Matching the heterotic string on orbifolds and their resolutions

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A l'alta fantasia qui mancò possa;
ma già volgeva il mio disio e 'l velle,
sì come rota ch'igualmente è mossa,
l'amor che move il sole e l'altre stelle.
Comedìa di Dante.

A Milagros, Alejandro, Alito,Tata y a todos mis abuelos.

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Chapter 1

Introduction: Why strings and compactifications?

1.1 Unification of all forces

Unification of fundamental forces is a long pursued goal in the history of physics. More than an aesthetical aspiration it lies in the core of a better understanding of the studied phenomena. One classical example of it are the Maxwell equations for Electromagnetism, in which electric and magnetic fields are understood as a single gauge interaction. At the beginning of the 20th century many experimental and theoretical developments came into play. On one hand special relativity, based on the fact that nothing can travel faster than light, describes new transformations of 4d spacetime between different observers. Later, general relativity based on the equivalence principle of inertial and gravitational mass, led to a revolutionary way to understand the gravitational interaction in terms of spacetime curvature. In those days, Quantum Mechanics explained new observations at the atomic scale. But its application to the relativistic theory of electromagnetism had the problem of divergencies. These problems were solved when the perturbation theory was complemented by a renormalization scheme, yielding quantum electrodynamics as the first consistent quantum field gauge theory. Dating back to the 1960s electromagnetic and weak interactions were unified in a single gauge theory. It was shown that a *Higgs mechanism* could break the electroweak symmetry spontaneously, give mass to the chiral fermions and yet ensure renormalizability. At the beginning of the 1970s the strong interaction was also understood in the frame of a gauge theory, called *Quantum Chromodynamics* (QCD). All these three interactions: electromagnetism, weak and strong are jointly described in the *Standard Model* (SM) of particle physics, which constitutes a successful description of all fundamental interactions at the quantum level, with the exception of gravity. The model has great experimental success. This summer, indications of the existence of its last missing block, the Higgs field, were found at the Large Hadron Collider in CERN [1].

1.2 The Standard Model and beyond

Particle content The Standard Model is a gauge theory with gauge group $G_{SM} = SU(3)_c \times SU(2)_L \times U(1)_Y$, describing strong and electroweak interactions. The matter content is given by three generations of the following repeating structure: a left handed quark $SU(2)_L$ doublet transforming in the fundamental representation of $SU(3)_c$ ($Q_L = (u_L, d_L)$), two right handed quarks $SU(2)_L$ singlets transforming in the fundamental representation of $SU(3)_c$ (u_R, d_R),[1] a left handed lepton doublet whose components are the electrically charged leptons (l_L) and uncharged leptons (ν_L) and an electrically charged right handed lepton (l_R). The quarks are denoted *up* (u) and *down* (d) according to their electric charge 2/3 and −1/3 respectively. The spectrum is chiral because left and right handed chiral fields have different quantum numbers. This chiral asymmetry is a restrictive feature when deriving the standard model from string theory, and we will see it in detail in the study of this thesis (Chapter 4 and 5). A non–chiral spectrum would allow mass terms for all the fermions, which are generically of the order of the string scale. So those particles would not be detectable. With the described particle spectrum and the hypercharge assignment SM is an anomaly free theory. The bosons of the SM are the Higgs scalar, the strong interaction gauge bosons (gluons) and the electroweak gauge bosons ($W^{\pm}$, Z and the photon).

The patterns in the SM Despite its amazing success there are open questions that arise in the SM. First, gravitational interactions are not included. In present high energy experiments this is irrelevant, but there are physical phenomena as black holes or the early universe evolution, in which quantum mechanics and general relativity are both required. There is also the issue of the many parameters which are not fixed in the theory: those are the three gauge coupling constants $\alpha_3, \alpha_2, \alpha_1$ of the gauge factors G_{SM}; the QCD θ parameter; the 9 masses of quarks, leptons and neutrinos; the CP violation phase and the parameters in the Higgs potential determining the *electroweak scale* $M_{\rm EW} \sim 100GeV$. In addition, some of these parameters have a quite interesting structure. The quarks and lepton masses possess what is called a *hierarchy*. Namely the fact that the mass of the first generation differs from the mass of the second the same order of magnitude than the second differs from the third. As the fermions of those three generations are not mass eigenstates, the mass matrix can be diagonalized leading to the Cabibbo-Kobayashi-Maskawa matrix, which describes the coupling to the electroweak bosons $W^{\pm}$. This matrix possesses an intriguing structure in which the diagonal elements are of order one and all the other elements are smaller. Also the measurements of neutrino masses show an hierarchical structure.[2] Another curious fact is that all the gauge couplings approximately unify at $10^{16}GeV$, when one evolves them with the *renormalization group equations* from the measured low energy values to a higher scale [2]. This assumes that no new physics appears between the electroweak scale $M_{\rm EW}$ and the unification scale. [3] This unification is natural if one assumes

[1] In this explanation we consider that the fields are Dirac fermions.

[2] The Pontecorvo-Maki-Nakagawa-Sakata matrix for neutrino states mixing has big angles with the same order of magnitude in all its entries.

[3] This is called the *desert hypothesis*.

that the SM is embedded in a *Grand Unification Theory* (GUT).

Grand Unification The GUT hypothesis says that at high energies the physics is described by a gauge theory with a bigger group (often simple) and multiplets accommodating the SM fields [3,4]. The SM is then obtained at lower energies through spontaneous symmetry breaking. The considered group should have at least rank four, and needs to have complex representations in order to ensure chirality. Then at the *GUT scale* M_{GUT} the interactions will be determined by a single coupling strength. There are many studied cases, the simplest of them is $SU(5)$ in which the matter content of one generation fits into the $\bar{\mathbf{5}} + \mathbf{10}$ representations. The gauge bosons are in the $SU(5)$ adjoint representation i.e. the $\mathbf{24}$ and the symmetry breaking is performed through a Higgs also in the adjoint, which gives masses to the extra gauge bosons. Evolving the Weinberg angle (θ_W) with $\sin^2\theta_W = \alpha_1/\sqrt{\alpha_1^2+\alpha_2^2}$ from its value at M_{GUT} to M_{EW} with the renormalization group, does not fits perfectly with the experimental results. In addition, this model predicts a proton decay which is faster than the experimental observation. Also the unification of quark and lepton Yukawa's at M_{GUT} can not explain its difference at M_{EW}. There are further popular GUT groups such as $SO(10), SU(6), SU(7), E_6$ and $SU(5) \times U(1)_X$, with different advantages and disadvantages. Different ways of breaking (different fields attaining vevs) and intermediate breaking steps can lead to different physics. E_6 is commonly obtained in four dimensional string theory, but also $SU(5) \times U(1)_X$ is an appealing option for string model building, because the breaking down to the SM does not require big representations. The failing of exact coupling unifications gives us a motivation for considering $\mathcal{N} = 1$ *Supersymmetry* [5–10]. This symmetry predicts to known bosons a fermionic partner and viceversa. In supersymmetric unification the three couplings meet much better and at a higher scale easing the problem of proton decay. There can be more than one supersymmetry, we will use the letter $\mathcal{N}$ to refer to the number of supersymmetries of a theory.

Naturalness There are further open questions, and they are related to the principle of *naturalness* [11]. This principle states that small parameters in a physical theory measure the deviation from a symmetry. One of this parameters is the *cosmological constant* which enters Einstein equations. Cosmological observations [12] suggest that the energy density of the gravitational vacuum is given by $\Lambda \sim (10^{-3}eV)^4$ [4]. In the SM there are potential sources of the vacuum energy as the minimum of the Higgs potential, and the one loop corrections that will give a contribution of the order of $(M_{\mathrm{cutoff}})^4$, where M_{cutoff} is the upper energy scale of the theory. It could be the electroweak or the Planck scale. But any of them is many orders of magnitude bigger than the measured value. Canceling very large contributions to obtain a very small experimental value is known as *fine tuning.* [5]

[4] This is what is called a De Sitter vacua with $\Lambda > 0$.

[5] One possible explanation is the *anthropic principle* [13], this states that in order to allow galaxy formation and the existence of observers the value of Λ has to be closed to the experimental one. This approach encounters an interpretation in the frame of string theory in which quantum fluctuations produced during *inflation* could create regions with different local vacuum energy, which due to expansion will evolve in

Another problem of this sort is the *strong CP problem* in which the CP violating term in the Lagrangian with parameter θ occurs. This parameter is expected to be of order one, but the CP-violation measurements indicate that it should be $< 10^{-10}$ [14]. This can be explained with the introduction of *axions* with respect to an additional $U(1)$ Peccei-Quinn global symmetry. The axions get a non-perturbative QCD correction to its potential which determines its mass in terms of the decay constant. The latter have been bounded experimentally to lie in the *axion window* [15]. String theory can provide axions that lie on that window.

Finally, there is the *electroweak hierarchy problem*. This is the problem of how the mass of the Higgs field can be of the order of M_{EW} if the corrections to its bare value coming from one-loop diagrams are of the order of the cutoff scale. There are many proposals to solve this problem, a really successful one is $\mathcal{N} = 1$ low energy supersymmetry. It gives additional diagrams with the supersymmetric partners in the loop, such that the total contribution to the bare mass of the Higgs cancels. Other approaches consider the mass generated dynamically by some gauge sectors of the theory, with "quarks" condensates which play the role of the Higgs. Among them the first and most famous proposals is *technicolor* [16, 17] but there are other approaches. In particular, the author contributed to the study of a modified version of QCD which addresses this problem [18]. [6]

Extra dimensions We have mentioned till now ideas on unification of the fundamental gauge interactions, and the problem of unification of gravity with quantum mechanics. But there is also the possibility of unifying gauge interactions with gravity at the classical level. This is achieved in a beautiful way by considering *extra dimensions*. This idea arose in the 1920s with the aim of unifying electromagnetism with general relativity. Kaluza and Klein proposed a 5d theory in which the additional spatial direction was compactified on a circle of radius R [19,20]. Considering as the starting point a five dimensional action with a kinetic term for scalars, and integrating out the 5th coordinate one obtains a zero mode and an infinite tower of massive states whose masses are quantized by $1/R$. The dimensional reduction of the 5d Ricci scalar gives in 4d the Ricci scalar term plus a scalar field without potential, called *modulus*, and a kinetic action for a $U(1)$ gauge field. This gauge field is the electromagnetic field. Despite this result, the theory is not viable because is not possible to incorporate chiral fermions. In addition, the scales for the masses is to low, so the tower of particles should have been observed. This process of integrating the extra dimensions to obtain a lower dimensional theory is called *compactification* or *dimensional reduction*. Despite its non immediate success this idea had further applications. In particular the idea that the dimensional reduction will give an effective gravity scale $M_{\text{P}} \sim M_5^3 R$ which depends on the initial scale M_5 and the compactification radius R. With the use of this mechanism it was proposed more recently to consider that the extra dimensions could render a scale for gravity which is almost of the order of M_{EW}. This example shows that the

different universes.

[6] This is done by large extra dimensions or by strongly warped ones. Finally there is also an *anthropic proposal* to explain why the electroweak scale has its value, basically stating that M_{EW} is required by electroweak symmetry breaking, which gives the needed interactions to create living systems.

particle content in 4d depends on the shape and size of the compact manifold. It has many valuable applications, among them is supergravity, a theory with local supersymmetry, which has been studied in various dimensions [21, 22]. It was shown that is not possible to obtain a chiral theory for fermions after compactification on a smooth space, if the original theory does not possess chirality. This difficulty can be overcome in the frame of string theory, either by starting with a chiral theory, or by obtaining the chiral fermions in special submanifolds of the compactification space. Our case of study will be the first one.

1.3 Smearing out point like interactions in gravity

Investigations of a quantum field theory of gravity show that the theory has unrenormalizable divergencies. There is a simple way to see that the problem is related to the dimensionful coupling constant $[G_{\rm N}] \sim 1/M^2$ [23]. If we consider the process of two freely propagating particles, a tree level correction to it in a quantum gravity theory will be given by the diagram in which a graviton is exchanged. This amplitude is proportional to the Newton gravitational constant $\sim G_{\rm N}$. The ratio between the original amplitude with characteristic energy E and the one-graviton corrected one will be given by the dimensionless combination $G_{\rm N}E^2\hbar^{-1}c^{-5}$. This combination also fixes the Planck scale to be the energy at which the one-graviton correction becomes relevant i.e. $G_{\rm N}M_{\rm P}^2\hbar^{-1}c^{-5} = 1$ such that[7]

$$M_{\rm P} = 1.22 \times 10^{19} GeV. \tag{1.1}$$

So the ratio of one graviton exchange- to the zero graviton exchange-amplitude is $(E/M_{\rm P})^2$. This quantity becomes weaker at low energies as $M_{\rm EW}$, but for $E \gg M_{\rm P}$ the perturbative approach breaks down. On dimensional grounds, the correction order by two gravitons exchange would be $\sim G_N^2 = 1/M_{\rm P}^4$ giving $\int dE_1 E_1^3/M_{\rm P}^4$, while the correction by three gravitons should be of the order $\int dE_1 E_1^2 \int dE_2 E_2^2/M_{\rm P}^6$, and so on. At arbitrary high energies $E \gg M_{\rm P}$ all of this contributions diverge, and the divergencies get stronger at subsequent orders in perturbation theory. One can ask whether those divergencies are merely a result of treating the theory perturbatively and not exactly. This is an open question related to the existence of a *non trivial ultraviolet fixed point* for gravity. Transforming the amplitudes to position space, arbitrary large intermediate energy corresponds to the limit in which the graviton vertices come arbitrarily close to each other. Therefore a possibility to solve this problem is to consider that beyond some energy the theory is modified such that the interaction is spread in space reducing the divergency. The only known way to spread the gravitational interaction in space while keeping the theory consistent is *string theory*. The scale of the spreading is the *string length* l_s, which is related to the *string scale* M_s by $M_s \sim 1/l_s$. In fact, in quantum field theory it is hard to spread the interactions in space, preserving Lorentz invariance, causality and unitarity.

[7] $M_{\rm P}$ is given in natural units where $c = \hbar = 1$. The Planck time $t_{\rm P} = \sqrt{\frac{\hbar G}{c^5}} = 5.39106(32) \times 10^{-44} s$ and the Planck length $l_{\rm P} = \sqrt{\frac{\hbar G}{c^3}} = 1.616199(97) \times 10^{-35} m$ are both given by $1/M_{\rm P}$ in natural units.

Strings String theory combines the ideas of supersymmetry, grand unification and extra dimensions and simultaneously gives a solution to the problem of quantum gravity divergencies. String theory is built from one-dimensional objects called *strings* [24–27]. Their spacetime trajectory describes a *world–sheet* parametrized by the proper time τ and the coordinate σ of the string. The world sheet can be open or closed, oriented or unoriented.

String theory was suggested in the 1960s as a model for strong interactions. Later QCD was established as the theory of strong interactions, but string theory, which has in its massless spectrum a spin 2 particle (identified with the graviton) was revived as a good candidate for a quantum theory of gravity. The effective theory for the gravitational sector gives general relativity equations corrected by effects of the order of $1/M_s$. In order to eliminate tachyons and to obtain spacetime fermions it is necessary to include world–sheet supersymmetry, yielding a *superstring theory.* Superstring theories have also space-time supersymmetry. In superstring theory consistency dictates a spacetime *critical dimension* of 10. This problem is solved by dimensional reduction.

Supergravities and Compactifications Effective descriptions of the strings, in which the M_s–scale massive fields are integrated out, and an action with the massless spectrum is obtained, are supergravity theories in 10d . In the 1980s it was shown that an *anomaly* of gauge symmetries in the 10d $\mathcal{N}=1$ supergravity can be canceled by the *Green-Schwarz mechanism* [28]. It restricts the allowed gauge group to $SO(32)$ or $E_8 \times E_8$. A string theory in which the $SO(32)$ gauge group is realized was already known. It is called *type I* string theory. This fact was an indication of the potential predictive power of the strings. It triggered the *first superstring revolution* on the 1980s, in which also the *heterotic string* was discovered. The heterotic string will be described in Chapter 2. It leads to a 10d $\mathcal{N}=1$ supergravity which can have gauge group $SO(32)$ or $E_8 \times E_8$, exactly the two gauge groups that were proven by Green and Schwarz to give anomaly free supergravity! Also the chiral type IIB and the non chiral type IIA supergravities arise from string theory. At the perturbative level the later theories have only abelian gauge groups. At the time of these discoveries it was realized that the extra dimensions can be compactified, keeping only $\mathcal{N}=1$ ($\mathcal{N}=2$) supersymmetry in 4d for type I and heterotic string (type IIA and type IIB). Smooth complex Kähler manifold spaces, with $SU(3)-holonomy$ achieving this goal are the so called *Calabi–Yau* (CY) manifolds [29]. It was then found that string theory was able to describe the known elementary particles and the fundamental gauge interactions in 4d. A compactification of the heterotic string on a 6d torus will leave $\mathcal{N}=4$ in 4d.[8] But a natural modification of the torus is a space constructed by modding out a symmetry from the lattice. The resulting variety is called an *orbifold.* It can be denoted as T^6/G_{orb}, where G_{orb} denotes the modded out symmetry. Conditions on G_{orb} can also be imposed to achieve $\mathcal{N}=1$ supersymmetry in 4d. This 6d quotient space is generically flat, but possesses subsets of higher codimension that are invariant under G_{orb} which constitute

[8]It is natural to explore for $\mathcal{N}=1$ supersymmetry in 4d, because this amount of SUSY offers a solution to the hierarchy problem and to the gauge coupling unification. Nevertheless, recently it was considered an example in which an $\mathcal{N}=2$ gauge sector seems to agree with the measured Higgs mass [30, 31].

curvature singularities. It was proven that in this backgrounds the world–sheet theory is well defined, and is possible to solve the string equations of motion and to obtain the string spectrum [32, 33]. It is precisely in the orbifold and Calabi–Yau compactifications of the heterotic string that our work will focus.

A unique theory of strings At this point there have already been found five superstring theories type IIA, type IIB, heterotic $SO(32)$, heterotic $E_8 \times E_8$ and type I. The type II and heterotic theories are *closed oriented string theories* and the type I is *closed plus open unoriented string theory*. As we are interested in a unified theory which reduction to low energies gives the known physics, it is required to understand why there are five theories. But then, non-perturbative dualities bringing together the five superstring theories were discovered. At the end of the 1980s *T-Duality* was discovered [34–36], it relates theories in different compactification geometries. For example it always includes a subgroup in which a theory at the compactification radius R is identified with another theory at the compactification radius α'/R. Under T-Duality transformations both type II theories as well as both heterotic theories are seen as different geometrical limits of the same theory. In the 1990s another duality called *S-Duality* was discovered [37–39], this was the beginning of the *second superstring revolution*. It relates a theory at string coupling g_s with a theory at coupling $1/g_s$. The duality implies that perturbation theory $g_s \ll 1$ gives information about the strong coupling behavior $g_s \gg 1$. Under this transformation type I goes to heterotic $SO(32)$ and type IIB is mapped to itself. Note that S-Duality is non-perturbative in g_s and T-Duality is non-perturbative in α'/R. Those coupling constants are dynamical quantities in string theory and they are given by the vevs of the *dilaton* or the moduli fields. S and T dualities pointed to the existence of a unique theory, called *M–Theory*, whose weak coupling limit is 11d supergravity [40]. This was found when exploring the strong coupling limit of type IIA and heterotic $E_8 \times E_8$ theories. Those theories grow an eleven dimension of size $g_s \alpha'^{1/2}$ in the strong coupling limit giving rise to a strongly coupled 11d theory. This last ingredient showed that all the five string theories can be seen as different limits of a unique *M–Theory*. M–Theory is expected to be the definitive theory of strings, but a microscopic description of it is not known, so much remains to be done on that path.

Branes, gauge/gravity and F–Theory On the course of the second revolution it was realized that the theory requires the inclusion of higher dimension objects *D-Branes* [41], whose existence opened the way to a whole new branch of studies with many applications for particle physics and cosmology. Microscopically D-branes are objects on which open strings can end. On the other hand they appear as solitonic (BPS) solutions in type I and II supergravities. The Yang Mills theory will arise in the *world volume* of these objects, and therefore this is were the SM can be encountered. There have been intensive studies on those models, in which configurations of branes are arranged to obtain the known particle physics [42]. The branes were also an ingredient of other discoveries at the end of the decade. Their existence permitted the construction of *black p-branes* which are generalizations of black holes, in the frame of string theory. They account for the microscopic entropy leading to a better understanding of black holes thermodynamics within string theory. The counting of the microstates and the computation of the entropy have been performed in many cases; obtaining corrections to the Bekenstein-Hawking formula. Another development related to D-branes [9] is the correspondence between type IIB strings on $AdS_5 \times S^5$ background and $\mathcal{N} = 4$, $SU(N)$ Super Yang Mills conformal field theory (CFT) on the boundary of AdS_5 for large N [43, 44][10]. This *AdS/CFT correspondence* has developed strongly in the last years giving rise to a more general *holographic principle*, studied in many different contexts and known as *gravity/gauge theory duality*. Finally we want to mention another theory that was discovered in the 1990s, this is *F–Theory* and it arose in the process of trying to connect type IIB theory with M–Theory (as it was done with type IIA and Heterotic $E_8 \times E_8$). This is based in an $SL(2, \mathbb{Z})$ symmetry of type IIB, which acts on the axion-dilaton τ. This leads to the interpretation of τ as the complex structure of an auxiliary two dimensional torus T^2 arriving at F–Theory as a 12d theory [45]. Many compactification models have been constructed for F–Theory on a 4-fold CY[11] to obtain 4d physics [46]. In particular there have been much work on the subject of F–Theory GUTs in which the local information on the compactification space is all what is needed. There exist also an F–Theory/Heterotic duality which we would like to explore in the future, maybe in the context of the present constructions. One of the reasons for it is the *moduli space problem*.

Moduli space The moduli space is parametrized by vevs of scalar fields *moduli* that typically regulate the geometry of the compactification space and affect the low energy physics, but are not fixed. The fact that these scalar fields appear in the massless spectrum of the strings and are not restricted by a potential is known as *moduli problem*. However by string *flux compactifications* one can create a natural potential for the moduli. Although M–Theory is unique, it admits a large number of solutions giving four macroscopic dimensions. Many of these vacua give good 4d physics, but is still an open problem the determination of the preferable one among all of them.

There are for sure important developments that we have skipped. But we hope that the general picture presented serves to understand the frame and scope of our work, on which we focus in the following.

1.4 Heterotic string orbifolds and resolutions

Heterotic orbifolds studies started after the first superstring revolution, and since then many promising models of the four dimensional world have been proposed. They constitute a fertile region of the landscape [47] in which the MSSM and GUT theories are

[9] The $AdS_5 \times S^5$ geometry is the near horizon limit of N coincident black $D3-$branes in type IIB theory with N units of F_5 flux on S^5.

[10] In this limit the planar diagrams of SYM are dominant. The propagators on those diagrams depending on the adjoint gauge field, possess two indices which give diagrams which in general can not be drawn on a plane.

[11] Constructed as a fibration of a 3-fold with T^2.

widely encountered. They have also additional appealing features. In particular due to the presence of fixed sets, fixed points and fixed tori, there are *twisted states* which are located at the fixed sets which are the singularities of the quotient space. Their properties depend on the subgroup of G_{orb} which leaves the set fixed. They can cause interesting local physics [48–51]. This fact has lead recently to the concept of *local grand unification* [52–54] that serves to explore many promising models. Heterotic orbifolds provide discrete symmetries [55], which serve to understand the hierarchy between M_{EW} and M_{GUT} [56], avoid proton decay [57,58], obtain *flavor symmetries* [59] and suppress the μ term [60–62].

The moduli space of the metric in CY manifolds consists of the *complex structure moduli* and the complexified *Kähler structure moduli*. There are well studied examples in which twisted states of orbifold models, which acquire vevs, smooth the singularities and can be identified with the moduli of the CY manifold [63]. This is expected because both CY and orbifolds preserve $\mathcal{N}=1$ supersymmetry. In fact, all $T^6/\mathbb{Z}_n$ orbifolds are known to be singular limits of a smooth CY manifolds [63–70]. In the last years there has been an intense work in the problem of understanding the transition between those two geometries. When studying the string on orbifolds the conformal field theory is free, the equations of motion are solvable and the interactions are computable. On the contrary, when working on smooth CY, the metric is not known but only the topological information of the manifold. Therefore one can not solve the conformal field theory[12]. The way to go is to compactify on the CY the effective heterotic 10d $\mathcal{N}=1$ theory, which consists of super Yang Mills coupled to supergravity. In this context it is possible to employ index theorems [71,72] to determine the massless fermionic modes in 4d.

The orbifold point in moduli space is very special. One encounters on it many exotics states, additional $U(1)$ symmetries and enhanced discrete symmetries. This abundance differs from what is found in the 4d world, nevertheless this can be fixed. Spontaneous symmetry breaking, giving vevs to twisted fields, can be used to decouple exotics from the spectrum. This would automatically reduce the abelian gauge sector and break partially global discrete symmetries[13]. In addition there exists generically an anomalous $U(1)_A$ symmetry which will generate a *Fayet-Iliopoulos D-term* (FI), which breaks supersymmetry by making the vacuum scalar potential non-zero [74]. Fortunately, the same mechanism which serves to decouple exotics and partially break symmetries can be used to cancel the FI term [75–77]. These twisted fields which attain vevs could correspond to moduli of the CY geometry, which vanish at the orbifold point. The previous arguments show that the transition (in the moduli space of CY manifolds) from an orbifold point to an smooth point, is well physically motivated. As at the orbifold point, the full spectrum, the interactions and the discrete symmetries can be easier determined, this can be used to extract information not known in the CY [78].

The techniques of algebraic geometry in toric varieties [79–81] have been applied to make the orbifold singularities smooth [65,66,68–70]. This process of removing the singularity

[12] With the exception of the Gepner point where also a CFT description is available.

[13] This partial breaking of discrete symmetries can be useful to create scales hierarchy, as for example in obtaining the pattern for quarks and lepton masses trough a Froggatt-Nielsen mechanism [73]

and adding exceptional divisors of finite size $\rho \neq 0$ is called *blow–up* or *resolution*, the inverse process of making $\rho \to 0$ is called *blow–down*. In the work by Groot-Nibbelink, Trapletti and Walter [69] non-compact orbifolds of the heterotic superstring $\mathbb{C}^3/\mathbb{Z}_n$ were resolved. In the *blow down* limit of these CY compactifications, in which an abelian gauge flux in 6d (vector bundle) is turned on, the vectors determining the vector bundle (over the $E_8 \times E_8$ Cartan subalgebra) correspond to shifts on the gauge degrees of freedom (d.o.f.) of the local orbifold action. In other words, if we want to identify the heterotic orbifold as the singular limit of the CY, it is necessary to construct the vector bundle in a way that orbifold rotation on the gauge d.o.f (*shifts*) are reproduced in the limit $\rho \to 0$. For the compact cases in which there are many local $\mathbb{C}^3/\mathbb{Z}_n$ singularities, the blow–down of the local resolutions fixes the vector bundle to reproduce the local shifts [14] [82, 83]. The described geometric blow–up can be identified with the process of giving vevs to twisted fields. Those twisted fields were interpreted as the CY Kähler moduli, by making an exponential field redefinition. This is supported by the fact that the gauge transformation of twisted fields coincides with the gauge transformation of the exponential of the Kähler moduli [83]. And also by the fact these Kähler moduli are local, because they appear on the cycles introduced in the resolutions. Furthermore, a way of identifying those *blow–up modes* on the orbifold with the components of the vector bundle was proposed. This was based on the fact that the *Bianchi Identities* (BI) giving a consistent gauge flux, possess strong similarities with the orbifold states mass equations. Those results, opened a way to study the transition in a more precise manner. If both the string theory on the blow–up geometry and the orbifold with vevs are coincident, then the massless spectrum should be completely identified.

In the attempts to describe the departure from the orbifold point within realistic compact orbifolds [83, 84] some difficulties were encountered. The models were the $\mathbb{Z}_{6-II}$ Mini–Landscape [47, 62, 85][15] and the $\mathbb{Z}_2 \times \mathbb{Z}_2$ Blaszczyk model of [87]. The problems have two sources. One is the absence of a unique way to perform the toric resolution. In fact, there are many different resolutions connected by *flop transitions* [84] . The second issue is the existence of discrete torsion [88, 89], which allows for brother models. For the particular case of $\mathbb{Z}_{6II}$, the brother models are disregarded once one considers only orbifold models whose physical states have consistent orbifold transformations [89,90]. Thus the ambiguity in the identification of the blow–up geometry and the corresponding orbifold deformation, arising because the identification of the vector bundle with the local orbifold shift is only up to lattice vectors is not present.

There is a complementary approach to explore the transition which was proposed in [91]. This method makes use of the fact that on the orbifold, localized anomalies [92] are understood in terms of chiral states at the fixed sets. On the blow–up, one can also talk about certain localization, on the cycles appearing in the resolution. Using the Green–Schwarz anomaly polynomial [28, 93] one can study the transition by comparing the anomaly in the blow–up and the anomaly on the orbifold deformed by vevs. At the first sight the

[14] Which will be the sum of the rotation shift V and local *Wilson lines* (embedding of lattice translations on gauge d.o.f.).

[15] There are other realistic orbifold construction as the one presented in [86].

anomaly cancellation mechanism is very different. On the orbifold there is only one axion needed to cancel an *universal anomaly.* Whereas in the blow–up there are many anomalous $U(1)$ and many axions which cancel them. It is important to mention that here we depart from immediate phenomenological applications, because in all the toric blow–ups known presently from standard $T^6/\mathbb{Z}_n$ all the gauge $U(1)s$ including the hypercharge turn out to be anomalous [16]. Nevertheless we can apply in the future the results of this investigation to phenomenologically more interesting schemes. The localization of the anomalies on the orbifold can be seen from the localized (twisted) chiral spectrum, whereas in the blow–up there are local axions, which descend from orbifold twisted fields attaining vevs (*blow–up mode*). Therefore, if the orbifold constitutes the blow–down limit of the toric blow–up, the anomaly polynomial encodes the complete information of the transition. This is directly related to the massless chiral spectrum, which has to be matched with the use of field redefinitions. [17] In our work, we will study two cases of toric resolutions of orbifolds. We will look at the transition from the two sides. First we will focus on achieving a match of the chiral massless spectrum in orbifold and in the blow–up. With that information at hand, we will study the transition through the match of the anomaly cancellation mechanism in both moduli regions.

Outline We proceed now to the outline of the subjects presented in the thesis. The Chapter 2 is devoted to review the heterotic $E_8 \times E_8$ string theory. We start with the action in the fermionic formulation, to show how the string vacuum is constructed. We explain the *GSO projection* which is necessary to project out the tachyon and get a consistent theory. Then, there is a section devoted to the bosonic formulation and its toroidal compactification. This has been proven to be quite useful in model building. Toroidal compactification already involves a *twist* of the theory by non-trivial boundary conditions which is generalized to the orbifold twist, as explained in Chapter 3. We end this Chapter 2 by writing the bosonic terms of the 10d $\mathcal{N} = 1$ supergravity, which illustrate how the different fields transform under the $E_8 \times E_8$ gauge symmetry.

In Chapter 3 we describe various aspects of 6d compactifications preserving $\mathcal{N} = 1$ in 4d. We start with 6d orbifolds of the heterotic $E_8 \times E_8$ theory, describing the orbifold group by its geometrical action and its embedding in the gauge degrees of freedom. We explain then the conditions on the orbifold twist to ensure $\mathcal{N} = 1$ supersymmetry in 4d. We write the modes expansions of the world–sheet fields, using the formula for the zero-point energy of bosonic or fermionic oscillators with twisted boundary conditions, to arrive at the level matching condition. We also give the consistency conditions that the twist and the Wilson lines have to fulfill. Then the $\mathbb{Z}_3$ orbifold is reviewed in some detail, because it is the first known example in which twisted moduli were identified with blow–up modes. Next, in section 3.4 is devoted to orbifold selection rules for the couplings. Some critical discussion of these rules based on an ongoing collaboration [94] are summarized. We report on an exploration of the orbifold automorphism group, focusing on subgroups

[16] A possible way out is to consider freely acting Wilson lines as in [87].

[17] This has to be done because on the orbifold the chiral states posses left moving momenta given in a distinct base as the momenta of massless chiral states of the supergravity.

which could lead to discrete R-symmetries or flavor symmetries in 4d. We come then to smooth compactifications describing Calabi–Yau manifolds. Poincaré duality, vector bundles, manifolds with $SU(3)$ structure and toric resolutions of non-compact and compact orbifold singularities are described. Section 3.7 is devoted to the implementation of the Green-Schwarz anomaly mechanism in the compactification of a generic CY with internal abelian gauge fluxes.

Chapter 4 is based on the collaboration [95]. In this work we analyze the orbifold $T^6/\mathbb{Z}_7$ and its resolution. In this case there are no ambiguities arising from flop transitions. We start by describing the geometry of the $T^6/\mathbb{Z}_7$ orbifold and its resolution. The relevant topological information on the toric CY is given, the supergravity on the resolution is reviewed, and it is explained how the CY Kähler moduli arise on the new cycles. Then, we analyze the spectrum of the supergravity on the CY and the heterotic string on the deformed orbifold. A perfect agreement is obtained. This study is done with the help of a local index theorem, which can be applied in this context because the BI are satisfied locally. The last section is devoted to the analysis of the anomaly. Here, we obtain the polynomial on the resolved space by dimensional reduction, and on the orbifold based on the non-chiral spectrum. To study the transition we have to apply field redefinitions and make a detailed analylisis of the massless spectrum. Finally, we check that the 4d anomalies are canceled on the resolved space. The non-universal blow–up axions are identified with the orbifold blow–up modes. The detailed identification of the states and the explicit anomaly formulas are given in the appendices.

The last chapter describes the study of a $T^6/\mathbb{Z}_{6II}$ orbifold and its resolution. We start again with the orbifold geometry, and then we describe the resolution. The present orbifold can have multiple local resolutions connected by flop transitions. We make a choice by selecting the same resolution at all local singularities. It turns out that two of the five possible resolutions are simpler and we focus on those, which we call A and B. We give the BI for these two cases, and explain how the search for solutions is performed. We consider a big sample of orbifold models. Those are the Mini-landscape models [47,62,85]. Selecting one of those orbifold models we search for candidate blow–up modes among the orbifold twisted singlets. For triangulation A the encountered solutions to the BI fail to match exactly the considered orbifold model. In triangulation B we find many solutions in which blow–up modes can be identified on the orbifold. We discuss also an exploration carried out over all different triangulations. Then for one set of blow–up modes, we study the matching between the deformed orbifold spectrum and the one in the resolution. First we find that is possible to make redefinitions in which a local index theorem is manifest. Imposing an agreement with orbifold mass terms, the allowed redefinitions are more restrictive, but we find at least one example in which the match works perfectly. Then, we study the anomaly cancelation mechanism in four dimensions. We compute the anomaly in the deformed orbifold by vevs and compare it with the dimensional reduction on the resolution of the 10d anomaly. We find a perfect agreement, and we are able to identify local blow–up modes as non–universal axions on the resolution. On the other hand the resolution universal axion turns to be a mixture of the single orbifold axion and the blow–up modes. This check helps to establish the vacuum away from the orbifold via twisted fields vevs as the CY manifold obtained by resolving the orbifold.

Chapter 2

Heterotic String Theory

In this chapter we review the heterotic string theory and its massless spectrum in 10d . We start with the fermionic construction of the theory, describing the spectrum. Then, we explain the concept of GSO projection, which leads to a consistent superstring theory. We discuss then the bosonic formulation of the theory and its compactification on toroidal spaces. We conclude with the bosonic action of the 10d $\mathcal{N} = 1$ supergravity theory, which constitutes an effective description of the heterotic string. This review is based on [96–101] and we use the notation of [96].

2.1 Heterotic String Theory

String theory at the classical level studies the propagation of one–dimensional objects. This is described by a map X from the *world–sheet* Σ into the space–time $\mathcal{M}$

$$X : \Sigma \to \mathcal{M}. \tag{2.1}$$

These configurations are weighted by an action whose bosonic part is essentially the area of the world–sheet. The world–sheet coordinates are given by σ and τ parametrizing world–sheet space and time. Left and right moving modes depend on the holomorphic coordinate $z = \sigma - \tau$ and the anti–holomorphic coordinate $\bar{z} = \sigma + \tau$ respectively. Here we use euclidean signature with τ purely imaginary. Closed string theories have independent left– and right–moving sectors, in which the fields depend only on z or $\bar{z}$.

The bosonic string considers only the modes X^μ of the map X. The spectrum of this theory contains only space–time bosons, including a tachyon. Therefore, to obtain a consistent theory with space–time fermions superconformal extensions of the action are needed. Fermions can have two different boundary conditions. That gives the vacuum a more complicated structure. Furthermore, there is only a finite number of superconformal theories, that can be used in string theory.

The heterotic string is a closed string theory whose world–sheet has a $(0,2)$ superconformal symmetry. The action is invariant under the conformal group and global world–sheet supersymmetry. Only an $\mathcal{N}=1$ world–sheet supersymmetry is local on the right hand side. In the fermionic formulation one has 10 left–moving bosons, 32 left–moving fermions,10 right–moving bosons and 10 right–moving fermions [23]:

$$X^\mu(z,\bar z), \lambda^A(z), \tilde\psi^\mu(\bar z), \quad \mu=0,...,9, A=1,...,32. \tag{2.2}$$

The gauge fixed action is given by

$$S=\frac{1}{4\pi^2}\int d^2z\left(\frac{2}{\alpha'}\partial X^\mu\bar\partial X_\mu+\lambda^A\bar\partial\lambda^A+\tilde\psi^\mu\partial\tilde\psi_\mu\right). \tag{2.3}$$

The central charge of a conformal theory measures an anomalous violation of the Weyl invariance by quantum effects. This reflects itself in the deviation of the transformation law of the energy–momentum–tensor from the tensor transformation law, under the conformal symmetry. A critical theory is Weyl invariant at the quantum level. This is achieved by canceling the total central charge. The gauge fixing requires a ghost system. On the left–moving side one has the b,c ghosts of the bosonic string. On the right–moving side we have in addition the β,γ ghosts coming from the right moving side of the type II string theory. These ghost systems contributes $(c^g,\tilde c^g)=(-26,-15)$ to the left– and right– central charges. As bosons contribute $+1$ to the central charge and fermions $+\frac{1}{2}$ one needs: 10 left–bosons $X^\mu(z)$, 32 left–fermions $\lambda^A(z)$, 10 right– bosons $X^\mu(\bar z)$ and 10 right–fermions $\tilde\psi^\mu(\bar z)$ in order to cancel the central charge. Therefore the critical dimension of the theory is 10. The world–sheet theory has symmetry $SO(9,1)\times SO(32)$. In addition, the constraints on physical states for the right–moving modes are the ones of the type II theory and the constraints on the left–moving modes are the ones of the bosonic string. This implies that the λ^A can not have a time-like signature, because negative-norm states can not be removed due to the absence of fermionic constraints.

We use world–sheet coordinates $z=e^{-iw}$ where $w=\sigma^1+i\sigma^2$ and $\sigma^1,\sigma^0=-i\sigma^2$ are the space- and time–like coordinates. The theory contents only closed strings, so is possible to consider different boundary conditions for left and right movers

$$\begin{aligned}
X^\mu(w+2\pi) &= X^\mu(w), &(2.4)\\
\tilde\psi^\mu(\bar w+2\pi) &= \pm\tilde\psi^\mu(\bar w), &(2.5)\\
\lambda^A(w+2\pi) &= \begin{pmatrix} \eta\lambda^A(w), & A=1...16, & \eta=\pm1 \\ \eta'\lambda^A(w), & A=17....32, & \eta'=\pm1\end{pmatrix}. &(2.6)
\end{aligned}$$

The plus and minus sign for the fermions denote *Ramond* (R) and *Neveu–Schwarz* (NS) boundary conditions. The periodicity of the λ^A fermions is only required up to a rotation in $SO(32)$. There are only two choices of boundary conditions which give space-time supersymmetry. The one given here yields the *heterotic* $E_8\times E_8$ *theory.* The other is the *heterotic* $SO(32)$ *theory.* These two choices are the only ones which lead to a modular invariant partition function for the 10d heterotic string as we will see. These choices are precisely the ones which make the 10d theory anomaly free. Later we will allow different periodicity conditions also for bosons, in the definition of a twisted orbifold theory. One can compute the zero point energies using the fact that periodic bosons contribute to it $-\frac{1}{24}$, while periodic and anti-periodic fermion will do it in $\frac{1}{24}$ and $-\frac{1}{48}$. The zero point energy is present in the equation for the mass levels of the string.

Let us consider the gauge-fixed form of the theory in old covariant quantization. After imposing the light cone-gauge, one can remove the negative-norm states, and there will be only 8 transverse bosons in the left and the right, and 8 transverse fermions $\tilde{\psi}$ on the right. The 32 fermions λ^A are still all present. The normal ordering constants for the sectors with different boundary conditions for the fermions are given by

$$\begin{aligned} \widetilde{NS} \quad &: \quad -\frac{8}{48}-\frac{8}{24}=-\frac{1}{2} \quad \widetilde{R}: \frac{8}{24}-\frac{8}{24}=0, \\ NS-NS' \quad &: \quad -\frac{8}{24}-\frac{32}{48}=-1, \quad R-NS': -\frac{8}{24}+\frac{16}{24}-\frac{16}{48}=0, \\ R-R' \quad &: \quad -\frac{8}{24}+\frac{32}{24}=1. \end{aligned} \tag{2.7}$$

Here we have used $\sim$ to denote the right–hand side sector. The two letters in the left–hand side denote the boundary conditions for $1 \leq A \leq 16$ and $17 \leq A \leq 32$ respectively. The last line in equations (2.7) corresponds to a sector that will not give rise to massless states.

2.2 Superstring vacuum

Let us briefly describe the *R–NS* vacuum for the fermionic levels of the string. We use the notation of the right moving modes, but the formalism also applies to the boundary conditions of the λ^A. Writing the boundary conditions as $\tilde{\psi}(\bar{w}+2\pi)=e^{-2\pi\tilde{\nu}}\tilde{\psi}^\mu(\bar{w})$ where $\tilde{\nu}=0,\frac{1}{2}$ represent R and NS boundary conditions respectively. After a transformation to the variable z the mode expansion for the fermions will be

$$\tilde{\psi}(\bar{z})=\sum_{r\in\mathbb{Z}+\tilde{\nu}}\frac{\tilde{\psi}^\mu_r}{\bar{z}^{r+1/2}},\ \{\tilde{\psi}^\mu_r,\tilde{\psi}^\nu_s\}=\eta^{\mu\nu}\delta_{r,-s}. \tag{2.8}$$

The NS sector has no zero mode, so the ground state is by definition annihilated by all the $r>0$ modes

$$\tilde{\psi_r}^\mu|0\rangle_{\rm NS}=0, \qquad r>0, \tag{2.9}$$

and it has no further structure. Instead the R ground state is by definition annihilated by all $r>0$ modes

$$\tilde{\psi_r}^\mu|\text{vac}\rangle_R=0, \qquad r>0. \tag{2.10}$$

The Ramond vacuum is therefore degenerated due to the relation $\{\tilde{\psi}^\mu_0,\tilde{\psi}^\mu_r\}=0$. So that the action of the zero modes $\tilde{\psi}^\mu_0$ on the ground state give another ground state. The modes $\Gamma^\mu=2^{1/2}\bar{\psi}^\mu$ can be represented by the gamma matrices. Such that the ground state forms a representation of the Clifford algebra. This representation is 32 dimensional in 10d . A convenient basis for the gamma matrices is given by

$$\Gamma^{0\pm} = \frac{1}{2}(\pm\Gamma^0 + \Gamma^1), \quad \Gamma^{a\pm} = \frac{1}{2}(\Gamma^{2a} \pm i\Gamma^{2a+1}). \tag{2.11}$$

The algebra in this basis reads

$$\{\Gamma^{a+}, \Gamma^{b-}\} = \delta^{ab}, \quad \{\Gamma^{a+}, \Gamma^{b+}\} = \{\Gamma^{a-}, \Gamma^{b-}\} = 0, \; a = 1, ..., 4. \tag{2.12}$$

Acting repeatedly with Γ^{a-} is possible to reach the state given as $\forall_a \; \Gamma^{a-}\zeta = 0$. Then by acting on such a state with Γ^{a+} in all the possible ways we get

$$\mid \mathbf{s}\rangle_{\mathrm{R}} = (\Gamma^{4+})^{s_4+1/2}(\Gamma^{3+})^{s_3+1/2}(\Gamma^{2+})^{s_2+1/2}(\Gamma^{1+})^{s_1+1/2}(\Gamma^{0+})^{s_0+1/2}\zeta, \tag{2.13}$$

where $s_a = \pm\frac{1}{2}$. These states are the Ramond ground states $\mid \mathbf{s}\rangle_{\mathrm{R}} = |s_0, s_1, s_2, s_3, s_4\rangle_{\mathrm{R}}$. They are eigenvectors of the spin operator $S_a = \Gamma^{a+}\Gamma^{a-} - \frac{1}{2}$ with eigenvalues

$$S_a \mid \mathbf{s}\rangle_{\mathrm{R}} = s_a \mid \mathbf{s}\rangle_{\mathrm{R}}. \tag{2.14}$$

This 32 dimensional representation (2.13) decomposes as $\mathbf{32} = \mathbf{16} + \mathbf{16}'$. The irreducible parts have eigenvalues 1 or -1 under $\Gamma = \prod_{\mu=0}^{9} \Gamma^\mu$. Note that $\{\Gamma, \tilde{\psi}_0^\mu\} = 0$. The space–time Lorentz generators which define the spin $S_a = i^{\delta_{a,0}}\Sigma^{2a,2a+1}$ are given by

$$\Sigma^{\mu\tau} = -\frac{i}{2} \sum_{r\in\mathbb{Z}+\nu} [\tilde{\psi}_r^\mu, \tilde{\psi}_{-r}^\tau]. \tag{2.15}$$

We can also define the *world–sheet fermion number* as

$$F = \sum_{a=0}^{4} S_a, \tag{2.16}$$

which obeys $\{e^{\pi i F}, \tilde{\psi}^\mu\} = 0$. So that $\tilde{\psi}^\mu$ changes the world–sheet fermion number by 1. The ghosts also contributes to the world–sheet fermion number: By -1 in the NS right sector and by i in the R right sector. There are no ghosts for the left–handed fermions.

From the above discussion we see that the states on the R sector will always have half-integer spin, because the vacuum has half-integer spin and the oscillators change the spin by one. The simplest vacuum is the NS one, there the ground state is annihilated by the $\Sigma^{\mu\lambda}$. This implies that it is a Lorentz singlet, so all other states have integer spin.

2.3 GSO projection

For the fermionic sectors of the superstring consistency requires, that there is only a subset of states in the theory. More precisely, this consistency ensures that the operator product expansions (OPE) of vertex operators are single valued. The subset is selected by a projection involving the fermion number operator as well as by specifying the R or NS boundary conditions. In the $E_8 \times E_8$ heterotic theory these boundary conditions were specified by (2.6) and one considers the projection

$$e^{\pi i F_1} = e^{\pi i F_1'} = e^{\pi i \tilde{F}} = 1, \tag{2.17}$$

where the first two operators anticommute with λ^A for $A = 1, ..., 16$ and $A = 17, ..., 32$, and the last operator anticommutes with $\tilde{\psi}^{\mu 1}$.

On the right–hand side the projection gives at the massless level a spinor $\mathbf{8}$ and a vector $\mathbf{8}_v$ of $SO(8)$. This group acts on the transversal degrees of freedom and is the *Little group* of the 10d Lorentz group for massless states. One obtains the above mentioned spinor and vector by applying the physical state condition on the massless states, see [96]

$$e_\mu \tilde{\psi}^\mu_{-1/2} |0; k\rangle_{\mathrm{NS}}, \quad |\mathbf{s}; k\rangle_R u_{\mathbf{s}}. \tag{2.18}$$

Where e_μ and $u_{\mathbf{s}}$ specify the polarization of the states and k is the ground state momentum. The first of those states is massless $\mathbf{8}_v$ vector boson. This states survives the GSO projection, while the NS tachyon $|0; k\rangle_{\mathrm{NS}}$ is projected out. The second state has to be decomposed and the component which survives the GSO projection $e^{\pi i \tilde{F}} = 1$ has to be selected. The physical state condition gives the massless Dirac equation

$$k \cdot \Gamma_{ss'} u_s \to \left(S_0 - \frac{1}{2} \right) |\mathbf{s}; k\rangle_{\mathrm{NS}} = 0, \tag{2.19}$$

such that only states with $s_0 = \frac{1}{2}$ survives. Using the decomposition of the Ramond ground state $\mathbf{32} = \mathbf{16} + \mathbf{16'}$ under $SO(9,1) \to SO(1,1) \times SO(8)$ one sees that only the state $\mathbf{8}$ with positive spin survives i.e.

$$\mathbf{16} \to \mathbf{8}_{\frac{1}{2}} + \mathbf{8'}_{-\frac{1}{2}}. \tag{2.20}$$

So the right hand side of the heterotic $E_8 \times E_8$ string has the vector $\mathbf{8}_v$ and the spinor $\mathbf{8}$ which will lead to the correct massless supersymmetric spectrum.

Now let us describe the left–moving massless states. On the NS-NS' sector the first excited states with $m = 0$ are

$$\alpha^i_{-1} |0\rangle_{NS,NS'}, \quad \lambda^A_{-\frac{1}{2}} \lambda^B_{-\frac{1}{2}} |0\rangle_{NS,NS'}, \tag{2.21}$$

where $1 \leq A, B \leq 16$ or $17 \leq A, B \leq 32$. This happens because of the GSO projections (2.17) separate the fields λ^A in two subsets. The boundary conditions (2.6) break the initial $SO(32) \to SO(16) \times SO(16)$. From the second states in (2.21) one gets an antisymmetric tensor $\mathbf{120}$.

The R-NS' sector has a massless ground state. In the R sector the zero modes of λ^A acting on a ground state will preserve the zero mass condition, so it is possible to construct rising

[1] The fermion number in the NS sector is given by $F = \sum_r \tilde{\psi}_{-r} \cdot \tilde{\psi}_r$.

and lowering operators in analogy to (2.11). Then, one obtains a $2^8 = 256 = 128 + 128'$ spinor representation. The 256 is divided in two copies, according to the eigenvalues of $e^{\pi i F}$. So after the projection only the **128** remains. The massless states for the left-movers are given by

$$(\mathbf{8}_v, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{120}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{120}) + (\mathbf{1}, \mathbf{128}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{128}). \tag{2.22}$$

Now one has to tensor this massless left moving states with the massless right moving states $(\mathbf{8}_v + \mathbf{8})$. Taking the 1st and the 3th term or the 2nd and the 4th term from (2.22) one sees that the product includes for every $SO(16)$ factor a vector bosons $(\mathbf{8}_v)_r(\mathbf{120} + \mathbf{128})_l$. One can easily see that this fits in the adjoint representation of E_8, which is the actual gauge group. Firstly the **248** adjoint representation of E_8 decomposes as $\mathbf{120} + \mathbf{128}$ under $SO(16) \subset E_8$. Secondly we only saw the symmetry $SO(8) \times SO(16) \times SO(16)$, one can construct additional currents which complete the world–sheet symmetry to $SO(8) \times E_8 \times E_8$. Those are obtained via bosonization of the fields λ^A as will be discussed in the next section. [2].

The massless spectrum of the heterotic string theory in the critical dimension is given by a 10d supergravity multiplet plus a gauge multiplet. One can now write the states (2.22) in terms of representations of $SO(8) \times E_8 \times E_8$. Performing the tensor product between the left and the right movers one gets the massless spectrum

$$\begin{aligned} (\mathbf{8}_v + \mathbf{8})_r \quad &\times \quad ((\mathbf{8}_v, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{248}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{248}))_l \\ &= \quad (\mathbf{1}, \mathbf{1}, \mathbf{1}) + (\mathbf{28}, \mathbf{1}, \mathbf{1}) + (\mathbf{35}, \mathbf{1}, \mathbf{1}) + (\mathbf{56}, \mathbf{1}, \mathbf{1}) + (\mathbf{8}', \mathbf{1}, \mathbf{1}) \\ &\quad\quad + (\mathbf{8}_v, \mathbf{248}, 1) + (\mathbf{8}, \mathbf{248}, \mathbf{1}) + (\mathbf{8}_v, \mathbf{1}, \mathbf{248}) + (\mathbf{8}, \mathbf{1}, \mathbf{248}). \end{aligned} \tag{2.23}$$

In the first line of the expanded formula we can see the $\mathcal{N} = 1$ supergravity multiplet. The second line collects the $\mathcal{N} = 1$ gauge multiplet. The fields are the *dilaton* $(\mathbf{1}, \mathbf{1}, \mathbf{1})$, the *antisymmetric tensor* $(\mathbf{28}, \mathbf{1}, \mathbf{1})$, the *metric* $(\mathbf{35}, \mathbf{1}, \mathbf{1})$, the *gravitino* $(\mathbf{56}, \mathbf{1}, \mathbf{1})$, the *neutral fermion* $(\mathbf{8}', \mathbf{1}, \mathbf{1})$, the *gauge boson* $(\mathbf{8}_v, \mathbf{248}, 1)$ and the *gaugino* $(\mathbf{8}, \mathbf{248}, \mathbf{1})$. There are two other gauge bosons and gauginos belonging to the second E_8 group.

2.4 Bosonic construction

Another description of the heterotic theories can be performed by considering a CFT with 26 bosonic left movers $X^\mu(z)$, $X^I(z)$, $\mu = 0, ..., 9$, $I = 1, ..., 16$, and 10 right–moving scalars and fermions $X^\mu(\bar{z}), \tilde{\psi}^\mu(\bar{z})$, $\mu = 0, ..., 9$. This theory has also central charge zero. If there are only $d < D$ non-compact dimensions, then the continuous momenta are denoted by k^μ. The compact momenta are denoted by (k_L^m, k_R^n) with $d \leq m \leq 25$, $d \leq n \leq 9$. The compact dimensionless momenta take values in a lattice $\Gamma_{m,n}$, they are related to the ordinary momenta k by $l = k(\alpha'/2)^{1/2}$.

[2]Bosonization is a way of describing the conformal field theory with certain fermions in terms of an identification with bosonic degrees of freedom, such that all the OPE of the original theory are reproduced, and both theories are equivalent.

The lattice should fulfill some conditions to have a consistent conformal theory, which is local and modular invariant. Here we explain how this occurs in the case of the bosonic theory. Locality means that the OPE of vertex operators is single valued. Writing the vertex operators for the winding states with momentum as : $e^{ik_L\cdot X_L(z)+ik_R\cdot X_R(\bar{z})}$: the condition of a single valued OPE of two vertex operators requires

$$l_L \cdot l'_L - l_R \cdot l'_R \equiv l \circ l' \in \mathbb{Z}. \tag{2.24}$$

This condition implies the lattice is included in the dual lattice $\Gamma \subset \Gamma^*$, because the dual lattice is defined as all set of points which have integer product with the lattice elements. The modular invariance condition can be seen by writing the one–loop partition function on the torus, with τ the complex structure modulus of the torus

$$Z_\Gamma \sim \sum_{l\in\Gamma} e^{\pi i\tau l_L^2 - \pi i\bar{\tau} l_R^2}. \tag{2.25}$$

Under the *T–transformation* $\tau \to \tau + 1$ we obtain

$$l \circ l \in 2\mathbb{Z}. \tag{2.26}$$

This last condition implies (2.24). While the *S–transformation* $\tau \to -1/\tau$ applied on (2.25) can be worked out with the Poisson resummation formula and gives

$$Z_\Gamma(\tau) = V_{\Gamma^*}^{-1} Z_{\Gamma^*}(-1/\tau), \tag{2.27}$$

which implies

$$\Gamma = \Gamma^*. \tag{2.28}$$

So, the consistency of the theory requires the lattice to be *even* (2.26) and *self-dual* (2.28). The signature for this even and self-dual lattice is $(26-d, 10-d)$.

The case of $d = 10$ non-compact dimensions is interesting. One has only the left part in (l_L, l_R) i.e. a lattice of dimension 16. There are only two of those even and self–dual lattices in dimension 16. Namely Γ_{16}, giving the heterotic $SO(32)$ theory and $\Gamma_8 \times \Gamma_8$ giving the heterotic $E_8 \times E_8$. With the choice in (2.6) we get only the latter, which is given by

$$\Gamma_8 \quad = \quad \left\{ \begin{array}{c} (n_1, ..., n_8) \\ (n_1 + \frac{1}{2}, ..., n_8 + \frac{1}{2}) \end{array} \right. \text{ with } n_i \in \mathbb{Z}, \sum_i n_i \in 2\mathbb{Z}. \tag{2.29}$$

At the massless level the vertex operators of the left–hand side contain the following currents

$$\partial X^m, \quad m = 1, ..., 16, \quad \partial X^\mu, \quad V_0(k) \sim : e^{\pi i k_L \cdot X(z)} : . \tag{2.30}$$

To have a massless vector boson the discrete momenta should satisfy $l_L^2 = 2$, and recall that $k_L = (2/\alpha')^{1/2} l_L$.The momenta p_L^m in the compact dimension are associated to the commuting currents $\partial X^m(z)$. The operators $V_0(k)$ have commutation relations with them

$$[p_L^m, V_0(k)] = l_L^m V_0(k). \tag{2.31}$$

The operators p_L^m and $V_0(k)$ generate a gauge group determined by the lattice Γ. This can be seen from the massless condition for the gauge bosons $l_L^2 = 2$ and the commutator (2.31), which imply that l_L are the roots of the gauge group. This associates p_L^m to the sixteen Cartan elements H^I and $V_0(k)$ to the generators E^α in the Cartan–Weyl basis of the Lie algebra[3].

The elements $l_L \in \Gamma_8$ in (2.29) with square $l_L^2 = 2$ are the roots of E_8. The same is true for the lattice Γ_{16} which will give the roots of $SO(32)$. So in this bosonic description with heterotic dimensions we have a gauge theory with gauge group $E_8 \times E_8$ or $SO(32)$.

2.5 Toroidal compactification

A toroidal compactification to d dimensions will give a lattice Γ with signature $(26-d, 10-d)$. The conditions (2.26) and (2.28) are invariant if the momenta product $\circ$ is preserved. The most general transformation which preserves this product is the boost $O(26-d, 10-d, \mathbb{R})$, but this is not a symmetry of the theory. This due to the fact that the mass-shell condition and the OPE depend on the separate products of the left and right parts of the momenta. Therefore only the $O(26-d, \mathbb{R})$ and $O(10-d, \mathbb{R})$ rotations will preserve l_L^2 and l_R^2 respectively, and be a symmetry of the theory. Denoting $O(26-d, 10-d, \mathbb{Z})$ the discrete subgroup of $O(26-d, 10-d, \mathbb{R})$ which takes a lattice Γ into itself, then the space of inequivalent compactifications or moduli space is

$$\frac{O(26-d, 10-d, \mathbb{R})}{O(26-d, \mathbb{R}) \times O(10-d, \mathbb{R}) \times O(26-d, 10-d, \mathbb{Z})}. \tag{2.32}$$

This space is (26-d)(10-d) dimensional and the corresponding moduli can be interpreted in terms of background fields, namely the metric, the antisymmetric tensor and the Wilson lines. The T–Duality group (2.32) includes the transformations

$$\begin{aligned} &R \to \alpha'/R, \ \text{on the radius of the different directions,} \\ &X'^m \to L_n^m X^n, \ L_n^m \in \mathbb{Z}, \ \det L = 1, \ \text{which preserve the lattice,} \\ &B_{mn} \to B_{mn} + N_{mn}, \ N_{mn} \in \mathbb{Z}. \end{aligned} \tag{2.33}$$

The unbroken gauge symmetry is given by the massless gauge bosons present in the spectrum. The vertex operators are given by

$$V_1 = \partial X^m \tilde{\psi}^\mu, \quad V_2 = \partial X^\mu \tilde{\psi}^m, \quad V_3 = e^{ik_L \cdot X_L} \tilde{\psi}^\mu, \ l_L^2 = 2, \ l_R = 0. \tag{2.34}$$

[3]As it can be seen in [102], one needs to use an integral form of $V_0(k)$ that commute with the L_n and this will guaranty to have the Lie algebra commutation relations.

The first two are $26-d$ and $10-d$ gauge bosons, from them 16 are the original gauge bosons of the Cartan group in ten dimensions (2.30), and $2(10-d)$ are Kaluza-Klein modes coming half of them from the metric and the antisymmetric tensor dimensional reduction. For a generic transformation $O(26-d, 10-d, \mathbb{R})$ there is a point in moduli space without $l_R = 0$, so there will be no bosons V_3 and the symmetry will be $U(1)^{36-2d}$. There are also points of enhanced gauge symmetries at specific points on the moduli space (2.32)

The supersymmetry preserved in a compactification to $d = 4$ dimensions can be understood by looking at the decomposition of the massless states in terms of the helicity $U(1)$, such that $U(1) \times SO(6) \subset SO(8)$. In performing the product $(\mathbf{8})_r \times (\mathbf{8}_v, 1, 1)_l$, taken from the massless spectrum (2.23) one obtains four gravitinos with helicity $\frac{3}{2}$, which tells that there are $\mathcal{N} = 4$ supersymmetry in $d = 4$. This decomposition can be made more explicitly as

$$\begin{aligned}
&\mathbf{8}_v \to \mathbf{6}_0 + \mathbf{1}_1 + \mathbf{1}_{-1}, \\
&\mathbf{8}_s \to \mathbf{4}_{1/2} + \bar{\mathbf{4}}_{-1/2}, \\
&\mathbf{8} \times \mathbf{8}_v \supset (\mathbf{56}, 1, 1) \to \mathbf{4}_{3/2} + \bar{\mathbf{4}}_{1/2} + \mathbf{4}_{-1/2} + \bar{\mathbf{4}}_{-3/2} + \mathbf{20}_{1/2} + \bar{\mathbf{20}}_{-1/2}.
\end{aligned} \tag{2.35}$$

This vacuum is not phenomenological appealing. In next chapter we will see two ways in which the supersymmetry in 4d can be reduced. One way is to take as starting point the torus, then the compactification space is constructed by modding out from the torus lattice one of its symmetries. This process defines a variety with curvature singularities which leads to $\mathcal{N} = 1$ in 4d.

2.6 $\mathcal{N} = 1$ Supergravity in $D = 10$

On our work there will be two approaches, one of them will be to study a compactification on a manifold from which we know the topological information but not the metric. For that purpose, we will study the dimensional reduction of the 10d effective heterotic theory to 4d. Thus, we need to describe the effective supergravity coupled to super Yang–Mills action, the one loop effect given by anomaly cancellation and the index theorem which serves to compute states multiplicities in 4d. Here we focus on the bosonic part of the 10d $\mathcal{N} = 1$ effective theory.

The bosonic part of the supergravity action for the heterotic string in 10d is given by

$$S_{het} = \frac{1}{2\kappa_{10}^2} \int d^{10}x (-G)^{1/2} e^{-2\Phi} \left[\mathfrak{R} + 4\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} |\widetilde{H}_3|^2 - \frac{\kappa_{10}^2}{g_{10}^2} \frac{1}{30} \mathrm{Tr}_a(|\mathfrak{F}|^2) \right]. \tag{2.36}$$

This action is composed by the bosonic fields of the theory. In the equation the trace runs over the adjoint of $E_8 \times E_8$, and the factor $\frac{1}{30}$ in the normalization is set in order to agree with the notation for the $SO(32)$ theory. The fields are the dilaton Φ, the antisymmetric tensor B_2 and the gauge potential $\mathfrak{A}$. We denote the 10d curvature and gauge field strengths with $\mathfrak{R}$ and $\mathfrak{F}$ respectively. The 3-form $\widetilde{H}_3$ is defined as

$$\widetilde{H}_3 = dB_2 - c\omega_{3Y} - c'\omega_{3L}. \tag{2.37}$$

The gauge and Lorentz *Chern-Simons 3-forms* are

$$\omega_{3Y} = \frac{1}{30}\mathrm{Tr}_a(\mathfrak{A}_1\mathfrak{A}_1 - \frac{2i}{3}\mathfrak{A}_1^3), \quad (2.38)$$

$$\omega_{3L} = \mathrm{tr}\,(\mathfrak{W}_1\mathfrak{W}_1 + \frac{2}{3}\mathfrak{W}_1^3). \quad (2.39)$$

The Lorentz-Chern Simons term includes the *spin connection* $\mathfrak{W}_1 = \mathfrak{W}^p_{\mu_q} dx^\mu$. The trace tr appearing in the expression for ω_{3L} is in the fundamental of $SO(10)$. In the viel–bein formalism the curvature tensor is expressed in terms of the spin connection as

$$\mathfrak{R}_{\mu\nu} = \partial_{[\mu}\mathfrak{W}_{\nu]} + \mathfrak{W}_{[\mu}\cdot\mathfrak{W}_{\nu]}, \quad (2.40)$$

where the energy momentum tensor $\mathfrak{R}^p_{\mu\nu_q}$ is a 2-form with respect to space-time index μ, ν and a $d \times d$ matrix on the fundamental representation of $SO(d-1,1)$ with indices p and q. This description involves two local symmetries, coordinate invariance and local Lorentz transformations. The gauge transformations χ and local Lorentz transformations Θ that leave invariant the action (2.36) are given by

$$\begin{aligned}
\delta\mathfrak{A}_1 &= d\chi - i[\mathfrak{A}_1, \chi], \\
\delta\mathfrak{W}_1 &= d\Theta + [\mathfrak{W}_1, \Theta], \\
\delta\omega_{3Y} &= \frac{1}{30} d\,\mathrm{Tr}_a(\chi d\mathfrak{A}_1), \\
\delta\omega_{3L} &= d\,\mathrm{tr}(\Theta d\,\mathfrak{W}_1), \\
\delta B_2 &= \frac{1}{30} c\,\mathrm{Tr}_a(\chi d\mathfrak{A}_1) + c'\mathrm{tr}_a(\Theta d\mathfrak{W}_1).
\end{aligned} \quad (2.41)$$

The Lorentz term in $\hat{H}_3$ is not a leading contribution at low energies. The minimal supergravity action is obtained by setting $c' \to 0$. Nevertheless for the consistency of the ten dimensional theory this Lorentz term is required.

Chapter 3

Compactification

In this chapter we describe the essential features of $\mathcal{N}=1$ 6d compactifications of the heterotic string. Most of the chapter is a review, but we will also present some of our results. We start describing the general features of orbifolds and give the example of $\mathbb{Z}_3$ in the standard embedding [32, 33, 96, 103]. We continue with an exploration of orbifold discrete symmetries, performed in a joint work [94]. Then, we describe Calabi–Yau compactifications [96, 104]. To provide the connection between orbifolds and CY compactifications, we present the toric geometric techniques applied to resolve non–compact and compact orbifolds [68, 79, 81, 83]. We conclude the chapter with the implementation of the 4d anomaly cancellation as descending from the 10d Green–Schwarz mechanism [95].

3.1 Orbifolds

As we discussed the toroidal compactification of the heterotic string leads to a four dimensional theory with $\mathcal{N}=4$ supersymmetry. It is possible to define a theory in which a symmetry of the toroidal lattice is modded out. This will be an *orbifold compactification*. Toroidal and orbifold compactifications are *twisted theories*. To obtain such a theory one starts with a CFT having a symmetry group H. One can construct then a new theory in the following way: First one adds twisted sectors, in which the fields are periodic up to some $h \in H$ i.e. $\phi(\sigma^1 + 2\pi) = h \cdot \phi(\sigma^1)$. Then one restricts the spectrum to invariant states under H. This ensures modular invariance. The conformal world–sheet theory of the heterotic string is consistent in orbifolds [32, 63, 105]. This theory at the perturbative level, leads to physics in which the Standard Model of particles can be obtained.

One starts with the heterotic $E_8 \times E_8$ theory in ten dimensions and takes H to be a discrete subgroup of the Poincaré×gauge group

$$H \subset (\mathbb{R}(9,1) \rtimes SO(9,1)) \times (E_8 \times E_8). \tag{3.1}$$

The group H has two components. Let us start with the six dimensional internal space and perform the toroidal compactification by identifying points under the translations Γ_6. This $\Gamma_6 \subset \mathbb{R}(9,1)$ is a subset of the Γ with dimensions $(6,22)$ which was described in section 2.5. In this way we obtain $T^6 = \mathbb{R}^6/\Gamma_6$. Now we take an isometry group P of Γ_6, and perform a *modding* of this symmetry to get T^6/P. P is called the *point group.*[1] The transformation $P \times \Gamma_6$ possesses also an *embedding in the gauge group* which we call G. The orbifold is defined by [106]

$$\Omega = \mathbb{R}^6/(\Gamma_6 \rtimes P) \times \Lambda/G. \tag{3.2}$$

In the bosonic representation for the gauge sector of the heterotic theory, $\Lambda = \Gamma_8 \times \Gamma_8$ represents the internal 16d torus. On the other hand, in the fermionic description, Λ represents the set of gauge rotations in the manifest $SO(16) \times SO(16)$ subgroup of $E_8 \times E_8$.

The world–sheet fields transform under H as

$$X^k \quad \to \theta^{kn} X^n + l^k,\ k = 5, ..., 10, \tag{3.3}$$
$$\tilde{\psi}^k \quad \to \theta^{kn} \tilde{\psi}^n, \tag{3.4}$$
$$\lambda^A \quad \to \gamma^{AB} \gamma'^{BC} \lambda^C, \tag{3.5}$$
$$X^I \quad \to X^I + V^I + A^I, I = 1, ..., 16. \tag{3.6}$$

Here $\theta \in P$, $\ l \in \Gamma_6$ and $\gamma, \gamma' \in G$, $V, A \in G$ in the fermionic and in the bosonic representation respectively.

Let us look at the gauge embedding of the orbifold action. In the fermionic description for the gauge d.o.f. γ corresponds to the spatial orbifold twist θ, while γ' represents the embedding of the lattice translations l. In the bosonic description, V and A represent the gauge embedding of the spatial rotations θ and lattice displacements l, respectively. The quantities V, γ and γ', A are denoted *shifts* and *Wilson lines* respectively. The simplest models, as the one we present as an example in this chapter, do not posses Wilson lines. However, Wilson lines turn out to be essential in order to break the gauge symmetry down to the Standard Model.

As there are six internal dimensions, vectors in the toroidal lattice Γ can be expressed in terms of a basis e_α, $\alpha = 1, ..., 6$. Such that

$$\forall l \in \Gamma,\ l = n_\alpha e_\alpha, \quad A^I = n_\alpha A^I_\alpha, \quad \gamma' = \prod_\alpha (\gamma_\alpha)^{n_\alpha}, \tag{3.7}$$

where A_α or γ'_α is the Wilson line corresponding to the lattice translation e_α.

The *space group* $S = (\theta, l)$ is defined as the subset of the orbifold (3.2) acting on the spacial internal dimensions X^k. Strings will propagate in the internal space given as $\mathbb{R}^6/S$. The space group multiplication law is given by

$$(\theta_1, l_1)(\theta_2, l_2)X = (\theta_1\theta_2, l_1 + \theta_1 l_2)X. \tag{3.8}$$

[1]When this group is (non-)abelian the orbifold is called (non-)abelian.

Every gauge embedding will correspond to a unique space group element $\gamma(\theta, l)$, $\gamma'(\theta, l)$ or $V(\theta, l), A(\theta, l)$ such that

$$\gamma(\theta_1, l_1).\gamma(\theta_1, l_1) = \gamma((\theta_1, l_1).(\theta_2, l_2)). \tag{3.9}$$

Analogous relations hold for γ', V and A. Note that the fermionic right modes $\tilde{\psi}^k$ share the orbifold rotation (3.4). Therefore world–sheet supersymmetry is preserved, because the twist commutes with supersymmetry generator. Furthermore, important objects are the *fixed sets* (*fixed points* and *fixed tori*) under the orbifold action. Those are defined by

$$X_{\mathbf{f}} = \theta X_{\mathbf{f}} + l, \tag{3.10}$$

where $X_{\mathbf{f}}$ are the 6d coordinates of the internal space. Fixed points correspond to the case in which $det(1-\theta) \neq 0$. When the determinant vanishes we encounter fixed tori.

$\mathcal{N} = 1$ susy, $\mathcal{N} = 2$ sectors. We considered orbifolds generated by $\mathbb{Z}_N$ rotations that preserve the lattice Γ. Let us choose the orbifold action to be of the form

$$\theta = \exp(2\pi i(v_1 J_{45} + v_2 J_{67} + v_3 J_{89})), \quad \theta \in \mathbb{Z}_N, \tag{3.11}$$

i.e. the transformation is block-diagonal in the internal Lorentz group $SO(6)$. The quantities J_{45}, J_{67}, J_{89} are the generators of rotations in three distinct planes. Let us impose that there is $\mathcal{N} = 1$ supersymmetry surviving. This can be done by looking at the transformation of the supersymmetry algebra generators:

$$\begin{aligned} Q_\alpha &\to D(\theta)_{\alpha\beta} Q_\beta, \\ Q_{\mathbf{s}} &\to \exp(2\pi i s \cdot v) Q_{\mathbf{s}}. \end{aligned} \tag{3.12}$$

The index $\mathbf{s}$ denotes the spinor representation of $SO(6)$, and is given by $(s_1, s_2, s_3) = (\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$. If the condition

$$\sum_i v_i = 0, \tag{3.13}$$

is fulfilled, the surviving generators are the ones with $s_1 = s_2 = s_3$. This condition implies that θ lies in an $SU(3)$ subgroup of $SO(6)$. This $SU(3)$ is embedded in the 10d Lorentz group as

$$SO(9,1) \to SO(3,1) \times SO(6) \to SO(3,1) \times SU(3). \tag{3.14}$$

The **16** spinor representation of $SO(9,1)$ decomposes into $\to$ $(\mathbf{2}, \mathbf{3}) + (\bar{\mathbf{2}}, \bar{\mathbf{3}}) + (\bar{\mathbf{2}}, \mathbf{1}) + (\mathbf{2}, \mathbf{1})$. Note that the susy generators must be $SU(3)$ singlets, so only the supersymmetry generators $(\bar{\mathbf{2}}, \mathbf{1})$ and $(\mathbf{2}, \mathbf{1})$ survive the orbifold projection. This gives $\mathcal{N} = 1$ in 4d.

If for some element of the point group θ one of the v_i is zero, then there are fixed tori. Look for example at the case $v_3 = 0$, this implies $v_1 + v_2 = 0$ such that θ satisfies

$$\theta \in SU(2) \subset SU(3) \subset SU(6), \tag{3.15}$$

what will give $\mathcal{N} = 2$ susy.

Another way of looking at the $\mathcal{N} = 1$ susy condition is to analyze the massless spectrum (2.35), and check how many gravitini survive the projection. Under the group decomposition $SO(6) \times U(1) \to SU(3) \times U(1)$, the gravitino $\mathbf{4}_{3/2}$ decomposes as

$$\mathbf{4}_{3/2} \to \mathbf{3}_{3/2} + \mathbf{1}_{3/2}, \tag{3.16}$$

and similarly for $\bar{\mathbf{4}}_{-3/2}$. Then the gravitino $\mathbf{1}_{\frac{3}{2}}$ survives the projection, giving $\mathcal{N} = 1$ susy in 4d. These states will appear in the *untwisted sector* of the string. It has been shown that $\mathcal{N} = 1$ condition and a crystallographic action in the lattice Γ, implies to have a $\mathbb{Z}_N$ point group.

3.2 Mode expansions and consistency conditions

Now we write the solutions to the string equations of motion for the world–sheet fields of the 6d space. The bosonic spatial coordinates X^k, $k = 1, ..., 6$, can be arranged to define a complex $\mathbb{C}^3$ basis

$$Z^i = \frac{1}{\sqrt{2}}(X^{2i+2} + iX^{2i+3}), \;\; i = 1, 2, 3. \tag{3.17}$$

The same basis can be defined for the right moving fermionic coordinates

$$\tilde{\psi}'^i = \frac{1}{\sqrt{2}}(\tilde{\psi}^{2i+2} + i\tilde{\psi}^{2i+3}), \;\; i = 1, 2, 3. \tag{3.18}$$

In the orbifold, closed strings allow more general boundary conditions that they do in the 10d space or in the toroidal compactification of the theory. These boundary conditions are given by

$$\begin{aligned} Z^i(\sigma + 2\pi) &= e^{2\pi i\phi_i} Z^i(\sigma) + l^i, \\ \tilde{\psi}^i(\sigma + 2\pi) &= e^{2\pi i(\phi_i + \nu)} \tilde{\psi}^i(\sigma), \end{aligned} \tag{3.19}$$

where $\nu = 0, \frac{1}{2}$ denotes the R or NS sector respectively. The quantities ϕ_i are multiples of the orbifold twist v_i as $(\phi_1, \phi_2, \phi_3) = n(v_1, v_2, v_3)$. The integer n denotes the twist of the different sectors. There are *untwisted sectors* with $n = 0$, in which the mode expansions correspond to the ones of the toroidal compactification. For generic *twisted sectors* the oscillator expansion for the internal bosonic coordinates is given by [106]

$$Z^i = z^i_{\mathbf{f}} + \frac{i}{2} \sum_{n \neq 0} \left[\frac{1}{n + \phi_i} \alpha^i_{n+\phi_i} e^{i(n+\phi_i)\omega} + \frac{1}{n - \phi_i} \tilde{\alpha}^i_{n-\phi_i} e^{-i(n-\phi_i)\bar{\omega}} \right]. \tag{3.20}$$

The quantity $z_{\mathbf{f}}$ denotes the coordinates of a fixed set with respect to the orbifold action. The complex conjugate mode expansion can be computed directly from last equation to give

$$\bar{Z}^{\bar{i}} = \bar{z}^{i}_{\mathbf{f}} + \frac{i}{2}\sum_{n\neq 0}\left[\frac{1}{n-\phi_i}\alpha^{\bar{i}}{}_{n-\phi_i}e^{i(n-\phi_i)\omega} + \frac{1}{n+\phi_i}\tilde{\alpha}^{\bar{i}}_{n+\phi_i}e^{-i(n+\phi_i)\bar{\omega}}\right]. \tag{3.21}$$

In the left–handed sector the creation operators will be $\alpha^{i}_{-n+\phi_i}$ with $-n+\phi_i < 0$ or $\alpha^{\bar{i}}{}_{-n-\phi_i}$ with $-n-\phi_i < 0$. The opposite sign of the indices defines annihilation operators. A similar result holds for the right part of the algebra. Computing the Poisson brackets and replacing them by Dirac brackets upon quantization, the modes algebra is obtained. The fermionic right–movers have the expansion

$$\widetilde{\psi}^{i} = \sum_{n\in\mathbb{Z}+\nu}\left[\tilde{\psi}^{i}_{n-\phi_i}e^{-i(n-\phi_i)\bar{\omega}}\right], \tag{3.22}$$

where as before, ν denotes the R or NS sectors. Creation operators will be $\psi_{-n-\phi_i}$ with $-n-\phi_i \leq 0$. The conjugate modes are

$$\widetilde{\psi}^{\bar{i}} = \sum_{n\in\mathbb{Z}+\nu}\left[\tilde{\psi}^{\bar{i}}_{n+\phi_i}e^{-i(n+\phi_i)\bar{\omega}}\right]. \tag{3.23}$$

We write the left–handed fermions in a basis defined as

$$\lambda^{K\pm} = \frac{1}{\sqrt{2}}(\lambda^{2K-1} \pm i\lambda^{2K}). \tag{3.24}$$

They transform under the gauge orbifold with the twist

$$\gamma = \mathrm{diag}(e^{2\pi i\beta_1}, e^{2\pi i\beta_2}...e^{2\pi i\beta_{16}}). \tag{3.25}$$

This gives the boundary conditions

$$\lambda^{K\pm}(\sigma + 2\pi) = e^{\pm 2\pi i\beta_K}\lambda^{K\pm}(\sigma). \tag{3.26}$$

Those boundary conditions lead to the mode expansion

$$\lambda^{K\pm} = \sum_{n\in\mathbb{Z}+\nu}\lambda^{K\pm}_{n\mp\beta_K}e^{-i\omega(n\mp\beta_K)}. \tag{3.27}$$

The transformation (3.25) can be set to be in the *standard embedding*, which means that it acts only on the 3 fermionic left movers λ^{K+}, $K = 1, 2, 3$, in the same way as on the fermionic right–movers $\tilde{\psi}^{i}$.

