that of an elliptic fibration.

The elliptic fibration structure is at the core of F-theory. Being the correct description of type IIB string theory with D7-branes, F-theory epitomizes the geometrization of the physics in type IIB compactifications (say on an n-fold B_n), by relating it to properties of an $(n+1)$-fold Y_{n+1} which is an elliptic fibration over B_n [99]. At the location of the 7-branes the complex structure diverges (see eq. (5.26)). In relation to the elliptic curve, the divergence of the complex structure implies that one of the cycles in the torus becomes shrinkable (i.e. the elliptic curve degenerates). Hence, 7-branes sit along divisors (codimension one surfaces) in the base manifold B_n where the elliptic curve pinches off. As observed already, the actual gauge symmetry exhibited by the diverse 7-branes in the compactification can be tracked by the τ monodromies in the vicinity of the divisor. The $[p,q]$ nature of the 7-brane is thus determined by the (p,q) cycle which shrinks to zero in the fiber.

Of particular interest for us are compactifications to four dimensions, hence we are interested in F-theory compactification on a four-fold Y_4. As it turns out, demanding $\mathcal{N}=1$ SUSY implies that Y_4 has to be a Calabi-Yau. In the following we aim at a more concrete understanding of the properties of this theory and its implications for particle physics, specifically, how one can obtain semi-realistic models in this constructions. However, it is important to recall that so far, F-theory does not have a description as a fundamental theory. Instead, it should be thought of as a non-perturbative description of a class of string vacua which in certain limits can be accessed via dualities to the heterotic string, type IIB and M-theory [96]. In our quest for model building within F-theory, these dualities will be reviewed in the spirit of extracting the relevant information concerning the (massless) particle spectrum and their corresponding interactions.

5.3.1 Elliptic fibers from the Weierstraß form

The torus $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ can be described as a hypersuface in a weighted projective space, this allows us to use some complex (algebraic) geometry techniques to study the fiber, and further, the whole elliptic fibration. Suitable coordinates in the torus must be functions which are doubly periodic in the complex plane. It was observed by Weierstraß that with the exception of the constant function over $\mathbb{C}$, a doubly periodic function must at least contain a double pole at the origin. It follows from the periodicity that the same pole should occur at any lattice point. After including additional terms needed to ensure convergence, a doubly periodic function with a double pole takes the form

$$\wp(z,\tau)=\frac{1}{z^2}+\sum_{m,n\neq 0}\left[\frac{1}{(z+m\tau+n)^2}-\frac{1}{(m\tau+n)^2}\right]. \tag{5.33}$$

A Laurent expansion for $\wp(z,\tau)$ leads to

$$\wp(z,\tau)=z^{-2}+\frac{\pi}{15}E_4(\tau)z^2+\frac{2\pi^6}{189}E_6(\tau)z^4+\mathcal{O}(z^6)+\dots, \tag{5.34}$$

in which $E_{2k}(\tau)$ are Eisenstein series (see e.g. [102])

$$E_{2k}(\tau)=\frac{1}{2\zeta(2k)}\sum_{m,n\neq 0}\frac{1}{(m\tau+n)^{2k}}. \tag{5.35}$$

After taking the derivative of $\wp(z,\tau)$ one can show that the following relation is met

$$(\wp')^2(z,\tau)-4\wp^3(z,\tau)+\frac{8\pi^4}{3}E_4(\tau)\wp(z,\tau)=-\frac{8\pi^6}{27}E_6(\tau)+\mathcal{O}(z)+\dots. \tag{5.36}$$

The crucial point is that the right side of the above equation is a doubly periodic holomorphic function, hence constant. With the aid of the following redefinitions

$$x=-\frac{\wp(z,\tau)}{\pi^2}\,,\quad y=\frac{\mathrm{i}\wp'(z,\tau)}{2\pi^3}\,,\quad f=-\frac{1}{3}E_4(\tau)\,,\quad g=\frac{2}{27}E_6(\tau) \tag{5.37}$$

one recovers the so called short Weierstraß form, which corresponds to the algebraic description of the elliptic curve

$$P_W = y^2 - x^3 - fx - g = 0\,. \tag{5.38}$$

Note that $\wp$ has been defined up to a complex scaling, i.e. the Weierstraß form is invariant up to a $\mathbb{C}^*$ action $1 \to \lambda 1$, $x \to \lambda^2 x$, $y \to \lambda^3 y$, such that $[1 : x : y]$ are homogeneous coordinates in the weighted projective space $\mathbb{P}_{1,2,3}$. The fiber degenerations occur whenever the relations $P_W = 0$ and $dP_W = 0$ hold. It turns out that these conditions are only met if the polynomial $x^3 + fx + g = 0$ has a zero of degree higher than one. This property is encoded in the vanishing of the discriminant

$$\Delta = 4f^3 + 27g^2\,. \tag{5.39}$$

Recall from eq. (5.37) that f and g depend on the complex structure τ, however they can not be used to read off the complex structure directly as they are not modular invariant: Note that by shifting τ to $(a\tau + b)/(c\tau + d)$ one gets a rescaling $E_{2k}(\tau) \to (c\tau + d)^{2k} E_{2k}(\tau)$ in the Eisenstein series. For this reason one has to take a suitable fraction in order to take the inversion. One such alternative is

$$j(\tau) = \frac{4(24f)^3}{\Delta}\,, \tag{5.40}$$

known as the $SL(2,\mathbb{Z})$ invariant Jacobi function [96]

$$j(\tau) = e^{-2\pi i \tau} + 744 + 196884 e^{2\pi i \tau} + \dots\,. \tag{5.41}$$

Having introduced the algebraic form of the elliptic fiber together with some useful tools to track its behavior, we find it convenient, at this point, to describe a more abstract approach to write elliptic curves as an algebraic variety, i.e. as the vanishing of a set of homogeneous polynomials in a projective space. We start with some generalities and then we specialize to the case of our interest, which is that of genus one curves.

To start with, consider an algebraic variety X with an open cover $\{U_\alpha\}_{\alpha \in A}$ and a given divisor $D \subset X$. Locally D can be written as the vanishing of a meromorphic function:

$$D \cap U_\alpha = \{\varphi_\alpha = 0\} \subseteq U_\alpha\,. \tag{5.42}$$

If φ_α factorizes as $\varphi_\alpha = \prod \varphi_{\alpha,a}^{m_a}$, then the divisor can be written as $D = m_a D_a$, in which D_a are defined by the vanishing of the $\varphi_{\alpha,a}$. The divisor can be used to define the line bundle $\mathcal{O}_{U_\alpha}(D)$ as the set of rational functions ψ such that $\psi\varphi_\alpha$ is regular over the whole of U_α. This means that any $\psi \in \mathcal{O}_{U_\alpha}(D)$ has its poles completely contained in D. Any element in $\mathcal{O}_{U_\alpha}(D)$ is called a (local) section of $\mathcal{O}_{U_\alpha}(D)$. Further one can see that the patching among open sets U_α, U_β is given by rational functions on $U_\alpha \cap U_\beta$, thus it makes sense to define the module $\mathcal{O}_X(D)$ as the set of global sections with poles on D. Note that the zeroes of a global section $z_i \in \mathcal{O}_X(D)$ define additional divisors D_i which are said to be linearly equivalent to D, i.e.

$$D_i \cap U_\alpha = \{z_i\varphi_\alpha = 0\} \in U_\alpha\,, \tag{5.43}$$

where we have used φ_α to make z_i regular on U_α. Similarly a global section z_k of $\mathcal{O}_{U_\alpha}(mD)$ is made regular on U_α upon multiplication by φ_α^m. This motivates the embedding of the global sections of $\mathcal{O}_{U_\alpha}(mD)$ as coordinates in a weighted projective space.

Now we specialize to the elliptic curve for which D corresponds to a generic point. In this particular case, the Riemman-Roch theorem [103] implies that the dimension of $\mathcal{O}_X(mD)$ is precisely m. To rephrase this in terms of the early analysis at the beginning of this section, The divisor D we considered was the point at the origin. Then we search for periodic functions with poles at these points, i.e. elements of $\mathcal{O}_X(mD)$. As we stressed already, there are no doubly periodic functions with a single pole, for which reason $\mathcal{O}_X(D)$ is spanned by the constant function over the whole $X = \mathbb{T}^2$

$$\mathcal{O}_X(D) = \langle 1 \rangle\,, \tag{5.44}$$

one step further, we found $x \sim \wp(z,\tau)$,

$$\mathcal{O}_X(2D) = \langle 1^2, x \rangle\,, \tag{5.45}$$

and with regards to poles of order 3 we found $y \sim \wp'(z,\tau)$, hence

$$\mathcal{O}_X(3D) = \langle 1^3,\, 1x,\, y \rangle\,. \tag{5.46}$$

It is then observed that 1, x and y can be used to generate all sections of $\mathcal{O}_X(4D)$ and $\mathcal{O}_X(5D)$

$$\mathcal{O}_X(4D) = \langle 1^4,\, 1^2x,\, 1y,\, x^2 \rangle\,, \tag{5.47}$$

$$\mathcal{O}_X(5D) = \langle 1^5,\, 1^3x,\, 1^2y,\, 1x^2,\, xy \rangle\,. \tag{5.48}$$

For $\mathcal{O}_X(6D)$, according to the Riemann-Roch theorem we expect six generators. However by taking products among 1, x and y we find seven sections

$$1^6,\, 1^4x,\, 1^3y,\, 1^2x^2,\, 1xy,\, x^3,\, y^2\,, \tag{5.49}$$

so that there must be a linear relation among them. This relation is known as the Tate form. It can be brought into the Weierstraß form after some redefinitions of x and y. These redefinitions are discussed in section 5.5.1.

The algorithm we have just discussed can be used to embed the elliptic curve into different weighted projective spaces, for example, as the vanishing of a cubic polynomial in $\mathbb{P}^3$. Even though many of these descriptions are equivalent to the Weierstraß form, some special properties of the fiber can be more transparently seen in a different ambient space. In particular, the embedding of the elliptic fiber in dP_2 will be of great relevance for the discussion of global U(1) symmetries in F-theory (sect. 6.1.1).

5.3.2 Elliptic Fibrations

In writing the elliptic curve as the vanishing of a degree six polynomial in $\mathbb{P}_{1,2,3}$, we have introduced the coefficients f and g, from which the complex structure τ can be read off. In an elliptic fibration we then expect f and g to depend on the base coordinates so that τ gets modulated over the base space B_n. One can straightforwardly see that the elliptic fibration encodes the correct behavior for the complex structure in the presence of 7-branes: Suppose that along a divisor $S \subset B_n$ given by $w = 0$, the discriminant Δ vanishes to order N. Taking the leading term in eq. (5.41), we see that in the vicinity of S the complex structure exhibits the behavior

$$\tau \sim \frac{N}{2\pi \mathrm{i}} \ln(w)\,, \tag{5.50}$$

which is in accordance with the observations made in the previous sections, and reveals the presence of N 7-branes wrapping the divisor S.

The elliptic fibrations of our interest are specified by a projection map

$$\pi\, :\, Y_{n+1} \to B_n\,, \tag{5.51}$$

such that for b, a generic point in B_n, $\pi^{-1}(b)$ is a genus one curve. Note that in the Weierstrasß form, the point at infinity $[0 : 1 : 1]$ is always a part of the fiber, and it is independent of f

and g. This fact is related to the fact that elliptic fibrations with the Weierstraß form include a holomorphic section, i.e. an embedding $\sigma_0 \,:\, B_n \hookrightarrow Y_{n+1}$, such that $\pi \circ \sigma_0$ is the identity map on B_n. This holomorphic map σ_0 is commonly referred to as the zero section. The presence of a section is what makes the Weierstraß form suitable as a description of a wide class of elliptic fibrations. This is the case because any elliptic fibration with a section[2] can be mapped to the Weierstraß model (with varying f and g) by means of a birational morphism.

Elliptic fibrations rendering some amount of supersymmetries unbroken are of outmost interest for phenomenology, for example in the case of Y_{n+1} being a Calabi-Yau manifold. It can be shown that the fiber degenerations introduce a curvature term, so that if Y_{n+1} is a CY manifold, such a curvature term must be counter accounted by the base. This relation reads[3]

$$c_1(TY_{n+1}) \sim \pi^* \left(c_1(TB_n) - \frac{1}{12}[\Delta] \right) , \tag{5.52}$$

in which $c_1(TY_{n+1})$ and $c_1(TB_n)$ are the first Chern classes of Y_{n+1}, and B_n, respectively, and π^* is the pullback from B_n to Y_{n+1}. The two form $[\Delta]$ is related to the vanishing of the discriminant,

$$[\Delta] = \mathrm{ord}(\Delta)|_{S_i}[S_i] \tag{5.53}$$

where $[S_i]$ are dual to base divisors where the fiber degenerates, and $\mathrm{ord}(\Delta)|_{S_i}$ denotes the vanishing order of Δ on S_i (see. table 5.1). With this machinery at hand we see that the Calabi-Yau condition on the elliptic fibration reads:

$$[\Delta] = \mathrm{ord}(\Delta)|_{S_i}[S_i] = 12 c_1(TB_n) , \tag{5.54}$$

which corresponds to the F-theory uplift of the D7/O7 tadpole cancelation conditions from the perturbative setup. Note also that in the perturbative regime, one would naïvely think that, as in the heterotic theory, in order to achieve $\mathcal{N} = 1$, the “physical” compactification space B_n, must be Calabi-Yau. From the previous discussion we have learnt that in the presence of 7-branes, some supersymmetry can be mantained, but the geometry of B_n is deformed away from the CY regime. This is another exemplification of the notorious backreaction that 7-branes have on the geometry, and adds to that of the cigar-shaped transverse directions described already in section 5.2. Note that eq. (5.54) also imposes constraints on the degree of f and g, which can be taken as

[2] Possible elliptic fibrations with a section arise from the vanishing of a restricted cubic on $\mathbb{P}^3$, or a degree four one in $\mathbb{P}_{1,1,2}$, both with varying coefficients, just to cite some celebrated examples [96, 104].

[3] The formula (5.52) is exact for $Y_2 = K3$ [105, 106], for higher dimensional base manifolds, an “error term” must be included. However, such an error term turns out to be irrelevant in the case Y_{n+1} is actually a CY.

sections of line bundles. As $c_1(TB_n) = -c_1(K_{B_n})$, with K_{B_n} the canonical bundle of the base, we then get

$$[27g^2 + 4f^3] = -12c_1(K_{B_n})\,. \tag{5.55}$$

Hence, it follows that f and g are sections of $K_{B_n}^{-4}$ and $K_{B_n}^{-6}$, respectively. Now we turn back to the short Weierstraß form (5.38), as in order to keep homogeneity 1, x, and y must be adjusted to include their base dependencies. Homogeneity is restored upon picking x out of $\mathcal{L}_2 \otimes K_{B_n}^{-2}$, y as a section of the bundle $\mathcal{L}_3 \otimes K_{B_n}^{-3}$, and 1 from $\mathcal{L}_1 \otimes \mathcal{O}$. Here $\mathcal{L}_i$ is a degree i line bundle on the fiber and $\mathcal{O}$ is the trivial bundle on the base.

5.4 F-theory as a dual to M-theory

The type IIB string theory is related by T-duality to the type IIA theory, which in turn, can be viewed as the reduction of 11 dimensional M-theory on a circle [16]. To see how the duality extends to F-theory, let us consider M-theory on $\mathbb{R}^{1,8} \times \mathbb{T}^2$, one can then shrink one circle S^1_A of the torus to obtain the perturbative type IIA theory on $\mathbb{R}^{1,8} \times S^1_B$, with coupling constant $g_s^{\mathrm{IIA}} \sim R_A$ (the radius of S^1_A). As a next step one can perform a T-duality transformation to obtain the IIB theory on $\mathbb{R}^{1,8} \times \tilde{S}^1_B$, where $\tilde{S}^1_B$ is the dual cycle. On the IIB side, the coupling constant is given by $g_s^{\mathrm{IIB}} \sim g_s^{\mathrm{IIA}}/R_B \sim R_A/R_B$. Note that, as before, the string coupling is related to the complex structure of a torus[4], which is physical on the M-theory side. In the limit $\mathrm{Vol}(\mathbb{T}^2) \to 0$ with τ fixed, one reaches the decompactification limit of $\tilde{S}^1_B$. The previous argument serves to further identify M-theory on $\mathbb{R}^{1,8-2n} \times Y_{n+1}$ as the dual of F-theory on $\mathbb{R}^{1,9-2n} \times Y_{n+1}$, in the limit of vanishing fiber volume. This duality is of great relevance to discuss fluxes and to understand the origin of the gauge symmetry, to which we devote the remainder of this section. Another very important application of this duality is related to the observation that M-theory on a Calabi-Yau fourfold leads to a three dimensional theory with four supercharges [108]. When shrinking the fiber, i.e. oxidating the theory to four dimensions, we see that the supercharges match the ammount needed for $\mathcal{N} = 1$ supersymmetry. This is why F-theory compactifications on CY fourfolds are so widely studied, as they lead to the ideal setup for particle model building.

5.4.1 Non-Abelian gauge symmetries from singular fibers

The M-theory approach to F-theory permits us to understand the emergence of the gauge symmetry from fiber degenerations, in terms the M2 brane spectrum, as well as reductions of the

[4] If the cycles R_A and R_B are perpendicular $R_A/R_B = g_s^{\mathrm{IIB}}$ is precisely the imaginary part of the complex structure. In more general cases one has to consider the T-duality transformation for the RR field C_0 to show that the relation $g_s^{\mathrm{IIB}} = \mathfrak{Im}(\tau)$ still holds [107].

