Essays on Dynamic Mechanism Design

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Contents

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Introduction

This thesis considers the problem of selling one or several units of a good in a dynamic environment. The environment is dynamic because the pool of potential buyers changes over time. It can grow if new buyers arrive, and it can shrink if buyers leave. More generally, in Chapter 3, the willingness to pay of every buyer in the pool may change over time. The allocation problem is dynamic, because the point in time when a unit is sold is not exogenously given and may depend on the available buyers and their characteristics.

Dynamic allocation problems are the rule rather than the exception. Dynamic decisions have to be made, even in situations where a static selling mechanism, like an auction with a reserve price, is used at a fixed date. The date at which the auction takes place and the time at which it is announced determines the pool of potential buyers. The reserve price, together with the valuations of the buyers, determines whether the object is sold in the auction, or later in some subsequent selling mechanism.

In dynamic allocation problems, it is crucial for the seller to know how the pool of potential buyers evolves over time. There may be many constraints that withhold some or all of this information from the seller. In Chapter 1, we will focus on two particular constraints: First, information realizes over time. There is a fundamental informational constraint that prevents the seller or any other person from looking into the future. Second, there may be asymmetric information about the departure of potential buyers.[1]

Without the first aspect, the problem would be essentially static. If agents could look into the future, the same information would be available at all times. Decisions today would not have to be based on *expectations* about the state of the world tomorrow, because the *actual* state is already known today.

If we consider the allocation of a single object, the fundamental difference between the static and the dynamic model becomes apparent. In a static world, i.e. a world with perfect foresight, the ex-post efficient allocation is trivially feasible. Already in the first period it is known at which point in the future the buyer with the highest valuation will arrive. In a dynamic world, on the other hand, the decision whether the object should be sold today or tomorrow, can at best be based on expectations about the arrival of future buyers. This necessarily leads to inefficiencies. If

[1]Another interesting constraint is asymmetric information about the arrival of new buyers. The analysis of this problem shares some similarities with the case of asymmetric information with respect to departure times. The conclusion of Chapter 1 will briefly comment on this.

the expected value of postponing the allocation is higher than the highest valuation of the buyers that are available today, it may still be possible that tomorrow, the state of the world turns out to be worse than expected, and the valuations of all new buyers are lower than the valuations of the buyers that were available today. With hindsight, it would have been more efficient in this case, to sell the object already today. Without perfect foresight, however, inefficiencies of this kind are unavoidable. Evidently, this fundamental informational constraint applies to a seller who maximizes revenue as well as to a seller who is interested in efficiency.

The second constraint—private information—turns out to be more stringent for revenue maximization than for efficiency. Consider a simple example of two buyers in an independent private values setting who arrive in two different time periods. The seller has a single indivisible object that he can sell either in period one or period two. If it is sold in the first period, it cannot be reallocated in the second period (think of a consumption good). Buyer one arrives in the first period. He has a privately known valuation for the object, and a deadline. The deadline determines his willingness to wait. If the deadline is period one, then he will not be interested in the object if it is offered to him in the second period. With deadline two, he is indifferent about the time of the allocation. Buyer two arrives in the second period. He has a privately known valuation which, by the lack of foresight, is not known to anybody in the first period.

Regardless of whether the seller wants to maximize revenue or efficiency, if he knew that the deadline were two, he would always postpone the allocation until period two. Waiting creates an efficiency gain because both buyers are available in the second period and the ex-post efficient allocation can be chosen. In the case of revenue maximization, the seller can extract enough of the efficiency gain from the buyers to make waiting worthwhile.

A difference between efficiency and revenue maximization arises, if we compare the incentives for buyer one to reveal his deadline truthfully. It is well known that the efficient allocation rule is implementable in quite general settings.[2] In particular, asymmetric information about the deadline does not pose a problem for the implementability of the efficient allocation rule. The picture is different in the case of revenue maximization. With asymmetric information about the valuation and symmetric information about the deadline, we show that there is a class of distributions for buyer one's valuation and deadline, for which the seller demands on average *higher prices if the deadline is two.* This destroys incentives to reveal the deadline truthfully. As a consequence, the revenue-maximizing mechanism does not fully separate buyers with different deadlines as is the case for the efficient

[2]In light of the informational constraint, the efficient allocation rule refers to the allocation rule that maximizes the expected surplus under symmetric information, subject to the informational constraint.
Parkes and Singh (2003) seem to be the first authors that have shown the implementability of the efficient allocation rule. See also Athey and Segal (2007) and Bergemann and Välimäki (2010).

allocation. In the example, the revenue-maximizing mechanism involves bunching at the top of the type space. The mechanism separates different deadlines if the valuation is sufficiently low, but for high valuations, buyer one receives the same allocation and the makes the same expected payment regardless of his deadline.

The fundamental reason for the difference between revenue-maximization and efficiency is that the deadline introduces multi-dimensional private information. Indeed the analysis of the optimal mechanism, in the case where the incentive compatibility constraint for the deadline is binding, is formally equivalent to a static mechanism design problem with multi-dimensional private information. Therefore, the methods used in Chapter 1 will also be useful in static auction problems with two dimensional private information (e.g. with buyers have privately known capacity requirements or budget constraints).

The dynamic nature of the problem, however, is reflected in the conditions under which the incentive constraint for the deadline is binding or slack. We analyze the revenue maximizing mechanism for the case of symmetric information about the deadline and show that there are two distinct effects that determine whether the incentive constraint for the deadline is fulfilled even though it was not explicitly imposed. The first effect which we call the *static pricing effect* can also be found in static models with two-dimensional private information.[3] The second effect, called the *dynamic pricing effect*, is genuinely dynamic and has not been documented in the literature so far. We will show that while different deadlines affect the *nature of competition* with competing buyers, *expected competition* that buyer one faces is independent of his deadline. If the deadline is two, buyer two has already arrived when the deadline is reached. The seller can therefore use the *realized* valuation of buyer two to decide whether buyer one gets the object and how much he has to pay, rather than forming an *expectation* about buyer two's valuation. Therefore, competition is more dispersed for a later deadline. We show that non-linearities in the hazard rate of the distribution of buyer one's valuation, together with the increased dispersion of competition, may lead to higher or lower payoffs for deadline two. If the payoff increases in the deadline, buyer one has an incentive to reveal his deadline truthfully. Conversely, if it decreases in the deadline, incentive compatibility is violated. In a static model, the second dimension of private information does not determine the amount of information that the mechanism uses to determine the allocation and payment. Therefore, different types do not lead to differences in the dispersion of competition and the dynamic pricing effect does not occur.

Chapter 1 generalizes these observations and makes them precise. For reasons of tractability, the analysis of the revenue-maximizing mechanism in the case of a binding incentive constraint for the deadline is confined to the example introduced above.

Chapter 2 contains a characterization of feasibility of asymmetric reduced form auctions. This is a technical result that is needed to formulate revenue-maximization

[3] See Chapter 1 for references to this literature.

problems in an auction context in terms of interim winning probabilities. The result is applied in Chapter 1 to solve the two-dimensional mechanism design problem that arises when the incentive constraint for the deadline is binding.

In Chapter 3, dynamic mechanism design is approached from a different angle, which is new to the literature. We will consider the efficient allocation rule, which is known to be implementable in quite general environments. We ask the question whether in a private values setting, dynamic incentive provision conflicts with the use of a *simple payment rule*. Inspired by standard static auctions, we require a simple payment rule to fulfill the following properties:

(A) Only the winner of an object should have to make a payment.

(B) The payment of the winner should not exceed his valuation.

(C) The mechanism should never transfer money to any buyer.

Moreover, motivated by the dynamic model, we require that

(D) payments can be made online, i.e. all information that is needed to determine the payment must be available at the time of allocation.

We argue in Chapter 3, that in a private values model, these properties are very convenient for the practical implementation of a mechanism.

The question whether properties (A)–(D) can be fulfilled without violating incentives is not trivial in a dynamic model. So far, none of the mechanism proposed to implement the efficient allocation rule satisfies all properties. In Chapter 3, a partial answer is given. For the allocation of a single object in a quite general dynamic model with independent private values, we demonstrate that the static Vickrey auction, when generalized to implement the dynamically efficient allocation rule, satisfies all properties (A)–(D).

In the standard static independent private values model, the payment rule of the Vickrey auction can be interpreted in different ways. On the one hand, it a Vickrey-Clarke-Groves mechanism. Therefore, the payment of a buyer corresponds to the externality that he imposes on the other buyers. On the other hand, it is a second-price auction. The payment of the winning bidder corresponds to his critical type, i.e. the lowest bid with which he could have won for a given profile of bids of the other buyers.

If we apply the first principle to the dynamic framework we obtain the mechanisms found in the existing literature. All of these violate at least one of the properties (A)–(C). If we apply the second principle, we get a mechanism (at least in the case of one object), that is incentive compatible and satisfies all properties (A)–(D). In contrast to the static model, however, the payments in the dynamic Vickrey auction do not always correspond to the externality imposed on other buyers.

The definition of Vickrey-payments according to the second principle is not trivial because the model has a multi-dimensional type-space. The critical type is not just the *lowest valuation* that suffices to win as in the static case. In Chapter 3, we show that for the efficient allocation of one object, there exists an order on the type space that allows to define critical types. We argue that the same construction is also possible for quasi-efficient allocations rules. Therefore, the existence of simple payment rules is not limited to the efficient allocation rule. If future research shows that the revenue maximizing allocation rules is quasi-efficient, then it will also be implementable by a simple payment rule. Whether a similar construction is possible for the case of more than one object remains an open question.

CHAPTER 1

Optimal Dynamic Mechanism Design with Deadlines

SUMMARY. A dynamic mechanism design problem with multi-dimensional private information is studied. One or several identical objects are sold to buyers who arrive over a finite number of periods. In addition to his privately known valuation, each buyer also has a privately known deadline for purchasing the object. The seller wants to maximize revenue.

Depending on the type distribution, the incentive compatibility constraint for the deadline may or may not be binding in the optimal mechanism. We identify a static and a dynamic pricing effect that drive incentive compatibility and violations thereof. Both effects are related to distinct properties of the type distribution and sufficient conditions are given under which each effect leads to a binding or slack incentive constraint for the deadline.

An optimal mechanism for the binding case is derived for the special case of one object, two periods and two buyers. It can be implemented by a fixed price in period one and an asymmetric auction in period two. The asymmetry prevails even if the valuations of both buyers are identically distributed. In order to prevent buyer one from buying in the first period when his deadline is two, the seller sets a reserve price that is lower than in the classic (Myerson, 1981) optimal auction and gives him a (non-linear) bonus. The bonus leads to robust bunching at the top of the type-space. The optimal mechanism can be characterized in terms of generalized virtual valuations which depend endogenously on the allocation rule.

1.1. Introduction

This chapter analyzes the problem of a seller, who wants to maximize revenue in a dynamic environment. The seller has a finite number of identical units of a good. Buyers have private values and unit demand, and arrive over a finite number of periods. They are privately informed about their valuations, and each buyer has a privately known deadline which determines the latest point in time at which he still values buying a unit. To focus on the role of deadlines, we assume that the seller observes new arrivals.

In many cases, buyers have deadlines that are imposed by third parties. Consider for example a company that needs to buy a good from a seller in order to enter a contractual relationship with a third party. The good could be a physical object, an option contract, a license, a patent, etc. It is conceivable that the third party sets a deadline after which the contractual relationship is no longer available. Therefore, the object becomes worthless for the company if it is purchased after the deadline. Other examples of dynamic allocation problems in which buyers can have deadlines are online auctions (think for example of buying a birthday present at eBay), the sale of airline tickets, hotel reservations or the sale of houses and real estate.

So far, most of the literature on dynamic mechanism design has abstracted from private information about deadlines, or more generally, from private information about time preferences (see Section 1.1.3). This abstraction typically leads to mechanism design problems with one-dimensional private information which are tractable under quite general assumptions about the dynamic arrival of new bidders and new objects. In this chapter, we take a different direction and allow for private information about the deadline.