To have a $\mathbb{Z}_N$ orbifold implies that ϕ_i and β_K are multiples of $1/N$. We obtained in (2.13) that the vacua of the Ramond sector form spinor representations of the symmetry group[2]. Therefore, they will transform under the orbifold action. Because the orbifold order is

[2]The Lorentz group for $\psi^{i,\bar{i}}$ and the manifest $SO(16)\times SO(16)$ for $\lambda^{K\pm}$.

N, acting N times on the R vacuum, it has to come back to itself. This imposes the conditions

$$N\sum_{i=1}^{3}\phi_i = N\sum_{K=1}^{8}\beta_K = N\sum_{K=9}^{16}\beta_K = 0 \mod 2. \tag{3.28}$$

Let us see which is the level mismatch for the string with the given boundary conditions. Modular invariance requires the *level matching* of the string levels. This restricts the difference between the zero modes of the energy momentum tensor to be an integer, i.e. $L_0 - \tilde{L}_0 \in \mathbb{Z}$. First, let us use the result for the zero point energy of a complex boson with mode expansion $n+\theta$, which is given by [23]

$$f(\theta) = \frac{1}{24} - \frac{1}{8}(2\theta - 1)^2. \tag{3.29}$$

A complex fermion will contribute $-f(\theta)$ to the zero point energy, while a real boson will contribute $f(\theta)/2$. With this information in hand, we can consider the sector (R,R,R). Which is the sector having R b.c. for the right modes and (R,R) b.c. for the left modes. This sector has zero point energies

$$\begin{aligned}
\delta_r &= 2\frac{f(0)}{2} - 2\frac{f(0)}{2} + \sum_i f(-\phi_i) - \sum_i f(-\phi_i) = 0, \qquad (3.30)\\
\delta_l &= 2\frac{f(0)}{2} + \sum_i f(\phi_i) - \sum_{K=1}^{16} f(\beta_K),\\
&= 1 + \frac{1}{2}\sum_{i=1}^{3}\phi_i(1-\phi_i) - \frac{1}{2}\sum_{K=1}^{16}\beta_K(1-\beta_K).
\end{aligned}$$

This gives a level mismatch of

$$\begin{aligned}
L_0 - \tilde{L}_0 &= -\frac{1}{2}\sum_{i=1}^{3}\phi_i(1-\phi_i) + \frac{1}{2}\sum_{K=1}^{16}\beta_K(1-\beta_K) \qquad (3.31)\\
&- \sum_{i=1}^{3}(N^i + \tilde{N}^i + \tilde{N}^i_\psi) - \sum_{K=1}^{16} N^K\beta_K = 0 \mod 1.
\end{aligned}$$

The oscillators numbers N^i, $\tilde{N}^i$, $\tilde{N}^i_\psi$ and N^K are defined as the difference between the number of a given excitation and its conjugate. For example: $\tilde{N}^i$ counts the number of $\tilde{\alpha}^i$ excitations minus the number of $\tilde{\alpha}^{\bar{i}}$ excitations and similarly for the other modes. Looking at the first line in (3.31), we see that zero point energy contribution has to be a multiple of $1/N$, to give an integer sum when adding the oscillator numbers, which are also multiples of $1/N$. Te equation (3.31) can be written as

$$\sum_{i=1}^{3}\phi_i^2 - \sum_{K=1}^{16}\beta_K^2 = 0 \mod \frac{2}{N}. \tag{3.32}$$

This last condition shows that the orbifold group can not be the point group alone, because the orbifold embedding in the gauge degrees of freedom is required to ensure modular invariance.

The level matching in all other sectors can be deduced from the invariance of the R vacuum expressed through the identity (3.28) and the condition (3.32). When one considers the embedding of the lattice displacements in the gauge d.o.f, i.e. when Wilson lines are present, the last results are generalized. Using bosonization to go to the bosonic formulation for the gauge degrees of freedom, the transformation relating λ^{I+} and X^I shows how boundary conditions are related in both cases. The correspondence reads

$$\lambda^{I+} =: \exp(iX^I) : . \tag{3.33}$$

This identification combined with (3.26) implies the boundary conditions for the bosonic coordinates :

$$X^I(\sigma + 2\pi) = X^I + 2\pi V^I, \quad V^I = \beta_I, \tag{3.34}$$

The shift β^I is called usually V^I, and the mode expansion for the left moving toridal coordinates is

$$X_L^I = x_L^I + (p_L^I + V^I)(\tau + \sigma) + \frac{i}{2}\sum_n \frac{1}{n}\alpha_n^I e^{in\omega}. \tag{3.35}$$

Acting with the orbifold N times we should recover the same expansion, so

$$NV^I \in T_{E_8 \times E_8}. \tag{3.36}$$

Working out the level matching condition for the theory in terms of X^I instead of λ^{I+}, equation (3.32) is also encountered but with the replacement $\beta_I \to V^I$.

Orbifold spectrum Let us resume the results needed to determine the orbifold massless spectrum starting with the bosonic formulation of heterotic strings. Consider the states with boundary conditions given by the constructing element $g = \big(\theta^k, m_a e_a\big)$. The untwisted and twisted mass modes correspond to solutions with $k = 0$ and $k \neq 0$ respectively.

The untwisted string states with constructing element $g = (1, 0)$ can be described by $|q\rangle_R \otimes \tilde{\alpha}|p\rangle_L$. In the formula $q = (q^0, q^1, q^2, q^3)$ represents the momentum of the bosonized right–moving fermion. This is a weight of the $SO(8)$ Lorentz symmetry manifest in the light cone gauge. The quantity p denotes the left moving momentum of the 16 gauge d.o.f. and takes values in the $\Gamma_8 \times \Gamma_8$ lattice. Whereas $\tilde{\alpha}$ denotes schematically the set of left moving oscillators.

For both twisted and untwisted states the mass shell conditions are given by

$$\begin{aligned} \frac{M_L^2}{8} &= \frac{(p + V_g)^2}{2} + N - 1 + \delta c, \\ \frac{M_R^2}{8} &= \frac{(q + \phi_g)^2}{2} - \frac{1}{2} + \delta c\,, \end{aligned} \tag{3.37}$$

where M_R and M_L are the masses of left and right movers. We have set the right oscillator numbers and the right–moving momentum to zero, to allow for massless right–movers. The level–matching requires that $M_L^2 = M_R^2$. The quantity $\phi_g = k\phi$ appearing in (3.37) is called the local twist.

V_g represents the embedding on the gauge d.o.f of the local constructing element g. The zero point energy in this scheme is given by $\delta c = \frac{1}{2}\sum_{i=1}^{3}\omega_i(1-\omega_i)$, with $\omega_i = (\phi_g)_i \mod 1$ such that $0 \leq \omega_i < 1$.

The $SO(8)$ transforming massless states $\mathbf{8}_v$ and $\mathbf{8}$ previously described, are identified here with the massless solutions $q = \underline{(0,0,0,\pm 1)}$ and $q = \underline{(-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})}$.

For twisted strings is convenient to define the shifted left–momentum of the state as $P_{sh} = p + V_g$. The weight P_{sh} gives the twisted string gauge transformations. An analogous definition is the shifted left–moving momentum $q_{sh} = q + v_g$. Then, twisted states with boundary conditions g can be written as $|q_{sh}\rangle_R \otimes \tilde{\alpha}|P_{sh}\rangle_L$. They will transform under another orbifold constructing element h as

$$|q_{sh}\rangle_R \otimes \tilde{\alpha}|P_{sh}\rangle_L \overset{h}{\mapsto} \Delta|q_{sh}\rangle_R \otimes \tilde{\alpha}|P_{sh}\rangle_L\ . \tag{3.38}$$

The transformation phase Δ reads

$$\Delta \;=\; e^{2\pi i\,[P_{sh}\cdot V_h - q_{sh}\cdot\phi_h - \frac{1}{2}(V_g\cdot V_h - \phi_g\cdot\phi_h)]}\ . \tag{3.39}$$

If the local twist is different from zero in every plane and q solves (3.37), then $q^0 = \pm\frac{1}{2}, 0$ defines the 4d chirality. This corresponds to a chiral multiplet of $\mathcal{N}=1$ supersymmetry and its CPT conjugate. If the twist action in one complex plane is trivial, i.e. $\phi_g^i = 0$ the orbifold is only four dimensional. The massless states are then hyper–multiplets of $\mathcal{N}=1$ supersymmetry in 6d. However, those multiplets are decomposed into chiral multiplets of 4d $\mathcal{N}=1$ susy when forming orbifold invariant states.

The set of all massless untwisted modes are [103]: the graviton $g_{\mu\nu}$, the antisymmetric tensor $B_{\mu\nu}$, the dilaton, the internal metric g_{mn}, the internal antisymmetric tensor B_{mn}, the gauge bosons and the fermionic partners of every of them. In addition there are chiral multiplets. All of the described states fill multiplets of 4d $\mathcal{N}=1$ supersymmetry.

The 4d $\mathcal{N}=1$ vector multiplet, composed by the gauge bosons and gauginos, is given by [103]

$$|\pm 1,0,0,0\rangle_R \quad \times\alpha_{-1}^I|0\rangle_L, \qquad |\pm(-1/2,1/2,1/2,1/2)\rangle_R \times \alpha_{-1}^I|0\rangle_L \tag{3.40}$$
$$|\pm 1,0,0,0\rangle_R \quad \times\alpha_{-1}^I|P^I\rangle_{P^2=2}, \qquad |\pm(-1/2,1/2,1/2,1/2)\rangle_R \times \alpha_{-1}^I|P\rangle_{P^2=2} \tag{3.41}$$

All the Cartan generators (3.40) survive the orbifold projection. But the charged generators (3.41) in 4d are only the ones which fulfill the orbifold projection $P\cdot V = 0$. This last condition determines the gauge group in 4d.

3.3 $\mathbb{Z}_3$ in standard embedding

Let us present next the first model considered in the literature of orbifolds. This is the $\mathbb{Z}_3$ orbifold in the *standard embedding* of the gauge connection into the space group [32]. For an orbifold with space twist v_i this embedding is defined by considering a twist

$$V^I = \beta_I = \left(v_1, v_2, v_3, 0^5, 0^8\right). \tag{3.42}$$

As previously mentioned, this implies that the orbifold acts in the same way on the left moving fermions $\lambda^{1+}, \lambda^{2+}$ and λ^{3+} as on the right moving fermions $\tilde{\psi}^1, \tilde{\psi}^2$ and $\tilde{\psi}^3$.

In the case of $\mathbb{Z}_3$ these twists are

$$\begin{aligned} v_i &= \left(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\right), \\ V^I &= \left(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}, 0^5, 0^8\right). \end{aligned} \tag{3.43}$$

The standard embedding will break the $E_8 \times E_8$ symmetry down to a product $G' \times E_6 \times E_8$. This happens because the shift V^I lies in the $SU(3)$ which is embedded in E_8 as

$$E_8 \supset SU(3) \times E_6. \tag{3.44}$$

One has then to check which gauge bosons survive the orbifold to determine which will be the factor G'. It turns out that for $\mathbb{Z}_3$ this factor is exactly $SU(3)$. Recalling that the Little group is broken by the space group action as in (3.14), on this model the breaking is given by

$$SO(8) \rightarrow SO(2) \times SU(3), \tag{3.45}$$

$$E_8 \times E_8 \rightarrow SU(3) \times E_6 \times E_8. \tag{3.46}$$

The T^6 lattice is (up to scalings) the root lattice of $SU(3) \times SU(3) \times SU(3)$. The point group is generated by $1, \theta$ and θ^2 transformations on Z^i. The action of θ is

$$\theta: \quad \left(Z^1, Z^2, Z^3\right) \rightarrow (e^{\frac{2\pi i}{3}} Z^1, e^{\frac{2\pi i}{3}} Z^2, e^{\frac{-4\pi i}{3}} Z^3). \tag{3.47}$$

The orbifold $T^6/\mathbb{Z}_3$ is represented in Figure 3.1. There are thee fixed points in every plane, giving a total of 27 fixed points. Those fixed points will content 27 identical copies of the twisted spectrum. We have selected a basis such that in every complex plane the vectors are $e_{2k-1} = (1,0)$ and $e_{2k} = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$ with $k = 1, 2, 3$.

Untwisted sector The untwisted sector contains the states of the toroidal compactification which survive the orbifold projection. States are composed in left and right parts. So, one needs to make combinations with eigenvalue 1 under θ. Therefore it is useful to

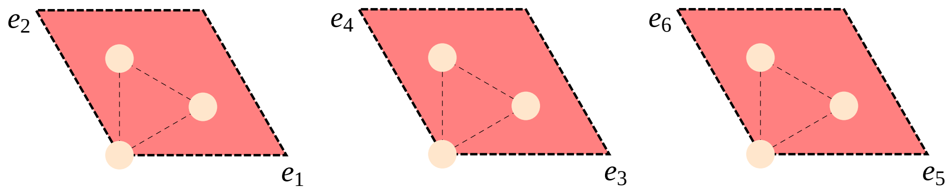

Figure 3.1: Orbifold $T^6/\mathbb{Z}_3$ on the torus lattice $SU(3) \times SU(3) \times SU(3)$. There is a total of 27 fixed points.

describe the left and right massless states with their different point group eigenvalues, and to combine them into invariant combinations [32]. Let us start with the right movers

$$\begin{aligned} \theta^0 &: \quad \tilde{\psi}^{\mu}_{-1/2}|0\rangle_{\mathrm{NS}}, \quad |\mathbf{1}, 1/2\rangle_{\mathrm{R}},, \quad |\bar{\mathbf{1}}, -1/2\rangle_{\mathrm{R}}, \\ \theta^1 &: \quad \tilde{\psi}^{i}_{-1/2}|0\rangle_{\mathrm{NS}}, \quad |\mathbf{3}, 1/2\rangle_{\mathrm{R}}, \\ \theta^2 &: \quad \tilde{\psi}^{\bar{i}}_{-1/2}|0\rangle_{\mathrm{NS}}, \quad |\bar{\mathbf{3}}, -1/2\rangle_{\mathrm{R}}. \end{aligned} \tag{3.48}$$

The left R ground states are denoted by its representation under the surviving $SU(3)\times U(1)$ subgroup of $SO(8)$. This is the decomposition of the $\mathbf{8}_s$ massless state of the toroidal compactification as

$$\mathbf{8}_s \to \mathbf{3}_{1/2} + \bar{\mathbf{3}}_{-1/2} + \mathbf{1}_{1/2} + \bar{\mathbf{1}}_{-1/2}. \tag{3.49}$$

The rest of the states constructed with right-fermionic oscillators are part of the $\mathbf{8}_v$. The index $\mu = 2, 3$ runs over the non-compact transverse directions after the gauge fixing of the world–sheet action. The left-moving massless modes have representations under the surviving gauge group (3.46). Those states are inherited from the 10d left-massless spectrum (2.23) i.e. the state $(\mathbf{1}, \mathbf{248}, \mathbf{1})$ is decomposed as

$$(\mathbf{248}, \mathbf{1}) \to (\mathbf{8}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{78}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{248}) + (\mathbf{3}, \mathbf{27}, \mathbf{1}) + (\bar{\mathbf{3}}, \overline{\mathbf{27}}, \mathbf{1}). \tag{3.50}$$

In the right hand side of the arrow the first, second and third entries are the quantum numbers under the gauge factors $SU(3), E_6$ and E_8 respectively. We omitted the first entry of $(\mathbf{1}, \mathbf{248}, \mathbf{1})$ because these states are singlets of the Lorentz group. The first three states of the decomposition have eigenvalue θ^0 under the orbifold. The last two have eigenvalues θ and $\theta^{-1} = \theta^2$ respectively. This is because we are in the standard embedding. Therefore, the massless left movers orbifold eigenstates are

$$\begin{aligned} \theta^0 &: \quad \alpha^{\mu}_{-1}|a_0\rangle,\ |a_0\rangle \in (\mathbf{8}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{78}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{248}), \\ \theta^1 &: \quad \alpha^{i}_{-1}|a_1\rangle,\ |a_1\rangle \in (\mathbf{3}, \mathbf{27}, \mathbf{1}), \\ \theta^2 &: \quad \alpha^{\bar{i}}_{-1}|a_2\rangle,\ |a_2\rangle \in (\bar{\mathbf{3}}, \overline{\mathbf{27}}, \mathbf{1}). \end{aligned} \tag{3.51}$$

Those states are constructed by acting with bosonic oscillators on the NS–NS' left vacuum. Now, the untwisted matter can be summarized in Table (3.1). We have put together right and left eigenvectors of the orbifold twist, and we have indicated the physical nature of the 4d states.

Table 3.1: Untwisted massless states of $\mathbb{Z}_3$ in the gauge standard embedding.

Particle	State	Product
4d Gauge bosons	$\widetilde{\psi}^{\mu}_{-1/2}\|a_0\rangle_{\mathrm{NS}}$	$\theta^0\cdot\theta^0$
4d graviton, dilaton and axion	$\alpha^{\mu}_{-1}\widetilde{\psi}^{\nu}_{-1/2}\|0\rangle_{\mathrm{NS}}$	$\theta^0\cdot\theta^0$
4d gravitino, dilatino and axino	$\alpha^{\mu}_{-1}\|\mathbf{1},s\rangle_{\mathrm{R}}$	$\theta^0\cdot\theta^0$
Neutral scalars, 6D Moduli	$\alpha^{i}_{-1}\widetilde{\psi}^{\bar{j}}_{-1/2}\|0,0\rangle_{\mathrm{NS}}$	$\theta^1\cdot\theta^2$
Scalars	$\widetilde{\psi}^{\bar{j}}_{-1/2}\|a_1\rangle$	$\theta^1\cdot\theta^2$
Spinors	$\|a_2,\mathbf{3},1/2\rangle_{\mathrm{R}}$	$\theta^2\cdot\theta$
Spinors	$\alpha^{i}_{-1}\|\mathbf{3},1/2\rangle_{\mathrm{R}}$	$\theta^2\cdot\theta$

Twisted sector To complete the orbifold description we should obtain the twisted sector states. There are 27 equivalence classes of fixed points, which are constructed by the products of two dimensional sets as

$$g=(\theta,n_\alpha e_\alpha),\ \text{with}\ (n_{2i-1},n_{2i})=\{(0,0),(1,2),(2,1)\},\ i=1,2,3. \tag{3.52}$$

They give the following boundary conditions for bosonic coordinates

$$Z^i(\sigma+2\pi)=\theta^i_j Z^j(\sigma)+n_\alpha e^i_\alpha. \tag{3.53}$$

As Wilson lines are absent, there will be $3^3=27$ copies of the spectrum, one at every fixed point. The CPT conjugated states are given by other 27 classes with θ^2 twist. We construct now the sectors for the right movers and left movers. In the right moving R sector the zero point energy is

$$2f(0)/2-2f(0)/2+3f(-1/3)-3f(-1/3)=0. \tag{3.54}$$

The first two terms come from the 4d real bosons and fermions $X^\mu(\bar{\omega})$ and $\widetilde{\psi}^\mu(\bar{\omega})$ which after gauge fixing are two real ones each. The last two come from the complex boson and fermions $Z^i(\bar{\omega})$ and $\widetilde{\psi}^i(\bar{\omega})$ with oscillators indices $n-1/3$ as seen in (3.20) and (3.22). The fermionic zero modes are the oscillators with $\widetilde{\psi}^\mu_0$, $\mu=2,3$ that can be redefined to $\widetilde{\psi}^{2\pm i3}_0=(\widetilde{\psi}^2_0\pm i\widetilde{\psi}^3_0)/\sqrt{2}$. Therefore, the possible R ground states in the fixed point g are denoted as

$$|\pm 1/2\rangle_{g,\mathrm{R}}. \tag{3.55}$$

They fulfill the conditions

$$\widetilde{\psi}^{2\pm i3}_0|\pm 1/2\rangle_{g,\mathrm{R}}=0,\quad \widetilde{\psi}^{i}_{n+2/3}|\pm 1/2\rangle_{g,\mathrm{R}}=0,\ n+2/3>0. \tag{3.56}$$

To implement the GSO projection, the vertex operator of the R–sector g–twisted ground states are determined via bosonization of the fermions $\widetilde{\psi}^i,\widetilde{\psi}^{2\pm i3}$, whose modes annihilate the vacuum $|\pm 1/2\rangle_{g,\mathrm{R}}$. The bosonization is given by[3]

$$\widetilde{\psi}^i\simeq e^{i\tilde{s}_i\widetilde{H}_i},\quad \tilde{s}_i=-\frac{1}{6},\qquad \widetilde{\psi}^{2\pm i3}\simeq e^{i\tilde{s}_0\widetilde{H}_0},\tilde{s}_0=\pm\frac{1}{2}. \tag{3.57}$$

[3] The twisted ground states of a spinor with periodicity $\widetilde{\psi}^i(\sigma+2\pi)=e^{2\pi i\zeta}\widetilde{\psi}^i(\sigma)$ fulfill $\widetilde{\psi}_{n+\zeta}|0\rangle_\zeta=\overline{\widetilde{\psi}}_{n+1-\zeta}|0\rangle_\zeta=0$. They have vertex operators given by $|0\rangle_\zeta\cong\mathcal{A}_\zeta=\exp\left(i(-1/2+\zeta)\tilde{H}\right)$.

The bosonization of fermions in equation (3.57) reproduces the result that a right–fermion with boundary condition phase ζ has a bosonization $e^{i(-1/2+\zeta)\widetilde{H}}$. The *spin field* or *vertex operator of the R ground state* is then

$$\Theta_R = \exp i \sum_a \tilde{s}_a \tilde{H}_a, \quad \tilde{s} = \left(\pm\frac{1}{2}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}\right). \tag{3.58}$$

This determines $|+1/2\rangle_{\mathrm{R}}$ as the state surviving the GSO projection $\sum \tilde{s}_a \in 2\mathbb{Z}$.

The right NS sector will have zero point energy given by

$$2f(0)/2 - 2f(1/2)/2 + 3f(2/3) - 3f(1/6) = 0. \tag{3.59}$$

The first two terms come from two space–time real periodic bosons and two space–time real anti–periodic fermions. The last two terms come from the three internal coordinates complex bosons with modes $n + 2/3$, and three internal complex fermions with modes $n + 1/2 - 1/3$ as seen in (3.22). Then, the only massless state in this sector is $|0\rangle_{\mathrm{NS}}$.

In the left-moving side the sectors R–NS and NS–NS will give massless states, with zero point energies

$$\begin{aligned} \text{R–NS} \quad &: \quad 2\frac{f(0)}{2} + 3f(1/3) - 3f(-1/3) - 10f(0)/2 - 16\frac{f(1/2)}{2} = 0, \\ \text{NS–NS} \quad &: \quad 2\frac{f(0)}{2} + 3f(1/3) - 3f(1/6) - 10f(1/2)/2 - 16\frac{f(1/2)}{2} = -1/2. \end{aligned} \tag{3.60}$$

The sector R–R has not massless states. In the R–NS sector there are fermionic zero modes $\lambda_0^I, I = 7, ..., 16$. Thus, as before, one can construct rising and lowering operators $\lambda_0^{K\pm}$ to get ground states which form $SO(10)$ representations. The lowest state is denoted by $|-1/2^5\rangle_{\mathrm{R}}$. The vertex operator of all the ground states will be

$$\Theta_L = \exp iq_K H_K, \quad \mathbf{q} = \left(1/6, 1/6, 1/6, \pm 1/2^5\right). \tag{3.61}$$

This implies that the $\overline{\mathbf{16}}$ representation with odd number of $-1/2$ survives the GSO projection $\sum_K q_k \in 2\mathbb{Z}$. To write (3.61) we use the result that a left twisted vacuum annihilated by modes of a spinor with phase ζ, has an associated vertex operator component $e^{i(1/2-\zeta)H}$. The spinors λ^I, $I = 1, 2, 3$, have phase $\zeta = 1/3$, so each of them will give the vertex operator component $e^{iH_I/6}$. Also the states $|... \pm 1/2...\rangle_{\mathrm{R}}$ annihilated by $\lambda_0^{K\pm}$ will have an associated vertex operator component $e^{\pm iH_K/2}$. The NS–NS sector has zero point energy $-1/2$, therefore this fixes three kind of massless sates. We can see the full twisted spectrum in Table 3.2.

3.3.1 Blow–up modes

The presented example serves also to describe an important concept in our work. It was precisely in the $\mathbb{Z}_3$ orbifold in which by the first time was discovered the presence of twisted

Table 3.2: Twisted $m = 0$ spectrum of $\mathbb{Z}_3$ in standard embedding.

irrep.	State
$(1, \overline{\mathbf{27}}, 1)$	$\left(\vert\overline{\mathbf{16}}\rangle_{R,NS} + \lambda^{1+}_{-1/6}\lambda^{2+}_{-1/6}\lambda^{3+}_{-1/6}\vert 0\rangle_{\mathrm{NS,NS}} + \lambda^{I}_{-1/2}\vert 0\rangle_{\mathrm{NS,NS}}\right)\vert 1/2\rangle_R$, $7 \le I \le 16$
$(1, \overline{\mathbf{27}}, 1)$	$\left(\vert\overline{\mathbf{16}}\rangle_{R,NS} + \lambda^{1+}_{-1/6}\lambda^{2+}_{-1/6}\lambda^{3+}_{-1/6}\vert 0\rangle_{\mathrm{NS,NS}} + \lambda^{I}_{-1/2}\vert 0\rangle_{\mathrm{NS,NS}}\right)\vert 0\rangle_{\mathrm{NS}}$, $7 \le I \le 16$
$(\mathbf{3}, 1, 1)^3$	$\left(\lambda^{K+}_{-1/6}\alpha^{\bar{j}}_{-1/3}\vert 0\rangle_{\mathrm{NS,NS}}\right)\vert 1/2\rangle_R, \quad K = 1, 2, 3.$
$(\mathbf{3}, 1, 1)^3$	$\left(\lambda^{K+}_{-1/6}\alpha^{\bar{j}}_{-1/3}\vert 0\rangle_{\mathrm{NS,NS}}\right)\vert 0\rangle_{\mathrm{NS}}, \quad K = 1, 2, 3.$

fields which vevs can be varied freely ensuring a vanishing D-term [63] . This means that a *flat direction* exists. In this case those twisted fields do not appear in the super-potential. The fields parametrizing the flat direction are the E_6 singlets $(\mathbf{3}, \mathbf{1}, \mathbf{1})$ presented in Table (3.2)

$$M_{K\bar{j}} = \lambda^{K+}_{-1/6}\alpha^{\bar{j}}_{-1/3}|0\rangle_{\mathrm{NS,NS}}. \tag{3.62}$$

The scalar potential for these modes will come from their D-term, which is in general given by

$$D^a(\phi, \phi^*) = -\frac{g_a^2}{2}(2\xi_a + \phi^{i*}t^a_{ij}\phi^j), \tag{3.63}$$

where ϕ^i are the scalar fields, g_a is the gauge coupling constant, ξ_a is the contribution of an $U(1)$ gauge symmetry called *Fayet-Iliopoulos term* (FI) and t^a_{ij} are the generators of the gauge group representation carried by the fields ϕ^i. In the present example there are no $U(1)$ symmetries, so the potential will be

$$D^a \propto M^*_{K\bar{j}}t^a_{KL}M_{L\bar{j}} = \mathrm{Tr}(M^\dagger t^a M), \quad t^a \in SU(3). \tag{3.64}$$

The condition for D^a to vanish for all a is that the matrices are unitary and satisfy

$$MM^\dagger = \rho^2\mathbf{1} \to \forall_a D^a \propto \mathrm{Tr}(t^a MM^\dagger) \propto \mathrm{Tr}(t^a) = 0. \tag{3.65}$$

The fields M can be taken to be proportional to the identity with a gauge rotation. This means that there is a one parameter family vacua in which $SU(3)$ is broken. They arise when the twisted fields take vacuum expectation values, such that those vacua can be understood as smooth *Calabi–Yau* (CY) manifolds with a curvature radius related to ρ [63]. The fields M are moduli of the CY, which turned off lead to the orbifold singularity. They are named as *blow–up modes*.

In the standard embedding the world–sheet theory possess $(2, 2)$ supersymmetry, because the gauge twist is equal to the space twist and this gives additional conserved left-currents. In our work we treat compactifications with $(0, 2)$ supersymmetry, coming from orbifolds which are not in the standard embedding. But on those models, similar effects have been studied [69]. Our aim is to identify the blow–up modes, and understand the massless states of the new vacuum. We will do this in terms of the original orbifold spectrum deformed by vevs, and in terms of the compactification of supergravity in the resolved orbifold. Differently to the situation encountered in $\mathbb{Z}_3$, the blow–up modes here will couple to other fields in the orbifold. They do appear in the super-potential. So, they will put at work a Higgs mechanism under which some orbifold chiral fermions get masses. Then, one needs to look also at the *F-Flatness* condition, and ensure that the family of vevs configuration satisfies it too.

This connection between orbifolds and smooth CY contributes to identify compactifications which belong to the same moduli space. In general, the CFTs obtained with generic backgrounds of twisted fields are not free, and therefore more difficult to treat. Due to that, the heterotic supergravity description coupled to super Yang–MIlls is employed to study the compactification on the smooth manifold. In our examples, there will be generically an anomalous $U(1)$ gauge symmetry such that a FI term ξ_a is generated at one–loop. Therefore, if we want to ensure a supersymmetric vacuum this term has to be canceled by vevs of twisted scalars. Consider the scalar components of the twisted chiral superfields ϕ_i attaining vevs $\langle\phi_i\rangle$ and only charged under the abelian gauge symmetries. Then, the total D–term can be written as [74]

$$D = \sum_{i,a} Q_a^i \langle\phi_i\rangle^2 + \xi, \;\; \xi = \frac{M_s^2}{192\pi^2}\, \mathrm{Tr}\, Q_{\mathrm{anom}}. \tag{3.66}$$

The index a runs over all the $U(1)_a$ symmetries, among them the anomalous one. Tr Q_{anom} denotes the trace of the charges under the anomalous symmetry. This FI term ξ has only one loop contributions, the higher loops contributions vanish [74]. A flat direction occurs, when the assigned vevs cancel this term, giving $D = 0$ and keeping also the F–term vanishing.

3.4 A look into selection rules

In order to determine the effective field theory arising from the orbifold compactification, we need to know which couplings are allowed in the theory. This can be determined by looking at the correlation function between vertex operators. The basic string selection rules as space group, gauge invariance and *H–momentum* conservation, are well known and well reviewed in many sources [107–110], so we will not treat them here.

Recently, the study of orbifold selection rules arising from instanton contributions to the world–sheet have been revisited. This lead to a reconsideration of the so called Rule 4 and the proposal of a new rule, Rule 5 [109], which have been recently debated. These developments motivated us [94] to study potential R-symmetries, which could arise from the instanton solutions. R–symmetries are expected to arise in the 4d theory as remnants from 6d Lorentz group $SO(6)$, which are also symmetries of the orbifolds.

Thus, we performed an investigation over the automorphism group of the toroidal lattice $Aut(\Gamma)$. Among the found generators we present in this section here the subgroup that leaves the *conjugacy classes* invariant. The most considered example of this subgroup, is the orbifold twist on a single plane[4]. The *conjugacy classes* for a fixed set characterized by boundary conditions $X^{\mathbf{f}}(e^{2\pi i}z) = \theta^k X^{\mathbf{f}}(z) + \lambda$ is given by

$$\bigcup_{r=0}^{l-1} \Big\{ (\theta^k, \theta^r \lambda + (1-\theta^k)\Lambda) \Big\} \,, \tag{3.67}$$

where equivalent fixed sets under the orbifold action have been identified under an element (θ^r, Λ). Another motivation for this study, was the blow–up of the $\mathbb{Z}_7$ orbifold, which will be discussed in chapter 4. In that project, initially we failed to match the spectrum on the deformed orbifold by vevs and in the resolved space. This happened because there were some Yukawa couplings to blow–up modes giving mass to apparently massless fields in blow–up. Unfortunately, the automorphism exploration doesn't give, any new result for the $T^6/\mathbb{Z}_7$ orbifold. Nevertheless, there could be some restrictions arising from the instanton selection rules, but this is still under study.

Let us describe the automorphisms that leave conjugacy classes invariants. For the factorizable case there is a nice way of identify them. Consider a fixed point of the sector θ^k given by the direct product of the coordinates in the three planes $f^{(k)} = g_1 \otimes g_2 \otimes g_3$. Every projection on a plane from the fixed point fulfills $(\theta_i)^k g_i = g_i + \lambda_i$, $\lambda_i \in \Gamma$. If the plane i has prime twist, then g_i also satisfies $\theta_i g_i = g_i + \lambda_i'$, $\lambda_i' \in \Gamma$. Dividing in the following itemized cases it is possible to identify which rotational symmetries leave the fixed points invariant.

(i) *All planes have prime order twists.* In this case, all the fixed points are fixed under the orbifold twist plane by plane:

$$\theta_i f^{(k)} = f^{(k)} + \lambda_i', \quad i = 1,2,3, \quad \bar{\psi}_{orall} f^{(k)} \,. \tag{3.68}$$

So we have the discrete symmetries generated by θ_1, θ_2 and θ_3.

(ii) *Only one plane is non-prime.* Here, all fixed points are fixed under the prime plane rotations, say θ_2 and θ_3. Moreover, considering the non-prime rotation, say θ_1, we have:

$$\begin{aligned} \theta_1 f^{(k)} &= \theta_1 g_1 \otimes g_2 \otimes g_3 \\ &= \theta_1 g_1 \otimes \theta_2 g_2 \otimes \theta_3 g_3 - \lambda_2' - \lambda_3' \\ &= \theta f^{(k)} - \lambda_2' - \lambda_3' \\ &\simeq f^{(k)} \,, \end{aligned} \tag{3.69}$$

where the last $\simeq$ indicates equivalence of the fixed points up to the conjugacy class. So we have again the symmetries, θ_1, θ_2 and θ_3.

(iii) *Two planes are non-prime.* Again, all the fixed points are fixed under the prime plane rotation, say θ_3. Moreover, they are invariant under the combined action of the

[4]This is motivated because invariance under twists in one plane leads to the standard orbifold R–charge.

Table 3.3: Orbifold automorphisms for some non-factorizable or partially factorizable orbifolds, counting independent discrete rotational symmetries that preserve the conjugacy classes, and labeling them with their generators. We refer to [111] for details of the torus lattice, orbifold twist and fixed points.

	Lattice	Twist	Orbifold Automorphisms
$\mathbb{Z}_4$	SU(4)⊗SU(4)	$\frac{1}{4}(1,1,-2)$	θ, $(\theta_1)^2$
$\mathbb{Z}_{6-\mathrm{II}}$	SU(6)⊗SU(2)	$\frac{1}{6}(1,2,-3)$	θ
$\mathbb{Z}_7$	SU(7)	$\frac{1}{7}(1,2,-3)$	θ
$\mathbb{Z}_{8-\mathrm{I}}$	SO(5)⊗SO(9)	$\frac{1}{8}(2,1,-3)$	θ, $(\theta_1)^2$
$\mathbb{Z}_{8-\mathrm{II}}$	SO(8)⊗SO(4)	$\frac{1}{8}(1,3,-4)$	θ, θ_3
$\mathbb{Z}_{12-\mathrm{I}}$	SU(3)⊗F_4	$\frac{1}{12}(4,1,-5)$	θ, θ_1
$\mathbb{Z}_{12-\mathrm{II}}$	F_4⊗SO(4)	$\frac{1}{12}(1,5,-6)$	θ, θ_3

non-prime rotations, say $\theta_1\theta_2$ since:

$$\begin{aligned} \theta_1\theta_2 f^{(k)} &= \theta_1 g_1 \otimes \theta_2 g_2 \otimes g_3 \\ &= \theta_1 g_1 \otimes \theta_2 g_2 \otimes \theta_3 g_3 - \lambda_3' \\ &= \theta f^{(k)} - \lambda_3' \\ &\simeq f^{(k)} \,. \end{aligned} \tag{3.70}$$

In this case, the symmetries are generated by $(\theta_1\theta_2)$ and θ_3.

A case not considered in what we described is the $\mathbb{Z}_4$ orbifold. This orbifold has twist $v = \frac{1}{4}(1,1,-2)$ has two independent twisted sectors θ^2 and θ^4. Here all the fixed points are fixed under θ_i^2 (the twist squared in every plane), and as a consequence under $(\theta)^2$. Then, it is clear that $(\theta_1)^2$ and $(\theta_2)^2$ are the two independent symmetries from the group $(\theta_i)^2$.

The computer scan performed for all factorizable orbifolds reveals, that the cases explained are all the possible ones.

For orbifolds whose underlying torus lattice and orbifold action have a factor in one plane only, we can perform a similar analysis. For non-factorizable orbifolds for which it is not possible to decompose the twist in the product of the three planes twists, we can still search for rotational discrete symmetries that preserve the orbifold and leave the conjugacy classes invariant. We have performed this exploration, and the main results of it can be seen in Table 3.3. In most of the situations the symmetry is the point group itself, but for $\mathbb{Z}_4$ and $\mathbb{Z}_{8I}$ there is a $\mathbb{Z}_2$ symmetry surviving and generated by θ_1^2.

In appendix I further details are given. There we present the study of two subgroups of the automorphism group. One is the group D and the other the group F. They are defined as

$$
\begin{aligned}
F &= \{\rho \in Aut(\Gamma),\, [\rho,\theta]=0,\, \theta\in P, \qquad (3.71)\\
&\quad \forall z_f \text{ F.P. of } S\, \nexists\, h\in S\, s.t.\, \rho z_\rho = h z_f\},\\
D &= \{\rho \in Aut(\Gamma),\, [\rho,\theta]=0,\, \theta\in P, \qquad (3.72)\\
&\quad \forall z_f \text{ F.P. of } S\, \exists\, h\in S\, s.t.\, \rho z_\rho = h z_f,\, \det(\rho)=1.\}
\end{aligned}
$$

Those are the most promising automorphism subgroups that could lead to 4d discrete symmetries. Group E is the one which was described in this section. But note that group F doesn't not preserve the conjugacy classes. However, our aim is to determine in which orbifolds this group maps among each other conjugacy classes with the same spectrum, which we call *degenerated conjugacy classes*. This group could lead to flavor 4d symmetries as described in [55, 59].

Currently, using the encountered symmetries of group D we perform an study of the CFT correlators, for non-prime orbifolds with the presence of gamma phases. We expect to present the results of this section together with the mentioned exploration elsewhere [94].

3.5 Calabi–Yau compactification

We compactify the extra six dimensions with the aim of preserve $\mathcal{N}=1$ supersymmetry in 4d. A supersymmetric background is ensured if the supersymmetry generators annihilate the vacuum. At the classical level, this implies that the variation of the Fermi fields has to be zero. In the 10d theory the Fermi fields are the gravitino, the dilatino the gaugino and the fermionic component of chiral superfields. The supersymmety transformation in the 10d supergravity depends on a Majorana-Weyl parameter ξ, which transforms in the **16** spinor representation of $SO(9,1)$. This parameter appears in the variation of the six dimensional gravitino

$$\delta\psi = (\partial_m + \frac{1}{4}(\omega_{mnp} - \frac{1}{2}H_{mnp})\Gamma^{np})\xi = \nabla_m\xi. \qquad (3.73)$$

Here ∇_m is the internal covariant derivative ∇_m and ω_{mnp} and H_{mnp} represent the spin connection and the 3-form field strength. Thus a supersymmetric vacuum implies that there should exist a covariantly constant spinor. Under the decomposition $SO(9,1)\to SO(3,1)\times SO(6)$, one has $\mathbf{16}\to(\mathbf{2},\mathbf{4})+(\bar{\mathbf{2}},\bar{\mathbf{4}})$. The spinor ξ in the **16**, decomposes as $\xi = \xi_{\alpha\beta}+\xi^*_{\alpha\beta}$, with indices α and β running over the **2** and the **4** representations respectively. Assuming there exists some unbroken supersymmetry, then the spinor ξ can be rotated in $SO(3,1)$ to achieve an structure $\xi_{\alpha\beta}=u_\alpha\zeta_\beta(x^m)$, with u_α constant, and ζ in the **4** of $SO(6)$. Then

$$\nabla_m\xi = 0 \to \nabla_m\zeta = 0. \qquad (3.74)$$

The latter is the parallel transport equation on the manifold. The question to ask, is which condition the manifold should satisfy in order to leave one component of the spinor invariant when performing parallel transport. This can be made explicit by looking at the deviation of the spinor from itself, after performing parallel transport in a closed loop

$$[\nabla_m, \nabla_n]\zeta = \frac{1}{4} R_{mnpq}\Gamma^{pq}\zeta = 0. \tag{3.75}$$

To achieve $\mathcal{N} = 1$ supersymmetry the rotations performed should leave one component of the spinor invariant. If the rotation $R_{mnpq}\Gamma^{pq}$ lies on $SU(3)$ this can be achieved, because in this case the spinor $\xi_{\alpha\beta} = u_\alpha \zeta_\beta$ decomposes as

$$(\mathbf{2}, \mathbf{4}) \to (\mathbf{2}, \mathbf{1}) + (\mathbf{2}, \mathbf{3}), \tag{3.76}$$

where the entries on the r.h.s are the representations under $SO(3,1)\times SU(3)$. The existence of a covariantly constant spinor can be formulated using the concept of the *holonomy group*, which is the group of rotations that a spinor or a vector experiences when it is parallel transported around a closed loop. The searched manifold has therefore $SU(3)$ *holonomy*. Recall that the $\mathcal{N} = 1$ supersymmetry constraint for orbifolds, imposed the orbifold rotation to lie in $SU(3)$, in this case the holonomy group coincides with the point group. The ten dimensional supercharges will give rise to four surviving supercharges in four dimensions

$$Q_\alpha \equiv (\mathbf{2}, \mathbf{1}), \qquad \bar{Q}_\alpha \equiv (\bar{\mathbf{2}}, \mathbf{1}). \tag{3.77}$$

Applying that same line of reasoning it is possible to see that the compactification on a T^6 which has trivial holonomy group will preserve all the supersymmetries. In the 10d $\mathcal{N} = 1$ theory there are 16 supercharges, compactification on a torus will therefore give $\mathcal{N} = 4$ sypersymmetry in 4d. Recall that on the $\mathcal{N} = 2$ sectors of the orbifold, the rotation lays on $SU(2)$. These orbifold sectors will preserve the same amount of supersymmetry that a manifold with $SU(2)$ holonomy, which has 8 surviving supercharges in 4d.

For more explicit formulas one can check the review in [96], but we want to mention that the zero variation of the 4d gravitino gives a 4d flat metric and the zero variation of the dilatino restricts the dilaton and the 3-form field to give torsion free compactifications i.e. $H_3 = 0$ a constant dilaton.

The variation of the gaugino gives a similar condition for the background flux in the internal dimensions, as the one obtained in the discussion of parallel transport. In particular requiring the vanishing of the gaugino variation $\delta\lambda = F_{mn}\Gamma^{mn}\xi$, restricts $F_{mn}\Gamma^{mn}$ to $SU(3)$ rotations. This restriction implies that

$$F_{ij} = F_{\bar{i}\bar{j}} = G^{i\bar{j}} F_{i\bar{j}} = 0, \tag{3.78}$$

where the indices i and $\bar{i}$ transform under $SU(3)$. Another requirement is the *Bianchi identity* (BI) for the 3-form field strength, which is given by

$$d\widetilde{H}_3 = \frac{\alpha'}{4}(\mathrm{tr}\mathcal{R}^2 - \mathrm{Tr}\mathcal{F}^2). \tag{3.79}$$

For vanishing torsion both terms on the r.h.s of the equation should coincide. This is achieved *embedding the spin connection into the gauge connection*: the gauge connection is set equal to the spin connection of the internal manifold. This we have seen in section 3.46 in which the $\mathbb{Z}_3$ orbifold in the standard embedding was discussed. In fact a blow–up of this model gives a CY manifold which possesses vanishing torsion and gauge group $E_6 \times E_8$. But in the models we are interested in, which are obtained from toric resolutions of orbifolds with $(0,2)$–world–sheet supersymmetry, the backgrounds have a gauge connection embedded in the $E_8 \times E_8$ Cartan subalgebra; so we have a *non standard embedding.* In this case for every $\widetilde{H}$ the equation (3.79) should be supplemented with the BI which are the equations obtained from integrating dH in a set of compact submanifolds of X.

Let us come now to the definition of Calabi–Yau manifold. To achieve a compactification manifold with the described properties, one starts with a complex manifold of complex dimension n. This is a $2n$ dimensional real manifold which allows for complex coordinates with holomorphic transition functions $\tilde{z}_i(z_k)$. One can also start with the existence of a *complex structure*, which is a globally defined tensor τ_n^m satisfying $\tau_n^m \tau_k^n = -\delta_k^m$, and can be used to define local coordinates s.t. $dz^i = dx^i + i\tau_k^i dy^k$. The holomorphic transition functions impose certain constraints in τ_k^i.

Now let us impose another constraint on the complex manifold. Define an Hermitian metric whose only non–zero components have mixed holomorphic and anti-holomorphic indices i.e. $G_{i\bar{j}} \neq 0$, $G_{ij} = G_{\bar{i}\bar{j}} = 0$. Then, the non–vanishing metric components can be casted in the 2-form

$$J = G_{i\bar{j}} dz^i d\bar{z}^{\bar{j}}, \tag{3.80}$$

which is called the *Kähler form* when is closed $dJ = 0$. A manifold with a Kähler form is called is Kähler. In a notation that a (p,q) form is a form with p holomorphic and q anti-holomorphic indices, J is of type $(1,1)$. The external derivative d is given by $d = \partial + \bar{\partial}$.

Another equivalent definition of a Kähler manifold is that parallel transport preserves holomorphic and anti–holomorphic indices giving an $U(n)$ *holonomy.*[5] This is because the condition $dJ = 0$ represents the parallel transport equation for the metric. From here we see that to arrive to our desired $SU(3)$ holonomy we need to impose a further constraint in the 3-fold, this is the Calabi–Yau condition which will be soon discussed.

(Co)Homology and Poincaré duality Let us recall some geometrical definitions which are useful in the frame of our work [112]. A p–form is closed when its exterior derivative is zero $d\omega = 0$, and it is exact when can be written as the exterior derivative of a $p-1$–form $\omega = d\tau$. This allows to define the *Rham cohomology* of a manifold X as

$$H_d^p(X) = \frac{\text{set of } d\text{-closed p-forms}}{\text{set of } d\text{–exact p-forms}}, \tag{3.81}$$

where $H_d^p(X)$ is the set of all closed forms modulo the equivalence relation $\omega_p \equiv \omega_p + d\tau_{p-1}$. This defines equivalence classes of p–forms which differ only by an exact form. The

[5] Also other equivalent condition is that the metric can be expressed in terms of *Kähler potential* as $G_{i\bar{j}} = \partial_i \partial_{\bar{j}} K(z,\bar{z})$.

dimension of $H^p_d(X)$ is the Betti number b_p, and the Euler number of the manifold is given by $\chi(X) = \sum_{p=0}^{d}(-1)^p b_p$. An *harmonic form* is a form satisfying $\Delta_d\omega = (d + *d*)^2\omega = 0$, where the symbol $*$ is the Hodge dual. Each of the equivalence classes of $H^p_d(X)$ contains one harmonic form. It is an important theorem that on a Kähler manifold, the cohomologies with respect to the derivatives $\partial, \bar{\partial}$ and d coincide [6]

$$H^{p,q}_d(X) = H^{p,q}_{\partial}(X) = H^{p,q}_{\bar{\partial}}(X). \tag{3.82}$$

The cohomologies with respect to ∂ and $\bar{\partial}$ are *Dolbeaut cohomologies*. The *Hodge numbers* are defined as $h^{p,q} = \dim H^{p,q}_{\bar{\partial}}(X)$. The cohomology class of the forms of total dimension $k = p + q$ decomposes as

$$H^k(X) = \sum_{i=0}^{k} H^{i-k,k}(X). \tag{3.83}$$

Therefore the Betti numbers are given by $b_k = \sum_{p=0}^{k} h^{p,k-p}$. Note that $\wedge^n J$ is proportional to the volume form. From this fact and the definitions above, it is clear that the Kähler form J belongs to the class $H^{1,1}$.

A similar classification can be done for the submanifolds of X. Let us consider a set of p-dimensional submanifolds of X denoted by N_i. Arbitrary linear combinations of N_i as $c_i N_i$ define a *$p-$chain* which has a direct meaning in terms of integration : $\sum_i c_i \int_{N_i}$. The boundary operator δ associates to a p–chain its $(p-1)$–dimensional boundary. The operator δ is also nilpotent, because the boundary of a submanifold has no boundary. A chain c_p is closed if it has no boundary i.e. $\delta c_p = 0$ and it is exact (trivial) if it can be written in terms of the boundary of a $p+1$-dimensional chain $c_p = \delta c_{p+1}$. A *$p-$cycle* is a closed p-chain. Non trivial (non exact) cycles can be casted in the concept of *Homology*. The p-homology group of X is defined as

$$H_p(X) = \frac{\text{set of } \delta\text{–closed p–chains}}{\text{set of } \delta\text{–exact p–chains}}. \tag{3.84}$$

So two cycles belong to the same equivalence class if they differ by a boundary i.e. $c_p \simeq c_p + \delta b_{p+1}$. Therefore, the operator δ plays a similar role for the homology theory as the exterior operator for cohomology. A central result which we will use through our studies is the one to one correspondence between $H^p(X)$ and $H_{d-p}(X)$ [113]. This is a consequence of *Stokes theorem*

$$\int_{\delta c_p} \alpha_{p-1} = \int_{c_p} d\alpha_{p-1}, \tag{3.85}$$

and *Poincaré duality*. Poincaré duality states that for every ω_p in $H^p(X)$ there is a $(d-p)$ form α_{d-p} in $H^{d-p}(X)$ which has compact support on a $(d-p)$ cycle c_{d-p}. This gives rise to the formula

$$\int_X \omega_p \wedge \alpha_{d-p} = \int_{c_{d-p}} \alpha_{d-p}. \tag{3.86}$$

[6] This is connected to the fact that there is a unique Laplacian $\Delta_d = 2\Delta_{\bar{\partial}} = 2\Delta_{\partial}$.

Calabi–Yau theorem When the manifold is Kähler the Ricci tensor is $(1,1)$ with only $\mathcal{R}_{i\bar{j}}$ as non-zero component and is closed i.e. $d\mathcal{R}_{1,1} = 0$. Because $\mathcal{R}_{1,1}$ is closed it gives an equivalence class of $H^{1,1}(X)$ and this class is the *first Chern-class*

$$c_1(X) = \frac{1}{2\pi}\mathcal{R}_{1,1}. \tag{3.87}$$

If the first Chern class vanishes the manifold is *Calabi–Yau.* A fact conjectured by Calabi and proved by Yau states that for any Kähler manifold with $c_1 = 0$ and a given complex and Kähler structure there exists a unique metric which is Ricci–flat. The converse is trivial. A Kähler manifold admits a Ricci-flat metric iff it has $SU(3)$–holonomy. Another useful theorem states that a Kähler manifold has $c_1 = 0$ iff it possesses a nowhere vanishing holomorphic (3,0) form.

There are nice relations for the Hodge numbers of such manifolds, those are $h^{p,0} = h^{3-p,0}$ and $h^{1,0} = h^{0,1}$. This information is encoded in the Hodge diamond

$$\begin{array}{ccccccc} & & & h^{3,3} & & & \\ & & h^{3,2} & & h^{2,3} & & \\ & h^{3,1} & & h^{2,2} & & h^{1,3} & \\ h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\ & h^{2,0} & & h^{1,1} & & h^{0,2} & \\ & & h^{1,0} & & h^{0,1} & & \\ & & & h^{0,0} & & & \end{array} = \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{2,1} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} . \tag{3.88}$$

$SU(3)$ **structure** The previous content of the section on manifolds with $SU(3)$ holonomy with respect to the Levi–Civita connection, giving rise to $\mathcal{N} = 1$ in 4d, it is valid for vanishing 3-form field strength $H_3 = 0$. When the background has torsion in the work by Strominger [114] similar conditions to ensure a 4d supersymmetric vacuum are obtained. Using the fact that a supersymmetric variation annihilates the fermi fields, it is proven that a nowhere vanishing holomorphic $(3,0)$ form ω exists and the requirements for 4d supersymmetry read:

- The internal manifold is complex.

- The form $J = G_{i\bar{j}} dz^i d\bar{z}^{\bar{j}}$ obeys

$$\begin{aligned} \partial\bar{\partial}J &= idH, \\ d^\dagger J &= i(\partial - \bar{\partial}) \ln ||\omega||. \end{aligned}$$

- The gauge field strength fullfils

$$\begin{gathered} J^{a\bar{b}} F_{a\bar{b}} = 0, \\ F_{ab} = F_{\bar{a}\bar{b}} = 0. \end{gathered}$$

The factor $||\omega||$ denotes the norm of the $(3,0)$ form ω. The *fundamental form* $(1,1)$ is $J_{mn} = J_m^p G_{pn}$, constructed from the complex structure J_m^p and the metric, for a Kähler manifold it is reduced to the Kähler form. As before, the $SU(3)$ holonomy condition (with the H-connection) comes from imposing the gravitino variation to zero. Also this compactification has vanishing first Chern-class $c_1 = 0$ and $h^{3,0} = 1$.

There is still a wider class of manifolds preserving $\mathcal{N} = 1$ supersymmetry in four dimensions, and those are *manifolds with SU(3) structure*. In addition to the non-vanishing H_3 they are constructed with a generic torsion one-form. There are different torsion classes defined by the set of forms $\mathcal{W}_i, i = 1, ..., 5$ which serve to classify them. Those forms parametrize the external derivative of J and the holomorphic nowhere vanishing 3-form ω as [115–121]

$$\begin{aligned} dJ &= -\frac{3}{2}\mathrm{Im}(\mathcal{W}_1\omega) + \mathcal{W}_4 J + \mathcal{W}_3, \\ d\omega &= \mathcal{W}_1 JJ + \mathcal{W}_2 J + \bar{\mathcal{W}}_5\omega. \end{aligned} \tag{3.89}$$

Strominger class posses $\mathcal{W}_1 = \mathcal{W}_2 = 0$. While the effective action for the cases $\mathcal{W}_4 = \mathcal{W}_5 = 0$ was computed in [120]. The compactifications we consider in which abelian fluxes are turned on is not in the standard embedding and therefore it has non–vanishing H_3.

Which of the five mentioned classes this compactifications belong to is an interesting problem that can be studied. As H is non vanishing the *Bianchi identities* (BI) ensure that dH is trivial in cohomology

$$\int_S dH = \int_S (\mathrm{tr}\mathcal{F}^2 - \mathrm{tr}\mathcal{R}^2) = 0. \tag{3.90}$$

In the previous equation an integration is performed for all S, being S the elements of a basis for the cycles of the manifold X.

Donaldson–Uhlenbeck–Yau theorem If we want to preserve supersymmetry the supersymmetric variation of the gaugino, has to vanish. As it was mentioned this gives rise to the Hermitian Yang–Mills equations. Let us write them together once more [104]

$$F_{\bar{a}\bar{b}} = F_{ab} = 0, \tag{3.91}$$

$$J^{a\bar{b}} F_{a\bar{b}} = 0. \tag{3.92}$$

They are valid both for the case of vanishing and non-vanishing torsion, in which J is the Kähler or the fundamental form of the manifold. Because the gauge fields is real A_a and $A_{\bar{a}}$ are hermitian conjugated to each other. Then, the equation (3.91) can be reduced to $F_{\bar{a}\bar{b}} = 0$. This last equation implies

$$A_{\bar{b}} = i\partial_{\bar{b}} V \cdot V^{-1}, \tag{3.93}$$

being V a function of the coordinates $z_a, z_{\bar{b}}$. The $(1,0)$ form can be obtained by conjugation. An holomorphic function with respect to the covariant derivative fulfills $D_{\bar{a}} f = 0$, which is a generalization of the concept of holomorphic function in complex analysis. Another important definition is the one of *holomorphic vector bundle*, which is a gauge bundle in which the transition functions can be chosen to be holomorphic. Let us explain the definition. In the more general case the manifold X is covered by open sets O_α with gauge field $A_{(\alpha)}$, this cover defines a vector bundle. In the overlap of two regions O_α and O_β the gauge fields are related by a gauge transformation $U_{\alpha\beta}$.[7] Here in each patch $A_{(\alpha)}$ can be written as in (3.93) with a particular V_α. Using this expression for A, the transition equation and the quantity $U'_{\alpha\beta} = V_\alpha^{-1} U_{\alpha\beta} V_\beta$ one finds that

$$\partial_{\bar{a}} U'_{\alpha\beta} \cdot U'^{-1}_{\alpha\beta} = 0. \tag{3.94}$$

The $U'_{\alpha\beta}$ form a new set of transition functions on the bundle after the application of a gauge transformation V_α to O_α with initial transition functions $U_{\alpha\beta}$. Thus, equation (3.94) says that if $F_{\bar{a}\bar{b}} = 0$ the transition functions can be chosen to be holomorphic, yielding an holomorphic vector bundle.

To keep the definition of holomorphic function in two vector bundles Y and Y' which transition functions are $U_{\alpha\beta}$ and $U'_{\alpha\beta}$ respectively, it should be possible to choose the function V_α to be holomorphic. A gauge field that also obeys (3.91) can be obtained using and hermitian matrix G[8]

$$A'_{\bar{a}} = G A_{\bar{a}} G^{-1} + i \partial_{\bar{a}} G \cdot G^{-1}. \tag{3.95}$$

If we want a solution of the system of Hermitian Yang Mills equations we need to determine if a connection can be chosen globally[9] such that in addition to $F_{\bar{a}\bar{b}} = 0$, also $J^{a\bar{b}} F_{a\bar{b}} = 0$ is fulfilled. We will consider a compactification with abelian gauge fluxes $\mathcal{F}$ in which these conditions have to be satisfied. For a Kähler manifold one can write

$$J^{a\bar{b}} F_{a\bar{b}} = \epsilon^{a_1 \ldots a_N} \epsilon^{\bar{b}_1 \ldots \bar{b}_N} F_{a_1\bar{b}_1} J_{a_2\bar{b}_2} \ldots . J_{a_N\bar{b}_N} / (N-1)!^2. \tag{3.96}$$

A necessary and sufficient condition for the previous equation to vanish is that $F \wedge J \ldots . \wedge J$ also vanishes. The gauge $(1,1)$ field strength represents the first Chern class of the line bundle Y

$$c_1(Y) = \mathcal{F}. \tag{3.97}$$

Because the $(1,1)$ cohomology class depends only of the topology of Y, is necessary and sufficient [104] that the invariant

$$\int_X \mathrm{tr}\mathcal{F} \wedge J \wedge J \ldots . \wedge J = \int_X (N-1)!^2 J^{a\bar{b}} \mathcal{F}_{a\bar{b}}, \tag{3.98}$$

vanishes. This invariant for a fixed bundle and fixed J class is independent of the choice of A. Therefore one needs to check that the abelian background in 6d $\mathcal{F}$ satisfies

$$\int_X \mathrm{tr}\mathcal{F} \wedge J \wedge J = 0. \tag{3.99}$$

[7] $A_{i(\alpha)} = U_{\alpha\beta} A_{i(\beta)} U^{-1}_{\alpha\beta} + i \partial_i U_{\alpha\beta} \cdot U^{-1}_{\alpha\beta}$, with $U_{\alpha\beta} U_{\beta\gamma} U_{\gamma\alpha} = 1$ and $U_{\alpha\beta} = U^{-1}_{\beta\alpha}$.

[8] This is not a gauge transformation, to be one G should be unitary.

[9] Locally there is enough freedom to select the hermitian G in order that the hermitian $J^{a\bar{b}} F_{a\bar{b}}$ vanishes.

In the non–abelian case, the abelian subgroup has to satisfy the resumed constraints, plus the requirement is that the bundle Y is *stable*. The BI together with the supersymmetric vacuum condition give rise to solutions that satisfy the equations of motion.

3.6 Toric resolution of orbifolds

The resolution of orbifold singularities have been studied in algebraic geometry. The subject of toric geometry [80] allows to describe the resolution of a local singularity in terms of combinatorial data. That knowledge has been applied to the singularities appearing in string theory in the works [67, 68]. In this section we present a short review of the application of toric geometry to the resolutions of orbifold singularities based on [67,79,81, 82]. This section is not aimed to be a comprehensive review of the subject, but a resume of the important mathematical results needed to resolve orbifolds.

A *toric variety* X of complex dimension r contains an algebraic torus $T = (\mathbb{C}^*)^r$ whose action on X is given by a multiplication law. As an example with $r = 2$ we can look at $\mathbb{CP}^2$ with homogeneous coordinates z_1, z_2, z_3 and the torus

$$T = \{\mu : \mu_i \neq 0, i = 1, 2, 3\} \subset \mathbb{CP}^2. \tag{3.100}$$

The torus action on X is given by

$$T(z_1, z_2, z_3) \to (\mu_1 z_1, \mu_2 z_2, \mu_3 z_3), \tag{3.101}$$

and one can see that under the torus action $\mathbb{CP}^2$ goes to $\mathbb{CP}^2$, so $\mathbb{CP}^2$ is a *T-invariant* variety.

Consider a lattice N of rank r, and the vector space $N_{\mathbb{R}} = N \otimes \mathbb{R}$. In some cases it is convenient to set an isomorphism $N \simeq \mathbb{Z}^r$ which implies an isomorphism $N_{\mathbb{R}} \simeq \mathbb{R}^r$. A *cone*[10] $\sigma \subset N_{\mathbb{R}}$ is a set

$$\sigma = \{a_1 v_1 + a_2 v_2 + ... + a_k v_k | a_i \geq 0\}, \tag{3.102}$$

generated by a finite set of vectors $v_1, v_2, ..., v_k$ in N such that $\sigma \cap (-\sigma) = \{0\}$.

A *collection* Σ *of cones in* $N_{\mathbb{R}}$ *is called a fan* if each face of a cone in Σ is also a cone in Σ and the intersection of two cones in Σ is a face of each. Starting from a given fan Σ one can construct the toric variety X_Σ. Denoting the set of one dimensional cones (edges) by $\Sigma(1)$. To each $\rho \in \Sigma(1)$ associate v_ρ been the unique generator of $\rho \cap N$.

The variety is constructed as a quotient of an open subset in $\mathbb{C}^n$ under a group G. This is done by associating to each edge ρ a coordinate x_ρ. For the set of edges $\{v_1, v_2, ..., v_n\}$ the corresponding coordinates will be $(x_1, x_2, ..., x_n)$. There will be a set $\mathcal{S} \subset \Sigma(1)$ that does not span a cone in Σ. Let $V(\mathcal{S}) \subset \mathbb{C}^n$ be the linear subspace defined by setting

[10] More exactly: a strongly convex rational polyhedral cone.

$\{\forall_{\rho\in\mathcal{S}} x_\rho = 0\}$, and $Z(\sigma)$ be the union of all the $V(\mathcal{S})$. The *toric variety* is defined to be

$$X_\Sigma = (\mathbb{C}^n - Z(\Sigma))/G. \tag{3.103}$$

Now let us look at the group G which is defined to be the kernel of a map ϕ given by

$$\phi : (\mathbb{C}^*)^n \to (\mathbb{C}^*)^r, \quad (t_1, ..., t_n) \to (\prod_{j=1}^n t_j^{v_{j1}}, ..., \prod_{j=1}^n t_j^{v_{jr}}). \tag{3.104}$$

This definition for G preserves $(\mathbb{C}^n - Z(\Sigma))$, and then the quotient (3.103) is well defined. The torus is given by $T = (\mathbb{C}^*)^n/G$ and acts on X_Σ multiplication–wise. Then all T-invariant subvarieties can be classified in an easy way. This is done by associating to a cone σ generated by the edges $\rho_1, ..., \rho_k$ the co–dimension k subvariety

$$Z_\sigma = \{x \in X_\Sigma | x_{\rho_1} = ... = x_{\rho_k} = 0\}. \tag{3.105}$$

The correspondence between a k-dimensional cone σ and the $r-k$-dimensional subvariety Z_σ reverses the inclusion order.[11] It is important to point out that a natural set of coordinates in Z_σ is given by the $G-$invariant variables

$$U^i = \prod_{j=1}^n (z^j)^{(v_j)_i}. \tag{3.106}$$

Example: Again $\mathbb{CP}^2$ is the simplest example. One has a fan Σ spanned by the edges generators $v_1 = (-1,-1)$, $v_2 = (1,0)$ and $v_3 = (0,1)$. In Figure 3.4 the toric diagram of $\mathbb{CP}^2$ is represented. To make manifest the correspondence of the ordinary divisors $D_i = \{z_i = 0\}$ with the edges generator v_i, we have written D_i at the end of the vector v_i.

The fan spanned by such generators has seven cones spanned by: $\{0\}$, $\{(-1,-1)\}$, $\{(1,0)\}$, $\{(0,1)\}$, $\{(1,0),(0,1)\}$, $\{(-1,-1),(0,1)\}$ and $\{(-1,-1),(1,0)\}$. The first is the trivial cone, the next three are the one-dimensional cones and the last three are the two dimensional cones. The set of edges that do not span a cone is $\mathcal{S} = \{(1,0),(0,1),(-1,-1)\}$, so $Z(\Sigma) = \{(0,0,0)\}$ and the group G is the kernel of the map

$$\phi : (\mathbb{C}^*)^3 \to (\mathbb{C}^*)^2, \quad (t_1, t_2, t_3) \to (t_1^{-1}t_2, t_1^{-1}t_3). \tag{3.107}$$

The kernel is given by $(t_1^{-1}t_2, t_1^{-1}t_3) = (1,1)$ such that $t_1 = t_2 = t_3 = t$ and

$$G = \{(t,t,t)| t \in \mathbb{C}^*\}, \tag{3.108}$$

with only the free parameter t such that $G \simeq \mathbb{C}^*$. So the standard definition of $\mathbb{CP}^2$ is obtained. The $\mathbb{CP}^2$ variety definition and the algebraic torus T are given by

$$\mathbb{CP}^2 = (\mathbb{C}^3 - \{(0,0,0)\})/\mathbb{C}^*, \quad T = (\mathbb{C}^*)^3/\mathbb{C}^*. \tag{3.109}$$

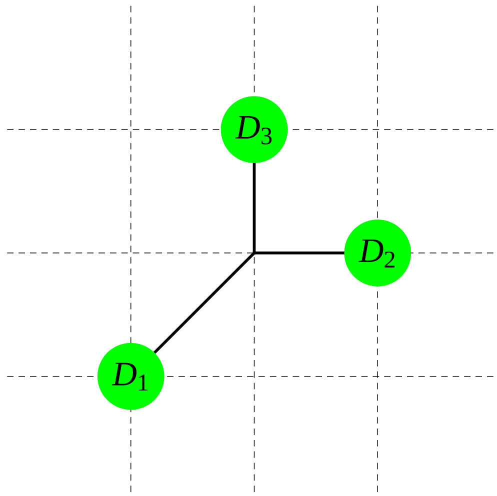

Figure 3.2: Toric diagram of $\mathbb{CP}^2$.

Table 3.4: Association between cones and T–invariant subvarieties of $\mathbb{CP}^2$.

(dim σ, dim Z_σ)	σ	Z_σ
(0,2)	$\{0\}$	$\mathbb{CP}^2$
(1,1)	$\{(-1,-1)\}$	$x_1 = 0$
(1,1)	$\{(1,0)\}$	$x_2 = 0$
(1,1)	$\{(0,1)\}$	$x_3 = 0$
(2,0)	$\{(1,0),(0,1)\}$	$\{(1,0,0)\}$
(2,0)	$\{(-1,-1),(0,1)\}$	$\{(0,1,0)\}$
(2,0)	$\{(-1,-1),(1,0)\}$	$\{(0,0,1)\}$

The association between cones and T-invariant subvarieties can be seen in Table 3.4 [81]

There are two important concepts that we need to apply, and those are compactness and smoothness. A toric variety is *compact* if the union of the cones $\sigma \in \Sigma$ is equal to all of $N_{\mathbb{R}}$, and a toric variety is *smooth* if and only if each $\sigma \in \Sigma$ is smooth. Further, σ is smooth if $\exists\, \mathbb{Z}$–basis $(n_1, n_2, ...n_r)$ of N [12] such that $\sigma = \mathbb{R}_{\geq 0} n_1 + ...\mathbb{R}_{\geq 0} n_s$, with $s \leq r$ [79]. This criteria can be reformulated in a more useful way i.e. the cone σ is smooth if every point in the sublattice $\sigma \cap N$ can be written as linear combination of the generators of the cone $v_1, ..., v_s$ with integer coefficients. From the given definitions it is clear that $\mathbb{CP}^2$ is smooth and compact.

It is also possible to go the other way around and construct a fan starting with the toric variety, and knowing the action of the torus T. T-invariant subvarieties are constructed as closure of T-orbits [13].

Let us define an *orbifold* in the language of toric varieties. A toric variety is an orbifold if and only if its fan Σ is *simplicial.* A cone is simplicial if it can be generated by a set of vectors $v_1, ..., v_k$ which constitute a basis for the vector space spanned by them, and a fan is simplicial if each cone on it is simplicial.

Let us look at the orbifold $\mathbb{C}^3/\mathbb{Z}_7$, which is one of the local singularities that we will later encounter. One can see that the set of vectors

$$v_1 = (2, 0, 1), \quad v_2 = (-1, 2, 1), \quad v_3 = (0, -1, 1), \tag{3.110}$$

constitute a fan for it by computing the group G. Recall that G is the kernel of the map ϕ which is here

$$\phi : (\mathbb{C}^*)^3 \to (\mathbb{C}^*)^3, \quad (t_1, t_2, t_3) \to (t_1^2 t_2^{-1}, t_2^2 t_3^{-1}, t_1 t_2 t_3). \tag{3.111}$$

This is the equation $(t_1^2 t_2^{-1}, t_2^2 t_3^{-1}, t_1 t_2 t_3) = (1, 1, 1)$ from which we obtain

$$G = \mathbb{Z}_7 = \{(t, t^2, t^4), \quad t = e^{2\pi i/7}\}. \tag{3.112}$$

The toric variety and the algebraic torus are given by

$$X = (\mathbb{C}^3)/\mathbb{Z}_7, \qquad T = (\mathbb{C}^*)^3/\mathbb{Z}_7. \tag{3.113}$$

The set $Z(\Sigma) = \{\}$ because there is no subset of vectors from v_1, v_2, v_3 not generating a cone.

One can see that the orbifold is non-compact, because the union of all cones, do not generate the full $N_{\mathbb{R}} = \mathbb{Z}^3 \otimes \mathbb{R}$. In addition it is also singular, for example: in the cone

[11] If $\sigma_1 \subset \sigma_2$ then $Z_{\sigma_2} \subset Z_{\sigma_1}$.

[12] A $\mathbb{Z}$–basis of the lattice N is a basis s.t. every vector of N can be expressed with integer coefficients in terms of the basis vectors.

[13] The closure of a set Y is the smallest subset closed under T that contains Y. A nice an simple example for $\mathbb{CP}^2$ can be read in page 110 from [81].

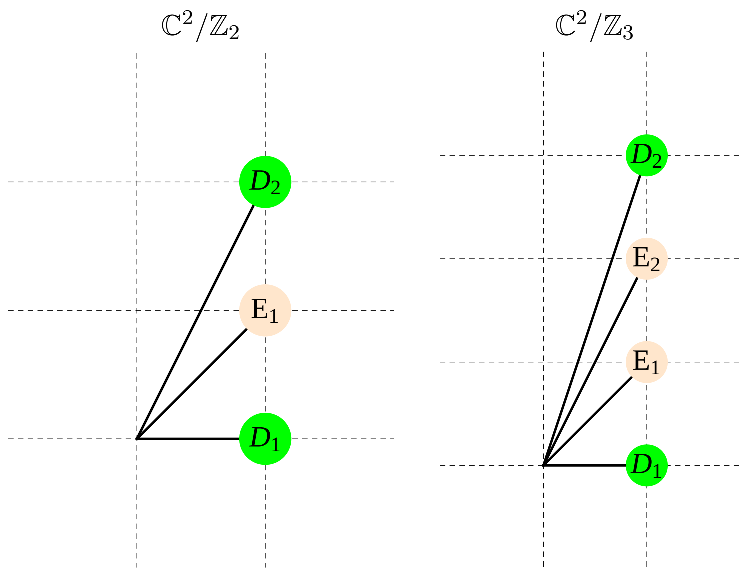

Figure 3.3: Resolution toric diagrams of $\mathbb{C}^2/\mathbb{Z}_2$ and $\mathbb{C}^2/\mathbb{Z}_3$.

generated by $\{v_1, v_2\}$ it is not possible to reach the point $(1,1,1) \in \sigma \cap N$ by performing a linear combination with integer coefficients $a_1 v_1 + a_2 v_2, \quad a_1, a_2 \in \mathbb{Z}$. Furthermore it is possible to write for the orbifold a set of G (θ in the notation of section 3.1) invariant local coordinates as

$$U^i = (z^1)^{(v_1)_i}(z^2)^{(v_2)_i}(z^3)^{(v_3)_i}. \tag{3.114}$$

A blow–up of the toric variety can be obtained by subdividing the fan. A fan Σ' subdivides Σ if $\Sigma(1) \subset \Sigma'(1)$ and each cone of Σ' is contained in some cone of Σ. Let us denote the initial and the final fan of the one-dimensional cones as: $\Sigma(1) = \{\rho_1, ..., \rho_n\}$ and $\Sigma'(1) = \{\rho_1, ..., \rho_m\}$ respectively. Here we consider that the ρ_i have the same order till the n position and that $m \geq n$. Then there is a (blown–down) map between the final and the initial variety $X_{\Sigma'} \to X_\Sigma$ which is performed by taking from the $(x_1, ..., x_m)$ homogeneous coordinates of $X_{\Sigma'}$ the first n coordinates $(x_1, ..., x_n)$.

The blow–up of a point corresponding to the cone σ with generators $v_1, ...v_r$ is performed by adding the edge $v_{r+1} = v_1 + ... + v_r$ and subdividing σ. The new fan Σ' is obtained combining the new cones with the cones in Σ.

The orbifold $\mathbb{C}^2/\mathbb{Z}_n$ has fan spanned by $v_1 = (1,0)$ and $v_2 = (1,n)$, and the group $G = \{(t, t^{n-1}), \;\; t^n = 1\}$. It can be blown–up by adding the vectors $(1,r), \;\; r = 1, .., n-1$, which correspond to $n-1$ exceptional divisors E_r. We can see that it is smooth because with the new added vectors every point in $\sigma \cap N$ can be spanned by integer coefficients in terms of the one–dimensional cones. This can be seen in Figure 3.3, where we show the toric diagrams for the resolved $\mathbb{C}^2/\mathbb{Z}_2$ and $\mathbb{C}^2/\mathbb{Z}_3$. We will encounter those local resolutions in the orbifold $T^6/\mathbb{Z}_{6II}$ which will be the subject of Chapter 5.

Let us talk a bit about the Calabi–Yau condition. We are interested in three complex dimensions orbifolds and resolutions which preserve $\mathcal{N}=1$. Consider an orbifold action given by $G=(e^{2\pi i n_1/N}, e^{2\pi i n_2/N}, e^{2\pi i n_3/N})$. An easy way to ensure the Calabi–Yau condition is looking at the $(3,0)$ holomorphic form $\Omega = dz_1 \wedge dz_2 \wedge dz_3$ which must be invariant under the orbifold action, i.e. $n_1+n_2+n_3=0 \mod N$. Now, from the invariance of (3.114) we obtain

$$n_1(v_1)_i + n_2(v_2)_i + n_3(v_3)_i = 0 \mod N. \tag{3.115}$$

We have three equations which allow to fix a basis in which all of the vectors have one of the coordinates set to one, because the equation $n_1+n_2+n_3=0 \mod N$ has to be fulfilled. This condition is the same as $t_1t_2t_3=1$ and this is obtained from the kernel of G by setting one (lets say the third) of the components of every vector equal to 1. A $GL(3)$ linear transformation of that solution is also possible; for example in $\mathbb{Z}_7$ case the set $v_1=(2,0,1), v_2=(-1,2,3), v_3=(0,-1,0)$ also gives $G=\{(t_1^2t_2^{-1}, t_2^2t_3^{-1}, t_1t_2^3\}$ which is $t_1=t, t_2=t^2, t_3=t^4$ and $t^7=1$.

After resolving the orbifold, the CY condition can be simply seen from evaluating Ω in every patch and checking that is no-where vanishing. This is ensured by adding the additional vector in the same hyperplane that the initial three vectors were lying [14]. Three dimensional $\mathbb{Z}_n$ orbifolds can be resolved by adding new generators [65–67]. Starting with the orbifold action

$$(z_1,z_2,z_3) \to (e^{2\pi i g_1}z_1, e^{2\pi i g_2}z_2, e^{2\pi i g_3}z_3), \quad i=1,...,n-1. \tag{3.116}$$

New vectors are added as

$$\omega_i = g_1^{(i)}v_1 + g_2^{(i)}v_2 + g_3^{(i)}v_3, \tag{3.117}$$

with $g^{(i)} = (g_1^{(i)}, g_2^{(i)}, g_3^{(i)})$ and $\mathrm{diag}(e^{2\pi i g_1^{(i)}}, e^{2\pi i g_2^{(i)}}, e^{2\pi i g_3^{(i)}}) = \{1,\theta,...,\theta^{n-1}\}$. Then the group G is determined to be the kernel of ϕ

$$\phi : (t_1,t_2,t_3,t_4,....,t_{3+d}) \to (\prod (t_j)^{(v_j)_1}, ..., \prod (t_j)^{(v_j)_{3+d}}), \tag{3.118}$$

with $v_{3+i}=\omega_i$ and the new toric variety will be given by

$$\tilde{X} = (\mathbb{C}^{3+d} - \tilde{Z}(\Sigma))/G, \tag{3.119}$$

where, as explained, $\tilde{Z}(\Sigma)$ is the union of all sets of generators not spanning a cone. When there are many possible triangulations the excluded set is what determines the different geometries. We can work out the $\mathbb{Z}_7$ example, for which there is only one triangulation. The added generators are determined from (3.117) to be

$$\begin{aligned} v_4 &= \omega_1 = (0,0,1), \\ v_5 &= \omega_2 = (0,1,1), \\ v_6 &= \omega_4 = (1,0,1). \end{aligned}$$

All of this information is contained in the toric diagram of Figure 3.4. The *ordinary divisor* $D_i = \{z_i = 0\}$ is associated to the vector v_i, with $i=1,2,3$. The *exceptional*

[14] In our basis this is implemented by setting the third component of every vector to one.

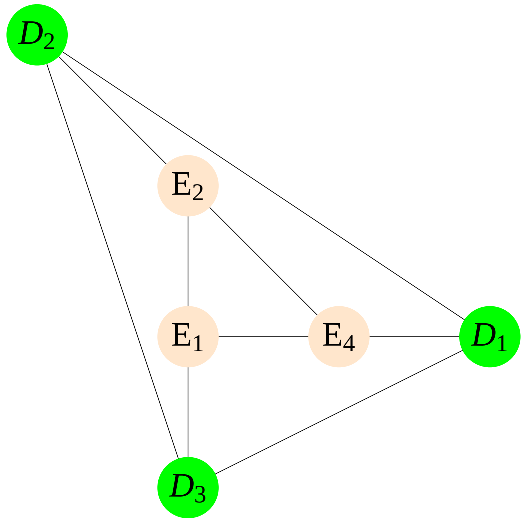

Figure 3.4: Resolution of the $\mathbb{C}^3/\mathbb{Z}_7$ orbifold.

divisor $E_r = \{y_r = 0\}$ is associated to the vector ω_r, with $r = 1, 2, 4$. The ordinary divisors are the subvarieties where one of the initial coordinates vanishes, whereas the exceptional divisors are the subvarieties where one of the new introduced coordinates vanishes.