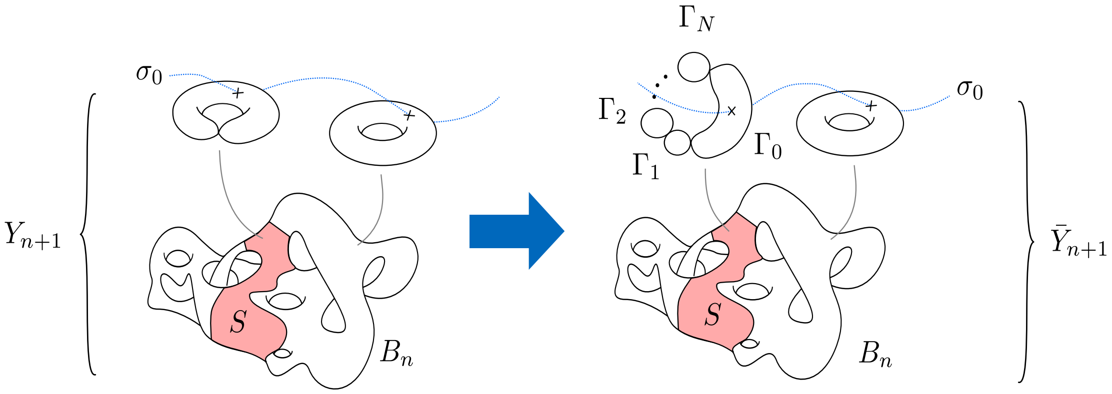


Figure 5.2: If Y_{n+1} exhibits a singularity over a certain divisor S. It can be blown up to $\bar{Y}_{n+1}$, where the singular fiber acquires the shape of a tree of $\mathbb{P}^1$'s over S. The chain depicted here corresponds to the affine Dynkin diagram of $\mathrm{SU}(N)$. We have also depicted the zero section as the point in the fiber which can be tracked over the whole of the base, and serves to identify the affine node Γ_0 from the singular regime.

three form potential C_3 on given cycles. To begin with, let us remark that the elliptic fiber can degenerate badly enough to make the full fibration Y_{n+1} become singular. In this case one can recur to standard methods of algebraic geometry to resolve the singularities. In many of the relevant cases, the smoothing of Y_{n+1} can be done while preserving its Calabi-Yau property.

Heuristically, the resolution can be seen as the singular fiber being replaced by a tree of $\mathbb{P}^1$s, which we denote by Γ_i, $i = 0, 1, ..., \mathrm{rk}(\mathfrak{g})$. Since the fibration has a section, it must intersect one of the $\mathbb{P}^1$s, which we set to be Γ_0 and referred to as the affine node. Note that in this sense, Γ_0 is related to the non-singular part of the fiber before it was blown up. The volume of Γ_0 is then the physical volume[5], so that Y_{n+1} is recovered from its smooth version $\bar{Y}_{n+1}$, in the limit when all other Γ_i shrink to zero [106].

The fiber resolutions follow a similar pattern as in the ADE classification, in the sense that the tree of $\mathbb{P}^1$'s intersects like the extended Dynkin diagram of a Lie algebra $\mathfrak{g}$. The amount of nodes (which we anticipatedly denoted by $\mathrm{rk}(\mathfrak{g})$) and the appropriate resolution are related to the vanishing order of f, g and the discriminant Δ. The classification of the singularities was carried out by Kodaira [105] for the specific case of $Y_2 = K3$ and it is summarized in table 5.1. This classification carries over to arbitrary Y_{n+1}. However, in higher dimensions, the fiber does not degenerate over points, but over entire surfaces (divisors) in the base. Because of that, and given the possibility for these surfaces to exhibit some monodromies, G_2, F_4 as well as some

[5] The presence of the zero section helps to track the fiber volume over the whole of the base, and this is relevant to take the F-theory limit. In general, elliptic fibrations without a section do not allow for a CY resolution [109].

$\mathrm{ord}(f)$	$\mathrm{ord}(g)$	$\mathrm{ord}(\Delta)$	Fiber type	Singularity type	Monodromy
≥ 0	≥ 0	0	smooth	-	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
0	0	n	I_n	A_{n-1} $(\mathrm{SU}(n))$	$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$
≥ 1	1	2	II	-	$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$
1	≥ 2	3	III	A_1	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
≥ 2	2	4	IV	A_2	$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$
2	≥ 3	$n+6$	I_n^*	D_{n+4} $(\mathrm{SO}(2n+8))$	$\begin{pmatrix} -1 & -n \\ 0 & -1 \end{pmatrix}$
≥ 3	2				
≥ 3	4	8	IV^*	E_6	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$
3	≥ 5	9	III^*	E_7	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
≥ 4	5	10	II^*	E_8	$\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$

Table 5.1: The Kodaira classification of singular fibers. In case $\mathrm{ord}(\Delta) \geq 10$, the fiber singularities become terminal in the fibration [106].

non-simply connected examples might occur if $n > 1$. At higher base codimensions the situation becomes even more critical, as some fibers appear which do not have a counterpart as a Dynkin diagram [110, 111].

Let us comment briefly on the compendium of fiber degenerations given in table 5.1. The simplest fiber singularity one can imagine is one of the I_1 type, where f and g are non-vanishing and Δ has a zero of order one. This type of singularity is not enough to produce a singularity in Y_{n+1}, so that the fiber is not blown up in this case. At higher vanishing orders in the discriminant Y_{n+1} becomes singular and one sees that in addition to the expected A_n- and D_n-type fibers, one also encounters the finite E-chain. Now we see why in type IIB they can not be perturbatively realized: Take for instance the E_8 singularity, associated to the following monodromy

$$M_{E_8} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} = A^7 B C^2\,, \tag{5.56}$$

so that, in terms of $[p, q]$-branes, the E_8 symmetry is achieved by putting seven D7-branes, an orientifold and a $[1, -1]$-brane (with no perturbative counterpart. Similarly, one can see that M_{E_8} maps the fundamental string to a $(1, 1)$ string, which implies that this object is crucial for the enhancement to an exceptional symmetry. In contrast to fundamental strings, these solitonic objects are multi-pronged so that they overcome the restrictions of two-index Chan-Paton factors [112, 113].

Lets assume that over a given codimension one surface $S \subset B_n$, the fiber degenerates badly enough to make Y_{n+1} become singular. This implies that the fiber in $\bar{Y}_{n+1}$ over S contains more than one irreducible node. Assume further that these nodes intersect as the Dynkin diagram of a Lie algebra $\mathfrak{g}$. Thus, after reducing the M-theory three form C_3 along the nodes Γ_i $i = 1, \ldots, \mathrm{rk}(\mathfrak{g})$ we obtain the following massless Abelian vector fields[6]

$$A_i = \int_{\Gamma_i} C_3\,, \tag{5.57}$$

The cycles Γ_i are dual to two forms in $H^2(\bar{Y}_{n+1})$ which do not belong to $H^2(B_n)$. The amount r of Abelian gauge potentials is counted by [114]

$$r = h^{1,1}(\bar{Y}_{n+1}) - h^{1,1}(B_n) - 1\,, \tag{5.58}$$

where we have substracted the contribution of the affine node Γ_0. As one can already anticipate from the presence of extra two forms in $\bar{Y}_{n+1}$, the resolution contains additional divisors D_i, which are fibrations of each of the $\mathbb{P}^1$ components over S, i.e. $D_i : \Gamma_i \to S$. Since the nodes intersect in the Dynkin diagram of $\mathfrak{g}$, the following relation is met

$$\Gamma_i \cdot D_j = -C_{ij}(\mathfrak{g})\,, \tag{5.59}$$

with $C_{ij}(\mathfrak{g})$ being the Cartan matrix of the Lie algebra $\mathfrak{g}$. The enhancement to a non-Abelian gauge symmetry can be explained as follows: M-theory on $\bar{Y}_{n+1}$ includes M2 branes wrapping chains of cycles $\alpha_k = \sum_k m_k \Gamma_k$. Taking into account all inequivalent combinations, one ends

[6] Recall that the affine node Γ_0 parameterizes the volume of the fiber. For this reason, the reduction of C_3 along Γ_0 does not give a gauge potential, but components of the metric in Y_{n+1}.

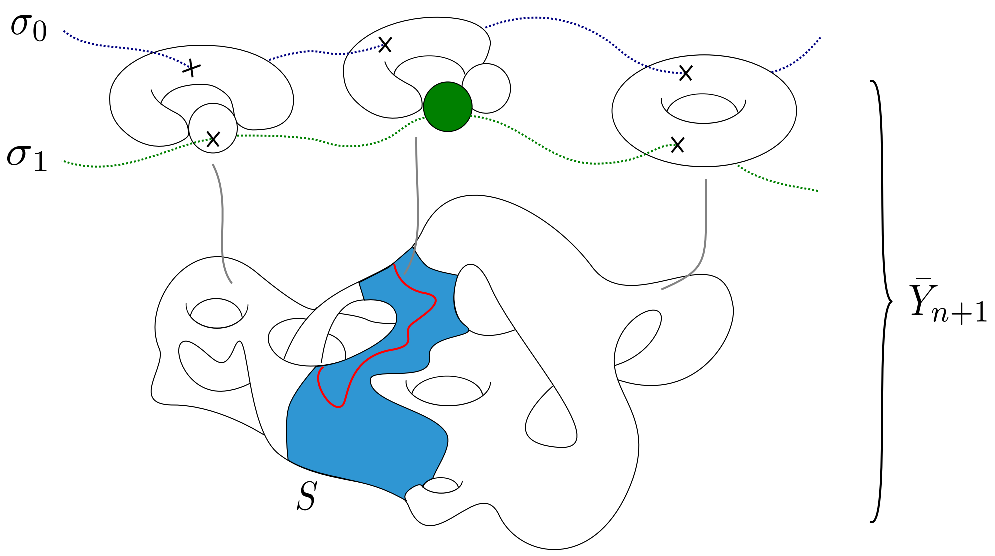


Figure 5.3: Additional sections need not to intersect the fiber precisely at the affine node Γ_0. In principle the might wrap entire fiber components at codimension two.

up exhausting the roots of $\mathfrak{g}$. These states have masses proportional to the volume of the Γ_i. In the limit $\bar{Y}_{n+1} \to Y_{n+1}$ the $\mathrm{U}(1)^{\mathrm{rk}(\mathfrak{g})}$ gets enhanced to G (the Lie algebra of which is precisely $\mathfrak{g}$).

All of the previous considerations were made at finite fiber size. The shrinking of Γ_0 along one cycle leaves us with the type IIA theory with gauge symmetry G. As T-duality does not affect the gauge symmetry, G is then the gauge symmetry one expects on the F-theory side.

5.4.2 The (not so) singular story of U(1)'s in F-theory

Previously we noted that F-theory on Y_{n+1} is equipped with r U(1) potentials (see eq. (5.58)). Out of those, not all need to belong to the Cartan subalgebra of a non-Abelian factor. From the Kodaira classification we saw that the simplest blow up of Y_{n+1} leads to an SU(2) symmetry. This implies that we have to develop further machinery in order to deal with U(1) factors. The reason for this is that, in contrast to the non-Abelian ones, U(1) symmetries are, in general, not localized over any base divisor but related to more global properties of the compactification. In fact, U(1) gauge factors are related to the presence of additional sections of the elliptic fibration [106, 114]. A pathological symptom of having an extra section is the presence of SU(2) singular fibers at codimension two. From the resolution of these one obtains an additional divisor associated with the U(1) potential.

As stressed already, elliptic fibrations governed by the Weierstraß form include a holomorphic section (the zero section σ_0) that specifies the base of the fibration[7], In terms of the Weierstraß model (5.38), σ_0 is related to the rational point $O = [0 : \lambda^2 : \lambda^3]$ on the torus. In [116, 117], additional sections were constructed by factorizing the Weierstraß equation. These extra sections need not be holomorphic. They simply have to be rational, so that in principle they can wrap entire fiber components at base codimensions greater than one.

The set of sections is known to form the so-called Mordell–Weil (MW) group of the compactification [118]. As rational sections can be thought of as rational points in the fiber, for many purposes it suffices to study the MW group of rational points. The first thing to fix is the identity element, which we take as a point O on the torus (the one related to the zero section). Addition of two rational points P and Q on the elliptic curve E is defined as follows [119]: A line $\mathbb{P}^1 \subset \mathbb{P}^2$ intersects the elliptic curve in three points (counted with multiplicities). In this way, given two points P and Q, one can define a third point R as the point of collision of the line PQ with E. Group addition is now defined by taking $P+Q$ as the point where the line OR intersects E. If we consider an elliptic curve to be specified as a cubic in $\mathbb{P}^2$ with coefficients from the field $\mathbb{Q}$, we see that the rational points on the elliptic curve indeed form a group, which is finitely generated. This can be applied to the previous discussion by mapping the cubic in $\mathbb{P}^2$ to the Weierstraß form (5.38) in $\mathbb{P}_{1,2,3}$. Then one observes that a non trivial MW group implies some restrictions for f and g in the Weierstraß form [120].

To proceed further, let us first review the kinds of divisors we have dealt with so far. First of all, we have the exceptional divisors D_i, related to the Cartan generators of the non-Abelian groups, and the zero section σ_0. In addition to those we have vertical divisors B_α which correspond to pullbacks of base divisors, and finally we will have the (non-trivial) generators of the MW group of rational sections σ_I. In the F-theory limit, the extra nodes Γ_i glued into the fiber are shrunk to zero, and this process must not affect the base of the fibration [121], i.e.

$$\Gamma_i \cdot \sigma_0 = \Gamma_i \cdot B_\alpha = 0\,. \tag{5.60}$$

In this way, the irreducible nodes have the correct interpretation as vector multiplets in the F-theory limit. Thus, in order to find out the correct U(1) charges under the Mordell–Weil generators, we need a sort of orthogonalization procedure in order to avoid mixtures with the base

[7] See [115] for a discussion with rational sections instead of holomorphic ones.

divisors, and also with the Cartan generators. This can by achieved by means of the Shioda map

$$s(\sigma_I) = \sigma_I - \sigma_0 - (\sigma_I \cdot \sigma_0 \cdot B_\alpha)B_\alpha - K_{B_n} + \sum_{\mathfrak{g}} (\sigma_I \cdot \Gamma_i)(C^{-1}(\mathfrak{g}))_{ij} D_j \,, \tag{5.61}$$

where the sum is implied over all non-Abelian factors $\mathfrak{g}$, with $C^{-1}(\mathfrak{g})$ being the inverse Cartan matrix. Now the $s(\sigma_I)$ are in one to one correspondence with the Abelian vector fields [121]. Note that this orthogonalization procedure also gets rid of the zero element of the MW $s(\sigma_0) = 0$, which, as mentioned already, contributes metric components but not U(1) potentials. The charges for the fields can be computed as intersections of the corresponding irreducible nodes with the Shioda map. Note that, by construction, all irreducible fibers at codimension one have a trivial intersection. This is so as to guarantee that all vector multiplets are neutral under under the U(1)'s. Charged matter appears at codimension two. For example, the singlet fields discussed at the beginning of this section. An specific example with non trivial MW group can be found in section 6.1.1. There we sketch how the singlet fields appear and compute their corresponding charges.

5.5 The heterotic F-theory duality

The heterotic $E_8 \times E_8$ theory (HE) can be obtained by compactifying M-theory on an interval $S^1/\mathbb{Z}_2$, with the E_8 factors living at the boundaries [122]. From this it follows that the eleven dimensional version of the HE theory on $\mathbb{T}^2$ must look like M-theory on $\mathbb{T}^2 \to S^1/\mathbb{Z}_2$. The presence of the torus allows us to relate the previous description to an F-theory background in which an additional dimension grows. This extra dimension can be envisaged as an S^1 fibered over the interval, so that these two dimensions fold into a $\mathbb{P}^1$. Hence, F-theory on an elliptic $K3 : \mathbb{T}^2 \to \mathbb{P}^1$ can be related to the HE theory on the torus. In fact, the matching of the moduli spaces for these two theories [99] provides more formal evidence for the duality between them.

To begin with, let us consider a $K3$ ($\mathbb{T}^2 \to \mathbb{P}^1$) compactification of F-theory given by the following fiber:

$$y^2 = x^3 + \alpha s^4 x + (s^5 + \beta s^6 + s^7) \,, \tag{5.62}$$

where we take $[1 : s]$ as inhomogeneous coordinates in $\mathbb{P}^1$, and α, β complex coefficients mapping to the torus data on the HE side [114]. From the Kodaira classification we see that the fiber exhibits an E_8 singularity at $s = 0$. Similarly we see that upon redefinitions $\tilde{y} = y/s^6$, $\tilde{x} = x/s^4$, and $\tilde{s} = 1/s$, we find another E_8 singularity living at $s = \infty$. This shows that the perturbative groups from the HE side localize at the poles of the $\mathbb{P}^1$ in the F-theory dual. More in

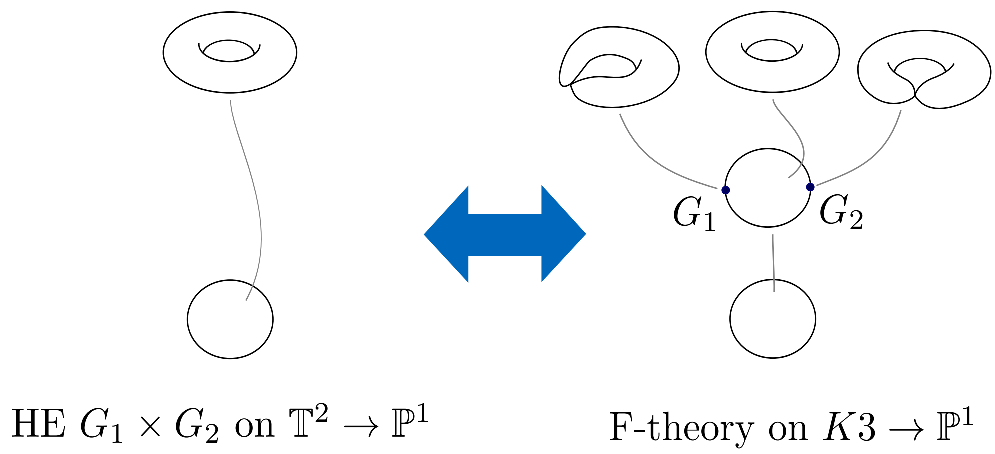


Figure 5.4: A schematic picture of the heterotic/F-theory duality. The HE string compactified on Z_2 : $\mathbb{T}2 \to \mathbb{P}^1$ with gauge group $G_1 \times G_2$ is dual to F-theory on Y_2 : $K3 \to \mathbb{P}^1$ with G_1 and G_2 localized at the poles of the intermediate $\mathbb{P}^1$.

general, F-theory on $K3$ fibrations Y_{n+1} : $K3 \to B_{n-1}$ will have a heterotic dual compactified on Z_n : $\mathbb{T}^2 \to B_{n-1}$. As discussed in section 5.5, the $\mathrm{E}_8 \times \mathrm{E}_8$ of the HE theory can be broken by means of a holomorphic vector bundle $V_1 \otimes V_2$, taking values in $H_1 \times H_2 \subset \mathrm{E}_8 \times \mathrm{E}_8$, to its commutant $G_1 \times G_2$. The resulting theory is generally chiral, for which reason the duality must map the bundle data, partially to the geometry, and partially to a flux responsible for the chirality on the F-theory side. This is one of the advantages of the duality: It helps the construction of fluxes in the F-theory side. A more concrete account of this is provided in section 6.2. For now, let us concentrate on the geometric data. Similarly as before, the groups G_1 and G_2 will localize again at the poles of the $\mathbb{P}^1$ base (in $K3$). Hence, the geometric part of the bundle data will serve to deform the E_8 singularity into one of the G_1 (G_2) type.