We derive conditions under which, in the optimal mechanism, the incentive constraint for the deadline is slack or binding, respectively. In the case of a slack incentive constraint, the seller's problem can be solved by standard techniques. For a special case that ensures tractability, we can solve the two-dimensional adverse selection problem that arises in the binding case. In contrast to the slack case, buyers with different types are not fully separated. We find robust bunching at the top of the type-space.

1.1.1. Summary of Results. As a benchmark, consider the *relaxed problem* of maximizing the seller's revenue when deadlines are commonly known. With commonly known deadlines, the seller can fully separate buyers with different deadlines at no cost. If valuations are not discounted, it is optimal to sell to a buyer only if his deadline is reached. Classic mechanism design theory can be used to deal with asymmetric information about valuations. Following Myerson (1981), buyers are compared in terms of their *virtual valuations*.[1] A buyer is awarded a unit if and only if his virtual valuation is higher than the opportunity cost of the seller, i.e. the highest virtual valuation among the other buyers or the option value of postponing the allocation of the unit to future periods. This allocation rule defines a *critical virtual valuation* which a buyer must overbid in order to get a unit of the good.

The critical virtual valuation can also be used to determine the payment of a winning bidder. Since the virtual valuation is a function of the true valuation, we can define the payment as the lowest valuation that suffices to overbid the critical virtual valuation. The seller uses the inverse of the virtual valuation as a *pricing rule* that maps critical virtual valuations to payments of the winning bidders. *Together* with the optimal allocation rule, this payment rule defines a revenue-maximizing mechanism that is incentive compatible in dominant strategies if deadlines are commonly known.[2]

With privately known deadlines, the seller can try to implement the relaxed solution by asking the buyers to report their deadlines. Buyers can therefore choose

[1]The virtual valuation of a buyer equals his true valuation, i.e. the price that the seller could charge from the buyer in the absence of asymmetric information, minus the expected incentive costs of selling to buyers with this valuation.

[2]We only require Bayes-Nash incentive compatibility in the seller's maximization problem but implementability in dominant strategies is automatically fulfilled for the relaxed solution.

in which period they buy. The relaxed solution is incentive compatible if no buyer has an incentive to buy earlier by misreporting his deadline. In principle, there could be three reasons why a buyer might want to buy earlier.

First, he might expect the critical virtual valuation to be lower in earlier periods. We call this effect the *competition effect* because the critical virtual valuation represents competition by other buyers. We show, however, that there is no competition effect in this model. The expected value of the critical virtual valuation is independent of the deadline.[3]

Second, the mapping between virtual valuations and true valuations could change with the reported deadline so that the seller uses a different pricing rule for different deadlines. This is called the *static pricing effect*.[4] The effect depends on stochastic dependencies between the deadline and the valuation of a buyer. For example, if valuations tend to be higher for earlier deadlines, than pricing will be more aggressive for earlier deadlines. This ensures incentive compatibility of the relaxed solution.

Finally, while expected competition is independent of the deadline, the distribution of the critical virtual valuation is less dispersed for earlier deadlines. Extending our result of equal expectations, we show that the critical virtual valuation for a later deadline is a mean preserving spread of the critical virtual valuation for an earlier deadline.[5] This can lead to different expected prices if the virtual valuation, and therefore the pricing rule used by the seller, is non-linear. This is called the *dynamic pricing effect* because the stochastic dominance results from the dynamic nature of the model. For example, if virtual valuations are convex, then the pricing rule is concave and more dispersed critical virtual valuations lead to lower expected prices. Since the dispersion increases in the deadline, this ensures incentive compatibility of the relaxed solution.

The static as well as the dynamic pricing effect can also work in the opposite direction. If valuations tend to be lower for earlier deadlines and if the virtual valuation is a concave function, then prices tend to be lower for earlier deadlines. This induces misreports of the deadline and the relaxed solution is not incentive compatible.

If the incentive constraint for the deadline is binding, the analysis of the optimal mechanism has to be restricted to get a tractable model. We consider the case of

[3]In models of dynamic learning informational externalities can lead to a competition effect because the report of a buyer conveys information about future buyers' type distributions and arrival time distributions. Therefore, competition is not independent of a buyer's private information (Gershkov and Moldovanu, 2009a,b).

[4]This effect can also arise in static models with a second dimension of private information such as a capacity requirement or a budget constraint.

[5]This result is shown for the case of one object and many time periods, and for the case of many objects and two time periods. I conjecture that the result extends to the case of many objects and many time periods. If this true, all results about the incentive compatibility of the relaxed solution carry over to the general case.

two buyers who arrive in two different time periods and assume that the seller has a single indivisible object.

In this case, the relaxed solution as well as the optimal mechanism have the following common structure: If buyer one reports that his deadline is one, the object is offered to him in the first period for a fixed price. If he declines the offer, the object is offered to the second buyer in the second period for a fixed price. If buyer one reports deadline two, then the seller conducts an auction that gathers both buyers in the second period.

With private information about the deadline, buyer two has the outside option of buying in the first period instead of participating in the auction, if his deadline is two. The seller has two instruments to make this outside option unprofitable. First, he can increase the price offered if buyer one reports deadline one. Second, he can distort the auction format in the second period in favor of buyer one, so that his expected payoff from the auction rises compared to the relaxed solution. We derive the optimal mechanism and show that the seller always uses both instruments.

The distortion of the auction format leads to an asymmetric auction that favors buyer one, even if both buyers have identically distributed valuations. More precisely, the optimal reserve price for buyer one is lower than in the relaxed solution, winning probabilities are higher for all valuations above the reserve price, and there is a non-trivial interval of valuations at the top of the type-space that win the auction with probability one. The expected price paid by buyers with these valuations equals the fixed price offered in the first period. Therefore, there is bunching in the valuation dimension as well as the deadline dimension. This is in contrast to the relaxed solution which fully separates buyers with different types for a large class of type distributions. We provide several examples of distributions for which the relaxed solution is not incentive compatible.

Finally, we propose a generalized virtual valuation for buyer one that allows to describe the optimal auction in the same way as the classic optimal auction (Myerson, 1981). A buyer wins if and only if his (generalized) virtual valuation is non-negative and higher than that of his opponent. In contrast to the classic model, the generalized virtual valuation has a parameter that depends endogenously on the allocation rule. This parameter determines the magnitude of the distortion compared to the relaxed solution (where the parameter vanishes). A simple procedure to compute the optimal distortion is provided. Using the generalized virtual valuation function, it is straight forward to define an ascending clock auction that implements the optimal mechanism in the second period.

1.1.2. Methods. In the auction problem, we have to deal with a type-dependent participation constraint for buyer one because he can choose to buy in the first period. The participation constraint is defined in terms of the interim expected utility. Therefore, Myerson's (1981) classic approach to solve the optimal auction problem by point-wise maximization is not applicable. Instead, the feasibility constraint, i.e. the condition that the object be allocated only once, is formulated in terms of interim winning probabilities. We use the characterization of feasibility of asymmetric reduced form allocation rules from Chapter 2 to solve the resulting control problem.[6]

If the interim winning probability of buyer one is absolutely continuous, the feasibility constraint can be substituted into the objective function. We get a standard control problem in which the winning probability is a state variable and its derivative is the control. (See Guesnerie and Laffont (1984) for an early application of this method). To allow for jumps in the winning probability, the control problem is first solved under the assumption that the winning probability is Lipschitz continuous (and hence absolutely continuous). The Lipschitz solutions converge to an optimal solution of the general problem if the Lipschitz constant approaches infinity. This method was pioneered by Reid (1968) and seems to be new to the mechanism design literature. It may be useful in other auction models where continuity of the winning probability is not guaranteed.[7]

Reid also provides a method to show that Myerson's ironing procedure can be applied to ensure monotonicity of the winning probability. This is important in the present context because the usual hazard rate assumption on the type distribution, which ensures a non-decreasing virtual valuation, does not guarantee monotonicity of the *generalized* virtual valuation.[8]

1.1.3. Related Literature. The literature on dynamic revenue maximization emerged from the literature on dynamic pricing and revenue management. For a survey, see for example Elmaghraby and Keskinocak (2003). McAfee and te Velde (2007) survey airline pricing. This literature typically assumes stochastic demand and abstracts from strategic buyers. If buyers are short-lived and only one buyer is present at the same time, this is a reasonable assumption and the optimal mechanism is a sequence of posted prices (Das Varma and Vettas, 2001). Gallien (2006) shows that a sequence of posted prices is the optimal strategy-proof mechanism for the sale of an inventory of identical objects to long-lived buyers with a commonly known discount factor. He gives conditions on the arrival time distribution that ensure that buyers are served only upon arrival, providing some justification for this assumption in the revenue management literature. More recently, Gershkov

[6]The characterization is a generalization of Border (1991) who studies symmetric allocation rules. Matthews (1984) conjectured the result proven by Border (see also Chen, 1986). For an early application of a special case of the result see Maskin and Riley (1984).

[7]Recently, Hellwig (2008) has derived a version of Pontriyagin's maximum principle that allows for a monotonicity constraint on the control variable without requiring absolute continuity. This is not applicable here, however, as we have to deal with the non-standard feasibility constraint.

[8]Except for Myerson's (1981) paper, which does not use control theory, there does not seem to be a full-fledged solution technique for the (valuation-)bunching case. Guesnerie and Laffont (1984) and earlier Mussa and Rosen (1978) derive necessary conditions for bunches, but do not give precise conditions on the location of bunches.

and Moldovanu (2008) derived the revenue maximizing policy for an inventory of heterogeneous but commonly ranked objects.

Another strand of literature considers discrete time models in which several buyers can arrive simultaneously. Vulcano, van Ryzin, and Maglaras (2002) show that a sequence of auctions with appropriately chosen reserve prices maximizes revenue if bidders are short-lived. Said (2008) considers a model with stochastic arrival and exit of bidders that have unit demand and a commonly known discount factor. A random number of perishable objects is available in each period. The optimal allocation rule awards the objects to the bidders with the highest virtual valuations. It can be implemented by a sequence of open auctions with suitable reserve prices.

Several papers have shown the implementability of the efficient allocation rule in dynamic settings (Parkes and Singh (2003), Bergemann and Välimäki (2010), Athey and Segal (2007)). Said (2008) also considers the efficient allocation rule for the model described above and shows that the mechanism derived in Bergemann and Välimäki (2010) can be implemented by a sequence of open auctions. Chapter 3 of this thesis shows that the efficient allocation of a single object with stochastically arriving long-lived bidders with privately known time preferences only requires transfers between the seller and the winning bidder.

Pavan, Segal, and Toikka (2008) consider a very general dynamic mechanism design model with a fixed set of agents who receive one-dimensional private information in every period. For a special case, the revenue-maximizing allocation rule can be described in terms of virtual surplus in this model.

While general multi-dimensional mechanism design models are very complex to analyze (see e.g. Armstrong (1996), Rochet and Choné (1998) and Jehiel, Moldovanu, and Stacchetti (1999)), several authors have analyzed two-dimensional models with additional structure on the second dimension of private information (the deadline in the present case). Firstly, in these models, agents only have feasible deviations in one direction of the second dimension (i.e. report earlier deadlines but not later deadlines). Secondly, the second parameter does not enter directly in the expected utility of an agent (i.e. the true deadline is immaterial as long as the agent receives the object before the deadline.) See Beaudry, Blackorby, and Szalay (2009) for an analysis of optimal taxation; Blackorby and Szalay (2008) and Szalay (2009) for regulation; Iyengar and Kumar (2008) for a static auction model with capacitated bidders; Dizdar, Gershkov, and Moldovanu (2009) for a dynamic model with capacitated bidders; and Che and Gale (2000) and Malakhov and Vohra (2005) for static models with budget constrained buyers.

Closest to this chapter is Pai and Vohra (2008b), who consider a slightly more general dynamic allocation problem with buyers who have privately known deadlines. They show that the relaxed solution is incentive compatible if the virtual valuation is "sufficiently monotone" in the deadline. This roughly corresponds to the static pricing effect we find in this chapter. Their condition, however, cannot be applied directly to the primitives of the model (i.e. the type distribution).