Here the group G is generated by the kernel of

$$\phi : (t_1, t_2, t_3, t_4, t_5, t_6) \to (t_1^2 t_2^{-1} t_6, t_2^2 t_3^{-1} t_5, t_1 t_2 t_3 t_4 t_5 t_6). \tag{3.120}$$

This is

$$G = \{(t_4^{-1/7} t_5^{-2/7} t_6^{-4/7}, t_4^{-2/7} t_5^{-4/7} t_6^{-1/7}, t_4^{-4/7} t_5^{-1/7} t_6^{-2/7}, t_4, t_5, t_6), t_4, t_5, t_6 \in \mathbb{C}^*\}. \tag{3.121}$$

The group G is isomorphic to $(\mathbb{C}^*)^3$ and its invariant coordinates are given by

$$\tilde{U}^1 = z_1^2 z_2^{-1} y_3, \quad \tilde{U}^2 = z_2^2 z_3^{-1} y_2, \quad \tilde{U}^3 = z_1 z_2 z_3 y_1 y_2 y_3. \tag{3.122}$$

The union of the sets of elements not generating a cone determines the excluded set

$$Z(\Sigma) = \{(z_3, y_2) = (0,0), (z_2, y_3) = (0,0), (z_1, y_1) = (0,0), (z_1, z_2, z_3) = (0,0,0)\}. \tag{3.123}$$

In [67] the detailed procedure to determine the exceptional divisor topologies employing the Mori cone is explained. One of the results is that divisors inside the toric diagram are compact and those ones on the edges are non-compact.

For our later study this is all the geometrical information that we need of the resolution of $\mathbb{Z}_7$. The prime orbifolds $\mathbb{Z}_3$ and $\mathbb{Z}_7$ have only one *triangulation* i.e. one possible way of choosing the set of cones in the fan Σ, this will give a unique exclusion set $Z(\Sigma)$. All the exceptional divisors of local resolutions are compact. Therefore, is enough to resolve all of the singularities locally and then the compact space $T^6/\mathbb{Z}_7$ is resolved. For the other case we are interested in the context of this thesis, which is the global resolution of the $T^6/\mathbb{Z}_{6II}$ orbifold, the situation is more complicated. For non-prime orbifolds where apart from fixed points there can be fixed lines, it happens often that fixed points are sited on the top of fixed lines. This situation requires a more careful analysis when determining the set of exceptional divisors.

In the local orbifold resolutions there are homological relations between the cycles that are obtained to be[15]

$$\sum_i (v_i)_j D_i + \sum_r (\omega_r)_j E_r \sim 0. \tag{3.124}$$

Another important topological information are intersection numbers between divisors. The intersection numbers of two different divisors is one if they belong to the same cone, and zero otherwise. For the $\mathbb{Z}_7$ orbifold we have the following equivalence relations

$$\begin{aligned} 7D_1 &\sim -E_1 - 2E_2 - 4E_4, \\ 7D_2 &\sim -2E_1 - 4E_2 - 4E_4, \\ 7D_3 &\sim -4E_1 - E_2 - 2E_4. \end{aligned} \tag{3.125}$$

These relations together with the non-vanishing intersections of three distinct divisors

$$E_1E_2E_4 = D_1D_3E_4 = D_1E_2E_4 = D_1D_2E_2 = D_3E_1E_4 = D_2D_3E_1 = D_2E_1E_2 = 1, \tag{3.126}$$

will give the set of triple intersections numbers

$$\begin{aligned} E_1^3 = E_2^3 = E_4^3 = 8\,, &\qquad E_1E_2^2 = E_2E_4^2 = E_4E_1^2 = 0\,, \\ E_1^2E_2 = E_2^2E_4 = E_4^2E_1 = -2\,, &\qquad E_1E_2E_4 = 1\,. \end{aligned} \tag{3.127}$$

To study the compact $T^6/\mathbb{Z}_7$ orbifold is enough to take into account the local information. The only specification to make in general is that in the compact variety a new set of divisors defined in the following will also appear.

Compact resolutions of the $T^6/\mathbb{Z}_{6II}$ We present now the resolution for the $T^6/\mathbb{Z}_{6II}$ orbifold. This orbifold has fixed points and fixed lines, and to perform a local resolution is necessary to add non-compact exceptional divisors. Here we make a resume of the procedure from [68] as presented in [83]. First we should say that as $\mathbb{Z}_{6II}$ is non-prime there will be local singularities of the kind $\mathbb{C}^3/\mathbb{Z}_{6II}$, $\mathbb{C}^2/\mathbb{Z}_3$, and $\mathbb{C}^2/\mathbb{Z}_2$. We already explained how the resolution of the $\mathbb{C}^2/\mathbb{Z}_n$ singularities is performed, so we describe now the local $\mathbb{C}^3/\mathbb{Z}_{6II}$.

The one-dimensional cones of the toric diagram are generated by

$$v_1 = (-2, 0, 1), \quad v_2 = (1, 0, 1), \quad v_3 = (0, 2, 1). \tag{3.128}$$

[15] In the works [67,68] the Mori cone is employed to determine the equivalence relations. This construction serves as well to study the topology of the divisors in detail.

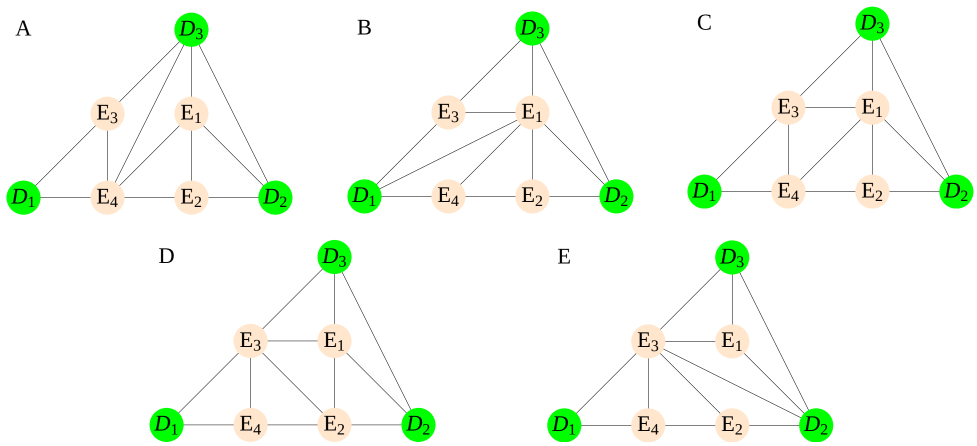

Figure 3.5: Local resolutions of the $\mathbb{C}^3/\mathbb{Z}_{6II}$ orbifold. There are five different ways of defining the fan Σ, what is represented in the five possible triangulations.

This gives a map from $(\mathbb{C}^*)^3$ to $(\mathbb{C}^*)^3$ as

$$\phi : (t_1, t_2, t_3) \to (t_1^{-2} t_2, t_3^2, t_1, t_2 t_3). \tag{3.129}$$

The kernel of the map will generate the group G

$$G = \{(t, t^2, t^{-3}), \qquad t = e^{2\pi i/6}\}. \tag{3.130}$$

The action of the torus can be casted as $T = (\mathbb{C}^*)^3/G$. It is easy to see that not all points in the lattice $\sigma \cap N$ can be reached with the cone generators, then one needs to resolve by subdividing the cone (adding new generators) what gives rise to the exceptional divisors E_1, E_2, E_3 and E_4. This is done following equation (3.117) to obtain

$$\begin{aligned} v_4 &= \omega_1 = (0,1,1), \\ v_5 &= \omega_2 = (0,0,1), \\ v_6 &= \omega_3 = (-1,0,1), \\ v_7 &= \omega_4 = (-1,1,1). \end{aligned} \tag{3.131}$$

This information is collected in the toric diagram of Figure 3.5. As before, ordinary divisors D_i are placed at the positions of vectors v_i, with $i = 1, 2, 3$; and exceptional divisors E_r are placed at the positions of ω_r with $r = 1, 2, 3, 4$. The set of Σ generators gives the map

$$\phi : (t_1, t_2, t_3, t_4, t_5, t_6, t_7) \to (t_1^{-2} t_2 t_6^{-1} t_7^{-1}, t_3^2 t_4 t_7, t_1 t_2 t_3 t_4 t_5 t_6), \tag{3.132}$$

whose kernel is the group

$$G = \{(t_4^{-1/6} t_5^{1/3} t_6^{-2/3} t_7^{-1/2}, t_4^{-1/3} t_5^{-2/3} t_6^{-1/3}, t_4^{-1/2} t_7^{-1/2}, t_4, t_5, t_6, t_7), \quad t_4, t_5, t_6, t_7 \in \mathbb{C}^*\}, \tag{3.133}$$

being isomorphic to $(\mathbb{C}^*)^4$. This gives as local G invariant coordinates

$$\tilde{U}_1 = \frac{z_2}{z_1^2 y_3 y_4}, \quad \tilde{U}_2 = z_3^2 y_1 y_3, \quad \tilde{U}_3 = z_1 z_2 z_3 y_1 y_2 y_3 y_4, \tag{3.134}$$

where the new divisors are given by $E_i = \{y_i = 0\}$ and in the diagram they correspond to the one-dimensional cones ω_i. Now, with the exceptional divisors introduced we have to define the fan Σ, this amounts to fix one triangulation for the diagram. This information will be implicit in the exclusion set of the variety $Z(\Sigma)$[16].

Let us focus now on the topological information that can be read from the diagram 3.5. A triple intersection of divisors is one if the corresponding generators form a cone of the fan Σ i.e. they are placed at the corners of a primitive triangle (that can not be subdivided). They are zero otherwise. For the triangulation A the set of not vanishing triple intersections with distinct divisors is given by

$$D_1 E_3 E_4 = D_3 E_3 E_4 = D_3 E_1 E_4 = E_1 E_2 E_4 = E_1 E_2 D_2 = D_3 D_2 E_1 = 1. \tag{3.135}$$

For the triangulation we B the intersection numbers are given by

$$D_1 E_1 E_4 = E_1 E_2 E_4 = D_2 E_1 E_2 = D_2 D_3 E_1 = D_3 E_1 E_3 = D_1 E_1 E_3 = 1. \tag{3.136}$$

We emphasize those because they are the ones relevant for our study. The rest of the triangulations can be found in Figure 3.5. From those intersections and the equivalence relations (3.124) it is possible to determine any triple intersection for the local resolutions.

However we are interested in resolving the global space $T^6/\mathbb{Z}_{6II}$. Let us address this problem now. In the thesis work [67] the procedure to determine the set of divisors on the global resolution is described in detail. For our case of interest this prescription is presented in [83]. We do not aim here to expose this method in all detail, rather we review it shortly.

First we have to consider the local resolutions of $\mathbb{C}^3/\mathbb{Z}_{6II}$, $\mathbb{C}^2/\mathbb{Z}_3$, and $\mathbb{C}^2/\mathbb{Z}_2$. Let us first comment on the ordinary divisors. An important fact is that fixed points which share one coordinate will have the same ordinary divisor in that coordinate. Furthermore, some divisors corresponding to fixed tori mapped into each other on the orbifold, will be united to form orbifold invariant combinations. Looking at the Figures 5.1-5.3 one can see that there are 12 fixed points of θ with indices $\alpha = 1$, $\beta = 1, 2, 3$ and $\gamma = 1, 2, 3, 4$; 6 fixed tori of θ^2 and θ^4 with indices $\alpha = 1, 3$ and $\beta = 1, 2, 3$; and 8 fixed tori of θ^3 with indices $\alpha = 1, 2$ and $\gamma = 1, 2, 3, 4$.

We will use $\widetilde{D}_{i,\rho}$ to denote the ordinary divisor of the local singularity ρ at the complex plane i. Thus, the ordinary divisors of the local $\mathbb{C}^3/\mathbb{Z}_{6II}$ singularity $(1, \beta, \gamma)$ are $\widetilde{D}_{1,1}$, $\widetilde{D}_{2,\beta}$ and $\widetilde{D}_{3,\gamma}$; the ordinary divisors from the local $\mathbb{C}^2/\mathbb{Z}_3$ singularity (α, β) are $\widetilde{D}_{1,\alpha}$, and $\widetilde{D}_{2,\beta}$; and the ordinary divisors of the local $\mathbb{C}^2/\mathbb{Z}_2$ singularity (α, γ) are $\widetilde{D}_{1,\alpha}$ and $D_{3,\gamma}$. We use tildes to denote that those divisors are on the orbifold cover. However we will form the

[16] With what has been here explained $Z(\Sigma)$ can be obtained easily.

ordinary divisors on the orbifold as invariant combinations of the divisors on the cover. This gives the following set of ordinary divisors for $T^6/\mathbb{Z}_{6II}$:

$$\begin{aligned} D_{1,1} &= \widetilde{D}_{1,1}, \\ D_{1,2} &= \widetilde{D}_{1,2} + \widetilde{D}_{1,4} + \widetilde{D}_{1,6}, \\ D_{1,3} &= \widetilde{D}_{1,3} + \widetilde{D}_{1,5}, \\ D_{2,\beta} &= \widetilde{D}_{2,\beta},\ \beta = 1,2,3, \\ D_{3,\gamma} &= \widetilde{D}_{3,\gamma},\ \gamma = 1,2,3,4. \end{aligned} \tag{3.137}$$

The divisor $D_{1,2}$ is constructed as a linear combination of divisors for fixed points with $\alpha = 2,4,6$. Also the divisor $D_{1,3}$ is constructed as a linear combination of the divisors for the fixed points with $\alpha = 3,5$.

To determine the total number of exceptional divisors we need to consider by separate the different kinds of local singularities. We denote the exceptional divisor by $\widetilde{E}_{k,\bar{\psi}_{ixedset}}$, where k represents the sector and fixed set stands for the localization. For the local $\mathbb{C}^3/\mathbb{Z}_{6II}$ singularity at $(1,\beta,\gamma)$, the divisors are $\widetilde{E}_{1,\beta\gamma}$, $\widetilde{E}_{2,1\beta}$, $\widetilde{E}_{4,1\beta}$ and $\widetilde{E}_{3,1\gamma}$. Looking at the singularities $\mathbb{C}^2/\mathbb{Z}_3$, they appear from the action of $\mathbb{Z}_3$ which is performed by θ^2 and θ^4. The fixed lines with $\alpha = 1$ were already considered, so the extra exceptional divisors are $\widetilde{E}_{2,\alpha\beta}$ and $\widetilde{E}_{4,\alpha\beta}$ with $\alpha = 3,5$. The singularities $\mathbb{C}^2/\mathbb{Z}_2$ give the exceptional divisors $\widetilde{E}_{3,\alpha\gamma}$ with $\alpha = 2,4,6$, also because already the divisors with $\alpha = 1$ were counted. The tildes notation implies that the listed divisors are on the cover. Orbifold invariant combinations of exceptional divisors will give

$$\begin{aligned} E_{1,\beta\gamma} &= \widetilde{E}_{1,\beta\gamma}, \\ E_{2,1\beta} &= \widetilde{E}_{2,1\beta}, \\ E_{3,1\gamma} &= \widetilde{E}_{3,1\gamma}, \\ E_{4,1\beta} &= \widetilde{E}_{4,1\beta}, \\ E_{2,3\beta} &= \widetilde{E}_{2,3\beta} + \widetilde{E}_{2,5\beta}, \quad \beta = 1,2,3, \\ E_{4,3\beta} &= \widetilde{E}_{4,3\beta} + \widetilde{E}_{4,5\beta}, \quad \beta = 1,2,3, \\ E_{3,2\gamma} &= \widetilde{E}_{3,2\gamma} + \widetilde{E}_{3,4\gamma} + \widetilde{E}_{3,6\gamma} \quad \gamma = 1,2,3,4. \end{aligned} \tag{3.138}$$

The fixed lines w.r.t θ^2 and θ^4 in $\alpha = 3,5$ are identified on the orbifold, and as well, the ones w.r.t θ^3 with $\alpha = 2,4,6$. In total there are 32 exceptional divisors on the compact blow–up.

The next ingredient to include are the *inherited divisors*. On the compact space the $(1,1)$ forms $dz^i \wedge d\bar{z}^{\bar{i}}$ are invariant under the orbifold action, and therefore exist on the orbifold. With the use of Poincaré duality we can associate a dual divisor R_i to each of them. In [68] it has been obtained in the orbifold the result $R_i \sim n_i D_i$ where n_i is the order of the orbifold action on the i torus. The recipe to obtain the global equivalence on the resolve space is substitute $n_i D_i$ by the summed local equivalence relations[17]. This leads to the complete

[17]The local equivalences can be obtained by the given formula.

set of equivalences

$$
\begin{aligned}
R_1 &\sim 6D_{1,1} + \sum_{\beta=1}^{3}\sum_{\gamma=1}^{4} E_{1,\beta\gamma} + \sum_{\beta=1}^{3}(2E_{2,1\,\beta} + 4E_{4,1\,\beta}) + 3\sum_{\gamma=1}^{4} E_{3,1\,\gamma}\ , \\
R_1 &\sim 2D_{1,2} + \sum_{\gamma=1}^{4} E_{3,2\,\gamma}\ , \\
R_1 &\sim 3D_{1,3} + \sum_{\beta=1}^{3}(E_{2,3\,\beta} + 2E_{4,3\,\beta})\ , \\
R_2 &\sim 3D_{2,\beta} + \sum_{\gamma=1}^{4} E_{1,\beta\gamma} + \sum_{\alpha=1,3}(2E_{2,\alpha\beta} + E_{4,\alpha\beta})\ ,\ \beta = 1,2,3, \\
R_3 &\sim 2D_{3,\gamma} + \sum_{\beta=1}^{3} E_{1,\beta\gamma} + \sum_{\alpha=1,2} E_{3,\alpha\gamma}\ ,\gamma = 1,2,3,4\ .
\end{aligned}
$$

Then, to obtain the intersection ring of all divisors there exists the method of constructing and *auxiliary polyhedra* [68]. We recall the method as presented in [83]. For the orbifold group G, select the lattice $N \simeq \mathbb{Z}^3$ such that $N = \{f_i = m_i e_i,\ m_i > 0,\ \ m_1m_2m_3 = n_1n_2n_3/|G|\}$. Rotate the full diagram $\mathbb{C}^3/G$ to a coordinate system in which the ordinary divisors D_i are placed at $v_{i+3} = n_i f_i$. The inherited divisors are placed on $v_i = -f_i$. Suppose that there are subgroups H that give not fixed points but higher dimensional singularities. Consider a subgroup H, to describe the singularities of the kind $\mathbb{C}^2/H$ one has to take an identical polyhedra to the previous one, just that the divisors absent in $\mathbb{C}^2/H$ are removed. For each local resolution a polyhedra has to be considered. If ordinary divisors are linear combinations of q divisors, the vector v_k should be divided by q. The triangulation of the polyhedra is performed in order to preserve the triangulation of the original toric diagram. Only divisors spanning a simplex on the same polyhedra will have non–zero intersection numbers given by

$$ABC = \mathrm{const}/|\det(v_A v_B v_C)|, \tag{3.139}$$

where the constant is fixed by the requirement that intersection numbers without inherited divisors coincide with the ones of the local resolution.

3.7 Anomalies and Green-Schwarz mechanism

Classical symmetries may be broken upon quantization giving rise to anomalies. Anomalies of local symmetries give an inconsistency, because they cause that the unphysical degrees of freedom are not decoupled from the theory. In string theory all the local anomalies are cancelled. Anomalies constitute short distances effects because they arise from the impossibility of regulating the theory in a form which is consistent with the symmetry. But also they are long distance effects, because this difficulty depends on the massless spectrum. Addition of massive degrees of freedom will not change the anomalies, because its contribution to the path integral is local at long distances and can be cancelled by a counterterm.

Heterotic string theory is not parity symmetric so may have potential anomalies. This is also the case for the remaining string theories with the exception of type IIA which is non–chiral. We will describe the anomaly from the low energy point of view, i.e. the 10d $\mathcal{N}=1$ supergravity theory. From this point of view its cancellation will include several apparent coincidences, that are really a consequence of defining a consistent string theory. The un-physical gauge and gravitational polarizations are decoupled by a cancellation with a counterterm. For the heterotic theory the diagrams are calculated on a torus world–sheet, the first one in the limit of $\tau_2 \to \infty$ and the second one in the limit of two vertex operators approaching.

Consider the anomalous variation of the path integral in the supergravity to be

$$\delta \ln Z = \frac{-i}{(2\pi)^5} \int I_d(\mathfrak{F}, \mathfrak{R}), \tag{3.140}$$

and the *anomaly polynomial* is defined to be a $d+2$- form in a formal sense such that [28]

$$I_{d+2} = dI_{d+1}, \quad \delta I_{d+1} = dI_d. \tag{3.141}$$

These are called *descend equations.* The polynomial I_{d+2} has the advantage that is written in terms of gauge invariant, closed and locally exact forms $\mathrm{tr}\mathfrak{F}^m$, $\mathrm{tr}\mathfrak{R}^n$. Then, the anomaly can be expressed as

$$\begin{aligned} I_{d+2} &= (c\mathrm{Tr}(\mathfrak{F})^2 + c'\mathrm{Tr}(\mathfrak{R})^2)X_8, \\ I_d &= (c\mathrm{Tr}(\chi\mathfrak{A}) + c'\mathrm{Tr}(\Theta d\mathfrak{W}))X_8, \end{aligned} \tag{3.142}$$

this anomalous variation $\delta \ln Z \sim \int I_d$ can be canceled by adding an interaction term of B with gluons and gravitons

$$S_B = \int B_2 X_8(\mathfrak{F}, \mathfrak{R}). \tag{3.143}$$

This can be checked directly from the local Lorentz and gauge transformations (2.41). This new term is an addition to the supergravity action (2.36).

The anomalous diagram and its counterterm in 10d are represented in Figure 3.6. The chiral fields of $\mathcal{N}=1$ supergravity are the gravitino $\mathbf{56}$, the neutral fermion $\mathbf{8}'$ and the gaugino $\mathbf{8}$. In the heterotic theory (2.23) those are the fields

$$(\mathbf{56}, \mathbf{1}, \mathbf{1}) + (\mathbf{8}', \mathbf{1}, \mathbf{1}) + (\mathbf{8}, \mathbf{248}, \mathbf{1}) + (\mathbf{8}, \mathbf{1}, \mathbf{248}). \tag{3.144}$$

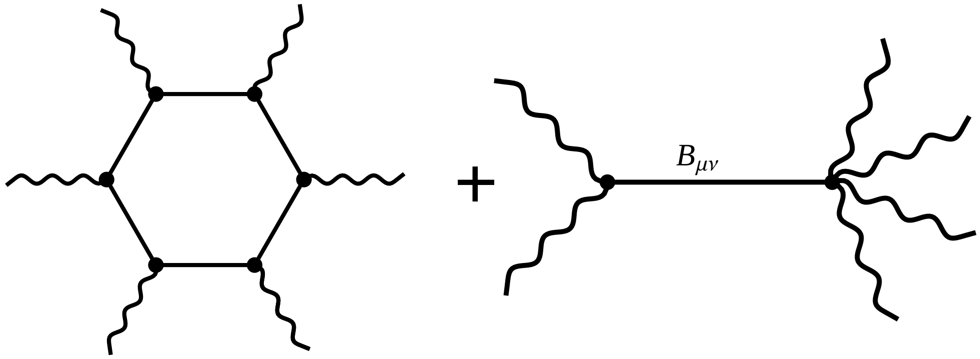

Figure 3.6: Hexagon diagram giving anomalies in 10d plus the counterterm canceling the anomaly. The external legs of the first diagram can be all gauge bosons, or all gravitons or 2 (4) gauge bosons and 4 (2) gravitons. In the second diagram the external legs of the left can be 2 gauge bosons (or 2 gravitons) and the ones from the right can be 4 gauge bosons (or 2 gauge bosons and 2 gravitons, or 4 gravitons).

The total anomaly polynomial for the $\mathcal{N} = 1$ supergravity coupled to super Yang–Mills with a gauge group of rank n is given by

$$\begin{aligned} I_{12} &= I_{\mathbf{56}}(\mathfrak{R}) - I_{\mathbf{8}'}(\mathfrak{R}) + I_{\mathbf{8}}(\mathfrak{F},\mathfrak{R}), \\ &= -\frac{1}{720}\mathrm{Tr}\mathfrak{F}^6 + \frac{1}{24\cdot 48}\mathrm{Tr}\mathfrak{F}^4\mathrm{tr}\mathfrak{R}^2 - \frac{1}{256}\mathrm{Tr}\mathfrak{F}^2\left[\frac{1}{45}\mathrm{tr}\mathfrak{R}^4 + \frac{1}{36}(\mathrm{tr}\mathfrak{R}^2)^2\right] \\ &+ \frac{n-496}{64}\left[\frac{1}{2\times 2835}\mathrm{tr}\mathfrak{R}^6 + \frac{1}{4\times 1080}\mathrm{tr}\mathfrak{R}^2\mathrm{tr}\mathfrak{R}^4 + \frac{1}{8\times 1296}(\mathrm{tr}\mathfrak{R}^2)^3\right] \\ &+ \frac{1}{384}\mathrm{tr}\mathfrak{R}^2\mathrm{tr}\mathfrak{R}^4 + \frac{1}{1536}(\mathrm{tr}\mathfrak{R}^2)^3. \end{aligned} \tag{3.145}$$

As we can see for the case $n = 496$ the anomaly gets reduced, this signals to choose groups with rank 496. Furthermore, the selection $SO(32)$ or $E_8 \times E_8$ implies identities of the traces $\mathrm{Tr}F^6$ such that the polynomial can be factorized like in (3.142) [28]. The trace Tr is given in the adjoint representation, and we define for $E_8 \times E_8$ the symbol $\mathrm{tr} = \frac{1}{30}\mathrm{Tr}$ which is an identity for $SO(32)$ relating the trace in the adjoint with the trace in the fundamental.

From the low energy description presented here, and used commonly, the anomaly cancelation involves the coincidences of: number of generators of the gauge group, identity of the traces and factorization of the polynomial. This apparent coincidences are explained due to the requirement of a consistent string theory. They are obtained as a consequence of the cancellation occurring between the anomalous world–sheet amplitudes and its counterterms.

In a given compactification, to understand the four dimensional anomaly cancelation we can make a dimensional reduction of I_{12} and S_B, obtaining the anomaly cancellation in terms of the various axions [91] descending from the B_2 expansion (4.23). The factorization of the polynomial I_{12} for the heterotic $E_8 \times E_8$ theory is given in terms of the forms X_4 and X_8 as

$$\begin{aligned}
I_{12} &= X_4 X_8 \,, && (3.146)\\
X_8 &= \frac{1}{4}\left((\mathrm{tr}\mathfrak{F}'^2)^2 + (\mathrm{tr}\mathfrak{F}''^2)^2\right) - \frac{1}{4}\mathrm{tr}\mathfrak{F}'^2\mathrm{tr}\mathfrak{F}''^2 - \frac{1}{8}(\mathrm{tr}\mathfrak{F}'^2 + \mathrm{tr}\mathfrak{F}''^2)\mathrm{tr}\mathfrak{R}^2 && (3.147)\\
&+ \frac{1}{8}\mathrm{tr}\mathfrak{R}^4 + \frac{1}{32}(\mathrm{tr}\mathfrak{R}^2)^2 \,, && (3.148)\\
X_4 &= \mathrm{tr}\mathfrak{R}^2 - \mathrm{tr}\mathfrak{F}'^2 - \mathrm{tr}\mathfrak{F}''^2 \,.
\end{aligned}$$

The 10d quantities are decomposed in terms of 6d internal and 4d components as $\mathfrak{W} = \mathcal{W} + \omega,\ \mathfrak{R} = \mathcal{R} + R,\ \mathfrak{A} = \mathcal{A} + A$ and $\mathfrak{F} = \mathcal{F} + F$, where the first term and the second term in the sums are the 6d and 4d components, respectively. When it is necessary to distinguish between the two E_8s, we mark the gauge fields and the field strengths in the first and second E_8 with $'$ and $''$, respectively. Starting from the anomaly (3.146) in ten dimensions, we will describe in the next section how the cancellation mechanism occurs in a smooth compactification.

3.7.1 Dimensional reduction of the anomaly polynomial

In this section we consider the anomaly cancellation mechanism of the four dimensional effective theory. This mechanism is understood in terms of the universal and the non–universal (localized) axions. In order to analyze anomalies in the 4d effective field theory we study the 4d anomaly polynomial I_6 and the 4d Green–Schwarz mechanism [28] derived from the one of ten dimensional supergravity.

We will show how the cancellation is implemented in a smooth Calabi–Yau in the presence of abelian gauge fluxes in the internal dimensions. We describe the way in which axions arise. This has been studied in [91] for a non–compact resolution with a single non–universal axion and in [122] for a more generic compactification. Our analysis applies these results to a case where multiple non–universal axions appear. We will be interested in a blow–up of an orbifold compactification, which is an example of it.

The change of the effective action due to gauge transformations (parameterized by χ) and Lorentz transformations (parameterized by Θ) is given by [104]

$$G(\chi, \Theta) = \int\limits_{\mathcal{M}\times M^4} I_{10} = \int\limits_{\mathcal{M}\times M^4} \left(\mathrm{tr}(\Theta d\mathfrak{W}) - \mathrm{tr}(\chi d\mathfrak{A})\right) X_8 \,, \qquad (3.149)$$

where we split up the 10d space into the 6d internal manifold $\mathcal{M}$ and 4d Minkowski space M^4, and omit a numerical factor arising in the dimensional reduction. The variation of the axion field $\delta_{\chi,\Theta} B_2 = -\mathrm{tr}(\Theta d\mathfrak{W}) + \mathrm{tr}(\chi d\mathfrak{A})$ induces a variation δS_B which exactly cancels $G(\chi, \Theta)$ (here we have set $c = 1,\ c' = -1$ in (2.41)). In the compactification to 4d, the anomaly cancellation arises from the variations of the B_2 components inherited from 10d variations, and from imposing the condition $\delta_{\chi_0} dB_2 = 0$, where χ_0 are gauge transformations on the gauge bundle $\mathcal{A} \to \mathcal{A} + \delta_{\chi_0}\mathcal{A}$, $[\mathcal{A}, \chi_0] = 0$. The B_2 field is expanded as

$$B_2 = b_2 + \alpha_i R_i - \beta_r E_r , \tag{3.150}$$

where R_i and E_r are exceptional divisors of the compact space, and the flux will be supported in the set E_r. The 4d *universal* axion a^{uni} and the *non–universal* axions β_r cancel the 4d anomaly. This can be seen from the reduction of G and S_B and by performing a field redefinition necessary to ensure $\delta_{\chi_0} dB_2 = 0$. The dimensional reduction of the variation of the effective action (3.149) reads

$$I_4 = \int\limits_{\mathcal{M}} \mathrm{tr}(\Theta d\mathfrak{W}) X_8 - \mathrm{tr}(\chi dA) \int\limits_{\mathcal{M}} X_{6,2} - \int\limits_{\mathcal{M}} \mathrm{tr}(\chi \mathcal{F}) X_{4,4} \, , \tag{3.151}$$

$$G = \int\limits_{M^4} I_4 = \int\limits_{M^4} \left[\mathrm{tr}(\Theta d\omega) X_2^{\mathrm{uni}} + \Theta_a (\mathcal{W}_a^r X_4^r + \mathcal{W}_a^i X_4^i) \right] - \int\limits_{M^4} \left[\mathrm{tr}(\chi dA) X_2^{\mathrm{uni}} + V_r^I \chi_I X_4^r \right] . \tag{3.152}$$

The forms $X_{2k,2l}$ with $2(k+l) = 8$ are the sum of all the terms in X_8 having $2k$ indices in the internal space, and $2l$ indices in the external 4d space, and $X_4^r = \int_{\mathcal{M}} E_r X_{4,4}$. Furthermore, $\Theta = \Theta_a T^a$ is the expansion of the Lorentz transformation in terms of $SO(9,1)$ generators T^a and $d\mathcal{W} = (\mathcal{W}_a^r E_r + \mathcal{W}_a^i R_i) T^a$ is the expansion of the derivative of the spin connection in T^a and in (1,1) forms on the internal manifold.
The whole anomaly variation of the action can be divided into a *universal* and a *non–universal* part given by

$$G_{\mathrm{uni}} = \int\limits_{M^4} \left(\mathrm{tr}(\Theta d\omega) - \mathrm{tr}(\chi dA) \right) X_2^{\mathrm{uni}} \, , \tag{3.153}$$

$$G_{\mathrm{non}} = \int\limits_{M^4} \Theta_a (\mathcal{W}_a^r X_4^r + \mathcal{W}_a^i X_4^i) - V_r^I \int\limits_{M^4} \chi_I X_4^r \, , \tag{3.154}$$

where $X_4^i = \int\limits_{\mathcal{M}} R_i X_{4,4}$. Along the same lines one can write the dimensional reduction of S_B as

$$S_B = \int\limits_{M^4 \times \mathcal{M}} B_2 X_8 = \int\limits_{M^4 \times \mathcal{M}} b_2 X_{6,2} + \int\limits_{M^4 \times \mathcal{M}} (\alpha_i R_i + \beta_r E_r) X_{4,4} = \int\limits_{M^4} b_2 X_2^{\mathrm{uni}} + \int\limits_{M^4} (\beta_r X_4^r + \alpha_i X_4^i) \, . \tag{3.155}$$

Now we can understand how the 4d transformations of a^{uni}, β_r, and α_i inherit the 10d anomalous variations of the B_2 field. Considering the 4d variations of the axions to be exactly the same as those of B_2, and without taking into account mixed index variations (which is equivalent to a redefinition of B_2 in order to achieve $\delta_{\chi_0} B_2 = 0$), anomaly cancellation in 4d is implemented by

$$\delta b_2 = \mathrm{tr}(\chi dA) - \mathrm{tr}(\Theta d\omega)\,, \tag{3.156}$$

$$\delta B_{1,1} = \mathrm{tr}(\chi\mathcal{F}) - \mathrm{tr}(\Theta d\mathcal{W}) = \chi^I V_r^I E_r - \Theta_a(\mathcal{W}_a^r E_r - \mathcal{W}_a^i R_i)\,, \tag{3.157}$$

where $B_{1,1} = \alpha_i R_i + \beta_r E_r$. The α_i and β_r satisfy

$$\delta\alpha_i = -\Theta_a \mathcal{W}_a^i, \quad \delta\beta_r = \chi^I V_r^I + \Theta_a \mathcal{W}_a^r\,, \tag{3.158}$$

which ensures

$$G_{\mathrm{non}} + G_{\mathrm{uni}} + \delta_\chi \int\limits_{M^4\times\mathcal{M}} B_2 X_8 = 0\,. \tag{3.159}$$

Let us now take a complementary approach, which proceeds via studying the reduction of H_3 and checking how δH_3 is canceled by the variation of the 4d axions. Let us consider gauge variations only. This will clarify why it is allowed to restrict the variation of B_2 to the 4d axions β_r or a^{uni}.
The three–form $\Omega_3^{\mathrm{YM}} = \mathrm{tr}(\mathfrak{A}\mathfrak{F} - \mathfrak{A}^3/3)$ can be expanded in terms of 4d and 6d parts as

$$\Omega_3^{\mathrm{YM}} = \Omega_3^{\mathrm{YM,4d}} + \mathrm{tr}(\mathcal{A}d\mathfrak{A}) + \mathrm{tr}(A\mathcal{F})\,. \tag{3.160}$$

The term $\mathrm{tr}(\mathcal{A}d\mathfrak{A})$ is used in the redefinition of dB_2. This procedure serves two purposes: it ensures $\delta_{\chi_0} dB_2 = 0$ and it fits with the dimensional reduction of B_2 which otherwise, due to the absence of mixed indices (between internal and 4d coordinates), does not cancel the $\mathrm{tr}(\mathcal{A}d\mathfrak{A})$ variation of H_3. The gauge anomalous variations of the *universal* axion a^{uni} cancels the one of $\Omega_3^{\mathrm{YM,4d}}$ and the gauge anomalous variations of the β_r cancels the one of $\mathrm{tr}(A\mathcal{F})$. A similar analysis can be done for the Lorentz part, but as we consider a space with vanishing Ricci–tensor in the internal dimensions, those variations are not present.
Finally, the field redefinition which ensures dB_2 invariance under bundle gauge transformation χ_0, is equivalent to the analysis where the decomposition of the 10d field B_2 in terms of b_2 and $B_{1,1}$ cancels the anomaly in 4d with a variation inherited from $\delta_\chi B_2$. This can be seen by noting that anomalous variations of the 4d axions which cancel the 4d anomaly make $\delta H_3 = 0$ only if the form $\mathrm{tr}(\mathcal{A}d\mathfrak{A})$ as well as the analog Lorentz form are absorbed in dB_2. By decomposing the 10d exterior derivative d as $d = d_4 + d_6$, the three–form field strength variation can be written as

$$\begin{aligned}\delta H_3 = \delta d_4 b_2 &+ [d_4(\mathrm{tr}\Theta d\omega) - d_4\mathrm{tr}(\chi dA)] + [d_4\delta\alpha_i R_i + d_4\delta\beta_r E_r] + [d_4(\mathrm{tr}\Theta d\mathcal{W}) - d_4\mathrm{tr}(\chi d\mathcal{A})]\\ &+ d_6[\mathrm{tr}(\Theta d_4\omega) - \mathrm{tr}(\chi d_4 A)] + d_6[\mathrm{tr}(\Theta d_6\mathcal{W}) - \mathrm{tr}(\chi d_6\mathcal{A})]\,.\end{aligned} \tag{3.161}$$

It is apparent that the second row, which can be written as $\delta\mathrm{tr}\mathcal{R}d\mathfrak{R} - \delta\mathrm{tr}\mathcal{A}d\mathfrak{A}$ has to be absorbed in the whole dB_2 because the index structure of its decomposition cannot cancel this variation. This is how we implement the Green–Schwarz mechanism in blow–up.

Chapter 4

The $\mathbb{Z}_7$ orbifold and its resolution

In this section we describe the study of a vacuum obtained from the heterotic theory on a $T^6/\mathbb{Z}_7$ orbifold via assigning vevs to singlets. Our aim is to interpret those singlets as blow–up modes which are moduli of the supergravity approximation compactified in the resolved orbifold. Because the string effects are suppressed in terms α'/R^2 powers, where R stands for the compactification scale, the approximation tends to be exact in the large volume limit. To resolve the orbifold the tools of toric geometry presented in section 3.6 are employed. First we will describe the resolution process in this particular model, then we will present the equivalence of the massless spectrum in both approaches. Finally we will show the connection between the anomaly cancellation mechanism in both descriptions.

4.1 Orbifold theory

The requirement to have a $\mathbb{Z}_7$ symmetry constraints strongly the lattice. It has to be the $SU(7)$ root lattice, in which two independent deformations are allowed. The $\mathbb{Z}_7$ action on the lattice vectors is given by

$$e_a \to e_{a+1} \forall a = 1, \ldots, 5, \quad e_6 \to -\sum_{i=1}^{6} e_i, \tag{4.1}$$

where the e_a are the group simple roots. One can then combine the orbifold group with the lattice shifts to obtain the space–group as $S = \mathbb{Z}_7 \rtimes \Lambda_6$ to define the orbifold as

$$\mathcal{O} = T^6/\mathbb{Z}_7 = \mathbb{C}^3/S\,. \tag{4.2}$$

The $\mathbb{Z}_7$ orbifold action is given by

$$\theta : (z_1, z_2, z_3) \to (\xi z_1, \xi^2 z_2, \xi^4 z_3) \qquad \text{with } \xi = e^{2\pi i/7}\,. \tag{4.3}$$

The shift satisfies the condition (3.13), lying on $SU(3) \subset SO(9,1)$, preserving one supersymmetry in four dimensions. This is equivalent to say that the holomorphic three-form $\Omega = dz_1 \wedge dz_2 \wedge dz_3$ is preserved. The fixed points are seven and constitute the weights of the anti–symmetric fundamental representations. In terms of the lattice vectors they are given by

$$\begin{aligned}
f_1 &= 0, \\
f_2 &= \frac{e_1}{7} + \frac{2e_2}{7} + \frac{3e_3}{7} + \frac{4e_4}{7} + \frac{5e_5}{7} + \frac{6e_6}{7}, \\
f_3 &= \frac{2e_1}{7} + \frac{4e_2}{7} + \frac{6e_3}{7} + \frac{e_4}{7} + \frac{3e_5}{7} + \frac{5e_6}{7}, \\
f_4 &= \frac{3e_1}{7} + \frac{6e_2}{7} + \frac{2e_3}{7} + \frac{5e_4}{7} + \frac{e_5}{7} + \frac{4e_6}{7}, \\
f_5 &= \frac{4e_1}{7} + \frac{e_2}{7} + \frac{5e_3}{7} + \frac{2e_4}{7} + \frac{6e_5}{7} + \frac{3e_6}{7}, \\
f_6 &= \frac{5}{7} + \frac{3e_2}{7} + \frac{e_3}{7} + \frac{6e_4}{7} + \frac{4e_5}{7} + \frac{2e_6}{7}, \\
f_7 &= \frac{6e_1}{7} + \frac{5e_2}{7} + \frac{4e_3}{7} + \frac{3e_4}{7} + \frac{2e_5}{7} + \frac{e_6}{7}.
\end{aligned} \tag{4.4}$$

Locally each of these singularities looks like $\mathbb{C}^3/\mathbb{Z}_7$.

As in the present orbifold all the six lattice vectors generating T^6 are equivalent, this causes that the embedding on the gauge d.o.f of the translations possesses only one independent Wilson line. The embeddings on the gauge degrees of freedom for the shift V and the Wilson line W should satisfy

$$7V, 7W \in \Lambda_{E_8 \times E_8}. \tag{4.5}$$

Recall that this happens because the orbifold order is 7 and thus acting seven times on the R-vacua the identity action should be obtained.

The untwisted strings are obtained from boundary conditions with the identity element. After projection, the four dimensional $\mathcal{N} = 1$ massless spectrum contains a SUGRA sector, a super Yang–Mills sector, and chiral superfields (untwisted moduli), charged fields and the axion–dilaton $a^{\text{orb}} - i\phi$. The gauge algebra is formed by the Cartan of $E_8 \times E_8$ together with the roots P satisfying

$$P \cdot V = P \cdot W = 0 \text{ mod } 7. \tag{4.6}$$

The charges of the chiral fields are given by the winding numbers around T^{16}.

The strings localized at the fixed points are characterized by the conjugacy classes given in Table 4.1, which can be written as

$$g = \left(\theta^k, (\sigma - 1)e_1\right). \tag{4.7}$$

We employ only the sectors θ, θ^2 and θ^4 because the sectors θ^6, θ^5 and θ^3 will contain the CPT partners. The left moving momentum of those twisted states is given by

$$V_g = kV + (\sigma - 1)W. \tag{4.8}$$

Table 4.1: Conjugacy classes for the $T^6/\mathbb{Z}_7$ orbifold vs. sectors.

F.P.	class θ	class θ^2	class θ^4
f_1	0	0	0
f_2	$e_1+e_2+e_3+e_4+e_5+e_6$	$e_2+e_3+e_4+e_5+e_6$	$e_4+e_5+e_6$
f_3	$e_1+e_2+e_3+e_5+e_6$	$e_2+e_3+e_6$	$e_1+e_2+e_3+e_4+e_5+e_6$
f_4	$e_1+e_2+e_4+e_6$	e_2	e_2+e_4
f_5	$e_1+e_3+e_5$	$e_1+e_2+e_3+e_4+e_5+e_6$	$e_1+e_3+e_4+e_5+e_6$
f_6	e_1+e_4	$e_1+e_2+e_4+e_5$	e_4
f_7	e_1	e_1+e_2	$e_1+e_2+e_3+e_4$

The orbifold model employed for our study has in the visible sector the SM gauge group, and three chiral families [123]. The orbifold twist and Wilson line are:

$$\begin{aligned} V &= \frac{1}{7}(0,0,-1,-1,-1,5,-2,6)\,(-1,-1,0,0,0,0,0,0)\ , \\ W &= \frac{1}{7}(-1,-1,-1,-1,-1,-10,2,-9)\,(4,3,-3,0,0,0,0,0)\ . \end{aligned} \tag{4.9}$$

The full gauge group is given by

$$SU(3)\times SU(2)\times U(1)^5\times[SO(10)\times U(1)^3]. \tag{4.10}$$

Next we give a summary of the massless charged states in terms of the non–abelian irreducible representations (irreps):

irrep	$(\mathbf{3},\mathbf{2},\mathbf{1})$	$(\mathbf{3},\mathbf{1},\mathbf{1})$	$(\overline{\mathbf{3}},\mathbf{1},\mathbf{1})$	$(\mathbf{1},\mathbf{2},\mathbf{1})$	$(\mathbf{1},\mathbf{1},\mathbf{10})$	$(\mathbf{1},\mathbf{1},\mathbf{1})$
multiplicity	3	12	18	21	1	133

4.2 Resolution of $T^6/\mathbb{Z}_7$

In this section we recall schematically how the singularities are resolved, for more details we refer to section 3.6. In order to resolve the singularities one has to add coordinates x_r together with appropriate $\mathbb{C}^*$ scalings λ_s, such that the dimensionality of the space is preserved. In this case the original z_i and additional coordinates x_r fulfill

$$(z_i,x_r)\sim(\lambda^{q_i}z_i,\lambda^{q_r}x_r)\,,\qquad \lambda^{q_i}=\prod_s\lambda_s^{q_i^s}\,, \tag{4.11}$$

such that the discrete orbifold action (4.3) is induced where $x_r\neq 0$. The charge assignment under the rotations is given by

	z_1	z_2	z_3	x_1	x_2	x_4
q^1	1	2	4	-7	0	0
q^2	2	4	1	0	-7	0
q^3	4	1	2	0	0	-7

The ordinary divisors are the hypersurfaces $D_i = \{z_i = 0\}$. The singular locus

$$D_1 \cap D_2 \cap D_3 = \{z_1 = z_2 = z_3 = 0\}, \tag{4.12}$$

is replaced by the exceptional divisors $E_r = \{x_r = 0\}$, making the space smooth. The geometrical orbifold is restored in the moduli space region when $\mathrm{Vol}(E_r) = 0$. As the $\mathbb{Z}_7$ is prime, after the resolution there is a unique topology, which is observed in the toric diagram as a unique triangulation. Thus, there will be no flop transitions. Using result derived from Poincaré duality (3.86) there is a correspondence between a $(1,1)$-form and every divisor E_r.

All the intersection numbers can be computed in terms of the following ones

$$\begin{aligned} E_1^3 = E_2^3 = E_4^3 = 8\,, \qquad & E_1E_2^2 = E_2E_4^2 = E_4E_1^2 = 0\,, \\ E_1^2E_2 = E_2^2E_4 = E_4^2E_1 = -2\,, \qquad & E_1E_2E_4 = 1\,. \end{aligned} \tag{4.13}$$

That set of intersection numbers allows us to compute integrals of wedge products of forms in the resolved space. We are interested in a global resolution of $T^6/\mathbb{Z}_7$ [68]. The global description of the resolution is complicated, but as the resolution of singularities happens locally, we can figure out the topological properties by hand. Starting with the orbifold and cutting out small open sets around the seven fixed points then we replace them by the resolved local singularities. This gives rise to a set of exceptional divisors, $E_{k,\sigma}$, $\sigma = 1, \ldots, 7$ which do not intersect when they are located at different fixed points

$$E_{k,\sigma}E_{l,\rho} = 0 \text{ if } \sigma \neq \rho. \tag{4.14}$$

There are in addition three inherited divisors R_i which are the duals of the orbifold invariant forms $dz_i \wedge d\bar{z}_i$ surviving the orbifold projections. The characteristic classes of the resolution and the gauge flux in the internal dimensions will be independent of the R_i. But a base for the $(1,1)$ forms in the compact space will be given by E_r and R_i, so those last ones will appear in the dimensional reduction of the antisymmetric field $B_{\mu\nu}$, and in the cancellation described in (3.7). There are equivalence relations, which will allow to express the local ordinary divisors $D_{k,\sigma}$ in terms of exceptional and inherited ones

$$\begin{aligned} D_{1,\sigma} &\sim 1/7(R_1 - E_{1,\sigma} - 2E_{2,\sigma} - 4E_{4,\sigma}), \\ D_{2,\sigma} &\sim 1/7(R_2 - 4E_{1,\sigma} - 2E_{3,\sigma}), \\ D_{3,\sigma} &\sim 1/7(R_3 - 2E_{1,\sigma} - 4E_{2,\sigma} - E_{4,\sigma}). \end{aligned} \tag{4.15}$$

The n-Chern class is obtained by taking the order of λ^n in the following expansion [83]

$$ch(X) = \prod_{k,\sigma}(1 + \lambda E_{k,\sigma}) \prod (1 + \lambda D_{i,\sigma}) \prod_i (1 - \lambda R_i). \tag{4.16}$$

This gives

$$c_1(X) = 0, \quad c_2(X) = -4, \quad c_3(X) = 48. \tag{4.17}$$

The $c_3(X)$ expression is clearly obtained from the fact that $R_1R_2R_3 = 49$ and the intersection for three different exceptional divisors (4.13).

Supergravity on the resolution The topological information that we have allows to perform the dimensional reduction of the ten dimensional heterotic supergravity coupled to super Yang-Mills. Because we don't have information about the metric, is not possible to write the world–sheet CFT so solve the string equations of motion in this background. In order to satisfy the Bianchi identities $\int_S dH_3 = 0$, where S denotes a compact divisor, is necessary to impose a flux on the internal space. This flux also reduces the gauge symmetry and leads to the appearance of chiral matter. We consider abelian fluxes supported in the exceptional divisors as

$$\mathcal{F} = H_I V_r^I E_r \,, \qquad r = (k, \sigma) \,, \quad k = 1, 2, 4 \,, \quad \sigma = 1, \dots, 7 \,. \tag{4.18}$$

The H_I are the Cartan generators of $E_8 \times E_8$. The bundle vectors V_r^I posses certain constraints. They should satisfy the flux quantization conditions such that

$$7V_r \equiv 0, \qquad V_{2k,\sigma} \equiv 2V_{k,\sigma}, \tag{4.19}$$

where the equivalence is up to lattice vectors. They also have to satisfy the BI, which constraint their length and scalar products

$$0 = \int\limits_S \left(\mathrm{tr}\mathcal{R}^2 - \mathrm{tr}\mathcal{F}^2\right) \,, \qquad S \in \{E_r, R_i\} \,. \tag{4.20}$$

With that information in hand we can compute the massless four dimensional spectrum. The gauge group will be spanned by the roots which are orthogonal to the flux, i.e. to the bundle vectors $\alpha_i \cdot V_r = 0$. The chiral field content can be obtained using the index theorem [69],

$$\hat{N} = \frac{1}{6} \int\limits_X \left(\mathcal{F}^3 - \frac{1}{4}\mathrm{tr}\mathcal{R}^2\mathcal{F}\right) \,. \tag{4.21}$$

The $E_8 \times E_8$ root lattice characterize the ten-dimensional states

$$(\mathbf{8}_v, \mathbf{248}, \mathbf{1}) + (\mathbf{8}, \mathbf{248}, \mathbf{1}), \tag{4.22}$$

and the analogous ones charged under the second E_8. Upon dimensional reduction the index theorem (4.21) says with which multiplicity the states (4.22) appear in the spectrum, and they will be forming representations of the 4d gauge group. The representation of a given state can be computed doing the product $\alpha_i \cdot P$ to obtain its Dynkin labels.

Massless states are also found from the reduction of the ten-dimensional states $(\mathbf{35}, \mathbf{1}, \mathbf{1})$ and $(\mathbf{28}, \mathbf{1}, \mathbf{1})$ which are the metric and the antisymmetric tensor respectively. The metric is given in terms of the Kähler form J, and the antisymmetric tensor is the Kalb–Ramond field B_2. Their expansions in the base for the internal $(1,1)$ forms is

$$J = a_i R_i - b_r E_r \,, \qquad B_2 = b_2 + \alpha_i R_i - \beta_r E_r \,. \tag{4.23}$$

In four dimensions J and B join to form the complex scalar components of the chiral multiplet

$$T_i|_{\theta=0} = a_i + i\alpha_i, \qquad T_r|_{\theta=0} = b_r + i\beta_r. \tag{4.24}$$

The real components a_i, b_r govern the size of the R_i and E_r cycles, respectively.

The four dimensional field b_2 is the dual of the blow–up universal axion a^{uni}. Let us recall the field strength H_3 in (2.37) with $c = c' = 1$

$$H_3 = dB_2 - \Omega_3^{\text{YM}} + \Omega_3^{\text{L}}. \tag{4.25}$$

The gauge invariance of H_3 under abelian gauge transformations implies equation (3.157), which restricted to gauge transformations reads

$$\delta\beta_r = V_r^I \chi^I, \qquad \delta\alpha_i = 0. \tag{4.26}$$

This is a particular case of the Lorentz and gauge transformations written in (2.41).

On the orbifold there are seven preserved $U(1)$s and one anomalous one. When performing the blow–up by giving vevs to twisted fields charged under the gauge symmetry, by the effect of the Higgs mechanism those gauge symmetries will be broken. In particular the abelian symmetries can be broken like that. When compactifying the supergravity plus SYM on the blow–up the non–abelian symmetries non-orthogonal to the flux gain masses at the classical level. However the abelian gauge symmetries non–orthogonal to the flux, will be broken only at the one–loop level due to the appearance of anomalies. There will be then many model dependent axions β_r which cancel the anomaly as described in (3.7), and this mechanism will create Stückelberg–like mass term for the $U(1)$ gauge bosons.

The bundle vectors which specify our resolution model are given as blow–up mode charges at the end of appendix A. The non–abelian gauge algebra is $SU(3) \times SU(2) \times SO(10)$, and a short summary of the charged spectrum is

irrep	$(\mathbf{3},\mathbf{2},\mathbf{1})$	$(\mathbf{3},\mathbf{1},\mathbf{1})$	$(\overline{\mathbf{3}},\mathbf{1},\mathbf{1})$	$(\mathbf{1},\mathbf{2},\mathbf{1})$	$(\mathbf{1},\mathbf{1},\mathbf{10})$	$(\mathbf{1},\mathbf{1},\mathbf{1})$
multiplicity	3	10	16	17	1	86

4.3 Spectrum comparison

We describe in this section the match between the massless spectrum in the resolved space, and in the deformed orbifold. We perform field redefinitions in the orbifold twisted fields, employing local blow–up modes. The resulting spectrum reproduces the chiral asymmetry of the supergravity on the resolution. It is possible to evaluate the index theorem locally, at each compact divisor. This makes easier to uncover vector-like pairs which are located at different fixed points.

4.3.1 Field redefinitions

In the comparison of the massless spectrum in blow–up with the one at the deformed orbifold, the first observation is that the orbifold and blow–up states transform under the gauge d.o.f with different charges. Orbifold states Φ_γ^{Orb} transform with the twisted momenta, while blow–up states Φ_γ^{BU} transform with momenta in the $E_8 \times E_8$ lattice. The index $\gamma = (k, \sigma, i)$ labels the state i localized at the orbifold fixed point (k, σ). Orbifold twisted fields which attain vevs we name as *blow–up modes*. The present study, aims to corroborate this interpretation. The axions localized in the exceptional divisors E_r, can be identified with complexified Kähler moduli as

$$\Phi_r^{\text{BU-Mode}} = e^{b_r + i\beta_r} , \tag{4.27}$$

where b_r are the Kähler moduli parameterizing the size of the blown–up cycle and β_r are the model dependent axions in (4.23). We perform field redefinitions, including integer powers of the blow–up modes as

$$\begin{aligned} \Phi_\gamma^{\text{BU}} &= \prod_k (\Phi_{k,\sigma}^{\text{BU-Mode}})^{-r_{k,\sigma}^\gamma} \Phi_\gamma^{\text{Orb}} & (4.28) \\ &= e^{-\sum_k r_{k,\sigma}^\gamma (b_{k,\sigma} + i\beta_{k,\sigma})} \Phi_\gamma^{\text{Orb}} . & (4.29) \end{aligned}$$

The coefficients $r_{k,\sigma}^\gamma$ will be specified below in (4.32). The considered redefinitions allow blow–up modes from all sectors $k = 1, 2, 4$, but localized at the same fixed point σ as Φ_γ^{Orb}. We denote the twisted momenta (charges) of the blow–up modes by $q_I^{k,\sigma}$, $I = 1, \ldots, 16$. Those coincide with the components of the abelian vector bundle $V_{k,\sigma}^I$. The charges of the other orbifold fields are denoted by Q_I^γ and the redefined charges in blow–up by $Q_I'^\gamma$. We define Δ_I^γ as

$$\Delta_I^\gamma = Q_I^\gamma - Q_I'^\gamma , \qquad \Delta_I^\gamma = \sum_{k=1,2,4} r_{k,\sigma}^\gamma q_I^{k,\sigma} . \tag{4.30}$$

The redefinition (4.29) gives the correct gauge transformation for the blow–up states

$$\beta_{k,\sigma} \to \beta_{k,\sigma} + V_{k,\sigma}^I \chi_I \qquad \Rightarrow \qquad \Phi_\gamma^{\text{BU}} \to e^{i\chi_I (Q_I^\gamma - \Delta_I^\gamma)} \Phi_\gamma^{\text{BU}} , \tag{4.31}$$

where χ_I is the gauge parameter. In addition, due to the exponential map the blow–down limit is recovered when the sizes of the exceptional cycles $b_{k,\sigma}$ go to $-\infty$ rather than to 0. The more intuitive behavior of $b_{k,\sigma} \to 0$ in blow–down, can be obtained using a different measure for the volume of curves [124].

All the redefinitions Δ_I^γ giving a blow–up state, should be such that the charge $Q_I'^\gamma$ is a root vector of $E_8 \times E_8$ lattice. Thus, we explored the following redefinitions

$$Q^{\text{Orb}}_{k,\sigma} \mapsto Q^{\text{BU}}_{k,\sigma} = Q^{\text{Orb}}_{k,\sigma} - V_{k,\sigma}\,, \tag{4.32a}$$

$$Q^{\text{Orb}}_{k,\sigma} \mapsto Q^{\text{BU}}_{k,\sigma} = Q^{\text{Orb}}_{k,\sigma} + V_{l,\sigma} + V_{m,\sigma}\,, \quad k \neq l \neq m \neq k\,, \tag{4.32b}$$

$$\begin{aligned} Q^{\text{Orb}}_{1,\sigma} &\mapsto Q^{\text{BU}}_{1,\sigma} = Q^{\text{Orb}}_{1,\sigma} + V_{1,\sigma} - V_{2,\sigma}\,, \\ Q^{\text{Orb}}_{2,\sigma} &\mapsto Q^{\text{BU}}_{2,\sigma} = Q^{\text{Orb}}_{2,\sigma} + V_{2,\sigma} - V_{4,\sigma}\,, \\ Q^{\text{Orb}}_{4,\sigma} &\mapsto Q^{\text{BU}}_{4,\sigma} = Q^{\text{Orb}}_{4,\sigma} - V_{1,\sigma} + V_{4,\sigma}\,, \end{aligned} \tag{4.32c}$$

$$\begin{aligned} Q^{\text{Orb}}_{1,\sigma} &\mapsto Q^{\text{BU}}_{1,\sigma} = Q^{\text{Orb}}_{1,\sigma} + V_{1,\sigma} + V_{2,\sigma} - V_{4,\sigma}\,, \\ Q^{\text{Orb}}_{2,\sigma} &\mapsto Q^{\text{BU}}_{2,\sigma} = Q^{\text{Orb}}_{2,\sigma} - V_{1,\sigma} + V_{2,\sigma} + V_{4,\sigma}\,, \\ Q^{\text{Orb}}_{4,\sigma} &\mapsto Q^{\text{BU}}_{4,\sigma} = Q^{\text{Orb}}_{4,\sigma} + V_{1,\sigma} - V_{2,\sigma} + V_{4,\sigma}\,. \end{aligned} \tag{4.32d}$$

The previous set allowed us to match the massless spectrum.

4.3.2 Local massless spectrum

Let us come now to the massless spectrum. On the orbifold we can compute it by determining the shifted momenta which satisfy the zero mass equation, the level matching and the projection conditions. Whereas in blow–up we make use of an index theorem (4.21).

On $T^6/\mathbb{Z}_7$ resolution the exceptional divisors supporting the gauge flux $\mathcal{F}$ are compact. This implies that the multiplicity $\hat{N}$ in (4.21) can be written as a sum of *local multiplicities* $\hat{N}(\sigma)$ at the seven fixed points as

$$\hat{N}(\sigma) = \frac{1}{3}\sum_{k=1,2,4}\left[4H^3_{k,\sigma} - H_{k,\sigma}\right] - H_{1,\sigma}H^2_{2,\sigma} - H^2_{1,\sigma}H_{4,\sigma} - H_{2,\sigma}H^2_{4,\sigma} + H_{1,\alpha}H_{2,\sigma}H_{4,\sigma}\,, \tag{4.33}$$

where we used the notation $H_{k,\sigma} = V^I_{k,\sigma}H_I$ with bundle vectors $V^I_{k,\sigma}$ and Cartan generators H_I. The total multiplicity is obtained as $\hat{N} = \sum_\sigma \hat{N}(\sigma)$. The expression (4.33) contains a sum over the twisted sectors k. Thus, this local index doesn't give information about the twisted sector of the orbifold pair corresponding to the blow–up state.

The particle spectrum is calculated by decomposing the 480 $E_8 \times E_8$ roots in irrep. of the unbroken gauge group. This means to determine how the 10d states $(\mathbf{8}_v, \mathbf{248}, 1)$ and $(\mathbf{8}, \mathbf{248}, 1)$ charged under the first E_8 (and also the ones charged under second E_8) decompose in 4d. To determine the global or local multiplicity of a state in blow–up one acts with N or $N(\sigma)$, respectively, on the corresponding $E_8 \times E_8$ root. This gives the multiplicity of each massless SUSY matter multiplet in blow–up. As (4.33) is an odd polynomial in H_I, $\hat{N}$ and $\hat{N}(\sigma)$ change sign for CPT conjugate states. In this section we evaluate the multiplicities by acting on the highest weight of the fundamental representations and on the lowest weight of the CPT conjugate of the anti–fundamental representations. Hence, states transforming in fundamental representations are assigned positive multiplicity and states transforming in anti–fundamental representations are assigned negative multiplicity.

4.3.3 Spectrum comparison

We perform redefinitions of the type (4.32) and compare the resulting spectrum. The details of the matching procedure are worked out in this subsection. A table of all $E_8 \times E_8$ root vectors, their redefinition, and the corresponding orbifold states is given in appendix A.

There are some facts that one should consider. First, the spectrum in blow–up is only known through its chiral asymmetry. The multiplicities only give the difference between the number of left-chiral states with certain charge, and the number of right-chiral states with the opposite charge. In contrast, for the orbifold we have access to the full spectrum, knowing all the left– and right– chiral states. Second, the vevs assignation that generates the blow–up also produces a *Higgs mechanism*. The gauge groups under which the blow–up modes are charged get broken, and fields which posses Yukawa couplings with blow–up modes get massive.

By choosing only non–abelian gauge singlets as blow–up modes we can preserve the rank of the gauge group on the orbifold variety. The constraints are the BI and the existence of non–abelian singlets in every fixed point. If one is forced to select blow–up modes which are charged under the non–abelian gauge sector then it would be necessary to reconstruct the breaking of the non–abelian gauge groups. In this case is also possible to study the matching, but the exposition is simpler with the use of singlets.

Vector–like states on the deformed orbifold are not captured by the index theorem (4.21). It occurs often that several different orbifold states are redefined via (4.32) to the same root vector. Also other roots have net multiplicity zero and don't appear in the redefinition process. If there are states which are redefined to the same root, while others are redefined to the negative root (i.e. the charge conjugate one), the multiplicity operator will only see the number of the first states minus the number of the second states. So we do not see vector–like pairs. This leads to the effect that there are apparently less states in blow–up than in the orbifold.

The advantage of the local multiplicity, is that even these vector–like pairs can be identified as long as they do not live at the same fixed point. Additionally, by checking the dependence of a vector–like pair on the Kähler parameters b_r, it can be seen which states are expected to get a mass in blow–up. We study also the Yukawa couplings on the orbifold, verifying that all involved states couple to one or more blow–up modes. The orbifold masses from Yukawa couplings to blow–up modes agree with the massless blow–up spectrum. The anomaly computation on both sides of the theory provides a very strong cross–check that the identified mass terms are correct.

Matching of massless states As an illustration of the matching method let us look at examples from the table of appendix A. Lets start with the 3 quark doublets $(\mathbf{3}, \mathbf{2}, \mathbf{1})$. The first field Q_1 lives in the untwisted sector. Hence it does not need to be redefined. The local multiplicity operator is 1/7 at each of the fixed points, i.e. the field is smeared out over all fixed points, as expected for an untwisted field. The fields Q_2 and Q_3 both live at the first fixed point. They are redefined to two root vectors via (4.32a) at the first fixed point (and at $k = 2$ and $k = 1$ respectively). The local multiplicity operator for those root vectors is one at the first fixed point. Hence the local multiplicity exactly sees the orbifold state. At the other fixed points, we see fractional multiplicities of $\pm 1/7$, which sum to zero giving one as net multiplicity. These non–existing states can be interpreted as untwisted states projected out on the orbifold. As long as they sum to zero, we ignore them in the following. If they do not sum to zero but to one, they indicate an untwisted field, as seen for Q_1.

There are $SU(3)$ charged states transforming in the fundamental $\mathbf{3}$ as well as in the anti–fundamental $\overline{\mathbf{3}}$. Our convention is only to look at the triplet weights since the anti–triplets weights correspond to their negatives[1]. Thus, a positive multiplicity indicates a triplet state whereas a negative multiplicity stands for the presence of an anti–triplet state. An example for this are the states $\bar{t}_7$ and t_6 which transform in the $(\overline{\mathbf{3}}, \mathbf{1}, \mathbf{1})$ and in the $(\mathbf{3}, \mathbf{1}, \mathbf{1})$. Their overall multiplicity is -1 and 1, and the local multiplicity operator reveals that these states live at fixed points 7 and 6, respectively. Something conceptually new happens for the orbifold states t_5, t_{12}, $\bar{t}_{11}$, and $\bar{t}_{18}$. Although those states are redefined to the same root the total multiplicity is zero. This happens because the multiplicity operator can only count the net number of states which is $2 - 2 = 0$. However, the local multiplicity operator gives some insight here. The three states t_5, t_{12}, $\bar{t}_{11}$ all live at fixed point 5 on the orbifold. As there are two left–chiral and one right–chiral state the local multiplicity is 1. For the one right–chiral state $\bar{t}_{18}$, there is a local multiplicity of -1 at fixed point 6. Hence the overall multiplicity is zero. The multiplicities of the other states can be worked out in a similar manner.

Matching of massive states Vector–like states can acquire a mass in the blow–up procedure from trilinear Yukawa couplings. The selection rules for allowed Yukawa couplings on the orbifold arise from requiring gauge invariance, compatibility with the space–group, and conservation of H–momentum. Conservation of R–charge will be discussed below. Gauge invariance amounts to the requirement that the sum of the left–moving shifted momenta of the strings involved in the coupling is zero.

The space–group selection rule implies that the product of the constructing space–group elements of the states involved in the coupling must be the identity $(\mathbb{1}, 0)$. Non–zero

[1] In the next chapter we take a different convention.

trilinear couplings should satisfy [125]

$$(k=1,\sigma_1)\circ(k=2,\sigma_2)\circ(k=4,\sigma_4)\,,\qquad \text{with } \sigma_1+2\sigma_2+4\sigma_4=0 \bmod 7\,. \tag{4.34}$$

This can be checked by using the conjugacy classes in Table (4.1). For example in the case $\sigma_1=\sigma_2=\sigma_4=1$ one obtains

$$(\theta,e_1+e_2+e_3+e_4+e_5+e_6)\cdot(\theta^2,e_2+e_3+e_4+e_5+e_6)(\theta^4,e_4+e_5+e_6)=(\mathbb{1},0).$$

If the coupling involves states which are all located at the same fixed point ($\sigma_1=\sigma_2=\sigma_4$), the space–group selection rule is trivially fulfilled. However, there are also solutions to (4.34) for states coming from three different fixed points. Since these couplings arise from world–sheet instantons [63, 105], they are suppressed by a factor of the form e^{-a_i} where a_i are the moduli which govern the sizes of the orbifold or Calabi–Yau (cf. (4.23)). Here, if the space–group selection rule is fulfilled for trilinear couplings then H–momentum is automatically conserved. As discussed in section 3.4 there can be other selection rules coming from the internal part of the Lorentz group. For a local orbifold $\mathbb{C}^3/\mathbb{Z}_N$ the rotation of the three individual complex planes is a continuous symmetry. Since the invariant spinor is charged under it, it will be an R–symmetry. The charges are given by

$$R^i_\gamma=q^i_{\text{sh},\gamma}+N^i_\gamma-\overline{N}^i_\gamma\,, \tag{4.35}$$

where q_{sh} are the shifted right–moving internal momenta of the orbifold state Φ^{Orb}_γ and N ($\overline{N}$) are the (anti–) holomorphic oscillator numbers. The conservation rule reads

$$\sum_\zeta R^i_\zeta=1\,, \tag{4.36}$$

where ζ runs over the three states involved in the Yukawa coupling. Equation (4.36) is trivially fulfilled for states without oscillators if the space–group rules are. However, in a compact orbifold this symmetry will be broken down to a subgroup by the torus lattice. Therefore the formerly forbidden couplings are expected to be supressed by the size of the lattice. If the lattice is factorizable, the remaining symmetry is the discrete rotation of the three two–tori. In this case the selection rule needs only to be satisfied up to multiples of the order of the orbifold group. For the non–factorizable $SU(7)$ lattice of the $\mathbb{Z}_7$ orbifold, we checked that the symmetry is broken completely except for the $\mathbb{Z}_7$ itself, so (4.36) should not be imposed on the compact orbifold.

The supergravity theory on the blow–up side is, however, only valid in the compactification space *large volume limit*, such that Kaluza–Klein excitations are absent in the spectrum. In particular, we expect that the R–charge selection rule (4.36), which is broken by the orbifold lattice, is still a valid symmetry in the large volume limit. Therefore we expect the states, which are supposed to get a mass via such suppressed couplings on the orbifold, to appear as massless states in the multiplicity operator in blow–up. By comparing the spectra we indeed find that the index theorem sees massless states for which the orbifold theory

predicts non–local mass terms, or mass terms which do not satisfy (4.36). To illustrate the absence of both types of mass terms in blow–up we look at suitable examples.
As an example for mass terms not satisfying (4.36) consider the singlet states s_{25}, s_{26}, s_{70}, s_{111}, s_{112} and s_{113}, see appendix A. These states are all oscillator states, what explains their degeneracy and makes them sensible to a possible R–symmetry. Together with the blow–up modes s_{68} and s_{27}, there are the following orbifold trilinear superpotential couplings when imposing only gauge– and space–group invariance and the H–momentum rule:

$$(s_{111}\ s_{112}\ s_{113}) \begin{pmatrix} a_{11}s_{68} & a_{12}s_{68} & a_{13}s_{27} \\ a_{21}s_{68} & a_{22}s_{68} & a_{23}s_{27} \\ a_{31}s_{68} & a_{32}s_{68} & a_{33}s_{27} \end{pmatrix} \begin{pmatrix} s_{25} \\ s_{26} \\ s_{70} \end{pmatrix}, \tag{4.37}$$

where the a_{ij} are coefficients of order one. Now when one gives a vev to the blow–up modes s_{68} and s_{27}, these couplings become a rank three mass matrix and thus one would expect all 6 singlets to become massive and disappear from the chiral spectrum in blow–up. However, when we look at the roots to which these singlets can be redefined, the local multiplicity operator reveals that there are four states at the resolved fixed point where the singlets in question were localized. Therefore four of these singlets must stay massless during blow–up. This means that the above mass matrix must only have rank one, such that just one pair of singlets is decoupled. One could explain this by assuming that all coefficients a_{ij} are equal, but this assumption is a priori not justified and would lead to mixing of the fields during redefinition. Our interpretation is that the local multiplicity operator sees states only in the large volume limit where the R–symmetry (4.36) is exact. Imposing R–symmetry here would set all coefficients to zero except for a_{21} and a_{23} and therefore naturally explain the rank one mass matrix.
To illustrate the non–local mass terms, we investigate the triplet states t_5, t_{12}, $\bar{t}_{11}$, and $\bar{t}_{18}$ encountered above. From the employed redefinitions we find

$$t_5^{\mathrm{BU}}\bar{t}_{11}^{\mathrm{BU}} = t_5^{\mathrm{Orb}}\bar{t}_{11}^{\mathrm{Orb}} e^{-b_{4,5}+b_{1,5}+b_{4,5}} = t_5^{\mathrm{Orb}}\bar{t}_{11}^{\mathrm{Orb}} e^{b_{1,5}}\,, \tag{4.38a}$$

$$t_{12}^{\mathrm{BU}}\bar{t}_{11}^{\mathrm{BU}} = t_{12}^{\mathrm{Orb}}\bar{t}_{11}^{\mathrm{Orb}} e^{-b_{1,5}+b_{1,5}+b_{4,5}} = t_{12}^{\mathrm{Orb}}\bar{t}_{11}^{\mathrm{Orb}} e^{b_{4,5}}\,. \tag{4.38b}$$

The coupling of t_5 and t_{12} with $\bar{t}_{18}$ is non–local as the states reside at different fixed points. Hence this coupling is not captured by the multiplicity operator. The redefinitions show that in blow–up where $b_{k,\sigma} \to \infty$, the couplings (4.38) provide a mass term which vanishes in the blow–down limit $b_{k,\sigma} \to -\infty$. This means that from the blow–up perspective a linear combination of t_5 and t_{12} pairs up with $\bar{t}_{11}$ and lifts the exotic state from the massless particle spectrum in blow–up. This behavior is also confirmed from the orbifold perspective. The appearance of $b_{1,5}$ (4.38a) shows that t_5 from the θ^4 sector and $\bar{t}_{11}$ from the θ^2 sector couple to the blow–up mode from the θ sector as dictated by the space–group selection rule. Likewise, for the second mass term (4.38b) we find a coupling between t_{12} from the θ sector, $\bar{t}_{11}$ from the θ^2 sector, and the blow–up mode from the θ^4 sector as indicated by $b_{4,5}$.
The local R–charge selection rule (4.36) is only relevant for oscillator states, as states satisfying the space–group selection rule have $\sum_\zeta q^i_{\mathrm{sh},\zeta} = 1$ and hence (4.36) is fulfilled for states without oscillators. Interestingly, the states which have oscillators often allow for

more than one possible redefinition (4.32). Imposing (4.36) in conjunction with consistency of the local blow–up spectra singles out a unique field redefinition. Using these redefinitions, we were able to establish a perfect match between the anomalies on the orbifold and in blow–up, which we take as a strong cross–check that the above discussion is valid. This will be explained in the next section.
The above analysis has been carried out in a similar fashion for all other $\mathcal{O}(200)$ states. Each time we find mass terms of the form (4.38) from the redefinitions on the blow–up side, they also constitute allowed couplings on the orbifold side and lead in the end to a perfect match of the anomaly computation. We expect also that there exists an orbifold mechanism explaining why a local R–charge can be applied in this case. This was a motivation for the study of orbifold selection rules presented in section 3.4, however this is still work in progress which we plan to address elsewhere.

4.4 Anomalies study

In this section we perform the study of the anomaly cancellation mechanism in the orbifold and in the resolution. We start with the dimensional reduction of the 10d anomaly polynomial using the technique presented in section 3.7. We then describe how to compute the anomaly in the orbifold deformed by vevs, which matches the dimensional reduced polynomial on the blow–up. To conclude, we obtain the relations between universal and non-universal axions in blow–up, with the universal orbifold axion, and the local axions which are identified with blow–up modes.

4.4.1 Anomalies in the resolved space

Now let us proceed to the calculation of the dimensional reduction of the 10d anomaly for our explicit blow–up model. First we give a general description of every term in the 4d anomaly polynomial. Then we investigate the pure $U(1)$, $U(1)\times \text{grav}^2$ and $U(1)\times G^2$ polynomials. As the pure gravitational anomalies are canceled by the presence of 496 gauginos in ten dimensions we do not include them in further discussions. After this we calculate the anomalies in blow–up in two different ways: from the coefficients appearing in the anomaly polynomial (4.39) and field–theoretically from the triangle anomaly graph given in figure 4.1. The fact that both results coincide provides a non–trivial cross–check for the spectrum computation and the field redefinitions explained in section 4.3. Expanding (3.148) in 6d and 4d fields, one obtains [83, 122]

$$I_6 = \int\limits_{\mathcal{M}} \Big\{ \frac{1}{6}\left(\text{tr}[\mathcal{F}'F']\right)^2 + \frac{1}{4}\left(\text{tr}\mathcal{F}'^2 - \frac{1}{2}\text{tr}\mathcal{R}^2\right)\text{tr}F'^2 - \frac{1}{8}\left(\text{tr}\mathcal{F}'^2 - \frac{5}{12}\text{tr}\mathcal{R}^2\right)\text{tr}R^2 \Big\}\text{tr}[\mathcal{F}'F'] + ('\to'')\,. \tag{4.39}$$

In both E_8s, the whole anomaly is multiplied by a factor $\text{tr}(\mathcal{F}F)$. This factor projects onto the $U(1)$ part of F, as our gauge background is by construction abelian. In addition, it is

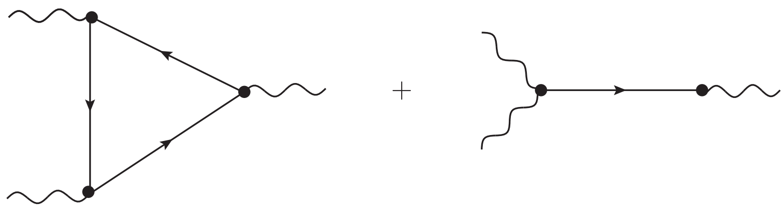

Figure 4.1: Triangle graph inducing the gauge 4d anomalies and the axionic Green–Schwarz counterterm.

generically only different from zero for anomalous $U(1)$s, as $T_{U(1)} \perp V_r$ for non–anomalous $U(1)$s. This means, that unless a miraculous cancellation occurs, the number of anomalous $U(1)$ is given by the rank of the 16×21 matrix V_r^I. In our example all $U(1)$s are anomalous in blow–up, so we get contributions for all abelian gauge group factors.

So let us discuss how the different $U(1)$ anomalies are encoded in (4.39) in detail:

- Term 1: As $\mathrm{tr}(\mathcal{F}F)$ projects onto the $U(1)$-part, only pure $U(1)$ anomalies can arise from this term. The whole term contains $[\mathrm{tr}(\mathcal{F}F)]^3 = E_r E_{r'} E_{r''} V_r^I V_{r'}^J V_{r''}^K F_I F_J F_K$. Depending on the values of I, J, and K, we get $U(1)^3$ anomalies if $I = J = K$, $U(1)^2 U(1)'$ anomalies if $I = J \neq K$, or $U(1)U(1)'U(1)''$ anomalies if $I \neq J \neq K \neq I$.
- Term 2: Here, we have a term $\mathrm{tr}(F)^2 \mathrm{tr}(\mathcal{F}F)$. The term $\mathrm{tr}(F)^2$ contains an inner product of the 4d field strength with itself, so from here we can get both abelian and non–abelian factors depending on the choice of the group element.
- Term 3: This term couples the 4d field strength to the 4d curvature. Hence, this term gives rise to the $U(1) \times \mathrm{grav}^2$ anomalies.

As mentioned above, the anomalies can also be evaluated in the 4d effective field–theory through the triangle Feynman graphs and counterterms arising from couplings between axions and fermions (cf. figure 4.1). The different anomalous contribution are given schematically by

$$
\begin{aligned}
U(1) \times U(1)' \times U(1)'' &: \quad \mathbf{sym} \sum_\lambda N(\lambda)(T_{U(1)} \cdot \lambda)(T_{U(1)'} \cdot \lambda)(T_{U(1)''} \cdot \lambda), \\
U(1) \times G^2 &: \quad k(\mathbf{r}(G)) \sum N(\mathbf{r}(G))(T_{U(1)} \cdot (\mathbf{r}(G)))\,, \\
U(1) \times \mathrm{grav}^2 &: \quad \sum_\lambda N(\lambda)(T_{U(1)} \cdot \lambda)\,.
\end{aligned}
\tag{4.40}
$$

Here $N(\cdot)$ denotes the multiplicity of the state in brackets and negative values indicate the conjugate representation as given by (4.21). $T_{U(1)} \cdot \lambda$ represents the charge of a given $E_8 \times E_8$ lattice vector λ, $k(\mathbf{r})$ is the Dynkin index of the irrep and **sym** accounts for the symmetry factor corresponding to the various $U(1)$ anomalies. For the first and last terms, the sum runs over all roots, whereas for the mixed $U(1) \times G^2$ anomalies, the sum runs over the roots transforming in the respective representation only. Taking into account the numerical factors, the values of these quantities should match the coefficients of the corresponding term in the anomaly polynomial. As will be discussed below, we have computed both the dimensional reduction of the anomaly (3.148) and the triangle anomalous graphs in the effective field theory, finding that the coefficients in (4.39) coincide with the result of (4.40). This agreement provides an important cross–check.