Before discussing the precise details, let us first sketch how the duality helps us to depart from the Kodaira classification in the general case of Y_{n+1} $(n > 2)$. In the heterotic duals discussed above, the gauge symmetries localize at the poles of the intermediate $\mathbb{P}^1$. The degenerate fiber over one of the poles will be reproduced over the whole of B_{n-1}. However, when moving around a point in the base, the fiber might exhibit monodromies [123], which will identify among irreducible fibers. This situation is analogous to the orbifolding procedure and implies that one obtains a smaller symmetry compared to that one naively expects from the Kodaira classification. In this way we get, for example, F_4 out of an E_6 Kodaira fiber.

Finally let us recall, once again, that our main interest are F-theory compactifications on Calabi-Yau fourfolds. Then in order to preserve the $\mathcal{N} = 1$ SUSY on the heterotic side, the elliptic fibration Z_3 needs to be a Calabi-Yau threefold (HE compactifications with $\mathcal{N} = 1$ SUSY were already discussed in section). The previous condition turns out to be very restrictive, leading to few possibilities for the base B_2: It can only be a $\mathrm{Bl}_r\mathbb{P}^2$ (a blow up of $\mathbb{P}^2$ at r points), a blow up of the Hirzebruch surface $\mathbb{F}_n$, or the Enriques surface $K3/\mathbb{Z}_2$ [96].

5.5.1 Tate models

The Tate algorithm [124, 125] is a refinement of the Kodaira classification. Its warhorse is the Tate form found at the end of section 5.3.1 which reads

$$P_T = x^3 - y^2 + a_1xy + a_2x^2z^2 + a_3yz^3 + a_4x + a_6 = 0\,. \tag{5.63}$$

The Tate form can be brought to the Weierstraß form by completing the square in y and the cube in x and subsequently redefining the fields. In this way the parameters f, g and the discriminant Δ of (5.39) can be recovered from the a_i by means of the following relations

$$\begin{aligned} f &= -\frac{1}{48}(\beta_2^2 - 24\beta_4)\,, \qquad g = -\frac{1}{864}(-\beta_2^3 + 36\beta_2\beta_4 - 216\beta_6)\,, \\ \Delta &= \frac{\beta_2^2}{4}(\beta_4^2 - \beta_2\beta_6) - 8\beta_4^3 - 27\beta_6^2 + 9\beta_2\beta_4\beta_6\ , \end{aligned} \tag{5.64}$$

where

$$\beta_2 = a_1^2 + 4a_2\,, \qquad \beta_4 = a_1a_3 + 2a_4\,, \qquad \beta_6 = a_3^2 + 4a_6\,. \tag{5.65}$$

First of all, let us remark that in contrast to f and g, the coefficients a_i need not to be globally well defined sections of the base. This means that, generically, the Tate form allows us to handle the singularities but does not necessarily capture all the global properties of the fibration [96]. With this clarification being made, assume that the fiber degenerates along a divisor S, defined by the local equation $w = 0$. Then the a_i can be written as $a_i = b_i w^{\mathrm{ord}(a_i)}$, where $\mathrm{ord}(a_i)$ denotes the corresponding vanishing order. Analogously as in the Kodaira classification, the type of singularity will be inferred by these vanishing orders. The complete Tate classification can be found elsewhere [96]. Here we only account for the types of singularities relevant for grand unified theories (see table 5.2).
With this machinery at hand we contemplate the possibility of an SU(5) singularity described by the following behavior of the a_i in the vicinity of the divisor S

$$a_1 = b_5\,, \quad a_2 = b_4w\,, \quad a_3 = b_3w^2\,, \quad a_4 = b_2w^3\,, \quad a_6 = b_0w^5\,. \tag{5.66}$$

Sing. Type	Fiber type	$\mathrm{ord}(a_1)$	$\mathrm{ord}(a_2)$	$\mathrm{ord}(a_3)$	$\mathrm{ord}(a_4)$	$\mathrm{ord}(a_5)$	$\mathrm{ord}(\Delta)$
I_5^s	SU(5)	0	1	2	3	5	5
I_6^s	SU(6)	0	1	3	3	6	6
I_7^s	SU(7)	0	1	3	4	7	7
I_1^{*s}	SO(10)	1	1	2	3	5	7
I_2^{*s}	SO(12)	1	1	3	3	5	8
VI^{*s}	E_6	1	2	2	3	5	8

Table 5.2: The Tate classification of singular fibers relevant for SU(5) GUTs. An exhaustive account of the classification can be found, e.g. in [124].

Inserting the parametrization (5.66) into (5.64), we find for the discriminant Δ

$$\Delta = -w^5 \left[P_{10}^4 P_5 + w P_{10}^2 (8 b_4 P_5 + P_{10} R) + O(w^2) \right] , \tag{5.67}$$

with

$$P_5 = (b_3^2 b_4 - b_2 b_3 b_5 + b_0 b_5^2) , \qquad P_{10} = b_5 , \qquad R = -b_3^3 - b_2^2 b_5 + 4 b_0 b_4 b_5 . \tag{5.68}$$

From (5.66) we see that the vanishing order of w in Δ is increased to 6 and 7 on the subloci where P_5 and P_{10} vanish, respectively. In analogy with intersecting D-brane models, these symmetry enhancements are related to the presence of chiral states[8]. According to table 5.2, the singularity structure at $P_5 = 0 \cap S$ and $P_{10} = 0 \cap S$ is consistent with an SU(6), and an SO(10) enhancement, respectively. The type of matter expected for each of these cases can be deduced from the decomposition of the adjoint representations into irreducible representations of SU(5) $\times$ U(1):

$$P_5 : \quad \mathbf{35} = \mathbf{24}_0 + \mathbf{5}_1 + \overline{\mathbf{5}}_{-1} + \mathbf{1}_0 , \tag{5.69}$$

$$P_{10} : \quad \mathbf{45} = \mathbf{24}_0 + \mathbf{10}_2 + \overline{\mathbf{10}}_{-2} + \mathbf{1}_0 , . \tag{5.70}$$

In fact, the previous observations exceed the range of validity of the Tate algorithm as it strictly holds for codimension one singularities [126]. It may occur that the symmetry enhancement is fictitious as described in the beginning of this section. However, the message to take from the above considerations is that in Y_4 matter is found at codimension two loci where the fiber exhibits further degenerations. Being already in the realms of the uncertain, we can not help ourselves but to consider the fiber behavior at points in the divisor S where two loci collide. In our SU(5)

[8] In the absence of flux, these states come in vector-like pairs

model we have the following alternatives

$$p_t = \{P_{10} = b_4 = 0\}\,, \quad p_u = \{P_{10} = b_3 = 0\}\,, \quad p_h = \{P_5 = R = 0\}\,. \tag{5.71}$$

The innocent expectation from table 5.2 is that at these points the singular fibers are of the E_6, SO(12) and SU(7) type, respectively. By looking at the adjoint decomposition into SU(5) multiplets

$$p_t \;:\quad \mathbf{78} = (\mathbf{5}+\overline{\mathbf{5}})_{-3,3} + (\mathbf{10}+\overline{\mathbf{10}})_{-1,-3} + (\mathbf{10}+\overline{\mathbf{10}})_{4,0} + \dots\,, \tag{5.72}$$

$$p_u \;:\quad \mathbf{66} = (\mathbf{5}+\overline{\mathbf{5}})_{-1,0} + (\mathbf{5}+\overline{\mathbf{5}})_{1,1} + (\mathbf{10}+\overline{\mathbf{10}})_{0,1} + \dots\,, \tag{5.73}$$

$$p_h \;:\quad \mathbf{48} = (\mathbf{5}+\overline{\mathbf{5}})_{-6,0} + (\mathbf{5}+\overline{\mathbf{5}})_{0,6} + (\mathbf{10}+\overline{\mathbf{10}})_{6,-6} + \dots\,, \tag{5.74}$$

we recognize the familiar Yukawa couplings

$$p_t \;:\; \mathbf{10}\,\mathbf{10}\,\mathbf{5}\,, \quad p_u \;:\; \overline{\mathbf{5}}\,\overline{\mathbf{5}}\,\mathbf{10}\,, \quad p_h \;:\; \mathbf{5}\,\overline{\mathbf{5}}\,\mathbf{1}\,. \tag{5.75}$$

Where in the last term we observe the presence of a singlet. As pointed out in section 5.4.2, singlets do not live on a locus of S, However, this locus intersects the divisor at the point p_h. The fiber resolutions at codimension three are worth a final remark. Note that we have based the previous arguments on the assumption that at the points of interest the fiber exhibits an ADE enhancement. This is not true in general. For instance, when approaching p_t along any of the loci, it is observed that no new $\mathbb{P}^1$ appears at p_t. Instead, the intersection pattern for the existing $\mathbb{P}^1$s is changed [110]. The $\mathbf{10}\,\mathbf{10}\,\mathbf{5}$ Yukawa coupling at p_t without an additional $\mathbb{P}^1$ is not in conflict with our M-theory intuition, since p_t corresponds to an interaction and not to the location of additional matter [126, 127].

5.5.2 The spectral cover

In eq. (5.62) we recognized an E_8 singularity from the fiber equation[9] $y^2 = x^3 + s^5$. Even though that was for the case of an elliptic $K3$, the same result holds if we think of s as a local coordinate whose vanishing defines a generic divisor S. We can then deform the E_8 fiber in order to recover an SU(5)-type degeneration over the divisor S [106, 129]

$$y^2 = x^3 + \prod_{i=1}^{5}(s - t_i)\,, \tag{5.76}$$

[9] The terms βs^6 and s^7 in (5.62) are subleading near the pole $s = 0$. Similarly, the term αs^4 corresponds to an irrelevant perturbation of the E_8 singularity [128].

in which the t_i are functions on S. They can be interpreted from two complementary perspectives: On the one hand, the SU(5) fiber arises from a partial blow up of the E_8 singularity, where the t_i parameterize the volume of the $\mathbb{P}^1$'s glued to the E_8 fiber. On the other hand, consideration of the group theoretic breaking $E_8 \to SU(5) \times SU(5)_\perp$, permits to understand the t_i as a diagonal VEV in the adjoint of $SU(5)_\perp$, so that they satisfy $b_1 = \sum_i t_i = 0$. Expanding the product in (5.76) we obtain

$$y^2 - x^3 = b_0 s^5 + b_2 s^3 + b_3 s^2 + b_4 s + b_5 \,, \tag{5.77}$$

in which the b_i are given by elementary symmetric polynomials of degree i in the t_i. Being pullbacks to S, the b_i in (5.77) must be sections of the bundle $\eta - ic_1(S)$ where $\eta = 6c_1(S) - t(S)$, with $c_1(S)$ and $-t(S)$ the first Chern class of the tangent and the normal bundle of S, respectively. It is not a coincidence that the polynomials b_i have the same name as those in the expansion (5.66), as it can be shown [127, 130] that they are the same in the limit $w = 0$, i.e. they are the leading order terms of the expansion of the Tate form coefficients a_i in powers of w. Due to the previous arguments, eq. (5.76) corresponds to an even more local description of the SU(5) fiber.

In the ultra-local approach we are considering, the divisor S can be interpreted as hypersurface in a projective non-CY threefold $K_S \to S$, with K_S the canonical bundle of S and s an affine (inhomogeneous) coordinate on K_S [131, 132]. In this space the spectral curve

$$\mathcal{C}^{(5)} \,:\, b_0 s^5 + b_2 s^3 + b_3 s^2 + b_4 s + b_5 = 0 \,, \tag{5.78}$$

defines a fivefold cover of S, also known as the spectral cover. This spectral cover construction captures the information of the various matter curves and gauge enhancements in the ultra-local zoom-in of S. The correspondence with the matter comes now as follows: If one thinks of the t_i as position dependent VEVs in the Cartan subalgebra of $SU(5)_\perp$, the resulting gauge symmetry one is left with is $SU(5) \times U(1)^4$. The matter curves will be loci on S where some of the t_i (or combinations of them) vanish, thus enhancing the SU(5) symmetry. For example, the $\mathbf{10}$-plets of SU(5) are determined by the intersection of S with the spectral curve $\mathcal{C}^{(5)}$ i.e.

$$P_{10} = b_5 = t_1 t_2 t_3 t_4 t_5 = 0 \,, \tag{5.79}$$

so that the spectral cover provides at most five $\mathbf{10}$ curves, given by

$$\Sigma_{\mathbf{10}_i} \,: \quad t_i = 0 \,. \tag{5.80}$$

This is in agreement with the decomposition of the adjoint of E_8 in terms of SU(5) $\times$ SU(5)$_\perp$

$$\mathbf{248} \;\rightarrow\; (\mathbf{24},\mathbf{1}_\perp)+(\mathbf{1},\mathbf{24}_\perp)+(\overline{\mathbf{5}},\mathbf{10}_\perp)+(\mathbf{5},\overline{\mathbf{10}}_\perp)+(\mathbf{10},\mathbf{5}_\perp)+(\overline{\mathbf{10}},\overline{\mathbf{5}}_\perp)\,. \tag{5.81}$$

The previous equation permits us also to infer the presence of at most ten $\mathbf{5}$ curves and 24 singlet curves $\Sigma_{\mathbf{1}_{ij}}$, which are given in terms of the t_i via

$$\begin{aligned} \Sigma_{\mathbf{5}_{ij}}:&\quad -(t_i+t_j)=0\,,\quad i\neq j \\ \Sigma_{\mathbf{1}_{ij}}:&\quad \pm(t_i-t_j)\,. \end{aligned} \tag{5.82}$$

In the simplest case, the spectral cover will have four U(1) symmetries resulting from the adjoint breaking of SU(5)$_\perp$. The U(1) generators can be taken as traceless linear combinations of t^i, satisfying the relation $t^i t_j = \delta^i_j$. Thus, the charges of a field can be read out of a five dimensional vector, for example, the charge vector of the $\mathbf{5}$-plet $-(t_i+t_k)$ is given by $-\delta^j_i-\delta^j_k$. The tracelessness condition implies that the vector $(1,1,1,1,1)$ is neutral under all U(1)'s.

Albeit concise, compared to the Tate algorithm, the spectral cover misses the information about possible monodromies for the t_i. In other words, some of the matter curves might be identified away from the E_8 point, leading to fewer curves [133–135]. Thus, depending on the monodromies, there can be zero to four extra U(1) symmetries appearing. Each U(1) is related to a polynomial of smaller degree n_j in the affine parameter s, which can be factored out of the spectral cover equation (5.78), such that all n_j's sum up to five and that the term s^4 does not occur. For one U(1), the spectral cover has to split into two polynomials, which leaves us with the two possibilities of either a linear and a quartic polynomial ($4+1$ factorization) or a quadratic and a cubic polynomial ($3+2$ factorization). In this case there is only one U(1), since the t_i's are identified by monodromies (e.g. $t_1 \leftrightarrow t_2 \leftrightarrow t_3$ and $t_4 \leftrightarrow t_5$ in the $3+2$ factorization). For these cases, the U(1) generators are given by

$$4+1\;:\quad t^1+t^2+t^3+t^4-t^5\,, \tag{5.83}$$

$$3+2\;:\quad 2(t^1+t^2+t^3)-3(t^4+t^5)\,. \tag{5.84}$$

Their corresponding matter spectra, including their U(1) charges are summarized in table 5.3. In the case of two U(1) symmetries one proceeds in a similar way, finding a $2+2+1$ factorization and a $3+1+1$ factorization. Higher factorizations of the spectral curve will lead to more U(1) factors. The spectral cover models exhibiting multiple U(1) symmetries can be found elsewhere [136]. In the ultra-local approach the U(1) symmetries observed in addition to the SU(5) factor

Curve	q
$\mathbf{10}_1$	1
$\mathbf{10}_5$	-4
$\bar{\mathbf{5}}_{11}$	2
$\bar{\mathbf{5}}_{15}$	-3

(a) 4+1 factorization

Curve	q
$\mathbf{10}_1$	2
$\mathbf{10}_4$	-3
$\bar{\mathbf{5}}_{11}$	4
$\bar{\mathbf{5}}_{14}$	-1
$\bar{\mathbf{5}}_{44}$	-6

(b) 3+2 factorization

Table 5.3: U(1) charges of SU(5) representations for the factorizations with a single U(1) factor. The indices specify the SU(5)$_\perp$ Cartan weights according to (5.82).

appear to be massless. Note that at this stage there is no way to show that these will be massless in the global description. If the U(1) are not present in the global description, they are analogous of the so-called geometrically massive U(1) symmetries of type IIB. In F-theory, this type of U(1) symmetries are associated to reductions of the M-theory three form along non-harmonic two forms w_A, which in turn, are subject to the condition

$$dw_A = C_A^a \alpha_a \,, \tag{5.85}$$

in which α_A is a basis of tree forms in the manifold. If any spectral cover U(1) happens to be massive in the global picture, this implies that its associated two form w_A is only harmonic over the 7-brane, but not over the whole fourfold [137]. Even if some (or all) spectral cover U(1) symmetries are massive, intuition from the type IIB side tells us that they could play an important role in the low energy. First of all, in type IIB is possible to tune flux along these factors in order to induce chirality. Secondly it is argued that geometrically massive U(1) symmetries could survive in the low energy as global symmetries.