Szalay (2009) is the only other paper that derives the optimal mechanism for the binding case. All other papers make assumptions that guarantee that the relaxed solution is incentive compatible. Jullien (2000) studies a principal-agent problem with type dependent participation constraints. The analysis of the auction model in the present chapter, however, requires different solution techniques than the models of Szalay (2009) and Jullien (2000) because of the (non-standard) feasibility constraint and the possibility of discontinuities in the optimal solution.

Organization of the Chapter. Section 1.2 describes the model. Section 1.3 presents a characterization of incentive compatibility. Section 1.4 states the sellers problem. Section 1.5 presents the relaxed solution, conditions for incentive compatibility and for violations thereof. Section 1.6 informally presents the general solution for the specialized model described above. Section 1.7 concludes and discusses limitations of the model and possible generalizations. The formal derivation of the results from Section 1.6 is developed in Appendix 1.A. Some other proofs are relegated to Appendix 1.B.

1.2. The Model

A seller wants to maximize the revenue from selling $K \in \mathbb{N}$ identical units of a good within $T \in \mathbb{N}$ time periods. The seller's valuation is normalized to zero. In each period, a random number of buyers $N_t \in \mathbb{N}_0$ arrives. To avoid measurability problems, we assume that there exists a finite upper bound $\bar{N} \in \mathbb{N}$ such that $N_t < \bar{N}$ for all t. The set of buyers who arrive in period t is denoted I_t and we write $I_{\leq t} = \bigcup_{\tau=1}^{t} I_\tau$ and $N_{\leq t} = |I_{\leq t}|$.

Each buyer is interested in buying at most one unit. A buyer $i \in I_t$ is characterized by his arrival time $a_i = t$, his valuation $v_i \in [0, \overline{v}]$, where $\overline{v} > 0$, and his deadline $d_i \in \{t, \ldots, T\}$. The object cannot be sold to a buyer before his arrival time.

Utility is quasi-linear. If buyer i has to make a total payment of y_i and gets (at least) one object in periods $a_i, \ldots, d_i$, then his payoff is $v_i - y_i$. If he only gets units in periods $d_i + 1, \ldots, T$, or if he does not get any unit, then his payoff is $-y_i$. Buyers are risk-neutral and maximize expected payoff. Neither the buyers nor the seller discount future payoffs.[9]

The number of arrivals in different periods are independently distributed. $\nu_{t,n}$ denotes the probability that n buyers arrive in period t. Deadline and valuation are jointly distributed for each buyer but independent for different buyers. Buyers with the same arrival period are ex-ante identical. For given arrival time a, the probability that the deadline of a buyer equals d is denoted $\rho_{a,d}$. Conditional on the deadline, the valuation has distribution function $F_a(v|d)$ and density $f_a(v|d)$.

[9]If only payments are discounted and all agents have a common discount factor, the results do not change. See also Section 1.7 for a discussion of discounting.

Information realizes over time. In period t, the numbers of future buyers N_{t+1}, $\dots, N_T$, and the types of buyers with $a_i > t$, are not known to anybody. In particular, the decision to sell a unit in period t cannot be based on this information. Upon arrival, each buyer privately observes his own valuation and his own deadline. In order to focus on the incentive issues of private information about deadlines, we assume that the seller observes the arrivals of all buyers.[10] The distributions $\nu_{t,n}$, $\rho_{a,d}$ and $F_a(.|d)$ are commonly known from the first period on.

We assume that for all a and all $d \geq a$, $f_a(v|d)$ is continuous in v and strictly positive for all $v \in [0, \overline{v}]$, continuously differentiable in v for $v \in (0, \overline{v})$, and that $f_1'(.|d)$ can be extended continuously to $[0, \overline{v}]$.

Assumption 1.2.1. *For all $a \in \{1, \dots T\}$ and all $d \in \{a, \dots T\}$, the* virtual valuation $J_a(v|d) := v - \frac{1-F_a(v|d)}{f_a(v|d)}$ *is strictly increasing in v.*

To avoid additional technicalities, Assumption 1.2.1 is maintained throughout the chapter. The zero of $J_a(.|d)$ is denoted $v_a^0|d$ and v_T^0 if $a = d = T$.

1.2.1. Allocation Rule. In the most general formulation, a *state* $s_t = (H_t, \xi_{<t})$ consist of the history of buyers' types $H_t = ((a_i, v_i, d_i))_{i \in I_{\leq t}}$, and the past allocation decisions $\xi_{<t} = (\xi_1, \dots, \xi_{t-1})$, where $\xi_\tau \in \{0,1\}^{N_{\leq \tau}}$. $\xi_{\tau,i} = 1$ means that buyer i gets a unit in period τ. For a given history, the number of available units is denoted $k_t = K - \sum_{\tau=1}^{t-1} \sum_{i \in I_{\leq \tau}} \xi_{\tau,i}$.

Definition 1.2.2. (i) The set of *feasible allocations* in state $s_t = (H_t, \xi_{<t})$ is defined as

$$\Phi_t(s_t) = \left\{ \xi_t \in \{0,1\}^{N_{\leq t}} \middle| \sum_{i \in I_{\leq t}} \xi_{t,i} \leq k_t \right\}, \tag{F}$$

and the set of *allocations at the deadline* in state s_t is defined as

$$\tilde{\Phi}_t(s_t) = \{\xi_t \in \Phi_t(s_t) | \forall i \in I_{\leq t} : \xi_{t,i} = 0 \text{ if } d_i \neq t\}.$$

(ii) Let $x_t(\xi_t|s_t)$ denote the probability that allocation ξ_t is chosen in state s_t. An *allocation rule* $x = (x_1, \dots, x_T)$ assigns a probability distribution over $\{0,1\}^{N_{\leq t}}$ to each state $s_t = (H_t, \xi_{<t})$, such that $x_t(\xi_t|s_t) = 0$ if $\xi_t \notin \Phi_t(s_t)$.

(iii) An allocation rule x *allocates only at the deadline* if $x_t(\xi_t|s_t) = 0$ for $\xi_t \notin \tilde{\Phi}_t(s_t)$.

(iv) An allocation rule is *symmetric* if for all t, all states s_t, all $\xi_t \in \Phi(s_t)$, and all $i, j \in I_{\leq t}$, such that $a_i = a_j$, $x_t(\xi_t|s_t) = x_t(\sigma_{i,j}(\xi_t)|\tilde{\sigma}_{i,j}(s_t))$.[11]

(v) A *payment rule* $y = (y_1, \dots, y_T)$ assigns to each state $s_t = (H_t, \xi_{<t})$ and each $\xi_t \in \{0,1\}^{N_{\leq t}}$, a payment $y_{t,i}(s_t, \xi_t) \in \mathbb{R}$ for each $i \in I_{\leq t}$. A payment rule is *symmetric* if for all t, all s_t, all ξ_t and all $i, j \in I_{\leq t}$, such that $a_i = a_j$, $y_t(s_{t,}\xi_t) = \sigma_{i,j}(y_t(\tilde{\sigma}_{i,j}(s_t), \sigma_{i,j}(\xi_t)))$

[10] See section 1.7 for a discussion of private information about arrival times.

[11] $\sigma_{i,j}$ is the permutation that interchanges the i^{th} and the j^{th} element of its argument and $\tilde{\sigma}_{i,j}(s_t) = (\sigma_{i,j}(H_t), (\sigma_{i,j}(\xi_1), \dots, \sigma_{i,j}(\xi_{t-1})))$.

1.2.2. Mechanisms. The seller's goal is to design a mechanism that has a Bayes-Nash-Equilibrium which maximizes his expected revenue. We assume that the seller can commit ex-ante to a mechanism. In general, a mechanism can be any game form with T stages, such that only buyers from $I_{\leq t}$ are active in stage t. We assume that the mechanism designer can choose to conceal any information about the first t stages from the buyers that arrive in stages $t+1, \ldots, T$.[12]

By the revelation principle, the seller can restrict attention to incentive compatible and individually rational direct mechanisms in which no information is revealed. Furthermore, since buyers who arrive in the same period are ex-ante identical, we can restrict attention to symmetric allocation and payment rules. In the following, we will dispense with the "symmetric" qualifier.

DEFINITION 1.2.3. A *direct mechanism* consists of message spaces $S_1 = [0, \overline{v}] \times \{1, \ldots, T\}$, ..., $S_T = [0, \overline{v}] \times \{T\}$, a symmetric allocation rule x, and a symmetric payment rule y.

The (reported) state in period t can be constructed from the reports until period t, which yield H_t, and the past allocations $\xi_{<t}$.

The *interim winning probability* for period t of a buyer $i \in I_a$ who reports (v', d'), if all other buyers (past, current and future) report their types truthfully, is given by

$$q_a^t(v', d') = \text{Prob}\{\xi_{t,i} = 1 | (a_i, v_i, d_i) = (a, v', d')\}.$$

The *interim expected payment* is given by

$$p_a(v', d') = E\left[\sum_{\tau=a}^{T} y_{\tau,i}(s_\tau, \xi_\tau) \middle| (a_i, v_i, d_i) = (a, v', d')\right],$$

where we aggregate payments from different periods. (q, p) is called the *reduced form* of (x, y). Explicit expressions can be found in Appendix 1.C. The *interim expected utility* from participating in a mechanism (x, y), with true type (v, d) and report (v', d') is given by

$$U_a(v, d, v', d') = \left[\sum_{\tau=a}^{d} q_a^\tau(v', d')\right] v - p_a(v', d'). \tag{1.2.1}$$

The expected utility from truth-telling is abbreviated $U_a(v, d) := U_a(v, d, v, d)$.

DEFINITION 1.2.4. (i) A direct mechanism (x, y) is *(Bayesian) incentive compatible* if for all $a \in \{1, \ldots, T\}$, all $v, v' \in [0, \overline{v}]$, and all $d, d' \in \{a, \ldots, T\}$,

$$U_a(v, d) \geq U_a(v, d, v', d'). \tag{IC}$$

[12]This assumption yields an upper bound on the revenue that can be achieved. We will see that this bound can also be achieved if buyers observe all information from past and current stages.

(ii) A direct mechanism (x, y) is *individually rational* if for all $a \in \{1, \ldots, T\}$, all $v, v' \in [0, \overline{v}]$, and all $d \in \{a, \ldots, T\}$,

$$U_a(v, d) \geq 0. \tag{IR}$$

1.3. Characterization of Incentive Compatibility

Since valuations are not discounted, the seller can restrict attention to direct mechanisms that allocate only at the deadline.

LEMMA 1.3.1. *Let (x, y) be a direct mechanism that satisfies (IC) and (IR). Then, there exists an allocation rule $\hat{x}$ that allocates only at the deadline, such that the direct mechanism $(\hat{x}, y)$ also satisfies (IC) and (IR). (x, y) and $(\hat{x}, y)$ yield the same expected revenue.*

PROOF. See Appendix 1.B. □

In the rest of the chapter, only mechanisms that allocate only at the deadline are considered and we write $q_a(v, d)$ instead of $q_a^d(v, d)$. Furthermore, with an allocation rule that allocates only at the deadline, the buyers who were assigned units in the past have deadlines $d_i < t$. Therefore, their identities are not relevant for current and future allocation decisions and we sometimes replace $\xi_{<t}$ by k_t in the state to simplify notation.

For the class of mechanisms that allocate only at the deadline, the two-dimensional incentive compatibility constraint (IC) is equivalent to two one-dimensional constraints.

THEOREM 1.3.2. *Let (x, y) be a direct mechanism with reduced form (q, p), that allocates only at the deadline.*

(i) (x, y) is incentive compatible if and only if for all $a \in \{1, \ldots, T\}$, all $d \in \{a, \ldots, T\}$, and all $v, v' \in [0, \overline{v}]$:

$$v > v' \Rightarrow q_a(v, d) \geq q_a(v', d), \tag{M}$$

$$U_a(v, d) = U_a(0, d) + \int_0^v q_a(s, d) ds, \tag{PE}$$

$$U_a(v, d) \leq U_a(v, d+1), \qquad \text{if } d < T, \tag{ICD$^\mathrm{d}$}$$

$$\text{and} \quad U_a(0, d) = U_a(0, d+1), \qquad \text{if } d < T. \tag{ICD$^\mathrm{u}$}$$

(ii) Suppose $K = 1$, $T = 2$ and $\nu_{1,1} = 1$. If for all $v_1 \in [0, \overline{v}]$:

$$x_1\left(1 \,|\, (H_1 = (1, v_1, 1), k_1 = 1)\right) \in \{0, 1\},$$

then (ICD$^\mathrm{d}$) holds for any v, if it is fulfilled for $v = 0$ and $v = \overline{v}$.