In order to obtain the three different kinds of anomalies explicitly, we choose an $E_8 \times E_8$ Cartan basis given in (B.1) in appendix B, in which the eight elements $T_j,\ j = 1, ..., 8$ are the $U(1)^8$ generators and the rest spans the Cartan subalgebra of the non–abelian part of the gauge group. The $U(1)$ generators have components in both E_8's.

$U(1) \times G^2$ anomalies Let us start the calculation of the anomalies with the explicit calculation of the $U(1) \times G^2$ contribution of (4.39) in the above basis. They are given by

$$I_G = (25F_1 - 20F_2 - 25F_3 - 4F_4 - 66F_5 + 18F_6 + 25F_7)\,\mathrm{tr}F'^2 - F_8\mathrm{tr}F''^2\,. \tag{4.41}$$

This is now compared with the anomalies $U(1)\times SU(2)^2$, $U(1)\times SU(3)^2$ and $U(1)\times SO(10)^2$ calculated from the triangle graph using the spectrum given in appendix A. The field strengths for $SU(2)$ and $SU(3)$ in the visible sector are in $\mathrm{tr}F'^2$ and the field strength of the hidden sector $SO(10)$ is in $\mathrm{tr}F''^2$. The dimensional reduced anomaly polynomial coefficients and the ones computed via the traces from the anomalous triangle diagram match exactly.

$U(1) \times \mathrm{grav}^2$ anomalies When comparing the coefficients in I_{grav} and the values of $\mathrm{tr}\,Q_i$ from the 4d effective spectrum for the $U(1) \times \mathrm{grav}^2$ anomalies, we obtain again exact agreement, after the normalization factor of $-1/24$ has been taken into account in the effective field theory computation. The polynomial reads

$$I_{\mathrm{grav}} = \frac{1}{12}\,(-166F_1 - 136F_2 + 292F_3 + 40F_4 + 464F_5 - 152F_6 - 187F_7 + 8F_8)\,\mathrm{tr}R^2\,. \tag{4.42}$$

Pure $U(1)$ anomalies Comparing the coefficients in I_{pure} with the values obtained from the 4d effective spectrum we find again a perfect agreement. Note that the symmetry factors **sym** of $1/1!$ for $\mathrm{tr}\,Q_IQ_JQ_K$ with $I \neq J \neq K \neq I$, $1/2!$ for $\mathrm{tr}\,Q_I^2Q_J$ with $I \neq J$, and $1/3!$ for $\mathrm{tr}\,Q_I^3$ have to be used in the 4d anomaly graph computation. The expression for the polynomial is more involved than the one of $U(1) \times G^2$ and $U(1) \times \mathrm{grav}^2$. It is of the schematic form

$$I_{\mathrm{pure}} = \sum a_I F_I^3 + k_{IJ} F_I^2 F_J + c_{IJK} F_I F_J F_K\,. \tag{4.43}$$

Anomaly universality in blow–up As explained above, on the orbifold we have only one axion to cancel the anomalies. Anomaly freedom then requires in particular that all three kinds of anomalies are proportional such that they can all be canceled with the same axion. In blow–up, this is generically not true. However, from (4.39) and the discussion thereafter, it is apparent that there are still partial anomaly universalities: one can find a $U(1)$ basis where one $U(1)$ captures all gravitational anomalies, and two further $U(1)$s capture all non–abelian anomalies of the visible and hidden sector, respectively. The rest of the $N-3$ $U(1)$s have only pure $U(1)$ anomalies.
In order to construct such a basis, the original basis is changed to $\{\bar{F}_J\}$ as given in (B.2) in appendix B. After performing the base change $F_I = K_I^J \bar{F}_J$, the relevant polynomials are given by

$$I_G = \bar{F}_1 \left(\mathrm{tr} F^2_{SU(2)} + \mathrm{tr} F^2_{SU(3)} \right) + \bar{F}_2 \mathrm{tr} F^2_{SO(10)} \,, \tag{4.44}$$

$$I_{\mathrm{grav}} = \bar{F}_3 \mathrm{tr} R^2 \,. \tag{4.45}$$

The expression for I_{pure} in terms of the new eight $U(1)$ directions is rather involved so we refrain from giving it explicitly here. While the non–abelian $U(1) \times SU(N)^2\,,\ N = 2,3$ and $U(1) \times SO(10)^2$ directions are orthogonal, the $U(1) \times \mathrm{grav}^2$ is not orthogonal to any of them.

4.4.2 Relating the anomalies on the orbifold and in blow–up

On the orbifold there is a single anomalous abelian gauge symmetry $U(1)_A$. This anomalous $U(1)_A$ induces an FI term which has to be canceled in a supersymmetric vacuum solution. This is done by assigning vevs to certain charged fields which in general are also charged under other $U(1)$s. Thus, once the vevs are given, we expect the breakdown of further $U(1)$s. This breakdown manifests itself from the blow–up perspective in $U(1)$s which become anomalous. The anomaly is canceled with the Green–Schwarz mechanism, which also gives a mass to the $U(1)$s. Thus we aim at investigating the 4d anomaly from the point of view of the orbifold and the blow–up. Via the descent equations, we get relations between the universal axion on the orbifold canceling the unique $U(1)_A$ anomaly and the axions in blow–up (universal and non–universal) canceling the multiple $U(1)$ anomalies.

4d anomaly from the orbifold point of view On the orbifold, our starting point is the anomaly polynomial I^{orb} which describes the single unique anomalous $U(1)$ on the orbifold. To this anomaly, we add the anomaly change which is due to the departure from the orbifold point when blowing up. These changes are induced by blow–up modes that acquire vevs and thus provide mass terms via Yukawa couplings, and by the field redefinitions. We call this contribution I^{red}. Thus, from the orbifold perspective, the 4d anomaly polynomial I_6, after assigning vevs to twisted fields, decomposes as

$$I_6 = I^{\mathrm{orb}} + I^{\mathrm{red}} \,. \tag{4.46}$$

4d anomaly from the blow–up point of view In blow–up, we start from the factorized anomaly polynomial in 10 dimensions (3.148), integrate out the internal space $\mathcal{M}$, and decompose the polynomial into a universal term I^{uni} plus a non–universal term I^{non}:

$$I_6 = I^{\text{uni}} + I^{\text{non}} = \int_{\mathcal{M}} X_{6,2} X_{0,4} + \int_{\mathcal{M}} X_{2,2} X_{4,4} \,. \tag{4.47}$$

The forms $X_{2k,2l}$ were defined in section 3.7.1. The explicit decomposition of X_4 and X_8 in terms of internal and four dimensional indices is given in appendix C. Note that the term $\int X_{2,6} X_{4,0}$ vanishes due to the Bianchi identities, and is thus not present. For later convenience, we introduce the short–hand expressions

$$X_2^{\text{uni}} := \int_{\mathcal{M}} X_{6,2} \,, \qquad X_4^{\text{uni}} := X_{0,4} \,, \qquad E_r X_2^r := \frac{1}{12} \cdot 2\,\mathrm{tr}(\mathcal{F} F) \,, \qquad X_4^r := \int_{\mathcal{M}} X_{4,4} E_r \,. \tag{4.48}$$

A factor $-1/12$ coming from the dimensional reduction is absorbed in the forms X_2^{uni} and X_2^r. This factor also rescales a^{uni} and β_r, and we use the same symbol to denote its new values. The expression $\int X_{6,2} X_{0,4}$ has terms mixing both E_8s ($'$ and $''$). This could also happen in $\int X_{4,4} X_{2,2}$. However, it turns out that these mixed terms are absent in the whole I_6 in (4.39), which has the first and the second E_8 anomalies fully separated [122].

Descent equations Putting together the pieces described above, we obtain a relation between the anomaly polynomials on the orbifold and in blow–up:

$$\begin{aligned} I^{\text{orb}} + I^{\text{red}} &= I^{\text{uni}} + I^{\text{non}} \,, \\ F^{\text{orb}} X_4^{\text{orb}} + \sum_a q_I^a F^I X_{4,a}^{\text{red}} &= X_2^{\text{uni}} X_4^{\text{uni}} + \sum_r X_2^r X_4^r \,. \end{aligned} \tag{4.49}$$

All the different factors in the polynomials X_2^r, X_4^r, X_2^{uni}, X_4^{uni} are given in appendix C. The counterterms of the axions involved in the cancellation of the anomalies described above are related via the descent equation as

$$a^{\text{orb}} X_4^{\text{orb}} + \sum_a \tau_a X_{4,a}^{\text{red}} = a^{\text{uni}} X_4^{\text{uni}} + \sum_r \beta_r X_4^r \,. \tag{4.50}$$

The left hand side contains the unique orbifold axion a^{orb} together with the blow–up modes τ_a, and the right hand side contains the universal axion a^{uni} in blow–up as well as the non–universal axions β_r. This last equation helps us to express the axions in terms of the blow–up modes. In (4.50) we have added a counterterm $\sum_a \tau_a X_{4,a}^{\text{red}}$ of blow–up modes whose variation accounts for the change of the orbifold anomaly. Our aim is to express β_r and a^{uni} in terms of a^{orb} and τ_a, in order to confirm the interpretation of the non–universal axions as phases of the blow–up modes [91]. We do so by calculating the four different anomalies I^{orb}, I^{red}, I^{uni}, and I^{non} of (4.49) separately. Then, we infer the relationship among the axions via the descent equations (4.50).

Universal orbifold anomaly I^{orb}

On the orbifold, we can choose a basis of $U(1)$ charges such that the single anomalous $U(1)_A$ is generated by

$$T_{\text{A}} = (3,3,1,1,1,5,-3,-3,0,-4,2,0,0,0,0,0) \tag{4.51}$$

in terms of an orthogonal standard base for the Cartan elements of $E_8 \times E_8$. With this anomalous $U(1)$ generator, the anomaly polynomial on the orbifold is

$$I^{\text{orb}} = 6F_1 \left(\text{tr}F^2_{SU(2)} + \text{tr}F^2_{SU(3)} + \text{tr}F^2_{SO(10)} - \text{tr}R^2 + \kappa^{IJ} \sum_{I,J} F_I F_J \right) . \tag{4.52}$$

The numerical factors κ^{IJ} are not given explicitly because they are not relevant in further discussions. The factor of 6 could be absorbed by changing the normalization of T_{A}. However, we prefer not to do so, as otherwise we find this factor of 6 in all field redefinitions in the next section.

Anomaly from field redefinition I^{red}

As explained in section 4.3, there is a field redefinition between the states on the orbifold and in blow–up. This field redefinition also induces a change of the anomaly polynomial described by I^{red}. We calculate this change by splitting up I^{red} into contributions from the three types of anomalies, $I^{\text{red}} = I^{\text{red}}_{\text{G}} + I^{\text{red}}_{\text{grav}} + I^{\text{red}}_{\text{pure}}$, which we will now compute.

$U(1) \times G^2$ anomaly redefinition In order to compute the redefinition of the $U(1)\times G^2$ anomaly polynomial we need to consider the change of $\text{tr}\, Q_I$ when going from the orbifold to blow–up, where the trace is taken over the fields charged under the non–abelian group. The change is due to the field redefinitions and to the fact that some fields become massive in blow–up and hence are not present in the massless spectrum anymore. Recall that Q^γ_I, Q'^γ_I, Δ^γ_I denote the charges of a state γ on the orbifold, the charges in blow–up, and the shift in the charge caused by field redefinitions, see (4.30).

The sum of the charges in blow–up $\text{tr}\,(Q_I)_{\text{BU}} = \sum_\alpha Q'^\alpha_I$ runs over the states α that remain massless after giving vevs to the blow–up modes. Hence, in order to recover the trace on the orbifold prior to having assigned vevs, we also have to include a sum over the states that gain a mass in blow–up, which we label by β. We thus obtain

$$\begin{aligned}
\text{tr}(Q_I)_{\text{BU}} &= \sum_\alpha Q^\alpha_I - \sum_\alpha \Delta^\alpha_I = \sum_\alpha Q^\alpha_I - \sum_\alpha \Delta^\alpha_I + \sum_\beta Q^\beta_I - \sum_\beta \Delta^\beta_I - \sum_\beta Q'^\beta_I \\
&= \text{tr}(Q_I)_{\text{orb}} - \sum_{\gamma=\alpha,\beta} \Delta^\gamma_I - \sum_\beta Q'^\beta_I \,,
\end{aligned} \tag{4.53}$$

where we added a 0 in the first step and rearranged the terms in the second step. Note that the last sum $\sum_\beta Q_I'^\beta$ which sums over all fields that became massive in blow–up, vanishes identically: all massive states are vector–like with respect to their charges, so the sum always contains pairs of opposite charges. Leaving out this last term, the contribution to the 4d anomaly polynomial and the redefinition part reads

$$\begin{aligned} I_G &= F^I \mathrm{tr} F_G^2 \sum_\alpha Q_I'^\alpha \,, \\ I_G^{\mathrm{red}} &\sim \sum_{G,I} \left(-\sum_\gamma \Delta_I^\gamma \right) F^I \mathrm{tr} F_G^2 \sim \sum_{G,I} c_I^G F^I \mathrm{tr} F_G^2 \,. \end{aligned} \tag{4.54}$$

In the sums G runs over $SU(2)$, $SU(3)$ and $SO(10)$. When evaluating the sum and comparing with the orbifold result, we obtain a perfect match of all $U(1) \times G^2$ anomalies of both theories. The anomaly coefficients c_I^G of (4.54) are given by

$$\begin{aligned} c_I^{SU(2),SU(3)} &= (19, -20, -25, -4, -66, 18, 25, 0) \ , \\ c_I^{SO(10)} &= (-6, 0, 0, 0, 0, 0, 0, -1) \ . \end{aligned} \tag{4.55}$$

$U(1) \times \mathrm{grav}^2$ anomaly redefinition For the $U(1) \times \mathrm{grav}^2$ anomaly one has to include all the massless fields in the trace. This means that, in contrast to the $U(1) \times G^2$ anomalies, one also has to add the contribution coming from the abelian blow–up mode charges q_I^a. The contribution to the 4d anomaly polynomial and the redefinition part is then given by

$$\begin{aligned} I_{\mathrm{grav}} &\sim F^I \mathrm{tr} R^2 \mathrm{tr}(Q_I')_{\mathrm{BU}} = F^I \mathrm{tr} R^2 \sum_\alpha Q_I'^\alpha \\ &= F^I \mathrm{tr} R^2 \left(\sum_\alpha Q_I^\alpha - \sum_\alpha \Delta_I^\alpha + \sum_\beta Q_I^\beta - \sum_\beta \Delta_I^\beta - \sum_\beta Q_I'^\beta + \sum_a q_I^a - \sum_a q_I^a \right) \,, \\ I_{\mathrm{grav}}^{\mathrm{red}} &\sim \left(-\sum_{\gamma=\alpha,\beta} \Delta_I^\gamma - \sum_a q_I^a \right) F^I \mathrm{tr} R^2 = c_I^{\mathrm{grav}} F^I \mathrm{tr} R^2 \,, \end{aligned} \tag{4.56}$$

where we again added the contributions from the massive fields and used that $\sum_\beta Q_I'^\beta = 0$. The index γ contains both α for massless and β for massive fields. The anomaly coefficients in (4.56) are

$$c_I^{\mathrm{grav}} = \left(-\frac{47}{6}, -\frac{34}{3}, \frac{73}{3}, \frac{10}{3}, \frac{116}{3}, -\frac{38}{3}, -\frac{187}{12}, \frac{2}{3} \right) \,. \tag{4.57}$$

We find again a perfect match between the blow–up polynomial and the redefined one, supporting the obtained field redefinitions (4.29).

Pure $U(1)$ anomaly redefinition A similar procedure can be applied to the pure $U(1)$ anomalies and in this case the field redefinitions change the polynomial via

$$
\begin{aligned}
I_{\text{pure}} &\sim \frac{1}{3!}\sum_{I,J,K} F^I F^J F^K \sum_\alpha Q_I'^\alpha Q_J'^\alpha Q_K'^\alpha \\
&= \frac{1}{3!}\sum_{I,J,K} F^I F^J F^K \left(\sum_\alpha Q_I^\alpha Q_J^\alpha Q_K^\alpha + \sum_a q_I^a q_J^a q_K^a + \sum_\beta Q_I^\beta Q_J^\beta Q_K^\beta\right) + I_{\text{pure}}^{\text{red}} \\
&= \frac{1}{3!}\sum_{I,J,K} F^I F^J F^K \text{tr}(Q_I Q_J Q_K)_{\text{orb}} + I_{\text{pure}}^{\text{red}}\,, \\
I_{\text{pure}}^{\text{red}} &\sim \frac{1}{3!}\sum_{I,J,K} F^I F^J F^K \Bigg(\sum_{\gamma=\alpha,\beta} (-3\Delta_I^\gamma Q_J^\gamma Q_K^\gamma + 3\Delta_I^\gamma \Delta_J^\gamma Q_K^\gamma - \sum_{\gamma=\alpha,\beta} \Delta_I^\gamma \Delta_J^\gamma \Delta_K^\gamma \\
&\qquad\qquad - \sum_a q_I^a q_J^a q_K^a - \sum_\beta Q_I'^\beta Q_J'^\beta Q_K'^\beta \Bigg)\,.
\end{aligned}
\tag{4.58}
$$

We have made explicit a factor of 1/3! coming from the symmetry factor **sym** and from permutation symmetries of the sum. The anomalies match perfectly when assuming the mass terms to have the structure explained in section 4.3. The coefficients of the anomaly terms turn out to be rather big. For example, the coefficients of the cubic anomaly term $\sum_I c_I^{\text{pure}} F_I^3$ are given by

$$
c_I^{\text{pure}} = \frac{1}{3!}\,(14576, 91184, -436928, -202064, -384592, 270832, 24026, -16)\,. \tag{4.59}
$$

The expression for I^{red} simplifies due to the fact that $\sum_\beta Q_I'^\beta Q_J'^\beta Q_K'^\beta = 0$ and $\sum_\beta Q_I'^\beta = 0$, thus we obtain

$$
\begin{aligned}
I^{\text{red}} \;=\; &-\sum_a q_I^a F^I \Bigg(\sum_{\gamma=\alpha,\beta} r_a^\gamma \text{tr} F_{\text{G}}^2 + (1 + \sum_{\gamma=\alpha,\beta} r_a^\gamma) \text{tr} R^2 \\
&+ \frac{1}{3!} F^J F^K \left[3\sum_\gamma r_a^\gamma Q_J^\gamma Q_K^\gamma - 3\sum_\gamma r_a^\gamma \Delta_J^\gamma Q_K^\gamma + \sum_\gamma r_a^\gamma \Delta_J^\gamma \Delta_K^\gamma + q_J^a q_K^a \right] \Bigg)\,.
\end{aligned}
\tag{4.60}
$$

In the sum running over $a = (k, \sigma)$, the factors r_a^γ not appearing in (4.30) are zero.

Universal blow–up anomaly I^{uni}

The universal anomaly in blow–up is given by

$$
\begin{aligned}
I^{\mathrm{uni}} &= \int\limits_{\mathcal{M}} X_2^{\mathrm{uni}} X_4^{\mathrm{uni}} \\
&= -\frac{1}{12}\int\limits_{\mathcal{M}} (\mathrm{tr}R^2 - \mathrm{tr}F^2)\left(\mathrm{tr}(\mathcal{F}'F')\mathrm{tr}\mathcal{F}'^2 - \frac{1}{2}\mathrm{tr}\mathcal{F}'^2\mathrm{tr}(\mathcal{F}''F'') - \frac{1}{4}\mathrm{tr}(\mathcal{F}'F')\mathrm{tr}\mathcal{R}^2 +' \leftrightarrow''\right) .
\end{aligned}
\tag{4.61}
$$

Using the intersection numbers and the expansion of the internal flux $\mathcal{F}$, we obtain for the universal anomaly in blow–up

$$
I^{\mathrm{uni}} = \frac{1}{2}(\mathrm{tr}R^2 - \mathrm{tr}F^2)\cdot(-25F_1 + 20F_2 + 25F_3 + 4F_4 + 66F_5 - 18F_6 - 25F_7 - F_8)) . \tag{4.62}
$$

Non–universal local anomalies I^{non}

Lastly, we have the non–universal axions β_r to cancel the other $U(1)$ anomalies. Their contributions are given by

$$
I^{\mathrm{non}} = \int\limits_{\mathcal{M}} X_2^r X_4^r \, . \tag{4.63}
$$

This expression is evaluated by using the Bianchi identities to express $\mathrm{tr}\mathcal{R}^2$ in terms of $\mathrm{tr}\mathcal{F}^2$ as

$$
\int\limits_{E_r} \mathrm{tr}\mathcal{R}^2 = \int\limits_{E_r} \mathrm{tr}\mathcal{F}^2 = V^I_{r_1} V^I_{r_2} E_{r_1} E_{r_2} E_r \, . \tag{4.64}
$$

In appendix C the expressions for X_4^r and X_2^r are given. The integration in (4.63) is performed by using the intersection numbers. We obtain

$$
\begin{aligned}
I^{\mathrm{non}} \; &= \frac{1}{2}(-25F_1 + 20F_2 + 25F_3 - 4F_4 - 66F_5 + 18F_6 + 25F_7 + F_8) \\
&\quad \cdot \left(\mathrm{tr}F^2_{SO(10)} - \mathrm{tr}F^2_{SU(2)} - \mathrm{tr}F^2_{SU(3)}\right) + \sum_{IJK} h^{IJK} F_I F_J F_K \\
&\quad + \frac{1}{12}(-16F_1 - 256F_2 + 142F_3 + 16F_4 + 68F_5 - 44F_6 - 37F_7 + 2F_8)\mathrm{tr}R^2 \, ,
\end{aligned}
\tag{4.65}
$$

where we have expressed the coefficients corresponding to pure $U(1)$ anomalies schematically as h^{IJK}. Now we have computed all 4 contributions to the anomalies in (4.49).

4.4.3 Relation among the axions

From the above results for I^{orb}, I^{red}, I^{uni}, and I^{non}, we can now establish the relation between the single orbifold axion, the axions in blow–up, and the blow–up modes using the descent equations (4.50). We need to make an ansatz to factorize I^{red} which is compatible with this interpretation. A given factorization $I^{\text{red}} = \sum_a q_I^a F_I X_{4,a}^{\text{red}}$ is canceled via the counterterm $\sum_a \tau_a X_{4,a}^{\text{red}}$. The indices a and r run over the same set, so we use only r. Considering $X_4^{\text{orb}} = -6 X_4^{\text{uni}}$ we make the following ansatz for relating the various axions

$$\beta_r = d_r \tau_r \,, \qquad a^{\text{uni}} = -6 a^{\text{orb}} + \sum_r c_r \tau_r \,. \tag{4.66}$$

Here, the c_r and d_r are coefficients in the linear combinations and the factor of -6 arises due to the normalization choice in (4.51). Substituting this ansatz into (4.50), the 4–form involved in the factorization is expressed as

$$X_4^{\text{red},r} = c_r X_4^{\text{uni}} + d_r X_4^r \,. \tag{4.67}$$

Substituting this last expression into I^{red} in (4.49) yields

$$I^{\text{red}} = \sum_r q_I^r F_I \left(c_r X_4^{\text{uni}} + d_r X_4^r \right) \,. \tag{4.68}$$

Looking at the whole anomaly polynomial (4.49), we impose equality of each factor on the left hand side and on the right hand side. As there are 8 anomalous $U(1)$s, we obtain 152 equations in total, where 8 equations arise from the 8 $U(1) \times \text{grav}^2$ anomalies, $8 \cdot 3 = 24$ equations arise from the mixed $U(1) \times G^2$ anomalies, and $8 + 8 \cdot 7 + 8 \cdot 7 \cdot 6/3! = 120$ equations arise from the pure $U(1)$ anomalies. At first sight, this system is highly over–constrained, as we only have $2 \cdot 21 = 42$ coefficients c_r, d_r. However, as it turns out, only 29 out of the 152 equations are independent. In particular, we find that part of the solution is $d_r = -1/6$ for all r. The factor of 6 arises again due to our normalization convention. From (4.66) we thus see that axions τ_r coming from field redefinitions are indeed the same as the non–universal axions β_r, which are responsible for canceling the non–universal anomalies in blow–up. This result allows us to interpret the blow–up modes as non–universal axions in a compact resolution of the $\mathbb{Z}_7$ orbifold.

However, choosing a common value for all c_r or grouping them by fixed points or by sectors turns out to be impossible. This implies that the universal axion in blow–up is a mixture of the unique orbifold axion and the blow–up modes.

The analysis of the chapter shows that a careful inspection of the blow–up mechanism reveals detailed information about the models away from the orbifold point. With the concept of local multiplicity operators the knowledge about orbifold properties can be carried over to the blow–up model. Within the framework of our $\mathbb{Z}_7$ example we can study the match of the spectrum in detail. All relevant states can be identified on both sides.

Masses can be compared and some subtleties (concerning masses in the large volume limit) can be clarified.

We have emphasized that the study of the Green–Schwarz anomaly polynomial is a key tool to understand the resolution of the orbifold point. In contrast to the single $U(1)_A$ of the orbifold model we find many anomalous $U(1)$s in blow–up and we identify the corresponding localized axions. Mixing of the axion in the anomaly polynomial is relevant for the interactions in the blow–up model. The match with the anomalies supports the reliability of the field theoretical methods used in the resolution procedure.

Our analysis shows that it pays off to study the blow–up mechanism in detail. It allows us to carry over the powerful computational techniques of orbifold compactification to smooth compactifications (where otherwise only effective field theory methods in the large volume limit are available). Here we have employed an example based on the $\mathbb{Z}_7$ orbifold which shares the complexity of realistic models but avoids some of the subtleties found e.g. in the models of the Mini–Landscape. These subtleties are not yet completely understood, but they seem to be no obstructions in principle. We hope that with the methods developed here these problems can be overcome.

Chapter 5

The $\mathbb{Z}_{6II}$ orbifold and its resolution

Confianza en el anteojo, nó en el ojo;
en la escalera, nunca en el peldaño;
en el ala, nó en el ave
y en ti sólo (...)

César Vallejo.

In this chapter we explore if a $T^6/\mathbb{Z}_{6II}$ orbifold deformed by vevs corresponds to a $\widehat{T^6/\mathbb{Z}_{6II}}$ CY compactification with $U(1)$ fluxes. We search for a $\widehat{T^6/\mathbb{Z}_{6II}}$, in which the blow–up modes (determining the $U(1)$ flux) can be identified with the twisted fields of the orbifold. First we present the orbifold geometry, and then we do a survey of the blow–up geometries, with the techniques introduced in section 3.6. We study in detail two topologies of the local resolution of $\mathbb{C}^3/\mathbb{Z}_{6II}$, corresponding to the triangulations A and B. For the first, we find a set of blow–up modes from which two modes can not be identified with orbifold twisted fields. After performing an exploration over possible topologies, from which we give in the appendix a generic one, we are lead to the case where the topologies of all $\mathbb{C}^3/\mathbb{Z}_{6II}$ resolutions is the one described by triangulation B. In that case, we find many solutions to the Bianchi identities, which can be used to identify orbifold twisted states with the blow–up modes. We select one identification and take the correspondence till its final consequences. In particular we perform field redefinitions which allow us to obtain the chiral asymmetry of the supergravity compactified on the smooth manifold. We analyze the superpotential at the orbifold point and find that is possible to select redefinitions which identify the massive orbifolds fields with non–chiral fields on the smooth CY. Finally we study the anomaly cancellation mechanism in 4d. A perfect agreement is found between the orbifold deformed by vevs and the $\widehat{T^6/\mathbb{Z}_{6II}}$ CY with abelian flux. Furthermore we identify blow–up modes with the non–universal axions on the resolution.

5.1 The $T^6/\mathbb{Z}_{6II}$ orbifold

In the work [47] many models of heterotic string compactified on $T^6/\mathbb{Z}_{6II}$ were studied. From 3×10^4 orbifolds they found of the order of 100 with the spectrum of the MSSM. This mini–landscape constitutes a fertile region of the $\mathcal{N} = 1$ heterotic compactifications landscape. The method employed consists in creating models with *local GUT* at the fixed sets. The corresponding local GUTs have gauge groups E_6 and $SO(10)$. The models we use are the ones with $SO(10)$ local GUT. In this case the orbifold shift is chosen to break $E_8 \times E_8$ down to $SO(10)$. Further breaking is performed by turning on the Wilson lines $A_3 \equiv A_4$ and A_5. Recall that with A_a we denote the Wilson line associated to a torus translation e_a.

A basis for the T^6 torus lattice is given by

$$\begin{aligned} e_1 &= ((-3+\sqrt{3})/2, (3+\sqrt{3})/2, 0, 0, 0, 0), \\ e_2 &= (1, -1, 0, 0, 0, 0), \\ e_3 &= (0, 0, (-1+\sqrt{3})/2, (1+\sqrt{3})/2, 0, 0), \\ e_4 &= (0, 0, 1, -1, 0, 0), \\ e_5 &= (0, 0, 0, 0, 1, -1), \\ e_6 &= (0, 0, 0, 0, 1, 1). \end{aligned} \tag{5.1}$$

In the figures 5.1, 5.2 and 5.3 we depict the geometry of the $T^6/\mathbb{Z}_{6II}$ orbifold. The T^6 lattice is the root lattice of $G_2 \times SU(3) \times SU(2)^2$. The geometrical twist in the basis used for the torus vectors is given by $v = (1/6, 1/3, -1/2)$. Let us denote the three complex coordinates by z_i with $i = 1, 2, 3$. The twists acts on them as $\theta : z_i \to e^{2\pi i v_i} z_i$. The shift on the gauge d.o.f. is given by $V_{SO(10),1}$ and the Wilson lines of a subset from these models can both be read in Tables F.10,F.11 and F.12, from appendix F.

The figure 5.1 corresponds to the first twisted sector θ, which has 12 fixed points. The fixed points in the complex planes $i = 1, 2, 3$ are denoted by α, β and γ respectively. The figure 5.2 corresponds to the fixed sets in the θ^2 and θ^4 sectors. In these sectors the coordinate z_3 is fixed under the orbifold action, so the twisted states are localized in points in the first two planes and in a torus in the third. Fixed tori with $\alpha = 3, 5$ are identified in the orbifold, so we have in total 6 fixed tori. The θ^3 sector is represented in figure 5.3, in this case the coordinate z_2 is fixed under orbifold rotations. Thus the twisted states are localized in points in the planes $i = 1, 3$ and in a torus in the plane $i = 2$. On the first plane the fixed tori with $\alpha = 2, 4, 6$ are identified in the orbifold. That gives us a total of 8 fixed tori. In Table F.13 of appendix F we give all the conjugacy classes with the corresponding fixed sets, together with the labels α, β and γ denoting their locus in the three complex planes.

The fixed points of θ are local singularities of the kind $\mathbb{C}^3/\mathbb{Z}_{6II}$. Whereas the fixed tori of the θ^2 and θ^4 sectors are singularities of the kind $\mathbb{C}^2/\mathbb{Z}_3$, and the θ^3 fixed tori are singularities of the kind $\mathbb{C}^2/\mathbb{Z}_2$. In section 3.6 we reviewed how toric geometry is used to resolve local singularities and a gluing procedure is performed to give a global smooth CY.

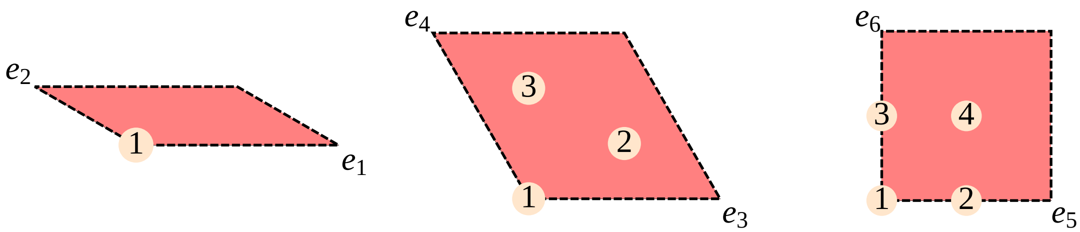

Figure 5.1: 12 fixed points of the θ sector from $T^6/\mathbb{Z}_{6II}$ orbifold. The labels of the fixed points in the planes 1, 2 and 3 denote α, β and γ, respectively.

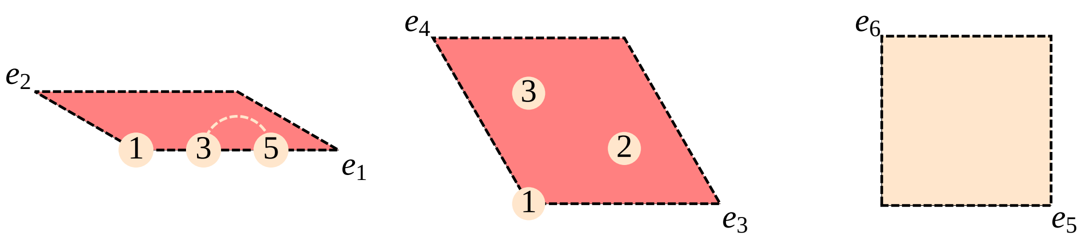

Figure 5.2: 6 fixed tori of the θ^2 and θ^4 sectors from $T^6/\mathbb{Z}_{6II}$ orbifold. The labels of the fixed points in the planes 1 and 2 denote α and β respectively. Points $\alpha = 3$ and $\alpha = 5$ joined by a line are identified under a θ^3 twist.

That procedure is used in next section in order to construct a smooth CY from the singular orbifold.

5.2 Exploring the resolutions of $T^6/\mathbb{Z}_{6II}$

The singularities are resolved first locally and then the local patches are glued to obtain the global resolution [68]. Let us first review the resolution of the local $\mathbb{C}^3/\mathbb{Z}_{6II}$ singularity. Two of the five possible resolution topologies are given in the toric diagrams of figures 5.4 and 5.5. In addition to the complex coordinates of the 6d internal space $z_i,\ i = 1, ..., 3$, as explained in section 3.6, new coordinates y_r and new scalings are introduced. Such that

$$U_j = \prod_{i,r} z_i^{(v_i)_j} y_r^{(\omega_r)_j}, \tag{5.2}$$

are local coordinates and invariant monomials under the new $(\mathbb{C}^*)^4$ action (3.133). Here $\omega_k = g_i v_i$, where z_i goes to $e^{2\pi g_i} z_i$ under θ^k. The dimension of the variety and the Calabi–Yau condition are preserved[1]. Every vector v_i and ω_r in the figure 5.5, is associated with a

[1] That the CY condition is preserved can be seen by checking explicitly that in the different patches a non-where vanishing $(3, 0)$ form is defined.

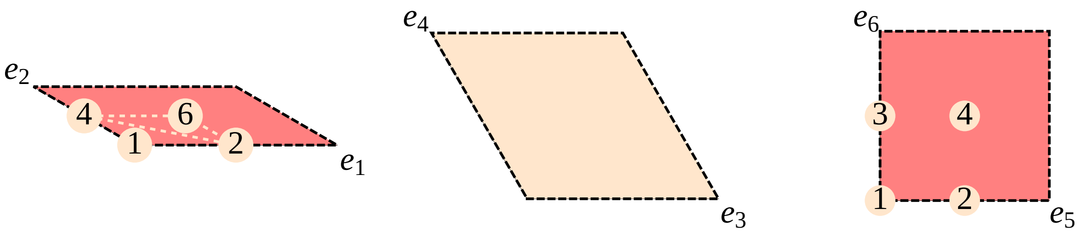

Figure 5.3: 8 fixed tori of the θ^3 sector from $T^6/\mathbb{Z}_{6II}$ orbifold. The labels of the fixed points in the planes 1 and 3 denote α and γ respectively. Points $\alpha = 2, 4, 6$ joined by a line are identified under a θ^2 twist.

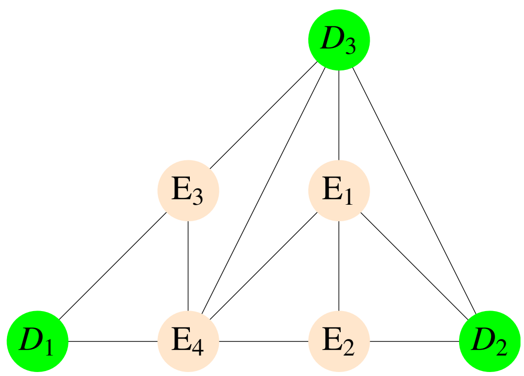

Figure 5.4: Local $\mathbb{C}^3/\mathbb{Z}_{6II}$ diagram for triangulation A.

codimension 1 hyper–surface i.e. divisor. We denote the divisors associated to the vectors v_i and ω_r by

$$D_i = \{z_i = 0\}, \qquad E_r = \{y_r = 0\}, \tag{5.3}$$

respectively.

Three divisors that correspond to the corners of a basic triangle have intersection 1. Triplets of divisors that do not have this property have intersection 0. Equivalence relations between the divisors are given by

$$\sum (v_i)_j D_i + \sum_k (\omega_k)_j E_k \sim 0. \tag{5.4}$$

Using Poincaré duality (3.86) and Stokes theorem (3.85) we relate cycles with closed–forms. Homology relations between the cycles translate into cohomology relations between the forms i.e. equivalences up to exact cycles translates into equivalences up to exact forms. The global information is obtained by taking into account all local resolutions. In addition, the divisors R_i which are the Poincaré duals of the (1,1) invariant orbifold forms $dz_i \wedge d\bar{z}_{\bar{i}}$ have to be included. An auxiliary polyhedron described in section 3.6 encodes all the triple intersections (3.139). In that way, new cohomology classes arise in the blow–up and we determine topological information from them. Taking the volume of the resolution

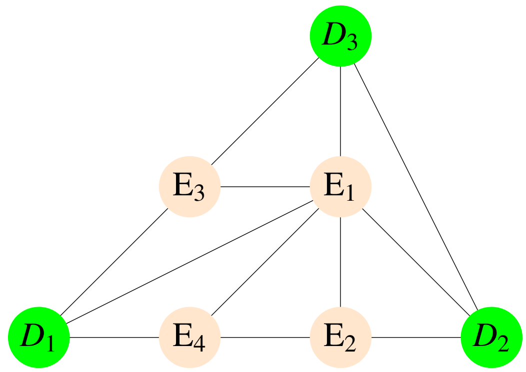

Figure 5.5: Local $\mathbb{C}^3/\mathbb{Z}_{6II}$ diagram for triangulation B.

cycles to zero the geometrical orbifold is recovered. There are different ways of resolving the orbifold encoded by the different triangulations of the toric diagram.

Information on the compact blow–ups of θ singularities for triangulation A and B can be read in the figures 5.6 and 5.7. The basic intersection numbers are given in section 3.6 and a detailed list for triangulation B can be found in appendix E. We explored mostly solutions in which all local $\mathbb{C}^3/\mathbb{Z}_{6II}$ fixed points have the same resolution. We searched for blow–up modes of the resolution for all the orbifold twisted states of a mini–landscape MSSM model. For that, we start with an orbifold model with non empty fixed–sets. We called a fixed set empty or non empty depending on whether it supports twisted matter or not. Then, we explore solutions of the Bianchi identities, which correspond to massless blow–up modes with no oscillators. Using triangulation A we found solutions with up to two modes projected out on the orbifold. [2] In triangulation B we found multiple sets of modes which fulfill the BI. By adjusting them is possible to break the $SU(6)$ hidden group gauge factor or to preserve it.

All of the encountered vacua possess moduli with different chirality on the orbifold theory. Also the modes in sectors θ^2 and θ^4 (these sectors contain the CPT conjugated states of the other) can not be conjugated to each other (have opposite vector V_r). Our exploration shows, that for the studied orbifold, the order of solutions with mass $m = 0$, and no oscillators $N = 0$, preserving the hidden group, is greater than 10^7. [3]

In the following we explain how the exploration is carried out. Then we focus on the triangulation B, and in one particular set of blow–up modes. For this case we match the chiral asymmetry of the 10d $\mathcal{N} = 1$ supergravity compactified in $\widehat{T^6/\mathbb{Z}_{6II}}$, by redefining the fields. We then study the mass terms generated from Yukawa couplings in the orbifold

[2] This blow–up could be connected to an orbifold brother–model with shift and Wilson lines fulfilling modular invariance constraints. However this model would have an inconsistent transformation for the physical states [89, 90]. This inconsistency would also appear in the partition function [126] which would not be single valued.

[3] Many of them are non-distinct because the blow–up modes of the $\mathcal{N} = 2$ orbifold sectors have always a pair.

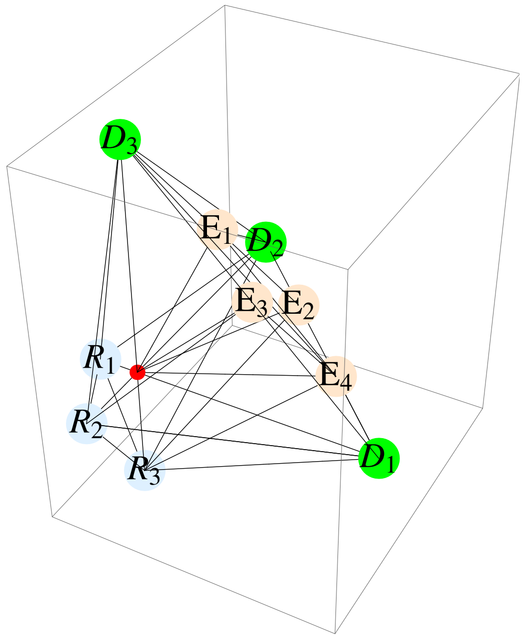

Figure 5.6: Global $\mathbb{C}^3/\mathbb{Z}_{6II}$ diagram for triangulation A.

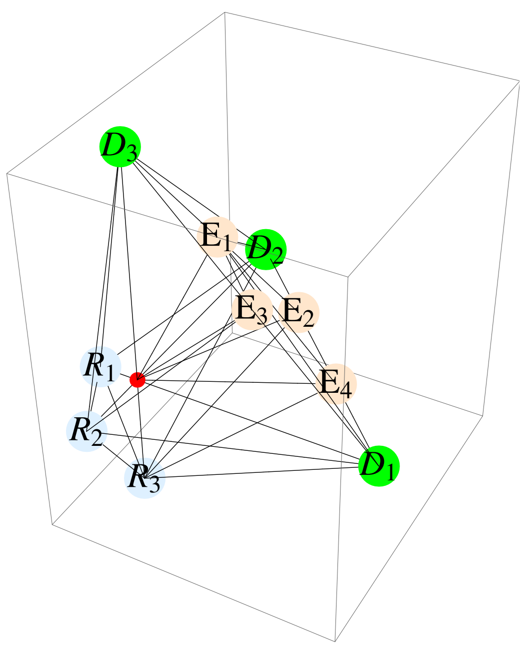

Figure 5.7: Global $\mathbb{C}^3/\mathbb{Z}_{6II}$ diagram for triangulation B.

superpotential, to determine which fields acquire mass. Finally, we describe the anomaly cancellation mechanism interpolation between the orbifold and the CY.

5.3 Blow–up modes for a resolution topology

One way to search candidates for blow–up modes is to fix the topology of the resolved manifold, by specifying the triangulation and then search for consistent assignations of vevs to the twisted fields. This is the exploration described in this section. We focus on the triangulations A and B presented in previous section. Those are the ones with more vanishing self–intersections, and lead to less restrictive equations for the vectors V_r, appearing in the abelian flux $\mathcal{F} = E_r V_r^I H_I$. More specifically the filed strength of the abelian vector bundle in 6d is given by

$$\mathcal{F} = H_I \left(\sum_{\beta=1}^{3} \sum_{\gamma=1}^{4} V^I_{1,\beta\gamma} E_{1\beta\gamma} + \sum_{k=2,4} \sum_{\alpha=1,3} \sum_{\beta=1}^{3} V^I_{k,\alpha\beta} E_{k\alpha\beta} + \sum_{\alpha=1}^{2} \sum_{\gamma=1}^{4} V^I_{3,\alpha\gamma} E_{3\alpha\gamma} \right). \tag{5.5}$$

To obtain the Bianchi identities and the multiplicity of the blow–up states we need all the intersections of exceptional divisors given in appendix E. Using (3.86) the intersection numbers of exceptional divisors are equivalent to the integrals on the manifold of the dual forms wedge products. Evaluating the Bianchi identities (4.20) gives the formulas

$$24 - \sum_{\gamma} V^2_{3,1\gamma} - 3 \sum_{\gamma} V^2_{3,2\gamma} = 0, \tag{5.6}$$

$$-2 + 2V^2_{3,1\gamma} - \sum_{\beta} V_{3,1\gamma} \cdot V_{4,1\beta} = 0, \tag{5.7}$$

$$12 - \sum_{\gamma} V^2_{1,\beta\gamma} + \sum_{\gamma} V_{1,\beta\gamma} \cdot V_{2,1\beta} - V^2_{2,1\beta} - \sum_{\gamma} V^2_{3,1\gamma} = 0, \tag{5.8}$$

$$4V^2_{1,\beta\gamma} - 2V_{1,\beta\gamma} \cdot V_{4,1\beta} + V^2_{4,1\beta} - (V_{2,1\beta}; V_{4,1\beta}) = 4, \tag{5.9}$$

$$-\quad 2 \sum_{\gamma} V_{1,\beta\gamma} \cdot V_{2,1\beta} + \sum_{\gamma} V_{1,\beta\gamma} \cdot V_{4,1\beta} + 2V^2_{2,1\beta} - 2V^2_{4,1\beta} + 2(V_{2,1\beta}; V_{4,1\beta}) = 4, \tag{5.10}$$

$$-\quad \sum_{\beta} (V_{2,1\beta}; V_{4,1\beta}) - 2 \sum_{\beta} (V_{2,3\beta}; V_{4,3\beta}) = -24, \tag{5.11}$$

for triangulation A in figure 5.7 and

$$24-\sum_{\gamma}V_{3,1\gamma}^2-3\sum_{\gamma}V_{3,2\gamma}^2=0, \tag{5.12}$$

$$3V_{1,\beta\gamma}^2-(V_{2,1\beta};V_{4,1\beta})-V_{3,1\gamma}^2=0, \tag{5.13}$$

$$-2-V_{3,1\gamma}\cdot\sum_{\beta}V_{1,\beta\gamma}+2V_{3,1\gamma}^2=0, \tag{5.14}$$

$$24-\sum_{\beta}(V_{2,1\beta};V_{4,1\beta})-2\sum_{\beta}(V_{2,3\beta};V_{4,3\beta})=0, \tag{5.15}$$

$$-12-3V_{4,1\beta}\cdot\sum_{\gamma}V_{1,\beta\gamma}+6V_{4,1\beta}^2+2(V_{2,1\beta};V_{4,1\beta})=0, \tag{5.16}$$

$$-12-3V_{2,1\beta}\cdot\sum_{\gamma}V_{1,\beta\gamma}+3V_{4,1\beta}^2+4(V_{2,1\beta};V_{4,1\beta})=0, \tag{5.17}$$

for triangulation B. These set of equations allow for an exploration of a given orbifold model, over a wide range of twisted singlets in a reasonable computing time. The equations (5.12) and (5.6) are automatically satisfied for all the states in the studied model. This is the *Model 28* with shift $V_{SO(10)}$ of the mini–landscape. The shift and Wilson lines of the Model 28 are

$$\begin{aligned}
V &= \left(\frac{1}{3},-\frac{1}{2},-\frac{1}{2},0^5,\frac{1}{2},-\frac{1}{6},-\frac{1}{2}^5,\frac{1}{2}\right), \qquad (5.18)\\
A_5 &= \left(-\frac{1}{2},-\frac{1}{2},0,\frac{1}{2},\frac{1}{2},0,0,0,\frac{15}{4},-\frac{19}{4},-\frac{15}{4},-\frac{15}{4},-\frac{15}{4},-\frac{15}{4},-\frac{11}{4},\frac{19}{4}\right),\\
A_3 &= A_4=\left(\frac{1}{6},\frac{1}{6},-\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{5}{3},-\frac{2}{3},-\frac{5}{3},-\frac{5}{3},-\frac{5}{3},-\frac{5}{3},-\frac{1}{3},\frac{8}{3}\right).
\end{aligned}$$

The shift breaks $E_8\times E_8$ down to $SO(10)$. Adding the Wilson lines the gauge group is broken further down to $SU(3)\times SU(2)\times SU(6)\times U(1)^8$. The fact that the equations involving $V_{3,\alpha\gamma}$ are automatically satisfied occurs because in all the fixed tori $(3,\alpha,\gamma)$ the singlets surviving the orbifold projection fulfill $P_{sh}^2=V_{3,\alpha\gamma}^2=\frac{3}{2}$, i.e. have oscillator number $N=0$. For triangulation A we fixed $V_{3,\alpha\gamma}$ and then search for all the possible sets which satisfy (5.11) and restricted further to the $V_{4,1\beta}$ obeying (5.7). This gives 15120 ways of solving the system of equations (5.6)(5.7)(5.11). As there are no restrictions for $V_{1,\beta\gamma}$ there are 2×10^{11} further possibilities to check, that obey equations (5.8–5.10). The described exploration was carried only for one fixed value of $V_{3,\alpha\gamma}$. The performed exploration shows that is not possible to select the blow–up modes from twisted states of Model 28.

Triangulation B is more promising. Here we use the same approach explained previously. First, for given values of $V_{3,\alpha\gamma}$ (recall that all the singlets fulfill (5.12)) we select first all the $V_{1,\beta\gamma}$ which obey (5.14). For a sample $V_{3,\alpha\gamma}$ there are 2401 $V_{1,\beta\gamma}$. There are 50400 $V_{2,\alpha\beta}$ and $V_{4,\alpha\beta}$ that satisfy (5.15). From this surviving set we explore which $V_{1,\beta\gamma},V_{2,\alpha\beta},V_{4,\alpha\beta}$ satisfy the equations (5.13) (5.16) and (5.17), which turn to be the more hardest to obey. An exploration for a fixed $V_{3,\alpha\gamma}$ requires 1.2×10^8 iterations, while a full exploration will require of the order of 3×10^{10} iterations. In the performed exploration we found multiple sets of blow–up modes which can be identified with twisted states of Model 28.

The twisted fields acquiring vevs have to ensure a D- and *F-flat* superpotential. To resume: It is possible for a smooth CY $\widehat{T^6/\mathbb{Z}_{6II}}$ compactification with abelian flux $\mathcal{F}$ and an orbifold $T^6/\mathbb{Z}_{6II}$ with certain gauge embedding to explore over the possible sets of twisted singlets acquiring vevs, such that they are identified as the blow–up modes. Triangulation B is very suitable for this search.

5.3.1 Abelian vector bundles for triangulation A

We want to explore if for the triangulation A there can be solutions of the BI with non-oscillatory massless modes i.e.$N = m = 0$. For Model 28 we found that is not possible to find a set of blow–up modes that will lead to a toric resolution with triangulation A. Nevertheless, we obtained a solution of the BI in which only two blow–up modes from the θ^3 sector are absent. This solution has length square of the vectors given by $V^2_{1,\beta\gamma} = \frac{25}{18}, V^2_{3,\alpha\gamma} = \frac{3}{2}$ and $V^2_{4,\alpha\beta} = V^2_{2,\alpha\beta} = \frac{14}{9}$. Imposing those length square values makes the BI easier to solve, by adding to an Ansatz V_r a $\lambda \in \Lambda_{E_8\times E_8}$ and solving the linear equations for this λ [83]. The set of identities obtained is given by

$$V_{3,1\gamma} \cdot \sum_\beta V_{4,1\beta} = 1, \tag{5.19}$$

$$V_{2,1\beta} \cdot \sum_\gamma V_{1,\beta\gamma} = \frac{10}{9}, \tag{5.20}$$

$$V_{4,1\beta} \cdot V_{1,\beta\gamma} - 2V_{2,1\beta} \cdot V_{4,1\beta} = 0, \tag{5.21}$$

$$4 + \sum_\beta V_{2,1\beta} \cdot V_{4,1\beta} + 2\sum_\beta V_{2,3\beta} \cdot V_{4,3\beta} = 0. \tag{5.22}$$

We explored if a brother model to Model 28 can be found such that it has all the modes that we found in the BI solution. This question is addressed in appendix D. We show that there are not brother models for $\mathbb{Z}_{6II}$ with consistent physical state transformations [90]. In the following we focus in checking if Model 28 can provide a set of blow–up modes for a different triangulation.

The hardest restriction to satisfy is equation (5.7) or after fixing the squares the equation (5.19). No set of orbifold states satisfies it. That's why the modes corresponding to $V_{3,12}, V_{3,14}$ are projected out, because we had to modify them to fulfill (5.19). In Table 5.1 one can read off the solution for the BI found in triangulation A. We used the recently released *Orbifolder* program [127] to compute the spectrum, and to simplify comparison we use the default numbering for the states used by that program. The right-(left-) chiral state is denoted in the Orbifolder output by bF_i (F_j) and we denote it by ψ_i ($\bar{\psi}_j$). The BI solution found doesn't have all its vectors identified with local shifts P_{sh} of orbifold twisted states (3.38).

Table 5.1: Blow–up modes for triangulation A.

Bundle vector V_r	Numerical expression for V_r	Model 28
$V_{1,11}$	$(-\frac{1}{6},0,0,-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},0,\frac{1}{3},0,0,0,0,0,0),$	ok
$V_{1,12}$	$(-\frac{1}{6},0,\frac{1}{2},-\frac{1}{2},-\frac{1}{2},0,0,0,-\frac{1}{4},-\frac{5}{12},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},-\frac{1}{4})$	ok
$V_{1,13}$	$(-\frac{1}{6},0,0,-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},0,\frac{1}{3},0,0,0,0,0,0)$	ok
$V_{1,14}$	$(-\frac{1}{6},0,\frac{1}{2},-\frac{1}{2},-\frac{1}{2},0,0,0,-\frac{1}{4},-\frac{5}{12},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},-\frac{1}{4})$	ok
$V_{1,21}$	$(-\frac{1}{2},-\frac{1}{3},0,\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},-\frac{5}{6},\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},\frac{1}{6},\frac{1}{6})$	ok
$V_{1,22}$	$(0,\frac{1}{6},0,-\frac{1}{3},-\frac{1}{3},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{5}{12},-\frac{1}{12},-\frac{5}{12},-\frac{5}{12},-\frac{5}{12},-\frac{5}{12},-\frac{1}{12},\frac{5}{12})$	ok
$V_{1,23}$	$(-\frac{1}{2},-\frac{1}{3},0,\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},-\frac{5}{6},\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},\frac{1}{6},\frac{1}{6})$	ok
$V_{1,24}$	$(0,\frac{1}{6},0,-\frac{1}{3},-\frac{1}{3},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{5}{12},-\frac{1}{12},-\frac{5}{12},-\frac{5}{12},-\frac{5}{12},-\frac{5}{12},-\frac{1}{12},\frac{5}{12})$	ok
$V_{1,31}$	$(-\frac{1}{3},-\frac{1}{6},-\frac{1}{2},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},-\frac{1}{6},-\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},-\frac{1}{6},-\frac{1}{6})$	ok
$V_{1,32}$	$(-\frac{1}{3},\frac{5}{6},0,\frac{1}{3},\frac{1}{3},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},\frac{1}{12},\frac{1}{4},-\frac{1}{12},-\frac{1}{12},-\frac{1}{12},-\frac{1}{12},-\frac{5}{12},\frac{1}{12})$	ok
$V_{1,33}$	$(-\frac{1}{3},-\frac{1}{6},-\frac{1}{2},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},-\frac{1}{6},-\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},-\frac{1}{6},-\frac{1}{6})$	ok
$V_{1,34}$	$(-\frac{1}{3},\frac{5}{6},0,\frac{1}{3},\frac{1}{3},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},\frac{1}{12},\frac{1}{4},-\frac{1}{12},-\frac{1}{12},-\frac{1}{12},-\frac{1}{12},-\frac{5}{12},\frac{1}{12})$	ok
$V_{2,11}$	$(-\frac{1}{3},0,1,0,0,0,0,0,0,\frac{2}{3},0,0,0,0,0,0)$	ok
$V_{2,12}$	$(\frac{1}{2},-\frac{1}{6},\frac{1}{2},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},\frac{1}{3},\frac{1}{3},-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},\frac{1}{3},\frac{1}{3})$	ok
$V_{2,13}$	$(-\frac{1}{6},\frac{1}{6},\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{2},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{5}{6},\frac{1}{6})$	ok
$V_{2,31}$	$(-\frac{1}{3},0,-1,0,0,0,0,0,0,\frac{2}{3},0,0,0,0,0,0)$	ok
$V_{2,32}$	$(-\frac{1}{2},\frac{5}{6},\frac{1}{2},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},-\frac{1}{6},-\frac{1}{6})$	ok
$V_{2,33}$	$(-\frac{2}{3},-\frac{1}{3},0,-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},\frac{1}{6},-\frac{1}{2},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},\frac{1}{6},\frac{1}{6})$	ok
$V_{4,11}$	$-\frac{2}{3},0,0,0,0,0,0,0,-1,\frac{1}{3},0,0,0,0,0,0)$	ok
$V_{4,12}$	$(\frac{1}{2},-\frac{5}{6},-\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},\frac{1}{6},\frac{1}{6})$	ok
$V_{4,13}$	$(\frac{2}{3},\frac{1}{3},0,\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},-\frac{1}{6},\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},-\frac{1}{6},-\frac{1}{6})$	ok
$V_{4,31}$	$(-\frac{2}{3},0,0,0,0,0,0,0,-1,\frac{1}{3},0,0,0,0,0,0)$	ok
$V_{4,32}$	$(-\frac{1}{2},\frac{1}{6},-\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},-\frac{1}{3},-\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},-\frac{1}{3},-\frac{1}{3})$	ok
$V_{4,33}$	$(\frac{1}{6},-\frac{1}{6},-\frac{1}{2},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{2}{3},0,-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},\frac{1}{3},\frac{1}{3})$	ok
$V_{3,11}$	$(0,-\frac{1}{2},\frac{1}{2},0,0,0,0,0,0,1,0,0,0,0,0,0)$	ok
$V_{3,12}$	$(-\frac{1}{2},0,-\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0,0,-\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},-\frac{1}{4})$	projected out
$V_{3,13}$	$(0,-\frac{1}{2},\frac{1}{2},0,0,0,0,0,0,1,0,0,0,0,0,0)$	ok
$V_{3,14}$	$(-\frac{1}{2},0,-\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0,0,-\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},-\frac{1}{4})$	projected out
$V_{3,21}$	$(0,-\frac{1}{2},\frac{1}{2},0,0,0,0,0,0,0,0,0,0,0,-1,0)$	ok
$V_{3,22}$	$(-\frac{1}{2},0,-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},0,0,0,-\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},-\frac{1}{4})$	ok
$V_{3,23}$	$(0,-\frac{1}{2},\frac{1}{2},0,0,0,0,0,0,0,0,0,0,0,-1,0)$	ok
$V_{3,24}$	$(-\frac{1}{2},0,-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},0,0,0,-\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},-\frac{1}{4})$	ok

5.3.2 Abelian vector bundles for triangulation B

Let us search for solutions of the Bianchi identities for triangulation B. Considering massless and non-oscillatory modes, the analogs of equations (5.12)-(5.17) are given by

$$(V_{2,1\beta}; V_{4;1\beta}) = \frac{8}{3}, \tag{5.23}$$

$$\sum_{\beta}(V_{2,3\beta}; V_{4,3\beta}) = 8, \tag{5.24}$$

$$V_{3,1\gamma} \cdot \sum_{\beta} V_{1,\beta\gamma} = 1, \tag{5.25}$$

$$V_{4,1\beta} \cdot \sum_{\gamma} V_{1,\beta\gamma} = \frac{8}{9}, \tag{5.26}$$

$$V_{2,1\beta} \cdot \sum_{\gamma} V_{1,\beta\gamma} = \frac{10}{9}. \tag{5.27}$$

Using this set of equations we searched for solutions in Model 28. We list as possible blow–up modes the states in the Table (F.1) of appendix F. This set of blow–up modes breaks the $SU(6)$ hidden group. The set of blow–up modes in the Table (F.2) of appendix F preserves the hidden group.

We explored for certain chirality assignations for a set of blow–up modes. For example: the modes in sector θ^2 and θ^4 can not be conjugate if modes in θ and θ^3 are massless and non-oscillatory. A possibility is that for the θ^2 all the modes are left handed and for the θ^4 all are right handed, but this can also not be achieved. For example in the case of $V_{2,11}$ and $V_{4,11}$ the only opposite chirality modes are $V_{2,11} = -V_{4,11}$ and this implies $(V_{2,11}; V_{2,11}) = \frac{14}{3}$ which violates the Bianchi Identities. As $(2, 1, 1)$ is the conjugated class of $(4, 1, 1)$, this means that is not possible to take a set of blow–up modes in which every component of a CPT pair is identified with one blow–up mode, which is a reasonable result.

Having a solution in which all the blow–up modes are right or left handed is also not possible here. This restriction is already explained by the fact that the fixed tori $(2, 1, 2)$ and $(4, 1, 2)$ don't possess right handed and left handed singlets respectively.

The modes $V_{2,3\beta}$ and $V_{4,3\beta}$ are easily adjusted, and one finds many different solutions. There are 107520 solutions of equation (5.24). If one requires that all the modes are left or right handed, there are 48 solutions. If instead one imposes that all the modes at same fixed tori from θ^2 and θ^4 have opposite chirality one obtains also 48 solutions. Later we will focus in a set of blow–up modes with some prescribed chirality properties, this is the set of Table 5.2. The spectrum of the CY compactification with abelian flux determined by the mentioned set of blow-up modes is given in appendix J.

If one considers the blow–up given in [83], the situation that not all the modes have the same chirality also arises. This can be examined in the Table (F.5) from the appendix F, in which we have recall the results of that paper explicitly, indicating also the chirality of the orbifold twisted states.

Another feature that appears in our solutions is that the blow–up modes can have states of coincident or opposite charges in the spectrum. This feature is also present in the solution of [83]. For Model 28 the following pairs have opposite charges

$$(\psi_{97}, \psi_{168}),\ (\psi_{98}, \psi_{167}),\ (\psi_{101}, \psi_{165}). \tag{5.28}$$

These states with the same and opposite charges are displayed in the Tables (F.3) and (F.4) in Appendix F respectively. Among the conjugated states, also the CPT pairs are shown, but they can be distinguished for having the opposite chirality.

Through this chapter we use the $\bar{\psi}$ and ψ to denote left and right orbifold chiral superfields respectively. We use the same notation to denote the fermionic components of the chiral super-field, whereas to denote the vev of the scalar component we will use $\langle\bar{\psi}\rangle$ or $\langle\psi\rangle$.

5.4 A blow–up for a given vev configuration

Another way to establish the orbifold-smooth CY transition is to start with a given orbifold vevs configuration, with zero D-term, and search for a resolution which allows to interpret the fields taking vevs as blow–up modes. To follow this strategy we computed with a program the self-intersections for all triangulations[4] $\sim 5^{12}$. Then, for a given set of vevs for the twisted orbifold states, we can explore if their weights P_{sh} can be solution of the BI in a given triangulation. This implies that the vacuum configuration, can be interpreted as the heterotic theory compactified on the smooth CY with abelian vector bundle, determined by the twisted fields. A sample of the BI from our computer exploration can be seen in Table E.6 of the appendix F. This exploration was not completed due to computing time. It turned out to be more efficient to concentrate in the triangulations A and B, which are the ones giving the less restrictive set of equations.

Another question is when the orbifold models can be blown-up at all. The first requirement is that they have twisted matter in every fixed point or fixed tori. We checked the Mini-landscape models with $SO(10)$ shift and two Wilson lines. From 80 of them this criteria was only fulfilled by 2. Most of those models have empty fixed sets. We observed that the fixed tori $(0,0,-,0,0),(0,0,-,0,1)$ on the θ^3 sector are usually empty. The fixed tori share projection conditions with $V_h = A_3(m_3+m_4)+kV, k = 0,...,5$. Those conditions are more restrictive than the ones of other fixed tori. For example the θ^3 fixed tori $(1,0,0,0,1,0)$, $(1,0,0,0,1,1)$ involve projections under $V_h = A_3(m_3+m_4), 3V + A_3(m_3+m_4) - A_5$. This circumstance makes hard not to project out all the states in the mentioned fixed tori.

5.5 Field redefinitions

Our aim is to interpret the deviation from the orbifold vacuum as a compactification in a smooth CY manifold. Then, after identifying the blow–up modes we need to compare the

[4]The exact number of inequivalent triangulations is explained in [83].

Table 5.2: Set of blow-up modes in triangulation B with almost all right-handed modes.

V_r^2	F.P.	Numerical value V_r	irrep.	Φ^{orb}
$\frac{25}{18}$	$\{1,1,1\}$	$\{-\frac{1}{6},0,0,-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},0,\frac{1}{3},0,0,0,0,0,0\}$	$\{\mathbf{1},r\}$	ψ_{57}
$\frac{25}{18}$	$\{1,1,2\}$	$\{-\frac{1}{6},0,\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0,0,-\frac{1}{4},-\frac{5}{12},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},-\frac{1}{4}\}$	$\{\mathbf{1},r\}$	ψ_{44}
$\frac{25}{18}$	$\{1,1,3\}$	$\{-\frac{1}{6},0,0,-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},0,\frac{1}{3},0,0,0,0,0,0\}$	$\{\mathbf{1},r\}$	ψ_{45}
$\frac{25}{18}$	$\{1,1,4\}$	$\{-\frac{1}{6},0,\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0,0,-\frac{1}{4},-\frac{5}{12},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},-\frac{1}{4}\}$	$\{\mathbf{1},r\}$	ψ_{41}
$\frac{25}{18}$	$\{1,2,1\}$	$\{-\frac{1}{2},-\frac{1}{3},0,\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{5}{6},\frac{1}{6}\}$	$\{\mathbf{1},r\}$	ψ_{88}
$\frac{25}{18}$	$\{1,2,2\}$	$\{0,\frac{1}{6},0,-\frac{1}{3},-\frac{1}{3},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{5}{12},-\frac{1}{12},-\frac{5}{12},-\frac{5}{12},-\frac{5}{12},-\frac{5}{12},-\frac{1}{12},\frac{5}{12}\}$	$\{\mathbf{1},r\}$	ψ_{77}
$\frac{25}{18}$	$\{1,2,3\}$	$\{-\frac{1}{2},-\frac{1}{3},0,\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{5}{6},\frac{1}{6}\}$	$\{\mathbf{1},r\}$	ψ_{85}
$\frac{25}{18}$	$\{1,2,4\}$	$\{0,\frac{1}{6},0,-\frac{1}{3},-\frac{1}{3},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{5}{12},-\frac{1}{12},-\frac{5}{12},-\frac{5}{12},-\frac{5}{12},-\frac{5}{12},-\frac{1}{12},\frac{5}{12}\}$	$\{\mathbf{1},r\}$	ψ_{70}
$\frac{25}{18}$	$\{1,3,1\}$	$\{\frac{1}{6},-\frac{2}{3},0,-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},\frac{1}{3},0,-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},\frac{1}{3},\frac{1}{3}\}$	$\{\mathbf{1},r\}$	ψ_{34}
$\frac{25}{18}$	$\{1,3,2\}$	$\{\frac{1}{6},-\frac{2}{3},\frac{1}{2},-\frac{1}{6},-\frac{1}{6},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{12},\frac{1}{4},-\frac{1}{12},-\frac{1}{12},-\frac{1}{12},-\frac{1}{12},-\frac{5}{12},\frac{1}{12}\}$	$\{\mathbf{1},r\}$	ψ_{22}
$\frac{25}{18}$	$\{1,3,3\}$	$\{\frac{1}{6},-\frac{2}{3},0,-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},\frac{1}{3},0,-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},\frac{1}{3},\frac{1}{3}\}$	$\{\mathbf{1},r\}$	ψ_{28}
$\frac{25}{18}$	$\{1,3,4\}$	$\{\frac{1}{6},-\frac{2}{3},\frac{1}{2},-\frac{1}{6},-\frac{1}{6},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{12},\frac{1}{4},-\frac{1}{12},-\frac{1}{12},-\frac{1}{12},-\frac{1}{12},-\frac{5}{12},\frac{1}{12}\}$	$\{\mathbf{1},r\}$	ψ_{15}
$\frac{14}{9}$	$\{2,1,1\}$	$\{-\frac{1}{3},0,1,0,0,0,0,0,0,\frac{2}{3},0,0,0,0,0,0\}$	$\{\mathbf{1},r\}$	ψ_{115}
$\frac{14}{9}$	$\{2,1,2\}$	$\{\frac{1}{2},-\frac{1}{6},\frac{1}{2},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},\frac{1}{3},\frac{1}{3},-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},\frac{1}{3},\frac{1}{3}\}$	$\{\mathbf{1},l\}$	$\bar{\psi}_{36}$
$\frac{14}{9}$	$\{2,1,3\}$	$\{-\frac{1}{6},\frac{1}{6},\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{2},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{5}{6},\frac{1}{6}\}$	$\{\mathbf{1},l\}$	$\bar{\psi}_{45}$
$\frac{14}{9}$	$\{4,1,1\}$	$\{-\frac{2}{3},0,0,0,0,0,0,0,0,\frac{1}{3},0,0,0,0,1,0\}$	$\{\mathbf{1},r\}$	ψ_{183}
$\frac{14}{9}$	$\{4,1,2\}$	$\{\frac{1}{2},-\frac{5}{6},-\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},\frac{1}{6},\frac{1}{6}\}$	$\{\mathbf{1},r\}$	ψ_{187}
$\frac{14}{9}$	$\{4,1,3\}$	$\{-\frac{1}{3},-\frac{2}{3},0,\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},-\frac{1}{6},\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},-\frac{1}{6},-\frac{1}{6}\}$	$\{\mathbf{1},l\}$	$\bar{\psi}_{106}$
$\frac{14}{9}$	$\{2,3,1\}$	$\{-\frac{1}{3},0,-1,0,0,0,0,0,0,\frac{2}{3},0,0,0,0,0,0\}$	$\{\mathbf{1},r\}$	ψ_{97}
$\frac{14}{9}$	$\{2,3,2\}$	$\{-\frac{1}{2},\frac{5}{6},\frac{1}{2},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},-\frac{1}{6},-\frac{1}{6}\}$	$\{\mathbf{1},r\}$	ψ_{90}
$\frac{14}{9}$	$\{2,3,3\}$	$\{-\frac{2}{3},-\frac{1}{3},0,-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},\frac{1}{6},-\frac{1}{2},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},\frac{1}{6},\frac{1}{6}\}$	$\{\mathbf{1},r\}$	ψ_{103}
$\frac{14}{9}$	$\{4,3,1\}$	$\{-\frac{2}{3},0,0,0,0,0,0,0,0,\frac{1}{3},0,0,0,0,1,0\}$	$\{\mathbf{1},r\}$	ψ_{165}
$\frac{14}{9}$	$\{4,3,2\}$	$\{-\frac{1}{2},\frac{1}{6},-\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},-\frac{1}{3},-\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},-\frac{1}{3},-\frac{1}{3}\}$	$\{\mathbf{1},r\}$	ψ_{170}
$\frac{14}{9}$	$\{4,3,3\}$	$\{\frac{1}{6},-\frac{1}{6},-\frac{1}{2},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{5}{6},-\frac{1}{6}\}$	$\{\mathbf{1},r\}$	ψ_{159}
$\frac{3}{2}$	$\{3,1,1\}$	$\{0,-\frac{1}{2},\frac{1}{2},0,0,0,0,0,0,1,0,0,0,0,0,0\}$	$\{\mathbf{1},r\}$	ψ_{155}
$\frac{3}{2}$	$\{3,1,2\}$	$\{\frac{1}{2},0,\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0,0,\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},\frac{1}{4}\}$	$\{\mathbf{1},r\}$	ψ_{153}
$\frac{3}{2}$	$\{3,1,3\}$	$\{0,-\frac{1}{2},\frac{1}{2},0,0,0,0,0,0,1,0,0,0,0,0,0\}$	$\{\mathbf{1},r\}$	ψ_{154}
$\frac{3}{2}$	$\{3,1,4\}$	$\{\frac{1}{2},0,\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0,0,\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},\frac{1}{4}\}$	$\{\mathbf{1},r\}$	ψ_{150}
$\frac{3}{2}$	$\{3,2,1\}$	$\{0,\frac{1}{2},-\frac{1}{2},0,0,0,0,0,0,-1,0,0,0,0,0,0\}$	$\{\mathbf{1},r\}$	ψ_{147}
$\frac{3}{2}$	$\{3,2,2\}$	$\{\frac{1}{2},0,\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0,0,\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},\frac{1}{4}\}$	$\{\mathbf{1},r\}$	ψ_{134}
$\frac{3}{2}$	$\{3,2,3\}$	$\{0,\frac{1}{2},-\frac{1}{2},0,0,0,0,0,0,-1,0,0,0,0,0,0\}$	$\{\mathbf{1},r\}$	ψ_{141}
$\frac{3}{2}$	$\{3,2,4\}$	$\{\frac{1}{2},0,\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0,0,\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},\frac{1}{4}\}$	$\{\mathbf{1},r\}$	ψ_{126}

massless spectrum. The massless chiral spectrum remaining after assigning vevs to some twisted fields should coincide with the massless spectrum encountered in the heterotic supergravity and super Yang-Mills on the resolution. However, states on the orbifold $\Phi_\gamma^{\rm orb}$ have weights P_{sh} which are not roots of $E_8 \times E_8$ lattice, and is natural to conjecture that field redefinitions should be performed. As in the $\mathbb{Z}_7$ case, we perform redefinitions in order to reproduce the chiral asymmetry of the supergravity on the blow–up, employing the blow–up modes $\Phi_i^{\rm BU-mode}$. The main requirement is that the sum of the left moving momenta of the states add up to a vector in the lattice. We consider redefinitions of the kind

$$\Phi_\gamma^{\rm BU} = \Phi_\gamma^{\rm orb} \prod_i (\Phi_i^{\rm BU-mode})^{c_i^\gamma}, \; c_i^\gamma \in \mathbb{Z}, \tag{5.29}$$

with integer coefficients c_i^γ such that they are single valued, and $\Phi_\gamma^{\rm BU}$ is a chiral state on the blow–up. We denote the conjugacy classes of $\Phi_\gamma^{\rm orb}$ and $\Phi_i^{\rm BU-mode}$ by $(\theta^k, n_\alpha e_\alpha)$ and $(\theta^{k_i}, m_\alpha^i e_\alpha)$ respectively. We can consider different number of blow–up modes in one redefinition. We studied the cases involving 1,2, or 3 blow–up modes.

Single field redefinitions Let us denote by λ the root system of $E_8 \times E_8$ and by Λ the corresponding root lattice. Then the left moving momentum of the blow–up state is $P_{BU} \in \lambda$. We denote as P_{sh} and P_{sh}^1 the left moving momentum of the twisted state and the blow–up mode respectively. They are given by

$$\begin{aligned} P_{sh} &= p + kV + n_\alpha A_\alpha, \\ P_{sh}^1 &= p_1 + k_1 V + m_\alpha A_\alpha, \end{aligned} \tag{5.30}$$

with $p, p_1 \in \Lambda$. Also the shift and Wilson lines have to satisfy $(6V, 3A_3, 3A_4, 2A_5, 2A_6) \subset \Lambda$. The momentum of a redefined state $\Phi_\gamma^{\rm BU} = \Phi_\gamma^{\rm orb}\Phi_1^c$ will be

$$P_{BU} = P_{sh} + cP_{sh}^1 \;\; = \;\; p + cp_1 + (k + ck_1)V + (n_\alpha + cm_\alpha)A_\alpha. \tag{5.31}$$

Thus, a necessary condition is [5]

$$\delta = (k + ck_1)V + (n_\alpha + cm_\alpha)A_\alpha \in \Lambda. \tag{5.32}$$

Then, the conjugacy class elements and the parameter c should satisfy

$$\begin{aligned} k + ck_1 &= 0 \mod 6, \\ n_3 + cm_3 + n_4 + cm_4 &= 0 \mod 3, \\ n_5 + cm_5 &= 0 \mod 2. \end{aligned} \tag{5.33}$$

where the condition for n_1, n_2, n_6 and m_1, m_2, m_6 are not present in the studied model in which the Wilson lines A_1, A_2 and A_6 vanish.

[5] This is not sufficient because the total P_{BU} should be in λ, thus summing $p + cp_1$ this condition may be satisfied.

Multiple fields redefinitions When more than one blow–up mode is employed in the redefinition, the procedure to follow is the same. We need to add a momentum that gives in total a vector of λ. Given the redefinition (5.29) we obtain for the blow–up state the momentum

$$\begin{aligned} P_{BU} &= P_{sh} + \sum_i c_i P^i_{sh} \\ &= p + \sum_i c_i p_i + \left(k + \sum_i c_i k_i\right) V + \left(n_\alpha + \sum_i c_i m^i_\alpha\right) A_\alpha. \end{aligned} \tag{5.34}$$

The sum (5.34) has to be in the lattice of $E_8 \times E_8$, which implies

$$\delta = (k + \sum_i c_i k_i) V + (n_\alpha + \sum_i c_i m^i_\alpha) A_\alpha \in \Lambda. \tag{5.35}$$

Then, redefinitions are restricted to have

$$\begin{aligned} (k + \sum_i c_i k_i) &= 0 \mod 6, \\ (n_3 + \sum_i c_i m^i_3 + n_4 + \sum_i c_i m^i_4) &= 0 \mod 3, \\ n_{5,6} + c m_{5,6} &= 0 \mod 2. \end{aligned} \tag{5.36}$$

In the $T^6/\mathbb{Z}_7$ case we allowed only for redefinitions of fields on the same fixed points. Here the situation is more complicated, because there are not only fixed points, but there are also fixed tori. In the resolved manifold occurs that exceptional divisors where blow-modes are localized have a compact intersection with other divisors in the manifold, such that every pair of exceptional divisors is connected. This fact motivates us to relax the redefinition conditions. Two examples of allowed redefinitions with 3 blow–up modes $\Phi_{(\theta^k,\alpha\beta\gamma)}$ are given by

$$\begin{aligned} \Phi^{\rm BU}_\gamma &= \Phi^{\rm orb}_{\gamma,111} \Phi^{-1}_{(\theta,113)} \Phi_{(\theta^5,114)} \Phi^{-1}_{(\theta^5,112)}, \\ \Phi^{\rm BU}_\gamma &= \Phi^{\rm orb}_{\gamma,111} \Phi^{-1}_{(\theta^2,121)} \Phi^{-1}_{(\theta^3,111)} \Phi^{-1}_{(\theta^4,131)}. \end{aligned} \tag{5.37}$$

The subindices on the blow–up modes denote the values of α, β and γ. That the redefinitions give a vector of Λ can be seen by checking the conjugacy classes in Table (F.13) of appendix F.

We choose to parametrize the redefinitions using the vector $(k_3, 3k_4 - k_3, 2k_5, 6m)$ which implies that a valid redefinition (5.35) is given by $\delta = (3k_4 A_{3,4} + 2k_5 A_5 + 6mV) \in \Lambda$, and this ensures that $P_{BU} \in \Lambda$.

For one and two blow–up modes we explore possible redefinitions with

$$-3 \le k_6 \le 3, \quad -3 \le k_4 \le 3, \quad -2 \le k_5 \le 2, \quad -1 \le m \le 1. \tag{5.38}$$

For three blow–up modes we explore possible redefinitions with

$$-6 \le k_3 \le 6, \quad -1 \le k_4 \le 1, \quad -2 \le k_5 \le 2, \quad -1 \le m \le 1. \tag{5.39}$$

Local multiplicities The multiplicity operator $\hat{N}$ (4.21) can be decomposed in a sum of terms $\hat{N} = \sum_r \hat{N}_r$, where $\hat{N}_r$ carries the index r from the exceptional divisor E_r. We explore here the ansatz of local multiplicity, to reduce ambiguity in the redefinition process. In the $\mathbb{Z}_7$ study, it turned out to be a powerful tool. Here things are different, because the θ^2, θ^4 and θ^3 twisted sectors, possess indices for its fixed tori, that do not appear in the sum at all. So, we don't impose a perfect agreement with local multiplicities. Rather in the search of a match, we test the interpretation that the $\hat{N}_r$ are contributions to $\hat{N}$ from orbifold twisted states localized around fixed sets.