CHAPTER 6

Model Building in F-Theory

A maggior forza e a miglior natura
liberi soggiacete; e quella cria
la mente in voi, che 'l ciel non ha in sua cura.
Però, se 'l mondo presente disvia,
in voi è la cagione, in voi si cheggia.

Dante Alighieri, Divina Commedia.

In this chapter we make efforts towards model building with F-theory GUTs. Particularly, we focus on a recent construction based on SU(5) tops. In contrast to heterotic orbifolds, in F-theory so far one does not have a toolbox of discrete symmetries which have a geometric origin. However, we can make use of additional U(1) symmetries accompanying the SU(5) in order to control the phenomenology of the models. We start by briefly discussing the geometric setup, which includes a fiber with multiple rational points. Then we make a brief review of how the matter spectrum of these theories can be made chiral by virtue of flux. Further we discuss the issue of anomaly cancelation in these models. With these ingredients at hand we proceed to analyze the matter spectra under the assumption that the SU(5) symmetry gets broken down to G_{SM} by means of the so-called hypercharge flux. Finally, we discuss the phenomenology of the models obtained, and present appealing models resulting from a bottom-up exploration. These bottom up models might allow for a geometric realization within F-theory.

6.1 Multiple U(1) s from a fiber in dP_2

In section 5.4.2 we have discussed how U(1) symmetries arise in elliptic fibrations with multiple sections. Here we discuss an explicit example of a fiber exhibiting a Mordell-Weil group of rank two, and illustrate how the U(1) charges of the fields are computed. The example we consider was worked out in refs. [104, 115, 138], which we partially reproduce.

In section 5.3.1 we showed how to obtain the Weierstraß form from a degree one line bundle. Now we aim to have three rational points which we denote by O, P and Q, then it suits to start with a degree three line bundle $\mathcal{O}_X(O+P+Q)$. This bundle contains three sections which we denote as u, v and w. The degree six line bundle $\mathcal{O}_X^2$ contains six monomials u^2, v^2, w^2, uv, uw, and vw. At degree nine, however, we obtain ten monomials u^3, v^3, w^3, uv^2, uw^2, uvw, u^2v, u^2w, vw^2 and wv^2, so that they are not all independent. One can see that after suitable redefinitions, the relation among them takes the form

$$p_T := vw(c_1 w + c_2 v) + u(b_0 v^2 + b_1 vw + b_2 w^2) + u^2(d_0 v + d_1 w + d_2 u) = 0 \tag{6.1}$$

This equation cuts an elliptic curve in $\mathbb{P}^2$. From the above equation we can identify the rational points

$$\begin{aligned} \sigma_0 :& \quad [0:0:w]\,, \\ \sigma_1 :& \quad [0:v:0]\,, \\ \sigma_2 :& \quad [0:-c_1:c_2]\,. \end{aligned} \tag{6.2}$$

In an would-be elliptic fibration the coefficients b_i, c_i, and d_i are promoted to sections of the base. Similarly the above rational points are expected to give us rational sections. In order to compute which singlets generically appear at codimension two, P_K has to be brought into the Weierstraß form. After this is done, we observes that the fiber develops SU(2) singularities whenever any of the following pairs of conditions is met

$$\begin{aligned} d_0 c_2^2 &= b_0 b_1 c_2 + b_0^2 c_1\,, \\ d_1 b_0 c_2 &= b_0^2 b_2 + c_2^2 d_2\,, \end{aligned} \tag{6.3}$$

$$\begin{aligned} d_1 c_1^2 &= b_1 b_2 c_1 - b_2^2 c_2\,, \\ d_0 b_2 c_1 &= b_0 b_2^2 + c_1^2 d_2\,, \end{aligned} \tag{6.4}$$

$$
\begin{aligned}
c_1^3(d_0c_2^2 - b_0b_1c_2 + b_0^2c_1) = c_2^3(d_1c_1^2 - b_1b_2c_1 + b_2^2c_2)\,,& \\
d_2c_1^4c_2^2 = (c_2(b_1c_1 - b_2c_2) - b_0c_1^2)& \\
(b_0b_2c_1^2 + c_2(d_1c_1^2 - b_1b_2c_1 + b_2^2c_2))\,.&
\end{aligned} \tag{6.5}
$$

Thus, one finds the following possible singlet solutions

$$\mathcal{C}_{\mathbf{1}_1}: \quad b_0 = c_2 = 0\,, \tag{6.6}$$

$$\mathcal{C}_{\mathbf{1}_2}: \quad \text{(6.3) with } (b_0, c_2) \neq (0,0)\,, \tag{6.7}$$

$$\mathcal{C}_{\mathbf{1}_3}: \quad b_2 = c_1 = 0\,, \tag{6.8}$$

$$\mathcal{C}_{\mathbf{1}_4}: \quad \text{(6.4) with } (b_2, c_1) \neq (0,0)\,, \tag{6.9}$$

$$\mathcal{C}_{\mathbf{1}_5}: \quad c_1 = c_2 = 0\,, \tag{6.10}$$

$$
\begin{aligned}
\mathcal{C}_{\mathbf{1}_6}: \quad & \text{(6.5) with } (c_1, c_2) \neq (0,0)\,, \\
& (b_0, c_2) \neq (0,0)\,, \text{ and } (b2, c_1) \neq (0,0)\,.
\end{aligned} \tag{6.11}
$$

Having the singlet curves, one can now look back and eq. (6.1). One see that at the points σ_1 and σ_0, the fiber becomes singular over $\mathcal{C}_{\mathbf{1}_1}$, and $\mathcal{C}_{\mathbf{1}_3}$, respectively. These can be cured by two blow ups of the form

$$
\begin{aligned}
u \to s_1 u\,, \quad w \to s_1 w\,, \\
u \to s_0 u\,, \quad v \to s_0 v\,.
\end{aligned} \tag{6.12}
$$

With this proper transform, equation (6.13) takes the following form

$$
\begin{aligned}
p_T :=& vw(c_1ws_1 + c_2vs_0) + u(b_0v^2s_0^2 + b_1vws_0s_1 + b_2w^2s_1^2) \\
&+ u^2s_0s_1(d_0vs_0 + d_1ws_1 + d_2us_0s_1) = 0\,.
\end{aligned} \tag{6.13}
$$

This fiber will live in the toric ambient space of dP_2, and the sections will be given by the vanishing of some of the toric coordinates. Making contact with the rational points given in (6.2). The section σ_0 defined by $\{s_0 = 0\}$, and can be viewed as the universal zero section. In addition to it, one has the following

$$\sigma_1 : \{s_1 = 0\}\,, \quad \sigma_2 : \{u = 0\}\,, \tag{6.14}$$

responsible for additional U(1) factors. In order to compute the charges for the singlets, we will also need their associated Shioda maps. According to (5.61), these maps are given by

$$s(\sigma_1) = 5(\sigma_1 - \sigma_0 + \ldots)\,, \tag{6.15}$$

$$s(\sigma_2) = 5(\sigma_2 - \sigma_0 + \ldots)\,, \tag{6.16}$$

where the dots represent contributions from base divisors. We have not included them here, since, by virtue of eq. (5.60), they do not make contributions to the singlet charges. The scaling by 5 in the above equations is convenient if we aim at elliptic fibrations with p_T which also contain an SU(5) singularity. To exemplify how the charges are computed, let us consider the fiber (6.13) over $\mathcal{C}_{1_1}$. Over this locus the fiber becomes $p_T = s_1 P$, with

$$\begin{aligned} P =& c_1 v w^2 + b_1 u^2 w s_0 + b_2 w^2 u s_1 \\ &+ u^2 s_0 (d_0 v s_0 + d_1 w s_1 + d_2 u s_0 s_1)\,, \end{aligned} \tag{6.17}$$

so that the fiber splits into two irreducible components

$$\Gamma_0 = \{P = 0\}\,, \Gamma_0 = \{s_1 = 0\}\,. \tag{6.18}$$

In the M-theory picture, the singlets will arise from M2 branes wrapping the irreducible fibers. We see that the sections σ_0 and σ_2 intersect the fiber Γ_0 at generic points, so that the intersection numbers $\sigma_0 \cdot \Gamma_0$ and $\sigma_2 \cdot \Gamma_0$ are equal to 1. This also implies that $\sigma_0 \cdot \Gamma_1$ and $\sigma_2 \cdot \Gamma_1$ are both zero. Furthermore, we see that σ_1 wraps the entire fiber Γ_1. Thus, σ_0 will intersect Γ_0 at two points, precisely those where Γ_0 and Γ_1 intersect. Since the sections have intersection number 1 with the whole elliptic fiber, we deduce that $\sigma_1 \cdot \Gamma_1 = -1$. The U(1) charges, as well as the corresponding intersection numbers have been summarized in table 6.1. There we see that the irreducible fibers over the codimension two loci give us a pair of vector-like states.

6.1.1 SU(5) completion

The sections described above result from embedding the elliptic fiber into a toric ambient space (that of dP_2). Sections of this type are known as toric sections and constitute a tractable subset of all rational sections. There are 16 possible ways to write the elliptic fiber as a hypersurface in a toric ambient space [139], each of which is characterized by its toric diagram or polygon. The toric sections of these polygons give rise to (the toric part of) their corresponding Mordell–

Curve		σ_0	σ_1	σ_2	$s(\sigma_1)$	$s(\sigma_2)$
$\mathcal{C}_{\mathbf{1}_1}$	Γ_0	1	2	0	5	-5
	Γ_1	0	-1	1	-5	5
$\mathcal{C}_{\mathbf{1}_2}$	Γ_0	0	1	0	5	0
	Γ_1	1	0	1	-5	0
$\mathcal{C}_{\mathbf{1}_3}$	Γ_0	2	1	0	-5	-10
	Γ_1	-1	0	1	5	10
$\mathcal{C}_{\mathbf{1}_4}$	Γ_0	1	0	0	-5	-5
	Γ_1	0	1	1	5	5
$\mathcal{C}_{\mathbf{1}_5}$	Γ_0	0	0	2	0	10
	Γ_1	1	1	-1	0	-10
$\mathcal{C}_{\mathbf{1}_6}$	Γ_0	0	0	1	0	5
	Γ_1	1	1	0	0	-5

Table 6.1: Intersection numbers and U(1) charges for the singlets of a dP_2 fiber.

Weil group[1] and have been analyzed in [119]. These polygons can lead to up to three toric U(1) symmetries.

The polygon gives us a description of the elliptic fiber and its sections. Next we have to include the information about the possible ways for desingularizing the SU(5) degenerations. In order to do so, we combine two polygons to form a three-dimensional polyhedron known as a *top*. We first place the fiber polygon into the plane $z = 0$. Parallel to it, at $z = 1$, we then introduce another polygon whose integer boundary points correspond to the five nodes of the affine Dynkin diagram of SU(5) (see Figure 6.1 (b) for an example). In this way, the facet at $z = 0$ encodes the generic fiber, whereas the one at $z = 1$ encodes its SU(5) resolution. The top completion is not unique and for every of the 16 polytopes there can be multiple tops. Note that tops can be related by symmetries or might not lead to a flat fibration (which means that there will be an infinite tower of massless fields), thus reducing the amount of viable tops.

As already mentioned, in order to compute the U(1) charges for the fields, one needs to find the intersections of the toric sections with the irreducible fiber components. Using the top, these intersections can simply be read off from the edges that are shared between a vertex of the fiber polygon that corresponds to the toric section and a vertex of the other polygon that corresponds to an irreducible $\mathbb{P}^1$. To exemplify this, the reader is referred to Figure 6.1, where details of the top $\tau_{5,2}$ are given. The red lines correspond to the intersections of the sections with the irreducible fibers. The upper facet of the top in Figure 6.1 (b) is shown in Figure 6.1 (d), from which one can

[1] In addition, there can be non-toric sections (i.e. sections that are not simply given by setting one fiber coordinate to zero) giving rise to further non-toric U(1) factors.

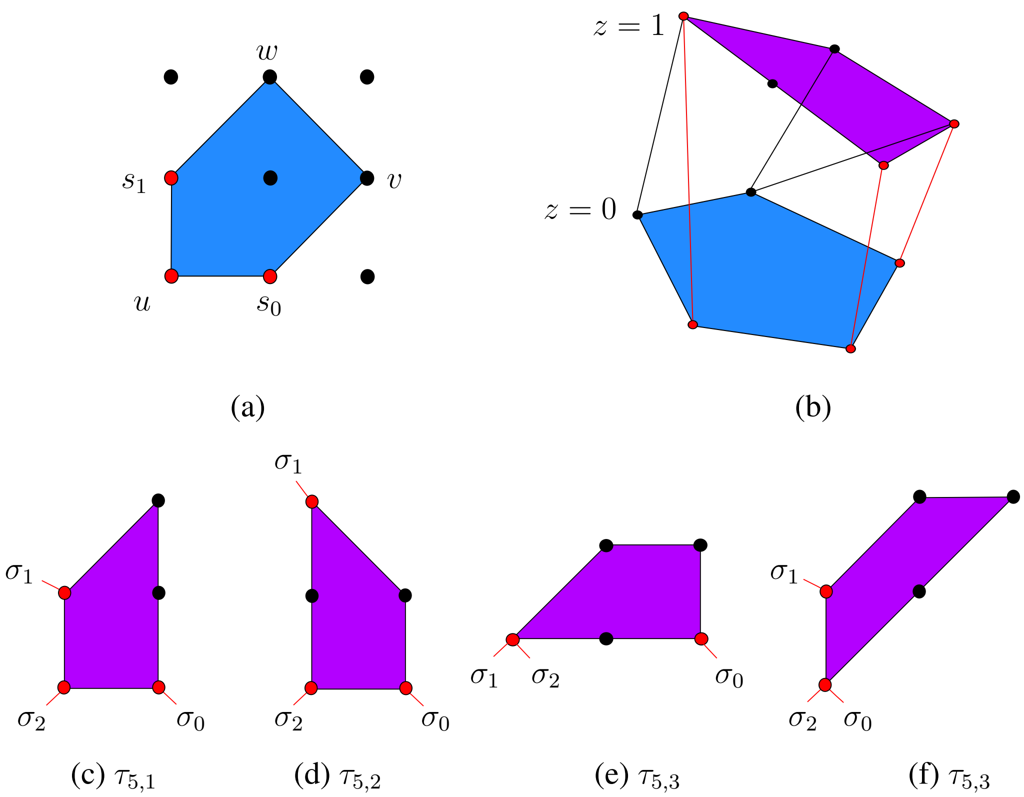


Figure 6.1: (a) Toric diagram for the polygon dP_2. It has the sections σ_0: $s_0 = 0$ (the zero section), σ_1: $s_1 = 0$ and σ_2: $u = 0$, which are marked as red points in the diagram. (b) The polygon is set as the basis for the SU(5) top at $z = 0$. The intersections of the sections with the tree of $\mathbb{P}^1$'s at $z = 1$ (see the red lines in the diagram) serve to compute the charges of the fields via the Shioda map. From the intersections one can also deduce the splitting for each of the inequivalent flat SU(5) tops allowed for F_5: (c) 2-3, 1-4 (d) 3-2, 1-4 (e) 2-3, 2-3 and (f) 1-4, 5-0.

see that this intersection pattern is consistent with a splitting of the form 3-2. By a 3-2 splitting we mean that the section σ_1 intersects a fiber component Γ_2 which is two nodes away from that which is intersected by the zero section Γ_0 (counted clockwise). Similarly, for the second U(1) we have a 1-4 splitting. As we will see in the following, the split is closely related to the charges of the matter fields under the corresponding U(1) symmetry. Given the presence of the SU(5) divisors the Shioda maps (6.15) and (6.16) must include their contributions. These will take the form

$$s(\sigma_I) = 5(\sigma_I - \sigma_0 - \mathfrak{b}_I) + D_i \begin{pmatrix} 4 & 3 & 2 & 1 \\ 3 & 6 & 4 & 2 \\ 2 & 4 & 6 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}_{ij} (\Gamma_i \cdot \sigma_I)\,, \tag{6.19}$$

Split	5-0	4-1	3-2	2-3	1-4
$Q_{\overline{\mathbf{5}}}$	0	1	2	3	4
$Q_{\mathbf{10}}$	0	3	1	4	2

Table 6.2: Charge assignments for $\overline{\mathbf{5}}$- and $\mathbf{10}$-curves for all possible splittings.

with $\mathfrak{b}_I$ are some base divisors [115, 138], and D_i are the exceptional divisors of the SU(5). Note that the intersections $\Gamma_i \cdot \sigma_I$ can be read out from the top, this is what fixes the given splitting. To find the charge of a given matter representation, we need to compute the intersection of the Shioda map with its associated curves. In particular, the intersection of these curves with the exceptional divisors D_i will give the Dynkin labels of the representation. Hence, we can see from eq. (6.19), that the SU(5) contribution to the U(1) charges of all $\mathbf{10}$-plets ($\mathbf{5}$-plets) is the same. This is the reason why, under a given split, the charges of *all* $\mathbf{10}$-plets, $\overline{\mathbf{5}}$-plets and singlets are subjected to the following relations:

$$q_{\overline{\mathbf{5}}} = Q_{\overline{\mathbf{5}}} + 5\mathbb{Z}\,, \quad q_{\mathbf{10}} = Q_{\mathbf{10}} + 5\mathbb{Z}\,, \quad q_{\mathbf{1}} = 0 + 5\mathbb{Z}\,, \tag{6.20}$$

where $Q_{\overline{\mathbf{5}}}$ and $Q_{\mathbf{10}}$ are given in Table 6.2 for all possible splits. This is a result that does not only hold in our example but applies to all types of sections. Note the charges for the fields in the different spectral cover factorizations also obey the relations (6.20). For example, the charges in the 4+1 factorization match those of a 3-2 splitting, cf. table 5.3 and 6.2. A very important feature about toric constructions is that the charges of the $\mathbf{10}$-plets under toric U(1) symmetries are very constrained: The $\mathbf{10}$ matter curves are given in terms of the triangulation of the facet of the polygon at height one. We want to choose this polygon such that it does not contain an interior point in order to maintain flatness of the fibration. Due to this we are left, up to isomorphisms, with only one such polygon (see Figure 6.1 (c)-(f)). This polygon admits two different triangulations corresponding to two possible degenerations of the SU(5) to an SO(10) Kodaira fiber. However, the choice is fixed by the top to be universal over the whole GUT divisor, such that the U(1) charges of all $\mathbf{10}$ matter curves coincide. This is in contrast to the $\mathbf{5}$ matter curves where the different degenerations can occur over different codimension two loci. Note that more general situations where different $\mathbf{10}$-curves carry different U(1) charges can be obtained by taking complete intersections instead of hypersurfaces. However, these constructions have not been studied in the literature up to now such that we do not include this possibility in our subsequent analysis.