Part (i) is the characterization of incentive compatibility for the general model. The condition that $q_a(v, d)$ be non-decreasing, together with the payoff equivalence formula (PE), is the standard characterization of one-dimensional incentive compatibility for the valuation (Myerson, 1981).

(ICD^d) rules out that underreporting the deadline is profitable. Together with (M) and (PE), this also rules out simultaneous misreports of an earlier deadline $d' < d$ and a valuation $v' \neq v$. For mechanisms that allocate only at the deadline, the constraint takes this simple form because the utility of a buyer who under-reports his deadline is independent of his true deadline (cf. (1.2.1)):

$$d' \leq d \quad \Rightarrow \quad U_a(v, d, v', d') = U_a(v, d', v', d').$$

Incentive compatibility for the valuation implies that $U_a(v, d', v', d')$, and therefore also $U_a(v, d, v', d')$, is maximized by $v' = v$. For $v' = v$, (ICD^d) rules out a downward deviation in the deadline. Therefore, simultaneous deviations in the deadline and the valuation are also ruled out. Necessity of (ICD^d) is obvious.

The downward incentive compatibility constraint for the deadline is similar to a type dependent participation constraint. A buyer with arrival time a and deadline d has the "outside option" to report $d' \in \{a, \ldots, d-1\}$. He only "participates" voluntarily with $d' = d$ if his payoff with $d' = d$ exceeds the payoff of his best outside option.

Finally, (ICD^u) rules out upward deviations in the deadline. A deviation to the outside option of reporting $d' > d$ can only be profitable if the mechanism pays a subsidy for a report d', i.e. if $p_a(v, d') < 0$. (PE) implies that subsidies are non-increasing in the valuation. Therefore, the highest subsidy (if any) is paid for $(0, d')$. By (PE), $v = 0$ is also the valuation for which over-reporting the deadline is most tempting. Hence, to rule out upward deviations in the deadline, it suffices that $U_a(0, d) = -p_a(0, d) \geq -p_a(0, d') = U_a(0, d')$. Together with ($ICD^d$) for $v = 0$, this is equivalent to (ICD^u).[13] Again, necessity is obvious.

Part (ii) of the theorem concerns the case of one unit and two periods. Furthermore, it is assumed that in period one, exactly one buyer arrives with probability one. The theorem states that the downward constraint for the deadline has to be checked only for the highest type if the allocation rule does not use lotteries in the first period. This result is very useful. It implies that the point where the constraint is binding is independent of the solution. The result is true because $q_1(v, 1)$ jumps from zero to one at $v = \overline{v} - U_1(\overline{v}, 1)$ if the allocation is deterministic and $N_1 = 1$. Therefore, the utility schedule for $d = 1$ is the lowest schedule that is consistent with $U_1(0, 1)$, $U_1(\overline{v}, 1)$ and (PE). If $U_1(0, 1) = U_1(0, 2)$ and $U_1(\overline{v}, 1) \leq U_1(\overline{v}, 2)$, then $U_1(v, 2)$ must necessarily be greater than $U_1(v, 2)$ for all $v \in [0, \overline{v}]$.

[13]Here, we use that the lower bound of the support of f_a is zero. If f_a has support $[\underline{v}, \overline{v}]$ with $\underline{v} > 0$, then the upward incentive compatibility constraint for the deadline would be $q_a(\underline{v}, d)\underline{v} - p_a(\underline{v}, d) \geq -p_a(\underline{v}, d+1)$. In this case, a subsidy could be used to separate buyers with different deadlines. One can show, however, that this instrument would not be used in the optimal mechanism unless the allocation rule is sufficiently distorted. The reason is that the cost of a subsidy is of first order whereas the cost of distorting the allocation rule is of second order.

1.4. The Seller's Problem

By the revelation principle and Lemma 1.3.1, the seller's problem is to choose an incentive compatible and individually rational direct mechanism that allocates only at the deadline, to maximize

$$\sum_{a=1}^{T} E[N_a]E[p_a(v,d)] = \sum_{a=1}^{T}\left[\left(\sum_{N_a=1}^{\bar{N}} N_a\,\nu_{a,N_a}\right)\sum_{d=a}^{T}\rho_{a,d}\int_0^{\overline{v}} p_a(v,d)f_a(v|d)dv\right].$$

Using (PE) to substitute the payment rule, integrating by parts and setting $U_a(0,d)=0$ for all $a \in \{0,\dots,T\}$ and all $d \in \{a,\dots,T\}$, the objective of the seller becomes

$$\sum_{a=1}^{T}\left[\left(\sum_{N_a=1}^{\bar{N}} N_a\,\nu_{a,N_a}\right)\sum_{d=a}^{T}\rho_{a,d}\int_0^{\overline{v}} q_a(v,d)J_a(v|d)f_a(v|d)dv\right].$$

If we substitute $q_1(v,d)$, this can be rearranged to[14]

$$E_{s_1}\left[\sum_{\xi_1\in\tilde{\Phi}_1(s_1)} x_1(\xi_1|s_1)\left(\sum_{i\in I_1}\xi_{1,i}J_{a_i}(v_i|1) + E_{s_2}\left[\sum_{\xi_2\in\tilde{\Phi}_2(s_2)} x_2(\xi_2|s_2)\left(\sum_{i\in I_{\le 2}}\xi_{2,i}J_{a_i}(v_i|2)+ \right.\right.\right.\right.$$
$$\left.\left.\left.\left.\dots\; E_{s_T}\left[\sum_{\xi_T\in\tilde{\Phi}_T(s_T)} x_T(\xi_T|s_T)\sum_{i\in I_{\le T}}\xi_{T,i}J_{a_i}(v_i|T)\middle| s_{T-1},\xi_{T-1}\right]\dots\right)\middle| s_1,\xi_1\right]\right)\right],$$

where E_{s_t} denotes the expectation with respect to s_t. It is more convenient to formulate the seller's problem as a recursive dynamic program $\mathcal{R}$:

$$V_T(s_T) := \max_{x_T}\sum_{\xi_T\in\tilde{\Phi}_T(s_T)} x_T(\xi_T|s_T)\left(\sum_{i\in I_{\le T}}\xi_{T,i}J_{a_i}(v_i|T)\right), \tag{$\mathcal{R}$}$$

$$\forall t<T:\; V_t(s_t) := \max_{x_t}\sum_{\xi_t\in\tilde{\Phi}_t(s_t)} x_t(\xi_t|s_t)\left(\sum_{i\in I_{\le t}}\xi_{t,i}J_{a_i}(v_i|t) + E_{s_{t+1}}\left[V_{t+1}(s_{t+1})|s_t,\xi_t\right]\right),$$

where the reduced form of the optimal policy must satisfy (M) and (ICD$^{\mathrm{d}}$) with $U_a(v,d)$ given by (PE) and $U_a(0,d)\equiv 0$.

1.5. The Relaxed Solution

In order to derive conditions under which the constraint (ICD$^{\mathrm{d}}$) is binding, we first solve $\mathcal{R}$ subject to (M) only. This is the *relaxed problem* and corresponds to the case where deadlines are observed by the seller.

As in the classic optimal auction problem, Assumption 1.2.1 guarantees that at the optimal policy for the relaxed problem, (M) is slack (Myerson, 1981). Therefore, we can ignore (M) in the derivation of the relaxed solution.

[14] Here, we use the assumption that $N_t \le \bar{N}$ for all t. See McAfee and McMillan (1987) for a similar derivation with a stochastic number of bidders in a static model.

For a given state s_t, define $c^t_{(1)} \geq \ldots \geq c^t_{(K)}$ as the K highest virtual valuations among the buyers $i \in I_{\leq t}$ with deadlines $d_i = t$. Let $i^t_{(1)}, \ldots, i^t_{(K)}$ denote the identities of buyers with these virtual valuations, i.e. for all $k = 1, \ldots, K$: $d_{i^t_{(k)}} = t$ and $J_{a_{i^t_{(k)}}}(v_{i^t_{(k)}}|t) = c^t_{(k)}$.[15] Furthermore, define $\Delta_{t+1}(s_t, k) = E_{s_{t+1}}[V_{t+1}(s_{t+1})|s_t, k_{t+1} = k] - E_{s_{t+1}}[V_{t+1}(s_{t+1})|s_t, k_{t+1} = k-1]$. $\Delta_{t+1}(s_t, k)$ is the marginal option value of retaining the k^{th} unit in period t. Note that the marginal option values are non-increasing in k. k^*_{t+1}, the optimal number of units that are retained for period $t+1$, is determined by the following conditions:[16]

$$\begin{aligned} & c^t_{(k_t - k^*_{t+1})} > \Delta_{t+1}(s_t, k^*_{t+1}+1) \quad \text{if } k^*_{t+1} < k_t, \\ \text{and} \qquad & c^t_{(k_t - k^*_{t+1}+1)} \leq \Delta_{t+1}(s_t, k^*_{t+1}) \quad \text{if } k^*_{t+1} > 0. \end{aligned}$$

The set of winning buyers is given by

$$W^*_t(s_t) := \left\{ i^t_{(1)}, \ldots, i^t_{(k_t - k^*_{t+1})} \right\}.$$

The optimal policy for the relaxed problem is deterministic and given by

$$x^{\text{rlx}}_t(\xi_t|s_t) = \begin{cases} 1 & \text{if } \xi_{t,i} = 1 \;\Leftrightarrow\; i \in W^*_t(s_t), \\ 0 & \text{otherwise.} \end{cases}$$

A buyer's type determines whether the buyer is in the set of winning bidders at his deadline, but it can also influence the number of units that are available at the deadline. Let $k^*_{a,d}(H_{d,-i}, (a, v, d), k_a)$ be the number of units that are available in period d if buyer i arrives in period a with type (a, v, d) and k_a units are available in the arrival period. Buyer i gets a unit if $i \in W^*_d((H_{d,-i}, (a, v, d)), k^*_{a,d}(H_{d,-i}, (a, v, d), k_a))$. Therefore, we define the *critical virtual valuation* of buyer i in state s_d for given k_a as

$$\zeta^i_{a,d}(H_d, k_a) := \inf \left\{ \zeta \middle| i \in W^*_d((H_{d,-i}, (a, J_a^{-1}(\zeta|d), d)), k^*_{a,d}(H_{d,-i}, (a, J_a^{-1}(\zeta|d), d), k_a)) \right\}.$$

With this definition, i gets a unit only if $J_{a_i}(v_i|d_i) \geq \zeta^i_{a,d}(H_d, k_a)$.[17]

The relaxed solution can be implemented by the following payment rule:

$$y^{\text{rlx}}_i(s_t, \xi_t) = \begin{cases} 0, & \text{if } \xi_{t,i} = 0, \\ J^{-1}_{a_i}(\zeta^i_{a_i,d_i}(H_{d_i}, k_a)|t), & \text{if } \xi_{t,i} = 1. \end{cases}$$

With this payment rule, the payment of a losing buyer is zero and each winner pays the lowest valuation with which he could have obtained a unit for given k_a and a given history of buyer arrivals in until period d. Thus, truth-telling is a weakly

[15] We assume that ties are broken in favor of buyers who arrive earlier, and randomly if there is a tie between two buyers with the same arrival time. Other tie-breaking rules yield the same expected revenue.

[16] Here we assume that ties are broken in favor of a later allocation. Again, other tie-breaking rules yield the same expected revenue.

[17] The converse is not necessarily true. If $J_{a_i}(v_i|d_i) = \zeta^i_{a,d}(H_d, k_a)$, the tie-breaking rule determines whether $i \in W^*_d((H_{d,-i}, (a, v, d)), k^*_{a,d}(H_{d,-i}, (a, v, d), k_a))$.

dominant strategy if the deadline is known to the sellers, and buyers only report their valuations.