The multiplicity of a given weight of $E_8 \times E_8$ denoted as w can be decomposed as

$$
\begin{aligned}
\hat{N}_{tot} &= \sum_{\beta\gamma} \hat{N}_{1,\beta\gamma}(\omega) + \sum_{\beta} \hat{N}_{2,\beta}(w) + \sum_{\gamma} \hat{N}_{3,\gamma}(\omega), \qquad (5.40)\\
\hat{N}_{1,\beta\gamma}(\omega) &= -V_{1\beta\gamma}.\omega((V_{21\beta}.w)^2 + (V_{41\beta}.\omega)^2 - (V_{21\beta}.w)(V_{41\beta}.\omega) - (V_{1\beta\gamma}.\omega)^2 + (V_{31\gamma}.\omega)^2),\\
\hat{N}_{2,\beta}(w) &= \frac{1}{3}\left(4(V_{2,1,\beta}.\omega)^3 + 4(V_{4,1,\beta}.\omega)^3 - V_{2,1,\beta}.\omega - V_{4,1,\beta}.\omega - 3(V_{2,1,\beta}.\omega)^2(V_{4,1\beta}.\omega)\right),\\
\hat{N}_{3,\gamma}(\omega) &= \frac{1}{3}\left(4(V_{31\gamma}.\omega)^3 - V_{31\gamma}.\omega\right).
\end{aligned}
$$

The multiplicity can also be written in terms of the multiplicities of the local $\mathbb{C}^3/\mathbb{Z}_{6II}$ resolutions as

$$
\hat{N}_{tot} = \sum_{\beta\gamma} N_{4d}(\widehat{\mathbb{C}^3/\mathbb{Z}_{6II}})|_{1\beta\gamma}, \qquad (5.41)
$$

with

$$
N_{4d}(\widehat{\mathbb{C}^3/\mathbb{Z}_{6II}})|_{1\beta\gamma} = \hat{N}_{1,\beta\gamma}(w) + \frac{1}{4}\hat{N}_{2,\beta}(w) + \frac{1}{3}\hat{N}_{3,\gamma}(w).
$$

This local multiplicity is the index of the Dirac operator in a compactification on a non-compact CY 3-fold $\widehat{\mathbb{C}^3/\mathbb{Z}_{6II}}$. Thus the decomposition of $\hat{N}_{tot}$ may give an indication of the identification of blow-up states with orbifold twisted states from the θ sector. However for fixed tori of $\theta^2, \theta^3, \theta^4$ there is no explicit dependence in $\hat{N}_{tot}$. In addition it is possible also to decompose $\hat{N}_{tot}$ in a sum of multiplicities for the different divisors, so that one can read from it extra information [128]. In the following we present a set of redefinitions obtained using 1,2 and 3 blow–up modes. We are able to reproduce the chiral asymmetry of the blow–up theory.

5.6 First orbifold–resolution spectrum match

In this section we describe an identification of the massless spectrum of the deformed orbifold by a vev configuration and the supergravity on the resolution. We search for field redefinitions that reproduce the chiral asymmetry of blow–up fermions. We explore for redefinitions using as a guideline the local multiplicities and we don't search for agreement with the orbifold superpotential mass terms. This last requirement will be explored in the next section.

Let us start with the field redefinitions for non-abelian charged fields. The charged matter under $SU(6)_{\text{hidden}}$ has representations $(\mathbf{1},\mathbf{1},\mathbf{6})$ and $(\mathbf{1},\mathbf{1},\bar{\mathbf{6}})$. The map can be seen in Table 5.3. We denote the blow–up states with charges in the first E_8 as I, in the second E_8 by II, and when they have zero multiplicity as III[6]. After obtaining the field redefinition we can check if it uses blow–up modes from the same fixed points.

Table 5.3: Orbifold-resolution identification for the $\mathbf{6}$ and $\bar{\mathbf{6}}$ representations of $SU(6)$.

Mult.	State blow–up	redef.	irrep.	irrep. blow–up
(-1,1)	$(\Phi_4^{II},\Phi_1^{II})$	$\psi_{182}\to\Phi_4^{II}$	$\mathbf{6}_r$	$(\mathbf{6},\bar{\mathbf{6}})$
(-1,1)	$(\Phi_2^{II},\Phi_6^{II})$	$\psi_2\equiv\Phi_2^{II}$	$\bar{\mathbf{6}}_r$	$(\bar{\mathbf{6}},\mathbf{6})$
(-1,1)	$(\Phi_9^{II},\Phi_{17}^{II})$	$\psi_9\equiv\Phi_9^{II}$, $\psi_{136},\psi_{142}\to\Phi_{17}^{II}$, $\psi_{137},\psi_{143}\to\Phi_9^{II}$	$\bar{\mathbf{6}}_r$	$(\bar{\mathbf{6}},\mathbf{6})$
(-1,1)	$(\Phi_{19}^{II},\Phi_{10}^{II})$	$\psi_{164}\to\Phi_{19}^{II}$, $\psi_{157}\to\Phi_{19}^{II}$, $\psi_{102}\to\Phi_{10}^{II}$	$\mathbf{6}_r,\mathbf{6}_r,\bar{\mathbf{6}}_r$	$(\mathbf{6},\bar{\mathbf{6}})$
(-2,2)	$(\Phi_{14}^{II},\Phi_{11}^{II})$	$(\psi_{106},\psi_{117})\to\Phi_{14}^{II}$	$\bar{\mathbf{6}}_r$	$(\bar{\mathbf{6}},\mathbf{6})$
(-2,2)	$(\Phi_{13}^{II},\Phi_{15}^{II})$	$(\psi_{86},\psi_{83})\to\Phi_{13}^{II}$	$\mathbf{6}_r$	$(\mathbf{6},\bar{\mathbf{6}})$

We have also explored at the mass terms coming from Yukawa couplings with blow–up modes. We don't consider higher order terms in the super-potential, because they are suppressed by M_s in comparison to the ones of order three, in addition we do not have access to those interactions in the smooth CY. The mass terms for right movers ψ_i and CPT conjugates $\bar{\psi}_j$ are

$$\psi_9\psi_{136}\langle\psi_{141}\rangle+\psi_9\psi_{142}\langle\psi_{147}\rangle\equiv\bar{\psi}_2\bar{\psi}_{65}\langle\bar{\psi}_{68}\rangle+\bar{\psi}_2\bar{\psi}_{71}\langle\bar{\psi}_{74}\rangle. \tag{5.42}$$

These superpotential terms are computed with the Orbifolder program [127] using the classical orbifold selection rules. The result (5.42) agrees with the fact that from the fields ψ_9,ψ_{136} and ψ_{142} there should be a massive pair. Furthermore away from the orbifold point another two pairs form to give a net field Φ_9^{II}.

Let us describe next the doublets redefinitions. In Table (5.4) we can read the blow–up state, its multiplicities and the redefinition to an orbifold field. This set of redefinitions matches the chiral asymmetry, but agreement of the masses requires to modify it.

The mass terms arising at tree level are given by

$$\begin{aligned}&\psi_{11}\psi_{178}\langle\psi_{118}\rangle+\psi_{158}\psi_{31}\langle\psi_{28}\rangle+\psi_{178}\psi_{31}\langle\psi_{28}\rangle+\\+\quad&\psi_{158}\psi_{37}\langle\psi_{34}\rangle+\psi_{178}\psi_{37}\langle\psi_{34}\rangle+\psi_{11}\psi_{175}\langle\psi_{90}\rangle.\end{aligned} \tag{5.43}$$

These are the trilinear couplings agreeing with orbifold selection rules.

A set of redefinitions consistent with the previous mass terms is given in Table (5.12), in order to obtain it we have to relax the guideline of the local multiplicity.

[6]Only the states charged under the surviving gauge symmetries in the first E_8 can have zero multiplicity.

Table 5.4: Orbifold–resolution map for doublets obtained using as guide local multiplicities.

Mult.	State blow–up	redef.	irrep.
(-2,2)	$(\Phi^I_{10}, \Phi^I_{33})$	$(\psi_{158}, \psi_{178}) \to \Phi^I_{10}$	$(\mathbf{1}, \mathbf{2}, \mathbf{1})_r$
(-2,2)	$(\Phi^I_{16}, \Phi^I_{40})$	$(\psi_{89}, \psi_{111}) \to \Phi^I_{40}$	$(\mathbf{1}, \mathbf{2}, \mathbf{1})_r$
(-2,2)	$(\Phi^I_{17}, \Phi^I_{41})$	$(\psi_{42}, \psi_{39}) \to \Phi^I_{41}$	$(\mathbf{1}, \mathbf{2}, \mathbf{1})_r$
(-1,1)	$(\Phi^I_{19}, \Phi^I_{29})$	$\psi_8 \equiv \Phi^I_{19}$	$(\mathbf{1}, \mathbf{2}, \mathbf{1})_r$
(-1,1)	$(\Phi^I_{21}, \Phi^I_{26})$	$\psi_4 \equiv \Phi^I_{26}$	$(\mathbf{1}, \mathbf{2}, \mathbf{1})_r$
(-1,1)	$(\Phi^I_{23}, \Phi^I_{32})$	$\psi_{10} \equiv \Phi^I_{23}$	$(\mathbf{1}, \mathbf{2}, \mathbf{1})_r$
(0,0)	$(\Phi^{II}_{4}, \Phi^{II}_{20})$	$(\psi_{61}, \psi_{49}) \to \Phi^{II}_4$, $(\psi_{37}, \psi_{31}) \to \Phi^{II}_{20}$	$(\mathbf{1}, \mathbf{2}, \mathbf{1})_r$
(0,0)	$(\Phi^{II}_{29}, \Phi^{II}_{31})$	$\psi_{11} \equiv \Phi^{II}_{31}, \psi_{12} \equiv \Phi^{II}_{29}$, $\psi_{108} \to \Phi^{II}_{31}, \psi_{175} \to \Phi^{II}_{29}$, $\psi_{24} \to \Phi^{II}_{31}, \psi_{17} \to \Phi^{II}_{29}$	$(\mathbf{1}, \mathbf{2}, \mathbf{1})_r$

Table 5.5: Doublets redefinition with correct orbifold mass terms.

Mult.	State blow–up	redef.	irrep.
(-2,2)	$(\Phi^I_{10}, \Phi^I_{33})$	$(\psi_{61}, \psi_{49}) \to \Phi^I_{10}$	$(\mathbf{1}, \mathbf{2}, \mathbf{1})_r$
(-2,2)	$(\Phi^I_{16}, \Phi^I_{40})$	$(\psi_{89}, \psi_{111}) \to \Phi^I_{40}$	$(\mathbf{1}, \mathbf{2}, \mathbf{1})_r$
(-2,2)	$(\Phi^I_{17}, \Phi^I_{41})$	$(\psi_{42}, \psi_{39}) \to \Phi^I_{41}$	$(\mathbf{1}, \mathbf{2}, \mathbf{1})_r$
(-1,1)	$(\Phi^I_{19}, \Phi^I_{29})$	$\psi_8 \equiv \Phi^I_{19}$	$(\mathbf{1}, \mathbf{2}, \mathbf{1})_r$
(-1,1)	$(\Phi^I_{21}, \Phi^I_{26})$	$\psi_4 \equiv \Phi^I_{26}$	$(\mathbf{1}, \mathbf{2}, \mathbf{1})_r$
(-1,1)	$(\Phi^I_{23}, \Phi^I_{32})$	$\psi_{10} \equiv \Phi^I_{23}$	$(\mathbf{1}, \mathbf{2}, \mathbf{1})_r$
(0,0)	$(\Phi^{II}_{29}, \Phi^{II}_{31})$	$\psi_{11} \equiv \Phi^{II}_{31}, \psi_{12} \equiv \Phi^{II}_{29}$, $\psi_{108} \to \Phi^{II}_{31}, \psi_{175} \to \Phi^{II}_{29}$, $\psi_{24}, (\psi_{31}, \psi_{37}) \to \Phi^{II}_{31}, \psi_{17}, (\psi_{158}, \psi_{178}) \to \Phi^{II}_{29}$	$(\mathbf{1}, \mathbf{2}, \mathbf{1})_r$

The redefinitions for triplets and anti–triplets are given in Table 5.11. For the states represented here, there is a perfect agreement with the intuition of the local multiplicity. However, the difference between the states mapped to Φ_x and $\bar{\Phi}_x$ determines the number of chiral states in blow–up. But we write in the first table the representatives with correct local multiplicities. The set in Table 5.7 will give a full rank mass matrix because conjugate

Table 5.6: Orbifold–resolution map for triplets.

Mult.	State blow–up	irrep.BU	redef.	irrep.
(-3,3)	(Φ^I_4, Φ^I_{47})	$(\mathbf{3}, \bar{\mathbf{3}})$	$(\psi_{121}, \psi_{129}) \to \Phi^I_4$, $(\psi_6)_r \equiv \Phi^I_4$	$\mathbf{3}_r$
(-2,2)	(Φ^I_7, Φ^I_{36})	$(\mathbf{3}, \bar{\mathbf{3}})$	$(\psi_{135}, \psi_{127}) \to \Phi^I_{36}$	$\bar{\mathbf{3}}_r$
(-2,2)	(Φ^I_8, Φ^I_{35})	$(\bar{\mathbf{3}}, \mathbf{3})$	$(\psi_{23}, \psi_{16}) \to \Phi^I_{35}$	$\mathbf{3}_r$
(-1,1)	$(\Phi^I_{12}, \Phi^I_{46})$	$(\mathbf{3}, \bar{\mathbf{3}})$	$(\psi_{58}, \psi_{46}) \to \Phi^I_{12}$	$\mathbf{3}_r$
(-1,1)	$(\Phi^I_{25}, \Phi^I_{31})$	$(\mathbf{3}, \bar{\mathbf{3}})$	$\psi_{99} \to \Phi^I_{25}$	$\mathbf{3}_r$

states are paired up. We postpone a detailed analysis of the orbifold superpotential mass terms for triplets to next section. To find an agreement with those terms leads to a different set of redefinitions.

Table 5.7: Orbifold–resolution map for triplets. States with total multiplicty zero.

Mult.	States blow–up	irrep.BU	redef.	irrep.
0	$(\Phi^{II}_5, \Phi^{II}_{21})$	$(\mathbf{3}, \bar{\mathbf{3}})$	$(\psi_{62}, \psi_{50}) \to \Phi^{II}_5$, $(\psi_{152}, \psi_{149}) \to \Phi^{II}_{21}$	$\mathbf{3}_r, \bar{\mathbf{3}}_r$
0	$(\Phi^{II}_{30}, \Phi^{II}_{32})$	$(\mathbf{3}, \bar{\mathbf{3}})$	$(\psi_{116}, \psi_{105}) \to \Phi^{II}_{30}$, $(\psi_{36}, \psi_{30}) \to \Phi^{II}_{32}$	$\mathbf{3}_r, \bar{\mathbf{3}}_r$
0	$(\Phi^{II}_{34}, \Phi^{II}_{36})$	$(\mathbf{3}, \bar{\mathbf{3}})$	$(\psi_{112}, \psi_{92}) \to \Phi^{II}_{34}$, $\psi_{100} \to \Phi^{II}_{36}$, $\psi_{94} \to \Phi^{II}_{36}$	$\mathbf{3}_r, \bar{\mathbf{3}}_r, \bar{\mathbf{3}}_r$
(-2,2)	$(\Phi^{I}_8, \Phi^{I}_{35})$	$(\bar{\mathbf{3}}, \mathbf{3})$	$(\psi_{20}, \psi_{13}) \to \Phi^{I}_8$, $(\psi_{185}, \psi_{169}) \to \Phi^{I}_8$, $(\psi_{151}, \psi_{132}, \psi_{148}, \psi_{124}) \to \Phi^{I}_{35}$	$\bar{\mathbf{3}}_r, \bar{\mathbf{3}}_r, \mathbf{3}_r$
(-3,3)	$(\Phi^{I}_4, \Phi^{I}_{47})$	$(\mathbf{3}, \bar{\mathbf{3}})$	$(\psi_{188}, \psi_{172}) \to \Phi^{I}_4$, $\psi_{174} \to \Phi^{I}_{47}$, $\psi_{161} \to \Phi^{I}_{47}$	$\mathbf{3}, \bar{\mathbf{3}}_r, \bar{\mathbf{3}}_r$
(-1,1)	$(\Phi^{I}_{25}, \Phi^{I}_{31})$	$(\mathbf{3}, \bar{\mathbf{3}})$	$(\psi_{184}, \psi_{166}) \to \Phi^{I}_{25}$, $(\psi_{125}, \psi_{133}) \to \Phi^{I}_{31}$	$\mathbf{3}_r, \bar{\mathbf{3}}_r$

Table 5.8: Triplets identification.

Mult.	Blow–up state	irrep.BU	redef.	irrep.
(-3,3)	$(\Phi^{I}_4, \Phi^{I}_{47})$	$(\mathbf{3}, \bar{\mathbf{3}})$	$(\psi_{121}, \psi_{129}) \to \Phi^{I}_4$, $(\psi_6)_r \equiv \Phi^{I}_4$	$\mathbf{3}_r$
(-2,2)	$(\Phi^{I}_7, \Phi^{I}_{36})$	$(\mathbf{3}, \bar{\mathbf{3}})$	$(\psi_{135}, \psi_{127}) \to \Phi^{I}_{36}$	$\bar{\mathbf{3}}_r$
(-2,2)	$(\Phi^{I}_8, \Phi^{I}_{35})$	$(\bar{\mathbf{3}}, \mathbf{3})$	$(\psi_{23}, \psi_{16}) \to \Phi^{I}_{35}$	$\mathbf{3}_r$
(-1,1)	$(\Phi^{I}_{12}, \Phi^{I}_{46})$	$(\mathbf{3}, \bar{\mathbf{3}})$	$(\psi_{58}, \psi_{46}) \to \Phi^{I}_{12}$	$\mathbf{3}_r$
(-1,1)	$(\Phi^{I}_{25}, \Phi^{I}_{31})$	$(\mathbf{3}, \bar{\mathbf{3}})$	$\psi_{99} \to \Phi^{I}_{25}$	$\mathbf{3}_r$

Now lets describe the match of the $(\mathbf{3}, \mathbf{2}, \mathbf{1})$ and $(\bar{\mathbf{3}}, \mathbf{2}, \mathbf{1})$ states in Table 5.9. The states with multiplicities different from zero also have local multiplicities compatible with the localization of the identified orbifold states. For example, the states (ψ_{48}, ψ_{60}) are located in the θ sector fixed points $(\beta, \gamma) = (1, 1), (1, 3)$ respectively. This agrees with the local multiplicities of Φ^{I}_{11} which are $N(\Phi^{I}_{11})_{1,1,1} = N(\Phi^{I}_{11})_{1,1,3} = -1$, and all the others are ~ 0. The same happens for the state ψ_{189} of θ^4 and fixed torus $(\alpha, \beta) = (1, 2)$. This state is mapped finally to Φ^{I}_{20} with local multiplicity $N(\Phi^{I}_{20})_{\beta=2} = -1$ and otherwise ~ 0. Let us mention that according to the orbifold selection rules, up to 5-point couplings there are no other mass terms appearing.

Table 5.9: Orbifold–resolution map for the $(\mathbf{3}, \mathbf{2}, \mathbf{1})$ representation.

Multip.	Blow–up state	Redefinition	irrep.
-2	$(\Phi^{I}_{11}, \Phi^{I}_{45})$	$(\psi_{48}, \psi_{60}) \to \Phi^{I}_{11}$	$(\bar{\mathbf{3}}, \mathbf{2}, \mathbf{1})$
-1	$(\Phi^{I}_{20}, \Phi^{I}_{28})$	$\psi_{189} \to \Phi^{I}_{20}$	$(\bar{\mathbf{3}}, \mathbf{2}, \mathbf{1})$
0	$(\Phi^{II}_{13}, \Phi^{II}_{10})$	$\psi_{93} \to \Phi^{II}_{13}$, $\psi_{173} \to \Phi^{II}_{10}$	$(\mathbf{3}, \mathbf{2}, \mathbf{1})$, $(\bar{\mathbf{3}}, \mathbf{2}, \mathbf{1})$

A map matching the singlets is given in Table 5.10. Here the interpretation in terms of the local multiplicity has more failures. This can be checked by using the redefinitions in appendix K and computing the local multiplicities from (5.40).

The results of this section show that is possible to implement field redefinitions such that the resulting spectrum in the orbifold deformed by vevs matches the chiral asymmetry of the supergravity on the resolution. As was done for the six, the doublets and the $(\mathbf{3}, \mathbf{2}, \mathbf{1})$, we need to identify which are the triplets and singlets fields which get masses from Yukawa

Table 5.10: Singlets identification.

Mult.	States blow–up	redef.
E_8^1 spectrum I		
$(-4,4)$	(Φ^I_1,Φ^I_{48})	$(\psi_{63},\psi_{66})\to\Phi^I_1$, $(\psi_{51},\psi_{54})\to\Phi^I_1$, $\psi_{163}\to\Phi^I_{48}$, $\psi_{160}\to\Phi^I_1$
$(-4,4)$	(Φ^I_2,Φ^I_{49})	$(\psi_{64},\psi_{52})\to\Phi^I_2$, $(\psi_{81},\psi_{75})\to\Phi^I_2$
$(-2,2)$	(Φ^I_5,Φ^I_{38})	$\psi_{65}\to\Phi^I_5$, $\psi_{55}\to\Phi^I_5$
$(-2,2)$	(Φ^I_6,Φ^I_{37})	$(\psi_{121},\psi_{14})\to\Phi^I_6$
$(-2,2)$	(Φ^I_9,Φ^I_{34})	$(\psi_{82},\psi_{74})\to\Phi^I_9$
$(-2,2)$	$(\Phi^I_{13},\Phi^I_{43})$	$(\psi_{87},\psi_{84})\to\Phi^I_{13}$
$(-2,2)$	$(\Phi^I_{14},\Phi^I_{44})$	$(\psi_{25},\psi_{18})\to\Phi^I_{14}$
$(-2,2)$	$(\Phi^I_{15},\Phi^I_{39})$	$(\psi_{130},\psi_{122}),\psi_{71}\to\Phi^I_{15}$, $\psi_{73}\to\Phi^I_{39}$
$(-4,4)$	(Φ^I_3,Φ^I_{50})	$\psi_1\equiv\Phi^I_3$, $\psi_{177}\to\Phi^I_3$, $\psi_{190}\to\Phi^I_3$, $\psi_{80}\to\Phi^I_3$
$(-2,2)$	$(\Phi^I_{18},\Phi^I_{42})$	$(\psi_{79},\psi_{74})\to\Phi^I_{42}$
$(-1,1)$	$(\Phi^I_{24},\Phi^I_{30})$	$\psi_{171}\to\Phi^I_{24}$, $(\psi_{33},\psi_{27})\to\Phi^I_{24}$, $(\psi_{26},\psi_{19})\to\Phi^I_{30}$, $\psi_{95},\psi_{96},\psi_{128},\psi_{167}\to\Phi^I_{24}$, $\psi_{91},\psi_{104},\psi_{120},\psi_{180}\to\Phi^I_{30}$
$(-1,1)$	$(\Phi^I_{22},\Phi^I_{27})$	$\psi_7\equiv\Phi^I_{22}$
E_8^2 spectrum II		
$(2,-2)$	$(\Phi^{II}_5,\Phi^{II}_7)$	$\psi_3\equiv\Phi^{II}_5$, $(\psi_{38},\psi_{32})\to\Phi^{II}_7$, $\psi_{113}\to\Phi^{II}_7$
$(-2,2)$	$(\Phi^{II}_3,\Phi^{II}_8)$	$(\psi_{76},\psi_{69})\to\Phi^{II}_3$
$(4,-4)$	$(\Phi^{II}_{12},\Phi^{II}_{20})$	$(\psi_{53},\psi_{67})\to\Phi^{II}_{12}$, $(\psi_{145},\psi_{139})\to\Phi^{II}_{12}$
$(4,-4)$	$(\Phi^{II}_{16},\Phi^{II}_{18})$	$(\psi_{43},\psi_{40})\equiv\Phi^{II}_{18}$, $\psi_{119}\to\Phi^{II}_{18}$, $\psi_{181}\to\Phi^{II}_{18}$, $(\psi_{47},\psi_{59})\to\Phi^{II}_{16}$, $(\psi_{109},\psi_{110})\to\Phi^{II}_{18}$
Non-chiral spectrum III		
$(0,0)$	$(\Phi^{III}_1,\Phi^I_{17})$	$\psi_5\equiv\Phi^{III}_{17},(\psi_{56},\psi_{68})\to\Phi^{III}_1$, $\psi_{114}\to\Phi^{III}_{17},\psi_{156}\to\Phi^{III}_1$, $\psi_{162}\to\Phi^{III}_{17}$, $\psi_{168},\psi_{176},\psi_{107}\to\Phi^{III}_1$, $\psi_{144},\psi_{138},\psi_{101}\to\Phi^{III}_{17}$
$(0,0)$	$(\Phi^I_{24},\Phi^I_{40})$	$\psi_{78},\psi_{98},\psi_{131},\psi_{146}\to\Phi^I_{24}$, $\psi_{35},\psi_{29},\psi_{123},\psi_{140}\to\Phi^I_{40}$

couplings with blow–up modes. Then we will explore which redefinition makes them non–chiral on the blow–up. Unfortunately the first set of redefinitions that we found can not accommodate those mass terms.

The fields redefinitions usually involve blow–up modes from different fixed sets as the orbifold field. It also occurs that only the local blow–up modes take part, but those are few cases. Nevertheless due to the complicated topology of the $\mathbb{Z}_{6II}$ orbifold and its resolution, this was expectable. In particular, as already mentioned in the resolved manifold occurs that exceptional divisors where blow-modes are localized have a compact intersection with other divisors in the manifold, such that every pair of exceptional divisors is connected.[7] It is an interesting question that we have not answered yet, if a suitable local multiplicity operator (for example in terms of 6d index theorems obtained integrating on the divisors) can see the blow–up modes employed in the redefinition.

5.7 Second spectrum match: Agreement with superpotential masses

In this section we analyze the mass terms for triplets and singlets generated by Yukawa couplings to blow–up modes, present in the orbifold superpotential W. We have to choose a different map to the one given in the previous section, if we want to match the spectrum. To accomplishing that, we give up on the interpretation of local multiplicities.

The triplets The orbifold-resolution map is summarized in Table 5.11. This map is compatible with the orbifold superpotential mass terms. In the Appendix K we explicitly give a set of redefinitions which realize it. We start by listing the mass terms in which triplets and blow-up modes are involved. The Yukawa couplings coefficients will be denoted by a_i, b_i, e_i, f_i and g_i, they have the same letter and index if they are equal. [8] A set of mass terms is given by

$$\begin{pmatrix} \psi_6 \\ \psi_{112} \\ \psi_{92} \\ \psi_{116} \\ \psi_{105} \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} 0 & 0 & a_1\langle\psi_{126}\rangle & a_2\langle\psi_{134}\rangle & a_3\langle\psi_{150}\rangle & a_4\langle\psi_{153}\rangle \\ 0 & 0 & a_5\langle\psi_{70}\rangle & a_5\langle\psi_{77}\rangle & a_6\langle\psi_{70}\rangle & a_6\langle\psi_{77}\rangle \\ 0 & 0 & a_7\langle\psi_{70}\rangle & a_7\langle\psi_{77}\rangle & a_8\langle\psi_{70}\rangle & a_8\langle\psi_{77}\rangle \\ e_1\langle\psi_{154}\rangle & e_1\langle\psi_{155}\rangle & e_2\langle\psi_{15}\rangle & e_2\langle\psi_{22}\rangle & e_1\langle\psi_{15}\rangle & e_1\langle\psi_{22}\rangle \\ e_3\langle\psi_{154}\rangle & e_3\langle\psi_{155}\rangle & e_4\langle\psi_{15}\rangle & e_4\langle\psi_{22}\rangle & e_3\langle\psi_{15}\rangle & e_3\langle\psi_{22}\rangle \end{pmatrix} \begin{pmatrix} \psi_{30} \\ \psi_{36} \\ \psi_{125} \\ \psi_{133} \\ \psi_{149} \\ \psi_{152} \end{pmatrix}. \tag{5.44}$$

[7] For example it is possible to go from the divisor $E_{1,34}$ to $E_{2,31}$ by first going to the compact curve $E_{1,34}D_{3,4}$ and then from $D_{3,4}$ to the compact curve $E_{2,31}D_{3,4}$.

[8] The Yukawa couplings [129,130] depend on the fixed points, the sectors and the fixed point degeneracy, so it is possible without calculating them to establish when they must be equal. However it could be that there are more equal coefficients than expected. This could happen for particular values of the orbifold moduli. So the rank of the mass matrices we give is a maximal bound.

The fields in the vector to the left are triplets and the ones in the vector to the right are the anti-triplets. This mass matrix has generically rank 5. The field ψ_6 is untwisted and its charges are exactly identified with Φ_4^I. We also redefine to Φ_4^I the remaning triplets ψ_{112}, ψ_{92}, ψ_{105} and ψ_{116} . If we want that the map transforms orbifold mass terms into blow-up mass terms we need to redefine conjugated pairs in (5.44) into conjugated pairs in blow-up. It is possible to perform a unitary transformation on the anti-triplets eliminating the last column obtaining one massless eigenstate. We can adjust the redefinitions to have all the triplets and anti-triplets in a given mass term redefined to conjugate pairs on the blow-up side. So we take ψ_{30}, ψ_{36}, ψ_{125}, ψ_{133}, ψ_{149} and ψ_{152} to $\bar{\Phi}_4^I$. Finally mapping ψ_{121}, ψ_{129}, ψ_{188} and ψ_{172} also to Φ_4^I a total of 3 massless states Φ_4^I is obtained in the CY.

Let us analyze now another set of states. The following masses agree easily with redefinitions

$$(\psi_{62}\,\psi_{50})\begin{pmatrix} b_1\langle\psi_{157}\rangle & b_2\langle\psi_{157}\rangle \\ b_1\langle\psi_{45}\rangle & b_2\langle\psi_{45}\rangle \end{pmatrix}\begin{pmatrix}\psi_{169}\\ \psi_{185}\end{pmatrix}. \tag{5.45}$$

The mass matrix has rank 1. Due to that there are two massless eigenstates in the orbifold formed with ψ_{62}, ψ_{50} and ψ_{169}, ψ_{185}. With the identifications $(\psi_{62}, \psi_{50}) \rightarrow \Phi_{16}^{III}$ and $(\psi_{169}, \psi_{185}) \rightarrow \bar{\Phi}_{16}^{III}$ we get a net zero number of blow-up states Φ_{16}^{III}. There are two orbifold massive linear combinations appearing in (5.45) that are conjugated pairs $\Phi_{16}^{III}\bar{\Phi}_{16}^{III}$. But also the two massless eigenstates from the orbifold perspective constitute a pair $\Phi_{16}^{III}\bar{\Phi}_{16}^{III}$ in blow-up.

The remaining mass terms are

$$\psi_{20}(\psi_{151}\langle\bar{\psi}_{106}\rangle + \psi_{132}\langle\bar{\psi}_{106}\rangle + \psi_{23}(\langle\bar{\psi}_{45}\rangle + \langle\psi_{159}\rangle)), \tag{5.46}$$

$$\psi_{13}(\psi_{148}\langle\bar{\psi}_{106}\rangle + \psi_{124}\langle\bar{\psi}_{106}\rangle + \psi_{16}(\langle\bar{\psi}_{45}\rangle + \langle\psi_{159}\rangle)). \tag{5.47}$$

The fields on them are redefined as shown in Table 5.11. In (5.46) and (5.47) we have omitted the Yukawa coefficients because the rank of both 1×3 mass matrices is clearly 1.

The exploration criterium was to search for a map that matches the spectrum. We give the redefinitions in Appendix K. The map transforms massive conjugated pairs to conjugated blow-up pairs. The redefined massive modes give a zero chiral asymmetry for a given blow-up state. In addition there are massless states from the orbifold superpotential perspective that are redefined to conjugated pairs in blow-up.

Let us conclude with the overall picture. In the orbifold there are 16 $(\mathbf{3},\mathbf{1},\mathbf{1})$ and 22 $(\bar{\mathbf{3}},\mathbf{1},\mathbf{1})$, whereas in blow-up there are 2 triplets and 8 anti-triplets. The redefinitions performed give a map in which 14 massive vector pairs are created and the chiral asymmetry of the Calabi-Yau compactification is reproduced.

The doublets Some of the mass terms arising from Yukawa couplings are

$$(\psi_{11}\,\psi_{31}\,\psi_{37})\begin{pmatrix} f_1\langle\psi_{118}\rangle & f_2\langle\psi_{90}\rangle & 0 \\ f_3\langle\psi_{28}\rangle & 0 & f_4\langle\psi_{28}\rangle \\ f_3\langle\psi_{34}\rangle & 0 & f_4\langle\psi_{34}\rangle \end{pmatrix}\begin{pmatrix}\psi_{178}\\ \psi_{175}\\ \psi_{158}\end{pmatrix} \tag{5.48}$$

Mult.	State blow-up	irrep.	redef.
-3	Φ_4^I	$\mathbf{3}$	$\psi_6 \equiv \Phi_4^I$ $\psi_{116}, \psi_{105}, \psi_{112}, \psi_{92} \to \Phi_4^I$ $\psi_{30},\ \psi_{36},\ \psi_{125},\ \psi_{133},\ \psi_{149},\ \psi_{152} \to \bar{\Phi}_4^I$ $(\psi_{121},\ \psi_{129}),\ (\psi_{188}, \psi_{172}) \to \Phi_4^I,$
-2	Φ_7^I	$\mathbf{3}$	$(\psi_{135}, \psi_{127}) \to \bar{\Phi}_7^I$
-2	Φ_8^I	$\bar{\mathbf{3}}$	$\psi_{20} \to \Phi_8^I,\ \psi_{151}, \psi_{132}, \psi_{23} \to \bar{\Phi}_8^I$
-2	Φ_{12}^I	$\mathbf{3}$	$(\psi_{58}, \psi_{46}) \to \Phi_{12}^I$
-1	Φ_{25}^I	$\mathbf{3}$	$\psi_{184}, \psi_{166},\ \psi_{99} \to \Phi_{25}^I$ $\psi_{100}, \psi_{174} \to \bar{\Phi}_{25}^I$
0	Φ_{16}^{III}	$\mathbf{3}$	$(\psi_{62}, \psi_{50}) \to \Phi_{16}^{III},\ (\psi_{169}, \psi_{185}) \to \bar{\Phi}_{16}^{III}$
0	Φ_{32}^{III}	$\bar{\mathbf{3}}$	$\psi_{148}, \psi_{124}, \psi_{16} \to \bar{\Phi}_{32}^{III}$ $\psi_{13},\ \psi_{94},\ \psi_{161} \to \Phi_{32}^{III}$

Table 5.11: Triplets identification in agreement with superpotential mass terms.

Mult.	State blow-up	redef.	irrep.
-2	Φ_{10}^I	$(\psi_{61}, \psi_{49}) \to \Phi_{10}^I$	$(\mathbf{1}, \mathbf{2}, \mathbf{1})$
-2	Φ_{16}^I	$\psi_{24},\ \psi_{108} \to \bar{\Phi}_{16}^I$	$(\mathbf{1}, \mathbf{2}, \mathbf{1})$
-2	Φ_{17}^I	$(\psi_{42}, \psi_{39}) \to \bar{\Phi}_{17}^I$	$(\mathbf{1}, \mathbf{2}, \mathbf{1})$
-1	Φ_{19}^I	$\psi_8 \equiv \Phi_{19}^I$	$(\mathbf{1}, \mathbf{2}, \mathbf{1})$
-1	Φ_{21}^I	$\psi_4 \equiv \bar{\Phi}_{21}^I$	$(\mathbf{1}, \mathbf{2}, \mathbf{1})$
-1	Φ_{23}^I	$\psi_{10} \equiv \Phi_{23}^I$	$(\mathbf{1}, \mathbf{2}, \mathbf{1})$
0	Φ_{29}^{III}	$\psi_{11} \equiv \bar{\Phi}_{29}^{III}$ $,\psi_{89},\ \psi_{111},\ (\psi_{31}, \psi_{37}) \to \bar{\Phi}_{29}^{III}$ $\psi_{12} \equiv \Phi_{29}^{III},\ \psi_{175},\ \psi_{17}, (\psi_{158}, \psi_{178}) \to \Phi_{29}^{III}$	$(\mathbf{1}, \mathbf{2}, \mathbf{1})$

Table 5.12: Doublets redefinition with correct orbifold mass terms.

The rank of the mass matrix (5.48) is 2. The remaining mass terms involving doublets are given by

$$\psi_{12}(\psi_{89}\langle\psi_{171}\rangle + \psi_{111}\langle\psi_{187}\rangle). \tag{5.49}$$

We omitted the Yukawa coupling coefficients because the rank of the 1×2 mass matrix is clearly 1. A set of redefinitions consistent with all given mass terms is given in Table 5.12. The untwisted field ψ_{12} is identified with Φ_{29}^{III}. The fields ψ_{89} and ψ_{111} are mapped to $\bar{\Phi}_{29}^{III}$. They form a massive linear combination and a massless one. In the orbifold there are 19 doublets and 10 of them form conjugated pairs in blow-up giving a total of 9 massless chiral fields.

The singlets At the orbifold all the untwisted singlets are massless and they only take part in Yukawa couplings with doublets. The twisted singlets instead have various mass terms coming from Yukawa couplings to blow-up modes. Those are

$$\psi_{160}\left(\psi_{84}\langle\psi_{45}\rangle g_1 + \psi_{87}\langle\psi_{57}\rangle g_1 + \psi_{27}\langle\psi_{28}\rangle g_2 + \psi_{33}\langle\psi_{34}\rangle g_2\right), \tag{5.50}$$
$$\psi_{180}(\psi_{84}\langle\psi_{45}\rangle g_3 + \psi_{87}\langle\psi_{57}\rangle g_3 + \psi_{27}\langle\psi_{28}\rangle g_4 + \psi_{33}\langle\psi_{34}\rangle g_4), \tag{5.51}$$
$$\psi_{40}(\psi_{114}(\langle\psi_{134}\rangle + \langle\psi_{153}\rangle) + \psi_{14}\langle\psi_{187}\rangle), \tag{5.52}$$
$$\psi_{43}(\psi_{114}(\langle\psi_{126}\rangle + \langle\psi_{150}\rangle) + \psi_{21}\langle\psi_{187}\rangle), \tag{5.53}$$
$$\psi_{35}(\psi_{146}\langle\bar{\psi}_{106}\rangle + \psi_{107}\langle\psi_{155}\rangle), \tag{5.54}$$
$$\psi_{29}(\psi_{140}\langle\bar{\psi}_{106}\rangle + \psi_{107}\langle\psi_{154}\rangle). \tag{5.55}$$

We only wrote explicitly the Yukawa coupling coefficients in (5.50) and (5.51) as g_i to illustrate that the mass matrix formed with those equations has rank 2. It is easy to check by looking at Table 5.13 that the identifications agree with the mass terms of the orbifold superpotential.

There is an ingredient not shown in the map presented so far. In the superpotential there are Yukawa couplings in which two blow-up modes are involved. We have checked up to trilinear order that the vevs can be assigned while ensuring F-flat vacua. In addition, only a pair of twisted singlets written as massless in the map of the Table 5.13 becomes massive due to those trilinear couplings. The map given above can be slightly modified to also reproduce the CY chiral asymmetry.[9] The number of singlets in the orbifold is 114 out of which 74 are redefined to conjugated states forming blow-up massive pairs, to give 40 massless states in blow-up.

This completes the matching of the heterotic string massless spectrum in the deformed orbifold and in the CY. At the level of the massless spectrum, the geometric resolution with abelian vector bundle constitutes a blow-up of the MSSM Mini-landscape Model 28, in which the twisted singlets in Table 5.2 are identified as the blow-up modes.

The field redefinitions in Appendix K usually involve blow-up modes from different fixed sets than those of the orbifold twisted fields. Although it also occurs that only the local blow-up modes take part in the redefinition. Due to the topology of the $T^6/\mathbb{Z}_{6II}$ orbifold and its resolution this was expectable.

5.8 Anomalies in the orbifold and its resolution

Choosing a basis in which the abelian factor $U(1)^8$ is explicit, we can express the anomaly polynomials in terms of it. In the blow–up model the $U(1) \times SU(6)^2$ anomalies cancel. We checked that the dimensionally reduced polynomial coincides with the one computed from the supergravity 4d spectrum. Details of the anomaly polynomials are given in this section. We write explicitily the anomaly polynomials of the orbifold (orb), blow–up (bu) and the polynomial variation due to field redefinitions (red). We use the symbols

[9] The fields ψ_{72}, ψ_{79} are the ones becoming massive due to the trilinear couplings with blow-up modes. The change in the map is to make $\psi_{95}, \psi_{96} \to \Phi^I_{18}$ via the redefinition $V_{122} - V_{312}$.

Mult.	States blow-up	redef.
E_8^1 spectrum I		
-4	Φ_1^I	$(\psi_{63},\psi_{66}) \to \Phi_1^I,\ (\psi_{51},\psi_{54}) \to \Phi_1^I$
-4	Φ_2^I	$(\psi_{64},\psi_{52}) \to \Phi_2^I,\ (\psi_{81},\psi_{75}) \to \Phi_2^I$
-2	Φ_5^I	$\psi_{123},\psi_{35},\psi_{29},\psi_{65},\psi_{55} \to \Phi_5^I,$ $\psi_{140},\psi_{146},\psi_{107} \to \bar{\Phi}_5^I$
-2	Φ_6^I	$\psi_{98},\psi_{114},\ (\psi_{21},\psi_{14}) \to \Phi_6^I,\ \psi_{40},\psi_{43} \to \bar{\Phi}_6^I$
-2	Φ_9^I	$(\psi_{82},\psi_{74}) \to \Phi_9^I$
-2	Φ_{13}^I	$\psi_{78},\psi_{163} \to \bar{\Phi}_{13}^I$
-2	Φ_{14}^I	$(\psi_{25},\psi_{18}) \to \Phi_{14}^I$
-2	Φ_{15}^I	$(\psi_{130},\psi_{122}),\psi_{71} \to \Phi_{15}^I,\ \psi_{73} \to \bar{\Phi}_{15}^I$
-4	Φ_3^I	$\psi_1 \equiv \Phi_3^I,\ \psi_{177},\ \psi_{190},\ \psi_{80} \to \Phi_3^I$
-2	Φ_{18}^I	$(\psi_{79},\psi_{72}) \to \bar{\Phi}_{18}^I$
-1	Φ_{24}^I	$\psi_{87},\psi_{84},\psi_{171},(\psi_{33},\psi_{27}) \to \Phi_{24}^I,$ $(\psi_{26},\psi_{19}) \to \bar{\Phi}_{24}^I,\ \psi_{95},\psi_{96},\psi_{128} \to \Phi_{24}^I,$ $\psi_{91},\psi_{104},\psi_{120},\psi_{180},\psi_{160} \to \bar{\Phi}_{24}^I$
-1	Φ_{22}^I	$\psi_7 \equiv \Phi_{22}^I$
E_8^2 spectrum II		
-2	Φ_7^{II}	$(\psi_{47},\psi_{59}) \to \bar{\Phi}_7^{II},\ \psi_3 \equiv \bar{\Phi}_7^{II},$ $(\psi_{38},\psi_{32}),\ \psi_{113},\psi_{168},\psi_{144} \to \Phi_7^{II}$
-2	Φ_3^{II}	$(\psi_{76},\psi_{69}) \to \Phi_3^{II}$
-4	Φ_{20}^{II}	$(\psi_{53},\psi_{67}) \to \bar{\Phi}_{20}^{II},\quad (\psi_{145},\psi_{139}) \to \bar{\Phi}_{20}^{II}$
-4	Φ_{18}^{II}	$\psi_{119},\ \psi_{181},\ (\psi_{109},\psi_{110}) \to \Phi_{18}^{II}$
Non-chiral III		
0	Φ_1^{III}	$\psi_5 \equiv \bar{\Phi}_1^{III},\ \psi_{162},\psi_{138},\psi_{101} \to \bar{\Phi}_1^{III}$ $(\psi_{56},\psi_{68}),\ \psi_{176},\psi_{156} \to \Phi_1^{III},$
0	Φ_{24}^{III}	$\psi_{131} \to \Phi_{24}^{III},\ \psi_{167} \to \bar{\Phi}_{24}^{III}$

Table 5.13: Singlets identification in agreement with superpotential mass terms.

$I_G^{\text{orb}}, I_G^{\text{red}}$ and I_G^{bu} to denote the anomaly polynomial for the gauge factors $U(1)$–G^2 with $G = SU(2), SU(3), SU(6)$. The other symbols are $I_{\text{grav}}^{\text{orb,bu,red}}$ to denote the $U(1)$–grav^2 anomalies, and $I_{\text{pure}}^{\text{orb,bu,red}}$ to denote the pure $U(1)$ anomalies.

The $U(1)$–$SU(3)^2$ anomalies are given by

$$
\begin{aligned}
I_{\text{su}(3)}^{\text{orb}} &= -\frac{52}{9}F_1 \text{tr}F_3^2, \qquad (5.56)\\
I_{\text{su}(3)}^{\text{bu}} &= \frac{1}{2}(11F_1 + 2F_2 - 30F_3 + 330F_4 + 1053F_5 - 243F_6 - 2087F_7 - 594F_8)\text{tr}F_3^2,\\
I_{\text{su}(3)}^{\text{red}} &= \frac{1}{6}\left(\frac{203}{3}F_1 + 6F_2 - 90F_3 + 990F_4 + 3159F_5 - 729F_6 - 6261F_7 - 1782F_8\right)\text{tr}F_3^2.
\end{aligned}
$$

It is clear from (5.56) that in the orbifold they are universal, with the unique axion canceling F_1, whereas in the blow–up all the $U(1)$ become anomalous. The $U(1)$–$SU(2)^2$ anomalies have an identical structure:

$$
\begin{aligned}
I_{\text{su}(2)}^{\text{orb}} &= -\frac{52}{9}F_1 \text{tr}F_2^2, \qquad (5.57)\\
I_{\text{su}(2)}^{\text{bu}} &= \frac{1}{2}(11F_1 + 2F_2 - 30F_3 + 330F_4 + 1053F_5 - 243F_6 - 2087F_7 - 594F_8)\text{tr}F_2^2,\\
I_{\text{su}(2)}^{\text{red}} &= \frac{1}{6}\left(\frac{203}{3}F_1 + 6F_2 - 90F_3 + 990F_4 + 3159F_5 - 729F_6 - 6261F_7 - 1782F_8\right)\text{tr}F_2^2.
\end{aligned}
$$

On the other hand the $U(1)$–$SU(6)^2$ anomaly has a very particular structure:

$$
\begin{aligned}
I_{\text{su}(6)}^{\text{orb}} &= -\frac{52}{9}F_1 \text{tr}F_6^2, \qquad (5.58)\\
I_{\text{su}(6)}^{\text{red}} &= \frac{52}{9}F_1 \text{tr}F_6^2, \qquad (5.59)\\
I_{\text{su}(6)}^{\text{bu}} &= 0. \qquad (5.60)
\end{aligned}
$$

As expected, in the orbifold it is universal, and in blow–up they turn out to be zero. The gravitational anomalies are given by

$$
\begin{aligned}
I_{\text{grav}}^{\text{orb}} &= \frac{52}{9}F_1 \text{tr}R^2, \qquad (5.61)\\
I_{\text{grav}}^{\text{bu}} &= -\frac{1}{12}(23F_1 + 7F_2 - 119F_3 + 1439F_4 + 3946F_5 + 6(-57F_6 - 967F_7 + F_8))\text{tr}R^2,\\
I_{\text{grav}}^{\text{red}} &= -\frac{1}{36}(277F_1 + 3(7F_2 - 119F_3 + 1439F_4 + 3946F_5 + 6(-57F_6 - 967F_7 + F_8)))\text{tr}R^2.
\end{aligned}
$$

The pure $U(1)$ anomalies have also a universal character in the blow–up:

$$
\begin{aligned}
I_{pure}^{\text{orb}} &= \frac{1}{6}\left(-\frac{10816}{27}F_1^3 - \frac{260}{9}F_1F_2^2 - \frac{13520}{3}F_1F_2^2 - \frac{1879280}{3}F_1F_4^2 - \frac{17809792}{3}F_1F_5^2\right)\\
&- \frac{1}{6}\left(\frac{40616576}{3}F_1F_6^2 - \frac{59672080}{3}F_1F_7^2 - 7830784F_1F_8^2\right). \qquad (5.62)
\end{aligned}
$$

On the blow–up the expression is much longer, so we refrain from giving it explicitly. It is important to mention the fact that all the $U(1)$s become anomalous.

We don't need the explicit field redefinitions obtained in order to match the anomalies in the supergravity and in the orbifold deformed by vevs. Any map that identifies the orbifold and blow–up massless spectrum gives the same I^{red}. Nevertheless, we give in the appendix K a list of the redefined orbifold fields and one of the many possible redefinitions that can be used to realized the considered map.

Blow–up modes and non–universal axions Let us explore how the orbifold axion and the blow–up modes are related to the blow–up universal– and non–universal axions. As in the $T^6/\mathbb{Z}_7$ study we want to determine if the local blow–up modes can be interpreted as the non–universal axions. For that purpose we write the anomaly change due to redefinitions as $I^{\text{red}} = \sum_r q_I^r F^I X_{4,r}^{red}$ i.e. as a factorization that can be canceled by a counterterm of blow–up modes. Then, the anomaly polynomial in the resolved space can be written as

$$I_6 = F_1 X_4^{orb} + \sum_r q_I^r F^I X_{4,r}^{red} = X_2^{uni} X_4^{uni} + \sum X_2^r X_4^r. \tag{5.63}$$

To describe the factorization use the formulas in appendix C.

We employ the ansatz

$$X_{4,r}^{red} = -\frac{1}{12}(c_r X_{4,r}^{\text{u}ni} + d_r X_4^r). \tag{5.64}$$

In which the $-1/12$ is introduced in order to simplify the normalization. In the appendix H we give the solutions for c_r and d_r. The results identify the blow–up modes τ_r as the non–universal axions β_r. The blow–up universal axion a^{uni} is given as a mixture of the blow–up modes and the orbifold axion a^{orb}. This can be seen in the following relations

$$a^{\text{uni}} = -\frac{1}{12}(a^{\text{orb}} + \sum_r c_r \tau_r) \tag{5.65}$$

$$\beta_r = -\frac{1}{12} d_r \tau_r. \tag{5.66}$$

The proportionality factor $-1/12 d_r$ can be chosen to be universal. It is $1/6$ for all the blow–up modes which are right–handed and $-1/6$ for the three blow–up modes which are left–handed. This results agrees exactly with the one encountered ins section 4.4 for the $T^6/\mathbb{Z}_7$ orbifold. In the appendix can also be seen that universal blow–up axion receives contributions from the unique orbifold axion a^{orb} and the blow–up modes. This one–loop computation establishes a perfect identification between the orbifold resolution and the deformed orbifold with vevs of twisted fields.

Chapter 6

Conclusions

En el momento en que el tenista lanza magistralmente su bala,
le posee una inocencia totalmente animal;
en el momento en que el filósofo sorprende una nueva verdad
es una bestia completa.
(....)Oh alma! Oh pensamiento! Oh Marx! Oh Feuerbach !

César Vallejo

Our work explains the transition, in the 6d Calabi–Yau moduli space, between a region of smooth geometry and a region with orbifold singularities. We understand the heterotic string theory on the deformed orbifold, by vevs of twisted fields, as the theory compactified on the orbifold resolution. For the cases $T^6/\mathbb{Z}_7$ and $T^6/\mathbb{Z}_{6II}$ the analysis has been carried out in detail. Our results show, that the mechanism which ensures 4d $\mathcal{N}=1$ supersymmetry and breaks the $U(1)$ gauge symmetries, also smooths the singularities and drives the space into a region of smooth Calabi–Yau. As a complementary project, we have studied automorphisms of all the $T^6/\mathbb{Z}_N$ orbifold varieties. The results found, can be applied to the study of 4d discrete symmetries in the future.

We initially selected the $T^6/\mathbb{Z}_7$ case because of the existence of a unique resolution and the absence of brother orbifolds simplifies the analysis. This model possesses the features of other realistic constructions, but on the orbifold it has extra exotics and it fails to give hypercharge with the normalization of $SU(5)$ GUT. However, it was a good starting point to perform a detailed analysis in a compact realistic model. In this collaboration, it was obtained a solution of the Bianchi Identities in which all the blow–up modes were identified with twisted states. A novel development was the use of a local index theorem, associated to a local multiplicity operator. This local multiplicity allows to identify the blow–up massless spectrum with the deformed orbifold massless spectrum. We also included in the analysis the masses coming from Yukawa couplings of orbifold states with blow–up modes. We found that the field redefinitions involve only blow–up modes localized in the same fixed set as the redefined twisted field. Then, we studied the 4d Green-Schwarz anomaly cancellation mechanism. This was done from two sides. First, we performed the dimensional reduction of the 10d anomaly in the Calabi–Yau, where the 4d anomaly cancellation follows automatically from the 10d cancellation. Then, starting from the orbifold universal anomaly, we considered how it changes due to field redefinitions and fields turning massive on the deformed orbifold. The factorization of the anomaly polynomial associated with redefinitions allowed to interpret blow–up modes as the non-universal axions in blow–up. Correspondingly, the unique orbifold axion was identified as a mixture of the blow–up universal and non-universal axions. We achieved a perfect anomaly match, which constitutes a one–loop effect, supporting the field theory approach to describe the physics on the resolved space.

As a following project we chose an orbifold model with more realistic features and more complexity. This is the orbifold $T^6/\mathbb{Z}_{6II}$ with gauge group $SU(3)\times SU(2)\times SU(6)\times U(1)^8$. It belongs to the MSSM Mini–landscape study and is phenomenologically appealing. This orbifold has the greatest complexity encountered in 6d heterotic orbifold constructions. There are fixed points and fixed tori, that gives local singularities: $\mathbb{C}^3/\mathbb{Z}_{6II}$, $\mathbb{C}^2/\mathbb{Z}_3$ and $\mathbb{C}^2/\mathbb{Z}_2$. The singularity $\mathbb{C}^3/\mathbb{Z}_{6II}$ brings with it part of the complexity of the model. A resolution of it can be performed in five different ways, giving many possibilities to resolve the compact variety. A further complexity would be the existence of brother models to the orbifold. However those models have no brothers in which the physical states transform in a consistent way under the orbifold. We scanned over the Mini–landscape, restricting the search to those models in which all fixed sets support chiral matter multiplets. Then, for a given orbifold model, we explored multiple resolutions. We observed, that the Bianchi identities were easier to fulfill by fixing the triangulation of all the local resolutions to be the same. We obtained Bianchi identities solutions for triangulation A in all local resolutions, in which we failed to identify two blow–up modes in the studied orbifold. However, for resolution B in all the fixed points, we identified many sets of twisted fields which can play the role of blow–up modes. Taking one of those resolutions we succeed to perform fields redefinitions that reproduce the chiral asymmetry of the supergravity on the toric Calabi–Yau. We considered masses generated by Yukawa couplings to blow–up modes and obtained that this restricts strongly the possible redefinitions. We found many equivalent redefinitions, which identify the orbifold spectrum with the blow–up spectrum with the same map. Another finding was that here the local index theorem does not seem to apply. That is expectable due to the presence of fixed sets, and the absence of some exceptional divisors on the triple intersections. Next, we obtained a match between the supergravity on the blow–up $m = 0$ spectrum and the orbifold $m = 0$ spectrum. A new observation in this study is that field redefinitions involve also non local blow–up modes. Intuitively this can be understood from the fact that in $\widehat{T^6/\mathbb{Z}_{6II}}$ every exceptional divisor has a compact intersection with other divisors on the manifold, such that every pair of exceptional divisors is connected. With this information in hand we carried a detailed analysis of the anomaly cancelation mechanism. On one side we computed the dimensional reduced anomaly polynomial in blow–up. We also obtained the orbifold anomaly polynomial and its variation due to field redefinitions and fields going massive in blow–up. The anomaly cancellation in 4d is inherited from the 10d cancellation, this is checked by obtaining the factorization of the 4d polynomial in blow–up. We were able to factorize the change of the orbifold anomaly polynomial, to obtain that the blow–up modes correspond to non–universal axions of the resolution. This study completes the identification of the smooth geometry with the deformed orbifold at the quantum level.

For the $T^6/\mathbb{Z}_7$ orbifold the automorphisms exploration doesn't give any new insight. So we don't expect modifications to its selection rules coming from rotations in any of the planes. However, for the $T^6/\mathbb{Z}_{6II}$ orbifold the inclusion of gamma phases can modify the discrete R–charge conservation, obtained by the rotations in the three planes. This fact is still under investigation. In addition, for some Wilson lines combination, there is an automorphism that maps fixed sets with the same spectrum among each other. This transformation could generate a flavor symmetry in 4d. We would like to study in a future work how those new developments on orbifold discrete symmetries affect the orbifold–toric CY transition [94].

Our results show the viability of using orbifold compactifications to obtain information about a smooth region of the Calabi–Yau moduli space. We have shown that the absence of a unique toric resolution is a problem that can be overcome by a careful analysis. We connected two regions of the moduli space from heterotic 3-fold Calabi–Yau at the level of the massless spectrum, and at the quantum level, by understanding the changes in the anomaly cancellation mechanism.

There are still open questions along this path. One of the questions which we would like treat in a future work, is how the Bianchi identities translate into the level–matching condition for blow–up modes. This has been studied in the Gauge Linear Sigma Models scheme. Another interesting check, would be to look at the partition function of the effective 10d orbifold and resolution theories, to explicitly check that the field redefinitions will give rise to a counterterm canceling the anomaly modification. Although during the realized exploration sometimes one has the impression of searching a needle in haystack, our results in the two worked examples, imply that there is something deeper than just coincidence. The case of the $T^6/\mathbb{Z}_{6II}$ orbifold is interesting. As mentioned, we found that there are degenerated redefinitions i.e. different redefinitions parametrize the same orbifold-resolution map. It seems also that various maps are possible, but we focused in finding one of them. Because of the mentioned properties, this orbifold offers a new insight: to understand the anomaly cancellation mechanism doesn't legitimize a particular set of field redefinitions. Of course the matching of the massless spectrum via field redefinitions is a check at the classical level, which is connected to the anomaly. But there are multiple ways to perform that match. The contribution that anomaly matching brings, is the observation that blow–up modes mutate into non–universal blow–up axions and the single orbifold axion depends on both universal and non–universal axions in blow–up. Then, our anomaly study constitutes a one–loop check that both theories can be identified.

Anomaly cancellation of the 10d heterotic string theory and of the 10d $\mathcal{N}=1$ supergravity is ensured by the fact that the gauge group is $E_8 \times E_8$ or $SO(32)$. This selection of the gauge group is also required by world–sheet consistency, via locality and modular invariance. So the consistency of the superconformal $(0,2)$ world–sheet theory expresses itself in the target space through the 4d anomaly cancellation. It is precisely this cancellation, that allowed us to give a quantum argument to identify the deformed orbifold and the toric Calabi–Yau manifold.

Appendix A

Orbifold and blow–up spectrum

This appendix contains a detailed list of all orbifold and blow–up states. For each state the local and global multiplicity is given, as well as the characteristic data (i.e. the $E_8 \times E_8$ roots for the blow–up states and the shifted momenta for the orbifold states) together with the field redefinition between these states. The organization of the table is as follows: it is divided into blocks where each block corresponds to an $E_8 \times E_8$ roots in blow–up. Below this root, we list all orbifold states which are redefined to this root, where the redefinition used is indicated in the last column.
We give the representation (of the blow–up root) or an auxiliary name (for the orbifold states) in the first column. The second column contains the twisted sector where the orbifold state lives (for the blow–up states this information is not defined anymore). The entry 1-7 indicates an untwisted state. The third column gives the local multiplicity, i.e. the multiplicity of each state at each fixed point. The "tot" column contains the total multiplicity, i.e. the sum of the local multiplicities over all fixed points. In our convention, we list only the highest states of non–abelian irreps, where a negative multiplicity indicates that the state belongs to the complex conjugate representation. The last block of the table contain the 21 orbifold states which were chosen as blow–up modes, they are denoted by BM.

State	Sector	Local multiplicity							tot	$E_8 \times E_8$ root / P_{sh}	Redef
		1	2	3	4	5	6	7			
(3,2,1)	–	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	1	$(1,0,0,0,-1,0,0,0)\,(0,0,0,0,0,0,0,0)$	–
Q_1	1-7	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	1	$(1,0,0,0,-1,0,0,0)\,(0,0,0,0,0,0,0,0)$	none
(3,2,1)	–	1	$-\frac{1}{7}$	$-\frac{1}{7}$	$\frac{1}{7}$	$-\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	1	$\left(\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right)(0,0,0,0,0,0,0,0)$	–
Q_2	2	1	0	0	0	0	0	0	1	$\left(\frac{1}{2},-\frac{1}{2},\frac{3}{14},\frac{3}{14},-\frac{11}{14},-\frac{1}{14},-\frac{1}{14},\frac{3}{14}\right)\left(-\frac{2}{7},-\frac{2}{7},0,0,0,0,0,0\right)$	(4.32a)
(3,2,1)	–	1	$-\frac{1}{7}$	$-\frac{1}{7}$	$\frac{1}{7}$	$-\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	1	$\left(\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right)(0,0,0,0,0,0,0,0)$	–
Q_3	1	1	0	0	0	0	0	0	1	$\left(\frac{1}{2},-\frac{1}{2},\frac{5}{14},\frac{5}{14},-\frac{9}{14},\frac{3}{14},\frac{3}{14},\frac{5}{14}\right)\left(-\frac{1}{7},-\frac{1}{7},0,0,0,0,0,0\right)$	(4.32a)
$(\overline{3},1,1)$	–	$\frac{1}{7}$	$\frac{1}{7}$	$-\frac{1}{7}$	$\frac{1}{7}$	$-\frac{1}{7}$	$-\frac{1}{7}$	-1	-1	$\left(\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)(0,0,0,0,0,0,0,0)$	–
$\bar{t}_7$	4	0	0	0	0	0	0	-1	-1	$\left(\frac{1}{14},\frac{1}{14},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{14},\frac{3}{14},\frac{1}{14}\right)\left(\frac{1}{7},\frac{5}{7},-\frac{2}{7},0,0,0,0,0\right)$	(4.32a)
(3,1,1)	–	$-\frac{1}{7}$	$\frac{1}{7}$	$-\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	1	$-\frac{1}{7}$	1	$(0,0,0,0,-1,0,1,0)\,(0,0,0,0,0,0,0,0)$	–
t_6	4	0	0	0	0	0	1	0	1	$\left(-\frac{5}{14},-\frac{5}{14},\frac{1}{14},\frac{1}{14},-\frac{13}{14},-\frac{3}{14},\frac{1}{14},\frac{3}{14}\right)\left(-\frac{1}{7},0,\frac{3}{7},0,0,0,0,0\right)$	(4.32a)
(3,1,1)	–	$-\frac{1}{7}$	$\frac{1}{7}$	$-\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	1	$-\frac{1}{7}$	1	$\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right)(0,0,0,0,0,0,0,0)$	–

State	Sector	Local multiplicity							tot	$E_8 \times E_8$ root / P_{sh}	Redef
		1	2	3	4	5	6	7			
t_7	4	0	0	0	0	0	1	0	1	$(\frac{1}{7},\frac{1}{7},\frac{4}{7},\frac{4}{7},-\frac{3}{7},\frac{2}{7},-\frac{3}{7},-\frac{2}{7})(-\frac{1}{7},0,\frac{3}{7},0,0,0,0,0)$	(4.32a)
$(\overline{3},1,1)$	–	$-\frac{1}{7}$	$-\frac{1}{7}$	$-\frac{1}{7}$	$-\frac{1}{7}$	$-\frac{1}{7}$	$-\frac{8}{7}$	$-\frac{8}{7}$	-3	$(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2})(0,0,0,0,0,0,0,0)$	–
$\bar{t}_1$	1-7	$-\frac{1}{7}$	$-\frac{1}{7}$	$-\frac{1}{7}$	$-\frac{1}{7}$	$-\frac{1}{7}$	$-\frac{1}{7}$	$-\frac{1}{7}$	-1	$(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2})(0,0,0,0,0,0,0,0)$	none
$\bar{t}_5$	4	0	0	0	0	0	-1	0	-1	$(\frac{1}{7},\frac{1}{7},-\frac{3}{7},-\frac{3}{7},\frac{4}{7},\frac{2}{7},\frac{4}{7},-\frac{2}{7})(-\frac{1}{7},0,\frac{3}{7},0,0,0,0,0)$	(4.32c)
$\bar{t}_6$	4	0	0	0	0	0	0	-1	-1	$(\frac{1}{14},\frac{1}{14},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{14},\frac{3}{14},\frac{1}{14})(\frac{1}{7},-\frac{2}{7},\frac{5}{7},0,0,0,0,0)$	(4.32c)
$(\overline{3},1,1)$	–	$-\frac{1}{7}$	$\frac{1}{7}$	$-\frac{1}{7}$	$-\frac{1}{7}$	-1	$\frac{1}{7}$	$\frac{1}{7}$	-1	$(\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2})(0,0,0,0,0,0,0,0)$	–
$\bar{t}_4$	4	0	0	0	0	-1	0	0	-1	$(\frac{3}{14},\frac{3}{14},-\frac{5}{14},-\frac{5}{14},\frac{9}{14},\frac{1}{2},-\frac{1}{14},\frac{5}{14})(-\frac{3}{7},\frac{2}{7},\frac{1}{7},0,0,0,0,0)$	(4.32a)
$(\overline{3},1,1)$	–	$-\frac{1}{7}$	$\frac{1}{7}$	$-\frac{1}{7}$	$-\frac{1}{7}$	-1	$\frac{1}{7}$	$\frac{1}{7}$	-1	$(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2})(0,0,0,0,0,0,0,0)$	–
$\bar{t}_{17}$	1	0	0	0	0	-1	0	0	-1	$(-\frac{1}{14},-\frac{1}{14},-\frac{3}{14},-\frac{3}{14},\frac{11}{14},-\frac{1}{2},\frac{5}{14},\frac{3}{14})(\frac{1}{7},-\frac{3}{7},\frac{2}{7},0,0,0,0,0)$	(4.32a)
$(3,1,1)$	–	$\frac{1}{7}$	$-\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	1	$-\frac{8}{7}$	$-\frac{1}{7}$	0	$(0,0,0,0,-1,0,0,-1)(0,0,0,0,0,0,0,0)$	–
t_5	4	0	0	0	0	1	0	0	1	$(-\frac{2}{7},-\frac{2}{7},\frac{1}{7},\frac{1}{7},-\frac{6}{7},0,\frac{3}{7},-\frac{1}{7})(-\frac{3}{7},\frac{2}{7},\frac{1}{7},0,0,0,0,0)$	(4.32a)
t_{12}	1	0	0	0	0	1	0	0	1	$(\frac{3}{7},\frac{3}{7},\frac{2}{7},\frac{2}{7},-\frac{5}{7},0,-\frac{1}{7},-\frac{2}{7})(\frac{1}{7},-\frac{3}{7},\frac{2}{7},0,0,0,0,0)$	(4.32a)
$\bar{t}_{11}$	2	0	0	0	0	-1	0	0	-1	$(-\frac{1}{7},-\frac{1}{7},\frac{4}{7},-\frac{3}{7},-\frac{3}{7},0,-\frac{2}{7},-\frac{4}{7})(\frac{2}{7},\frac{1}{7},-\frac{3}{7},0,0,0,0,0)$	(4.32b)
$\bar{t}_{18}$	1	0	0	0	0	0	-1	0	-1	$(-\frac{3}{14},-\frac{3}{14},\frac{9}{14},-\frac{5}{14},-\frac{5}{14},\frac{1}{14},-\frac{5}{14},-\frac{1}{14})(\frac{5}{7},0,-\frac{1}{7},0,0,0,0,0)$	(4.32c)
$(\overline{3},1,1)$	–	$-\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	-1	$-\frac{1}{7}$	$-\frac{1}{7}$	$\frac{1}{7}$	-1	$(0,0,0,0,1,0,1,0)(0,0,0,0,0,0,0,0)$	–
$\bar{t}_{16}$	1	0	0	0	-1	0	0	0	-1	$(\frac{1}{14},\frac{1}{14},-\frac{1}{14},-\frac{1}{14},\frac{13}{14},-\frac{1}{14},\frac{1}{14},\frac{1}{2})(-\frac{3}{7},\frac{1}{7},-\frac{2}{7},0,0,0,0,0)$	(4.32a)
$(\overline{3},1,1)$	–	$-\frac{1}{7}$	$-\frac{1}{7}$	-1	$\frac{1}{7}$	$\frac{1}{7}$	$-\frac{1}{7}$	$\frac{1}{7}$	-1	$(\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2})(0,0,0,0,0,0,0,0)$	–
$\bar{t}_3$	4	0	0	-1	0	0	0	0	-1	$(\frac{5}{14},\frac{5}{14},-\frac{3}{14},-\frac{3}{14},\frac{11}{14},-\frac{1}{14},-\frac{5}{14},-\frac{5}{14})(0,-\frac{1}{7},-\frac{3}{7},0,0,0,0,0)$	(4.32a)
$(3,1,1)$	–	$-\frac{1}{7}$	$-\frac{1}{7}$	1	$\frac{1}{7}$	$\frac{1}{7}$	$-\frac{8}{7}$	$\frac{1}{7}$	0	$(-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2})(0,0,0,0,0,0,0,0)$	–
t_9	2	0	0	1	0	0	0	0	1	$(-\frac{1}{14},-\frac{1}{14},\frac{9}{14},\frac{9}{14},-\frac{5}{14},\frac{3}{14},\frac{1}{14},\frac{1}{14})(0,-\frac{4}{7},\frac{2}{7},0,0,0,0,0)$	(4.32a)
$\bar{t}_{12}$	2	0	0	0	0	0	-1	0	-1	$(\frac{1}{14},\frac{1}{14},\frac{11}{14},-\frac{3}{14},-\frac{3}{14},-\frac{5}{14},-\frac{3}{14},\frac{5}{14})(-\frac{4}{7},0,-\frac{2}{7},0,0,0,0,0)$	(4.32c)
$(3,1,1)$	–	$\frac{1}{7}$	$\frac{8}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{8}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	3	$(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2})(0,0,0,0,0,0,0,0)$	–
t_1	1-7	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	1	$(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2})(0,0,0,0,0,0,0,0)$	none
t_8	2	0	1	0	0	0	0	0	1	$(-\frac{2}{7},-\frac{2}{7},\frac{3}{7},\frac{3}{7},-\frac{4}{7},-\frac{3}{7},0,\frac{1}{7})(-\frac{1}{7},\frac{4}{7},\frac{1}{7},0,0,0,0,0)$	(4.32c)
t_{10}	2	0	0	0	0	1	0	0	1	$(-\frac{1}{7},-\frac{1}{7},\frac{4}{7},\frac{4}{7},-\frac{3}{7},0,-\frac{2}{7},\frac{3}{7})(\frac{2}{7},\frac{1}{7},-\frac{3}{7},0,0,0,0,0)$	(4.32c)
$(3,1,1)$	–	-1	$-\frac{1}{7}$	$\frac{6}{7}$	$-\frac{6}{7}$	$-\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	-1	$(0,0,0,0,-1,0,0,1)(0,0,0,0,0,0,0,0)$	–
t_2	4	-1	0	0	0	0	0	0	-1	$(0,0,\frac{3}{7},\frac{3}{7},-\frac{4}{7},-\frac{1}{7},-\frac{1}{7},-\frac{4}{7})(\frac{3}{7},\frac{3}{7},0,0,0,0,0,0)$	(4.32b)
t_3	4	0	0	-1	0	0	0	0	-1	$(-\frac{1}{7},-\frac{1}{7},\frac{2}{7},\frac{2}{7},-\frac{5}{7},\frac{3}{7},\frac{1}{7},\frac{1}{7})(0,-\frac{1}{7},-\frac{3}{7},0,0,0,0,0)$	(4.32a)
$\bar{t}_8$	2	1	0	0	0	0	0	0	1	$(0,0,\frac{5}{7},-\frac{2}{7},-\frac{2}{7},-\frac{4}{7},\frac{3}{7},-\frac{2}{7})(-\frac{2}{7},-\frac{2}{7},0,0,0,0,0,0)$	(4.32a)
$\bar{t}_{10}$	2	0	0	0	1	0	0	0	1	$(\frac{1}{7},\frac{1}{7},\frac{6}{7},-\frac{1}{7},-\frac{1}{7},-\frac{1}{7},\frac{1}{7},0)(\frac{1}{7},\frac{2}{7},\frac{3}{7},0,0,0,0,0)$	(4.32a)
$\bar{t}_{14}$	1	1	0	0	0	0	0	0	1	$(0,0,\frac{6}{7},-\frac{1}{7},-\frac{1}{7},\frac{5}{7},-\frac{2}{7},-\frac{1}{7})(-\frac{1}{7},-\frac{1}{7},0,0,0,0,0,0)$	(4.32a)
$(3,1,1)$	–	$-\frac{8}{7}$	$-\frac{1}{7}$	$\frac{1}{7}$	$-\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	1	0	$(0,0,1,1,0,0,0,0)(0,0,0,0,0,0,0,0)$	–
t_{11}	2	0	0	0	0	0	0	1	1	$(-\frac{3}{14},-\frac{3}{14},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{3}{14},\frac{5}{14},-\frac{3}{14})(\frac{4}{7},-\frac{1}{7},-\frac{1}{7},0,0,0,0,0)$	(4.32a)
$\bar{t}_9$	2	1	0	0	0	0	0	0	-1	$(0,0,\frac{5}{7},-\frac{2}{7},-\frac{2}{7},\frac{3}{7},-\frac{4}{7},-\frac{2}{7})(-\frac{2}{7},-\frac{2}{7},0,0,0,0,0,0)$	(4.32c)
$(3,1,1)$	–	$-\frac{8}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	1	$-\frac{1}{7}$	$-\frac{1}{7}$	$\frac{1}{7}$	0	$(0,0,0,0,-1,1,0,0)(0,0,0,0,0,0,0,0)$	–
t_4	4	0	0	0	1	0	0	0	1	$(-\frac{3}{14},-\frac{3}{14},\frac{3}{14},\frac{3}{14},-\frac{11}{14},\frac{3}{14},-\frac{3}{14},\frac{1}{2})(\frac{2}{7},-\frac{3}{7},-\frac{1}{7},0,0,0,0,0)$	(4.32a)
$\bar{t}_2$	4	1	0	0	0	0	0	0	-1	$(0,0,\frac{3}{7},-\frac{4}{7},-\frac{4}{7},-\frac{1}{7},-\frac{1}{7},\frac{3}{7})(\frac{3}{7},\frac{3}{7},0,0,0,0,0,0)$	(4.32c)
$(\overline{3},1,1)$	–	$-\frac{8}{7}$	-1	$\frac{1}{7}$	$\frac{1}{7}$	$-\frac{1}{7}$	$\frac{1}{7}$	$-\frac{1}{7}$	-2	$(0,0,0,0,1,1,0,0)(0,0,0,0,0,0,0,0)$	–
$\bar{t}_{13}$	1	-1	0	0	0	0	0	0	-1	$(0,0,-\frac{1}{7},-\frac{1}{7},\frac{6}{7},-\frac{2}{7},\frac{5}{7},-\frac{1}{7})(-\frac{1}{7},-\frac{1}{7},0,0,0,0,0,0)$	(4.32c)
$\bar{t}_{15}$	1	0	-1	0	0	0	0	0	-1	$(-\frac{1}{7},-\frac{1}{7},-\frac{2}{7},-\frac{2}{7},\frac{5}{7},\frac{2}{7},0,-\frac{3}{7})(\frac{3}{7},\frac{2}{7},-\frac{3}{7},0,0,0,0,0)$	(4.32a)
$(1,2,1)$	–	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	1	$(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2})(0,0,0,0,0,0,0,0)$	–
h_2	1-7	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	1	$(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2})(0,0,0,0,0,0,0,0)$	none
$(1,2,1)$	–	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	1	$(\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2})(0,0,0,0,0,0,0,0)$	–
h_1	1-7	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	1	$(\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2})(0,0,0,0,0,0,0,0)$	none
h_4	4	1	0	0	0	0	0	0	1	$(\frac{1}{2},-\frac{1}{2},-\frac{1}{14},-\frac{1}{14},-\frac{1}{14},\frac{5}{14},-\frac{9}{14},-\frac{1}{14})(\frac{3}{7},\frac{3}{7},0,0,0,0,0,0)$	(4.32c)
h_{17}	1	-1	0	0	0	0	0	0	-1	$(\frac{1}{2},-\frac{1}{2},\frac{5}{14},\frac{5}{14},\frac{5}{14},\frac{3}{14},\frac{3}{14},-\frac{9}{14})(-\frac{1}{7},-\frac{1}{7},0,0,0,0,0,0)$	(4.32d)