In [138, 140] five of the 16 possible reflexive polytopes were completed using all inequivalent tops that lead to SU(5). Out of these five polygons only the polygon of dP_2, exhibits two U(1) gauge factors while the others exhibit only one or less toric U(1) symmetries. After the

Curve	q_1	q_2
$\mathbf{10}_1$	-1	2
$\overline{\mathbf{5}}_1$	3	-1
$\overline{\mathbf{5}}_2$	-2	4
$\overline{\mathbf{5}}_3$	-2	-6
$\overline{\mathbf{5}}_4$	3	4
$\overline{\mathbf{5}}_5$	-2	-1

(a) Top $\tau_{5,1}$.

Curve	q_1	q_2
$\mathbf{10}_1$	1	2
$\overline{\mathbf{5}}_1$	-3	4
$\overline{\mathbf{5}}_2$	-3	-6
$\overline{\mathbf{5}}_3$	-3	-1
$\overline{\mathbf{5}}_4$	2	4
$\overline{\mathbf{5}}_5$	2	-1

(b) Top $\tau_{5,2}$.

Curve	q_1	q_2
$\mathbf{10}_1$	-1	-1
$\overline{\mathbf{5}}_1$	3	-2
$\overline{\mathbf{5}}_2$	-2	-7
$\overline{\mathbf{5}}_3$	-2	3
$\overline{\mathbf{5}}_4$	3	3
$\overline{\mathbf{5}}_5$	-2	-2

(c) Top $\tau_{5,3}$.

Curve	q_1	q_2
$\mathbf{10}_1$	2	0
$\overline{\mathbf{5}}_1$	4	5
$\overline{\mathbf{5}}_2$	4	0
$\overline{\mathbf{5}}_3$	-1	5
$\overline{\mathbf{5}}_4$	-1	-5
$\overline{\mathbf{5}}_5$	-1	0

(d) Top $\tau_{5,4}$.

Table 6.3: U(1) charges of the four inequivalent tops based on the fiber polygon F_5. The singlet charges are the same for all tops.

completion to SU(5), matter curves and their U(1) charges have been calculated for every of these tops. As dP_2 will be of main phenomenological interest we show the matter content and U(1) charges for its four possible inequivalent tops that can lead to a flat fibration in Table 6.3.

So far we have not specified the whole CY fourfold or the base space. Thus, as a next step we would have to complete the top to a polygon which describes the complete CY fourfold. Indeed, whether or not a given base polytope can be combined with any of the tops in such a way that the fibration is flat depends on the choice of the base [119, 138, 141]. In particular, in ref. [138] it is shown that the tops $\tau_{5,1}$-$\tau_{5,4}$ generically exhibit a non-flat Yukawa point. Thus one has to make a special choice for the base in which either, one of the curves involved in the coupling is removed, or the curves involved in the Yukawa do not intersect. A detailed study of the bases that complete the top is beyond the scope of this work. Here we simply assume that there exists a choice for the base such that the fibration is flat with the maximum amount of matter curves. Given this assumption, the other properties like the amount of toric U(1) symmetries and non-Abelian gauge factors or the U(1) charges can be studied from the top alone without specifying a base. Since it will be our main concern to satisfy the anomaly cancelation constraints, it is sufficient for us to know the number of U(1) symmetries, how many curves we can expect, and what their U(1) charge pattern can be. As we will show in the following, anomaly cancelation places very strong constraints on the fluxes.

6.2 Flux configurations

Flux is a crucial ingredient to achieve chiral spectra in F-theory compactifications. In type IIB, one has closed string fluxes such as $G_3 = F_3 - \tau H_3$ (see eq. (5.37)), as well as those corresponding to the background value for the Yang-Mills field strength of a given U(1) factor. In the uplift to F-theory, both kinds of fluxes are incorporated into the M-theory four form flux G_4 [142]. With regards to U(1) fluxes, these arise from the decomposition

$$G_4 = f^i \wedge w_i + \dots\,, \tag{6.21}$$

where the two forms w^i are dual, either to the exceptional divisors D_i or to the sections σ_I discussed in section 5.4.2. In the first case flux will be turned along the Cartan generators of some non-Abelian factor G. This means that G will be broken by flux to the commutant of the corresponding Cartan element. This Cartan generator will become Stückelberg massive, because the flux generically induces couplings to the axionic sector. A similar situation occurs for pure U(1) symmetries but in this case, all non-Abelian factors remain unbroken.

Of course G_4 allows for more exotic types of flux. These are, however, harder to deal with, because their description needs of non-Abelian gauge bundles [96]. An special case of this is the universal spectral cover flux, which can be accessed via the heterotic duality [127, 143, 144].

Having appreciated the very basic features of G_4 flux, we must discuss now how chirality is induced in the spectrum. In F-theory G_4 integrates naturally over four cycles. These correspond to matter curves as we will see in the following. Recall that one can fiber the resolution of a given codimension two singularity over the curve in the base supporting it. This defines a two complex dimensional surface. Since at codimension two we find the matter representations, the net chirality of a given matter representation $\mathbf{R}$ will then result from integrating the G_4 flux over its corresponding matter curve $\Sigma_{\mathbf{R}}$.

We can rephrase this statements in terms of the SU(5) models of our interest. In engineering the standard model we would need of two types of fluxes: One which must act transversely to the SU(5) symmetry which could help us to achieve three net generations. The second type of flux would be set along the hypercharge generator[2]. This would break the SU(5) down to G_{SM} and could help us to lift the color-triplets accompanying the Higgses. This seems counterintuitive as we have argued that fluxes along the U(1) factors would make them massive. In order for the hypercharge to remain massless, one has to ensure that it does not have axionic couplings. Axions come from the closed RR sector and are counted by the cohomology of the fourfold. Thus, in order to avoid a Stückelberg mass for the hypercharge generator, *hypercharge flux* must

[2] In F-theory it is also possible to break the SU(5) by means of a Higgs mechanism as in field theoretic models. However, constructions of this type would suffer of the traditional problems of 4D GUTs. Wilson lines are also an alternative. Some efforts in this direction have been made, but the resulting models remain phenomenologically unappealing [145].

be tuned along a cycle which is non-trivial in the homology of the GUT surface S, but homologically trivial in the full CY. After switching on the fluxes previously discussed, the amount of chiral matter in a given representation $\mathbf{R}$ originating from a matter curve Σ (the base of the matter surface $\Sigma_{\mathbf{R}}$) is counted via the following index theorem

$$\chi(\mathbf{R}) = \int_{\Sigma} c_1\left(V \otimes \mathcal{L}_Y^{Y_{\mathbf{R}}}\right) = \int_{\Sigma} \left[c_1(V_{\Sigma}) + \mathrm{rk}(V) c_1(\mathcal{L}^{Y_{\mathbf{R}}})\right], \tag{6.22}$$

where V is the bundle responsible for the SU(5) chirality, $\mathcal{L}_Y$ is a line bundle used to specify the hypercharge flux, and $Y_{\mathbf{R}}$ denotes the hypercharge carried by $\mathbf{R}$. We can split the previous relation as

$$\chi(\mathbf{R}) = \underbrace{\int_{\Sigma} c_1(V)}_{\mathcal{M}_{\Sigma}} + Y_{\mathbf{R}} \underbrace{\left[\mathrm{rk}(V) \int_{\Sigma} f_Y\right]}_{\mathcal{N}_{\Sigma}}, \tag{6.23}$$

were have introduced the $(1,1)$ form $f_Y \sim c_1(L_Y)$ together with the quantities $\mathcal{M}_{\Sigma}$ and $\mathcal{N}_{\Sigma}$, which are the same for all representations steming from Σ. With this we arrive at the following chiralities for the SM components originating from a $\mathbf{10}$- or a $\overline{\mathbf{5}}$-curve

$$\begin{array}{llll} \Sigma_{\mathbf{10}^{(a)}}: & (\mathbf{3},\mathbf{2})_{1/6} & : & M_a \\ & (\overline{\mathbf{3}},\mathbf{1})_{-2/3} & : & M_a - N_a \\ & (\mathbf{1},\mathbf{1})_1 & : & M_a + N_a\,. \end{array} \qquad \begin{array}{llll} \Sigma_{\overline{\mathbf{5}}^{(i)}}: & (\overline{\mathbf{3}},\mathbf{1})_{1/3} & : & M_i \\ & (\mathbf{1},\mathbf{2})_{-1/2} & : & M_i + N_i \\ & & & \end{array} \tag{6.24}$$

where the indices a and i label $\mathbf{10}$- and $\overline{\mathbf{5}}$-curves, respectively. The non curly M and N are integer numbers related to the quantities in eq. (6.23) via

$$\begin{aligned} M_a &= \mathcal{M}_a + \frac{1}{6}\mathcal{N}_a\,, \qquad M_i = \mathcal{M}_i + \frac{1}{3}\mathcal{N}_i\,, \\ M_a &= \frac{5}{6}\mathcal{N}_a\,, \qquad N_i = -\frac{5}{6}\mathcal{N}_i\,. \end{aligned} \tag{6.25}$$

Of course, the field multiplicities are not arbitrary, but very dependent on the fourfold geometry as well as the bundles V and $\mathcal{L}_Y$, which in turn must obey certain quantization conditions (see e.g. [127, 146]. However, for the models which can be described in terms of the spectral cover there is a prescription on how to assign the necessary flux quanta. In this context it was observed that the flux distribution is subject to certain constraints which are generic to all consistent models, and are related to the masslessness condition on the hypercharge generator. These are known as the Dudas-Palti (DP) relations [147, 148]

$$\sum_i M_i - \sum_a M_a = 0, \quad \sum_i N_i = \sum_a N_a = 0\,, \tag{6.26}$$

$$\sum_a q_a^A N_a + \sum_i q_i^A N_i = 0\,, \tag{6.27}$$

with q^A being the charges for the fields under any of the additional U(1) symmetries. One sees from eq. (6.26) that the hypercharge flux induces no net chirality, as a consequence of it being globally trivial. Furthermore, it was shown that these conditions are nothing but the mere requirement of anomaly cancelation [149]. The eqs. (6.26) can be obtained from demanding all SM anomalies to vanish. Similarly eq. (6.27) arises from imposing that hypercharge flux does not modify the mechanisms canceling anomalies of the type $G_{\mathrm{SM}} - G_{\mathrm{SM}} - \mathrm{U}(1)_A$. This is a very compelling result. However, it faces us with a puzzle: If anomalies are not lifted by hypercharge flux, then anomalies of the form $\mathrm{U}(1)_Y - \mathrm{U}(1)_A - \mathrm{U}(1)_B$ must also vanish. This is so because they descend from the SU(5) $-$ $\mathrm{U}(1)_A - \mathrm{U}(1)_B$ anomaly. This, in turn, imposes an additional constraint on the matter multiplicities [150]

$$3\sum_a q_a^A q_a^b N_a + \sum_i q_i^A q_i^B N_i = 0\,. \tag{6.28}$$

However, no such a condition has been found from homology relations in the spectral cover. Even more surprisingly, there are no phenomenologically appealing models which satisfy this condition. Due to our scarce understanding of G_4 fluxes, there remains the naïve possibility of a mechanism which is yet to be discovered, which allows for a shift in the $\mathrm{U}(1)_Y - \mathrm{U}(1)_A - \mathrm{U}(1)_B$ anomalies, while leaving the Dudas-Palti relations unaffected. On a more earthy note, observations from the type IIB side have opened room for an orientifold odd Green-Schwarz mechanism [151] to cancel the anomaly (6.28). However, it is unknown to date how this mechanism is uplifted to F-theory and why is it not captured by the local picture. We come back to these matters in section 6.4.

6.3 Towards realistic models

In this section we aim at constructing models with the exact MSSM spectrum (plus singlet extensions) based on SU(5) F-theory GUTs with additional U(1) symmetries emerging from the constructions with rational sections discussed in section 5.4.2. In this spirit, we review the results from previous model building attempts in the context of the spectral cover. Next we discuss phenomenological constraints we implement along our search. After that we present the results of our exploration. Finally we apply the same criteria in order to find bottom-up models which share the same overall features regarding the U(1) charges but go beyond the explicit rational section models considered to date.

6.3.1 Status of spectral cover model building

As mentioned above the spectral cover constructions yield an explicit framework for realistic model building attempts within F-theory [136, 145, 152–155]. However, when building GUT models within F-theory one has to deal with similar problems as those faced in standard SUSY GUTs. First, one needs to ensure that the triplets accompanying the Higgs multiplets are decoupled from the low energy theory. This is achieved by breaking the GUT group via hypercharge flux; with this flux it is possible to project out the triplets in the Higgs multiplets. Second, one needs to guarantee that all couplings are well under control, such that, for example dangerous operators which mediate fast proton decay are sufficiently suppressed. For this purpose additional U(1) symmetries are used as they appear naturally in this framework.

Let us start the discussion on F-theory model building by introducing the MSSM superpotential at the level of SU(5)

$$\begin{aligned}\mathcal{W} = \; & \mu \mathbf{5}_{H_u}\overline{\mathbf{5}}_{H_d} + \beta_i \overline{\mathbf{5}}_i \mathbf{5}_{H_u} \\ & + Y^u_{ij}\mathbf{10}_i\mathbf{10}_j\mathbf{5}_{H_u} + Y^d_{ij}\overline{\mathbf{5}}_i\mathbf{10}_j\overline{\mathbf{5}}_{H_d} + W_{ij}\overline{\mathbf{5}}_i\overline{\mathbf{5}}_j\mathbf{5}_{H_u}\mathbf{5}_{H_u} \\ & + \lambda_{ijk}\overline{\mathbf{5}}_i\overline{\mathbf{5}}_j\mathbf{10}_k + \delta_{ijk}\mathbf{10}_i\mathbf{10}_j\mathbf{10}_k\overline{\mathbf{5}}_{H_d} + \gamma_i\overline{\mathbf{5}}_i\overline{\mathbf{5}}_{H_d}\mathbf{5}_{H_u}\mathbf{5}_{H_u} \\ & + \omega_{ijkl}\mathbf{10}_i\mathbf{10}_j\mathbf{10}_k\overline{\mathbf{5}}_l \,,\end{aligned} \tag{6.29}$$

where we have included operators up to dimension five. The representations $\overline{\mathbf{5}}_i$ and $\mathbf{10}_i$ correspond to the i-th family and $\mathbf{5}_{H_u}$, $\overline{\mathbf{5}}_{H_d}$ are the SU(5) multiplets giving rise to the up- and down-type Higgs respectively. The operators in the first line of (6.29) are those leading to the μ-term and the bilinears between H_u and the lepton doublets. In the second line we have the Yukawa couplings $Y^{u,d}_{ij}$ and the Weinberg operator W_{ij}. The operators given in the third line violate baryon and lepton number and are forbidden in models with matter parity. At dimension five one encounters another proton decay operator which is allowed by matter parity. One also has dangerous proton decay operators arising from the Kähler potential [136, 156], namely

$$\mathcal{K} \supset \kappa_{ijk}\mathbf{10}_i\mathbf{10}_j\mathbf{5}_k + \overline{\kappa}_i\overline{\mathbf{5}}_{H_u}\overline{\mathbf{5}}_{H_d}\mathbf{10}_i \,. \tag{6.30}$$

In engineering semi-realistic models from F-theory, one expects the U(1) symmetries to provide sufficient suppression for the dangerous operators present in (6.29) and (6.30). Similarly, one expects a small μ-term, so that it is desirable if if is forbidden by the U(1) s and only generated upon their breaking. As mentioned already the U(1) symmetries from the spectral cover are typically GS massive, but remain as global symmetries in the effective field theory. These global factors can be broken for example by singlet VEVs or instanton effects [157].

Previous searches for models in the spectral cover constructions were based on the idea that the three families arise from complete SU(5) representations (i.e. from curves on which the hypercharge flux acts trivially). In addition to that one has two extra **5**-curves on which the hypercharge flux acts by projecting out the triplet components so that one ends up with only one pair of doublets (namely the Higgses). In these models, the breakdown of the U(1) s has been used generate the flavor structure in the quark and lepton sector [136, 148]. The relevant couplings are generated by singlet VEVs while keeping the dangerous operators under control. This follows the spirit of the Froggatt-Nielsen mechanism [158].

After choosing the factorization, the matter representations with their corresponding U(1) charges are fixed, and the model building is based on tuning the flux quanta carried by the different curves. This is not arbitrary and one has to ensure that the anomaly cancelation conditions (6.26)-(6.27) are satisfied. It turns out that these restrictions are far from trivial and severely constrain the possibilities for promising models. Previous explorations have lead to the following two observations:

- If one insists on the exact MSSM spectrum, there is only one flavor-blind U(1) symmetry available. This symmetry corresponds to a linear combination of hypercharge and $\mathrm{U}(1)_{B-L}$. As it allows for a μ-term as well as dimension five proton decay [148], this construction is not suitable for phenomenology.

- If one requires the presence of a U(1) symmetry which explicitly forbids the μ-term, i.e. a so-called Peccei–Quinn (PQ) symmetry, this implies the existence of exotic fields which are vector-like under the Standard Model gauge group and come as incomplete representations of the underlying SU(5) [147].