Now we turn to the question whether the relaxed solution is incentive compatible if the deadline is privately known. By (PE), expected payoffs are determined by the allocation rule except for the constant $U_a(0,d)$. (ICD$^\mathrm{u}$) implies that $U_a(0,d) = U_a(0,d')$ for all $a \leq d' \leq d$. Therefore, the constants $U_a(0,d)$ cannot be used to separate buyers with different deadlines. It suffices to check whether the expected payoffs for the payment rule y^{rlx} defined above satisfy (ICD$^\mathrm{d}$).

In order to compare expected payoffs for different deadlines, we make the following crucial observation:

LEMMA 1.5.1. *Suppose that $K = 1$ or $T \leq 2$. Let $a < T$. For all states s_a, all $i \in I_a$ with deadline $d > a$, and for all $d' \in \{a, \ldots, d-1\}$,*

$$E_{H_{d'}}\left[\zeta^i_{a,d'}(H_{d'},k_a)\middle|s_a\right] = E_{H_d}\left[\zeta^i_{a,d}(H_d,k_a)\middle|s_a\right]$$
$$\text{and} \qquad \left[\zeta^i_{a,d'}(H_{d'},k_a)\middle|s_a\right] \succ_{SSD} \left[\zeta^i_{a,d}(H_d,k_a)\middle|s_a\right],$$

where $\succ_{SSD}$ denotes second-order stochastic dominance.

PROOF. See Appendix 1.B. □

The following example illustrates the lemma. Suppose that $T = 2$, $K = 1$, $\nu_{1,2} = 1$ and $\nu_{2,1} = 1$. Let the sets of new buyers in period one and two be $I_1 = \{1,2\}$ and $I_2 = \{3\}$, respectively. In this case, the critical virtual valuations of buyer one for $d_1 = 1$ and $d_1 = 2$ are given by

$$\zeta^1_{1,1}(H_1,1) = \begin{cases} \max\left\{J_1(v_2|1), E_{v_3}\left[\max\left\{0, J_2(v_3|2)\right\}\right]\right\}, & \text{if } d_2 = 1, \\ E_{v_3}\left[\max\left\{0, J_1(v_2|2), J_2(v_3|2)\right\}\right], & \text{if } d_2 = 2, \end{cases}$$

$$\zeta^1_{1,2}(H_2,1) = \begin{cases} \max\left\{z(J_1(v_2|1)), J_2(v_3|2)\right\}, & \text{if } d_2 = 1, \\ \max\left\{0, J_1(v_2|2), J_2(v_3|2)\right\}, & \text{if } d_2 = 2, \end{cases}$$

$$\text{where} \quad z(J_1(v_2|1)) = \min\left\{z \geq 0 \,\middle|\, E_{v_3}\left[\max\left\{z, J_2(v_3|2)\right\}\right] \geq J_1(v_2|1)\right\}.$$

If $d_1 = 2$ and $d_2 = 1$, then the object is retained in period one if and only if buyer 1's virtual valuation is greater or equal than $z(J_1(v_2|1))$. In other words, buyer one must have a virtual valuation $J_1(v_1|2) \geq z(J_1(v_2|1))$ to overbid buyer two in the first period. Since $\max\left\{J_1(v_2|1), E\left[\max\left\{0, J_2(v_3|2)\right\}\right]\right\} = E\left[\max\left\{z(J_1(v_2|1)), J_2(v_3|2)\right\}\right]$, we have that $\zeta^1_{1,1}(H_1,k_1) = E_{v_3}[\zeta^1_{1,2}(H_2,k_1)|s_1]$ as stated in the lemma.

Buyer one faces competition by both buyers, no matter whether his deadline is $d_1 = 1$ or $d_1 = 2$. In both cases, he competes directly with buyer two. Competition with buyer 3 is direct if $d_1 = 2$ and indirect, through the option value of retaining the object, if $d_1 = 1$. A later deadline has two effects, first it lowers the virtual valuation needed to overbid buyer two, because $z(J_1(v_2|1)) < J_1(v_2|1)$ if $v_2 < \overline{v}$ and $J_1(v_2|2) < E_{v_3}\left[\max\left\{J_1(v_2|2), J_2(v_3|2)\right\}\right]$. Second, a higher virtual valuation is needed to overbid buyer three whenever $J_2(v_3|2) > E_{v_3}\left[\max\left\{0, J_2(v_3|2)\right\}\right]$. The lemma shows that the two effects cancel in expectation. If we interpret the critical virtual valuation as a measure of competition by other buyers, this shows that expected competition is independent of the reported deadline. Hence, differences in expected payoffs for different deadlines are not caused by a competition effect.

In this example, second order stochastic dominance is obvious because conditional on s_1, $\zeta_1^1(s_1)$ is constant and therefore dominates $\zeta_2^1(s_2)$ which has the same expectation.

The following theorem gives sufficient conditions under which the static and the dynamic pricing effects lead to incentive compatibility of the relaxed solution and violations of incentive compatibility, respectively.

THEOREM 1.5.2. *Suppose that* $K = 1$ *or that* $T \leq 2$*. Then, in the relaxed solution,*

(i) (ICD$^{\mathrm{d}}$) *is violated for type* (a, v, d) *if there exits* $d' \in \{a, \dots, d-1\}$*, such that*
   (a) $J_a(v|d') \geq J_a(v|d)$ *for all* $v \in [v_a^0|d', \overline{v}]$*, and*
   (b) $J_a(v|d)$ *or* $J_a(v|d')$ *is strictly concave as a function of* v*.*
   If $J_a(v|d') > J_a(v|d)$ *for all* $v \in [v_a^0|d, \overline{v})$*, strict concavity can be replaced by weak concavity.*

(ii) (ICD$^{\mathrm{d}}$) *is satisfied for type* (a, v, d) *if for all* $d' \in \{a, \dots, d-1\}$,
   (a) $J_a(v|d') \leq J_a(v|d)$ *for all* $v \in [v_a^0|d, \overline{v}]$*, and*
   (b) $J_a(v|d)$ *or* $J_a(v|d')$ *is weakly convex as a function of* v*.*

PROOF. See Appendix 1.B. □

Conditions (a) in both parts of the theorem correspond to the static pricing effect. By the definition of the virtual valuation, these conditions can also be formulated as monotonicity conditions on the conditional hazard rate of the type distribution. Conditions (b) correspond to the dynamic pricing effect.

Let us consider a buyer with the highest possible valuation $v = \overline{v}$. Such a buyer wins with probability one, regardless of his deadline. Therefore, his expected payoff only depends on the expected price he has to pay. The static pricing effect is caused by dependencies between deadlines and valuations. If $J_a(v|d) \leq J_a(v|d')$ for $d > d'$, then v tends to be higher for the later deadline. This leads to more aggressive pricing for the later deadline ($J_a^{-1}(v|d) > J_a^{-1}(v|d)$). Therefore, the buyer would like to pretend to have the earlier deadline in order to avoid higher prices—incentive compatibility is violated.

Note that this effect does not depend on the stochastic dominance result in Lemma 1.5.1. Indeed, it can also be found in static models where the second dimension of private information (e.g. a capacity requirement) is correlated with the valuation.

The dynamic pricing effect is caused by stochastic dominance of the critical virtual valuations, which arises because later allocation decisions are based on more information than earlier decisions. Therefore, the effect is genuinely dynamic. Moreover, it does not depend on correlations and also occurs if the deadline and the valuation are independently distributed. Suppose that the virtual valuation is concave. Then the pricing rule $J_a^{-1}(\zeta|d)$ used by the seller is convex and expected prices

density (support: $[0,1]$)	$J(v)$	$J''(v)$
$2v$	$\frac{1}{2}\frac{3v^2-1}{v}$	$-\frac{1}{v^3} < 0$
$1-k+2kv$ ($k \in (0,1]$)	$\frac{2v-2kv+3kv^2-1}{1-k+2kv}$	$-\frac{2k(1+k)^2}{(1-k+2kv)^3} < 0$
$(k+1)v^k$ ($k > 0$)	$\frac{vk+2v-v^{-k}}{k+1}$	$-v^{-2-k}k < 0$
$12(v-\frac{1}{2})^2$	$\frac{2}{3}\frac{v^2(4v-3)}{(2v-1)^2}$	$-\frac{4}{(2v-1)^4} < 0$
$\frac{3}{2}-6(v-\frac{1}{2})^2$	$\frac{8v^2-v-1}{6v}$	$-\frac{1}{3v^3} < 0$
$2-2v$	$\frac{3v}{2}-\frac{1}{2}$	0
1 (uniform)	$2v-1$	0
$(1+k)(1-v)^k$	$\frac{(k+2)v-1}{k+1}$	0
$1-k+2kv$ ($k \in [-1,0)$)	$\frac{2v-2kv+3kv-1}{1-k+2kv}$	$-\frac{2k(1+k)^2}{(1-k+2kv)^3} > 0$

TABLE 1. Distributions with strictly concave, linear, and strictly convex virtual valuations.

are higher if the distribution of the critical virtual valuation is more dispersed. By Lemma 1.5.1 this is the case for higher deadlines. Therefore, the buyer prefers to report the lower deadline to avoid higher prices. Again, incentive compatibility is violated.

Both effects work in the opposite direction if the virtual valuation is increasing in the deadline, and convex in the valuation, respectively. To show incentive compatibility of the relaxed solution, however, (ICD$^{\text{d}}$) also has to be checked for $v < \overline{v}$. Details are given in the proof.

Table 1 shows densities and virtual valuations for several distributions. For the first group, the virtual valuation is strictly concave wherever it is non-negative. For the second group, it is linear and for the third group it is convex. If valuation and deadline of a buyer are independently distributed, the relaxed solution violates incentive compatibility for all distributions in the first group and satisfies incentive compatibility for all other examples. An example for a violation of incentive compatibility for the dependent case is $f_1(v|1) = 2-2v$ and $f_1(v|2) = 1$. In this case, the virtual valuation is linear for both distributions but strictly decreasing in the deadline. If we exchange $f_1(v|1)$ and $f_1(v|2)$, incentive compatibility is satisfied for buyers with types $(1, v, 2)$. Other examples are easily constructed.

Remark: Lemma 1.5.1 conditions on the state in the arrival period. Therefore, the incentive compatibility result of Theorem 1.5.2 also holds if buyers can condition their reports on the state at their arrival time. In other words, under the conditions of part (ii) of the theorem, the relaxed solution is periodic ex-post incentive compatible. This shows that the optimal solution does not rely on the seller's ability to conceal information from earlier periods.

1.6. The General Solution

In cases where the relaxed solution is not incentive compatible, the analysis is significantly more complex. For tractability, we solve the general problem for the case of two periods ($T = 2$), one object ($K = 1$) and assume deterministic arrival of one buyer in each period ($\nu_{1,1} = \nu_{2,1} = 1$). Furthermore, we will make an assumption that ensures that the optimal mechanism does not use lotteries in the first period (Assumption 1.6.1 below). For this case, we solve $\mathcal{R}$ subject to (M), (ICD^{d}) and (PE). Assumption 1.2.1 guarantees that in the optimal solution, (M) is slack for buyer two. For buyer one, however, this assumption is not sufficient to guarantee monotonicity of the optimal solution. In Section 1.6.4, we show how the mechanism has to be ironed if (M) is binding at the optimal solution.

In the following section, we will simplify the notation and decompose the seller's problem into two subproblems; one for $d_1 = 1$ and one for $d_1 = 2$. These problems are only linked by the incentive compatibility constraint for the deadline (ICD^{d}). In Section 1.6.2, we impose an assumption that rules out lotteries and solve the revenue maximization problem for $d_1 = 1$. Section 1.6.3 deals with the problem for $d_1 = 2$ in the regular case where the monotonicity constraint is slack. Section 1.6.4 gives the general solution that also applies to the irregular case of a binding monotonicity constraint. The reader may want to skip section 1.6.4 at the first read. Finally, we combine the solutions to a solution for the general problem.