State	Sector	Local multiplicity							tot	$E_8 \times E_8$ root / P_{sh}	Redef
		1	2	3	4	5	6	7			
(1,2,1)	–	$-\frac{1}{7}$	$-\frac{1}{7}$	$\frac{1}{7}$	$-\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	1	1	$(0,-1,0,0,0,1,0,0)(0,0,0,0,0,0,0,0)$	–
h_{21}	1	0	0	0	0	0	0	1	1	$(\frac{1}{7},-\frac{6}{7},0,0,0,\frac{1}{7},\frac{3}{7},\frac{1}{7})(\frac{2}{7},\frac{3}{7},\frac{3}{7},0,0,0,0,0)$	(4.32a)
(1,2,1)	–	$-\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	1	$-\frac{1}{7}$	$-\frac{1}{7}$	$\frac{1}{7}$	1	$(1,0,0,0,0,1,0,0)(0,0,0,0,0,0,0,0)$	–
h_7	4	0	0	0	1	0	0	0	1	$(\frac{11}{14},-\frac{3}{14},\frac{3}{14},\frac{3}{14},\frac{3}{14},\frac{3}{14},-\frac{3}{14},\frac{1}{2})(\frac{2}{7},-\frac{3}{7},-\frac{1}{7},0,0,0,0,0)$	(4.32a)
(1,2,1)	–	$-\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	1	$-\frac{1}{7}$	$-\frac{1}{7}$	$\frac{1}{7}$	1	$(0,-1,0,0,0,0,1,0)(0,0,0,0,0,0,0,0)$	–
h_{19}	1	0	0	0	1	0	0	0	1	$(\frac{1}{14},-\frac{13}{14},-\frac{1}{14},-\frac{1}{14},-\frac{1}{14},-\frac{1}{14},\frac{1}{14},\frac{1}{2})(-\frac{3}{7},\frac{1}{7},-\frac{2}{7},0,0,0,0,0)$	(4.32a)
(1,2,1)	–	$-\frac{1}{7}$	$\frac{1}{7}$	$\frac{6}{7}$	$-\frac{1}{7}$	0	$\frac{1}{7}$	$\frac{1}{7}$	1	$(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2})(0,0,0,0,0,0,0,0)$	–
h_5	4	0	0	1	0	0	0	0	1	$(\frac{5}{14},-\frac{9}{14},-\frac{3}{14},-\frac{3}{14},-\frac{3}{14},-\frac{1}{14},-\frac{5}{14},-\frac{5}{14})(0,-\frac{1}{7},-\frac{3}{7},0,0,0,0,0)$	(4.32a)
(1,2,1)	–	$-\frac{1}{7}$	$-\frac{1}{7}$	-1	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{13}{7}$	$\frac{1}{7}$	1	$(1,0,0,0,0,0,1,0)(0,0,0,0,0,0,0,0)$	–
h_9	4	0	0	0	0	0	1	0	1	$(\frac{9}{14},-\frac{5}{14},\frac{1}{14},\frac{1}{14},\frac{1}{14},-\frac{3}{14},\frac{1}{14},\frac{3}{14})(-\frac{1}{7},0,\frac{3}{7},0,0,0,0,0)$	(4.32a)
h_{10}	4	0	0	0	0	0	1	0	1	$(\frac{9}{14},-\frac{5}{14},\frac{1}{14},\frac{1}{14},\frac{1}{14},-\frac{3}{14},\frac{1}{14},\frac{3}{14})(-\frac{1}{7},0,\frac{3}{7},0,0,0,0,0)$	(4.32a)
h_{13}	2	0	0	-1	0	0	0	0	-1	$(\frac{3}{7},-\frac{4}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7},-\frac{2}{7},-\frac{3}{7},-\frac{3}{7})(0,-\frac{4}{7},\frac{2}{7},0,0,0,0,0)$	(4.32a)
(1,2,1)	–	$\frac{1}{7}$	$\frac{1}{7}$	-1	$\frac{6}{7}$	$\frac{6}{7}$	$\frac{1}{7}$	$-\frac{1}{7}$	1	$(0,-1,0,0,0,0,0,-1)(0,0,0,0,0,0,0,0)$	–
h_6	4	0	0	-1	0	0	0	0	-1	$(\frac{6}{7},-\frac{1}{7},\frac{2}{7},\frac{2}{7},\frac{2}{7},\frac{3}{7},\frac{1}{7},\frac{1}{7})(0,-\frac{1}{7},-\frac{3}{7},0,0,0,0,0)$	(4.32a)
h_{14}	2	0	0	0	1	0	0	0	1	$(\frac{1}{7},-\frac{6}{7},-\frac{1}{7},-\frac{1}{7},-\frac{1}{7},-\frac{1}{7},\frac{1}{7},0)(\frac{1}{7},\frac{2}{7},\frac{3}{7},0,0,0,0,0)$	(4.32a)
h_{20}	1	0	0	0	0	1	0	0	1	$(\frac{3}{7},-\frac{4}{7},\frac{2}{7},\frac{2}{7},\frac{2}{7},0,-\frac{1}{7},-\frac{2}{7})(\frac{1}{7},-\frac{3}{7},\frac{2}{7},0,0,0,0,0)$	(4.32a)
(1,2,1)	–	$-\frac{1}{7}$	1	$\frac{1}{7}$	$\frac{1}{7}$	$-\frac{1}{7}$	$\frac{1}{7}$	$\frac{6}{7}$	2	$(\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2})(0,0,0,0,0,0,0,0)$	–
h_3	4	1	0	0	0	0	0	0	1	$(\frac{1}{2},-\frac{1}{2},-\frac{1}{14},-\frac{1}{14},-\frac{1}{14},-\frac{9}{14},\frac{5}{14},-\frac{1}{14})(\frac{3}{7},\frac{3}{7},0,0,0,0,0,0)$	(4.32d)
h_{11}	2	-1	0	0	0	0	0	0	-1	$(\frac{1}{2},-\frac{1}{2},\frac{3}{14},\frac{3}{14},\frac{3}{14},-\frac{1}{14},-\frac{1}{14},-\frac{11}{14})(-\frac{2}{7},-\frac{2}{7},0,0,0,0,0,0)$	(4.32c)
h_{16}	2	0	0	0	0	0	0	1	1	$(\frac{2}{7},-\frac{5}{7},0,0,0,\frac{2}{7},-\frac{1}{7},\frac{2}{7})(\frac{4}{7},-\frac{1}{7},-\frac{1}{7},0,0,0,0,0)$	(4.32a)
h_{18}	1	0	1	0	0	0	0	0	1	$(\frac{5}{14},-\frac{9}{14},\frac{3}{14},\frac{3}{14},\frac{3}{14},-\frac{3}{14},-\frac{1}{2},\frac{1}{14})(\frac{3}{7},\frac{2}{7},-\frac{3}{7},0,0,0,0,0)$	(4.32a)
(1,2,1)	–	$\frac{1}{7}$	$\frac{8}{7}$	$-\frac{1}{7}$	$\frac{1}{7}$	$\frac{13}{7}$	$-\frac{1}{7}$	0	3	$(1,0,0,0,0,0,0,-1)(0,0,0,0,0,0,0,0)$	–
h_8	4	0	0	0	0	1	0	0	1	$(\frac{5}{7},-\frac{2}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7},0,\frac{3}{7},-\frac{1}{7})(-\frac{3}{7},\frac{2}{7},\frac{1}{7},0,0,0,0,0)$	(4.32a)
h_{12}	2	0	1	0	0	0	0	0	1	$(\frac{3}{14},-\frac{11}{14},-\frac{1}{14},-\frac{1}{14},-\frac{1}{14},\frac{1}{14},\frac{1}{2},-\frac{5}{14})(-\frac{1}{7},\frac{4}{7},\frac{1}{7},0,0,0,0,0)$	(4.32c)
h_{15}	2	0	0	0	0	1	0	0	1	$(\frac{5}{14},-\frac{9}{14},\frac{1}{14},\frac{1}{14},\frac{1}{14},\frac{1}{2},\frac{3}{14},-\frac{1}{14})(\frac{2}{7},\frac{1}{7},-\frac{3}{7},0,0,0,0,0)$	(4.32c)
(1,1,10)	–	$\frac{1}{7}$	$\frac{1}{7}$	1	$-\frac{1}{7}$	$-\frac{1}{7}$	$\frac{1}{7}$	$-\frac{1}{7}$	1	$(0,0,0,0,0,0,0,0)(-1,0,0,-1,0,0,0,0)$	–
X_1	1	0	0	1	0	0	0	0	1	$(\frac{3}{14},\frac{3}{14},\frac{1}{14},\frac{1}{14},\frac{1}{14},\frac{5}{14},-\frac{3}{14},-\frac{3}{14})(0,-\frac{2}{7},\frac{1}{7},1,0,0,0,0)$	(4.32a)
(1,1,1)	–	0	$\frac{13}{7}$	$-\frac{1}{7}$	$\frac{1}{7}$	$-\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	2	$(1,1,0,0,0,0,0,0)(0,0,0,0,0,0,0,0)$	–
s_{17}	4	0	1	0	0	0	0	0	1	$(\frac{3}{7},\frac{3}{7},-\frac{1}{7},-\frac{1}{7},-\frac{1}{7},\frac{1}{7},0,\frac{2}{7})(\frac{5}{7},\frac{1}{7},\frac{2}{7},0,0,0,0,0)$	(4.32a)
s_{34}	4	0	0	0	0	-1	0	0	-1	$(-\frac{2}{7},-\frac{2}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7},0,-\frac{4}{7},-\frac{1}{7})(\frac{4}{7},-\frac{5}{7},\frac{1}{7},0,0,0,0,0)$	(4.32c)
s_{56}	2	0	1	0	0	0	0	0	1	$(\frac{3}{14},\frac{3}{14},-\frac{1}{14},-\frac{1}{14},-\frac{1}{14},\frac{1}{14},\frac{1}{2},\frac{9}{14})(-\frac{1}{7},\frac{4}{7},\frac{1}{7},0,0,0,0,0)$	(4.32c)
s_{120}	1	0	0	0	0	1	0	0	1	$(-\frac{1}{14},-\frac{1}{14},-\frac{3}{14},-\frac{3}{14},-\frac{3}{14},\frac{1}{2},\frac{5}{14},\frac{3}{14})(\frac{1}{7},\frac{4}{7},-\frac{5}{7},0,0,0,0,0)$	(4.32d)
(1,1,1)	–	0	$\frac{13}{7}$	$\frac{13}{7}$	$\frac{1}{7}$	$-\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	4	$(0,0,0,0,0,0,0,0)(-1,1,0,0,0,0,0,0)$	–
s_{55}	2	0	1	0	0	0	0	0	1	$(\frac{3}{14},\frac{3}{14},-\frac{1}{14},-\frac{1}{14},-\frac{1}{14},\frac{1}{14},-\frac{1}{2},-\frac{5}{14})(-\frac{1}{7},\frac{4}{7},\frac{1}{7},0,0,0,0,0)$	(4.32a)
s_{57}	2	0	1	0	0	0	0	0	1	$(\frac{3}{14},\frac{3}{14},-\frac{1}{14},-\frac{1}{14},-\frac{1}{14},\frac{1}{14},-\frac{1}{2},-\frac{5}{14})(-\frac{1}{7},\frac{4}{7},\frac{1}{7},0,0,0,0,0)$	(4.32a)
s_{72}	2	0	0	0	0	1	0	0	1	$(\frac{5}{14},\frac{5}{14},\frac{1}{14},\frac{1}{14},\frac{1}{14},\frac{1}{2},-\frac{11}{14},-\frac{1}{14})(\frac{2}{7},\frac{1}{7},-\frac{3}{7},0,0,0,0,0)$	(4.32d)
s_{106}	1	0	0	1	0	0	0	0	1	$(\frac{3}{14},\frac{3}{14},\frac{1}{14},\frac{1}{14},\frac{1}{14},\frac{5}{14},-\frac{3}{14},-\frac{3}{14})(0,\frac{5}{7},\frac{1}{7},0,0,0,0,0)$	(4.32a)
s_{107}	1	0	0	1	0	0	0	0	1	$(\frac{3}{14},\frac{3}{14},\frac{1}{14},\frac{1}{14},\frac{1}{14},\frac{5}{14},-\frac{3}{14},-\frac{3}{14})(0,\frac{5}{7},\frac{1}{7},0,0,0,0,0)$	(4.32a)
s_{117}	1	0	0	0	0	-1	0	0	-1	$(-\frac{1}{14},-\frac{1}{14},-\frac{3}{14},-\frac{3}{14},-\frac{3}{14},-\frac{1}{2},\frac{5}{14},-\frac{11}{14})(\frac{1}{7},-\frac{3}{7},\frac{2}{7},0,0,0,0,0)$	(4.32c)
(1,1,1)	–	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	1	$(0,0,0,0,0,0,0,0)(0,-1,-1,0,0,0,0,0)$	–
s_2	1-7	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	1	$(0,0,0,0,0,0,0,0)(0,-1,-1,0,0,0,0,0)$	none
(1,1,1)	–	$\frac{1}{7}$	$\frac{1}{7}$	$-\frac{1}{7}$	$\frac{1}{7}$	$-\frac{1}{7}$	$-\frac{1}{7}$	1	1	$(0,0,0,0,0,0,0,0)(0,-1,1,0,0,0,0,0)$	–
s_{42}	4	0	0	0	0	0	0	1	1	$(-\frac{3}{7},-\frac{3}{7},0,0,0,-\frac{3}{7},-\frac{2}{7},-\frac{3}{7})(\frac{1}{7},-\frac{2}{7},\frac{5}{7},0,0,0,0,0)$	(4.32a)
(1,1,1)	–	$-\frac{1}{7}$	$-\frac{1}{7}$	$-\frac{1}{7}$	$-\frac{1}{7}$	$\frac{13}{7}$	$\frac{6}{7}$	$-\frac{1}{7}$	2	$(0,0,0,0,0,0,0,0)(1,0,-1,0,0,0,0,0)$	–
s_1	1-7	$-\frac{1}{7}$	$-\frac{1}{7}$	$-\frac{1}{7}$	$-\frac{1}{7}$	$-\frac{1}{7}$	$-\frac{1}{7}$	$-\frac{1}{7}$	-1	$(0,0,0,0,0,0,0,0)(-1,0,1,0,0,0,0,0)$	none
s_{73}	2	0	0	0	0	1	0	0	1	$(\frac{5}{14},\frac{5}{14},\frac{1}{14},\frac{1}{14},\frac{1}{14},-\frac{1}{2},\frac{3}{14},-\frac{1}{14})(\frac{2}{7},\frac{1}{7},-\frac{3}{7},0,0,0,0,0)$	(4.32a)

State	Sector	Local multiplicity							tot	$E_8 \times E_8$ root / P_{sh}	Redef
		1	2	3	4	5	6	7			
s_{74}	2	0	0	0	0	1	0	0	1	$(\frac{5}{14},\frac{5}{14},\frac{1}{14},\frac{1}{14},\frac{1}{14},-\frac{1}{2},\frac{3}{14},-\frac{1}{14})(\frac{2}{7},\frac{1}{7},-\frac{3}{7},0,0,0,0,0)$	(4.32a)
s_{124}	1	0	0	0	0	0	1	0	1	$(\frac{2}{7},\frac{2}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7},-\frac{3}{7},\frac{1}{7},\frac{3}{7})(\frac{5}{7},0,-\frac{1}{7},0,0,0,0,0)$	(4.32a)
(1,1,1)	–	$\frac{1}{7}$	$-\frac{1}{7}$	$\frac{1}{7}$	$-\frac{1}{7}$	$\frac{13}{7}$	-1	$\frac{1}{7}$	1	$(\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2})(0,0,0,0,0,0,0,0)$	–
s_{14}	4	0	-1	0	0	0	0	0	-1	$(-\frac{1}{14},-\frac{1}{14},\frac{5}{14},\frac{5}{14},\frac{5}{14},-\frac{5}{14},\frac{1}{2},-\frac{3}{14})(-\frac{2}{7},\frac{1}{7},-\frac{5}{7},0,0,0,0,0)$	(4.32c)
s_{37}	4	0	0	0	0	1	0	0	1	$(\frac{3}{14},\frac{3}{14},-\frac{5}{14},-\frac{5}{14},-\frac{5}{14},-\frac{1}{2},-\frac{1}{14},\frac{5}{14})(-\frac{3}{7},\frac{2}{7},\frac{1}{7},0,0,0,0,0)$	(4.32a)
s_{76}	2	0	0	0	0	1	0	0	1	$(-\frac{1}{7},-\frac{1}{7},-\frac{3}{7},-\frac{3}{7},-\frac{3}{7},0,-\frac{2}{7},\frac{3}{7})(\frac{2}{7},\frac{1}{7},-\frac{3}{7},0,0,0,0,0)$	(4.32c)
s_{77}	2	0	0	0	0	0	-1	0	-1	$(-\frac{3}{7},-\frac{3}{7},\frac{2}{7},\frac{2}{7},\frac{2}{7},\frac{1}{7},\frac{2}{7},-\frac{1}{7})(\frac{3}{7},0,\frac{5}{7},0,0,0,0,0)$	(4.32a)
s_{99}	1	0	1	0	0	0	0	0	1	$(-\frac{1}{7},-\frac{1}{7},-\frac{2}{7},-\frac{2}{7},-\frac{2}{7},\frac{2}{7},0,\frac{4}{7})(-\frac{4}{7},\frac{2}{7},\frac{4}{7},0,0,0,0,0)$	(4.32d)
(1,1,1)	–	$-\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	1	$-\frac{1}{7}$	$\frac{6}{7}$	$\frac{1}{7}$	2	$(0,0,0,0,0,1,1,0)(0,0,0,0,0,0,0,0)$	–
s_{29}	4	0	0	0	1	0	0	0	1	$(-\frac{3}{14},-\frac{3}{14},\frac{3}{14},\frac{3}{14},\frac{3}{14},\frac{3}{14},\frac{11}{14},\frac{1}{2})(\frac{2}{7},-\frac{3}{7},-\frac{1}{7},0,0,0,0,0)$	(4.32a)
s_{40}	4	0	0	0	0	0	1	0	1	$(-\frac{5}{14},-\frac{5}{14},\frac{1}{14},\frac{1}{14},\frac{1}{14},\frac{11}{14},\frac{1}{14},\frac{3}{14})(-\frac{1}{7},0,\frac{3}{7},0,0,0,0,0)$	(4.32a)
s_{67}	2	0	0	0	-1	0	0	0	-1	$(\frac{1}{7},\frac{1}{7},-\frac{1}{7},-\frac{1}{7},-\frac{1}{7},-\frac{1}{7},\frac{1}{7},-1)(\frac{1}{7},\frac{2}{7},\frac{3}{7},0,0,0,0,0)$	(4.32b)
s_{116}	1	0	0	0	1	0	0	0	1	$(\frac{1}{14},\frac{1}{14},-\frac{1}{14},-\frac{1}{14},-\frac{1}{14},\frac{13}{14},\frac{1}{14},\frac{1}{2})(-\frac{3}{7},\frac{1}{7},-\frac{2}{7},0,0,0,0,0)$	(4.32a)
(1,1,1)	–	$-\frac{1}{7}$	$\frac{6}{7}$	$\frac{1}{7}$	$-\frac{1}{7}$	$-\frac{6}{7}$	$\frac{1}{7}$	1	1	$(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})(0,0,0,0,0,0,0,0)$	–
s_{45}	4	0	0	0	0	0	0	1	1	$(\frac{1}{14},\frac{1}{14},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{14},\frac{3}{14},\frac{1}{14})(\frac{1}{7},\frac{5}{7},-\frac{2}{7},0,0,0,0,0)$	(4.32a)
s_{80}	2	0	0	0	0	0	-1	0	-1	$(\frac{1}{14},\frac{1}{14},-\frac{3}{14},-\frac{3}{14},-\frac{3}{14},-\frac{5}{14},\frac{11}{14},\frac{5}{14})(-\frac{4}{7},0,-\frac{2}{7},0,0,0,0,0)$	(4.32d)
s_{86}	2	0	0	0	0	0	0	1	1	$(\frac{2}{7},\frac{2}{7},0,0,0,\frac{2}{7},\frac{6}{7},\frac{2}{7})(\frac{4}{7},-\frac{1}{7},-\frac{1}{7},0,0,0,0,0)$	(4.32a)
s_{103}	1	0	1	0	0	0	0	0	1	$(\frac{5}{14},\frac{5}{14},\frac{3}{14},\frac{3}{14},\frac{3}{14},-\frac{3}{14},\frac{1}{2},\frac{1}{14})(\frac{3}{7},\frac{2}{7},-\frac{3}{7},0,0,0,0,0)$	(4.32a)
s_{122}	1	0	0	0	0	-1	0	0	-1	$(-\frac{1}{14},-\frac{1}{14},-\frac{3}{14},-\frac{3}{14},-\frac{3}{14},-\frac{1}{2},-\frac{9}{14},\frac{3}{14})(\frac{1}{7},-\frac{3}{7},\frac{2}{7},0,0,0,0,0)$	(4.32a)
s_{126}	1	0	0	0	0	0	1	0	1	$(\frac{2}{7},\frac{2}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7},\frac{4}{7},\frac{1}{7},-\frac{4}{7})(\frac{5}{7},0,-\frac{1}{7},0,0,0,0,0)$	(4.32c)
s_{131}	1	0	0	0	0	0	0	-1	-1	$(\frac{1}{7},\frac{1}{7},0,0,0,\frac{1}{7},-\frac{4}{7},\frac{1}{7})(-\frac{5}{7},-\frac{4}{7},\frac{3}{7},0,0,0,0,0)$	(4.32b)
(1,1,1)	–	$\frac{1}{7}$	1	$\frac{13}{7}$	$-\frac{1}{7}$	$\frac{1}{7}$	$\frac{6}{7}$	$\frac{1}{7}$	4	$(0,0,0,0,0,0,0,0)(-1,0,-1,0,0,0,0,0)$	–
s_{11}	4	0	1	0	0	0	0	0	1	$(-\frac{4}{7},-\frac{4}{7},-\frac{1}{7},-\frac{1}{7},-\frac{1}{7},\frac{1}{7},0,\frac{2}{7})(-\frac{2}{7},\frac{1}{7},-\frac{5}{7},0,0,0,0,0)$	(4.32a)
s_{22}	4	0	0	1	0	0	0	0	1	$(\frac{5}{14},\frac{5}{14},-\frac{3}{14},-\frac{3}{14},-\frac{3}{14},-\frac{1}{14},-\frac{5}{14},\frac{9}{14})(0,-\frac{1}{7},-\frac{3}{7},0,0,0,0,0)$	(4.32c)
s_{59}	2	0	1	0	0	0	0	0	1	$(\frac{3}{14},\frac{3}{14},-\frac{1}{14},-\frac{1}{14},-\frac{1}{14},\frac{1}{14},-\frac{1}{2},-\frac{5}{14})(-\frac{1}{7},-\frac{3}{7},-\frac{6}{7},0,0,0,0,0)$	(4.32a)
s_{83}	2	0	0	0	0	0	1	0	1	$(\frac{1}{14},\frac{1}{14},-\frac{3}{14},-\frac{3}{14},-\frac{3}{14},-\frac{5}{14},-\frac{3}{14},-\frac{9}{14})(-\frac{4}{7},0,-\frac{2}{7},0,0,0,0,0)$	(4.32a)
s_{102}	1	0	-1	0	0	0	0	0	-1	$(\frac{5}{14},\frac{5}{14},\frac{3}{14},\frac{3}{14},\frac{3}{14},-\frac{3}{14},\frac{1}{2},\frac{1}{14})(-\frac{4}{7},\frac{2}{7},\frac{4}{7},0,0,0,0,0)$	(4.32b)
s_{105}	1	0	0	1	0	0	0	0	1	$(\frac{3}{14},\frac{3}{14},\frac{1}{14},\frac{1}{14},\frac{1}{14},\frac{5}{14},-\frac{3}{14},-\frac{3}{14})(0,-\frac{2}{7},-\frac{6}{7},0,0,0,0,0)$	(4.32a)
(1,1,1)	–	$\frac{6}{7}$	$-\frac{1}{7}$	$-\frac{1}{7}$	$-\frac{1}{7}$	$-\frac{1}{7}$	$-\frac{1}{7}$	$-\frac{1}{7}$	0	$(0,0,0,0,0,0,0,0)(1,1,0,0,0,0,0,0)$	–
s_3	1-7	$-\frac{1}{7}$	$-\frac{1}{7}$	$-\frac{1}{7}$	$-\frac{1}{7}$	$-\frac{1}{7}$	$-\frac{1}{7}$	$-\frac{1}{7}$	-1	$(0,0,0,0,0,0,0,0)(-1,-1,0,0,0,0,0,0)$	none
s_9	4	1	0	0	0	0	0	0	1	$(0,0,\frac{3}{7},\frac{3}{7},\frac{3}{7},-\frac{1}{7},-\frac{1}{7},\frac{3}{7})(\frac{3}{7},\frac{3}{7},0,0,0,0,0,0)$	(4.32a)
(1,1,1)	–	$\frac{6}{7}$	$-\frac{1}{7}$	$-\frac{1}{7}$	$\frac{13}{7}$	$\frac{13}{7}$	$-\frac{1}{7}$	$-\frac{1}{7}$	4	$(0,0,0,0,0,0,-1,-1)(0,0,0,0,0,0,0,0)$	–
s_4	1-7	$-\frac{1}{7}$	$-\frac{1}{7}$	$-\frac{1}{7}$	$-\frac{1}{7}$	$-\frac{1}{7}$	$-\frac{1}{7}$	$-\frac{1}{7}$	-1	$(0,0,0,0,0,0,1,1)(0,0,0,0,0,0,0,0)$	none
s_{32}	4	0	0	0	0	1	0	0	1	$(-\frac{2}{7},-\frac{2}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7},0,-\frac{4}{7},-\frac{1}{7})(-\frac{3}{7},\frac{2}{7},\frac{1}{7},0,0,0,0,0)$	(4.32a)
s_{33}	4	0	0	0	0	1	0	0	1	$(-\frac{2}{7},-\frac{2}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7},0,-\frac{4}{7},-\frac{1}{7})(-\frac{3}{7},\frac{2}{7},\frac{1}{7},0,0,0,0,0)$	(4.32a)
s_{47}	2	1	0	0	0	0	0	0	1	$(0,0,-\frac{2}{7},-\frac{2}{7},-\frac{2}{7},-\frac{4}{7},-\frac{4}{7},-\frac{2}{7})(-\frac{2}{7},-\frac{2}{7},0,0,0,0,0,0)$	(4.32a)
s_{69}	2	0	0	0	1	0	0	0	1	$(\frac{1}{7},\frac{1}{7},-\frac{1}{7},-\frac{1}{7},-\frac{1}{7},-\frac{1}{7},-\frac{6}{7},0)(\frac{1}{7},\frac{2}{7},\frac{3}{7},0,0,0,0,0)$	(4.32a)
s_{82}	2	0	0	0	0	0	1	0	1	$(\frac{4}{7},\frac{4}{7},\frac{2}{7},\frac{2}{7},\frac{2}{7},\frac{1}{7},\frac{2}{7},-\frac{1}{7})(-\frac{4}{7},0,-\frac{2}{7},0,0,0,0,0)$	(4.32d)
s_{110}	1	0	0	0	1	0	0	0	1	$(\frac{1}{14},\frac{1}{14},-\frac{1}{14},-\frac{1}{14},-\frac{1}{14},-\frac{1}{14},\frac{1}{14},-\frac{1}{2})(\frac{4}{7},\frac{1}{7},\frac{5}{7},0,0,0,0,0)$	(4.32c)
s_{123}	1	0	0	0	0	0	-1	0	-1	$(-\frac{3}{14},-\frac{3}{14},-\frac{5}{14},-\frac{5}{14},-\frac{5}{14},\frac{1}{14},\frac{9}{14},-\frac{1}{14})(\frac{5}{7},0,-\frac{1}{7},0,0,0,0,0)$	(4.32c)
(1,1,1)	–	$\frac{6}{7}$	$-\frac{1}{7}$	-1	$\frac{15}{7}$	$\frac{1}{7}$	$\frac{13}{7}$	$\frac{1}{7}$	4	$(0,0,0,0,0,0,1,-1)(0,0,0,0,0,0,0,0)$	–
s_{18}	4	0	0	-1	0	0	0	0	-1	$(-\frac{1}{7},-\frac{1}{7},\frac{2}{7},\frac{2}{7},\frac{2}{7},\frac{3}{7},-\frac{6}{7},\frac{1}{7})(0,-\frac{1}{7},-\frac{3}{7},0,0,0,0,0)$	(4.32a)
s_{38}	4	0	0	0	0	0	1	0	1	$(-\frac{5}{14},-\frac{5}{14},\frac{1}{14},\frac{1}{14},\frac{1}{14},-\frac{3}{14},\frac{1}{14},-\frac{11}{14})(-\frac{1}{7},0,\frac{3}{7},0,0,0,0,0)$	(4.32a)
s_{64}	2	0	0	-1	0	0	0	0	-1	$(\frac{3}{7},\frac{3}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7},-\frac{2}{7},-\frac{3}{7},\frac{4}{7})(0,-\frac{4}{7},\frac{2}{7},0,0,0,0,0)$	(4.32a)
s_{78}	2	0	0	0	0	0	1	0	1	$(-\frac{3}{7},-\frac{3}{7},\frac{2}{7},\frac{2}{7},\frac{2}{7},\frac{1}{7},\frac{2}{7},-\frac{1}{7})(-\frac{4}{7},0,-\frac{2}{7},0,0,0,0,0)$	(4.32c)
s_{97}	1	1	0	0	0	0	0	0	1	$(0,0,-\frac{1}{7},-\frac{1}{7},-\frac{1}{7},\frac{5}{7},\frac{5}{7},-\frac{1}{7})(-\frac{1}{7},-\frac{1}{7},0,0,0,0,0,0)$	(4.32a)
s_{104}	1	0	0	1	0	0	0	0	1	$(-\frac{2}{7},-\frac{2}{7},-\frac{3}{7},-\frac{3}{7},-\frac{3}{7},-\frac{1}{7},\frac{2}{7},\frac{2}{7})(0,\frac{5}{7},\frac{1}{7},0,0,0,0,0)$	(4.32b)
s_{111}	1	0	0	0	1	0	0	0	1	$(\frac{1}{14},\frac{1}{14},-\frac{1}{14},-\frac{1}{14},-\frac{1}{14},-\frac{1}{14},\frac{1}{14},-\frac{1}{2})(-\frac{3}{7},\frac{1}{7},-\frac{2}{7},0,0,0,0,0)$	(4.32a)

State	Sector	Local multiplicity							tot	$E_8 \times E_8$ root / P_{sh}	Redef
		1	2	3	4	5	6	7			
s_{113}	1	0	0	0	1	0	0	0	1	$(\frac{1}{14},\frac{1}{14},-\frac{1}{14},-\frac{1}{14},-\frac{1}{14},-\frac{1}{14},\frac{1}{14},-\frac{1}{2})(-\frac{3}{7},\frac{1}{7},-\frac{2}{7},0,0,0,0,0)$	(4.32a)
(1,1,1)	–	1	$-\frac{1}{7}$	$-\frac{1}{7}$	$\frac{1}{7}$	$\frac{13}{7}$	$\frac{1}{7}$	$-\frac{13}{7}$	1	$(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2})(0,0,0,0,0,0,0,0)$	–
s_{5}	4	1	0	0	0	0	0	0	1	$(-\frac{1}{2},-\frac{1}{2},-\frac{1}{14},-\frac{1}{14},-\frac{1}{14},\frac{5}{14},\frac{5}{14},-\frac{1}{14})(-\frac{4}{7},-\frac{4}{7},0,0,0,0,0,0)$	(4.32a)
s_{87}	2	0	0	0	0	0	0	-1	-1	$(\frac{2}{7},\frac{2}{7},0,0,0,-\frac{5}{7},-\frac{1}{7},\frac{2}{7})(\frac{4}{7},-\frac{1}{7},-\frac{1}{7},0,0,0,0,0)$	(4.32a)
s_{118}	1	0	0	0	0	1	0	0	1	$(-\frac{1}{14},-\frac{1}{14},-\frac{3}{14},-\frac{3}{14},-\frac{3}{14},\frac{1}{2},\frac{5}{14},\frac{3}{14})(\frac{1}{7},-\frac{3}{7},\frac{2}{7},0,0,0,0,0)$	(4.32a)
s_{119}	1	0	0	0	0	1	0	0	1	$(-\frac{1}{14},-\frac{1}{14},-\frac{3}{14},-\frac{3}{14},-\frac{3}{14},\frac{1}{2},\frac{5}{14},\frac{3}{14})(\frac{1}{7},-\frac{3}{7},\frac{2}{7},0,0,0,0,0)$	(4.32a)
s_{127}	1	0	0	0	0	0	0	-1	-1	$(\frac{1}{7},\frac{1}{7},0,0,0,\frac{1}{7},-\frac{4}{7},\frac{1}{7})(\frac{2}{7},-\frac{4}{7},-\frac{4}{7},0,0,0,0,0)$	(4.32c)
(1,1,1)	–	1	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{13}{7}$	$\frac{1}{7}$	$\frac{13}{7}$	$-\frac{1}{7}$	5	$(\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2})(0,0,0,0,0,0,0,0)$	–
s_{8}	4	1	0	0	0	0	0	0	1	$(\frac{1}{2},\frac{1}{2},-\frac{1}{14},-\frac{1}{14},-\frac{1}{14},\frac{5}{14},\frac{5}{14},-\frac{1}{14})(-\frac{4}{7},-\frac{4}{7},0,0,0,0,0,0)$	(4.32a)
s_{30}	4	0	0	0	1	0	0	0	1	$(\frac{2}{7},\frac{2}{7},-\frac{2}{7},-\frac{2}{7},-\frac{2}{7},-\frac{2}{7},\frac{2}{7},0)(\frac{2}{7},-\frac{3}{7},-\frac{1}{7},0,0,0,0,0)$	(4.32a)
s_{31}	4	0	0	0	1	0	0	0	1	$(\frac{2}{7},\frac{2}{7},-\frac{2}{7},-\frac{2}{7},-\frac{2}{7},-\frac{2}{7},\frac{2}{7},0)(\frac{2}{7},-\frac{3}{7},-\frac{1}{7},0,0,0,0,0)$	(4.32a)
s_{41}	4	0	0	0	0	0	1	0	1	$(\frac{1}{7},\frac{1}{7},-\frac{3}{7},-\frac{3}{7},-\frac{3}{7},\frac{2}{7},-\frac{3}{7},-\frac{2}{7})(-\frac{1}{7},0,\frac{3}{7},0,0,0,0,0)$	(4.32a)
s_{84}	2	0	0	0	0	0	1	0	1	$(\frac{1}{14},\frac{1}{14},-\frac{3}{14},-\frac{3}{14},-\frac{3}{14},\frac{9}{14},-\frac{3}{14},\frac{5}{14})(-\frac{4}{7},0,-\frac{2}{7},0,0,0,0,0)$	(4.32c)
(1,1,1)	–	1	$\frac{1}{7}$	$\frac{8}{7}$	$-\frac{8}{7}$	$\frac{1}{7}$	$-\frac{1}{7}$	$\frac{13}{7}$	3	$(0,0,0,0,0,1,-1,0)(0,0,0,0,0,0,0,0)$	–
s_{28}	4	0	0	0	-1	0	0	0	-1	$(\frac{2}{7},\frac{2}{7},-\frac{2}{7},-\frac{2}{7},-\frac{2}{7},-\frac{2}{7},\frac{2}{7},0)(-\frac{5}{7},\frac{4}{7},-\frac{1}{7},0,0,0,0,0)$	(4.32c)
s_{50}	2	1	0	0	0	0	0	0	1	$(0,0,-\frac{2}{7},-\frac{2}{7},-\frac{2}{7},\frac{3}{7},-\frac{4}{7},\frac{5}{7})(-\frac{2}{7},-\frac{2}{7},0,0,0,0,0,0)$	(4.32a)
s_{108}	1	0	0	1	0	0	0	0	1	$(\frac{3}{14},\frac{3}{14},\frac{1}{14},\frac{1}{14},\frac{1}{14},\frac{5}{14},-\frac{3}{14},-\frac{3}{14})(-1,-\frac{2}{7},\frac{1}{7},0,0,0,0,0)$	(4.32c)
s_{128}	1	0	0	0	0	0	0	1	1	$(\frac{1}{7},\frac{1}{7},0,0,0,\frac{1}{7},-\frac{4}{7},\frac{1}{7})(\frac{2}{7},\frac{3}{7},\frac{3}{7},0,0,0,0,0)$	(4.32a)
s_{129}	1	0	0	0	0	0	0	1	1	$(\frac{1}{7},\frac{1}{7},0,0,0,\frac{1}{7},-\frac{4}{7},\frac{1}{7})(\frac{2}{7},\frac{3}{7},\frac{3}{7},0,0,0,0,0)$	(4.32a)
(1,1,1)	–	$\frac{13}{7}$	$\frac{1}{7}$	$-\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	0	$-\frac{1}{7}$	2	$(-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2})(0,0,0,0,0,0,0,0)$	–
s_{6}	4	1	0	0	0	0	0	0	1	$(-\frac{1}{2},-\frac{1}{2},-\frac{1}{14},-\frac{1}{14},-\frac{1}{14},\frac{5}{14},\frac{5}{14},-\frac{1}{14})(\frac{3}{7},\frac{3}{7},0,0,0,0,0,0)$	(4.32c)
s_{24}	4	0	0	0	1	0	0	0	1	$(-\frac{3}{14},-\frac{3}{14},\frac{3}{14},\frac{3}{14},\frac{3}{14},\frac{3}{14},-\frac{3}{14},-\frac{1}{2})(-\frac{5}{7},\frac{4}{7},-\frac{1}{7},0,0,0,0,0)$	(4.32c)
s_{89}	1	1	0	0	0	0	0	0	1	$(-\frac{1}{2},-\frac{1}{2},\frac{5}{14},\frac{5}{14},\frac{5}{14},\frac{3}{14},\frac{3}{14},\frac{5}{14})(-\frac{1}{7},-\frac{1}{7},0,0,0,0,0,0)$	(4.32a)
s_{115}	1	0	0	0	-1	0	0	0	-1	$(\frac{1}{14},\frac{1}{14},-\frac{1}{14},-\frac{1}{14},-\frac{1}{14},-\frac{1}{14},\frac{1}{14},-\frac{1}{2})(\frac{4}{7},-\frac{6}{7},-\frac{2}{7},0,0,0,0,0)$	(4.32d)
(1,1,1)	–	$\frac{13}{7}$	$\frac{1}{7}$	$-\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	0	$\frac{6}{7}$	3	$(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2})(0,0,0,0,0,0,0,0)$	–
s_{16}	4	0	-1	0	0	0	0	0	-1	$(\frac{3}{7},\frac{3}{7},-\frac{1}{7},-\frac{1}{7},-\frac{1}{7},\frac{1}{7},0,\frac{2}{7})(-\frac{2}{7},-\frac{6}{7},\frac{2}{7},0,0,0,0,0)$	(4.32d)
s_{52}	2	1	0	0	0	0	0	0	1	$(\frac{1}{2},\frac{1}{2},\frac{3}{14},\frac{3}{14},\frac{3}{14},-\frac{1}{14},-\frac{1}{14},\frac{3}{14})(-\frac{2}{7},-\frac{2}{7},0,0,0,0,0,0)$	(4.32a)
s_{54}	2	1	0	0	0	0	0	0	1	$(\frac{1}{2},\frac{1}{2},\frac{3}{14},\frac{3}{14},\frac{3}{14},-\frac{1}{14},-\frac{1}{14},\frac{3}{14})(-\frac{2}{7},-\frac{2}{7},0,0,0,0,0,0)$	(4.32a)
s_{58}	2	0	1	0	0	0	0	0	1	$(-\frac{2}{7},-\frac{2}{7},\frac{3}{7},\frac{3}{7},\frac{3}{7},\frac{4}{7},0,\frac{1}{7})(-\frac{1}{7},\frac{4}{7},\frac{1}{7},0,0,0,0,0)$	(4.32c)
s_{88}	2	0	0	0	0	0	0	1	1	$(\frac{2}{7},\frac{2}{7},0,0,0,\frac{2}{7},-\frac{1}{7},-\frac{5}{7})(\frac{4}{7},-\frac{1}{7},-\frac{1}{7},0,0,0,0,0)$	(4.32a)
(1,1,1)	–	$\frac{13}{7}$	$\frac{1}{7}$	$-\frac{13}{7}$	1	$-\frac{1}{7}$	$-\frac{1}{7}$	$\frac{1}{7}$	1	$(-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2})(0,0,0,0,0,0,0,0)$	–
s_{23}	4	0	0	-1	0	0	0	0	-1	$(\frac{5}{14},\frac{5}{14},-\frac{3}{14},-\frac{3}{14},-\frac{3}{14},-\frac{1}{14},\frac{9}{14},-\frac{5}{14})(0,-\frac{1}{7},-\frac{3}{7},0,0,0,0,0)$	(4.32a)
s_{44}	4	0	0	0	0	0	0	1	1	$(\frac{1}{14},\frac{1}{14},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{14},\frac{3}{14},\frac{1}{14})(\frac{1}{7},-\frac{2}{7},\frac{5}{7},0,0,0,0,0)$	(4.32c)
s_{46}	2	1	0	0	0	0	0	0	1	$(-\frac{1}{2},-\frac{1}{2},\frac{3}{14},\frac{3}{14},\frac{3}{14},-\frac{1}{14},-\frac{1}{14},\frac{3}{14})(-\frac{2}{7},-\frac{2}{7},0,0,0,0,0,0)$	(4.32a)
s_{48}	2	1	0	0	0	0	0	0	1	$(-\frac{1}{2},-\frac{1}{2},\frac{3}{14},\frac{3}{14},\frac{3}{14},-\frac{1}{14},-\frac{1}{14},\frac{3}{14})(-\frac{2}{7},-\frac{2}{7},0,0,0,0,0,0)$	(4.32a)
s_{62}	2	0	0	-1	0	0	0	0	-1	$(-\frac{1}{14},-\frac{1}{14},-\frac{5}{14},-\frac{5}{14},-\frac{5}{14},\frac{3}{14},\frac{1}{14},\frac{1}{14})(0,\frac{3}{7},-\frac{5}{7},0,0,0,0,0)$	(4.32c)
s_{66}	2	0	0	0	1	0	0	0	1	$(-\frac{5}{14},-\frac{5}{14},\frac{5}{14},\frac{5}{14},\frac{5}{14},\frac{5}{14},-\frac{5}{14},\frac{1}{2})(\frac{1}{7},\frac{2}{7},\frac{3}{7},0,0,0,0,0)$	(4.32a)
s_{132}	1	0	0	0	0	0	0	-1	-1	$(\frac{1}{7},\frac{1}{7},0,0,0,\frac{1}{7},-\frac{4}{7},\frac{1}{7})(-\frac{5}{7},\frac{3}{7},-\frac{4}{7},0,0,0,0,0)$	(4.32d)
(1,1,1)	–	$\frac{13}{7}$	$-\frac{1}{7}$	$-\frac{13}{7}$	$\frac{13}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	1	3	$(0,0,0,0,0,1,0,-1)(0,0,0,0,0,0,0,0)$	–
s_{19}	4	0	0	-1	0	0	0	0	-1	$(-\frac{1}{7},-\frac{1}{7},\frac{2}{7},\frac{2}{7},\frac{2}{7},-\frac{4}{7},\frac{1}{7},\frac{1}{7})(0,-\frac{1}{7},-\frac{3}{7},0,0,0,0,0)$	(4.32a)
s_{20}	4	0	0	-1	0	0	0	0	-1	$(-\frac{1}{7},-\frac{1}{7},\frac{2}{7},\frac{2}{7},\frac{2}{7},-\frac{4}{7},\frac{1}{7},\frac{1}{7})(0,-\frac{1}{7},-\frac{3}{7},0,0,0,0,0)$	(4.32a)
s_{25}	4	0	0	0	1	0	0	0	1	$(-\frac{3}{14},-\frac{3}{14},\frac{3}{14},\frac{3}{14},\frac{3}{14},\frac{3}{14},-\frac{3}{14},-\frac{1}{2})(\frac{2}{7},-\frac{3}{7},-\frac{1}{7},0,0,0,0,0)$	(4.32a)
s_{26}	4	0	0	0	1	0	0	0	1	$(-\frac{3}{14},-\frac{3}{14},\frac{3}{14},\frac{3}{14},\frac{3}{14},\frac{3}{14},-\frac{3}{14},-\frac{1}{2})(\frac{2}{7},-\frac{3}{7},-\frac{1}{7},0,0,0,0,0)$	(4.32a)
s_{51}	2	1	0	0	0	0	0	0	1	$(0,0,-\frac{2}{7},-\frac{2}{7},-\frac{2}{7},\frac{3}{7},\frac{3}{7},-\frac{2}{7})(-\frac{2}{7},-\frac{2}{7},0,0,0,0,0,0)$	(4.32a)
s_{53}	2	1	0	0	0	0	0	0	1	$(0,0,-\frac{2}{7},-\frac{2}{7},-\frac{2}{7},\frac{3}{7},\frac{3}{7},-\frac{2}{7})(-\frac{2}{7},-\frac{2}{7},0,0,0,0,0,0)$	(4.32a)
s_{70}	2	0	0	0	1	0	0	0	1	$(\frac{1}{7},\frac{1}{7},-\frac{1}{7},-\frac{1}{7},-\frac{1}{7},\frac{6}{7},\frac{1}{7},0)(\frac{1}{7},\frac{2}{7},\frac{3}{7},0,0,0,0,0)$	(4.32a)
s_{112}	1	0	0	0	-1	0	0	0	-1	$(\frac{1}{14},\frac{1}{14},-\frac{1}{14},-\frac{1}{14},-\frac{1}{14},-\frac{1}{14},\frac{1}{14},-\frac{1}{2})(-\frac{3}{7},\frac{1}{7},-\frac{2}{7},0,0,0,0,0)$	(4.32b)
s_{133}	1	0	0	0	0	0	0	1	1	$(\frac{1}{7},\frac{1}{7},0,0,0,\frac{1}{7},\frac{3}{7},-\frac{6}{7})(\frac{2}{7},\frac{3}{7},\frac{3}{7},0,0,0,0,0)$	(4.32a)

State	Sector	Local multiplicity							tot	$E_8 \times E_8$ root / P_{sh}	Redef
		1	2	3	4	5	6	7			
(1,1,1)	–	$\frac{13}{7}$	1	-$\frac{13}{7}$	$\frac{1}{7}$	-$\frac{1}{7}$	$\frac{1}{7}$	-$\frac{1}{7}$	1	$(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2})(0,0,0,0,0,0,0,0)$	–
s_{10}	4	1	0	0	0	0	0	0	1	$(\frac{1}{2},\frac{1}{2},-\frac{1}{14},-\frac{1}{14},-\frac{1}{14},\frac{5}{14},\frac{5}{14},-\frac{1}{14})(\frac{3}{7},\frac{3}{7},0,0,0,0,0,0)$	(4.32c)
s_{15}	4	0	1	0	0	0	0	0	1	$(-\frac{1}{14},-\frac{1}{14},\frac{5}{14},\frac{5}{14},\frac{5}{14},-\frac{5}{14},\frac{1}{2},-\frac{3}{14})(\frac{5}{7},\frac{1}{7},\frac{2}{7},0,0,0,0,0)$	(4.32a)
s_{61}	2	0	0	-1	0	0	0	0	-1	$(-\frac{1}{14},-\frac{1}{14},-\frac{5}{14},-\frac{5}{14},-\frac{5}{14},\frac{3}{14},\frac{1}{14},\frac{1}{14})(0,-\frac{4}{7},\frac{2}{7},0,0,0,0,0)$	(4.32a)
s_{63}	2	0	0	-1	0	0	0	0	-1	$(-\frac{1}{14},-\frac{1}{14},-\frac{5}{14},-\frac{5}{14},-\frac{5}{14},\frac{3}{14},\frac{1}{14},\frac{1}{14})(0,-\frac{4}{7},\frac{2}{7},0,0,0,0,0)$	(4.32a)
s_{98}	1	1	0	0	0	0	0	0	1	$(\frac{1}{2},\frac{1}{2},\frac{5}{14},\frac{5}{14},\frac{5}{14},\frac{3}{14},\frac{3}{14},\frac{5}{14})(-\frac{1}{7},-\frac{1}{7},0,0,0,0,0,0)$	(4.32a)
(1,1,1)	–	$\frac{27}{7}$	-$\frac{13}{7}$	-$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	-1	-$\frac{1}{7}$	1	$(0,0,0,0,0,-1,0,-1)(0,0,0,0,0,0,0,0)$	–
s_{12}	4	0	-1	0	0	0	0	0	-1	$(\frac{3}{7},\frac{3}{7},-\frac{1}{7},-\frac{1}{7},-\frac{1}{7},\frac{1}{7},0,\frac{2}{7})(-\frac{2}{7},\frac{1}{7},-\frac{5}{7},0,0,0,0,0)$	(4.32c)
s_{35}	4	0	0	0	0	-1	0	0	-1	$(\frac{3}{14},\frac{3}{14},-\frac{5}{14},-\frac{5}{14},-\frac{5}{14},\frac{1}{2},-\frac{1}{14},-\frac{9}{14})(-\frac{3}{7},\frac{2}{7},\frac{1}{7},0,0,0,0,0)$	(4.32d)
s_{71}	2	0	0	0	0	1	0	0	1	$(-\frac{9}{14},-\frac{9}{14},\frac{1}{14},\frac{1}{14},\frac{1}{14},-\frac{1}{2},\frac{3}{14},-\frac{1}{14})(\frac{2}{7},\frac{1}{7},-\frac{3}{7},0,0,0,0,0)$	(4.32c)
s_{81}	2	0	0	0	0	0	-1	0	-1	$(\frac{1}{14},\frac{1}{14},-\frac{3}{14},-\frac{3}{14},-\frac{3}{14},\frac{9}{14},-\frac{3}{14},\frac{5}{14})(\frac{3}{7},0,\frac{5}{7},0,0,0,0,0)$	(4.32a)
s_{90}	1	1	0	0	0	0	0	0	1	$(0,0,-\frac{1}{7},-\frac{1}{7},-\frac{1}{7},-\frac{2}{7},-\frac{2}{7},-\frac{1}{7})(-\frac{1}{7},-\frac{1}{7},0,0,0,0,0,0)$	(4.32a)
s_{91}	1	1	0	0	0	0	0	0	1	$(0,0,-\frac{1}{7},-\frac{1}{7},-\frac{1}{7},-\frac{2}{7},-\frac{2}{7},-\frac{1}{7})(-\frac{1}{7},-\frac{1}{7},0,0,0,0,0,0)$	(4.32a)
s_{92}	1	1	0	0	0	0	0	0	1	$(0,0,-\frac{1}{7},-\frac{1}{7},-\frac{1}{7},-\frac{2}{7},-\frac{2}{7},-\frac{1}{7})(-\frac{1}{7},-\frac{1}{7},0,0,0,0,0,0)$	(4.32a)
s_{93}	1	1	0	0	0	0	0	0	1	$(0,0,-\frac{1}{7},-\frac{1}{7},-\frac{1}{7},-\frac{2}{7},-\frac{2}{7},-\frac{1}{7})(-\frac{1}{7},-\frac{1}{7},0,0,0,0,0,0)$	(4.32a)
s_{94}	1	1	0	0	0	0	0	0	1	$(0,0,-\frac{1}{7},-\frac{1}{7},-\frac{1}{7},-\frac{2}{7},-\frac{2}{7},-\frac{1}{7})(-\frac{1}{7},-\frac{1}{7},0,0,0,0,0,0)$	(4.32a)
s_{95}	1	-1	0	0	0	0	0	0	-1	$(0,0,-\frac{1}{7},-\frac{1}{7},-\frac{1}{7},-\frac{2}{7},\frac{5}{7},\frac{6}{7})(-\frac{1}{7},-\frac{1}{7},0,0,0,0,0,0)$	(4.32c)
s_{100}	1	0	-1	0	0	0	0	0	-1	$(-\frac{1}{7},-\frac{1}{7},-\frac{2}{7},-\frac{2}{7},-\frac{2}{7},\frac{2}{7},0,\frac{4}{7})(\frac{3}{7},\frac{2}{7},-\frac{3}{7},0,0,0,0,0)$	(4.32a)
s_{96}	1	1	0	0	0	0	0	0	1	$(0,0,-\frac{1}{7},-\frac{1}{7},-\frac{1}{7},\frac{5}{7},-\frac{2}{7},\frac{6}{7})(-\frac{1}{7},-\frac{1}{7},0,0,0,0,0,0)$	BM
s_{101}	1	0	1	0	0	0	0	0	1	$(-\frac{1}{7},-\frac{1}{7},-\frac{2}{7},-\frac{2}{7},-\frac{2}{7},-\frac{5}{7},0,-\frac{3}{7})(\frac{3}{7},\frac{2}{7},-\frac{3}{7},0,0,0,0,0)$	BM
s_{109}	1	0	0	1	0	0	0	0	1	$(\frac{3}{14},\frac{3}{14},\frac{1}{14},\frac{1}{14},\frac{1}{14},\frac{5}{14},-\frac{3}{14},-\frac{3}{14})(1,-\frac{2}{7},\frac{1}{7},0,0,0,0,0)$	BM
s_{114}	1	0	0	0	1	0	0	0	1	$(\frac{1}{14},\frac{1}{14},-\frac{1}{14},-\frac{1}{14},-\frac{1}{14},-\frac{1}{14},-\frac{13}{14},\frac{1}{2})(-\frac{3}{7},\frac{1}{7},-\frac{2}{7},0,0,0,0,0)$	BM
s_{121}	1	0	0	0	0	1	0	0	1	$(\frac{3}{7},\frac{3}{7},\frac{2}{7},\frac{2}{7},\frac{2}{7},0,-\frac{1}{7},\frac{5}{7})(\frac{1}{7},-\frac{3}{7},\frac{2}{7},0,0,0,0,0)$	BM
s_{125}	1	0	0	0	0	0	1	0	1	$(\frac{2}{7},\frac{2}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7},-\frac{3}{7},\frac{1}{7},\frac{3}{7})(-\frac{2}{7},0,\frac{6}{7},0,0,0,0,0)$	BM
s_{130}	1	0	0	0	0	0	0	1	1	$(\frac{1}{7},\frac{1}{7},0,0,0,-\frac{6}{7},\frac{3}{7},\frac{1}{7})(\frac{2}{7},\frac{3}{7},\frac{3}{7},0,0,0,0,0)$	BM
s_{49}	2	1	0	0	0	0	0	0	1	$(0,0,-\frac{2}{7},-\frac{2}{7},-\frac{2}{7},-\frac{4}{7},\frac{3}{7},\frac{5}{7})(-\frac{2}{7},-\frac{2}{7},0,0,0,0,0,0)$	BM
s_{60}	2	0	1	0	0	0	0	0	1	$(\frac{3}{14},\frac{3}{14},-\frac{1}{14},-\frac{1}{14},-\frac{1}{14},\frac{1}{14},-\frac{1}{2},-\frac{5}{14})(\frac{6}{7},-\frac{3}{7},\frac{1}{7},0,0,0,0,0)$	BM
s_{65}	2	0	0	1	0	0	0	0	1	$(\frac{3}{7},\frac{3}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7},-\frac{2}{7},\frac{4}{7},-\frac{3}{7})(0,-\frac{4}{7},\frac{2}{7},0,0,0,0,0)$	BM
s_{68}	2	0	0	0	1	0	0	0	1	$(\frac{1}{7},\frac{1}{7},-\frac{1}{7},-\frac{1}{7},-\frac{1}{7},-\frac{1}{7},\frac{1}{7},1)(\frac{1}{7},\frac{2}{7},\frac{3}{7},0,0,0,0,0)$	BM
s_{75}	2	0	0	0	0	1	0	0	1	$(\frac{5}{14},\frac{5}{14},\frac{1}{14},\frac{1}{14},\frac{1}{14},-\frac{1}{2},\frac{3}{14},-\frac{1}{14})(-\frac{5}{7},\frac{1}{7},\frac{4}{7},0,0,0,0,0)$	BM
s_{79}	2	0	0	0	0	0	1	0	1	$(\frac{1}{14},\frac{1}{14},-\frac{3}{14},-\frac{3}{14},-\frac{3}{14},-\frac{5}{14},-\frac{3}{14},-\frac{9}{14})(\frac{3}{7},0,\frac{5}{7},0,0,0,0,0)$	BM
s_{85}	2	0	0	0	0	0	0	1	1	$(-\frac{3}{14},-\frac{3}{14},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{3}{14},\frac{5}{14},-\frac{3}{14})(\frac{4}{7},-\frac{1}{7},-\frac{1}{7},0,0,0,0,0)$	BM
s_{7}	4	1	0	0	0	0	0	0	1	$(0,0,\frac{3}{7},\frac{3}{7},\frac{3}{7},-\frac{1}{7},-\frac{1}{7},\frac{3}{7})(-\frac{4}{7},-\frac{4}{7},0,0,0,0,0,0)$	BM
s_{13}	4	0	1	0	0	0	0	0	1	$(-\frac{4}{7},-\frac{4}{7},-\frac{1}{7},-\frac{1}{7},-\frac{1}{7},\frac{1}{7},0,\frac{2}{7})(\frac{5}{7},\frac{1}{7},\frac{2}{7},0,0,0,0,0)$	BM
s_{21}	4	0	0	1	0	0	0	0	1	$(-\frac{1}{7},-\frac{1}{7},\frac{2}{7},\frac{2}{7},\frac{2}{7},\frac{3}{7},\frac{1}{7},-\frac{6}{7})(0,-\frac{1}{7},-\frac{3}{7},0,0,0,0,0)$	BM
s_{27}	4	0	0	0	1	0	0	0	1	$(-\frac{3}{14},-\frac{3}{14},\frac{3}{14},\frac{3}{14},\frac{3}{14},-\frac{11}{14},-\frac{3}{14},\frac{1}{2})(\frac{2}{7},-\frac{3}{7},-\frac{1}{7},0,0,0,0,0)$	BM
s_{36}	4	0	0	0	0	1	0	0	1	$(-\frac{2}{7},-\frac{2}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7},0,\frac{3}{7},\frac{6}{7})(-\frac{3}{7},\frac{2}{7},\frac{1}{7},0,0,0,0,0)$	BM
s_{39}	4	0	0	0	0	0	1	0	1	$(-\frac{5}{14},-\frac{5}{14},\frac{1}{14},\frac{1}{14},\frac{1}{14},-\frac{3}{14},-\frac{13}{14},\frac{3}{14})(-\frac{1}{7},0,\frac{3}{7},0,0,0,0,0)$	BM
s_{43}	4	0	0	0	0	0	0	1	1	$(-\frac{3}{7},-\frac{3}{7},0,0,0,-\frac{3}{7},-\frac{2}{7},-\frac{3}{7})(\frac{1}{7},\frac{5}{7},-\frac{2}{7},0,0,0,0,0)$	BM

Appendix B

$U(1)$ basis for $T^6/\mathbb{Z}_7$ and its resolution

We use two different Cartan bases in the chapter. In the first basis, the anomalous direction on the orbifold is singled out. In the second basis, the gravity and non–abelian anomalies in blow–up are singled out. In the first basis:

$$Q^I_K = \begin{pmatrix}
3 & 3 & 1 & 1 & 1 & 5 & -3 & -3 & 0 & -4 & 2 & 0 & 0 & 0 & 0 & 0 \\
-15 & -15 & -5 & -5 & -5 & 59 & 15 & 15 & 0 & 20 & -10 & 0 & 0 & 0 & 0 & 0 \\
-3 & -3 & -1 & -1 & -1 & -5 & 3 & 3 & 0 & 4 & 40 & 0 & 0 & 0 & 0 & 0 \\
-3 & -3 & 27 & 27 & 27 & -5 & 3 & 3 & 0 & 4 & -2 & 0 & 0 & 0 & 0 & 0 \\
3 & 3 & 1 & 1 & 1 & 5 & -3 & 25 & 0 & -4 & 2 & 0 & 0 & 0 & 0 & 0 \\
3 & 3 & 1 & 1 & 1 & 5 & 25 & -3 & 0 & -4 & 2 & 0 & 0 & 0 & 0 & 0 \\
3 & 3 & 1 & 1 & 1 & 5 & -3 & -3 & 0 & 17 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1
\end{pmatrix}, \tag{B.1}$$

the components of the Cartan subalgebra are chosen such that the first row corresponds to the anomalous $U(1)$ generator (4.51). The next 7 rows correspond to other $U(1)$ generators perpendicular to $U(1)_A$. The last 8 rows are the Cartan basis of the non–abelian group factors $SU(3) \times SU(2) \times SO(10)$. The basis is given as $T_K = Q^I_K H_I$, where the H_I form an orthogonal basis for $E_8 \times E_8$ fulfilling $\mathrm{tr}(H_I H_J) = \delta_{IJ}$.

The second basis for the eight $U(1)$s in Section 3.7 is given by generators $T_I (K^{-1})^I_J$. The field strengths are related via $\bar{F}_I = K_I^J F_J$. The matrix K is given by

$$K_I^J = \begin{pmatrix}
25 & -20 & -25 & -4 & -66 & 18 & 25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{83}{6} & -\frac{34}{3} & \frac{73}{3} & \frac{10}{3} & \frac{116}{3} & -\frac{38}{3} & -\frac{187}{12} & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -2 & \frac{75}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 8 & -55 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{14}{3} & -\frac{137}{3} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{2}{3} & \frac{26}{3} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{3}{2} & \frac{125}{8} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} . \tag{B.2}$$

Appendix C

Anomaly polynomials in 4d

To obtain the factorization of the polynomial, one has to start with the expressions

$$\begin{aligned} X_{6,2} &= -\frac{1}{12}\left[\frac{3}{2}\mathrm{tr}(F'\mathcal{F}')\mathrm{tr}\mathcal{F}'^2 - \frac{3}{4}\mathrm{tr}(F'\mathcal{F}')\mathrm{tr}\mathcal{R}^2 +' \leftrightarrow''\right], && \text{(C.1)} \\ X_{4,4} &= \mathrm{tr}(\mathcal{F}'F')^2 + \frac{3}{4}\mathrm{tr}F'^2\mathrm{tr}\mathcal{F}'^2 - \frac{3}{8}\mathrm{tr}F'^2\mathrm{tr}\mathcal{R}^2 - \frac{1}{8}\mathrm{tr}\mathcal{F}'^2\mathrm{tr}R^2 +' \leftrightarrow'' \\ &\quad + \frac{1}{16}\mathrm{tr}R^2\mathrm{tr}\mathcal{R}^2 - \mathrm{tr}(\mathcal{F}'F')\mathrm{tr}(\mathcal{F}''F''). && \text{(C.2)} \end{aligned}$$

The anomaly polynomial factorization is given in terms of the following 2– and 4–forms

$$\begin{aligned} X_4^{\mathrm{uni}} &= X_{0,4} = (\mathrm{tr}R^2 - \mathrm{tr}F^2), && \text{(C.3)} \\ X_4^r &= \int\limits_{\mathcal{M}} E_{r_1}E_{r_2}E_r\left(V_{r_1}^{I'}V_{r_2}^{J'}F'_{I'}F'_{J'} + V_{r_1}^{I'}V_{r_2}^{I'}\left(\frac{3}{4}\mathrm{tr}F'^2 - \frac{1}{8}\mathrm{tr}R^2\right) +' \leftrightarrow'' - V_{r_1}^{I'}V_{r_2}^{I''}F_{I'}F_{I''}\right) \\ &\quad + \int\limits_{\mathcal{M}} \mathrm{tr}\mathcal{R}^2 E_r\left(\frac{1}{16}\mathrm{tr}R^2 - \frac{3}{8}\mathrm{tr}F'^2 - \frac{3}{8}\mathrm{tr}F''^2\right). && \text{(C.4)} \end{aligned}$$

Using the Bianchi identities (4.64) we obtain

$$\begin{aligned} X_4^r &= \int\limits_{\mathcal{M}} E_{r_1}E_{r_2}E_r\left(V_{r_1}^{I'}V_{r_2}^{J'}F'_{I'}F'_{J'} + V_{r_1}^{I'}V_{r_2}^{I'}\left(\frac{3}{4}\mathrm{tr}F'^2 - \frac{1}{8}\mathrm{tr}R^2\right) +' \leftrightarrow'' - V_{r_1}^{I'}V_{r_2}^{I''}F_{I'}F_{I''}\right) \\ &\quad + \int\limits_{\mathcal{M}} E_{r_1}E_{r_2}E_r V_{r_1}^{I}V_{r_2}^{I}\left(\frac{1}{16}\mathrm{tr}R^2 - \frac{3}{8}\mathrm{tr}F'^2 - \frac{3}{8}\mathrm{tr}F''^2\right). && \text{(C.5)} \end{aligned}$$

The 2–forms are given by

$$X_2^{\mathrm{uni}} = \int\limits_{\mathcal{M}} X_{6,2} = -\frac{1}{12}\int\limits_{\mathcal{M}}(\mathrm{tr}(\mathcal{F}'F')\mathrm{tr}\mathcal{F}'^2 - \frac{1}{2}\mathrm{tr}\mathcal{F}'^2\mathrm{tr}(\mathcal{F}''F'') - \frac{1}{4}\mathrm{tr}(\mathcal{F}'F')\mathrm{tr}\mathcal{R}^2 +' \leftrightarrow''), \tag{C.6}$$

$$X_2^r = \frac{1}{12}V_r^I F_I\,. \tag{C.7}$$

Appendix D

Brother models of a $T^6/\mathbb{Z}_{6II}$ heterotic orbifold?

Here we put together all the equations blow–up modes need to satisfy to be such, both from the orbifold and from the supergravity picture with abelian fluxes. The aim is to explore brother models of a given $\mathbb{Z}_{6II}$ model. We see that the physical states transformation consistency rules out the brothers for $\mathbb{Z}_{6II}$ [90].

First, there are the orbifold projection conditions. Looking at Tables F.8 and F.9, one can read for every V_r (i.e. its conjugacy class) which projection conditions need to be satisfied. The projection conditions for commuting elements read

$$V_r \cdot V_h - (q_{sh} + N^i - N^{*i})v_h^i - \frac{1}{2}(V_g \cdot V_h - v_g \cdot v_h) = 0 \text{ mod } 1, \tag{D.1}$$
$$N = \omega_i N^i + \bar{\omega}_i N^{*i}. \tag{D.2}$$

Second the mass equations for $M_L = 0$ left modes gives

$$V^2_{1,\beta\gamma} = \frac{25}{18} - 2N, \tag{D.3}$$
$$V^2_{3,\alpha\gamma} = \frac{3}{2} - 2N, \tag{D.4}$$
$$V^2_{2,\alpha\beta} = V^2_{4,\alpha\beta} = \frac{14}{9} - 2N. \tag{D.5}$$

And third, the mass equation for right–moving mode q_{sh} with $M_R = 0$ is

$$\frac{q^2_{sh}}{2} - \frac{1}{2} + \delta c_r = 0. \tag{D.6}$$

These relations together with the Bianchi identities (5.12)-(5.17) should be satisfied, if we fix the triangulation. Lets explore how to interpret a solution of Bianchi identities as a blow–up of a brother orbifold model to one of the V_{so10} Mini–landscape. Let us suppose we have a solution for equations (5.12)-(5.17), then knowing that we want the vectors $V_r = V_{3,11}, V_{3,13}$ to be present in the massless spectrum, they should obey the orbifold projection

$$V_r \cdot V_h - \left(0, \frac{1}{2}, 0, \frac{1}{2}\right) \cdot v_h = 0 \text{ mod } 1, \tag{D.7}$$

The vacuum phase $-\frac{1}{2}(V_g \cdot V_h - v_g \cdot v_h)$ is trivial here. We have made explicit the q_{sh} for θ^3 with the quantities $q = (0,0,0,1)$ and $3v = (0, \frac{1}{2}, 0, -\frac{1}{2})$. We assume that the left-moving oscillator number of these modes is $N = 0$. This is reasonable to guess since $V_{3,\alpha\gamma}^2 = \frac{3}{2}$ (equation for massless modes with no oscillators) solves the Bianchi identities and in the studied model the surviving states have this P_{sh}^2. Further we know from Tables F.8 and F.9 that the commuting elements for both fixed tori are $V_h = A_3(m_3 + m_4) + kV$ with $k = 0, ..., 5$ and that $v_h = k(0, \frac{1}{6}, \frac{1}{3}, -\frac{1}{2})$. So we have

$$\begin{aligned} V_r \cdot A_3(m_3 + m_4) + (V_r \cdot V_{so10}) + 1/6 = 0 \text{ mod } 1, \\ V_r \cdot A_3(m_3 + m_4) + 2(V_r \cdot V_{so10}) + 1/3 = 0 \text{ mod } 1, \\ V_r \cdot A_3(m_3 + m_4) + 3(V_r \cdot V_{so10}) + 1/2 = 0 \text{ mod } 1. \end{aligned} \tag{D.8}$$

Unfortunately for most of the Mini–landscape models the Bianchi identities solution in [83] has $V_{3,11}$ and $V_{3,13}$ projected out. So if that solution can be interpreted as an orbifold blow–up we should go to brothers or to a different set of orbifold models. For a brother model defined by

$$A_3' = A_3 + \Delta A_3, \quad V' = V_{so10} + \Delta V, \quad \Delta V \in \Lambda, \quad \Delta A_3 \in \Lambda, \tag{D.9}$$

the ΔA_3 and ΔV can be plugged in (D.8) and the set of linear equations solved. From these simple considerations we see that is feasible an exploration to obtain a solution of the Bianchi identities which has only massless modes, and search for a brother model to ones of the Mini–landscape, in which the blow–up modes are not projected out. However those brother models will be "bad" ones. In the following we will explain in which sense. This is directly related with the fact that the construction of physical states imposes a restriction on shifts and Wilson lines if consistent transformation under orbifold elements are required. This can be rephrased in imposing that the partition function [126] is singled valued.

Let us consider a twisted state $|\text{phys}\rangle_g$ with left–moving momentum $p_{sh} = p + V_g$ located at $g = (\theta^m, m_\alpha e_\alpha)$ with commuting elements denoted by $h = (\theta^k, n_\alpha e_\alpha)$. If the state is kept or projected in a model with shift $V + \Delta V$ is determined by the equation

$$kp \cdot \Delta V - \frac{mk}{2}\Delta V^2 + \frac{km_3 - mn_3}{3 \times 2}(3A_3 \cdot \Delta V) + \frac{km_5 - mn_5}{2 \times 2}(2A_5 \cdot \Delta V). \tag{D.10}$$

The requirement that physical states have a proper transformation [89] imposes

$$(3A_3 \cdot \Delta V) = (2A_5 \cdot \Delta V) = 0 \quad \text{mod } 2. \tag{D.11}$$

Taking into account that ΔV^2 is even ($\Gamma_8 \times \Gamma_8$ is an even lattice) and checking over all of the fixed points of $\mathbb{Z}_{6II}$ with its respective projection conditions (determined by the commuting elements) is not possible that an state existing in model with shift V is projected out in the "brother model" with shift $V + \Delta V$. So, there are no brother models of Z_{6II} differing in a lattice vector of the shift. In the same way we have checked that is also not possible to construct different models by changing the Wilson lines to be $A_3 + \Delta A_3$ or $A_5 + \Delta A_5$.

The restriction for the transformation phase of a physical state $|\mathrm{phys}\rangle_g$ at fixed point g under h to be consistent i.e. $\Phi(g,h) = \Phi(g^{n+1}, h)$ [89] eliminates the possibility of brother models in $\mathbb{Z}_{6II}$. However with the traditional restrictions for shifts and Wilson lines coming from modular invariance conditions $(6A_3 \cdot \Delta V) = (6A_5 \cdot \Delta V) = 0 \mod 1$ brother models are allowed. An example is a model generated with ΔV given by

$$\Delta V = (-4, -1, 2, -1, -1, -1, 0, 0, -7, 2, 5, 5, 4, 4, 1, -18). \tag{D.12}$$

Which satisfies $3A_3 \cdot \Delta \in 2\mathbb{Z} + 1$ which circumvents the prohibition imposed by D.11, projecting out various states.

Appendix E

$T^6/\mathbb{Z}_{6II}$ divisors intersections

The set of the intersection numbers coming from the $\mathbb{C}^3/\mathbb{Z}_{6-II}$ singularity, can be divided in the ones independent from the triangulation and the ones dependent. Here the triangulation independent intersections are

$$\begin{gathered} R_1R_2R_3 = 6,\ R_1R_2D_{3,\gamma} = 3,\ R_1R_3D_{2,\beta} = 2,\ R_1D_{2,\beta}D_{3,\gamma} = 1, \\ R_1D_{2,\beta}D_{3,\gamma} = R_3D_{2,\beta}E_{2,1\beta} = R_3E_{2,1\beta}E_{4,1\beta} = R_3E_{4,1\beta}D_{1,1} = 1, \\ R_3E_{4,1\beta}D_{1,1} = R_3R_2D_{1,1} = R_2D_{1,1}E_{3,\gamma} = R_2 = E_{3,1\gamma}D_{3,\gamma} = 1. \end{gathered} \tag{E.1}$$

The triangulation dependent ones for this local singularity can be read in Table E.1.

Table E.1: Triangulation dependent intersections of distinct divisors.

Triangulation	Intersections
1	$D_{2,\beta}E_{2,1\beta}E_{1,\beta\gamma} = D_{2,\beta}E_{1,\beta\gamma}D_{3,\gamma} = E_{2,1\beta}E_{4,1\beta}E_{1,\beta\gamma} = 1,$ $E_{4,1\beta}E_{3,1\gamma}D_{1,1} = D_{3,\gamma}E_{1,\beta\gamma} = E_{4,1\beta} = E_{3,1\gamma}E_{4,1\beta}D_{3,\gamma} = 1.$

The intersections for the local singularity $\mathbb{C}^2/\mathbb{Z}_2$ are given by

$$\begin{gathered} R_2R_3D_{1,2} = 3, D_{3,\gamma}R_2E_{3,2\gamma} = 3, E_{3,2\gamma}D_{1,2}R_2 = 3, \\ D_{1,2}D_{2,\beta}R_3 = 1, E_{3,2\gamma}D_{2,\beta}D_{1,2} = 1, D_{3,\gamma}D_{2,\beta}E_{3,2\gamma} = 1. \end{gathered} \tag{E.2}$$

while the intersection for the local $\mathbb{C}^2/\mathbb{Z}_3$ singularity are

$$\begin{aligned} D_{3,\gamma}D_{1,3}E_{4,3\beta} &= 1, D_{3,\gamma}E_{4,3\beta}E_{2,3\beta} = 1, D_{3,\gamma}E_{2,3\beta}D_{2,\beta} = 1, \\ D_{1,3}E_{4,3\beta}R_3 &= 2, E_{4,3\beta}E_{2,3\beta}R_3 = 2, E_{2,3\beta}D_{2,\beta}R_3 = 2, \\ D_{1,3}R_2R_3 &= 1, D_{1,3}R_2D_{3,\gamma} = 1. \end{aligned} \tag{E.3}$$

Starting with the intersection numbers of distinct divisors, and computing from them all self intersections [83] for triangulation B one obtains

$$\begin{aligned} E_{1,\beta\gamma}^3 &= 6, E_{2,1\beta}^3 = 8, E_{3,1\gamma}^3 = 8, E_{4,1\beta}^3 = 8, E_{1,\beta\gamma}E_{2,1\beta}^2 = -2, \\ E_{1,\beta\gamma}E_{3,1\gamma}^2 &= -2, E_{1,\beta\gamma}E_{4,1\beta}^2 = -2, E_{1,\beta\gamma}E_{2,1\beta}E_{4,1\beta} = 1, E_{2,1\beta}^2 E_{4,1\beta} = -2, \\ c_2(\mathcal{M})E_{2,1\beta} &= c_2(\mathcal{M})E_{4,1\beta} = c_2(\mathcal{M})E_{3,1\gamma} = -4,\ c_2(\mathcal{M})R_2 = c_2(\mathcal{M})R_3 = 24. \end{aligned} \tag{E.4}$$

The second Chern-class of the manifold $c_2(\mathcal{M})$ is the piece of degree two in the formal variables D_J, E_r and R_i in the total Chern-class [83] according to

$$c(\mathcal{M}) = \prod_{J,r}(1 + D_J)(1 + E_r)(1 - R_1)(1 - R_2)(1 - R_3)^2. \tag{E.5}$$

The Bianchi identities selecting a different triangulation in all local resolutions are more difficult to fulfill. Here, one can see a sample of this computation as obtained by our computer scan:

$$\begin{aligned}
0 &= -8 + 8V_{1,1,1}.V_{1,1,1} - 4V_{1,1,1}.V_{4,1,1} - 2V_{2,1,1}.V_{2,1,1} + 2V_{2,1,1}.V_{4,1,1}, \\
0 &= -8 + 8V_{1,1,2}.V_{1,1,2} - 2V_{1,1,2}.V_{2,1,1} - 4V_{1,1,2}.V_{3,1,2} - V_{2,1,1}.V_{2,1,1} + 2V_{2,1,1}.V_{3,1,2}, -4 + 7V_{1,1,3}.V_{1,1,3} - \\
&\quad 2V_{1,1,3}.V_{3,1,3} - 2V_{1,1,3}.V_{4,1,1} - 2V_{2,1,1}.V_{2,1,1} + 2V_{2,1,1}.V_{4,1,1} - V_{3,1,3}.V_{3,1,3} + 2V_{3,1,3}.V_{4,1,1} - V_{4,1,1}.V_{4,1,1}, \\
0 &= -4 + 7V_{1,1,4}.V_{1,1,4} - 2V_{1,1,4}.V_{3,1,4} - 2V_{1,1,4}.V_{4,1,1} - \\
&\quad 2V_{2,1,1}.V_{2,1,1} + 2V_{2,1,1}.V_{4,1,1} - V_{3,1,4}.V_{3,1,4} + 2V_{3,1,4}.V_{4,1,1} - V_{4,1,1}.V_{4,1,1}, \\
0 &= -8 + 8V_{1,2,1}.V_{1,2,1} - 2V_{1,2,1}.V_{2,1,2} - 4V_{1,2,1}.V_{3,1,1} - V_{2,1,2}.V_{2,1,2} + 2V_{2,1,2}.V_{3,1,1}, \\
0 &= 6V_{1,2,2}.V_{1,2,2} - 2V_{2,1,2}.V_{2,1,2} + 2V_{2,1,2}.V_{4,1,2} - 2V_{3,1,2}.V_{3,1,2} - 2V_{4,1,2}.V_{4,1,2}, \\
0 &= -8 + 8V_{1,2,3}.V_{1,2,3} - 4V_{1,2,3}.V_{4,1,2} - 2V_{2,1,2}.V_{2,1,2} + 2V_{2,1,2}.V_{4,1,2}, \\
0 &= 6V_{1,2,4}.V_{1,2,4} - 2V_{2,1,2}.V_{2,1,2} + 2V_{2,1,2}.V_{4,1,2} - 2V_{3,1,4}.V_{3,1,4} - 2V_{4,1,2}.V_{4,1,2}, \\
0 &= -4 + 7V_{1,3,1}.V_{1,3,1} - 2V_{1,3,1}.V_{3,1,1} - 2V_{1,3,1}.V_{4,1,3} \\
&\quad -2V_{2,1,3}.V_{2,1,3} + 2V_{2,1,3}.V_{4,1,3} - V_{3,1,1}.V_{3,1,1} + 2V_{3,1,1}.V_{4,1,3} - V_{4,1,3}.V_{4,1,3}, \\
0 &= -4 + 7V_{1,3,2}.V_{1,3,2} - 2V_{1,3,2}.V_{3,1,2} - 2V_{1,3,2}.V_{4,1,3} \\
&\quad -2V_{2,1,3}.V_{2,1,3} + 2V_{2,1,3}.V_{4,1,3} - V_{3,1,2}.V_{3,1,2} + 2V_{3,1,2}.V_{4,1,3} - V_{4,1,3}.V_{4,1,3}, \\
0 &= -12 + 9V_{1,3,3}.V_{1,3,3} - 6V_{1,3,3}.V_{3,1,3} + V_{3,1,3}.V_{3,1,3},
\end{aligned}$$

$$
\begin{aligned}
0 &= -8 + 8V_{1,3,4}.V_{1,3,4} - 2V_{1,3,4}.V_{2,1,3} - 4V_{1,3,4}.V_{3,1,4} - V_{2,1,3}.V_{2,1,3} + 2V_{2,1,3}.V_{3,1,4},\\
0 &= -4 - 4V_{1,1,1}.V_{2,1,1} + 2V_{1,1,1}.V_{4,1,1} - V_{1,1,2}.V_{1,1,2} - 2V_{1,1,2}.V_{2,1,1} + 2V_{1,1,2}.V_{3,1,2} - 4V_{1,1,3}.V_{2,1,1} + 2V_{1,1,3}.V_{4,1,1}\\
&\quad -4V_{1,1,4}.V_{2,1,1} + 2V_{1,1,4}.V_{4,1,1} + 7V_{2,1,1}.V_{2,1,1} - 2V_{2,1,1}.V_{3,1,2}\\
&\quad -2V_{2,1,1}.V_{4,1,1} - V_{3,1,2}.V_{3,1,2} + 2V_{3,1,2}.V_{4,1,1} - V_{4,1,1}.V_{4,1,1},\\
0 &= -4 - V_{1,2,1}.V_{1,2,1} - 2V_{1,2,1}.V_{2,1,2} + 2V_{1,2,1}.V_{3,1,1} - 4V_{1,2,2}.V_{2,1,2} + 2V_{1,2,2}.V_{4,1,2} - 4V_{1,2,3}.V_{2,1,2} +\\
&\quad 2V_{1,2,3}.V_{4,1,2} - 4V_{1,2,4}.V_{2,1,2} + 2V_{1,2,4}.V_{4,1,2} + 7V_{2,1,2}.V_{2,1,2} - 2V_{2,1,2}.V_{3,1,1} - 2V_{2,1,2}.V_{4,1,2} -\\
&\quad V_{3,1,1}.V_{3,1,1} + 2V_{3,1,1}.V_{4,1,2} - V_{4,1,2}.V_{4,1,2},\\
0 &= -4 - 4V_{1,3,1}.V_{2,1,3} + 2V_{1,3,1}.V_{4,1,3} - 4V_{1,3,2}.V_{2,1,3} + 2V_{1,3,2}.V_{4,1,3} - V_{1,3,4}.V_{1,3,4} - 2V_{1,3,4}.V_{2,1,3} + 2V_{1,3,4}.V_{3,1,4} +\\
&\quad 7V_{2,1,3}.V_{2,1,3} - 4V_{2,1,3}.V_{3,1,3} - 2V_{2,1,3}.V_{3,1,4} + 2V_{3,1,3}.V_{4,1,3} -\\
&\quad V_{3,1,4}.V_{3,1,4} + 2V_{3,1,4}.V_{4,1,3} - 2V_{4,1,3}.V_{4,1,3},\\
0 &= 8 - 2V_{1,1,1}.V_{1,1,1} + 2V_{1,1,1}.V_{2,1,1} - V_{1,1,3}.V_{1,1,3} + 2V_{1,1,3}.V_{2,1,1} + 2V_{1,1,3}.V_{3,1,3} - 2V_{1,1,3}.V_{4,1,1} -\\
&\quad V_{1,1,4}.V_{1,1,4} + 2V_{1,1,4}.V_{2,1,1} + 2V_{1,1,4}.V_{3,1,4} - 2V_{1,1,4}.V_{4,1,1} - V_{2,1,1}.V_{2,1,1} + 2V_{2,1,1}.V_{3,1,2} -\\
&\quad 2V_{2,1,1}.V_{4,1,1} - 2V_{3,1,1}.V_{3,1,1} - 4V_{3,1,2}.V_{4,1,1} - V_{3,1,3}.V_{3,1,3} - 2V_{3,1,3}.V_{4,1,1} - V_{3,1,4}.V_{3,1,4}\\
&\quad -2V_{3,1,4}.V_{4,1,1} + 4V_{4,1,1}.V_{4,1,1},\\
0 &= 2V_{1,2,2}.V_{2,1,2} - 4V_{1,2,2}.V_{4,1,2} - 2V_{1,2,3}.V_{1,2,3} + 2V_{1,2,3}.V_{2,1,2} + 2V_{1,2,4}.V_{2,1,2} - 4V_{1,2,4}.V_{4,1,2} -\\
&\quad V_{2,1,2}.V_{2,1,2} + 2V_{2,1,2}.V_{3,1,1} - 2V_{2,1,2}.V_{4,1,2} - 4V_{3,1,1}.V_{4,1,2} - 2V_{3,1,3}.V_{3,1,3} + 6V_{4,1,2}.V_{4,1,2},
\end{aligned}
$$

$$
\begin{aligned}
0 &= -V_{1,3,1}.V_{1,3,1} + 2V_{1,3,1}.V_{2,1,3} + 2V_{1,3,1}.V_{3,1,1} - 2V_{1,3,1}.V_{4,1,3} - V_{1,3,2}.V_{1,3,2} +\\
&\quad 2V_{1,3,2}.V_{2,1,3} + 2V_{1,3,2}.V_{3,1,2} - 2V_{1,3,2}.V_{4,1,3} + 2V_{2,1,3}.V_{3,1,3} + 2V_{2,1,3}.V_{3,1,4} - 4V_{2,1,3}.V_{4,1,3} -\\
&\quad V_{3,1,1}.V_{3,1,1} - 2V_{3,1,1}.V_{4,1,3} - V_{3,1,2}.V_{3,1,2} - 2V_{3,1,2}.V_{4,1,3} - 4V_{3,1,3}.V_{4,1,3} - 4V_{3,1,4}.V_{4,1,3} + 6V_{4,1,3}.V_{4,1,3},\\
0 &= 4 - 2V_{1,2,1}.V_{1,2,1} + 2V_{1,2,1}.V_{2,1,2} - V_{1,3,1}.V_{1,3,1} - 2V_{1,3,1}.V_{3,1,1} + 2V_{1,3,1}.V_{4,1,3} - V_{2,1,2}.V_{2,1,2} -\\
&\quad 2V_{2,1,2}.V_{3,1,1} + 2V_{2,1,2}.V_{4,1,2} + 5V_{3,1,1}.V_{3,1,1} - 4V_{3,1,1}.V_{4,1,1} - 2V_{3,1,1}.V_{4,1,3} - 2V_{4,1,2}.V_{4,1,2} - V_{4,1,3}.V_{4,1,3},\\
0 &= 4 - 2V_{1,1,2}.V_{1,1,2} + 2V_{1,1,2}.V_{2,1,1} - 4V_{1,2,2}.V_{3,1,2} - V_{1,3,2}.V_{1,3,2} - 2V_{1,3,2}.V_{3,1,2} + 2V_{1,3,2}.V_{4,1,3} -\\
&\quad V_{2,1,1}.V_{2,1,1} - 2V_{2,1,1}.V_{3,1,2} + 2V_{2,1,1}.V_{4,1,1} + 5V_{3,1,2}.V_{3,1,2} - 2V_{3,1,2}.V_{4,1,3} - 2V_{4,1,1}.V_{4,1,1} - V_{4,1,3}.V_{4,1,3},\\
0 &= 8 - V_{1,1,3}.V_{1,1,3} - 2V_{1,1,3}.V_{3,1,3} + 2V_{1,1,3}.V_{4,1,1} - 3V_{1,3,3}.V_{1,3,3} + 2V_{1,3,3}.V_{3,1,3} - 2V_{2,1,3}.V_{2,1,3} +\\
&\quad 2V_{2,1,3}.V_{4,1,3} + 4V_{3,1,3}.V_{3,1,3} - 2V_{3,1,3}.V_{4,1,1} - 4V_{3,1,3}.V_{4,1,2} - V_{4,1,1}.V_{4,1,1} - 2V_{4,1,3}.V_{4,1,3},\\
0 &= 48 - 2V_{3,1,1}.V_{3,1,1} - 2V_{3,1,2}.V_{3,1,2} - 2V_{3,1,3}.V_{3,1,3} - 2V_{3,1,4}.V_{3,1,4} -\\
&\quad 6V_{3,2,1}.V_{3,2,1} - 6V_{3,2,2}.V_{3,2,2} - 6V_{3,2,3}.V_{3,2,3} - 6V_{3,2,4}.V_{3,2,4},\\
0 &= 48 - 2V_{2,1,1}.V_{2,1,1} + 2V_{2,1,1}.V_{4,1,1} - 2V_{2,1,2}.V_{2,1,2} + 2V_{2,1,2}.V_{4,1,2} - 2V_{2,1,3}.V_{2,1,3} + 2V_{2,1,3}.V_{4,1,3} -\\
&\quad 4V_{2,3,1}.V_{2,3,1} + 4V_{2,3,1}.V_{4,3,1} - 4V_{2,3,2}.V_{2,3,2} + 4V_{2,3,2}.V_{4,3,2} - 4V_{2,3,3}.V_{2,3,3} + 4V_{2,3,3}.V_{4,3,3} -\\
&\quad 2V_{4,1,1}.V_{4,1,1} - 2V_{4,1,2}.V_{4,1,2} - 2V_{4,1,3}.V_{4,1,3} - 4V_{4,3,1}.V_{4,3,1} - 4V_{4,3,2}.V_{4,3,2} - 4V_{4,3,3}.V_{4,3,3},
\end{aligned}
$$

Appendix F

Orbifold tables

Table F.1: Set of blow–up modes for Model 28 in triangulation B.