These observations suggest a strong tension between a solution to the μ-problem and the absence of light exotics in the spectrum. As already mentioned, in these models the matter arise from SU(5) representations which are not split by hypercharge flux. Here we explore whether in models with split multiplets these tensions can be avoided and whether, in principle, they permit us to obtain an exotic free spectrum with appealing operator structure.

As a final remark let us comment that in these models, the anomalies of the type $\mathrm{U}(1)_Y - \mathrm{U}(1)_A - \mathrm{U}(1)_B$ do not comply with condition (6.28). Thus, given the deep physical meaning of this condition, the models obtained so far are consistent only if there is an F-theoretical mechanism to cancel the anomalies.

6.3.2 Search strategy

The first requirement of our search is that the models have the MSSM matter content (i.e. three families of quarks and leptons and one pair of Higgses) up to possible SM singlets and the extra U(1) gauge bosons. Thus, from the beginning we impose the absence of extra states charged under the Standard Model gauge group. We also require the spectrum to satisfy the four-dimensional anomaly cancelation conditions (6.26)-(6.28) such that the masslessness of the hypercharge is guaranteed.

After fixing the matter curves and their U(1) charges from a any of the models given in table (6.3), the only freedom left is to switch on the fluxes such that the desired MSSM content is obtained and the four-dimensional anomaly cancelation conditions are satisfied. Using the notation of section 6.2, we obtain the following requirements for the flux choices:
Requiring **three chiral families** imposes that the chirality flux has to satisfy

$$\sum_{\Sigma_{\mathbf{10}}} M^a = \sum_{\Sigma_{\overline{\mathbf{5}}}} M^i = 3 \quad \text{with} \quad M^a\,,\, M^i \geq 0\,. \tag{6.31}$$

This relation guarantees that the first anomaly constraint from (6.26) is satisfied. We demand that in addition to the MSSM matter content, **no exotics** are present in the spectrum (with the exception of singlet fields neutral under the SM gauge group). This requirement constrains the quanta of hypercharge flux to obey the following relations

$$\sum_{\Sigma_{\mathbf{10}}} N^a = 0 \quad \text{with} \quad -M^a \leq N^a \leq M^a\,, \tag{6.32}$$

$$\sum_{\Sigma_{\overline{\mathbf{5}}}} N^i = 0 \quad \text{with} \quad -M^i - 1 \leq N^i \leq 3\,. \tag{6.33}$$

Note that this flux configuration automatically satisfies the condition (6.26). The additional flux constraint on the **5**-curves

$$\sum_{\Sigma_{\overline{\mathbf{5}}}} |M^i + N^i| = 5\,, \tag{6.34}$$

guarantees three lepton doublets together with *exactly one pair of Higgses*. For a configuration with such a spectrum, one also has to check whether the flux choices satisfy the additional constraints (6.27) and (6.28).from anomaly cancelation.

We also have to constrain the operators of the effective field theory in order to obtain a realistic model. As the charges for the fields are given ab initio, we have to ensure that dangerous operators are not generated at tree level, so that the U(1) charges for these operators have to be non-zero. Since we are not dealing with complete SU(5) multiplets, it is necessary to decompose the SU(5) couplings (6.29) and (6.30) in terms of the SM fields. In order to have a heavy top quark we demand the presence of the top Yukawa coupling at tree level, i.e. we require that at least one of the up-type Yukawa couplings

$$\mathbf{10}_i\mathbf{10}_j\mathbf{5}_{H_u} \supset Q_i\bar{u}_jH_u \tag{6.35}$$

is allowed by all U(1) symmetries. In the case where all Q_i and $\bar{u}_j$ descend from only one **10**-curve the up-quark Yukawa matrix is of rank one. Nevertheless, this matrix can acquire full rank when appropriate flux or non-commutative deformations [133, 159–164] away from the E_6 Yukawa point are included.

In order for low energy SUSY to *solve the μ-problem*, we require the μ-term

$$\mu\mathbf{5}_{H_u}\overline{\mathbf{5}}_{H_d} \supset \mu H_dH_u \tag{6.36}$$

to be forbidden by any of the U(1) symmetries. The above coupling will be generated upon breakdown of these symmetries. Let us also remark that in addition to the expected suppressions in the couplings due to singlet VEVs (or instantons) some additional suppression is expected when the couplings arise from the Kähler potential. This is fact is particularly appealing for the generation of the μ-term as it can be sufficiently small, if induced from the Kähler potential along the lines of the so-called Giudice-Masiero (GM) mechanism [165].

The μ-term is closely linked to the presence of dimension five B-L invariant operators

$$\omega_{ijkl}\mathbf{10}_i\mathbf{10}_j\mathbf{10}_k\overline{\mathbf{5}}_l \supset \omega^1_{ijkl}Q_iQ_jQ_kL_l + \omega^2_{ijkl}\bar{u}_i\bar{u}_j\bar{e}_k\bar{d}_l + \omega^3_{ijkl}Q_i\bar{u}_j\bar{e}_kL_l\,. \tag{6.37}$$

We demand these operators to be forbidden by the U(1) charges, keeping in mind that they can be generated in a similar fashion as the μ-term.

In order to *avoid fast proton decay*, the U(1) symmetries must also forbid the following superpotential and Kähler potential couplings:

$$
\begin{aligned}
&\beta_i \quad \overline{\mathbf{5}}_i \mathbf{5}_{H_u} && \supset \beta_i L_i H_u \,, \\
&\lambda_{ijk} \overline{\mathbf{5}}_i \overline{\mathbf{5}}_j \mathbf{10}_k && \supset \lambda^0_{ijk} L_i L_j \bar{e}_k + \lambda^1_{ijk} \bar{d}_i L_j Q_k + \lambda^2_{ijk} \bar{d}_i \bar{d}_j \bar{u}_k \,, \\
&\delta_{ijk}\, \mathbf{10}_i \mathbf{10}_j \mathbf{10}_k \overline{\mathbf{5}}_{H_d} && \supset \delta^1_{ijk} Q_i Q_j Q_k H_d + \delta^1_{ijk} Q_i \bar{u}_j \bar{e}_k H_d \,, \\
&\gamma_i \quad \overline{\mathbf{5}}_i \overline{\mathbf{5}}_{H_d} \mathbf{5}_{H_u} \mathbf{5}_{H_u} && \supset \gamma_i L_i H_d H_u H_u \,, \\
&\kappa_{ijk} \mathbf{10}_i \mathbf{10}_j \mathbf{5}_k && \supset \kappa^1_{ijk} Q_i \bar{u}_j \bar{L}_k + \kappa^2_{ijk} \bar{e}_i \bar{u}_j d_k + \kappa^3_{ijk} Q_i Q_j d_k \,, \\
&\overline{\kappa}_i \quad \overline{\mathbf{5}}_{H_u} \overline{\mathbf{5}}_{H_d} \mathbf{10}_i && \supset \overline{\kappa}^1_i H^*_u H_d \bar{e}_i \,.
\end{aligned}
\tag{6.38}
$$

For a consistent model we need to require that upon breakdown of the U(1) symmetries these operators are not generated. This is for example achieved by demanding the presence of an effective matter parity symmetry.

We also expect that while the operators in (6.38) remain absent, it is possible to generate *full rank Yukawa matrices*[3] Y^u_{ij}, Y^d_{ij} and Y^L_{ij}. This necessarily implies that the charges of the desired operators must differ in comparison to the undesired ones. As an immediate consequence of this, the field H_d has to come from a different curve than all the other leptons and triplets to guarantee that dimension four operators (such as λ^0 and λ^1 in (6.38)) are not introduced together with the Yukawa entries.

6.3.3 U(1) Charge pattern

As we observe from the models discussed in section 6.1.1, all $\mathbf{10}$ matter curves carry the same U(1) charges[4]. Hence we cannot put hypercharge flux along those. For that reason, we suppress the family indices of their corresponding Standard Model representations. This restricted setup also allows us to find some analytic relations between the U(1) charges of certain operators which substantially influence the phenomenological properties of the models. First of all, the presence of a tree level top Yukawa fixes the H_u charge to be

$$q(H_u) = -2q(\mathbf{10})\,. \tag{6.39}$$

[3] Recall that, as the matter fields need not to arise from complete SU(5) multiplets, so that the down and lepton Yukawas do not necessarily coincide.

[4] Note that this restriction is true for all models considered here but is not a generic feature of models with extra section. The reason for which we have a single $\mathbf{10}$-plet is due to the toric treatment of the fiber and its resolution, as we already stressed in section 6.1.1.

For the subsequent discussion, we introduce the following notation for the charges of the operators:

$$
\begin{aligned}
\mu \;&:\; q(H_d H_u) \;= q(H_d) + q(H_u) := q^{\mu}\,, \\
Y^L \;&:\; q(\bar{e} H_d L_i) := q^{Y_i^L}\,, \\
Y^d \;&:\; q(\bar{u} H_d \bar{d}_i) \;:= q^{Y_i^d}\,, \\
\beta_i \;&:\; q(L_i H_u) \;\; = q(L_i) + q(H_u) := q^{\beta_j}\,.
\end{aligned}
\tag{6.40}
$$

Among these, all but the β_i terms should be induced upon breakdown of the U(1) symmetries.

Now we can express the charges of all unwanted operators in terms of the charges defined above. The dangerous dimension four proton decay operators are:

$$
\begin{aligned}
\lambda^0_{ij} \;&:\; q(Q\bar{d}_i L_j) = q^{Y_i^d} + q(H_d) - q(L_j) = q^{Y_i^d} + q^{\mu} - q^{\beta_j} \\
\lambda^1_{ij} \;&:\; q(\bar{e} L_i L_j) \;= q^{Y_i^L} + q(H_d) - q(L_j) = q^{Y_i^L} + q^{\mu} - q^{\beta_j} \\
\lambda^2_{ij} \;&:\; q(\bar{u}\bar{d}_i\bar{d}_j) \;= q^{Y_i^d} + q(H_d) - q(\bar{d}_j) \;= q^{Y_i^d} + q^{\mu} - q(H_u) - q(\bar{d}_j)\,.
\end{aligned}
\tag{6.41}
$$

Since we want to generate the down-type Yukawa matrices, we see that the previous couplings are only forbidden due to the charge difference between the H_d- and L_j-curves in the case of the λ^0_{ij} and λ^1_{ij} couplings, and due to the charge difference between H_d- and $\bar{d}_j$-curves in the case of the λ^2_{ij}. Thus, as already pointed out, it is necessary that the $\mathbf{5}_{H_d}$-curve contains only the down-type Higgs, since any lepton or down-type quark with identical charge will automatically induce a dangerous operator. As we also want to obtain a μ-term, no $\bar{d}_i$ field can arise from the H_u-curve either.[5]

Overall, note that the charges can also be written in terms of those of the forbidden operators β_i. Thus, if we find a configuration such that the Yukawa couplings and the μ-term is induced but the β_i-terms stay forbidden, the dimension four operators stay forbidden as well. Furthermore, we observe that the dimension five operators in the superpotential

$$
\begin{aligned}
\omega^1_i\,,\omega^3_i \;&:\; q(QQQL_i) = q(Q\bar{u}\bar{e}L_i) = -q^{\mu} + q^{Y_i^L} := q(\mathbf{10}\,\mathbf{10}\,\mathbf{10}\,L_i)\,, \\
\omega^2_i \;&:\; q(QQ\bar{u}\bar{d}_i) \;=\; q(\bar{u}\bar{u}\bar{e}\bar{d}_i) \;=-q^{\mu} + q^{Y_i^d} := q(\mathbf{10}\,\mathbf{10}\,\mathbf{10}\,\bar{d}_i)\,,
\end{aligned}
\tag{6.42}
$$

will be unavoidably induced together with the Yukawa couplings and the μ-term. It should be noted that the μ-term charge enters with a minus sign in the previous equations. This implies that

[5] Note that if H_u and $\bar{d}_j$ come from the same curve their U(1) charges carry opposite signs.

the mechanism (such as a singlet VEV) which induces the ω^i-terms will not induce the μ-term directly in the superpotential but can generate it from the Kähler potential. Note also that it is possible to induce a Weinberg operator

$$W_{ij}: \quad q(L_i L_j H_u H_u) = q^{\beta_i} + q^{\beta_j} \tag{6.43}$$

without inducing the β_i-terms by using, for example, singlet VEVs with charge $q(s_i) = -2q^{\beta_i}$. In a similar fashion, we observe that the operators

$$\begin{aligned} \delta^1\,,\delta^2: \quad q(QQQH_d) &= q(Q\bar{u}\bar{e}H_d) = -q^{\beta_i} + q^{Y_i^L}\,, \\ \gamma_i: \; q(L_i H_d H_u H_u) &= q^{\mu} + q^{\beta_i}\,, \end{aligned} \tag{6.44}$$

will remain absent as long as the β_i-terms are not induced. The same holds for the Kähler potential terms

$$\begin{aligned} \kappa_i^1: \quad q(Q\bar{u}L_i^*) &= -q^{\beta_i}\,, \\ \overline{\kappa}: \; q(\bar{e}H_u^* H_d) &= q^{\mu} + q^{\beta_i}\,, \end{aligned} \tag{6.45}$$

with the exception of

$$\kappa_i^2\,,\kappa_i^3: \quad q(\mathbf{10}\,\mathbf{10}\,\bar{d}_i^*) = -q(H_u) - q(\bar{d}_j) = -q^{\mu} + q(H_d) - q(\bar{d}_i)\,, \tag{6.46}$$

were we have defined $q(\mathbf{10}\,\mathbf{10}\,\bar{d}_i^*) := q(QQ\bar{d}_i^*) = q(\bar{u}\bar{e}\bar{d}_i^*)$. For this kind of couplings one has to ensure that no triplets emerge from the Higgs curves as a necessary (but not sufficient) condition.

Note that the above observations are independent of the number of $\mathbf{5}$-curves and U(1) symmetries. However, there remains a crucial interplay between the Higgs charges compared to those of the down-type quarks and those of the singlet fields, which have to be checked on a case by case analysis.

6.3.4 Results of the scan

Based on the previous considerations, we have scanned over the models. The result is that there is no solution which satisfies all anomaly conditions (even in the cases where one allows for exotic matter). Since this is similar to what occurs in the spectral cover, we could assume that there is a mechanism which lifts the anomaly (6.28), which is the critical one. From only demanding eqs. (6.26) and (6.27) we find four models which met all of our criteria. These are based on tops $\tau_{5,1}$ and $\tau_{5,2}$. They exhibit very similar features, so that a benchmark model suffices for the discussion of the phenomenology. In table 6.4 we have summarized the spectrum, there we have also included the singlet fields needed to generate the desired couplings. We also give the charges of those dangerous operators which are not automatically absent as long as the β_i-terms are forbidden. For this model we see that a singlet s_1 with charge $(0,5)$ that develops a VEV will generate a μ-term from the Kähler potential and induce the Yukawa couplings and dimension five operators in the superpotential, while all dimension four operators stay forbidden. The orders of magnitude for these couplings are

$$\begin{aligned} Y_i^L &\sim \frac{\langle s_1\rangle}{\Lambda}\,, & Y_i^d &\sim \delta_{1,i} + \frac{\langle s_1\rangle}{\Lambda}(\delta_{2,i}+\delta_{3,i})\,, \\ \omega_i^1\,,\omega_i^3 &\sim \frac{\langle s_1\rangle^2}{\Lambda^3}\,, & \omega_i^2 &\sim \frac{\langle s_1\rangle}{\Lambda^2}\delta_{1,i} + \frac{\langle s_1\rangle^2}{\Lambda^3}(\delta_{2,i}+\delta_{3,i})\,, \end{aligned} \tag{6.47}$$

where Λ is the appropriate cutoff scale, which depends on the global embedding of the local model. In the absence of dimension four proton decay operators, the bounds on dimension five couplings are [166, 167]

$$\omega^1 \lesssim \frac{10^{-7}}{M_{\mathrm{P}}} \text{ and } \omega^2 \lesssim \frac{10^{-7}}{M_{\mathrm{P}}}\,. \tag{6.48}$$

The coupling ω^2 is only to be taken seriously if there is a non-diagonal degeneracy of quark and squark masses. The ω^1-operator, however, puts constraints on the size of the VEV singlet s_1 that induces the operator after two singlet insertions, i.e.

$$\frac{\langle s_1\rangle^2}{\Lambda^3} \lesssim \frac{10^{-7}}{M_{\mathrm{P}}}\,. \tag{6.49}$$

Such a size of the VEV seems compatible with down-quark and lepton-Yukawa couplings at the weak scale [168].

As mentioned before, it is possible to generate the Weinberg operator while keeping dangerous couplings absent. In the singlet spectrum of the dP_2 fiber (see table 6.1) we see that there is no singlet with charge $(\pm 10, 0)$, whose VEV can introduce that operator. However, one might envisage a non-perturbative effect (e.g. via instantons) which generates this coupling. Note again that for our purposes, such an instanton has the same effect of a singlet VEV and we will para-

1. Spectrum

Curve	q_1	q_2	M	N	Matter
$\mathbf{10}$	-1	2	3	0	$(Q+\bar{u}+\bar{e})_{1,2,3}$
$\overline{\mathbf{5}}_1$	3	-1	1	-1	$\bar{d}_1$
$\overline{\mathbf{5}}_2$	-2	4	0	-1	H_u
$\overline{\mathbf{5}}_4$	3	4	2	1	$L_{1,2,3}+\bar{d}_{2,3}$
$\overline{\mathbf{5}}_5$	-2	-1	0	1	H_d

2. Singlet VEVs: s_1 , a

$$q(s_1)=(0,5)\,,\quad q(a)=(10,0)\,.$$

3. μ- and β_i-terms

$$q(H_u\bar{L}_i)=(5,0)\,,\quad q(H_uH_d)=(0,-5)\,.$$

4. Yukawa couplings

$$q(Q_i\bar{u}_jH_u)=(0,0)\,,\qquad q(Q_i\bar{d}_jH_d)=\begin{pmatrix}(0,0)\\(0,5)\\(0,5)\end{pmatrix}_j\,,\qquad q(\bar{e}_iL_jH_d)=(0,5)\,.$$

5. Allowed dimension five proton decay and Weinberg operators

$$q(\mathbf{10}\,\mathbf{10}\,\mathbf{10}\,L_i)=(0,10)\,,\qquad q(\mathbf{10}\,\mathbf{10}\,\mathbf{10}\,\bar{d}_i)=\begin{pmatrix}(0,5)\\(0,10)\\(0,10)\end{pmatrix}_j\,,\qquad q(L_i\,L_j\,H_u\,H_u)=(10,0)\,.$$

6. Forbidden operators

$$q(\bar{u}\bar{d}_i\bar{d}_j)=\begin{pmatrix}(5,0)&(5,0)&(5,0)\\(5,0)&(5,10)&(5,10)\\(5,0)&(5,10)&(5,10)\end{pmatrix}_{i,j}\,,\qquad q(\mathbf{10}\,\mathbf{10}\,\bar{d}_i^*)=\begin{pmatrix}(-5,5)\\(-5,0)\\(-5,0)\end{pmatrix}\,.$$

Table 6.4: Details of benchmark model A. We give the charges for the operators λ^2_{ij} and κ^3_{ij}, κ^2_{ij} discussed in (6.41) and (6.46).