1.6.1. Decomposition of the seller's problem. Since $N_1 = N_2 = 1$ we write d, ρ, $f_2(v_2)$, and $F_2(v_2)$ instead of d_1, $\rho_{1,1}$, $f_2(v_2|2)$, and $F_2(v_2|2)$, respectively. Slightly abusing notation, we write winning probabilities as

$$\begin{aligned} x_1(v_1, 1) &= x_1\left(\, \xi_1 = (1) \,|\, s_1 = ((1, v_1, 1), 1)\,\right), \\ x_1(v_1, 2, v_2) &= x_1\left(\, \xi_2 = (1, 0) \,|\, s_2 = (((1, v_1, 2), (2, v_2, 2)), 1)\,\right), \\ \text{and} \qquad x_2(v_1, d, v_2) &= x_2\left(\, \xi_2 = (0, 1) \,|\, s_2 = (((1, v_1, d), (2, v_2, 2)), 1)\,\right). \end{aligned}$$

$x_1(v_1, 1)$ is the probability that buyer one gets the object if his deadline is one. $x_i(v_1, d, v_2)$ is the probability that buyer i gets the object for a given type-profile and conditional on the event that the object has not been allocated in period one. Note that x is feasible if and only if for all $v_1, v_2 \in [0, \overline{v}]$, $d \in \{1, 2\}$, and $i \in \{1, 2\}$,

$$x_1(v_1, 1), x_i(v_1, d, v_2) \in [0, 1] \quad \text{and} \quad x_1(v_1, 2, v_2) + x_2(v_1, 2, v_2) \leq 1. \tag{F}$$

The feasibility constraint for $d = 1$ is fulfilled automatically because $x_2(v_1, 1, v_2)$ is the winning probability of buyer two conditional on the event that the object has not been allocated in the first period.

Interim winning probabilities of buyer one are given by:

$$\begin{aligned} q_1(v_1, 1) &= x_1(v_1, 1), \\ \text{and} \quad q_1(v_1, 2) &= \int_0^{\overline{v}} x_1(v_1, 2, v_2) f_2(v_2) dv_2. \end{aligned}$$

Furthermore, we define the interim winning probability of buyer two, conditional on the deadline of buyer one and the event that the object has not been allocated in period one as:

$$q_2(v_2, d) = \int_0^{\overline{v}} x_2(v_1, d, v_2) f_1(v_1|d) dv_1.$$

Hence, we have

$$q_2(v_2) = \rho \left(\int_0^{\overline{v}} (1 - x_1(v_1, 1)) f_1(v_1|1) d_1 \right) q_2(v_2, 1) + (1 - \rho) q_2(v_2, 2).$$

With these definitions, $\mathcal{R}$ subject to (ICD$^{\text{d}}$), (PE) and (M) for buyer one can be rewritten as the maximization problem $\mathcal{P}$:

$$\max_q \rho \int_0^{\overline{v}} \left[q_1(v_1, 1) J_1(v_1|1) + (1 - q_1(v_1, 1)) \int_0^{\overline{v}} q_2(v_2, 1) J_2(v_2) f_2(v_2) dv_2 \right] f_1(v_1|1) dv_1$$

$$+ (1 - \rho) \int_0^{\overline{v}} q_1(v, 2)\, J_1(v|2)\, f_1(v|2) + q_2(v, 2)\, J_2(v)\, f_2(v)\, dv \qquad (\mathcal{P})$$

such that q is the reduced form of a feasible allocation rule and subject to

$$\forall d \in \{1, 2\}, \forall v, v' \in [0, \overline{v}] : \qquad v > v' \Rightarrow q_1(v, d) \geq q_1(v', d) \qquad (\text{M}_1)$$

$$\forall d \in \{1, 2\}, \forall v \in [0, \overline{v}] : \qquad U_1(v, d) = \int_0^{v} q_1(s, d) ds, \qquad (\text{PE}_1)$$

$$\forall v \in [0, \overline{v}] : \qquad U_1(v, 1) \leq U_1(v, 2). \qquad (\text{ICD}^{\text{d}}_1)$$

Except for the incentive constraint for the deadline (ICD$^{\text{d}}_1$), the expected revenue for $d = 1$ (first line in the objective) and $d = 2$ (second line) can be maximized independently. In order to decompose the seller's problem, we introduce a function $U : [0, \overline{v}] \to [0, \overline{v}]$ that separates $U_1(., 1)$ from $U_1(., 2)$:

$$\forall v \in [0, \overline{v}] : \qquad U_1(v, 1) \leq U(v) \leq U_2(v, 2). \qquad (\text{ICD}^{\text{d}}_{\text{U}})$$

Using U as a parameter, the maximization problem can be rewritten as $\mathcal{P}'$:

$$\max_U \rho\, \pi_1[U] + (1 - \rho)\, \pi_2[U] \qquad (\mathcal{P}')$$

$\pi_1[U]$ is defined as the maximal expected revenue that can be achieved if the deadline is one and the expected payoff of the first buyer is constrained by $U_1(v, 1) \leq U(v)$ for all $v \in [0, \overline{v}]$. This maximization problem is called $\mathcal{P}_1$:

$$\pi_1[U] := \max_{q_i(.,1)} \int_0^{\overline{v}} \Bigg[q_1(v_1, 1) J_1(v_1|1) + \qquad (\mathcal{P}_1)$$

$$(1 - q_1(v_1, 1)) \int_0^{\overline{v}} q_2(v_2, 1) J_2(v_2) f_2(v_2) dv_2 \Bigg] f_1(v_1|1) dv_1$$

$$\text{s.t.} \quad q_i(v, 1) \in [0, 1],\ (\text{PE}_1),\ (\text{M}_1) \text{ and } (\text{ICD}^{\text{d}}_{\text{U}})$$

$\pi_2[U]$ is defined as the maximal expected revenue that can be achieved if the deadline is two and the utility of the first buyer is constrained by $U_1(v, 2) \geq U(v)$

for all $v \in [0, \overline{v}]$. This maximization problem is called $\mathcal{P}_2$:

$$\pi_2(U) := \max_{q_i(.,2)} \int_0^{\overline{v}} q_1(v,2)J_1(v|2)f_1(v|2) + q_2(v,2)J_2(v)f_2(v)dv \qquad (\mathcal{P}_2)$$
$$\text{s.t.} \quad \text{(F)},(\text{PE}_1),\ (\text{M}_1) \text{ and } (\text{ICD}^{\text{d}}_{\text{U}}).$$

If $\mathcal{P}_1$ and $\mathcal{P}_2$ are solved for the same U, we get a solution for $\mathcal{P}$. Therefore, $\mathcal{P}$ can be reformulated as a problem of choosing U optimally as in $\mathcal{P}'$.

1.6.2. Solution to $\mathcal{P}_1$. If $(\text{ICD}^{\text{d}}_{\text{U}})$ is ignored, $\mathcal{P}_1$ is equivalent to the problem of finding the optimal selling strategy for a sequence of short-lived buyers. The optimal solution is a sequence of fixed prices (Riley and Zeckhauser, 1983). Optimal prices are determined working backwards in time. If the object was not sold in the first period, the optimal price in the second period is $r_2 = v_2^0$. This implies an option value of postponing the allocation of $V_2^{\text{opt}} := \int_{v_2^0}^{\overline{v}} J_2(v_2)f_2(v_2)dv_2 = v_2^0(1 - F_2(v_2^0))$. Consequently, the optimal price in the first period, r_1, is given by $J_1(r_1|1) = V_2^{\text{opt}}$. This is the relaxed solution of $\mathcal{P}_1$.

If constraint $(\text{ICD}^{\text{d}}_{\text{U}})$ is imposed, the optimal solution to $\mathcal{P}_1$ may involve lotteries.[18] To rule out this possibility we make

Assumption 1.6.1. *$J_1(v|1)f_1(v|1)$ is strictly increasing for all $v \in [v_1^0|1, \overline{v}]$.*

Theorem 1.3.2 implies that if the allocation rule is deterministic in the first period, $(\text{ICD}^{\text{d}}_{\text{U}})$ reduces to $U_1(\overline{v}, 1) \leq \bar{U}$, where we define $\bar{U} := U(\overline{v})$. We will therefore treat π_1 as a function of $\bar{U}$ and write $\pi_1(\bar{U})$ instead of $\pi_1[U]$ in this case. The optimal fixed price in period one is now given by the lowest price that satisfies $J_1(r_1|1) \geq V_2^{\text{opt}}$ and $\overline{v} - r_1 \leq \bar{U}$. The optimal fixed price in period two, r_2, is not affected by constraint $(\text{ICD}^{\text{d}}_{\text{U}})$.

Theorem 1.6.2. *Suppose f_1 satisfies Assumption 1.6.1. Then*

(i) the optimal solution of $\mathcal{P}_1$ does not use lotteries in the first period and is given by

$$q_1(v_1, 1) = \begin{cases} 0, & \text{if } J_1(v_1|1) < \max\{V_2^{\text{opt}}, J_1(\overline{v} - \bar{U}|1)\}, \\ 1, & \text{otherwise}, \end{cases}$$

$$q_2(v_2, 1) = \begin{cases} 0, & \text{if } J_2(v_2) < 0, \\ 1, & \text{otherwise}. \end{cases}$$

(ii) $\pi_1(\bar{U})$ is continuously differentiable for $\bar{U} \in (0, \overline{v})$ and strictly concave in $\bar{U}$ for $\bar{U} < \overline{v} - J_1^{-1}(V_2^{\text{opt}})$.

[18]The no-haggling result of Riley and Zeckhauser (1983) is a consequence of a special structure of the feasible set of the maximization problem. Manelli and Vincent (2007) show that the set of extremal points of the feasible set, which contains the maximizers, is equal to the set of deterministic allocation rules. Due to the additional constraint $(\text{ICD}^{\text{d}}_{\text{U}})$, the set of extremal points changes. Rather than trying to extend the results of Manelli and Vincent here, we use Assumption 1.6.1 as a sufficient condition for a deterministic mechanism.

Proof. See Appendix 1.B. □

To understand the role of Assumption 1.6.1, note that in the constraint $U(v) \geq \int_0^v q_1(s,1)ds$, winning probabilities are not weighted in the integral because incentive compatibility constraints are independent of the buyer's own distribution function. In the objective, however, $q_1(v_1,1)$ is weighted by $(J_1(v_1|1) - V_2^{\text{opt}})f_1(v_1|1)$. Increasing the winning probability $q_1(v_1,1)$ for valuations in $[v, v+\varepsilon]$ and decreasing it by the same amount on $[v', v'+\varepsilon]$, with $v'+\varepsilon \leq v$, decreases $U_1(v_1,1)$ for $v_1 \in [v', v+\varepsilon]$ and leaves $U_1(v_1,1)$ unchanged otherwise. Hence, such a change in q_1 does not destroy incentive compatibility. On the other hand, this shift of winning probability from low to high types increases the seller's revenue if $(J_1(v_1) - V_2^{\text{opt}})f_1(v_1)$ is increasing. Assumption 1.6.1 guarantees that $(J_1(v_1|1) - V_2^{\text{opt}})f_1(v_1|1)$ is increasing whenever $J_1(v_1|1) - V_2^{\text{opt}} \geq 0$. Therefore, the winning probability must jump from zero to one at some point and the allocation is deterministic.

If Assumption 1.6.1 does not hold, raising the winning probability for a lower valuation may be more profitable than for a higher valuation because it is sufficiently more likely that buyer one has the low valuation. For this to be the case, the decrease in the density must outweigh the increase in expected revenue, i.e. the virtual valuation. Finally, note that Assumption 1.6.1 is a sufficient condition. Presumably, a necessary and sufficient condition cannot be stated as a local condition.

1.6.3. Solution to $\mathcal{P}_2$ – The Regular Case. In this section, we solve $\mathcal{P}_2$, imposing ($\text{ICD}_\text{U}^\text{d}$) only for $v = \overline{v}$. By Theorems 1.3.2 and 1.6.2, this is sufficient for the general problem if Assumption 1.6.1 is fulfilled. In the derivation of the optimal solution of $\mathcal{P}_2$, however, Assumption 1.6.1 is not used. Therefore, the results of this and the following section also apply if the mechanism designer is exogenously restricted to set a fixed price in the first period.