V_r	Numerical value for V_r	irrep.	Orbifold state
$V_{3,11}$	$(0,-\frac{1}{2},\frac{1}{2},0,0,0,0,0,0,1,0,0,0,0,0,0)$	$\{\mathbf{1},r\}$	ψ_{155}
$V_{3,12}$	$(\frac{1}{2},0,\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0,0,\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},\frac{1}{4})$	$\{\mathbf{1},r\}$	ψ_{153}
$V_{3,13}$	$(0,-\frac{1}{2},\frac{1}{2},0,0,0,0,0,0,1,0,0,0,0,0,0)$	$\{\mathbf{1},r\}$	ψ_{154}
$V_{3,14}$	$(\frac{1}{2},0,\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0,0,\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},\frac{1}{4})$	$\{\mathbf{1},r\}$	ψ_{150}
$V_{1,11}$	$(-\frac{1}{6},0,0,-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},0,\frac{1}{3},0,0,0,0,0,0)$	$\{\mathbf{1},r\}$	ψ_{57}
$V_{1,21}$	$(-\frac{1}{2},-\frac{1}{3},0,\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},-\frac{5}{6},\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},\frac{1}{6},\frac{1}{6})$	$\{\{1,1,6\},r\}$	ψ_{86}
$V_{1,31}$	$(\frac{1}{6},-\frac{2}{3},0,-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},\frac{1}{3},0,-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},\frac{1}{3},\frac{1}{3})$	$\{\mathbf{1},r\}$	ψ_{34}
$V_{1,12}$	$(-\frac{1}{6},0,\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0,0,-\frac{1}{4},-\frac{5}{12},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},-\frac{1}{4})$	$\{\mathbf{1},r\}$	ψ_{44}
$V_{1,22}$	$(0,\frac{1}{6},0,-\frac{1}{3},-\frac{1}{3},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{5}{12},-\frac{1}{12},-\frac{5}{12},-\frac{5}{12},-\frac{5}{12},-\frac{5}{12},-\frac{1}{12},\frac{5}{12})$	$\{\mathbf{1},r\}$	ψ_{77}
$V_{1,32}$	$(-\frac{1}{3},\frac{5}{6},0,\frac{1}{3},\frac{1}{3},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},\frac{1}{12},\frac{1}{4},-\frac{1}{12},-\frac{1}{12},-\frac{1}{12},-\frac{1}{12},-\frac{5}{12},\frac{1}{12})$	$\{\mathbf{1},r\}$	ψ_{21}
$V_{1,13}$	$(-\frac{1}{6},0,0,-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},0,\frac{1}{3},0,0,0,0,0,0)$	$\{\mathbf{1},r\}$	ψ_{45}
$V_{1,23}$	$(-\frac{1}{2},-\frac{1}{3},0,\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},-\frac{5}{6},\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},\frac{1}{6},\frac{1}{6})$	$\{\{1,1,6\},r\}$	ψ_{83}
$V_{1,33}$	$(\frac{1}{6},-\frac{2}{3},0,-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},\frac{1}{3},0,-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},\frac{1}{3},\frac{1}{3})$	$\{\mathbf{1},r\}$	ψ_{28}
$V_{1,14}$	$(-\frac{1}{6},0,\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0,0,-\frac{1}{4},-\frac{5}{12},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},-\frac{1}{4})$	$\{\mathbf{1},r\}$	ψ_{41}
$V_{1,24}$	$(0,\frac{1}{6},0,-\frac{1}{3},-\frac{1}{3},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{5}{12},-\frac{1}{12},-\frac{5}{12},-\frac{5}{12},-\frac{5}{12},-\frac{5}{12},-\frac{1}{12},\frac{5}{12})$	$\{\mathbf{1},r\}$	ψ_{70}
$V_{1,34}$	$(-\frac{1}{3},\frac{5}{6},0,\frac{1}{3},\frac{1}{3},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},\frac{1}{12},\frac{1}{4},-\frac{1}{12},-\frac{1}{12},-\frac{1}{12},-\frac{1}{12},-\frac{5}{12},\frac{1}{12})$	$\{\mathbf{1},r\}$	ψ_{14}
$V_{2,11}$	$(-\frac{1}{3},0,1,0,0,0,0,0,0,\frac{2}{3},0,0,0,0,0,0)$	$\{\mathbf{1},r\}$	ψ_{115}
$V_{4,11}$	$(-\frac{2}{3},0,0,0,0,0,0,0,-1,\frac{1}{3},0,0,0,0,0,0)$	$\{\{1,1,6\},r\}$	ψ_{182}
$V_{2,12}$	$(\frac{1}{2},-\frac{1}{6},\frac{1}{2},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},\frac{1}{3},\frac{1}{3},-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},\frac{1}{3},\frac{1}{3})$	$\{\mathbf{1},l\}$	$\bar{\psi}_{36}$
$V_{4,12}$	$(\frac{1}{2},-\frac{5}{6},-\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},\frac{1}{6},\frac{1}{6})$	$\{\mathbf{1},r\}$	ψ_{187}
$V_{2,13}$	$(-\frac{2}{3},-\frac{1}{3},0,-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},\frac{1}{6},-\frac{1}{2},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},\frac{1}{6},\frac{1}{6})$	$\{\mathbf{1},l\}$	$\bar{\psi}_{44}$
$V_{4,13}$	$(\frac{1}{6},-\frac{1}{6},-\frac{1}{2},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{2}{3},0,-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},\frac{1}{3},\frac{1}{3})$	$\{\{1,1,6\},l\}$	$\bar{\psi}_{107}$
$V_{2,31}$	$(-\frac{1}{3},0,-1,0,0,0,0,0,0,\frac{2}{3},0,0,0,0,0,0)$	$\{\mathbf{1},l\}$	$\bar{\psi}_{21}$
$V_{2,32}$	$(-\frac{1}{2},\frac{5}{6},\frac{1}{2},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},-\frac{1}{6},-\frac{1}{6})$	$\{\mathbf{1},l\}$	$\bar{\psi}_{14}$
$V_{2,33}$	$(-\frac{2}{3},-\frac{1}{3},0,-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},\frac{1}{6},-\frac{1}{2},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},\frac{1}{6},\frac{1}{6})$	$\{\mathbf{1},l\}$	$\bar{\psi}_{27}$
$V_{4,31}$	$(-\frac{2}{3},0,0,0,0,0,0,0,-1,\frac{1}{3},0,0,0,0,0,0)$	$\{\{1,1,6\},l\}$	$\bar{\psi}_{92}$
$V_{4,32}$	$(-\frac{1}{2},\frac{1}{6},-\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},-\frac{1}{3},-\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},-\frac{1}{3},-\frac{1}{3})$	$\{\mathbf{1},l\}$	$\bar{\psi}_{98}$
$V_{4,33}$	$(\frac{1}{6},-\frac{1}{6},-\frac{1}{2},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{2}{3},0,-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},\frac{1}{3},\frac{1}{3})$	$\{\{1,1,6\},l\}$	$\bar{\psi}_{85}$
$V_{3,21}$	$(0,\frac{1}{2},-\frac{1}{2},0,0,0,0,0,0,-1,0,0,0,0,0,0)$	$\{\mathbf{1},r\}$	ψ_{147}
$V_{3,22}$	$(\frac{1}{2},0,\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0,0,\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},\frac{1}{4})$	$\{\mathbf{1},r\}$	ψ_{134}
$V_{3,23}$	$(0,\frac{1}{2},-\frac{1}{2},0,0,0,0,0,0,-1,0,0,0,0,0,0)$	$\{\mathbf{1},r\}$	ψ_{141}
$V_{3,24}$	$(\frac{1}{2},0,\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0,0,\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},\frac{1}{4})$	$\{\mathbf{1},r\}$	ψ_{126}

Table F.2: Set of blow–up modes for Model 28 in triangulation B.

V_r^2	Fixed set	Numerical value for V_r	irrep.	Φ^{orb}
$\frac{25}{18}$	$\{1,1,1\}$	$\{-\frac{1}{6},0,0,-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},0,\frac{1}{3},0,0,0,0,0,0\}$	$\{\mathbf{1},r\}$	ψ_{57}
$\frac{25}{18}$	$\{1,1,2\}$	$\{-\frac{1}{6},0,\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0,0,-\frac{1}{4},-\frac{5}{12},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},-\frac{1}{4}\}$	$\{\mathbf{1},r\}$	ψ_{44}
$\frac{25}{18}$	$\{1,1,3\}$	$\{-\frac{1}{6},0,0,-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},0,\frac{1}{3},0,0,0,0,0,0\}$	$\{\mathbf{1},r\}$	ψ_{45}
$\frac{25}{18}$	$\{1,1,4\}$	$\{-\frac{1}{6},0,\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0,0,-\frac{1}{4},-\frac{5}{12},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},-\frac{1}{4}\}$	$\{\mathbf{1},r\}$	ψ_{41}
$\frac{25}{18}$	$\{1,2,1\}$	$\{-\frac{1}{2},-\frac{1}{3},0,\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},-\frac{5}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},\frac{1}{6},\frac{1}{6}\}$	$\{\mathbf{1},r\}$	ψ_{87}
$\frac{25}{18}$	$\{1,2,2\}$	$\{0,\frac{1}{6},0,-\frac{1}{3},-\frac{1}{3},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{5}{12},-\frac{1}{12},-\frac{5}{12},-\frac{5}{12},-\frac{5}{12},-\frac{5}{12},-\frac{1}{12},\frac{5}{12}\}$	$\{\mathbf{1},r\}$	ψ_{77}
$\frac{25}{18}$	$\{1,2,3\}$	$\{-\frac{1}{2},-\frac{1}{3},0,\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},-\frac{5}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},\frac{1}{6},\frac{1}{6}\}$	$\{\mathbf{1},r\}$	ψ_{84}
$\frac{25}{18}$	$\{1,2,4\}$	$\{0,\frac{1}{6},0,-\frac{1}{3},-\frac{1}{3},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{5}{12},-\frac{1}{12},-\frac{5}{12},-\frac{5}{12},-\frac{5}{12},-\frac{5}{12},-\frac{1}{12},\frac{5}{12}\}$	$\{\mathbf{1},r\}$	ψ_{70}
$\frac{25}{18}$	$\{1,3,1\}$	$\{-\frac{5}{6},\frac{1}{3},0,-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},-\frac{1}{6},-\frac{1}{6}\}$	$\{\mathbf{1},r\}$	ψ_{33}
$\frac{25}{18}$	$\{1,3,2\}$	$\{-\frac{1}{3},\frac{5}{6},0,\frac{1}{3},\frac{1}{3},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},\frac{1}{12},\frac{1}{4},-\frac{1}{12},-\frac{1}{12},-\frac{1}{12},-\frac{1}{12},-\frac{5}{12},\frac{1}{12}\}$	$\{\mathbf{1},r\}$	ψ_{21}
$\frac{25}{18}$	$\{1,3,3\}$	$\{-\frac{1}{3},-\frac{1}{6},-\frac{1}{2},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},-\frac{1}{6},-\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},-\frac{1}{6},-\frac{1}{6}\}$	$\{\mathbf{1},r\}$	ψ_{29}
$\frac{25}{18}$	$\{1,3,4\}$	$\{\frac{1}{6},-\frac{2}{3},\frac{1}{2},-\frac{1}{6},-\frac{1}{6},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{12},\frac{1}{4},-\frac{1}{12},-\frac{1}{12},-\frac{1}{12},-\frac{1}{12},-\frac{5}{12},\frac{1}{12}\}$	$\{\mathbf{1},r\}$	ψ_{15}
$\frac{14}{9}$	$\{2,1,1\}$	$\{-\frac{1}{3},0,1,0,0,0,0,0,0,\frac{2}{3},0,0,0,0,0,0\}$	$\{\mathbf{1},r\}$	ψ_{115}
$\frac{14}{9}$	$\{2,1,2\}$	$\{\frac{1}{2},-\frac{1}{6},\frac{1}{2},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},\frac{1}{3},\frac{1}{3},-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},\frac{1}{3},\frac{1}{3}\}$	$\{\mathbf{1},l\}$	$\bar{\psi}_{36}$
$\frac{14}{9}$	$\{2,1,3\}$	$\{-\frac{1}{6},\frac{1}{6},\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{2},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{5}{6},\frac{1}{6}\}$	$\{\mathbf{1},l\}$	$\bar{\psi}_{45}$
$\frac{14}{9}$	$\{4,1,1\}$	$\{-\frac{2}{3},0,0,0,0,0,0,0,0,\frac{1}{3},0,0,0,0,1,0\}$	$\{\mathbf{1},r\}$	ψ_{183}
$\frac{14}{9}$	$\{4,1,2\}$	$\{\frac{1}{2},-\frac{5}{6},-\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},\frac{1}{6},\frac{1}{6}\}$	$\{\mathbf{1},r\}$	ψ_{187}
$\frac{14}{9}$	$\{4,1,3\}$	$\{-\frac{1}{3},-\frac{2}{3},0,\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},-\frac{1}{6},\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},-\frac{1}{6},-\frac{1}{6}\}$	$\{\mathbf{1},l\}$	$\bar{\psi}_{106}$
$\frac{14}{9}$	$\{2,3,1\}$	$\{-\frac{1}{3},0,-1,0,0,0,0,0,0,\frac{2}{3},0,0,0,0,0,0\}$	$\{\mathbf{1},l\}$	$\bar{\psi}_{21}$
$\frac{14}{9}$	$\{2,3,2\}$	$\{-\frac{1}{2},\frac{5}{6},\frac{1}{2},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},-\frac{1}{6},-\frac{1}{6}\}$	$\{\mathbf{1},l\}$	$\bar{\psi}_{14}$
$\frac{14}{9}$	$\{2,3,3\}$	$\{-\frac{2}{3},-\frac{1}{3},0,-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},\frac{1}{6},-\frac{1}{2},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},\frac{1}{6},\frac{1}{6}\}$	$\{\mathbf{1},l\}$	$\bar{\psi}_{27}$
$\frac{14}{9}$	$\{4,3,1\}$	$\{-\frac{2}{3},0,0,0,0,0,0,0,0,\frac{1}{3},0,0,0,0,1,0\}$	$\{\mathbf{1},l\}$	$\bar{\psi}_{93}$
$\frac{14}{9}$	$\{4,3,2\}$	$\{-\frac{1}{2},\frac{1}{6},-\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},-\frac{1}{3},-\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},-\frac{1}{3},-\frac{1}{3}\}$	$\{\mathbf{1},l\}$	$\bar{\psi}_{98}$
$\frac{14}{9}$	$\{4,3,3\}$	$\{\frac{1}{6},-\frac{1}{6},-\frac{1}{2},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{5}{6},-\frac{1}{6}\}$	$\{\mathbf{1},l\}$	$\bar{\psi}_{87}$
$\frac{3}{2}$	$\{3,1,1\}$	$\{0,\frac{1}{2},-\frac{1}{2},0,0,0,0,0,0,-1,0,0,0,0,0,0\}$	$\{\mathbf{1},l\}$	$\bar{\psi}_{83}$
$\frac{3}{2}$	$\{3,1,2\}$	$\{\frac{1}{2},0,\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0,0,\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},\frac{1}{4}\}$	$\{\mathbf{1},r\}$	ψ_{153}
$\frac{3}{2}$	$\{3,1,3\}$	$\{0,\frac{1}{2},-\frac{1}{2},0,0,0,0,0,0,-1,0,0,0,0,0,0\}$	$\{\mathbf{1},l\}$	$\bar{\psi}_{82}$
$\frac{3}{2}$	$\{3,1,4\}$	$\{\frac{1}{2},0,\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0,0,\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},\frac{1}{4}\}$	$\{\mathbf{1},r\}$	ψ_{150}
$\frac{3}{2}$	$\{3,2,1\}$	$\{0,\frac{1}{2},-\frac{1}{2},0,0,0,0,0,0,-1,0,0,0,0,0,0\}$	$\{\mathbf{1},r\}$	ψ_{147}
$\frac{3}{2}$	$\{3,2,2\}$	$\{\frac{1}{2},0,\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0,0,\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},\frac{1}{4}\}$	$\{\mathbf{1},r\}$	ψ_{134}
$\frac{3}{2}$	$\{3,2,3\}$	$\{0,\frac{1}{2},-\frac{1}{2},0,0,0,0,0,0,-1,0,0,0,0,0,0\}$	$\{\mathbf{1},r\}$	ψ_{141}
$\frac{3}{2}$	$\{3,2,4\}$	$\{\frac{1}{2},0,\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0,0,\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},\frac{1}{4}\}$	$\{\mathbf{1},r\}$	ψ_{126}

Table F.3: Blow–up modes of table (F.1) and states with same charges.

Fixed sets	States
$(1,1,1)$	$(((1,1,1),\psi_{57}),((1,1,3),\psi_{45}))$
$(1,1,2)$	$(((1,1,2),\psi_{44}),((1,1,4),\psi_{41}))$
$(1,1,3)$	$(((1,1,1),\psi_{57}),((1,1,3),\psi_{45}))$
$(1,1,4)$	$(((1,1,2),\psi_{44}),((1,1,4),\psi_{41}))$
$(1,2,1)$	$(((1,2,1),\psi_{88}),((1,2,3),\psi_{85}))$
$(1,2,2)$	$(((1,2,2),\psi_{77}),((1,2,4),\psi_{70}))$
$(1,2,3)$	$(((1,2,1),\psi_{88}),((1,2,3),\psi_{85}))$
$(1,2,4)$	$(((1,2,2),\psi_{77}),((1,2,4),\psi_{70}))$
$(1,3,1)$	$(((1,3,1),\psi_{34}),((1,3,3),\psi_{28}))$
$(1,3,2)$	$(((1,3,2),\psi_{22}),((1,3,4),\psi_{15}))$
$(1,3,3)$	$(((1,3,1),\psi_{34}),((1,3,3),\psi_{28}))$
$(1,3,4)$	$(((1,3,2),\psi_{22}),((1,3,4),\psi_{15}))$
$(2,1,1)$	$(((2,1,1),\psi_{115}),((2,3,1),\bar\psi_{22}),((2,3,1),\psi_{98}))$
$(2,1,2)$	$(((2,1,2),\bar\psi_{36}),((2,3,2),\bar\psi_{15}),((2,3,2),\psi_{91}))$
$(2,1,3)$	$(((2,1,3),\bar\psi_{45}),((2,3,3),\bar\psi_{28}),((2,3,3),\psi_{104}))$
$(2,3,1)$	$(((2,1,1),\psi_{114}),((2,3,1),\bar\psi_{21}),((2,3,1),\psi_{97}))$
$(2,3,2)$	$(((2,1,2),\bar\psi_{35}),((2,3,2),\bar\psi_{14}),((2,3,2),\psi_{90}))$
$(2,3,3)$	$(((2,1,3),\bar\psi_{44}),((2,3,3),\bar\psi_{27}),((2,3,3),\psi_{103}))$
$(4,1,1)$	$(((4,1,1),\psi_{183}),((4,3,1),\bar\psi_{93}),((4,3,1),\psi_{165}))$
$(4,1,2)$	$(((4,1,2),\psi_{187}),((4,3,2),\bar\psi_{99}),((4,3,2),\psi_{171}))$
$(4,1,3)$	$(((4,1,3),\bar\psi_{106}),((4,3,3),\bar\psi_{84}),((4,3,3),\psi_{156}))$
$(4,3,1)$	$(((4,1,1),\psi_{183}),((4,3,1),\bar\psi_{93}),((4,3,1),\psi_{165}))$
$(4,3,2)$	$(((4,1,2),\psi_{186}),((4,3,2),\bar\psi_{98}),((4,3,2),\psi_{170}))$
$(4,3,3)$	$(((4,1,3),\psi_{179}),((4,3,3),\bar\psi_{87}),((4,3,3),\psi_{159}))$
$(3,1,1)$	$(((3,1,1),\psi_{155}),((3,1,3),\psi_{154}),((3,2,1),\bar\psi_{74})$
	$((3,2,1),\psi_{146}),((3,2,3),\bar\psi_{68}),((3,2,3),\psi_{140}))$
$(3,1,2)$	$(((3,1,2),\psi_{153}),((3,1,4),\psi_{150}),((3,2,2),\bar\psi_{62})$
	$((3,2,2),\psi_{134}),((3,2,4),\bar\psi_{54}),((3,2,4),\psi_{126}))$
$(3,1,3)$	$(((3,1,1),\psi_{155}),((3,1,3),\psi_{154}),((3,2,1),\bar\psi_{74})$
	$((3,2,1),\psi_{146}),((3,2,3),\bar\psi_{68}),((3,2,3),\psi_{140}))$
$(3,2,1)$	$(((3,1,1),\bar\psi_{83}),((3,1,3),\bar\psi_{82}),((3,2,1),\bar\psi_{75})$
	$((3,2,1),\psi_{147}),((3,2,3),\bar\psi_{69}),((3,2,3),\psi_{141}))$
$(3,2,2)$	$(((3,1,2),\psi_{153}),((3,1,4),\psi_{150}),((3,2,2),\bar\psi_{62})$
	$((3,2,2),\psi_{134}),((3,2,4),\bar\psi_{54}),((3,2,4),\psi_{126}))$
$(3,2,3)$	$(((3,1,1),\bar\psi_{83}),((3,1,3),\bar\psi_{82}),((3,2,1),\bar\psi_{75})$
	$((3,2,1),\psi_{147}),((3,2,3),\bar\psi_{69}),((3,2,3),\psi_{141}))$

Table F.4: Conjugated states of table (F.1) modes. The conjugated fields to the ones in the θ sector, all in θ^5 are not given.

Fixed sets	States
$(2,1,1)$	$(((4,1,1),\bar{\psi}_{110}),((4,3,1),\bar{\psi}_{95}),((4,3,1),\psi_{167}))$
$(2,1,2)$	$(((4,1,2),\psi_{186}),((4,3,2),\bar{\psi}_{98}),((4,3,2),\psi_{170}))$
$(2,1,3)$	$(((4,1,3),\psi_{179}),((4,3,3),\bar{\psi}_{87}),((4,3,3),\psi_{159}))$
$(2,3,1)$	$(((4,1,1),\bar{\psi}_{111}),((4,3,1),\bar{\psi}_{96}),((4,3,1),\psi_{168}))$
$(2,3,2)$	$(((4,1,2),\psi_{187}),((4,3,2),\bar{\psi}_{99}),((4,3,2),\psi_{171}))$
$(2,3,3)$	$(((4,1,3),\psi_{180}),((4,3,3),\bar{\psi}_{88}),((4,3,3),\psi_{160}))$
$(4,1,1)$	$(((2,1,1),\bar{\psi}_{42}),((2,3,1),\bar{\psi}_{25}),((2,3,1),\psi_{101}))$
$(4,1,2)$	$(((2,1,2),\bar{\psi}_{35}),((2,3,2),\bar{\psi}_{14}),((2,3,2),\psi_{90}))$
$(4,1,3)$	$(((2,1,3),\psi_{118}),((2,3,3),\bar{\psi}_{31}),((2,3,3),\psi_{107}))$
$(4,3,1)$	$(((2,1,1),\bar{\psi}_{42}),((2,3,1),\bar{\psi}_{25}),((2,3,1),\psi_{101}))$
$(4,3,2)$	$(((2,1,2),\bar{\psi}_{36}),((2,3,2),\bar{\psi}_{15}),((2,3,2),\psi_{91}))$
$(4,3,3)$	$(((2,1,3),\bar{\psi}_{45}),((2,3,3),\bar{\psi}_{28}),((2,3,3),\psi_{104}))$
$(3,1,1)$	$(((3,1,1),\bar{\psi}_{83}),((3,1,3),\bar{\psi}_{82}),((3,2,1),\bar{\psi}_{75}),$
	$((3,2,1),\psi_{147}),((3,2,3),\bar{\psi}_{69}),((3,2,3),\psi_{141}))$
$(3,1,2)$	$(((3,1,2),\bar{\psi}_{79}),((3,1,4),\bar{\psi}_{76}),((3,2,2),\bar{\psi}_{56}),$
	$((3,2,2),\psi_{128}),((3,2,4),\bar{\psi}_{48}),((3,2,4),\psi_{120}))$
$(3,1,3)$	$(((3,1,1),\bar{\psi}_{83}),((3,1,3),\bar{\psi}_{82}),((3,2,1),\bar{\psi}_{75}),$
	$((3,2,1),\psi_{147}),((3,2,3),\bar{\psi}_{69}),((3,2,3),\psi_{141}))$
$(3,2,1)$	$(((3,1,1),\psi_{155}),((3,1,3),\psi_{154}),((3,2,1),\bar{\psi}_{74}),$
	$((3,2,1),\psi_{146}),((3,2,3),\bar{\psi}_{68}),((3,2,3),\psi_{140}))$
$(3,2,2)$	$(((3,1,2),\bar{\psi}_{79}),((3,1,4),\bar{\psi}_{76}),((3,2,2),\bar{\psi}_{56}),$
	$((3,2,2),\psi_{128}),((3,2,4),\bar{\psi}_{48}),((3,2,4),\psi_{120}))$
$(3,2,3)$	$(((3,1,1),\psi_{155}),((3,1,3),\psi_{154}),((3,2,1),\bar{\psi}_{74}),$
	$((3,2,1),\psi_{146}),((3,2,3),\bar{\psi}_{68}),((3,2,3),\psi_{140}))$

Table F.5: Blow–up modes from [83] for Benchmark Model 2.

F.P.	Numerical value for V_r	irrep.	Φ^{orb}
$\{1,1,1\}$	$\{-\frac{1}{6},0,0,-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},0,\frac{1}{3},0,0,0,0,0,0\}$	$\{\{1,1,1,1\},r\}$	ψ_{62}
$\{1,1,2\}$	$\{-\frac{5}{12},\frac{1}{4},-\frac{3}{4},-\frac{1}{4},-\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},-\frac{1}{2},-\frac{1}{6},0,0,0,0,0,0\}$	$\{\{1,1,1,1\},r\}$	ψ_{43}
$\{1,1,3\}$	$\{-\frac{1}{6},0,0,-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},0,\frac{1}{3},0,0,0,0,0,0\}$	$\{\{1,1,1,1\},r\}$	ψ_{50}
$\{1,1,4\}$	$\{-\frac{5}{12},\frac{1}{4},-\frac{3}{4},-\frac{1}{4},-\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},-\frac{1}{2},-\frac{1}{6},0,0,0,0,0,0\}$	$\{\{1,1,1,1\},r\}$	ψ_{35}
$\{1,2,1\}$	$\{-\frac{1}{6},0,\frac{2}{3},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{3},-\frac{2}{3},-\frac{1}{3},-\frac{1}{3},0,0,0,0\}$	$\{\{1,1,1,1\},r\}$	ψ_{84}
$\{1,2,2\}$	$\{\frac{1}{12},-\frac{1}{4},\frac{5}{12},-\frac{1}{12},-\frac{1}{12},\frac{5}{12},\frac{5}{12},\frac{5}{12},-\frac{1}{6},-\frac{1}{6},-\frac{1}{3},\frac{2}{3},0,0,0,0\}$	$\{\{1,1,1,2\},r\}$	ψ_{78}
$\{1,2,3\}$	$\{-\frac{1}{6},0,\frac{2}{3},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{3},-\frac{2}{3},-\frac{1}{3},-\frac{1}{3},0,0,0,0\}$	$\{\{1,1,1,1\},r\}$	ψ_{81}
$\{1,2,4\}$	$\{\frac{1}{12},-\frac{1}{4},\frac{5}{12},-\frac{1}{12},-\frac{1}{12},\frac{5}{12},\frac{5}{12},\frac{5}{12},-\frac{1}{6},-\frac{1}{6},-\frac{1}{3},\frac{2}{3},0,0,0,0\}$	$\{\{1,1,1,2\},r\}$	ψ_{75}
$\{1,3,1\}$	$\{-\frac{1}{6},0,\frac{1}{3},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	$\{\{1,1,8s,1\},r\}$	ψ_{30}
$\{1,3,2\}$	$\{\frac{1}{12},-\frac{1}{4},\frac{1}{12},-\frac{5}{12},-\frac{5}{12},\frac{1}{12},\frac{1}{12},\frac{1}{12},\frac{1}{6},-\frac{1}{6},-\frac{2}{3},-\frac{2}{3},0,0,0,0\}$	$\{\{1,1,1,1\},r\}$	ψ_{20}
$\{1,3,3\}$	$\{-\frac{1}{6},0,\frac{1}{3},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	$\{\{1,1,8s,1\},r\}$	ψ_{26}
$\{1,3,4\}$	$\{\frac{1}{12},-\frac{1}{4},\frac{1}{12},-\frac{5}{12},-\frac{5}{12},\frac{1}{12},\frac{1}{12},\frac{1}{12},\frac{1}{6},-\frac{1}{6},-\frac{2}{3},-\frac{2}{3},0,0,0,0\}$	$\{\{1,1,1,1\},r\}$	ψ_{12}
$\{2,1,3\}$	$\{-\frac{5}{6},\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{3},-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},0,0,0,0\}$	$\{\{1,1,1,1\},r\}$	ψ_{112}
$\{2,3,1\}$	$\{\frac{2}{3},0,0,0,0,0,0,0,0,-\frac{1}{3},0,-1,0,0,0,0\}$	$\{\{1,1,1,2\},l\}$	$\bar{\psi}_{18}$
$\{2,3,1\}$	$\{\frac{2}{3},0,0,0,0,0,0,0,0,-\frac{1}{3},0,-1,0,0,0,0\}$	$\{\{1,1,1,2\},r\}$	ψ_{94}
$\{2,3,2\}$	$\{\frac{1}{6},-\frac{1}{2},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},\frac{2}{3},-\frac{1}{3},\frac{1}{3},-\frac{2}{3},0,0,0,0\}$	$\{\{1,1,1,2\},l\}$	$\bar{\psi}_{14}$
$\{2,3,2\}$	$\{\frac{1}{6},-\frac{1}{2},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},\frac{2}{3},-\frac{1}{3},\frac{1}{3},-\frac{2}{3},0,0,0,0\}$	$\{\{1,1,1,2\},r\}$	ψ_{90}
$\{2,3,3\}$	$\{\frac{1}{6},-\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{3},\frac{2}{3},\frac{2}{3},-\frac{1}{3},0,0,0,0\}$	$\{\{1,1,1,2\},l\}$	$\bar{\psi}_{27}$
$\{2,3,3\}$	$\{\frac{1}{6},-\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{3},\frac{2}{3},\frac{2}{3},-\frac{1}{3},0,0,0,0\}$	$\{\{1,1,1,2\},r\}$	ψ_{103}
$\{4,1,1\}$	$\{\frac{1}{3},0,-1,0,0,0,0,0,0,-\frac{2}{3},0,0,0,0,0,0\}$	$\{\{1,1,1,1\},l\}$	$\bar{\psi}_{93}$
$\{4,1,2\}$	$\{-\frac{1}{6},\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{3},-\frac{2}{3},\frac{2}{3},-\frac{1}{3},0,0,0,0\}$	$\{\{1,1,1,2\},l\}$	$\bar{\psi}_{96}$
$\{4,1,3\}$	$\{-\frac{1}{6},\frac{1}{2},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{3},-\frac{2}{3},\frac{1}{3},-\frac{2}{3},0,0,0,0\}$	$\{\{1,1,1,2\},r\}$	ψ_{163}
$\{4,3,1\}$	$\{\frac{1}{3},0,-1,0,0,0,0,0,0,-\frac{2}{3},0,0,0,0,0,0\}$	$\{\{1,1,1,1\},l\}$	$\bar{\psi}_{81}$
$\{4,3,1\}$	$\{\frac{1}{3},0,-1,0,0,0,0,0,0,-\frac{2}{3},0,0,0,0,0,0\}$	$\{\{1,1,1,1\},r\}$	ψ_{155}
$\{4,3,2\}$	$\{-\frac{1}{6},\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{3},-\frac{2}{3},\frac{2}{3},-\frac{1}{3},0,0,0,0\}$	$\{\{1,1,1,2\},l\}$	$\bar{\psi}_{87}$
$\{4,3,2\}$	$\{-\frac{1}{6},\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{3},-\frac{2}{3},\frac{2}{3},-\frac{1}{3},0,0,0,0\}$	$\{\{1,1,1,2\},r\}$	ψ_{161}
$\{4,3,3\}$	$\{\frac{5}{6},-\frac{1}{2},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},0,0,0,0\}$	$\{\{1,1,1,1\},l\}$	$\bar{\psi}_{74}$
$\{4,3,3\}$	$\{\frac{5}{6},-\frac{1}{2},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},0,0,0,0\}$	$\{\{1,1,1,1\},r\}$	ψ_{148}
$\{3,1,2\}$	$\{\frac{1}{4},-\frac{3}{4},\frac{1}{4},-\frac{1}{4},-\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{2},-\frac{1}{2},0,0,0,0,0,0\}$	$\{\{1,1,1,1\},l\}$	$\bar{\psi}_{67}$
$\{3,1,4\}$	$\{\frac{1}{4},-\frac{3}{4},\frac{1}{4},-\frac{1}{4},-\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{2},-\frac{1}{2},0,0,0,0,0,0\}$	$\{\{1,1,1,1\},l\}$	$\bar{\psi}_{65}$
$\{3,2,2\}$	$\{\frac{1}{4},-\frac{3}{4},\frac{1}{4},-\frac{1}{4},-\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{2},-\frac{1}{2},0,0,0,0,0,0\}$	$\{\{1,1,1,1\},l\}$	$\bar{\psi}_{55}$
$\{3,2,2\}$	$\{\frac{1}{4},-\frac{3}{4},\frac{1}{4},-\frac{1}{4},-\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{2},-\frac{1}{2},0,0,0,0,0,0\}$	$\{\{1,1,1,1\},r\}$	ψ_{129}
$\{3,2,4\}$	$\{\frac{1}{4},-\frac{3}{4},\frac{1}{4},-\frac{1}{4},-\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{2},-\frac{1}{2},0,0,0,0,0,0\}$	$\{\{1,1,1,1\},l\}$	$\bar{\psi}_{47}$
$\{3,2,4\}$	$\{\frac{1}{4},-\frac{3}{4},\frac{1}{4},-\frac{1}{4},-\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{2},-\frac{1}{2},0,0,0,0,0,0\}$	$\{\{1,1,1,1\},r\}$	ψ_{121}
$\{2,1,1\}$	$\{-\frac{1}{3},0,-1,0,0,0,0,0,0,-\frac{1}{3},0,-1,0,0,0,0\}$	Massive	–
$\{2,1,2\}$	$\{-\frac{1}{3},0,-1,0,0,0,0,0,0,-\frac{1}{3},0,-1,0,0,0,0\}$	Massive	–
$\{3,1,1\}$	$\{0,-\frac{1}{2},\frac{1}{2},0,0,0,0,0,0,-1,0,0,0,0,0,0\}$	projected out	–
$\{3,1,3\}$	$\{0,-\frac{1}{2},\frac{1}{2},0,0,0,0,0,0,-1,0,0,0,0,0,0\}$	projected out	–
$\{3,2,1\}$	$\{0,-\frac{1}{2},\frac{1}{2},0,0,0,0,0,0,-1,0,0,0,0,0,0\}$	projected out	–
$\{3,2,3\}$	$\{0,-\frac{1}{2},\frac{1}{2},0,0,0,0,0,0,-1,0,0,0,0,0,0\}$	projected out	–

Table F.6: Blow–up modes and same charged fields from Bianchi identities solution in [83].

Fixed sets	States
$(1,1,1)$	$(((1,1,3),\psi_{50}),((1,1,1),\psi_{62}))$
$(1,1,2)$	$(((1,1,4),\psi_{35}),((1,1,2),\psi_{43}))$
$(1,1,3)$	$(((1,1,3),\psi_{50}),((1,1,1),\psi_{62}))$
$(1,1,4)$	$(((1,1,4),\psi_{35}),((1,1,2),\psi_{43}))$
$(1,2,1)$	$(((1,2,3),\psi_{81}),((1,2,1),\psi_{84}))$
$(1,2,2)$	$(((1,2,4),\psi_{75}),((1,2,2),\psi_{78}))$
$(1,2,3)$	$(((1,2,3),\psi_{81}),((1,2,1),\psi_{84}))$
$(1,2,4)$	$(((1,2,4),\psi_{75}),((1,2,2),\psi_{78}))$
$(1,3,1)$	$(((1,3,3),\psi_{26}),((1,3,1),\psi_{30}))$
$(1,3,2)$	$(((1,3,4),\psi_{12}),((1,3,2),\psi_{20}))$
$(1,3,3)$	$(((1,3,3),\psi_{26}),((1,3,1),\psi_{30}))$
$(1,3,4)$	$(((1,3,4),\psi_{12}),((1,3,2),\psi_{20}))$
$(2,1,1)$	$()$
$(2,1,2)$	$()$
$(2,1,3)$	$(((2,3,3),\bar{\psi}_{24}),((2,3,3),\psi_{100}),((2,1,3),\psi_{112}))$
$(2,3,1)$	$(((2,3,1),\bar{\psi}_{18}),((2,3,1),\psi_{94}),((2,1,1),\bar{\psi}_{34}))$
$(2,3,2)$	$(((2,3,2),\bar{\psi}_{14}),((2,3,2),\psi_{90}),((2,1,2),\psi_{108}))$
$(2,3,3)$	$(((2,3,3),\bar{\psi}_{27}),((2,3,3),\psi_{103}),((2,1,3),\bar{\psi}_{39}))$
$(4,1,1)$	$(((4,3,1),\bar{\psi}_{81}),((4,3,1),\psi_{155}),((4,1,1),\bar{\psi}_{93}))$
$(4,1,2)$	$(((4,3,2),\bar{\psi}_{87}),((4,3,2),\psi_{161}),((4,1,2),\bar{\psi}_{96}))$
$(4,1,3)$	$(((4,3,3),\bar{\psi}_{70}),((4,3,3),\psi_{144}),((4,1,3),\psi_{163}))$
$(4,3,1)$	$(((4,3,1),\bar{\psi}_{81}),((4,3,1),\psi_{155}),((4,1,1),\bar{\psi}_{93}))$
$(4,3,2)$	$(((4,3,2),\bar{\psi}_{87}),((4,3,2),\psi_{161}),((4,1,2),\bar{\psi}_{96}))$
$(4,3,3)$	$(((4,3,3),\bar{\psi}_{74}),((4,3,3),\psi_{148}),((4,1,3),\bar{\psi}_{91}))$
$(3,1,1)$	$()$
$(3,1,2)$	$(((3,2,4),\bar{\psi}_{47}),((3,2,4),\psi_{121}),((3,2,2),\bar{\psi}_{55}),$ $((3,2,2),\psi_{129}),((3,1,4),\bar{\psi}_{65}),((3,1,2),\bar{\psi}_{67}))$
$(3,1,3)$	$()$
$(3,2,1)$	$()$
$(3,2,2)$	$(((3,2,4),\bar{\psi}_{47}),((3,2,4),\psi_{121}),((3,2,2),\bar{\psi}_{55}),$ $((3,2,2),\psi_{129}),((3,1,4),\bar{\psi}_{65}),((3,1,2),\bar{\psi}_{67}))$
$(3,2,3)$	$()$

Table F.7: Conjugated states of blow–up modes from Bianchi identities solution in [83].

Fixed sets	States
$(1,1,1)$	$(((1,1,3),\bar{\psi}_{131}),((1,1,1),\bar{\psi}_{143}))$
$(1,1,2)$	$(((1,1,4),\bar{\psi}_{113}),((1,1,2),\bar{\psi}_{121}))$
$(1,1,3)$	$(((1,1,3),\bar{\psi}_{131}),((1,1,1),\bar{\psi}_{143}))$
$(1,1,4)$	$(((1,1,4),\bar{\psi}_{113}),((1,1,2),\bar{\psi}_{121}))$
$(1,2,1)$	$(((1,2,3),\bar{\psi}_{105}),((1,2,1),\bar{\psi}_{108}))$
$(1,2,2)$	$(((1,2,4),\bar{\psi}_{99}),((1,2,2),\bar{\psi}_{102}))$
$(1,2,3)$	$(((1,2,3),\bar{\psi}_{105}),((1,2,1),\bar{\psi}_{108}))$
$(1,2,4)$	$(((1,2,4),\bar{\psi}_{99}),((1,2,2),\bar{\psi}_{102}))$
$(1,3,1)$	$(((1,3,3),\bar{\psi}_{166}),((1,3,1),\bar{\psi}_{170}))$
$(1,3,2)$	$(((1,3,4),\bar{\psi}_{152}),((1,3,2),\bar{\psi}_{160}))$
$(1,3,3)$	$(((1,3,3),\bar{\psi}_{166}),((1,3,1),\bar{\psi}_{170}))$
$(1,3,4)$	$(((1,3,4),\bar{\psi}_{152}),((1,3,2),\bar{\psi}_{160}))$
$(2,1,1)$	$()$
$(2,1,2)$	$()$
$(2,1,3)$	$(((4,3,3),\bar{\psi}_{74}),((4,3,3),\psi_{148}),((4,1,3),\bar{\psi}_{91}))$
$(2,3,1)$	$(((4,3,1),\bar{\psi}_{78}),((4,3,1),\psi_{152}),((4,1,1),\psi_{169}))$
$(2,3,2)$	$(((4,3,2),\bar{\psi}_{83}),((4,3,2),\psi_{157}),((4,1,2),\bar{\psi}_{95}))$
$(2,3,3)$	$(((4,3,3),\bar{\psi}_{70}),((4,3,3),\psi_{144}),((4,1,3),\psi_{163}))$
$(4,1,1)$	$(((2,3,1),\bar{\psi}_{21}),((2,3,1),\psi_{97}),((2,1,1),\psi_{110}))$
$(4,1,2)$	$(((2,3,2),\bar{\psi}_{13}),((2,3,2),\psi_{89}),((2,1,2),\psi_{107}))$
$(4,1,3)$	$(((2,3,3),\bar{\psi}_{27}),((2,3,3),\psi_{103}),((2,1,3),\bar{\psi}_{39}))$
$(4,3,1)$	$(((2,3,1),\bar{\psi}_{21}),((2,3,1),\psi_{97}),((2,1,1),\psi_{110}))$
$(4,3,2)$	$(((2,3,2),\bar{\psi}_{13}),((2,3,2),\psi_{89}),((2,1,2),\psi_{107}))$
$(4,3,3)$	$(((2,3,3),\bar{\psi}_{24}),((2,3,3),\psi_{100}),((2,1,3),\psi_{112}))$
$(3,1,1)$	$()$
$(3,1,2)$	$(((3,2,4),\bar{\psi}_{46}),((3,2,4),\psi_{120}),((3,2,2),\bar{\psi}_{54})$
	$((3,2,2),\psi_{128}),((3,1,4),\psi_{140}),((3,1,2),\psi_{142}))$
$(3,1,3)$	$()$
$(3,2,1)$	$()$
$(3,2,2)$	$(((3,2,4),\bar{\psi}_{46}),((3,2,4),\psi_{120}),((3,2,2),\bar{\psi}_{54})$
	$((3,2,2),\psi_{128}),((3,1,4),\psi_{140}),((3,1,2),\psi_{142}))$
$(3,2,3)$	$()$

Table F.8: Projection conditions: Commuting elements to every conjugacy class with Wilson line A_3 on.

fixed point	conjugacy class	projector V_h
$\{0,0,0,0,0,0\}$	$\{\{0,0,0,0,0,0\},1\}$	$V,2V,3V,4V,5V$
$\left\{0,0,0,0,\frac{1}{2},0\right\}$	$\{\{0,0,0,0,1,0\},1\}$	$V,2V,3V,4V,5V$
$\left\{0,0,0,0,0,\frac{1}{2}\right\}$	$\{\{0,0,0,0,0,1\},1\}$	$V,2V,3V,4V,5V$
$\left\{0,0,0,0,\frac{1}{2},\frac{1}{2}\right\}$	$\{\{0,0,0,0,1,1\},1\}$	$V,2V,3V,4V,5V$
$\left\{0,0,\frac{2}{3},\frac{1}{3},0,0\right\}$	$\{\{0,0,1,0,0,0\},1\}$	$3V,A+V,2A+2V,A+4V,2A+5V$
$\left\{0,0,\frac{2}{3},\frac{1}{3},\frac{1}{2},0\right\}$	$\{\{0,0,1,0,1,0\},1\}$	$3V,A+V,2A+2V,A+4V,2A+5V$
$\left\{0,0,\frac{2}{3},\frac{1}{3},0,\frac{1}{2}\right\}$	$\{\{0,0,1,0,0,1\},1\}$	$3V,A+V,2A+2V,A+4V,2A+5V$
$\left\{0,0,\frac{2}{3},\frac{1}{3},\frac{1}{2},\frac{1}{2}\right\}$	$\{\{0,0,1,0,1,1\},1\}$	$3V,A+V,2A+2V,A+4V,2A+5V$
$\left\{0,0,\frac{1}{3},\frac{2}{3},0,0\right\}$	$\{\{0,0,1,1,0,0\},1\}$	$3V,2A+V,A+2V,2A+4V,A+5V$
$\left\{0,0,\frac{1}{3},\frac{2}{3},\frac{1}{2},0\right\}$	$\{\{0,0,1,1,1,0\},1\}$	$3V,2A+V,A+2V,2A+4V,A+5V$
$\left\{0,0,\frac{1}{3},\frac{2}{3},0,\frac{1}{2}\right\}$	$\{\{0,0,1,1,0,1\},1\}$	$3V,2A+V,A+2V,2A+4V,A+5V$
$\left\{0,0,\frac{1}{3},\frac{2}{3},\frac{1}{2},\frac{1}{2}\right\}$	$\{\{0,0,1,1,1,1\},1\}$	$3V,2A+V,A+2V,2A+4V,A+5V$
$\{0,0,0,0,0,0\}$	$\{\{0,0,0,0,0,0\},2\}$	$V,2V,3V,4V,5V$
$\left\{\frac{1}{3},0,0,0,0,0\right\}$	$\{\{0,-1,0,0,0,0\},2\}$	$2V,4V$
$\left\{0,0,\frac{2}{3},\frac{1}{3},0,0\right\}$	$\{\{0,0,1,1,0,0\},2\}$	$3V,A+V,2A+2V,A+4V,2A+5V$
$\left\{\frac{1}{3},0,\frac{2}{3},\frac{1}{3},0,0\right\}$	$\{\{0,-1,1,1,0,0\},2\}$	$2A+2V,A+4V$
$\left\{0,0,\frac{1}{3},\frac{2}{3},0,0\right\}$	$\{\{0,0,0,1,0,0\},2\}$	$3V,2A+V,A+2V,2A+4V,A+5V$
$\left\{\frac{1}{3},0,\frac{1}{3},\frac{2}{3},0,0\right\}$	$\{\{0,-1,0,1,0,0\},2\}$	$A+2V,2A+4V$
$\{0,0,0,0,0,0\}$	$\{\{0,0,0,0,0,0\},3\}$	$A(m_3+m_4)+kV,k=0,...,5$
$\left\{\frac{1}{2},0,0,0,0,0\right\}$	$\{\{1,0,0,0,0,0\},3\}$	$A(m_3+m_4),A(m_3+m_4)+3V$
$\left\{0,0,0,0,\frac{1}{2},0\right\}$	$\{\{0,0,0,0,1,0\},3\}$	$A(m_3+m_4)+kV,k=0,...,5$
$\left\{\frac{1}{2},0,0,0,\frac{1}{2},0\right\}$	$\{\{1,0,0,0,1,0\},3\}$	$A(m_3+m_4),A(m_3+m_4)+3V$
$\left\{0,0,0,0,0,\frac{1}{2}\right\}$	$\{\{0,0,0,0,0,1\},3\}$	$A(m_3+m_4)+kV,k=0,...,5$
$\left\{\frac{1}{2},0,0,0,0,\frac{1}{2}\right\}$	$\{\{1,0,0,0,0,1\},3\}$	$A(m_3+m_4),A(m_3+m_4)+3V$
$\left\{0,0,0,0,\frac{1}{2},\frac{1}{2}\right\}$	$\{\{0,0,0,0,1,1\},3\}$	$A(m_3+m_4)+kV,k=0,...,5$
$\left\{\frac{1}{2},0,0,0,\frac{1}{2},\frac{1}{2}\right\}$	$\{\{1,0,0,0,1,1\},3\}$	$A(m_3+m_4),A(m_3+m_4)+3V$

Table F.9: Projection conditions: Commuting elements to every conjugacy class with Wilson line A_5 on.

fixed point	conjugacy class	projector V_h
$\{0,0,0,0,0,0\}$	$\{\{0,0,0,0,0,0\},1\}$	$V,2V,3V,5V$
$\{0,0,\frac{2}{3},\frac{1}{3},0,0\}$	$\{\{0,0,1,0,0,0\},1\}$	$V,2V,3V,5V$
$\{0,0,\frac{1}{3},\frac{2}{3},0,0\}$	$\{\{0,0,1,1,0,0\},1\}$	$V,2V,3V,5V$
$\{0,0,0,0,0,\frac{1}{2}\}$	$\{\{0,0,0,0,0,1\},1\}$	$V,2V,3V,5V$
$\{0,0,\frac{2}{3},\frac{1}{3},0,\frac{1}{2}\}$	$\{\{0,0,1,0,0,1\},1\}$	$V,2V,3V,5V$
$\{0,0,\frac{1}{3},\frac{2}{3},0,\frac{1}{2}\}$	$\{\{0,0,1,1,0,1\},1\}$	$V,2V,3V,5V$
$\{0,0,0,0,\frac{1}{2},0\}$	$\{\{0,0,0,0,1,0\},1\}$	$2V,4V,A_5+V,A_5+3V,A_5+5V$
$\{0,0,\frac{2}{3},\frac{1}{3},\frac{1}{2},0\}$	$\{\{0,0,1,0,1,0\},1\}$	$2V,4V,A_5+V,A_5+3V,A_5+5V$
$\{0,0,\frac{1}{3},\frac{2}{3},\frac{1}{2},0\}$	$\{\{0,0,1,1,1,0\},1\}$	$2V,4V,A_5+V,A_5+3V,A_5+5V$
$\{0,0,0,0,\frac{1}{2},\frac{1}{2}\}$	$\{\{0,0,0,0,1,1\},1\}$	$2V,4V,A_5+V,A_5+3V,A_5+5V$
$\{0,0,\frac{2}{3},\frac{1}{3},\frac{1}{2},\frac{1}{2}\}$	$\{\{0,0,1,0,1,1\},1\}$	$2V,4V,A_5+V,A_5+3V,A_5+5V$
$\{0,0,\frac{1}{3},\frac{2}{3},\frac{1}{2},\frac{1}{2}\}$	$\{\{0,0,1,1,1,1\},1\}$	$2V,4V,A_5+V,A_5+3V,A_5+5V$
$\{0,0,0,0,0,0\}$	$\{\{0,0,0,0,0,0\},2\}$	$m_5A_5+kv, 0\leq k\leq 5,$
$\{\frac{1}{3},0,0,0,0,0\}$	$\{\{0,-1,0,0,0,0\},2\}$	$m_5A_5+kv, k=0,2,4,$
$\{0,0,\frac{2}{3},\frac{1}{3},0,0\}$	$\{\{0,0,1,1,0,0\},2\}$	$m_5A_5+kv, 0\leq k\leq 5,$
$\{\frac{1}{3},0,\frac{2}{3},\frac{1}{3},0,0\}$	$\{\{0,-1,1,1,0,0\},2\}$	$m_5A_5+kv, k=0,2,4,$
$\{0,0,\frac{1}{3},\frac{2}{3},0,0\}$	$\{\{0,0,0,1,0,0\},2\}$	$m_5A_5+kv, 0\leq k\leq 5,$
$\{\frac{1}{3},0,\frac{1}{3},\frac{2}{3},0,0\}$	$\{\{0,-1,0,1,0,0\},2\}$	$m_5A_5+kv, k=0,2,4,$
$\{0,0,0,0,0,0\}$	$\{\{0,0,0,0,0,0\},3\}$	$V,2V,3V,4V,5V$
$\{\frac{1}{2},0,0,0,0,0\}$	$\{\{1,0,0,0,0,0\},3\}$	$3V$
$\{0,0,0,0,0,\frac{1}{2}\}$	$\{\{0,0,0,0,0,1\},3\}$	$V,2V,3V,4V,5V$
$\{\frac{1}{2},0,0,0,0,\frac{1}{2}\}$	$\{\{1,0,0,0,0,1\},3\}$	$3V$
$\{0,0,0,0,\frac{1}{2},0\}$	$\{\{0,0,0,0,1,0\},3\}$	$2V,4V,A_5+V,A_5+3V,A_5+5V$
$\{\frac{1}{2},0,0,0,\frac{1}{2},0\}$	$\{\{1,0,0,0,1,0\},3\}$	A_5+3V
$\{0,0,0,0,\frac{1}{2},\frac{1}{2}\}$	$\{\{0,0,0,0,1,1\},3\}$	$2V,4V,A_5+V,A_5+3V,A_5+5V$
$\{\frac{1}{2},0,0,0,\frac{1}{2},\frac{1}{2}\}$	$\{\{1,0,0,0,1,1\},3\}$	A_5+3V

Table F.10: Empty fixed sets of the Mini–landscape models with $V_{SO(10),1}$

Shift
$V = \left(\frac{1}{3}, -\frac{1}{2}, -\frac{1}{2}, 0^5, \frac{1}{2}, -\frac{1}{6}, -\frac{1}{2}^5, \frac{1}{2}\right)$
Model 1
$WL1 = (-\frac{1}{4}, \frac{3}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{3}{2}, -\frac{1}{2}, -1, \frac{1}{2}, -\frac{7}{2}, -1, -1, 1)$ $WL2 = (\frac{1}{6}, \frac{1}{6}, \frac{1}{2}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, -\frac{3}{2}, \frac{1}{6}, \frac{3}{2}, \frac{23}{6}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, -\frac{3}{2})$
$SU(3) \times SU(2) \times U(1)^5$ and $SO(8) \times SU(2) \times SU(2) \times U(1)^2$ No singlets in $\theta^5(0,0,0,1,1,0),(0,0,0,1,1,1)$
Model 2
WL1=(3/4, 1/4, 1/4, 1/4, 1/4, -1/4, -1/4, -1/4 , 5/2, -3/2, -2/1, -5/2, -5/2, -2/1, -2/1, 2/1) WL2=(-1/2, -1/2, 1/6,1/6, 1/6, 1/6, 1/6, 1/6 , 1/1, 0/1, -1/3, -2/3, 2/3, -5/3, -2/3, 1/3)
SU3 x SU2 x U15 AND SU4 x SU2 x U14 No singlets in $\theta^5(0,0,0,1,1,0)(0,0,0,1,1,1)$. Empty $\theta^3(0^6),(0^5 1)$
Model 3
WL1=(3/4, 1/4, 1/4, 1/4, 1/4, -1/4, -1/4, -1/4 , 5/2, -3/2, -2/1, -5/2, -5/2, -2, -2, 2) WL2=(-1/2, -1/2, 1/6, 1/6, 1/6, 1/6, 1/6, 1/6 , 1/3, 0/1, 0/1, 2/3, 32/3, -11, 0, 0)
SU3 x SU2 x U15 AND SO8 x SU3 x U12 Empty fixed branes $\theta^3(0^6),(0^5 1),(0^4 10),(0^4 11)$
Model 4
WL1=(3/4, 1/4, 1/4, 1/4, 1/4, -1/4, -1/4, -1/4 , 5/2, -3/2, -2/1, -5/2, -5/2, -2/1, 0/1, 4/1) WL2=(-1/2, -1/2, 1/6, 1/6, 1/6, 1/6, 1/6, 1/6 , 1/3, 0/1, 0/1, 0/1, 2/1, -4/3, 2/3, 1/1)
SU3 x SU2 x U15 AND SU2 x SU2 x SU2 x SU2 x SU2 x U13 No singlets in $\theta^5,(0,0,0,1,1,0),(0,0,0,1,1,1)$. Empty branes $\theta^3(0^6),(0^5 1)$
Model 5
WL1=(3/4, 1/4, 1/4, 1/4, 1/4, -1/4, -1/4, -1/4 , 5/2, -3/2, -2/1, -5/2, -5/2, -2/1, -2/1, 2/1) WL2=(-1/2, -1/2, 1/6, 1/6, 1/6, 1/6, 1/6, 1/6,2/3, -1/1, -1/1, -1/3, 5/3, -7/3, -1/3, 2/3)
SU3 x SU2 x U15 AND SU3 x SU3 x U14 Empty branes $\theta^3(0^6),(0^5 1)$
Model 6
WL1= (3/4, 1/4, 1/4, 1/4, 1/4, -1/4, -1/4, -1/4 , 5/2, -3/2, -2/1, -5/2, -5/2, -2/1, -2/1, 2/1) WL2=(-1/2, -1/2, 1/6, 1/6, 1/6, 1/6, 13/6, 13/6 , 0/1, 0/1, 0/1, 0/1, 1/1, -1/1, 0/1, 0/1)
SU3 x SU2 x U15 AND SO8 x SU4 x U1 Empty θ^3. No singlets in $\theta^5(0,0,0,1,1,0),(0,0,0,1,1,1)$.
Model 7
WL1=(3/4, 1/4, 1/4, 1/4, 1/4, -1/4, -1/4, -1/4) (5/2, -3/2, -2/1, -5/2, 3/2, -6/1, -2/1, 2/1) WL2=(-1/2, -1/2, 1/6, 1/6, 1/6, 1/6, 1/6, 1/6) (-5/3, 1/1, 1/1, 4/3, 23/3, -6/1, 1/1, -1/1)
SU3 x SU2 x U15 AND SO8 x SU3 x U12 No singlets in $\theta^5,(0,0,0,1,1,0),(0,0,0,1,1,1)$. Empty branes $\theta^3(0^6),(0^5 1)$

Table F.11: Empty fixed sets of the Mini–landscape models with $V_{SO(10),1}$.

Shift
$V = \left(\frac{1}{3}, -\frac{1}{2}, -\frac{1}{2}, 0^5, \frac{1}{2}, -\frac{1}{6}, -\frac{1}{2}^5, \frac{1}{2}\right)$
Model 8
WL1=(3/4, 1/4, 1/4, 1/4, 1/4, -1/4, -1/4, -1/4 , 5/2, -3/2, -2/1, -5/2, -5/2, -2/1, -2/1, 2/1) WL2=(-1/6, -1/2, 1/2, 1/6, 1/6, 1/6, 1/6, 1/6 , 7/6, -5/6, -3/2, -5/6, -1/2, -11/6, 1/6, 13/6)
SU3 x SU2 x SU2 x U14 AND SU3 x SU2 x U15 Empty branes $\theta^3(0^6), (0^5 1)$
Model 9
WL1=(3/4, 1/4, 1/4, 1/4, 1/4, -1/4, -1/4, -1/4 ,5/2, -3/2, -2/1, -5/2, -5/2, -1/1, -3/1, 2/1) WL2=(-1/6, -1/2, 1/2, 1/6, 1/6, 1/6, 1/6, 1/6 , -11/6, 13/6, 13/6, 13/6, 23/6, 13/6, 1/2, -5/2)
SU3 x SU2 x SU2 x U14 AND SU2 x SU2 x SU3 x SU2 x U13 Empty branes $\theta^3(0^6), (0^5 1)$
Model 10
WL1=(3/4, 1/4, 1/4, 1/4, 1/4, -1/4, -1/4, -1/4 , 5/2, -3/2, -2/1, -5/2, -5/2, -2/1, 0/1, 4/1) WL2=(-1/6, -1/2, 1/2, 1/6, 1/6, 1/6, 1/6, 1/6 , 5/2, -11/6, -13/6, -13/6, -3/2, -19/6, -13/6, 11/6)
SU3 x SU2 x SU2 x U14 AND SU4 x SU2 x SU2 x U13 Empty branes $\theta^3(0^6), (0^5 1)$
Model 11
WL1=(-1/4, 3/4, -1/4, 1/4, 1/4, -1/4, -1/4, -1/4 , 3/2, -1/2, 0/1, -5/2, -1/2, -2/1, -1/1, 1/1) WL2=(-1/6, 1/2, 1/6, 1/6, 1/6, 1/6, 1/6, 1/6 , -1/2, 1/6, 1/6, 1/2, 1/2, 1/2, 1/2, -1/2)
SU3 x SU2 x U15 AND SU4 x SU4 x U12 Empty fixed points $\theta^5(0^3, 1, 1, 0), (0^3 1, 1, 1)$
Model 12
WL1=(1/4, -3/4, 1/4, -1/4, -1/4, 1/4, 5/4, 5/4 , 7/2, -5/2, -5/2, -5/2, -5/2, -5/2, -5/2, 5/2) WL2=(1/6, 1/6, 1/2, -1/6, -1/6, -1/6, 5/6, 5/6 , -1/2, 11/6, -1/2, 11/6, -1/2, 1/2, 1/2, -1/2)
SU3 x SU2 x U15 AND SO12 x U12 No singlets in $\theta, (0^6)$
Model 13
WL1=(-3/4, -1/4, -1/4, -1/4, -1/4, 1/4, 5/4, 5/4 , 5/2, -3/2, -3/2, -3/2, -3/2, -3/2, -1/2, 5/2) WL2=(-1/2, -1/2, 1/6, 1/6, 1/6, 1/6, 7/6, 7/6 , 2/1, -1/1, -5/3, -5/3, -5/3, -5/3, -5/3, 4/3)
SU3 x SU2 x U15 AND SU6 x U13 Empty fixed branes $\theta^3, (0^6)(0^5, 1)$ and $\theta^4(001100)$. No singlets in $\theta^2, (0^6)$ and $\theta^3, (0^4, 1, 0)(0^4, 1, 1)(010010)(010011)$.
Model 14
WL1=(-3/4, -1/4, -1/4, -1/4, -1/4, 1/4, 5/4, 5/4 , 5/2, -3/2, -3/2, -3/2, 1/2, -7/2, -3/2, 3/2) WL2=(-1/2, -1/2, 1/6, 1/6, 1/6, 1/6, 7/6, 7/6 , 4/3, 0/1, -2/1, -1/3, -4/3, -1/1, -1/1, 1/1)
SU3 x SU2 x U15 AND SO8 x SU3 x U12 Empty fixed branes $\theta^3, (0^6)(0^5, 1)$. No singlets in $\theta^2, (0^6)$ and $\theta^3, (0^4, 1, 0)(0^4, 1, 1)(010010)(010011)$.

Table F.12: Empty fixed sets of the Mini–landscape models with $V_{SO(10),1}$.

Shift
$V = \left(\frac{1}{3}, -\frac{1}{2}, -\frac{1}{2}, 0^5, \frac{1}{2}, -\frac{1}{6}, -\frac{1}{2}^5, \frac{1}{2}\right)$
Model 15
WL1=(-3/4, -1/4, -1/4, -1/4, -1/4, 1/4, 13/4, 13/4 , 5/2, -3/2, -3/2, -3/2, -3/2, -3/2, -3/2, 3/2) WL2=(-1/2, -1/2, 1/6, 1/6, 1/6, 1/6, 31/6, 31/6 , 0, 0, 0, 0, 0, 0, 0, 0)
SU3 x SU2 x U15 AND SO14 x U1 No singlets in $\theta^2(0^6)(0,0,1,1,0,0)$. Empty branes $\theta^3, (0^6)(0^5,1), (01,0^4), (010001)$ no singlets in oder θ^3 fixed sets.
Model 27
WL1=(-1/2, -1/2, 0, 1/2, 1/2, 0, 0, 0 , 13/4, -17/4, -13/4, -13/4, -13/4, -13/4, -9/4, 9/4) WL2=(-1/6, -5/6, -1/2, 1/6, 1/6, 1/6, 1/6, 1/6 , 9/2, -25/6, -9/2, -9/2, -9/2, -9/2, -9/2, 17/6)
SU3 x SU2 x U15 AND SU6 x U13 No empty fixed branes
Model 28
WL1=(-1/2, -1/2, 0, 1/2, 1/2, 0, 0, 0 ,15/4, -19/4, -15/4, -15/4, -15/4, -15/4, -11/4, 19/4) WL2=(1/6, 1/6, -1/2, 1/6, 1/6, 1/6, 1/6, 1/6 , 5/3, -2/3, -5/3, -5/3, -5/3, -5/3, -1/3, 8/3)
SU3 x SU2 x U15 AND SU6 x U13 No empty fixed branes.

Table F.13: Conjugacy classes of the $\mathbb{Z}_{6II}$ orbifold.

$n_\alpha e_\alpha$	$\{n_1, n_2, n_3, n_4, n_5, n_6\}$	k	$\{\alpha, \beta, \gamma\}$	F.P. coordinates
0	{0, 0, 0, 0, 0, 0}	1	{1, 1, 1}	{0, 0, 0, 0, 0, 0}
e6	{0, 0, 0, 0, 0, 1}	1	{1, 1, 3}	{0, 0, 0, 0, 0, 1/2}
e5	{0, 0, 0, 0, 1, 0}	1	{1, 1, 2}	{0, 0, 0, 0, 1/2, 0}
e5 + e6	{0, 0, 0, 0, 1, 1}	1	{1, 1, 4}	{0, 0, 0, 0, 1/2, 1/2}
e3	{0, 0, 1, 0, 0, 0}	1	{1, 2, 1}	{0, 0, 2/3, 1/3, 0, 0}
e3 + e6	{0, 0, 1, 0, 0, 1}	1	{1, 2, 3}	{0, 0, 2/3, 1/3, 0, 1/2}
e3 + e5	{0, 0, 1, 0, 1, 0}	1	{1, 2, 2}	{0, 0, 2/3, 1/3, 1/2, 0}
e3 + e5 + e6	{0, 0, 1, 0, 1, 1}	1	{1, 2, 4}	{0, 0, 2/3, 1/3, 1/2, 1/2}
e3 + e4	{0, 0, 1, 1, 0, 0}	1	{1, 3, 1}	{0, 0, 1/3, 2/3, 0, 0}
e3 + e4 + e6	{0, 0, 1, 1, 0, 1}	1	{1, 3, 3}	{0, 0, 1/3, 2/3, 0, 1/2}
e3 + e4 + e5	{0, 0, 1, 1, 1, 0}	1	{1, 3, 2}	{0, 0, 1/3, 2/3, 1/2, 0}
e3 + e4 + e5 + e6	{0, 0, 1, 1, 1, 1}	1	{1, 3, 4}	{0, 0, 1/3, 2/3, 1/2, 1/2}
-2 e2	{0, -2, 0, 0, 0, 0}	2	{5, 1, 1}	{2/3, 0, 0, 0, 0, 0}
-2 e2 + e4	{0, -2, 0, 1, 0, 0}	2	{5, 3, 1}	{2/3, 0, 1/3, 2/3, 0, 0}
-2 e2 + e3 + e4	{0, -2, 1, 1, 0, 0}	2	{5, 2, 1}	{2/3, 0, 2/3, 1/3, 0, 0}
0	{0, 0, 0, 0, 0, 0}	2	{1, 1, 1}	{0, 0, 0, 0, 0, 0}
e4	{0, 0, 0, 1, 0, 0}	2	{1, 3, 1}	{0, 0, 1/3, 2/3, 0, 0}
e3 + e4	{0, 0, 1, 1, 0, 0}	2	{1, 2, 1}	{0, 0, 2/3, 1/3, 0, 0}
0	{0, 0, 0, 0, 0, 0}	3	{1, 1, 1}	{0, 0, 0, 0, 0, 0}
e6	{0, 0, 0, 0, 0, 1}	3	{1, 1, 3}	{0, 0, 0, 0, 0, 1/2}
e5	{0, 0, 0, 0, 1, 0}	3	{1, 1, 2}	{0, 0, 0, 0, 1/2, 0}
e5 + e6	{0, 0, 0, 0, 1, 1}	3	{1, 1, 4}	{0, 0, 0, 0, 1/2, 1/2}
e2	{0, 1, 0, 0, 0, 0}	3	{4, 1, 1}	{0, 1/2, 0, 0, 0, 0}
e2 + e6	{0, 1, 0, 0, 0, 1}	3	{4, 1, 3}	{0, 1/2, 0, 0, 0, 1/2}
e2 + e5	{0, 1, 0, 0, 1, 0}	3	{4, 1, 2}	{0, 1/2, 0, 0, 1/2, 0}
e2 + e5 + e6	{0, 1, 0, 0, 1, 1}	3	{4, 1, 4}	{0, 1/2, 0, 0, 1/2, 1/2}
0	{0, 0, 0, 0, 0, 0}	4	{1, 1, 1}	{0, 0, 0, 0, 0, 0}
e3	{0, 0, 1, 0, 0, 0}	4	{1, 2, 1}	{0, 0, 2/3, 1/3, 0, 0}
e3 + e4	{0, 0, 1, 1, 0, 0}	4	{1, 3, 1}	{0, 0, 1/3, 2/3, 0, 0}
e1 + e2	{1, 1, 0, 0, 0, 0}	4	{3, 1, 1}	{1/3, 0, 0, 0, 0, 0}
e1 + e2 + e3	{1, 1, 1, 0, 0, 0}	4	{3, 2, 1}	{1/3, 0, 2/3, 1/3, 0, 0}
e1 + e2 + e3 + e4	{1, 1, 1, 1, 0, 0}	4	{3, 3, 1}	{1/3, 0, 1/3, 2/3, 0, 0}

Appendix G

$U(1)$ base for $T^6/\mathbb{Z}_{6II}$ blow–up

We start with a basis for a set of Cartan generators H_I such that $\mathrm{tr} H_I H_J = \delta_{IJ}$. There are 8 $U(1)$ symmetries, and we write the generator of each of them as $U(1)_a = \sum_I c_a^I H_I$ what is summarized in the following

$$
\begin{aligned}
U(1)_1 &= \{\frac{11}{6}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{13}{6}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\},\\
U(1)_2 &= \{0, 0, 0, \frac{1}{2}, \frac{1}{2}, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, 0, 0, 0, 0, 0, 0, 0, 0\},\\
U(1)_3 &= \{-3, 11, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\},\\
U(1)_4 &= \{33, 9, 130, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\},\\
U(1)_5 &= \{286, 78, -78, 0, 0, 0, 0, 0, 0, 278, 0, 0, 0, 0, 0, 0\},\\
U(1)_6 &= \{-66, -18, 18, 0, 0, 0, 0, 0, 0, 78, 0, 0, 0, 0, 616, 0\},\\
U(1)_7 &= \{165, 45, -45, 317, 317, 317, 317, 317, 0, -195, 0, 0, 0, 0, 45, 0\},\\
U(1)_8 &= \{-99, -27, 27, 27, 27, 27, 27, 27, 181, 117, -181, -181, -181, -181, -27, 181\}.
\end{aligned}
$$

Every entrance I in the vector $U(1)_a$ represents the coefficient c_a^I. The generator $U(1)_a$ is the generator of the anomalous $U(1)$.

Appendix H

Axions in blow–up versus orbifold axions

Here we give the solutions for the coefficients c_r and d_r relating the orbifold axion $a^{\rm orb}$ and the blow–up modes τ_r with the universal $a^{\rm uni}$ and non–universal axions β_r in the resolution. The relations are

$$
\begin{aligned}
a^{\rm uni} &= -\frac{1}{12}(a^{\rm orb} + \sum_r c_r \tau_r) && \text{(H.1)}\\
\beta_r &= -\frac{1}{12} d_r \tau_r. && \text{(H.2)}
\end{aligned}
$$

The following set correspond to solutions such that $I^{\rm red}$ is factorizable and therefore can be canceled by a counterterm:

$$
\begin{aligned}
c_1 &= d_{19} + \frac{45}{4}(-4 - d_{28}) + \frac{1}{4}(4 - 4c_3 + 16c_{17} + 16c_{19} - 8c_{20} - 20c_{21} + 10c_{25} && \text{(H.3)}\\
&+ 12c_{26} + 10c_{27} + 12c_{28} - 2c_{29} + 12c_{30} - 2c_{31} + 12c_{32} + 4d_{20} + 4d_{21} + 4d_{22}\\
&+ 4d_{23} + 4d_{24} + 45d_{28} + 4d_{29} + 4d_{30} + 4d_{31} + 4d_{32},\\
c_2 &= 1 - c_4 + 4c_{17} + 5c_{19} - 2c_{20} - 5c_{21} + 3c_{25} + 2c_{26} + 3c_{27} + 2c_{28} - c_{29} && \text{(H.4)}\\
&+ 2c_{30} - c_{31} + 2c_{32} + d_{19} + d_{20} + d_{21} + d_{22} + d_{23} + d_{24} + \frac{21}{2}(-4 - d_{28})\\
&+ \frac{21 d_{28}}{2} + d_{29} + d_{30} + d_{31} + d_{32},
\end{aligned}
$$

$$
\begin{aligned}
c_9 &= -1 - c_5 - c_7 - c_{11} - 4c_{17} - 4c_{19} + 2c_{20} + 3c_{21} - \frac{5c_{25}}{2} - 3c_{26} - \frac{5c_{27}}{2} && \text{(H.5)}\\
&- 3c_{28} + \frac{c_{29}}{2} - 3c_{30} + \frac{c_{31}}{2} - 3c_{32} - d_{19} - d_{20} - d_{21} - d_{22} - d_{23} - d_{24} - \frac{51}{4}(-4 - d_{28})\\
&- \frac{51d_{28}}{4} - d_{29} - d_{30} - d_{31} - d_{32}\\
c_{10} &= 1 - c_6 - c_8 - c_{12} + 4c_{17} + 5c_{19} - 2c_{20} - 5c_{21} + 3c_{25} + 3c_{26} + 3c_{27} + 3c_{28} && \text{(H.6)}\\
&- c_{29} + 3c_{30} - c_{31} + 3c_{32} + d_{19} + d_{20} + d_{21} + d_{22} + d_{23} + d_{24} + \frac{27}{2}(-4 - d_{28})\\
&+ \frac{27d_{28}}{2} + d_{29} + d_{30} + d_{31} + d_{32}\\
c_{13} &= -2 + c_6 + c_8 - 7c_{17} - 8c_{19} + 3c_{20} + 9c_{21} - 6c_{25} - 5c_{26} - 6c_{27} - 5c_{28} && \text{(H.7)}\\
&+ 2c_{29} - 5c_{30} + 2c_{31} - 5c_{32} - 2d_{19} - 2d_{20} - 2d_{21} - 2d_{22} - 2d_{23} - 2d_{24}\\
&- 24(-4 - d_{28}) - 24d_{28} - 2d_{29} - 2d_{30} - 2d_{31} - 2d_{32},\\
c_{14} &= -1 - c_5 - c_7 - 3c_{17} - 4c_{19} + c_{20} + 3c_{21} - c_{23} - 2c_{25} - 2c_{26} - 2c_{27} - 2c_{28} && \text{(H.8)}\\
&- 2c_{30} - 2c_{32} - d_{19} - d_{20} - d_{21} - d_{22} - d_{23} - d_{24} - \frac{15}{2}(-4 - d_{28})\\
&- \frac{15d_{28}}{2} - d_{29} - d_{30} - d_{31} - d_{32},\\
c_{15} &= -1 + c_5 + c_6 + c_7 + c_8 - 4c_{17} - 4c_{19} + 2c_{20} + 5c_{21} - c_{24} - 3c_{25} - 2c_{26} && \text{(H.9)}\\
&- 3c_{27} - 2c_{28} + c_{29} - 2c_{30} + c_{31} - 2c_{32} - d_{19} - d_{20} - d_{21} - d_{22} - d_{23} - d_{24}\\
&- \frac{63}{4}(-4 - d_{28}) - \frac{63d_{28}}{4} - d_{29} - d_{30} - d_{31} - d_{32}\\
c_{16} &= 1 - c_6 - c_8 + 4c_{17} + 4c_{19} - 2c_{20} - 5c_{21} - c_{22} + 3c_{25} + 3c_{26} + 3c_{27} + 3c_{28} && \text{(H.10)}\\
&- c_{29} + 3c_{30} - c_{31} + 3c_{32} + d_{19} + d_{20} + d_{21} + d_{22} + d_{23} + d_{24}\\
&+ \frac{69}{4}(-4 - d_{28}) + \frac{69d_{28}}{4} + d_{29} + d_{30} + d_{31} + d_{32}\\
c_{18} &= -c_6 - c_8 + c_{17} + c_{19} - c_{20} - c_{21} + c_{25} + c_{27} - c_{29} - c_{31} && \text{(H.11)}\\
&- \frac{3}{4}(-4 - d_{28}) - \frac{3d_{28}}{4}.\\
d_1 &= -4 - d_3, && \text{(H.12)}\\
d_2 &= -4 - d_4, && \text{(H.13)}\\
d_5 &= -4 - d_7, && \text{(H.14)}\\
d_6 &= -4 - d_8, && \text{(H.15)}\\
& && \text{(H.16)}
\end{aligned}
$$

$$d_9 = -4 - d_{11}, \tag{H.17}$$
$$d_{10} = -4 - d_{12}, \tag{H.18}$$
$$d_{13} = -2, \tag{H.19}$$
$$d_{14} = 2, \tag{H.20}$$
$$d_{15} = 2, \tag{H.21}$$
$$d_{16} = -2, \tag{H.22}$$
$$d_{17} = -2, \tag{H.23}$$
$$d_{18} = 2, \tag{H.24}$$
$$d_{25} = -4 - d_{27}, \tag{H.25}$$
$$d_{26} = -4 - d_{28}. \tag{H.26}$$

Appendix I

Automorphisms for $\mathbb{Z}_N$ orbifolds

In order to investigate the possible discrete symmetries present in a model and their effects, we have constructed the automorphism groups for the lattices listed in Tables I.1 and I.2 [94]. We present here the symmetries involving the subgroups D and F, defined in section 3.4.