	$(Q+\bar{u}+\bar{e})_{1,2,3}$	$\bar{d}_1$	H_u	$L_{1,2,3}+\bar{d}_{2,3}$	H_d	s_1	a
q_1	-1	3	2	3	-2	0	-10
q_2	2	-1	-4	4	-1	-5	0
$q_1'=(q_1-2q_2)/5$	-1	1	2	-1	0	2	-2
$q_2'=(2q_1+q_1)/5$	0	1	0	2	-1	-1	-4

Table 6.5: The U(1) charges for the benchmark model in a rotated U(1) basis. Note that after giving VEVs, the charges q_1' are those of matter parity, whereas the charges q_2' become all trivial since s_1 has charge -1.

meterize it by $\langle a\rangle$ such that the operator is introduced by[6]

$$W_{ij} \sim \frac{\langle a\rangle}{\Lambda^2}\,. \tag{6.50}$$

Another interesting question is which symmetries remain after the U(1) symmetries are broken. For this purpose it is more convenient to rotate the U(1) generators as specified in table 6.5. There we see that after appropriate normalization, the charges of the singlets break the U(1) symmetries to a $\mathbb{Z}_2$ subgroup under which the charges q_1' coincide with those of matter parity. On the other hand, we see that the second U(1) with charges q_2' gets broken completely by the VEV of s_1.

6.3.5 Beyond available constructions

In the previous section we only considered the subclass of SU(5) tops which have been studied in the literature so far. For example, the polytopes F_7, F_9 and F_{12} still remain to be analyzed. There are many possibilities left to exhaust: On the one hand we can consider top models from fibers in ambient spaces different than dP_2. On the other hand, special choices of the base, may lead to additional sections with any of the fibers considered already. However, in our exploration we have found a similar problem as in the spectral cover, namely, no realistic model allows for anomalies of the type $\mathrm{U}(1)_Y-\mathrm{U}(1)_A-\mathrm{U}(1)_B$ to be cancelled. This problem could be pathological to a broader class of constructions. Motivated by this problem we want to explore possible bottom-up models in which all anomalies vanish. The aim is to provide a guideline for future geometric engineering efforts and it would be very interesting to obtain explicit realizations of the promising models. Based on the observation that the U(1) charges are all fixed (up to multiples of five) to a certain value given by the corresponding splitting, we consider the possibility of having further models with two U(1) symmetries whose splitting gives rise to any of those charge

[6] Note that the expected order of magnitude for this operator is the same for all generations as all lepton doublets in this model are found to arise from the same matter curve.

1. Spectrum

Curve	q_1	q_2	M	N	Matter
$\mathbf{10}$	-3	-1	3	0	$(Q+\bar{u}+\bar{e})_{1,2,3}$
$\overline{\mathbf{5}}_1$	9	-2	0	1	L_1
$\overline{\mathbf{5}}_2$	9	-7	1	-1	$\bar{d}_1$
$\overline{\mathbf{5}}_3$	-1	8	2	-1	$L_2+\bar{d}_{1,2}$
$\overline{\mathbf{5}}_4$	-1	-7	0	1	L_3
$\overline{\mathbf{5}}_5$	-6	8	0	1	H_d
$\overline{\mathbf{5}}_6$	-6	-2	0	-1	H_u

2. Singlet VEVs: s_1 , s_2

$$q(s_1)=(0,\pm 5)\,,\quad q(s_2)=(\pm 10,0)\,.$$

3. μ- and β_i-terms

$$q(H_uL_i)=\begin{pmatrix}(15,0)\\(5,10)\\(5,-5)\end{pmatrix}\,,\quad q(H_uH_d)=(0,10)\ .$$

4. Yukawa couplings

$$q(Q\bar{u}H_u)=(0,0)\,,\qquad q(Q\bar{d}_jH_d)=\begin{pmatrix}(0,0)\\(-10,15)\\(-10,15)\end{pmatrix}\,,\qquad q(\bar{e}L_jH_d)=\begin{pmatrix}(0,5)\\(-10,15)\\(-10,0)\end{pmatrix}\,.$$

5. Allowed dimension five proton decay and Weinberg operators

$$q(\mathbf{10}\,\mathbf{10}\,\mathbf{10}\,L_i)=\begin{pmatrix}(0,-5)\\(-10,5)\\(-10,-10)\end{pmatrix}\,,\quad q(\mathbf{10}\,\mathbf{10}\,\mathbf{10}\,\bar{d}_i)=\begin{pmatrix}(0,-10)\\(-10,5)\\(-10,5)\end{pmatrix}\,,$$

$$q(L_i\,L_j\,H_u\,H_u)=\begin{pmatrix}(30,0)&(20,10)&(20,-5)\\(20,10)&(10,20)&(10,5)\\(20,-5)&(10,5)&(10,-10)\end{pmatrix}\,.$$

6. Forbidden operators

$$q(\bar{u}\bar{d}_i\bar{d}_j)=\begin{pmatrix}(15,-15)&(5,0)&(5,0)\\(5,0)&(-5,15)&(-5,15)\\(5,0)&(-5,15)&(-5,15)\end{pmatrix}\,,\qquad q(\mathbf{10}\mathbf{10}\bar{d}_i^*)=\begin{pmatrix}(-15,5)\\(-5,-10)\\(-5,-10)\end{pmatrix}\,.$$

Table 6.6: Details of a benchmark model beyond the toric sections. We give the charges for the operators λ^2_{ij} and κ^3_{ij}, κ^2_{ij} discussed in (6.41) and (6.46).

assignments in table 6.2. Furthermore, we assume that such models allow for more $\mathbf{5}$-curves, but only one $\mathbf{10}$-curve as before. These additional $\mathbf{5}$-curves allow for more freedom to satisfy the anomaly constraints. The $\mathbf{5}$-curves are chosen to have charges

$$q_{1,\overline{\mathbf{5}}} = Q_{1,\overline{\mathbf{5}}} + 5\,n_{1,i}\,, \quad q_{2,\overline{\mathbf{5}}} = Q_{2,\overline{\mathbf{5}}} + 5\,n_{2,i}\,, \tag{6.51}$$

where $Q_{1,\overline{\mathbf{5}}}$ and $Q_{2,\overline{\mathbf{5}}}$ are fixed by the splitting that is chosen for each U(1). The integer valued $n_{1,i}, n_{2,i}$ are in the range

$$-2 \leq n_{1,i}\,, n_{2,i} \leq 2\,, \tag{6.52}$$

as these seem to be realistic intersection numbers for the sections with the irreducible fiber components. The charge of the $\mathbf{10}$-curve is chosen such that it fits the structure of the chosen split, see Table 6.2. The flux distribution and the search strategy follow as described in Section 6.3.2.

For example, in models where the U(1) generators follow the 4-1, 3-2 splitting, we find $\mathcal{O}(10^3)$ models which satisfy all anomaly conditions, in particular also (6.28), and have all unwanted operators forbidden at tree level. Out of those, some are found to allow for suitable U(1) charges which lead to the desired operator structure. An example is the model given in Table 6.6. Note that the analysis of the previous section still applies, since all $\mathbf{10}$-curves have the same charge. In this model, two singlets (or correspondingly appropriate instanton effects), denoted by s_1 and s_2, could generate the μ-term and all Yukawa couplings. The charges for these fields have to be

$$q(s_1) = (0, 5)\,, \quad q(s_2) = (10, 0)\,, \tag{6.53}$$

which is similar to charges appearing in the benchmark models of the previous section. One slight difference is that one expects a higher suppression for the μ-term since it is generated after two singlet insertions. Note that again we need a singlet with charge 10 in the first U(1) to generate the desired coupling structure. Similarly as before, we observe that in this model, the U(1) symmetries are broken in such a way that there is a surviving matter parity.

6.4 Anomaly cancelation revisited

Having observed that the issue of anomaly cancellation is not entirely settled in F-theory model building. Let us discuss a proposal to cancel the anomaly (6.28), which is inspired by type IIB setups with orientifolds and D7-branes [151]. The first remark to make is that orientifolds split the cohomologies of the internal space $\mathcal{M}_6$ into their orientifold odd and even parts. Thus, the decomposition of the RR potentials C_{2p} is given by

$$C_6 = c_a^2 \wedge \tilde{w}^a + \dots \,, \tag{6.54}$$

$$C_4 = c_2^\alpha \wedge w_\alpha + c_\alpha^0 \tilde{w}^\alpha + \dots \,, \tag{6.55}$$

$$C_2 = c_0^a w_a + \dots \,, \tag{6.56}$$

with $\tilde{w}^a \in H_-^4(\mathcal{M}_6)$, $\tilde{w}^\alpha \in H_+^4(\mathcal{M}_6)$, $w_a \in H_-^2(\mathcal{M}_6)$ und $w_\alpha \in H_+^2(\mathcal{M}_6)$, where we have used the subindices $+$ and $-$ to denote orientifold even and odd cohomologies. Recall that in four dimensions, the forms c_α^2 and c_0^α are dual to each other (the same holds true for c_a^2 and c_0^a). They are interpreted as the orientifold even (odd) axions of the theory. In the following we assume that a D7-brane stack wraps a certain cycle S in internal space. We denote by G the gauge symmetry associated to this stack and consider the presence of an internal flux f. The terms which are relevant for anomaly cancellation can be found from expanding the D7 brane action, these read

$$S_1 = \mathrm{tr}(T_I) C_S^a \int_{\mathbb{R}^{3,1}} F_I \wedge c_a^2 \,, \tag{6.57}$$

$$S_2 = \int_{\mathbb{R}^{3,1}} \mathrm{tr}(T_I T_J) F_I \wedge c_2^\alpha \int_S f_J \wedge \iota^* w_\alpha \,, \tag{6.58}$$

$$S_3 = \int_{\mathbb{R}^{3,1}} \mathrm{tr}(T_I T_J T_K) c_a^0 F_I \wedge F_J \int_S f_K \wedge \iota^* w_a \tag{6.59}$$

and

$$S_4 = \mathrm{tr}(T_I T_J) C_S^\alpha c_\alpha^0 F_I \wedge F_J \,. \tag{6.60}$$

where the T_I are the generators of the Lie algebra associated to G, similarly f_I are the components of the flux along the generators T_I, and $\iota^* w$ denotes the pullback of w to the divisor S. Similarly the coefficients C_S^a and C_S^α can be obtained from the decomposition

$$[S] = C_S^a w_a + C_S^\alpha w_\alpha \,, \tag{6.61}$$

where $[S]$ is the two form dual to S. Note firstly that conditions (6.57) and (6.58) will induce a mass for the gauge fields A_I (that whose field strength is F_I). In the first case, the field is lifted even without flux, in that sense, the gauge field acquires its mass thanks to a geometrical mechanism. Furthermore, thanks to the duality between zero- and two-forms, eqs. (6.57) to

(6.60) are the building blocks for the anomaly cancelling terms involved in the Green-Schwarz mechanism. We want to use these results to investigate how hypercharge flux modifies anomaly relations. The flux of our interest is of the form $f_Y T_Y$. The first observation comes from eq. (6.58): In order for U(1)$_Y$ to remain massless, the following condition must hold

$$\int_S f_Y \wedge \iota^* w_\alpha = 0\,, \quad \forall\, w_\alpha \in H^2_+(\mathcal{M}_6)\,. \tag{6.62}$$

This is equivalent to the more familiar observation that hypercharge flux must be tuned on a trivial cycle in $\mathcal{M}_6$ which is non-trivial in S. Thus, if hypercharge is to modify any kind of anomaly this must occur via the orientifold odd sector.

Let us now discuss anomalies of the type U(1)$_Y$ $-$ U(1)$_A$ $-$ U(1)$_B$, from eqs. (6.57) and (6.59), we observe that the GS canceling term will be proportional to

$$\rho^Y_A \mathrm{tr}(T_B T_Y^2) + \rho^Y_B \mathrm{tr}(T_A T_Y^2)\,, \tag{6.63}$$

where $\rho^Y_{A,B}$ have been defined as

$$\rho^Y_{A,B} = \phi_a C^a_S \mathrm{tr}(T_{A,B}) \int_S f_Y \wedge \iota^* w_a\,, \tag{6.64}$$

with ϕ_a parameterizing the axion shifts responsible for the cancelation of the anomalies. Note that the contribution (6.63) is non vanishing only if U(1)$_A$ or U(1)$_B$ are geometrically massive. Similar terms can be found for the anomalies U(1)$_Y$ $-$ U(1)^2_A and U(1)$_Y$ $-$ U(1)^2_B. Even though the uplift of the GS mechanism in not known yet, the previous result implies the possibility that eq. (6.28) does not hold in F-theory. However, let us recall that in order for the mechanism to work, some of the U(1) symmetries must be geometrically massive. This could occur in the spectral cover, but not in the more global constructions with rational sections where the U(1) s are massless by construction.

One step further we can consider anomalies of the type G^2_{SM} $-$ U(1)$_A$, we also see that these also get modified by the hypercharge flux, i.e.

$$\phi_a \rho^Y_{A,B} \mathrm{tr}(T_I^2 T_Y)\,, \tag{6.65}$$

where T_I is a generator in SU(3), SU(2), or U(1)$_Y$.

With this observations at hand we can compute the anomaly coefficients in our F-theory models. This we can do using the multiplicities discussed in section 6.2. As an exercise, we can start with the pure SM anomalies, these read

$$\mathrm{SU}(3)^2 - \mathrm{U}(1)_Y : \quad \sum_a \left[2 \cdot \frac{1}{6} M_a - \frac{2}{3}(M_a - N_a)\right] + \frac{1}{3} \sum_i M_i = 0 \,, \tag{6.66}$$

$$\mathrm{SU}(2)^2 - \mathrm{U}(1)_Y : \quad 3 \cdot \frac{1}{6} \sum_a M_a - \frac{1}{2} \sum_i (M_i + N_i) = 0 \,, \tag{6.67}$$

$$\begin{aligned} \mathrm{U}(1)_Y^3 : \quad & \sum_a \left[6 \cdot \left(\frac{1}{6}\right)^3 M_a + 3 \cdot \left(-\frac{2}{3}\right)^3 (M_a - N_a) + (M_a + N_a)\right] \\ & + \sum_i \left[3 \cdot \left(\frac{1}{3}\right)^3 M_i + 2 \cdot \left(-\frac{1}{2}\right)^3 (M_i + N_i)\right] = 0 \,, \end{aligned} \tag{6.68}$$

where we have omitted some overall normalization factors for simplicity. One can show that these equations will reduce to the Dudas-Palti relation (6.26).
For anomalies of the type $G_{\mathrm{SM}}^2 - \mathrm{U}(1)_A$ we obtain

$$\mathrm{SU}(3)^2 - \mathrm{U}(1)_A : \quad \sum_a q_a^A [2M_a + (M_a - N_a)] + \sum_i q_i^A M_i = \rho + \frac{1}{3}\rho_A^Y \,, \tag{6.69}$$

$$\mathrm{SU}(2)^2 - \mathrm{U}(1)_A : \quad 3 \sum_a q_a^A M_a + \sum_i q_i^A (M_i + N_i) = \rho - \frac{1}{2}\rho_A^Y \,, \tag{6.70}$$

$$\begin{aligned} \mathrm{U}(1)_Y^2 - \mathrm{U}(1)_A : \quad & \sum_a q_a^A \left[6 \cdot \left(\frac{1}{6}\right)^2 M_a + 3 \cdot \left(-\frac{2}{3}\right)^2 (M_a - N_a) + (M_a + N_a)\right] \\ & + \sum_i q_i^A \left[3 \cdot \left(\frac{1}{3}\right)^2 M_i + 2 \cdot \left(-\frac{1}{2}\right)^2 (M_i + N_i)\right] = \frac{5}{6}\left(\rho - \frac{1}{6}\rho_A^Y\right) \end{aligned} \tag{6.71}$$

where we have included the piece ρ to account for flux contributions different from that in the hypercharge direction. The different scalings in the anomalies are due to the traces over the various generators. From the previous anomaly relations one obtain

$$A_A := \sum_a q_a^A N_a + \sum_i q_i^A N_i \sim \rho_Y = -\frac{5}{6}\rho_A^Y \,, \tag{6.72}$$

so that the Dudas Palti relation (6.27) is shifted by hypercharge flux. Finally, let us compute the anomaly coefficients for $\mathrm{U}(1)_Y - \mathrm{U}(1)_A - \mathrm{U}(1)_B$

$$\sum_a q_a^A q_a^A \left[6 \cdot \left(\frac{1}{6}\right) M_a + 3 \cdot \left(-\frac{2}{3}\right)(M_a - N_a) + (M_a + N_a)\right] + \sum_i q_i^A q_i^B \left[3 \cdot \left(\frac{1}{3}\right) M_i + 2 \cdot \left(-\frac{1}{2}\right)(M_i + N_i)\right] = 0\,, \tag{6.73}$$

from which follows

$$A_{AB} := 3\sum_a q_a^A q_a^B N_a + \sum_i q_i^A q_i^B N_i = \rho_A^Y \mathrm{tr}(T_B T_Y^2) + \rho_B^Y \mathrm{tr}(T_A T_Y^2)\,. \tag{6.74}$$

Note that no fluxes outside SU(5) can contribute to this anomaly. Recall that we also have similar relations for the anomalies $\mathrm{U}(1)_Y - (\mathrm{U}(1)_A)^2$ (A_{AA}) and $\mathrm{U}(1)_Y - (\mathrm{U}(1)_B)^2$ (A_{BB}), i.e.

$$A_{AA,BB} = 2\rho_{A,B}^Y \mathrm{tr}(T_{A,B} T_Y^2)\,. \tag{6.75}$$

Thus we can combine the relations (6.74) and (6.75) in the following manner,

$$2A_{AB} = A_{BB}\frac{\rho_A^Y}{\rho_B^Y} + A_{AA}\frac{\rho_B^Y}{\rho_A^Y}\,, \tag{6.76}$$

but from equation (6.72) we see that $A_A/A_B = \rho_A^Y/\rho_B^Y$. With this we can finally arrive at an expression which involves only the spectrum of chiral fields

$$2A_{AB} = A_{BB}A_A^2 + A_{AA}A_B^2\,. \tag{6.77}$$

Therefore, if the orientifold odd GS mechanism is to cancel all anomalies in the model, this implies that the Dudas-Palti relation (6.26) together with eq. (6.77) which involves linear and quadratic anomalies in the U(1) factors, simultaneously. The F-theory origin of this conditions and its implications for particle model building are yet to be worked out.