To state the optimal solution, we define the *generalized virtual valuation* of buyer one:

$$J_1^{p_U}(v) := J_1(v|1) + \frac{p_U}{f_1(v|1)}.$$

The parameter p_U determines the magnitude of the distortion of the allocation rule away from Myerson's (1981) solution for $\mathcal{P}_2$ without ($\text{ICD}_\text{U}^\text{d}$). ($p_U$ is the multiplier of constraint ($\text{ICD}_\text{U}^\text{d}$) in the underlying control problem.) Suppose we already know the optimal p_U. Then, the optimal allocation rule is given by

$$\begin{aligned} x_1(v_1,2,v_2) &= \begin{cases} 0, & \text{if } J_1^{p_U}(v_1) < \max\{0, J_2(v_2)\} \\ 1, & \text{otherwise,} \end{cases} \\ x_2(v_1,2,v_2) &= \begin{cases} 0, & \text{if } J_2(v_2) \leq \max\{0, J_1^{p_U}(v_1)\} \\ 1, & \text{otherwise.} \end{cases} \end{aligned} \tag{1.6.1}$$

For every $\bar{U} \in [0, \overline{v})$, let $p_{\bar{U}}^*$ be the lowest value $p_U \geq 0$, such that the reduced form of (1.6.1) satisfies $\int_0^{\overline{v}} q_1(v,2)dv \geq \bar{U}$.

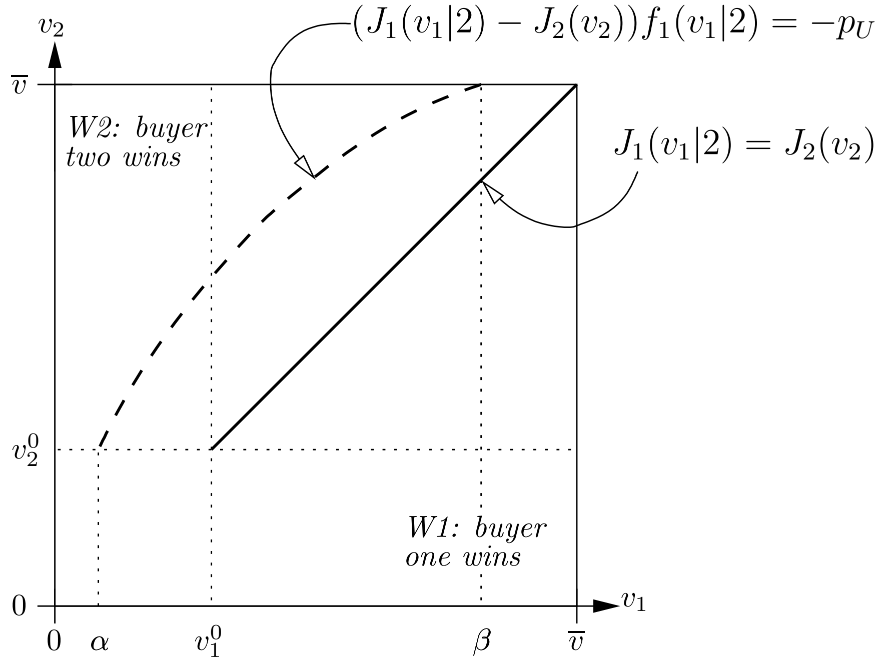

FIGURE 1.6.1. Optimal allocation rule

THEOREM 1.6.3. *Fix $\bar{U}$ and suppose $J_1^{p^*_{\bar{U}}}(v_1)$ is strictly increasing in v_1. Then*

(i) the reduced form of (1.6.1) for $p_U = p^*_{\bar{U}}$ is an optimal solution of $\mathcal{P}_2$ subject to (M_1), (PE_1), and (ICD^d_U) for $v = \overline{v}$.

(ii) $p^*_U = -\pi_2'(\bar{U})$.

(iii) π_2 is weakly concave.

PROOF. Theorem 1.6.3 is a special case of Theorem 1.6.5 below. □

If the relaxed solution is incentive compatible, p_U is zero and valuations (v_1, v_2) tie if $J_1(v_1|2) = J_2(v_2)$ as in Myerson's solution. If the relaxed solution is not incentive compatible, p_U is strictly positive and valuations tie if $J_1^{p_U}(v_1) = J_2(v_2)$, which is equivalent to

$$(J_1(v_1|2) - J_2(v_2))f_1(v_1|2) = -p_U. \tag{1.6.2}$$

Figure 1.6.1 sketches both cases for identically distributed valuations ($f_1(.|2) = f_2$). The solid line is the Myerson-line at which valuations tie in the relaxed solution. The dashed line is the distorted Myerson-line at which valuations tie in the general solution. Note that for $p_U > 0$, valuations tie in an area where the (standard) virtual valuation of buyer one is strictly smaller than the virtual valuation of buyer two.

To understand condition (1.6.2), consider the effect on π_2 of an increase of $q_1(.,2)$. Fix any (v_1, v_2) on the distorted Myerson-line, such that $0 \leq J_1^{p_U}(v_1) \leq \overline{v}$. In the figure, this corresponds to $\alpha \leq v_1 \leq \beta$. In order to increase $q_1(v_1, 2)$, the allocation has to be changed from buyer two to buyer one at (v_1, v_2). This leads to a marginal change of π_2 by $J_1(v_1|2) - J_2(v_2) < 0$ per mass of type profiles for which the allocation is changed. This mass of type profiles is proportional to $f_1(v_1|2)$. Hence, the left-hand side of (1.6.2) quantifies the marginal cost of increasing $q_1(v_1, 2)$.

The marginal cost of increasing $q_1(v_1, 2)$ must be independent of v_1. The reason is that winning probabilities are not weighted in the constraint $\int_0^{\overline{v}} q_1(s,2)ds \geq \bar{U}$. If the marginal cost of changing $q_1(v_1, 2)$ varied with v_1, we could increase $q_1(v_1, 2)$ where the marginal cost is small and decrease it where the marginal cost is big. If we chose this variation such that $U_1(\overline{v}, 2) = \int_0^{\overline{v}} q_1(s,2)ds$ were not changed, we could increase the objective function without violating the constraints—a contradiction. Hence, the marginal cost of increasing $q_1(.,2)$ must be constant and equal to p_U for all $v_1 \in [\alpha, \beta]$. As the utility of the highest type is given by $U_1(\overline{v}, 2) = \int_0^{\overline{v}} q_1(s,2)ds$, p_U can also be interpreted as the marginal cost of the constraint $U_1(\overline{v}, 2) \geq \bar{U}$.

Furthermore, note that the distortion is increasing in p_U, and that by Assumption 1.2.1, the marginal cost of a distortion is increasing in the distance from the Myerson solution (the LHS of (1.6.2) is decreasing in v_2). Therefore, it is optimal to choose the lowest p_U such that ($\mathrm{ICD}_{\mathrm{U}}^{\mathrm{d}}$) is satisfied, and the cost of distortions is convex, which implies concavity of π_2 in $\bar{U}$.

(1.6.2) also implies that the distortion of the Myerson-line is bigger for types with lower densities. This is intuitive because the expected cost of a distortion is lower for types that are less frequent. But this also means that an increasing density can lead to non-monotonicities of the winning-probability.

1.6.4. Solution to $\mathcal{P}_2$ – The Irregular Case. Theorem 1.6.3 requires that $J_1^{p_U^*}$ is strictly increasing because otherwise, the winning probability of buyer one would be decreasing. This is a condition on an endogenous object and Assumption 1.2.1 does not guarantee monotonicity of $J_1^{p_U}$ for all values of p_U. A decreasing density $f_1(v|2)$ together with Assumption 1.2.1 would be sufficient, but this is quite restrictive and rules out most of the examples of concave virtual valuations in Table 1. To give a complete solution without further assumptions, we show that Myerson's ironing procedure can be used to deal with non-monotonicities of $J_1^{p_U}$.

Definition 1.6.4 (Ironing; Myerson, 1981). (i) For every $t \in [0,1]$, define

$$M_1^{p_U}(t) := J_1(F_1^{-1}(t|2)|2) + \frac{p_U}{f_1(F_1^{-1}(t|2)|2)},$$

as the generalized virtual valuation at the t-quantile of $F_1(.|2)$.

(ii) Integrate this function:

$$H^{p_U}(t) := \int_0^t M_1^{p_U}(s)ds.$$

(iii) Take the convex hull (i.e. the greatest convex function G such that $G(t) \leq H^{p_U}(t)$ for all t):

$$\bar{H}^{p_U}(t) := \operatorname{conv} H^{p_U}(t).$$

(iv) Since $\bar{H}^{p_U}$ is convex, it is almost everywhere differentiable and any selection $\bar{M}_1^{p_U}(t)$ from the sub-gradient is non-decreasing.

(v) Reverse the change of variables made in (i) to obtain the *ironed generalized virtual valuation*
$$\bar{J}_1^{p_U}(v_1) := \bar{M}_1^{p_U}(F_1(v_1|2)).$$

In the irregular case, the optimal allocation rule depends on two parameters, p_U and $\underline{x}_1^0$, and has the following structure:

$$\bar{x}_1(v_1,2,v_2) = \begin{cases} 1, & \text{if } \bar{J}_1^{p_U}(v_1) > 0 \text{ and } \bar{J}_1^{p_U}(v_1) \geq J_2(v_2) \\ \underline{x}_1^0, & \text{if } \bar{J}_1^{p_U}(v_1) = 0 \text{ and } J_2(v_2) \leq 0, \\ 0, & \text{otherwise,} \end{cases} , \qquad (1.6.3)$$
$$\bar{x}_2(v_1,2,v_2) = \begin{cases} 0, & \text{if } J_2(v_2) \leq \max\{0, \bar{J}_1^{p_U}(v_1)\}, \\ 1, & \text{otherwise.} \end{cases}$$

The parameters are determined as follows. First, let p_U^* be the minimal value $p_U \geq 0$ such that the reduced form of (1.6.3) with $\underline{x}_1^0 = 1$ satisfies $\int_0^{\overline{v}} q_1(v,2)dv \geq \bar{U}$. Second, if $p_U^* > 0$, select $\underline{x}_1^{0*} \in [0,1]$ such that $\int_0^{\overline{v}} q_1(v,2)dv = \bar{U}$, otherwise set $\underline{x}_1^{0*} = 1$.

The additional parameter $\underline{x}_1^0$ is only needed if $\bar{J}_1^{p_U}(v_1) = 0$ on an interval $[\underline{v}_1^0, \bar{v}_1^0]$ with $\underline{v}_1^0 < \bar{v}_1^0$. In this case, $\int_{\underline{v}_1^0}^{\bar{v}_1^0} J_1^{p_U}(v)dv = 0$ and hence, $U_1(\overline{v},2)$ can be varied at constant marginal cost p_U by changing the winning probability for all valuations in the interval $[\underline{v}_1^0, \bar{v}_1^0]$. Therefore, the same value of p_U defines the ironed generalized virtual valuation for different values $\bar{U}$ in a non-empty interval $[a,b]$. $\underline{x}_1^0$ is varied to achieve different values of $U_1(\overline{v},2) \in [a,b]$.

The allocation rule in (1.6.3) excludes buyer one if his valuation is smaller than $\underline{v}_1^0$. With a valuation in $[\underline{v}_1^0, \bar{v}_1^0]$, he can win against buyer two if $v_2 \leq v_2^0$, but he gets the object only with probability $\underline{x}_1^0$.[19] To summarize, we have

THEOREM 1.6.5. *(i) The reduced form of (1.6.3) for p_U^* and $\underline{x}_1^{0*}$ is an optimal solution of $\mathcal{P}_2$ subject to (M₁), (PE₁), and (ICD_U^d) for $v = \overline{v}$.*
(ii) For almost every $\bar{U}$, $\pi_2'(\bar{U}) = -p_U^*$
(iii) π_2 is weakly concave in $\bar{U}$ and strictly concave if $p_U > 0$ and $\bar{J}_1^{p_U}(v) = 0$ has a unique solution.