First we give our findings for the torus lattices which factorize along the complex coordinates and are the most broadly studied in the literature. Then we proceed to the non-factorizable ones. For each case the R–symmetries resulting from the generators on D are presented, and the generators in F are listed, analyzing when they survive in 4d.

Working with the group F we study the symmetries of the 4d theory coming from transformations which map conjugacy classes with the same spectrum among each other. This subgroup of the Lorentz group in 10d is maximal when no Wilson are on, because in this case all the fixed points of the same sector are degenerated. A *degeneration class* $\mathcal{C}$ is composed by a subset of all the conjugacy classes in which every two elements

$$g_1 = (\theta^{k_1}, (n_1)_\alpha e_\alpha), g_2 = (\theta^{k_2}, (n_2)_\alpha e_\alpha), g_1, g_2 \in \mathcal{C}. \tag{I.1}$$

Those are representatives from the conjugacy classes $[g_1]$ and $[g_2]$, satisfy $V_{g_1} = V_{g_2}$ and have the same orbifold projection conditions. The first requirement reads $k_1 = k_2$ and $(n_1)_\alpha A_\alpha = (n_2)_\alpha A_\alpha$. To fulfill the second requirement, we checked if the set of all commuting elements $\{h_1\}$ and $\{h_2\}$ s.t.$[g_1, h_1] = 0$ and $[g_2, h_2] = 0$ give the same set of projection conditions.

The orbifold $\mathbb{Z}_7$ with Wilson line on has no degenerated conjugacy classes, while the orbifold $\mathbb{Z}_{8II}$ posses no element in the group F, due to that we don't list them. The elements of the group F giving 4d symmetries map conjugacy classes belonging to the same set $\mathcal{C}$. We analyze every case, specifying which generators survive each Wilson line configuration. For that we use a set of generators $\{G_i\}$ of F s.t. every element $\rho \in F$ can be written as $\rho = \prod_i G_i^{n_i} \mathbb{1}$ where $\mathbb{1}$ denotes the unity, an element of the subgroup D. This selection of generators is not unique, thus we give some representatives. As the tables describing the fixed sets degeneracies will give a too big extension to the appendices, we just give them for few of the presented examples.

I.0.1 Factorizable Lattices

Table I.1: Factorizable $\mathbb{Z}_N$ orbifolds under our consideration, the lattice, the shift and the number of inequivalent fixed points for each twisted sector are given.

	Lattice	Shift	# of F.P.		
$\mathbb{Z}_3$	SU(3)⊗SU(3)⊗SU(3)	$\frac{1}{3}(1,1,-2)$	27	27	
$\mathbb{Z}_4$	SO(4)⊗SO(4)⊗SU(2)^2	$\frac{1}{4}(1,1,-2)$	16	10	
$\mathbb{Z}_{6-\mathrm{I}}$	$G_2 \otimes G_2$⊗SU(3)	$\frac{1}{6}(1,1,-2)$	3	15	6
$\mathbb{Z}_{6-\mathrm{II}}$	G_2⊗SU(3)⊗SO(4)	$\frac{1}{6}(1,2,-3)$	12	6	8

$\mathbb{Z}_3$ on $SU(3) \times SU(3) \times SU(3)$

Group D $\mathbb{Z}_3$ has three prime twists composing the orbifold action. The constraints they impose on the L point couplings are of the form:

$$\sum_{\alpha=1}^{L} R^{\alpha}_{\theta_i} = 1 \bmod 3, \quad i = 1, 2, 3,$$

these twists are enough to generate the subgroup D.

Group F The Coxeter element of $SU(3) \times SU(3) \times SU(3)$ gives the identifications $A_1 \equiv A_2$, $A_3 \equiv A_4$, $A_5 \equiv A_6$. The order 3 for all the independent Wilson lines is $3A_{1,3,5} \in \Lambda$. Two constructing elements g_1, g_2 that posses equal V_g should satisfy conditions

$$\begin{aligned}(n_1 + n_2) &= (m_1 + m_2),\\ (n_3 + n_4) &= (m_3 + m_4),\\ (n_5 + n_6) &= (m_5 + m_6).\end{aligned} \tag{I.2}$$

Then we compute all the projection conditions with elements $[g, h] \neq 0$ to construct the degeneracy classes. Then we check which generators preserve them for every Wilson line configuration.

In terms of the three complex coordinates basis, the three generators of F are given by

$$G_1 = \mathrm{diag}(-1, 1, 1), \quad G_2 = \mathrm{diag}(1, -1, 1), \; G_3 = \mathrm{diag}(1, 1, -1).$$

When only the Wilson line A_5 is turned on, the rotations $\rho = G_1^{a_1} G_2^{a_2}$, map inside the fixed point degeneracies, this can be checked in Table I.3. Similarly for the Wilson line A_1 (A_3) the rotations that map inside the degeneracies are $\rho = G_2^{a_2} G_3^{a_3}$ ($\rho = G_1^{a_1} G_3^{a_3}$). When the set of turned Wilson lines are the pairs A_1A_5, A_1A_3 and A_3A_5 the rotations which map conjugacy classes inside the same degeneracy are $G_2^{a_2}$, $G_3^{a_3}$ and $G_1^{a_1}$ respectively. The powers of the generators are given by $a_i = 1, 2$. The multiplication by an element of D which is $\mathbb{1}$ in F is implicit.

$\mathbb{Z}_4$ **on** $SO(4) \times SO(4) \times SO(4)$

Group D $\mathbb{Z}_4$ contains only one prime twist so that only $\theta_1\theta_2$ and θ_3 leave the fixed points invariant. From the side of the automorphism group one can find that the symmetry $(\theta_2)^2$ also fulfills this property, such that one has the following selection rules

$$\sum_{\alpha=1}^{L} R^\alpha_{\theta_1\theta_2} = 1 \bmod 2, \quad \sum_{\alpha=1}^{L} R^\alpha_{\theta_1^2} = 1 \bmod 2, \quad \sum_{\alpha=1}^{L} R^\alpha_{\theta_3} = -1 \bmod 2. \tag{I.3}$$

Group F From the Coxeter elements, the conditions for the Wilson lines are

$$A_1 \equiv A_2,\ A_3 \equiv A_4,\ 2A_1, 2A_3, 2A_5, 2A_6 \in \Lambda. \tag{I.4}$$

Thus, two constructing elements are identified if

$$\begin{aligned} n_1 + n_2 &= m_1 + m_2, \\ n_3 + n_4 &= m_3 + m_4, \\ n_5 &= m_5, \\ n_6 &= m_6, \end{aligned} \tag{I.5}$$

and possess equal projection conditions. The generator of F is $G_1 = \theta_1$. It maps elements of a degeneration to others in the same degeneration in the sector θ^2. When only one Wilson line from A_1, A_5, A_3 and A_6 is on, the action of G_1 preserves the degeneration class. As a sample we give the Table I.4 for degeneracy classes of A_1. The mapped conjugacy classes are degenerated with every Wilson line, so with any combination of Wilson lines the powers of the generator G_1 multiplying an element of D i.e. $G_1^{n_1}\mathbb{1}$ will give a symmetry in 4d.

G_1 which corresponds to the orbifold twist in the first plane, doesn't leaves the fixed points invariant, and should give rise to the standard R_1-charge. In the second plane also $\theta_2 = (1, e^{2\pi i v_2}, 1)$ will change conjugacy classes but respecting the degeneration, there the element θ_2 can be written as $\theta_2 = G_1^3\theta_3\theta$. Where $\mathbb{1} \sim \theta_3\theta$ belonging to D identifies G_1 with θ_2 .

$\mathbb{Z}_{6I}$ **on** $G_2 \times G_2 \times SU(3)$

Group D $\mathbb{Z}_{6-\mathrm{I}}$ also contains only one twist which is prime, so that only the symmetries described in section 3.4 for this case are present. They will give the 4d R-symmetries:

$$\sum_{\alpha=1}^{L} R^\alpha_{\theta_1\theta_2} = 1 \bmod 3, \qquad \sum_{\alpha=1}^{L} R^\alpha_{\theta_3} = -1 \bmod 3\,. \tag{I.6}$$

Group F The conditions for the Wilson lines are

$$A_5 \equiv A_6,\ 3A_5 \in \Lambda,\ A_1, A_2, A_3, A_4 \in \Lambda, \tag{I.7}$$

which tells us that two elements with the same V_g follow the identification

$$n_5 + n_6 = m_5 + m_6. \tag{I.8}$$

The generator $G_1 = \theta_1^5$ maps conjugacy classes to other in the same degeneration. There are two other generators in F that only give a 4d symmetry when non Wilson lines are on, those are $G_2 = \mathrm{diag}(1, 1, -1)\theta_3^2$ and $G_3 = \theta_2 G_2$.

$\mathbb{Z}_{6II}$ **in** $G_2 \times SU(3) \times SO(4)$

Group D For the $\mathbb{Z}_{6-\mathrm{II}}$ orbifold the presence of two prime twists permits to recover the independent action of all the three twists leading to the symmetries found in [49]

$$\sum_{\alpha=1}^{L}(R^{\alpha}_{\theta_1}) = 1 \bmod 6, \qquad \sum_{\alpha=1}^{L} R^{\alpha}_{\theta_2} = 1 \bmod 3, \qquad \sum_{\alpha=1}^{L} R^{\alpha}_{\theta_3} = 1 \bmod 2\,. \tag{I.9}$$

This corresponds to the only set of independent constraints one can impose from the automorphism group of $G_2 \otimes SU(3) \otimes SO(4)$.

Group F The independent Wilson lines are $A = A_3 = A_4$ and A_5, A_6. They fulfill $2A_5, 2A_6, 3A \in \Lambda$. Then two conjugacy classes will give same spectrum if they have same projection conditions and equal shift, we require

$$n_3 + n_4 = m_3 + m_4, n_5 = m_5, n_6 = m_6. \tag{I.10}$$

The generator of F is $G_1 = \mathrm{diag}(1, -1, 1)\theta_2^2$. When the Wilson line A is on, G_1 breaks the degeneration. With Wilson lines A_5 or A_6 turned on alone, G_1 maps conjugacy classes in the same degeneration. As example we give the degeneration classes when A_3, A_5 Wilson lines are on in Tables F.8, F.9. Also with the pair A_5A_6 on, G_1 preserves the degeneracies. When all Wilson lines are on, or the pairs A_3, A_5 or A_3, A_6 then G_1 breaks the degeneracy.

I.0.2 Non Factorizable Lattices

Contrary to the approach we followed in section I.0.1, we will study some non–factorizable orbifolds (see Table I.2) in a more model dependent fashion[1] by studying the automorphism group of the given lattice. For not factorizable torus lattices we can still not draw conclusions of the specific 4d constraints for couplings, nevertheless we present a complete study of the symmetries in the groups D and F.

[1]Given that in such cases the set of fixed points do not factorize along the complex planes, it is hard to tell wether or not the twists of the orbifold can be incorporated as global R-symmetries.

Table I.2: Some basic information concerning the non-factorizable $\mathbb{Z}_N$ orbifolds we investigated [111].

	Lattice	Shift	# of F.P.					
$\mathbb{Z}_4$	SU(4)⊗SU(4)	$\frac{1}{4}(1,1,-2)$	16	4				
$\mathbb{Z}_{6-\mathrm{II}}$	SU(6)⊗SU(2)	$\frac{1}{6}(1,2,-3)$	12	3	4			
$\mathbb{Z}_7$	SU(7)	$\frac{1}{7}(1,2,-3)$	7	7	7	7		
$\mathbb{Z}_{8-\mathrm{I}}$	SO(5)⊗SO(9)	$\frac{1}{8}(2,1,-3)$	4	10	4	6		
$\mathbb{Z}_{8-\mathrm{II}}$	SO(8)⊗SO(4)	$\frac{1}{8}(1,3,-4)$	8	3	8	6		
$\mathbb{Z}_{12-\mathrm{I}}$	SU(3)⊗F_4	$\frac{1}{12}(4,1,-5)$	3	3	2	9	3	4
$\mathbb{Z}_{12-\mathrm{II}}$	F_4⊗SO(4)	$\frac{1}{12}(1,5,-6)$	4	1	8	3	4	4

$\mathbb{Z}_4$ on $SU(4) \times SU(4)$

Group D In the case of $\mathbb{Z}_4$, the automorphism group for the lattice of SU(4)⊗SU(4) is generated by considering all inequivalent products of automorphisms each factor with the operator which exchanges them. From this group only 128 elements commute with the orbifold action and only 16 preserve the structure of the conjugacy classes. Any element of the former group can be written as a product of the point group elements with powers of $(\theta_1)^2$.

Group F The relation between Wilson lines is

$$\begin{aligned} A_3 &\equiv -A_2,\ A_1 \equiv -2A_2,\ 2A_1, 4A_2, 4A_3 \in \Lambda, \\ A_6 &\equiv -A_5,\ A_4 \equiv -2A_5,\ 2A_4,\ 4A_4, 4A_5 \in \Lambda, \end{aligned} \tag{I.11}$$

which fixing the equivalences without lattice vectors differences translates into

$$\begin{aligned} -2n_1 + n_2 - n_3 &= -2m_1 + m_2 - m_3, \\ -2n_4 + n_5 - n_6 &= -2m_4 + m_5 - m_6. \end{aligned} \tag{I.12}$$

Lets decompose the action of θ by $\theta = \theta_I \theta_{II}$ where the two factors represent the rotations on the first three coordinates and in the last three. Lets denote a diagonal matrix in two blocks of 3×3 by $\mathrm{diag}(a, b)$. Then, the generators of the group F can be written as

$$G_1 = \mathrm{diag}(-1, 1)\theta_I^2,\ G_2 = \mathrm{diag}(1, -1)\theta_{II}^2.$$

When Wilson line A_5 is on, G_1 preserves the fixed point degeneracies. When only Wilson line A_2 is on, G_2 preserves the fixed points degeneracies. We give here as example of the A_5 degeneracies the Table I.6. When both independent Wilson lines are on here are not degenerated classes. As before we can construct symmetries with powers of the generators multiplied by $\mathbb{1}$.

$\mathbb{Z}_{6II}$ in $SU(6) \times SU(2)$

Group D As found in [107] the automorphism group of $SU(6) \otimes SU(2)$ does not give rise to any R-symmetry for the non-factorizable $\mathbb{Z}_{6-\mathrm{II}}$. This is also the case of $\mathbb{Z}_7$ where only 14 elements were found to commute with the point group, among those, the only transformations leaving the fixed points untouched were found to belong to the point group.

Group F There are two independent Wilson lines $A = A_1 \equiv A_2 \equiv A_3 \equiv A_4 \equiv A_5$ and A_6, of order 6 and 2 respectively. Then two conjugacy classes will have equal shift V_g if they fulfill

$$\sum_{i=1}^{5} n_i = \sum_{i=1}^{5} m_i, n_6 = m_6. \tag{I.13}$$

The generators are $G_1 = \mathrm{diag}(-1,1)\theta$, where $\mathrm{diag}(-1,1)$ represents a π rotation in all the first five planes. When both Wilson lines are on there is no degeneracy. When only the Wilson line A is on, G_1 breaks the degeneracy. When Wilson line A_6 is on, G_1 maps conjugacy classes inside the same degeneracy.

$\mathbb{Z}_{8I}$ in $SO(5) \times SO(9)$.

Group D For the $\mathbb{Z}_{8-\mathrm{I}}$ orbifold one finds that there is only one generator θ_1^2.

Group F The relations between Wilson lines are

$$A_1, 2A_2, 2A_6 \in \Lambda, A_3 \equiv A_4 \equiv A_5 \in \Lambda.$$

Two conjugacy class elements are identified if $n_2 = m_2,\ n_6 = m_6$ and have same projection conditions. The orbifold action can be written in terms of the action in the two factorizable pieces I $SO(5)$ and II $SO(9)$ as $\theta = \theta_I \theta_{II}$, then the generator of the group F is $G_1 = \theta_I$. With only one Wilson line A_2 or A_6 is on, G_1 maps two conjugacy classes in the same degeneracy. With both A_2 and A_6 G_1 still maps between conjugacy classes in the same degeneration I.7.

$\mathbb{Z}_{8II}$ in $SO(4) \times SO(8)$.

In this case the twist θ_3 leaves the fixed points invariant, but the group F is empty.

$\mathbb{Z}_7$ in $SU(7)$.

The group D is empty, and in the group F we find as generator $G_1 = -\theta^2$. Which is $-\theta^2$, equivalent to the minus the identity matrix. However, when the single independent Wilson line is on there are no degeneracy classes, so no surviving 4d symmetry.

$\mathbb{Z}_{12I}$ in $SU(3) \times F_4$

Group D The only symmetry in this group is given by θ_1.

Group F The relations between Wilson lines are

$$A_1 \equiv A_2,\; 3A_1 \in \Lambda, A_3 \equiv A_4 \in \Lambda, A_5 \equiv A_6 \in \Lambda.$$

Then two conjugacy class elements are identified if $(n_1 + n_2) = (m_1 + m_2)$. The generator is $G_1 = \mathrm{diag}(-1, 1)\theta_I^2$. When the Wilson line A_1 is on G_1 transformation breaks the degeneration. For example: it maps the fixed point of θ^4 $\{\frac{1}{3}, \frac{2}{3}, 0, 0, 0, 0\}$ to $\{\frac{2}{3}, \frac{1}{3}, 0, 0, 0, 0\}$ which conjugacy classes have $(n_1 + n_2) = 2$ and $(n_1 + n_2) = 1$ respectively. So there is no 4d symmetry surviving when the Wilson line is on.

$\mathbb{Z}_{12II}$ in $F_4 \times SO(4)$

Group D In $\mathbb{Z}_{12-\mathrm{II}}$ the only symmetry belonging to D is given by θ_3. This is due to the $\mathbb{Z}_2$ symmetry of the sublattice in the third complex plane.

Group F The Wilson lines are related by

$$A_1 \equiv A_2 \in \Lambda, A_3 \equiv A_4 \in \Lambda, 2A_5 \in \Lambda, 2A_6 \in \Lambda.$$

Two conjugacy classes are degenerated if

$$n_5 = m_5,\; n_6 = m_6.$$

$$G_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

When only A_5 or A_6 is on, G_1 doesn't respects the degeneration of the conjugacy classes. For example: it maps the fixed point in the θ^3 sector $\{0,0,0,0,\frac{1}{2},0\}$ to $\{0,0,0,0,0,\frac{1}{2}\}$ which have $n_5 = 1, n_6 = 0$ and $n_5 = 0, n_6 = 1$ respectively.

Table I.3: Z_3 on $SU(3)\times SU(3)\times SU(3)$ with Wilson line A_5 on.

fixed point	conjugacy class	projector V_h
$\{0,0,0,0,0,0\}$	$\{\{0,0,0,0,0,0\},1\}$	V
$\{0,0,\frac{1}{3},\frac{2}{3},0,0\}$	$\{\{0,0,1,1,0,0\},1\}$	V
$\{0,0,\frac{2}{3},\frac{1}{3},0,0\}$	$\{\{0,0,1,0,0,0\},1\}$	V
$\{\frac{1}{3},\frac{2}{3},0,0,0,0\}$	$\{\{1,1,0,0,0,0\},1\}$	V
$\{\frac{1}{3},\frac{2}{3},\frac{1}{3},\frac{2}{3},0,0\}$	$\{\{1,1,1,1,0,0\},1\}$	V
$\{\frac{1}{3},\frac{2}{3},\frac{2}{3},\frac{1}{3},0,0\}$	$\{\{1,1,1,0,0,0\},1\}$	V
$\{\frac{2}{3},\frac{1}{3},0,0,0,0\}$	$\{\{1,0,0,0,0,0\},1\}$	V
$\{\frac{2}{3},\frac{1}{3},\frac{1}{3},\frac{2}{3},0,0\}$	$\{\{1,0,1,1,0,0\},1\}$	V
$\{\frac{2}{3},\frac{1}{3},\frac{2}{3},\frac{1}{3},0,0\}$	$\{\{1,0,1,0,0,0\},1\}$	V
$\{0,0,0,0,\frac{1}{3},\frac{2}{3}\}$	$\{\{0,0,0,0,1,1\},1\}$	$2A_5+V$
$\{0,0,\frac{1}{3},\frac{2}{3},\frac{1}{3},\frac{2}{3}\}$	$\{\{0,0,1,1,1,1\},1\}$	$2A_5+V$
$\{0,0,\frac{2}{3},\frac{1}{3},\frac{1}{3},\frac{2}{3}\}$	$\{\{0,0,1,0,1,1\},1\}$	$2A_5+V$
$\{\frac{1}{3},\frac{2}{3},0,0,\frac{1}{3},\frac{2}{3}\}$	$\{\{1,1,0,0,1,1\},1\}$	$2A_5+V$
$\{\frac{1}{3},\frac{2}{3},\frac{1}{3},\frac{2}{3},\frac{1}{3},\frac{2}{3}\}$	$\{\{1,1,1,1,1,1\},1\}$	$2A_5+V$
$\{\frac{1}{3},\frac{2}{3},\frac{2}{3},\frac{1}{3},\frac{1}{3},\frac{2}{3}\}$	$\{\{1,1,1,0,1,1\},1\}$	$2A_5+V$
$\{\frac{2}{3},\frac{1}{3},0,0,\frac{1}{3},\frac{2}{3}\}$	$\{\{1,0,0,0,1,1\},1\}$	$2A_5+V$
$\{\frac{2}{3},\frac{1}{3},\frac{1}{3},\frac{2}{3},\frac{1}{3},\frac{2}{3}\}$	$\{\{1,0,1,1,1,1\},1\}$	$2A_5+V$
$\{\frac{2}{3},\frac{1}{3},\frac{2}{3},\frac{1}{3},\frac{1}{3},\frac{2}{3}\}$	$\{\{1,0,1,0,1,1\},1\}$	$2A_5+V$
$\{0,0,0,0,\frac{2}{3},\frac{1}{3}\}$	$\{\{0,0,0,0,1,0\},1\}$	A_5+V
$\{0,0,\frac{1}{3},\frac{2}{3},\frac{2}{3},\frac{1}{3}\}$	$\{\{0,0,1,1,1,0\},1\}$	A_5+V
$\{0,0,\frac{2}{3},\frac{1}{3},\frac{2}{3},\frac{1}{3}\}$	$\{\{0,0,1,0,1,0\},1\}$	A_5+V
$\{\frac{1}{3},\frac{2}{3},0,0,\frac{2}{3},\frac{1}{3}\}$	$\{\{1,1,0,0,1,0\},1\}$	A_5+V
$\{\frac{1}{3},\frac{2}{3},\frac{1}{3},\frac{2}{3},\frac{2}{3},\frac{1}{3}\}$	$\{\{1,1,1,1,1,0\},1\}$	A_5+V
$\{\frac{1}{3},\frac{2}{3},\frac{2}{3},\frac{1}{3},\frac{2}{3},\frac{1}{3}\}$	$\{\{1,1,1,0,1,0\},1\}$	A_5+V
$\{\frac{2}{3},\frac{1}{3},0,0,\frac{2}{3},\frac{1}{3}\}$	$\{\{1,0,0,0,1,0\},1\}$	A_5+V
$\{\frac{2}{3},\frac{1}{3},\frac{1}{3},\frac{2}{3},\frac{2}{3},\frac{1}{3}\}$	$\{\{1,0,1,1,1,0\},1\}$	A_5+V
$\{\frac{2}{3},\frac{1}{3},\frac{2}{3},\frac{1}{3},\frac{2}{3},\frac{1}{3}\}$	$\{\{1,0,1,0,1,0\},1\}$	A_5+V

Table I.4: Z_4 on $SO(4)\times SO(4)\times SO(4)$ with Wilson line A_1 on.

fixed point	conjugacy class	projector V_h
$\{0,0,0,0,0,0\}$	$\{\{0,0,0,0,0,0\},1\}$	$V,2V$
$\{0,0,0,0,0,\frac{1}{2}\}$	$\{\{0,0,0,0,0,1\},1\}$	$V,2V$
$\{0,0,0,0,\frac{1}{2},0\}$	$\{\{0,0,0,0,1,0\},1\}$	$V,2V$
$\{0,0,0,0,\frac{1}{2},\frac{1}{2}\}$	$\{\{0,0,0,0,1,1\},1\}$	$V,2V$
$\{0,0,\frac{1}{2},\frac{1}{2},0,0\}$	$\{\{0,0,1,0,0,0\},1\}$	$V,2V$
$\{0,0,\frac{1}{2},\frac{1}{2},0,\frac{1}{2}\}$	$\{\{0,0,1,0,0,1\},1\}$	$V,2V$
$\{0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},0\}$	$\{\{0,0,1,0,1,0\},1\}$	$V,2V$
$\{0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	$\{\{0,0,1,0,1,1\},1\}$	$V,2V$
$\{\frac{1}{2},\frac{1}{2},0,0,0,0\}$	$\{\{1,0,0,0,0,0\},1\}$	$A_1+V,2A_1+2V$
$\{\frac{1}{2},\frac{1}{2},0,0,0,\frac{1}{2}\}$	$\{\{1,0,0,0,0,1\},1\}$	$A_1+V,2A_1+2V$
$\{\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2},0\}$	$\{\{1,0,0,0,1,0\},1\}$	$A_1+V,2A_1+2V$
$\{\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2},\frac{1}{2}\}$	$\{\{1,0,0,0,1,1\},1\}$	$A_1+V,2A_1+2V$
$\{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0\}$	$\{\{1,0,1,0,0,0\},1\}$	$A_1+V,2A_1+2V$
$\{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},0,\frac{1}{2}\}$	$\{\{1,0,1,0,0,1\},1\}$	$A_1+V,2A_1+2V$
$\{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},0\}$	$\{\{1,0,1,0,1,0\},1\}$	$A_1+V,2A_1+2V$
$\{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	$\{\{1,0,1,0,1,1\},1\}$	$A_1+V,2A_1+2V$
$\{0,0,0,0,0,0\}$	$\{\{0,0,0,0,0,0\},2\}$	$V,2V$
$\{0,0,\frac{1}{2},0,0,0\}$	$\{\{0,0,1,0,0,0\},2\}$	$V,2V$
$\{0,0,\frac{1}{2},\frac{1}{2},0,0\}$	$\{\{0,0,1,1,0,0\},2\}$	$V,2V$
$\{\frac{1}{2},0,0,0,0,0\}$	$\{\{1,0,0,0,0,0\},2\}$	A_1+2V
$\{\frac{1}{2},0,\frac{1}{2},0,0,0\}$	$\{\{1,0,1,0,0,0\},2\}$	A_1+2V
$\{\frac{1}{2},0,\frac{1}{2},\frac{1}{2},0,0\}$	$\{\{1,0,1,1,0,0\},2\}$	A_1+2V
$\{0,\frac{1}{2},\frac{1}{2},0,0,0\}$	$\{\{0,1,1,0,0,0\},2\}$	A_1+2V
$\{\frac{1}{2},\frac{1}{2},0,0,0,0\}$	$\{\{1,1,0,0,0,0\},2\}$	$A_1+V,2A_1+2V$
$\{\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0,0\}$	$\{\{1,1,1,0,0,0\},2\}$	$2A_1+2V$
$\{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0\}$	$\{\{1,1,1,1,0,0\},2\}$	$A_1+V,2A_1+2V$

Table I.5: Z_4 on $SO(4)\times SO(4)\times SO(4)$ with all Wilson lines A_1,A_3,A_5,A_6 on.

fixed point	conjugacy class	projector V_h
$\{\frac{1}{2},0,\frac{1}{2},0,0,0\}$	$\{\{1,0,1,0,0,0\},2\}$	$A_5m_5+A_6m_6,A_1+A_3+A_5m_5+A_6m_6+2V$
$\{0,\frac{1}{2},\frac{1}{2},0,0,0\}$	$\{\{0,1,1,0,0,0\},2\}$	$A_5m_5+A_6m_6,A_1+A_3+A_5m_5+A_6m_6+2V$

Table I.6: Z_4 on $SU(4) \times SU(4)$ with Wilson line A_5 on.

fixed point	conjugacy class	projector V_h
$\{0,0,0,0,0,0\}$	$\{\{0,0,0,0,0,0\},1\}$	$V, 2V$
$\{\frac{1}{2},\frac{3}{4},\frac{1}{4},0,0,0\}$	$\{\{0,1,0,0,0,0\},1\}$	$V, 2V$
$\{0,\frac{1}{2},\frac{1}{2},0,0,0\}$	$\{\{-1,0,0,0,0,0\},1\}$	$V, 2V$
$\{\frac{1}{2},\frac{1}{4},\frac{3}{4},0,0,0\}$	$\{\{0,0,1,0,0,0\},1\}$	$V, 2V$
$\{0,0,0,\frac{1}{2},\frac{3}{4},\frac{1}{4}\}$	$\{\{0,0,0,0,1,0\},1\}$	$A_5+V, 2A_5+2V$
$\{\frac{1}{2},\frac{3}{4},\frac{1}{4},\frac{1}{2},\frac{3}{4},\frac{1}{4}\}$	$\{\{0,1,0,0,1,0\},1\}$	$A_5+V, 2A_5+2V$
$\{0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{4},\frac{1}{4}\}$	$\{\{-1,0,0,0,1,0\},1\}$	$A_5+V, 2A_5+2V$
$\{\frac{1}{2},\frac{1}{4},\frac{3}{4},\frac{1}{2},\frac{3}{4},\frac{1}{4}\}$	$\{\{0,0,1,0,1,0\},1\}$	$A_5+V, 2A_5+2V$
$\{0,0,0,0,\frac{1}{2},\frac{1}{2}\}$	$\{\{0,0,0,-1,0,0\},1\}$	$2V, -2A_5+V$
$\{\frac{1}{2},\frac{3}{4},\frac{1}{4},0,\frac{1}{2},\frac{1}{2}\}$	$\{\{0,1,0,-1,0,0\},1\}$	$2V, -2A_5+V$
$\{0,\frac{1}{2},\frac{1}{2},0,\frac{1}{2},\frac{1}{2}\}$	$\{\{-1,0,0,-1,0,0\},1\}$	$2V, -2A_5+V$
$\{\frac{1}{2},\frac{1}{4},\frac{3}{4},0,\frac{1}{2},\frac{1}{2}\}$	$\{\{0,0,1,-1,0,0\},1\}$	$2V, -2A_5+V$
$\{0,0,0,\frac{1}{2},\frac{1}{4},\frac{3}{4}\}$	$\{\{0,0,0,0,0,1\},1\}$	$-A_5+V, 2A_5+2V$
$\{\frac{1}{2},\frac{3}{4},\frac{1}{4},\frac{1}{2},\frac{1}{4},\frac{3}{4}\}$	$\{\{0,1,0,0,0,1\},1\}$	$-A_5+V, 2A_5+2V$
$\{0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{3}{4}\}$	$\{\{-1,0,0,0,0,1\},1\}$	$-A_5+V, 2A_5+2V$
$\{\frac{1}{2},\frac{1}{4},\frac{3}{4},\frac{1}{2},\frac{1}{4},\frac{3}{4}\}$	$\{\{0,0,1,0,0,1\},1\}$	$-A_5+V, 2A_5+2V$
$\{0,0,0,0,0,0\}$	$\{\{0,0,0,0,0,0\},2\}$	$2A_5m_5, 2A_5m_5+V, 2A_5m_5+2V$
$\{\frac{1}{2},0,0,0,0,0\}$	$\{\{1,0,0,0,0,0\},2\}$	$2A_5m_5, 2A_5m_5+V, 2A_5m_5+2V$
$\{0,0,0,\frac{1}{2},0,0\}$	$\{\{0,0,0,1,0,0\},2\}$	$2A_5m_5, A_5(1+2m_5)+V, A_5(2+2m_5)+2V$
$\{\frac{1}{2},0,0,\frac{1}{2},0,0\}$	$\{\{1,0,0,1,0,0\},2\}$	$2A_5m_5, A_5(1+2m_5)+V, A_5(2+2m_5)+2V$

Table I.7: Z_{8I} on $SO(5) \times SO(9)$ with Wilson lines A_2 and A_6 on.

fixed point	conjugacy class	projector V_h
$\{0,0,0,0,0,0\}$	$\{\{0,0,0,0,0,0\},2\}$	$V, 2V, 3V, 4V, 5V, 6V, 7V$
$\{\frac{1}{2},0,0,0,0,0\}$	$\{\{1,0,0,0,0,0\},2\}$	$2V, 4V, 6V, -A_2+V, A_2+3V, -A_2+5V, A_2+7V$
$\{0,0,\frac{1}{2},0,\frac{1}{2},0\}$	$\{\{0,0,1,0,0,0\},2\}$	$2V, 4V, 6V, -A_6+V, -A_6+3V, A_6+5V, A_6+7V$
$\{\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},0\}$	$\{\{1,0,1,0,0,0\},2\}$	$2kV, k=1,...,3,\ -A_2-A_6+V, A_2-A_6+3V,$
		$-A_2+A_6+5V, A_2+A_6+7V$
$\{0,\frac{1}{2},0,0,0,0\}$	$\{\{0,1,0,0,0,0\},2\}$	$4V, A_2+2V, A_2+6V$
$\{0,\frac{1}{2},\frac{1}{2},0,\frac{1}{2},0\}$	$\{\{0,1,1,0,0,0\},2\}$	$4V, A_2+2V, A_2+6V$
$\{0,0,\frac{1}{2},\frac{1}{2},0,0\}$	$\{\{0,0,0,0,-1,-1\},2\}$	$4V, -A_6+2V, A_6+6V$
$\{\frac{1}{2},0,\frac{1}{2},\frac{1}{2},0,0\}$	$\{\{1,0,0,0,-1,-1\},2\}$	$4V, -A_6+2V, A_6+6V$
$\{0,\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0\}$	$\{\{0,1,0,0,-1,-1\},2\}$	$4V, A_2-A_6+2V, A_2+A_6+6V$
$\{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0\}$	$\{\{1,1,0,0,-1,-1\},2\}$	$4V, A_2-A_6+2V, A_2+A_6+6V$
$\{0,0,0,0,0,0\}$	$\{\{0,0,0,0,0,0\},4\}$	$A_2m_2+kV, k=0,...,7$
$\{0,0,\frac{1}{2},0,0,0\}$	$\{\{0,0,1,0,0,0\},4\}$	A_2m_2, A_2m_2+4V
$\{0,0,\frac{1}{2},\frac{1}{2},0,0\}$	$\{\{0,0,1,1,0,0\},4\}$	$A_2m_2, -A_6+A_2m_2+2V, A_2m_2+4V, A_6+A_2m_2+6V$
$\{0,0,\frac{1}{2},0,\frac{1}{2},0\}$	$\{\{0,0,1,0,1,0\},4\}$	$A_2m_2, -A_6+A_2m_2+V, A_2m_2+2V, -A_6+A_2m_2+3V,$
		$A_2m_2+4V, A_6+A_2m_2+5V, A_2m_2+6V, A_6+A_2m_2+7V$
$\{0,0,0,0,0,\frac{1}{2}\}$	$\{\{0,0,0,0,0,1\},4\}$	$A_2m_2, A_6+A_2m_2+4V$
$\{0,0,\frac{1}{2},0,0,\frac{1}{2}\}$	$\{\{0,0,1,0,0,1\},4\}$	$A_2m_2, A_6+A_2m_2+4V$

Appendix J

Blow-up spectrum

Table J.1: Here we give all the blow-up states representations, together with one of its roots.

Φ^I			
1	(1,1)	-4	$(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0,0,0,0,0,0,0)$
2	(1,1)	-4	$(-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0,0,0,0,0,0,0)$
3	(1,1)	-4	(-1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0)
4	(3,1)	-3	$(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},0,0,0,0,0,0,0,0)$
5	(1,1)	-2	$(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0,0,0,0,0,0,0)$
6	(1,1)	-2	$(\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},0,0,0,0,0,0,0,0)$
7	(3,1)	-2	$(-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},0,0,0,0,0,0,0,0)$
8	$(\bar{3},1)$	-2	$(-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},0,0,0,0,0,0,0,0)$
9	(1,1)	-2	$(-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0,0,0,0,0,0,0)$
10	(1,2)	-2	$(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0,0,0,0,0,0,0)$
11	$(\bar{3},2)$	-2	(0,0,0,1,0,1,0,0,0,0,0,0,0,0,0,0)
12	(3,1)	-2	(0,0,0,0,0,1,1,0,0,0,0,0,0,0,0,0)
13	(1,1)	-2	(-1,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0)
14	(1,1)	-2	(0,1,-1,0,0,0,0,0,0,0,0,0,0,0,0,0)
15	(1,1)	-2	(-1,0,-1,0,0,0,0,0,0,0,0,0,0,0,0,0)
16	(1,2)	-2	(-1,0,0,-1,0,0,0,0,0,0,0,0,0,0,0,0)
17	(1,2)	-2	(0,0,-1,-1,0,0,0,0,0,0,0,0,0,0,0,0)
18	(1,1)	-2	(0,0,0,-1,-1,0,0,0,0,0,0,0,0,0,0,0)
19	(1,2)	-1	$(-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0,0,0,0,0,0,0)$
20	$(\bar{3},2)$	-1	$(-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},0,0,0,0,0,0,0,0)$
21	(1,2)	-1	$(-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},0,0,0,0,0,0,0,0)$

22	(1,1)	-1	$(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},0,0,0,0,0,0,0,0)$
23	(1,2)	-1	(1,0,0,-1,0,0,0,0,0,0,0,0,0,0,0,0)
24	(1,1)	-1	(-1,-1,0,0,0,0,0,0,0,0,0,0,0,0,0,0)
25	(3,1)	-1	(0,0,-1,0,0,-1,0,0,0,0,0,0,0,0,0,0)
Φ^{II}			
2	$\bar{6}$	-1	$(0,0,0,0,0,0,0,0,-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$
3	1	-2	$(0,0,0,0,0,0,0,0,-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2})$
4	6	-1	$(0,0,0,0,0,0,0,0,\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2})$
7	1	-2	$(0,0,0,0,0,0,0,0,-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$
9	$\bar{6}$	-1	(0,0,0,0,0,0,0,0,0,1,-1,0,0,0,0,0)
13	6	-2	(0,0,0,0,0,0,0,0,0,0,0,1,0,0,1,0)
14	$\bar{6}$	-2	(0,0,0,0,0,0,0,0,0,-1,-1,0,0,0,0,0)
18	1	-4	(0,0,0,0,0,0,0,0,0,-1,0,0,0,0,1,0)
19	6	-1	(0,0,0,0,0,0,0,0,0,0,0,1,0,0,-1,0)
20	1	-4	(0,0,0,0,0,0,0,0,0,-1,0,0,0,0,-1,0)
Φ^{III}			
1	(1,1)	0	$(-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0,0,0,0,0,0,0)$
13	(3,2)	0	$(-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},0,0,0,0,0,0,0,0)$
16	(3,1)	0	$(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},0,0,0,0,0,0,0,0)$
24	(1,1)	0	(0,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0)
29	(1,2)	0	(0,1,0,-1,0,0,0,0,0,0,0,0,0,0,0,0)
32	$(\bar{3},1)$	0	(0,-1,0,0,0,1,0,0,0,0,0,0,0,0,0,0)

Appendix K

Field redefinitions for $T^6/\mathbb{Z}_{6II}$

Here we present a sample of the field redefinitions found which perform the presented map from orbifold to blow-up states. The first column of the table denotes the orbifold field to be redefined ψ_γ, the second column represents its fixed point and in the third column one can read off the redefinition.

Field	fixed point	redefinition
ψ_{61}	$(1,1,1)$	$V_{1,3,1} - V_{2,1,2}$
ψ_{60}	$(1,1,1)$	$-V_{1,1,1}$
ψ_{62}	$(1,1,1)$	$V_{1,3,1} - V_{2,1,2}$
ψ_{58}	$(1,1,1)$	$-V_{1,1,1}$
ψ_{68}	$(1,1,1)$	$-V_{1,1,1} + V_{2,3,3} + V_{4,1,3}$
ψ_{67}	$(1,1,1)$	$-V_{1,2,1} + V_{4,1,1} - V_{4,3,3}$
ψ_{59}	$(1,1,1)$	$2V_{1,2,2} - V_{1,2,3} + V_{4,1,3}$
ψ_{65}	$(1,1,1)$	$-V_{1,1,1}$
ψ_{64}	$(1,1,1)$	$-V_{1,1,1}$
ψ_{66}	$(1,1,1)$	$-V_{1,1,1}$
ψ_{63}	$(1,1,1)$	$-V_{1,1,1} - V_{2,3,3} - V_{4,1,3}$
ψ_{42}	$(1,1,2)$	$V_{1,1,2} + V_{2,1,3} - V_{4,3,2}$
ψ_{43}	$(1,1,2)$	$V_{1,2,2} + V_{4,1,3}$
ψ_{49}	$(1,1,3)$	$V_{1,3,3} - V_{2,1,2}$
ψ_{48}	$(1,1,3)$	$-V_{1,1,3}$
ψ_{50}	$(1,1,3)$	$V_{1,3,3} - V_{2,1,2}$
ψ_{46}	$(1,1,3)$	$-V_{1,1,3}$
ψ_{56}	$(1,1,3)$	$-V_{1,1,3} + V_{2,3,3} + V_{4,1,3}$
ψ_{53}	$(1,1,3)$	$-V_{1,2,3} + V_{4,1,1} - V_{4,3,3}$
ψ_{47}	$(1,1,3)$	$2V_{1,2,2} - V_{1,2,3} + V_{4,1,3}$
ψ_{55}	$(1,1,3)$	$-V_{1,1,3}$
ψ_{52}	$(1,1,3)$	$-V_{1,1,3}$
ψ_{54}	$(1,1,3)$	$-V_{1,1,3}$
ψ_{51}	$(1,1,3)$	$-V_{1,1,3} - V_{2,3,3} - V_{4,1,3}$
ψ_{39}	$(1,1,4)$	$V_{1,1,4} + V_{2,1,3} - V_{4,3,2}$
ψ_{40}	$(1,1,4)$	$V_{1,2,4} + V_{4,1,3}$
ψ_{86}	$(1,2,1)$	$-V_{1,2,1}$
ψ_{87}	$(1,2,1)$	$V_{1,1,1} + V_{4,1,3}$

ψ_{76}	$(1,2,2)$	$-V_{1,2,1}-V_{1,3,1}+V_{1,3,2}$
ψ_{79}	$(1,2,2)$	$-V_{1,1,1}+V_{1,2,1}-V_{1,3,2}$
ψ_{80}	$(1,2,2)$	$-V_{1,3,2}+V_{2,1,3}+V_{4,1,1}$
ψ_{78}	$(1,2,2)$	$-V_{1,1,2}-V_{2,1,3}-V_{4,1,1}$
ψ_{82}	$(1,2,2)$	$V_{1,2,2}+V_{4,3,2}$
ψ_{81}	$(1,2,2)$	$-V_{1,1,1}+V_{1,2,1}-V_{1,3,2}$
ψ_{83}	$(1,2,3)$	$-V_{1,2,3}$
ψ_{84}	$(1,2,3)$	$V_{1,1,3}+V_{4,1,3}$
ψ_{69}	$(1,2,4)$	$-V_{1,2,1}-V_{1,3,1}+V_{1,3,4}$
ψ_{72}	$(1,2,4)$	$-V_{1,1,1}+V_{1,2,1}-V_{1,3,4}$
ψ_{73}	$(1,2,4)$	$-V_{1,2,1}+V_{3,1,4}-V_{3,2,1}$
ψ_{71}	$(1,2,4)$	$-V_{1,1,4}-V_{2,1,2}+V_{2,3,3}$
ψ_{74}	$(1,2,4)$	$V_{1,2,4}-V_{2,1,2}$
ψ_{75}	$(1,2,4)$	$-V_{1,1,1}+V_{1,2,1}-V_{1,3,4}$
ψ_{36}	$(1,3,1)$	$-V_{1,3,2}+V_{3,1,2}-V_{3,2,1}$
ψ_{37}	$(1,3,1)$	$V_{1,3,1}+V_{4,1,3}$
ψ_{38}	$(1,3,1)$	$-V_{1,3,1}$
ψ_{33}	$(1,3,1)$	$V_{1,3,1}+V_{4,1,3}$
ψ_{35}	$(1,3,1)$	$V_{1,3,1}-V_{2,3,3}$
ψ_{24}	$(1,3,2)$	$V_{3,1,2}-V_{2,3,3}+V_{4,3,3}$
ψ_{23}	$(1,3,2)$	$V_{1,3,2}-V_{2,1,3}$
ψ_{20}	$(1,3,2)$	$-V_{1,3,2}$
ψ_{26}	$(1,3,2)$	$V_{1,2,2}-V_{1,2,3}-V_{1,3,3}$
ψ_{25}	$(1,3,2)$	$-V_{1,3,2}$
ψ_{21}	$(1,3,2)$	$-V_{1,2,2}-V_{2,3,2}-V_{4,1,3}$
ψ_{31}	$(1,3,3)$	$V_{1,3,3}+V_{4,1,3}$
ψ_{32}	$(1,3,3)$	$-V_{1,3,3}$
ψ_{27}	$(1,3,3)$	$V_{1,3,3}+V_{4,1,3}$
ψ_{29}	$(1,3,3)$	$V_{1,3,3}-V_{2,3,3}$
ψ_{30}	$(1,3,3)$	$-V_{1,3,4}+V_{3,1,3}+V_{3,1,4}$
ψ_{17}	$(1,3,4)$	$-V_{1,2,1}+V_{1,2,4}-V_{1,3,1}$
ψ_{16}	$(1,3,4)$	$-V_{1,3,4}$
ψ_{13}	$(1,3,4)$	$V_{1,3,4}-V_{2,1,3}$
ψ_{19}	$(1,3,4)$	$-V_{1,2,1}+V_{1,2,4}-V_{1,3,1}$
ψ_{18}	$(1,3,4)$	$-V_{1,3,4}$
ψ_{14}	$(1,3,4)$	$-V_{1,2,4}-V_{2,3,2}-V_{4,1,3}$
ψ_{114}	$(2,1,1)$	$-V_{1,2,2}+V_{3,1,2}-V_{4,1,3}$
ψ_{111}	$(2,1,2)$	$-V_{2,3,2}$
ψ_{112}	$(2,1,2)$	$V_{1,2,2}-V_{3,1,2}$
ψ_{113}	$(2,1,2)$	$-V_{2,1,2}$
ψ_{117}	$(2,1,3)$	$V_{1,3,3}-V_{3,1,3}$
ψ_{116}	$(2,1,3)$	$V_{1,3,2}-V_{3,1,2}$
ψ_{119}	$(2,1,3)$	$-V_{2,1,3}$
ψ_{102}	$(2,3,1)$	$V_{4,3,1}$
ψ_{100}	$(2,3,1)$	$-V_{2,3,1}$
ψ_{99}	$(2,3,1)$	$-V_{2,1,1}$
ψ_{101}	$(2,3,1)$	$-V_{1,3,2}-V_{3,1,2}-V_{4,3,2}$
ψ_{98}	$(2,3,1)$	$-2V_{1,3,4}-V_{2,3,2}-V_{4,3,1}$
ψ_{89}	$(2,3,2)$	$-V_{2,3,2}$
ψ_{93}	$(2,3,2)$	$V_{1,1,2}+V_{1,2,2}+V_{2,3,1}$
ψ_{94}	$(2,3,2)$	$-V_{1,1,1}+V_{2,3,3}+V_{3,1,1}$
ψ_{92}	$(2,3,2)$	$V_{1,2,2}-V_{3,1,2}$

ψ_{91}	$(2,3,2)$	$-2V_{1,3,1} - V_{2,3,2} - V_{4,1,3}$
ψ_{96}	$(2,3,2)$	$2V_{1,2,1} - V_{2,1,3} + V_{4,3,1}$
ψ_{95}	$(2,3,2)$	$2V_{1,2,1} - V_{2,1,3} + V_{4,3,1}$
ψ_{106}	$(2,3,3)$	$V_{1,3,1} + V_{3,2,1}$
ψ_{108}	$(2,3,3)$	$-V_{2,3,3}$
ψ_{105}	$(2,3,3)$	$V_{1,3,2} - V_{3,1,2}$
ψ_{110}	$(2,3,3)$	$V_{4,3,3}$
ψ_{109}	$(2,3,3)$	$V_{4,3,3}$
ψ_{104}	$(2,3,3)$	$-2V_{1,2,1} - V_{2,1,1} - V_{4,3,3}$
ψ_{107}	$(2,3,3)$	$-V_{1,3,1} + V_{2,3,3} - V_{3,2,1}$
ψ_{182}	$(4,1,1)$	$-V_{2,1,1} + V_{2,1,2} + V_{2,1,3}$
ψ_{185}	$(4,1,1)$	$V_{1,1,1} - V_{1,2,1} + V_{2,1,3}$
ψ_{184}	$(4,1,1)$	$-V_{1,1,2} - V_{1,3,2} + V_{4,1,3}$
ψ_{189}	$(4,1,2)$	$-V_{4,1,2}$
ψ_{188}	$(4,1,2)$	$V_{1,3,2} - V_{2,1,1} - V_{3,1,2}$
ψ_{190}	$(4,1,2)$	$-V_{4,1,2}$
ψ_{178}	$(4,1,3)$	$-V_{4,1,3}$
ψ_{181}	$(4,1,3)$	$2V_{1,1,2} + 2V_{1,2,2} - V_{2,1,3}$
ψ_{180}	$(4,1,3)$	$-V_{4,1,3}$
ψ_{164}	$(4,3,1)$	$-V_{4,3,1}$
ψ_{169}	$(4,3,1)$	$V_{1,1,1} - V_{1,3,3} - V_{4,3,2}$
ψ_{166}	$(4,3,1)$	$V_{2,3,1}$
ψ_{167}	$(4,3,1)$	$V_{1,3,1} - V_{3,2,3} + V_{4,3,2}$
ψ_{168}	$(4,3,1)$	$-V_{1,1,4} - V_{1,2,2} + V_{4,3,2}$
ψ_{175}	$(4,3,2)$	$V_{2,3,2}$
ψ_{173}	$(4,3,2)$	$-V_{1,1,2} - V_{1,2,4} - V_{2,3,1}$
ψ_{174}	$(4,3,2)$	$V_{1,1,2} + V_{1,3,2}$
ψ_{172}	$(4,3,2)$	$V_{1,3,2} - V_{2,1,1} - V_{3,1,2}$
ψ_{176}	$(4,3,2)$	$-V_{2,3,1} + V_{4,1,3}$
ψ_{171}	$(4,3,2)$	$V_{2,3,3} - V_{4,1,2} + V_{4,1,3}$
ψ_{177}	$(4,3,2)$	$V_{2,3,2}$
ψ_{157}	$(4,3,3)$	$-V_{4,1,2} + V_{4,1,3} - V_{4,3,1}$
ψ_{158}	$(4,3,3)$	$-V_{4,1,3}$
ψ_{161}	$(4,3,3)$	$V_{1,3,1} - V_{3,1,1} + V_{4,1,3}$
ψ_{162}	$(4,3,3)$	$-V_{4,1,3}$
ψ_{156}	$(4,3,3)$	$-V_{1,1,1} + V_{1,3,1} + V_{2,3,2}$
ψ_{160}	$(4,3,3)$	$-V_{4,1,3}$
ψ_{163}	$(4,3,3)$	$V_{1,3,1} - V_{2,3,3} - V_{3,1,1}$
ψ_{151}	$(3,1,2)$	$V_{1,3,2} - V_{4,1,3}$
ψ_{152}	$(3,1,2)$	$V_{3,1,2}$
ψ_{148}	$(3,1,4)$	$-V_{1,3,4} + V_{2,1,3} - V_{4,1,3}$
ψ_{149}	$(3,1,4)$	$V_{3,1,4}$
ψ_{143}	$(3,2,1)$	$-V_{3,2,1}$
ψ_{142}	$(3,2,1)$	$V_{3,2,1}$
ψ_{145}	$(3,2,1)$	$-V_{3,2,1}$
ψ_{144}	$(3,2,1)$	$-V_{1,3,1} - V_{2,1,3}$
ψ_{146}	$(3,2,1)$	$-V_{1,3,1} + V_{2,3,3} - V_{4,1,3}$
ψ_{132}	$(3,2,2)$	$-V_{1,1,2} - V_{2,3,1}$
ψ_{133}	$(3,2,2)$	$V_{3,2,2}$
ψ_{135}	$(3,2,2)$	$-V_{1,2,2} - V_{2,3,2}$
ψ_{133}	$(3,2,2)$	$V_{3,2,2}$
ψ_{129}	$(3,2,2)$	$-V_{3,2,2}$

ψ_{131}	$(3,2,2)$	$V_{1,2,2}+V_{2,3,2}$
ψ_{128}	$(3,2,2)$	$V_{1,1,2}+V_{1,2,1}+V_{1,3,1}$
ψ_{130}	$(3,2,2)$	$V_{1,1,2}+V_{2,3,1}$
ψ_{137}	$(3,2,3)$	$-V_{3,2,3}$
ψ_{136}	$(3,2,3)$	$V_{3,2,3}$
ψ_{138}	$(3,2,3)$	$-V_{1,1,2}-V_{1,2,3}-V_{1,3,2}$
ψ_{139}	$(3,2,3)$	$-V_{3,2,3}$
ψ_{140}	$(3,2,3)$	$-V_{1,3,3}+V_{2,3,3}-V_{4,1,3}$
ψ_{124}	$(3,2,4)$	$-V_{1,3,4}+V_{2,1,3}-V_{4,1,3}$
ψ_{127}	$(3,2,4)$	$-V_{1,2,4}-V_{2,3,2}$
ψ_{121}	$(3,2,4)$	$-V_{3,2,4}$
ψ_{120}	$(3,2,4)$	$-V_{2,3,3}+V_{3,2,4}-V_{4,1,3}$
ψ_{122}	$(3,2,4)$	$V_{1,1,4}+V_{2,3,1}$
ψ_{123}	$(3,2,4)$	$V_{1,2,4}-V_{2,1,2}+V_{4,1,2}$
ψ_{125}	$(3,2,4)$	$V_{3,2,4}$

Bibliography

[1] **ATLAS Collaboration** Collaboration, G. Aad *et al.*, "Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC," *Phys.Lett.* **B716** (2012) 1–29, `arXiv:1207.7214 [hep-ex]`.

[2] H. Georgi, H. R. Quinn, and S. Weinberg, "Hierarchy of Interactions in Unified Gauge Theories," *Phys.Rev.Lett.* **33** (1974) 451–454.

[3] J. C. Pati and A. Salam, "Unified Lepton-Hadron Symmetry and a Gauge Theory of the Basic Interactions," *Phys.Rev.* **D8** (1973) 1240–1251.

[4] H. Georgi and S. L. Glashow, "Unity of All Elementary Particle Forces," *Phys. Rev. Lett.* **32** (1974) 438–441.

[5] P. Ramond, "Dual Theory for Free Fermions," *Phys.Rev.* **D3** (1971) 2415–2418.

[6] Y. Golfand and E. Likhtman, "Extension of the Algebra of Poincare Group Generators and Violation of p Invariance," *JETP Lett.* **13** (1971) 323–326.

[7] D. Volkov and V. Akulov, "Is the Neutrino a Goldstone Particle?," *Phys.Lett.* **B46** (1973) 109–110.

[8] J. Wess and B. Zumino, "Supergauge Transformations in Four-Dimensions," *Nucl.Phys.* **B70** (1974) 39–50.

[9] H. P. Nilles, "Supersymmetry, Supergravity and Particle Physics," *Phys. Rept.* **110** (1984) 1–162.

[10] J. Wess and J. Bagger, *Supersymmetry and supergravity.* Princeton University Press, 1992.

[11] N. Seiberg, "Naturalness versus supersymmetric nonrenormalization theorems," *Phys.Lett.* **B318** (1993) 469–475, `arXiv:hep-ph/9309335 [hep-ph]`.

[12] **Supernova Search Team** Collaboration, A. G. Riess *et al.*, "Observational evidence from supernovae for an accelerating universe and a cosmological constant," *Astron.J.* **116** (1998) 1009–1038, `arXiv:9805201 [astro-ph]`.

[13] S. Weinberg, "Anthropic Bound on the Cosmological Constant," *Phys.Rev.Lett.* **59** (1987) 2607.

[14] C. Baker, D. Doyle, P. Geltenbort, K. Green, M. van der Grinten, *et al.*, "An Improved experimental limit on the electric dipole moment of the neutron," *Phys.Rev.Lett.* **97** (2006) 131801, `arXiv:hep-ex/0602020 [hep-ex]`.

[15] G. Raffelt and L.J.Rosenberg, "Axions and other similar particles," tech. rep., Particle Data Group, 2012.

[16] S. Weinberg, "Implications of Dynamical Symmetry Breaking," *Phys.Rev.* **D13** (1976) 974–996.

[17] L. Susskind, "Dynamics of Spontaneous Symmetry Breaking in the Weinberg-Salam Theory," *Phys.Rev.* **D20** (1979) 2619–2625.

[18] A. Cabo Montes de Oca, N. Cabo-Bizet, and A. Cabo-Bizet, "A path integral formula for quark condensate states in a modified PQCD," *Eur. Phys. J.* **C64** (2009) 133–145.

[19] T. Kaluza, "Zum Unitätsproblem in der Physik.," *Sitzungsber. Preuss. Akad. Wiss. Berlin.* **96** (1921) 69–72.

[20] O. Klein, "Quantentheorie und fünfdimensionale Relativitätstheorie," *Zeitschrift für Physik A* **37(12)** (1926) 895–906.

[21] D. Z. Freedman, P. van Nieuwenhuizen, and S. Ferrara, "Progress Toward a Theory of Supergravity," *Phys.Rev.* **D13** (1976) 3214–3218.

[22] D. Z. Freedman and A. Van Proeyen, *Supergravity.* Cambridge University Press, 2012.

[23] J. Polchinski, *String theory. Vol. 1: An introduction to the bosonic string.* Cambridge University Press, 1998.

[24] G. Veneziano, "Construction of a crossing - symmetric, Regge behaved amplitude for linearly rising trajectories," *Nuovo Cim.* **A57** (1968) 190–197.

[25] Y. Nambu, "Quark model and the factorization of the Veneziano amplitude,".

[26] H. Nielsen, "An almost physical interpretation of the integrand of the n–point Veneziano amplitude,". 15th conference on High Energy Physics, Kiev (1870).

[27] L. Susskind, "Dual-symmetric theory of hadrons. 1.," *Nuovo Cim.* **A69S10** (1970) 457–496.

[28] M. B. Green and J. H. Schwarz, "Anomaly Cancellation in Supersymmetric D=10 Gauge Theory and Superstring Theory," *Phys. Lett.* **B149** (1984) 117–122.

[29] S.-T. Yau, "Calabi's Conjecture and some new results in algebraic geometry," *Proc.Nat.Acad.Sci.* **74** (1977) 1798–1799.

[30] G. Bhattacharyya and T. S. Ray, "Pushing the SUSY Higgs mass towards 125 GeV with a color adjoint," `arXiv:1210.0594 [hep-ph]`.

[31] L. Motl, "SUSY with a colored adjoint chiral multiplet," *Blog: The Reference Frame* (2012) .

[32] L. J. Dixon, J. A. Harvey, C. Vafa, and E. Witten, "Strings on Orbifolds," *Nucl. Phys.* **B261** (1985) 678–686.

[33] L. J. Dixon, J. A. Harvey, C. Vafa, and E. Witten, "Strings on Orbifolds. 2," *Nucl. Phys.* **B274** (1986) 285–314.

[34] M. B. Green, J. H. Schwarz, and L. Brink, "N=4 Yang-Mills and N=8 Supergravity as Limits of String Theories," *Nucl.Phys.* **B198** (1982) 474–492.

[35] K. Kikkawa and M. Yamasaki, "Casimir Effects in Superstring Theories," *Phys.Lett.* **B149** (1984) 357.

[36] T. Buscher, "A Symmetry of the String Background Field Equations," *Phys.Lett.* **B194** (1987) 59.

[37] A. Font, L. E. Ibanez, D. Lust, and F. Quevedo, "Strong - weak coupling duality and nonperturbative effects in string theory," *Phys.Lett.* **B249** (1990) 35–43.

[38] J. H. Schwarz, "Does string theory have a duality symmetry relating weak and strong coupling?," `arXiv:9307121 [hep-th]`.

[39] A. Sen, "Strong - weak coupling duality in four-dimensional string theory," *Int.J.Mod.Phys.* **A9** (1994) 3707–3750, `arXiv:9402002 [hep-th]`.

[40] E. Witten, "String theory dynamics in various dimensions," *Nucl.Phys.* **B443** (1995) 85–126, `arXiv:9503124 [hep-th]`.

[41] J. Polchinski, "Dirichlet Branes and Ramond-Ramond charges," *Phys.Rev.Lett.* **75** (1995) 4724–4727, `arXiv:9510017 [hep-th]`.

[42] G. Aldazabal, L. E. Ibanez, F. Quevedo, and A. Uranga, "D-branes at singularities: A Bottom up approach to the string embedding of the standard model," *JHEP* **0008** (2000) 002, `arXiv:0005067 [hep-th]`.

[43] J. M. Maldacena, "The Large N limit of superconformal field theories and supergravity," *Adv.Theor.Math.Phys.* **2** (1998) 231–252, `arXiv:9711200 [hep-th]`.

[44] E. Witten, "Anti-de Sitter space and holography," *Adv.Theor.Math.Phys.* **2** (1998) 253–291, `arXiv:9802150 [hep-th]`.

[45] C. Vafa, "Evidence for F theory," *Nucl.Phys.* **B469** (1996) 403–418, `arXiv:9602022 [hep-th]`.

[46] F. Denef, "Les Houches Lectures on Constructing String Vacua," `arXiv:0803.1194 [hep-th]`.

[47] O. Lebedev, H. P. Nilles, S. Raby, S. Ramos-Sanchez, M. Ratz, *et al.*, "A Mini-landscape of exact MSSM spectra in heterotic orbifolds," *Phys.Lett.* **B645** (2007) 88–94, `arXiv:0611095 [hep-th]`.

[48] S. Forste, H. P. Nilles, P. K. S. Vaudrevange, and A. Wingerter, "Heterotic brane world," *Phys. Rev.* **D70** (2004) 106008, `arXiv:0406208`.

[49] T. Kobayashi, S. Raby, and R.-J. Zhang, "Searching for realistic 4d string models with a Pati-Salam symmetry: Orbifold grand unified theories from heterotic string compactification on a Z(6) orbifold," *Nucl. Phys.* **B704** (2005) 3–55, `arXiv:hep-ph/0409098`.

[50] W. Buchmuller, K. Hamaguchi, O. Lebedev, and M. Ratz, "Dual models of gauge unification in various dimensions," *Nucl. Phys.* **B712** (2005) 139–156, `arXiv:hep-ph/0412318`.

[51] W. Buchmuller, K. Hamaguchi, O. Lebedev, and M. Ratz, "Supersymmetric standard model from the heterotic string," *Phys. Rev. Lett.* **96** (2006) 121602, `arXiv:hep-ph/0511035`.

[52] H. P. Nilles, S. Ramos-Sanchez, and P. K. S. Vaudrevange, "Local Grand Unification and String Theory," *AIP Conf. Proc.* **1200** (2010) 226–234, `arXiv:0909.3948 [hep-th]`.

[53] H. P. Nilles, S. Ramos-Sanchez, M. Ratz, and P. K. S. Vaudrevange, "From strings to the MSSM," *Eur. Phys. J.* **C59** (2009) 249–267, `arXiv:0806.3905 [hep-th]`.

[54] W. Buchmuller, K. Hamaguchi, O. Lebedev, and M. Ratz, "Local grand unification," `arXiv:hep-ph/0512326`.

[55] T. Kobayashi, H. P. Nilles, F. Ploger, S. Raby, and M. Ratz, "Stringy origin of non-Abelian discrete flavor symmetries," *Nucl. Phys.* **B768** (2007) 135–156, `arXiv:hep-ph/0611020`.

[56] R. Kappl *et al.*, "Large hierarchies from approximate R symmetries," *Phys. Rev. Lett.* **102** (2009) 121602, `arXiv:0812.2120 [hep-th]`.

[57] S. Forste, H. P. Nilles, S. Ramos-Sanchez, and P. K. S. Vaudrevange, "Proton Hexality in Local Grand Unification," *Phys. Lett.* **B693** (2010) 386–392, `arXiv:1007.3915 [hep-ph]`.

[58] H. M. Lee *et al.*, "A unique Z_4^R symmetry for the MSSM," *Phys. Lett.* **B694** (2011) 491–495, `arXiv:1009.0905 [hep-ph]`.

[59] P. Ko, T. Kobayashi, J.-h. Park, and S. Raby, "String-derived D4 flavor symmetry and phenomenological implications," *Phys. Rev.* **D76** (2007) 035005, `arXiv:0704.2807 [hep-ph]`.

[60] J. A. Casas and C. Munoz, "A Natural Solution to the MU Problem," *Phys. Lett.* **B306** (1993) 288–294, `arXiv:hep-ph/9302227`.

[61] I. Antoniadis, E. Gava, K. S. Narain, and T. R. Taylor, "Effective mu term in superstring theory," *Nucl. Phys.* **B432** (1994) 187–204, `arXiv:9405024`.

[62] O. Lebedev *et al.*, "The Heterotic Road to the MSSM with R parity," *Phys. Rev.* **D77** (2008) 046013, `arXiv:0708.2691 [hep-th]`.

[63] S. Hamidi and C. Vafa, "Interactions on Orbifolds," *Nucl. Phys.* **B279** (1987) 465.

[64] P. Candelas, G. T. Horowitz, A. Strominger, and E. Witten, "Vacuum Configurations for Superstrings," *Nucl. Phys.* **B258** (1985) 46–74.

[65] J. Erler and A. Klemm, "Comment on the generation number in orbifold compactifications," *Commun. Math. Phys.* **153** (1993) 579–604, `arXiv:9207111 [hep-th]`.

[66] P. S. Aspinwall, "Resolution of orbifold singularities in string theory," `arXiv:9403123 [hep-th]`. To appear in 'Essays on Mirror Manifolds 2'.

[67] S. Reffert, "Toroidal Orbifolds: Resolutions, Orientifolds and Applications in String Phenomenology," `arXiv:0609040 [hep-th]`.

[68] D. Lust, S. Reffert, E. Scheidegger, and S. Stieberger, "Resolved toroidal orbifolds and their orientifolds," *Adv. Theor. Math. Phys.* **12** (2008) 67–183, `arXiv:0609014`.

[69] S. G. Nibbelink, M. Trapletti, and M. Walter, "Resolutions of Cn/Zn Orbifolds, their U(1) Bundles, and Applications to String Model Building," *JHEP* **03** (2007) 035, `arXiv:0701227`.

[70] S. G. Nibbelink, T.-W. Ha, and M. Trapletti, "Toric Resolutions of Heterotic Orbifolds," *Phys. Rev.* **D77** (2008) 026002, `arXiv:0707.1597 [hep-th]`.

[71] E. Witten, "Search for a Realistic Kaluza-Klein Theory," *Nucl.Phys.* **B186** (1981) 412.

[72] E. Witten, "Some Properties of O(32) Superstrings," *Phys. Lett.* **B149** (1984) 351–356.

[73] C. D. Froggatt and H. B. Nielsen, "Hierarchy of Quark Masses, Cabibbo Angles and CP Violation," *Nucl. Phys.* **B147** (1979) 277.

[74] W. Fischler, H. P. Nilles, J. Polchinski, S. Raby, and L. Susskind, "Vanishing Renormalization of the D Term in Supersymmetric U(1) Theories," *Phys.Rev.Lett.* **47** (1981) 757.

[75] J. J. Atick, L. J. Dixon, and A. Sen, "String Calculation of Fayet-Iliopoulos d Terms in Arbitrary Supersymmetric Compactifications," *Nucl. Phys.* **B292** (1987) 109–149.

[76] M. Dine, N. Seiberg, and E. Witten, "Fayet-Iliopoulos Terms in String Theory," *Nucl. Phys.* **B289** (1987) 589.

[77] A. Font, L. E. Ibanez, H. P. Nilles, and F. Quevedo, "Yukawa Couplings in Degenerate Orbifolds: Towards a Realistic SU(3) x SU(2) x U(1) Superstring," *Phys. Lett.* **210B** (1988) 101.

[78] C. Ludeling, F. Ruehle, and C. Wieck, "Non-Universal Anomalies in Heterotic String Constructions," *Phys. Rev.* **D85** (2012) 106010, `arXiv:1203.5789 [hep-th]`.

[79] T. Oda, *Convex Bodies and Algebraic Geometry.* Springer–Verlag, 1988.

[80] W. Fulton, *Introduction to toric varieties.* Annals of mathematics studies ; 131 The William H. Roever lectures in geometry. Princeton University Press, 1997.

[81] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, *et al.*, "Mirror symmetry,".

[82] S. G. Nibbelink, D. Klevers, F. Ploger, M. Trapletti, and P. K. S. Vaudrevange, "Compact heterotic orbifolds in blow-up," *JHEP* **04** (2008) 060, `arXiv:0802.2809 [hep-th]`.

[83] S. G. Nibbelink, J. Held, F. Ruehle, M. Trapletti, and P. K. S. Vaudrevange, "Heterotic Z6-II MSSM Orbifolds in Blowup," *JHEP* **03** (2009) 005, `arXiv:0901.3059 [hep-th]`.

[84] M. Blaszczyk, S. Nibbelink Groot, F. Ruehle, M. Trapletti, and P. K. Vaudrevange, "Heterotic MSSM on a Resolved Orbifold," *JHEP* **1009** (2010) 065, `arXiv:1007.0203 [hep-th]`.

[85] O. Lebedev, H. P. Nilles, S. Ramos-Sanchez, M. Ratz, and P. K. S. Vaudrevange, "Heterotic mini-landscape (II): completing the search for MSSM vacua in a Z_6 orbifold," *Phys. Lett.* **B668** (2008) 331–335, `arXiv:0807.4384 [hep-th]`.

[86] J. E. Kim, J.-H. Kim, and B. Kyae, "Superstring standard model from Z(12-I) orbifold compactification with and without exotics, and effective R- parity," *JHEP* **06** (2007) 034, `arXiv:hep-ph/0702278`.

[87] M. Blaszczyk *et al.*, "A Z2xZ2 standard model," *Phys. Lett.* **B683** (2010) 340–348, `arXiv:0911.4905 [hep-th]`.

[88] C. Vafa and E. Witten, "On orbifolds with discrete torsion," *J. Geom. Phys.* **15** (1995) 189–214, `arXiv:9409188`.

[89] F. Ploger, S. Ramos-Sanchez, M. Ratz, and P. K. S. Vaudrevange, "Mirage Torsion," *JHEP* **04** (2007) 063, `arXiv:0702176`.

[90] S. Ramos Sánchez Private communication, 2013.

[91] S. Groot Nibbelink, H. P. Nilles, and M. Trapletti, "Multiple anomalous U(1)s in heterotic blow-ups," *Phys. Lett.* **B652** (2007) 124–127, `arXiv:0703211`.

[92] F. Gmeiner, S. Groot Nibbelink, H. P. Nilles, M. Olechowski, and M. G. A. Walter, "Localized anomalies in heterotic orbifolds," *Nucl. Phys.* **B648** (2003) 35–68, `arXiv:0208146`.

[93] A. N. Schellekens and N. P. Warner, "Anomalies, Characters and Strings," *Nucl. Phys.* **B287** (1987) 317.

[94] N. G. Cabo Bizet, T. Kobayashi, D. K. Mayorga Peña, S. L. Parameswaran, M. Schmitz, *et al.*, "R-charge Conservation and More in Factorizable and Non-Factorizable Orbifolds," `arXiv:1301.2322 [hep-th]`.

[95] M. Blaszczyk, N. G. Cabo Bizet, H. P. Nilles, and F. Ruhle, "A perfect match of MSSM-like orbifold and resolution models via anomalies," *JHEP* **1110** (2011) 117, `arXiv:1108.0667 [hep-th]`.

[96] J. Polchinski, *String theory. Vol. 2: Superstring theory and beyond.* Cambridge University Press, 1998.

[97] F. Gliozzi, J. Scherk, and D. I. Olive, "Supersymmetry, Supergravity Theories and the Dual Spinor Model," *Nucl. Phys.* **B122** (1977) 253–290.

[98] D. J. Gross, J. A. Harvey, E. J. Martinec, and R. Rohm, "The Heterotic String," *Phys. Rev. Lett.* **54** (1985) 502–505.

[99] D. J. Gross, J. A. Harvey, E. J. Martinec, and R. Rohm, "Heterotic String Theory. 1. The Free Heterotic String," *Nucl. Phys.* **B256** (1985) 253.

[100] K. Narain, "TOROIDAL COMPACTIFICATION OF HETEROTIC STRING. (ABSTRACT ONLY),".

[101] K. Narain, M. Sarmadi, and E. Witten, "A Note on Toroidal Compactification of Heterotic String Theory," *Nucl. Phys.* **B279** (1987) 369.

[102] M. B. Green, J. Schwarz, and E. Witten, *SUPERSTRING THEORY. VOL. 1: INTRODUCTION.* Cambridge University Press, 1987.

[103] L. Ibañez and A. Uranga, *String theory and particle physics.* Cambridge University Press, 2012.

[104] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory*, vol. 2. Cambridge University Press, 1999.

[105] L. J. Dixon, D. Friedan, E. J. Martinec, and S. H. Shenker, "The Conformal Field Theory of Orbifolds," *Nucl. Phys.* **B282** (1987) 13–73.

[106] D. Bailin and A. Love, "Orbifold compactifications of string theory," *Phys.Rept.* **315** (1999) 285–408.

[107] S. Ramos-Sanchez, "Towards Low Energy Physics from the Heterotic String," *Fortsch.Phys.* **10** (2009) 907–1036, `arXiv:0812.3560 [hep-th]`.

[108] P. K. Vaudrevange, "Grand Unification in the Heterotic Brane World," `arXiv:0812.3503 [hep-th]`.

[109] T. Kobayashi, S. L. Parameswaran, S. Ramos-Sanchez, and I. Zavala, "Revisiting Coupling Selection Rules in Heterotic Orbifold Models," *JHEP* **1205** (2012) 008, `arXiv:1107.2137 [hep-th]`.

[110] D. Mayorga Peña, "Towards Realistic Spectra from the $\mathbb{Z}_2 \times \mathbb{Z}_4$ Orbifold Model," 2011.

[111] J. Casas, F. Gomez, and C. Munoz, "Complete structure of Z(n) Yukawa couplings," *Int.J.Mod.Phys.* **A8** (1993) 455–506, `arXiv:9110060 [hep-th]`.

[112] K. Becker, M. Becker, and J. Schwarz, *String theory and M-theory: A modern introduction.* Cambridge University Press, 2007.

[113] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, vol. 52 of Wiley Classics Library. Whiley, 1999.

[114] A. Strominger, "Superstrings with Torsion," *Nucl.Phys.* **B274** (1986) 253.

[115] C. Hull, "Superstring compactifications with torsion and spacetime supersymmetry," Tech. Rep. Print-86-0251, Cambridge University, 1986.

[116] C. Hull, "Compactifications of the heterotic superstring," *Phys. Lett.* **B178** (1986) 357.

[117] J. Gates, S.J., C. Hull, and M. Rocek, "Twisted Multiplets and New Supersymmetric Nonlinear Sigma Models," *Nucl.Phys.* **B248** (1984) 157.

[118] M. Rocek, "Modified Calabi-Yau manifolds with torsion,".

[119] P. Koerber, "Lectures on Generalized Complex Geometry for Physicists," *Fortsch.Phys.* **59** (2011) 169–242, `arXiv:1006.1536 [hep-th]`.

[120] I. Benmachiche, J. Louis, and D. Martinez-Pedrera, "The Effective action of the heterotic string compactified on manifolds with SU(3) structure," *Class.Quant.Grav.* **25** (2008) 135006, `arXiv:0802.0410 [hep-th]`.

[121] A. Lukas and C. Matti, "G-structures and Domain Walls in Heterotic Theories," *JHEP* **1101** (2011) 151, `arXiv:1005.5302 [hep-th]`.

[122] R. Blumenhagen, G. Honecker, and T. Weigand, "Loop-corrected compactifications of the heterotic string with line bundles," *JHEP* **0506** (2005) 020, `arXiv:0504232 [hep-th]`.

[123] J. A. Casas, A. de la Macorra, M. Mondragon, and C. Munoz, "Z(7) phenomenology," *Phys. Lett.* **B247** (1990) 50–56.

[124] P. S. Aspinwall, B. R. Greene, and D. R. Morrison, "Measuring small distances in N=2 sigma models," *Nucl. Phys.* **B420** (1994) 184–242, `arXiv:9311042`.

[125] Y. Katsuki, Y. Kawamura, T. Kobayashi, and N. Ohtsubo, "Z(7) ORBIFOLD MODELS," *Phys.Lett.* **B212** (1988) 339.

[126] T. Kobayashi and N. Ohtsubo, "Analysis on the Wilson lines of Z(N) orbifold models," *Phys.Lett.* **B257** (1991) 56–62.

[127] H. P. Nilles, S. Ramos-Sanchez, P. K. Vaudrevange, and A. Wingerter, "The Orbifolder: A Tool to study the Low Energy Effective Theory of Heterotic Orbifolds," *Comput.Phys.Commun.* **183** (2012) 1363–1380, `arXiv:1110.5229 [hep-th]`.

[128] S. Groot Nibbelink and P. K. Vaudrevange, "Schoen manifold with line bundles as resolved magnetized orbifolds," *JHEP* **1303** (2013) 142, `arXiv:1212.4033 [hep-th]`.

[129] K.-S. Choi and J. E. Kim, "Quarks and leptons from orbifolded superstring," *Lect.Notes Phys.* **696** (2006) 1–406.

[130] S. L. Parameswaran, S. Ramos-Sanchez, and I. Zavala, "On Moduli Stabilisation and de Sitter Vacua in MSSM Heterotic Orbifolds," *JHEP* **1101** (2011) 071, `arXiv:1009.3931 [hep-th]`.