CHAPTER 7

Conclusions

So, naturalists observe, a flea
Hath smaller fleas that on him prey;
And these have smaller still to bite 'em;
And so proceed ad infinitum.
Thus every poet, in his kind,
Is bit by him that comes behind.

Jonathan Swift, On Poetry, A Rhapsody.

In this work we aimed to study discrete symmetries originating from string compactifications, and we observed that these are in close relation with the geometry of the internal space. We focussed our analysis on heterotic orbifolds, where an intuitive picture, as well as powerful computational tools are available. We see that in the simplest case we obtain Abelian discrete factors, which can be enhanced to non-Abelian symmetries due to operator degeneracies in the orbifold CFT. We also considered surviving remnants of the Lorentz group in internal space. Since these remnants treat bosons and fermions in a different way, these symmetries can be readily interpreted as R-symmetries in the effective theory. In the orbifold, the remnants of the Lorentz group are lattice automorphisms respecting the point group symmetry. There we identified two possible sources: In the first place we have transformations leaving all fixed points invariant. In the second case, we have transformations which do not leave all fixed points invariant, but instead, exchange fixed points which can not be distinguished from the CFT point of view. While the elements of the first class are present in all orbifolds studied, those from the second class are more rare (they occur for example in the $\mathbb{Z}_2$ and the $\mathbb{Z}_2 \times \mathbb{Z}_4$ orbifolds). From invariance of the correlators under these elements, we were able to infer the corresponding R-symmetries induced in the low energy, and to compute the R-charges of the physical states.

The anomalies of the resulting R-symmetries were then computed, and a scan over thousands of randomly generated models showed that they were all universal. Also for the more exotic R-symmetries of the $\mathbb{Z}_4$ orbifold. The universality of these anomalies is a beautiful and compelling result, hinting at the correctness of our results. Despite of the fact that the orbifolds analyzed here are the simplest of all, we expect our findings to hold more generally (including the more sophisticated orbifold constructions discussed in refs. [169, 170])

A very important observation about the R-charges we derived, is that they do not only include geometrical information about the internal space, but also some information regarding the gauge symmetries (which is encoded in the gamma phases). This we can understand in terms of the Bianchi identity (2.28), connecting the tangent and gauge bundles. Owed to this relation, one would expect that both of these objects contribute when considering a Lorentz transformation of the physical states. Despite of this observation, we are still far from understanding how R-symmetries arise in smooth compactifications. A possibility to extend our results is to consider gauge linear sigma models in the transition from the orbifold to the smooth phase. First steps towards addressing this problem have been taken in ref. [56].

Next we considered the effects of discrete symmetries in realistic models, specifically how can they help us keep the phenomenology under control. The charges of the fields under these discrete symmetries serve to track their location in the extra dimensions. Thus, it is not entirely surprising that as a result from previous model building efforts in the $\mathbb{Z}_{6-\mathrm{II}}$ mini-landscape, one finds a preferred configuration for the SM fields to allocate in the internal space. This configuration plays a crucial role in solving field theory issues such as the doublet-triplet splitting, flavor and μ- problems. The lessons we learned from this SM Zip code can be summarized as follows

1. A scenario in which the three families arise from complete SO(10) multiplets is not consistent with a hierarchy for the mass of the SM fields.

2. A completely untwisted top-Yukawa coupling seems to be the most favored situation, leading to the familiar gauge-top unification scheme.

3. In most of the cases the down-type Higgs lives in the bulk as well. If the untwisted Higgs pair remains massless the model will enjoy gauge-Higgs unification.

4. The two light families usually arise from the twisted sectors. They can appear as complete multiplets of the underlying local GUT.

In this work we describe general results from model building on the $\mathbb{Z}_2 \times \mathbb{Z}_4$ orbifold. From analysis of the spectrum at the GUT level we observe that there a similar Zip-code arises. Furthermore, we presented an explicit example where the matter content of the MSSM is achieved. We did not construct a large class of this models but hinted at the possibility that the diversity of gauge groups in $\mathbb{Z}_2 \times \mathbb{Z}_4$ provides many more MSSM-like models in comparison with $\mathbb{Z}_{6-\mathrm{II}}$. This conjecture was confirmed recently [171] in an exhaustive exploration aided by the C++ orbifolder. There it is shown that in the $\mathbb{Z}_2 \times \mathbb{Z}_4$ there are about 3632 models which resemble the MSSM, in contrast to the $\mathbb{Z}_{6-\mathrm{II}}$ where only 348 models are found. This observation turns the $\mathbb{Z}_2 \times \mathbb{Z}_4$ into the most promising orbifold alternative for particle phenomenology known to date.

Of course, there is a long way down to a realistic model. In between, the issues of moduli stabilization and the cancelation of FI terms have to be addressed. In this processes, many discrete symmetries get broken, so that it is not obvious which of them actually survive in the low energy. Since many symmetries have been proposed to control the phenomenology of models beyond the SM, we expect that suitable vacuum configurations leave some discrete symmetry unbroken. If so, these symmetry would be in general, a mixture of R and non-R symmetries as well as discrete remnants of gauge factors.

Finally, we contemplated the possibility of model building in the context of F-theory. Since in this framework the discrete symmetries are less understood, we followed a different strategy: We concentrated on F-theory SU(5) models which have up to two additional U(1) symmetries and allow for the three generations of quarks and leptons to arise from incomplete SU(5) representations. To find these appealing models we analyzed a class of models with toric sections whose geometric setup has been discussed recently in the literature, and can be embedded into the spectral cover. A special feature of these models is that all matter from $\mathbf{10}$-curves stay in complete multiplets which share the same charges under the additional U(1) symmetries, while in contrast the $\overline{\mathbf{5}}$-plets are usually split. This “half-complete” multiplet structure makes it possible to relate the charges of all operators to each other and is sufficient for generating phenomenologically interesting couplings. In particular, both types of models feature the top quark Yukawa coupling at tree level whereas the μ-term and all baryon and lepton number violating operators are forbidden. We identified a singlet VEV configuration which promises to induce a realistic Yukawa structure, while all dimension four proton decay operators stay forbidden. We could further relate this situation to the presence of a residual matter parity.

Nevertheless, the presence of additional U(1) symmetries in these models forces us to consider their corresponding anomalies. In particular, it is shown that in the models discussed, anomalies of the form $\mathrm{U}(1)_Y - \mathrm{U}(1)_A - \mathrm{U}(1)_B$ can not be cancelled. Further consideration of the anomaly cancellation mechanisms in the type IIB theory show that the critical anomalies could be cancelled by virtue of an orientifold odd Green-Schwarz mechanism. This, however, induces a shift in the Dudas-Palti relations. Inspired by the U(1) charge pattern observed in F-theory, we considered bottom-up models. There we showed that a suitable phenomenology is possible, and that by adding an additional matter curve, all anomaly cancelation conditions are satisfied. It would be intriguing to find explicit geometric realizations of the bottom-up models presented here.

There are various interesting future research directions. It would be intriguing to find an explicit geometric realization of the bottom-up models presented here, and to construct compactifications that realize the above assumptions on hypercharge flux, gauge coupling unification and Green-Schwarz anomaly-cancelation. Given such a realization one of the next avenues might be to combine these local models with moduli stabilization. We hope to return to some of these questions in the near future.

APPENDIX A

The space group elements h_g in the $\mathbb{Z}_4$ orbifold

g	$h_g^{\rho_1}$	$h_g^{\varrho_2}$	$h_g^{\varrho_3}$	h_g^{ζ}
$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$
$\theta,(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$
$\theta,(0,0,0,0,0,1)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,-1)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$
$\theta,(0,0,1,0,0,0)$	$\mathbb{1},(0,0,-1,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$
$\theta,(0,0,1,0,0,1)$	$\mathbb{1},(0,0,-1,0,0,0)$	$\mathbb{1},(0,0,0,0,0,-1)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$
$\theta,(1,0,0,0,0,0)$	$\mathbb{1},(-1,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(-1,-1,0,0,0,0)$	$\mathbb{1},(-1,0,0,0,0,0)$
$\theta,(1,0,0,0,0,1)$	$\mathbb{1},(-1,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,-1)$	$\mathbb{1},(-1,-1,0,0,0,0)$	$\mathbb{1},(-1,0,0,0,0,0)$
$\theta,(1,0,1,0,0,0)$	$\mathbb{1},(-1,0,-1,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(-1,-1,0,0,0,0)$	$\mathbb{1},(-1,0,0,0,0,0)$
$\theta,(1,0,1,0,0,1)$	$\mathbb{1},(-1,0,-1,0,0,0)$	$\mathbb{1},(0,0,0,0,0,-1)$	$\mathbb{1},(-1,-1,0,0,0,0)$	$\mathbb{1},(-1,0,0,0,0,0)$
$\theta,(0,0,0,0,1,0)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,-1,0)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$
$\theta,(0,0,0,0,1,1)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,-1,-1)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$
$\theta,(0,0,1,0,1,0)$	$\mathbb{1},(0,0,-1,0,0,0)$	$\mathbb{1},(0,0,0,0,-1,0)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$
$\theta,(0,0,1,0,1,1)$	$\mathbb{1},(0,0,-1,0,0,0)$	$\mathbb{1},(0,0,0,0,-1,-1)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$
$\theta,(1,0,0,0,1,0)$	$\mathbb{1},(-1,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,-1,0)$	$\mathbb{1},(-1,-1,0,0,0,0)$	$\mathbb{1},(-1,0,0,0,0,0)$
$\theta,(1,0,0,0,1,1)$	$\mathbb{1},(-1,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,-1,-1)$	$\mathbb{1},(-1,-1,0,0,0,0)$	$\mathbb{1},(-1,0,0,0,0,0)$
$\theta,(1,0,1,0,1,0)$	$\mathbb{1},(-1,0,-1,0,0,0)$	$\mathbb{1},(0,0,0,0,-1,0)$	$\mathbb{1},(-1,-1,0,0,0,0)$	$\mathbb{1},(-1,0,0,0,0,0)$
$\theta,(1,0,1,0,1,1)$	$\mathbb{1},(-1,0,-1,0,0,0)$	$\mathbb{1},(0,0,0,0,-1,-1)$	$\mathbb{1},(-1,-1,0,0,0,0)$	$\mathbb{1},(-1,0,0,0,0,0)$
$\theta^2,(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$
$\theta^2,(0,0,0,1,0,0)$	$\theta,(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$
$\theta^2,(0,0,1,1,0,0)$	$\mathbb{1},(0,0,-1,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$
$\theta^2,(0,1,0,0,0,0)$	$\theta,(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,-1,0,0,0,0)$	$\theta,(0,0,0,0,0,0)$
$\theta^2,(0,1,0,1,0,0)$	$\theta,(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,-1,0,0,0,0)$	$\theta,(0,0,0,0,0,0)^\dagger$
$\theta^2,(0,1,1,0,0,0)$	$\theta,(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,-1,0,0,0,0)$	$\theta,(0,0,1,0,0,0)^\dagger$
$\theta^2,(0,1,1,1,0,0)$	$\theta,(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,-1,0,0,0,0)$	$\theta,(0,0,1,0,0,0)$
$\theta^2,(1,1,0,0,0,0)$	$\mathbb{1},(-1,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(-1,-1,0,0,0,0)$	$\mathbb{1},(-1,0,0,0,0,0)$

g	$h_g^{\rho_1}$	$h_g^{\varrho_2}$	$h_g^{\varrho_3}$	h_g^{ζ}
$\theta^2,(1,1,0,1,0,0)$	$\theta,(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(-1,-1,0,0,0,0)$	$\mathbb{1},(-1,0,0,0,0,0)$
$\theta^2,(1,1,1,1,0,0)$	$\mathbb{1},(-1,0,-1,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(-1,-1,0,0,0,0)$	$\mathbb{1},(-1,0,0,0,0,0)$
$\theta^3,(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$
$\theta^3,(0,0,0,0,0,1)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,-1)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$
$\theta^3,(0,0,1,0,0,0)$	$\mathbb{1},(0,0,0,1,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$
$\theta^3,(0,0,1,0,0,1)$	$\mathbb{1},(0,0,0,1,0,0)$	$\mathbb{1},(0,0,0,0,0,-1)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$
$\theta^3,(1,0,0,0,0,0)$	$\mathbb{1},(0,1,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(-1,1,0,0,0,0)$	$\mathbb{1},(0,1,0,0,0,0)$
$\theta^3,(1,0,0,0,0,1)$	$\mathbb{1},(0,1,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,-1)$	$\mathbb{1},(-1,1,0,0,0,0)$	$\mathbb{1},(0,1,0,0,0,0)$
$\theta^3,(1,0,1,0,0,0)$	$\mathbb{1},(0,1,0,1,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(-1,1,0,0,0,0)$	$\mathbb{1},(0,1,0,0,0,0)$
$\theta^3,(1,0,1,0,0,1)$	$\mathbb{1},(0,1,0,1,0,0)$	$\mathbb{1},(0,0,0,0,0,-1)$	$\mathbb{1},(-1,1,0,0,0,0)$	$\mathbb{1},(0,1,0,0,0,0)$
$\theta^3,(0,0,0,0,1,0)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,-1,0)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$
$\theta^3,(0,0,0,0,1,1)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,-1,-1)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$
$\theta^3,(0,0,1,0,1,0)$	$\mathbb{1},(0,0,0,1,0,0)$	$\mathbb{1},(0,0,0,0,-1,0)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$
$\theta^3,(0,0,1,0,1,1)$	$\mathbb{1},(0,0,0,1,0,0)$	$\mathbb{1},(0,0,0,0,-1,-1)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$
$\theta^3,(1,0,0,0,1,0)$	$\mathbb{1},(0,1,0,0,0,0)$	$\mathbb{1},(0,0,0,0,-1,0)$	$\mathbb{1},(-1,1,0,0,0,0)$	$\mathbb{1},(0,1,0,0,0,0)$
$\theta^3,(1,0,0,0,1,1)$	$\mathbb{1},(0,1,0,0,0,0)$	$\mathbb{1},(0,0,0,0,-1,-1)$	$\mathbb{1},(-1,1,0,0,0,0)$	$\mathbb{1},(0,1,0,0,0,0)$
$\theta^3,(1,0,1,0,1,0)$	$\mathbb{1},(0,1,0,1,0,0)$	$\mathbb{1},(0,0,0,0,-1,0)$	$\mathbb{1},(-1,1,0,0,0,0)$	$\mathbb{1},(0,1,0,0,0,0)$
$\theta^3,(1,0,1,0,1,1)$	$\mathbb{1},(0,1,0,1,0,0)$	$\mathbb{1},(0,0,0,0,-1,-1)$	$\mathbb{1},(-1,1,0,0,0,0)$	$\mathbb{1},(0,1,0,0,0,0)$

APPENDIX B

Promising E_6 embeddings in the $\mathbb{Z}_2 \times \mathbb{Z}_4$ orbifold

Here we present a selection of E_6 models with a Wilson line configuration leading to two complete families ($\mathbf{27}$) and an untwisted trilinear coupling. All of these models work with all Wilson lines on except W_1 or W_2.

For the shift embedding

$$
\begin{aligned}
4V_4 &= (0,0,0,0,0,0,0,0)(1,1,0,0,0,0,0,0)\,,\\
2V_3 &= (-1,-1,-1,0,0,0,0,1)(0,1,0,0,0,0,1,0)\,,
\end{aligned}
$$

if we take $W_{1,2} \neq 0$ to break E_6 to SO(10) the surviving pieces from the split multiplets may lead to complete families, provided the $\mathbf{16}-\mathrm{plets} \subset \mathbf{27}$ are not projected out. In the case of

$$
\begin{aligned}
4V_4 &= (2,2,0,0,0,0,0,0)(1,1,0,0,0,0,0,0)\,,\\
2V_2 &= (2,2,0,0,0,0,0,0)(0,1,0,0,0,0,1,0)
\end{aligned}
$$

we have to switch on the WLs in the same manner as before, taking again $W_{1,2}$ to break to SO(10), then we have the chance to obtain two $\mathbf{16}$-plets at the $T_{(1,2)}$ sector. The model given by

$$
\begin{aligned}
4V_4 &= (1,1,0,0,0,0,0,0)(1,1,1,1,1,1,1,-1)\,,\\
2V_2 &= (0,1,0,0,0,0,1,0)(1,1,1,1,1,1,1,-1)\,,
\end{aligned}
$$

may lead to two families at the $T_{(1,2)}$ sector, provided W_3 breaks to SO(10), this picture is similar to that one gets from the embedding

$$4V_4 = (1,1,0,0,0,0,0,0)(1,1,1,1,1,1,1,-1)\,,$$
$$2V_2 = (0,1,0,0,0,0,1,0)(2,0,0,0,0,-2,0,0)\,,$$

where the only difference is that the families will appear at the $T_{(1,0)}$ sector.

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