PROOF. See Appendix 1.A. □

Note that if $J_1^{p_U}$ is increasing, $\bar{J}_1^{p_U}$ equals $J_1^{p_U}$. Therefore, Theorem 1.6.3 is a special case of Theorem 1.6.5.

1.6.5. Global Solution and Discussion. Under Assumption 1.6.1, $\mathcal{P}'$ reduces to the problem of choosing $\bar{U}$ optimally. The first order necessary condition is
$$\rho\, \pi_1'(\bar{U}) = -(1-\rho)\, \pi_2'(\bar{U}).$$

[19] It is also possible to construct a deterministic allocation rule with the same reduced form. Choose $\hat{v}_2$ such that $\underline{x}_1^0 = \frac{F_2(\hat{v}_2)}{F_2(v_2^0)}$. For $v_1 \in [\underline{v}_1^0, \bar{v}_1^0]$, set $x_1(v_1,2,v_2) = 1$ if $v_2 \leq \hat{v}_2$ and $x_1(v_1,2,v_2) = 0$ otherwise. This construction, however, has the disadvantage that the allocation decision for buyer one depends on truthful reports of buyer two in cases when he can never win the object.

By Theorem 1.6.5, π_2 is concave and by Theorem 1.6.2 and Assumption 1.6.1, π_1 is concave. Therefore, the first-order condition is also a sufficient. To determine the optimal distortion, it suffices to compute the unique solution $p_U \geq 0$ of

$$p_U = \frac{\rho}{1-\rho}\,\pi_1'(\bar{U}),$$

$$\text{and} \qquad \bar{U} \leq \int_0^{\overline{v}} q_1^{p_U}(v,2)dv_1, \quad \text{with equality if } p_U > 0,$$

where q^{p_U} is the reduced form of (1.6.1) for given value of p_U.[20] An explicit form of the solution is not available. However, for given p_U, $U_1(\overline{v},2) = \int_0^{\overline{v}} q_1^{p_U}(v,2)dv_1$ is easy to calculate and an explicit expression for π_1' is given in the proof of Theorem 1.6.2. Hence, it is easy to compute the optimal p_U numerically. If Assumption 1.6.1 is violated, π_1 may fail to be concave and it may be necessary to compute all local maxima to find the global solution.

We will now discuss several properties of the general solution.

Monotonicity of q_2. $q_2(v_2,1)$, defined by the fixed price r_2, and $q_2(v_2,2)$, defined by the reduced form of (1.6.3), are non-decreasing. This follows from Assumption 1.2.1. Therefore, $q_2(v_2)$ is also non-decreasing and the optimal solutions of $\mathcal{P}_1$ and $\mathcal{P}_2$ together fulfill all constraints of $\mathcal{P}$. We have derived an optimal solution of $\mathcal{P}$.

Distortions in Both Periods. By Theorem 1.6.2, $\pi_1(\bar{U})$ is continuously differentiable. Therefore, $p_U > 0$ implies that the allocation for $d = 1$ is distorted. Hence, in all cases where the relaxed solution is not incentive compatible, the general solution involves a distortion for both deadlines. The relative magnitude of the distortion depends on ρ. If $d = 1$ is relatively unlikely (ρ small), then the distortion of the fixed price is bigger and the auction is closer to Myerson's solution. The reason is that distortions are more costly at the deadline which occurs more frequently.

Distortions. In the first period, the fixed price is $\max\{\bar{U} - \overline{v}, J_1^{-1}(V_2^{\text{opt}}|1)\}$. It is distorted upwards compared to the relaxed solution to make the fixed price less attractive.

To analyze the distortions in the auction in period two, note that

$$\forall v_1 \in [0,\overline{v}]: \quad J_1^{p_U}(v_1) = J_1(v_1|2) + \frac{p_U}{f_1(v_1|2)} > J_1(v_1|2),$$

if the relaxed solution is not incentive compatible ($p_U > 0$). Therefore, the reserve price for buyer one is smaller than in the relaxed solution. Secondly, for all valuations above the reserve price, the winning probability is higher than in the relaxed solution because v_1 ties with a higher valuation v_2. Finally, in contrast to the relaxed solution, the winning probability of bidder two is strictly smaller than one for all $v_2 \in [0,\overline{v}]$. The reason is that for every $p_U > 0$, there is a non-empty interval $(c,\overline{v}]$ such that

[20]We only discuss the global solution for the regular case. The irregular case is similar.

$J_1^{pu}(v_1) > \overline{v}$ for all valuations $v_1 \in (c, \overline{v}]$. Buyer two cannot win against buyer one with valuation $v_1 > c$.

Bunching. We find that for the optimal allocation rule, there is bunching at the top of the type-space if the incentive compatibility constraint for the deadline is binding. The bunching region has full dimension. The optimal mechanism does not separate different types of buyer one if their valuations are very high ($v_1 > c$). In the auction, these types win with probability one and have to make an expected payment equal to the fixed price in the first period. Therefore, we have bunching of valuations as well as deadlines. This finding is robust: a (small) bunch occurs even if the allocation is only slightly distorted.

Dominant Strategies and Indirect Implementation. There are several ways to implement the optimal auction in period two. For example, it can be implemented by a generalized Vickrey auction. In this auction, the winning bidder pays the valuation for which his (generalized) virtual valuation ties with the (generalized) virtual valuation of the losing bidder. For buyer two, this mechanism is incentive compatible in dominant strategies.[21] Hence, the optimal mechanism does not rely on the seller's ability to conceal information about period one.

As in the standard auction model, there is also an open format that corresponds to this direct mechanism. Consider the following ascending clock auction. The auctioneer has a clock that runs from zero to $\overline{v}$. For each bidder i, the auctioneer's clock value c_a is translated into a bidder-specific clock value c_i. For bidder one, this is $c_1 = \left(J_1^{pu}\right)^{-1}(c_a|2)$. For bidder two, this is $c_2 = J_2^{-1}(c_a)$. The auctioneer raises c_a continuously and bidders can drop out at any time. If bidder i drops out, the clock stops immediately. Bidder $j \neq i$ wins the object and has to make a payment equal to his bidder-specific clock-value c_j. Given the informational assumptions made in this chapter, this auction is strategically equivalent to the generalized Vickrey auction. It has the advantage that the winning bidder does not have to reveal his true valuation to the auctioneer.

1.7. Conclusion

We have analyzed a dynamic mechanism design model, in which a seller wants to maximize the revenue from selling one or multiple identical units of a good to buyers that arrive over time, within a finite time horizon. The main innovation of the model is that buyers are privately informed about their deadlines for buying a unit of the good.

First, we have studied the case of full separation in which the additional incentive compatibility constraint for the deadline is slack in the seller-optimal mechanism. We found sufficient conditions for the incentive compatibility of the relaxed solution and

[21] If the auction is considered in isolation, it is also a dominant strategy for buyer one, to bid his true valuation. In the dynamic context, however, it is not a dominant strategy to report the deadline truthfully.

sufficient conditions for the incentive compatibility constraint to be violated. Both conditions exploit (a) a static pricing effect that depends on stochastic dependencies between the deadline and the valuation of a buyer, and (b) a dynamic pricing effect that depends on non-linearities in the virtual valuation function of a buyer. While the former effect can also be found in static models with two-dimensional private information, the latter effect is due to the dynamic nature of the allocation problem. The critical virtual valuation that buyer has to overbid in order to get a unit is a martingale with respect to the information about all buyer's types. Therefore, critical virtual valuations for later periods are mean preserving spreads of critical virtual valuations for earlier periods. This leads to lower (higher) payoffs for later deadlines in the case of concave (convex) virtual valuations and destroys (guarantees) incentive compatibility.

Second, we have studied the case of bunching. If the relaxed solution is not incentive compatible, the incentive constraint for the deadline is binding in the optimal mechanism. Therefore, we had to solve a mechanism design problem with two dimensional private information. The fact that that the second dimension is a deadline puts some structure on the model. The two dimensional problem is similar to a standard one-dimensional mechanism design problem with a type-dependent outside option. We solve this model for the case of two time periods, one object and deterministic arrival of one buyer in each period. We show that the optimal mechanism has a very similar structure as the relaxed solution, but the allocation rule is distorted in favor of buyers with later deadlines and earlier arrival. This provides incentives to report the deadline truthfully. The optimal mechanism can be described in terms of a generalized virtual valuation.

Several assumptions have been made to ensure tractability or to simplify the exposition.

Discounting. Throughout the chapter, we have abstracted from discounting. This assumption can be relaxed. If only payments are discounted and buyers and the seller use a common discount factor, the analysis is almost identical. On the other hand, if the whole payoff is discounted, Lemma 1.3.1 may not be valid. For example, it may be optimal to allocate a unit in the first period even if the deadline of the winner is two, because the waiting cost due to discounting is too high. In this case, it is more complicated to rule out upward deviations in the deadline.

Which modelling assumption is more pertinent depends on the application. In the example given in the introduction, the value of the object for the buyer is the present discounted value of the revenue stream from the contractual relationship with the third party. This could for example be a production contract. If production starts after the deadline and is independent of the time at which the firm obtains the object (as long as it gets it before the deadline), it seems reasonable that the firm only discounts payments. Similar arguments apply in any situation where the buyer plans to use the object at a fixed time after the deadline as in the case of flight tickets of hotel reservations.

Stochastic Exit. We have implicitly assumed that buyers are available until their deadline. In some situations, however, buyers may find other opportunities to purchase a similar object if the seller does not sell in the period of arrival. Therefore, stochastic exit, random participation or competition with other sellers would be interesting extensions for future research.

Incentive Compatibility of the Relaxed Solution with Many Objects. The proof of the martingale property of the critical virtual valuation uses a property of the optimal allocation rule that was shown in Chapter 3 for the case of a single object. With one object, there is a unique bidder in each period that has a positive probability of winning. This greatly simplifies the analysis because in each state, the type of only one buyer is relevant for the allocation rule and buyers who are irrelevant in period t will not be recalled in the future. Unfortunately, a generalization of this property to the case of many objects is not available. I conjecture, however, that the martingale property of the critical virtual valuation generalizes to the case of many objects. If this conjecture is true, then the dynamic as well as the static pricing effect, and the absence of a competition effect will carry over to the case of multiple objects and more than two time periods. Therefore the sufficient conditions for incentive compatibility of the relaxed solution will also apply to the more general model.

Privately Known Arrival Times. The arrival time has similar properties as the deadline. Misreports are feasible only in one direction and it does not directly enter the utility functions because units cannot be allocated before the arrival. Therefore, the analysis of a model with privately known arrival times will be similar as the analysis in the present chapter. Incentive compatibility of the relaxed solution, however, is guaranteed for the arrival time under weaker conditions than incentive compatibility for the deadline. As in the case of deadlines, there is a static pricing effect if the valuation depends stochastically on the arrival time (Pai and Vohra, 2008b). The dynamic pricing effect, however, does not arise with arrival times because the arrival time does not influence the time of the allocation. Furthermore, independent of the type distribution there is an additional effect, that relaxes the incentive compatibility constraint for the arrival time. By delaying the report of his arrival, a buyer runs the risk that units are allocated to buyers that he could have overbid if he had reported his arrival truthfully. Therefore, an adverse static pricing effect does not automatically destroy incentive compatibility.

Generalizing the Bunching Case: More Bidders. Introducing more bidders who arrive in the second period is straight forward. The assumption that there is only one bidder in the first period is more important. It was used to show that the object is offered to buyer one for a fixed price if he reports deadline one. We have shown that in this case, misreporting deadline one instead of deadline two is most profitable for the buyer with the highest valuation. Hence, we know exactly where the incentive compatibility constraint for the deadline binds. If more than one buyer arrives in the first period, a fixed price is no longer optimal and the incentive compatibility constraint for the deadline may bind for interior types. The exact points where it binds arise endogenously in the optimal solution.

Generalizing the Bunching Case: Number of Periods. Increasing the number of periods introduces several complications. Consider for example a model with three periods. Suppose that in each period a single bidder arrives, whose deadline can be any period after his arrival. Now, from period two onwards, there is more than one bidder who participates in the mechanism. This introduces similar problems as the introduction of more bidders in the first period discussed in the preceding paragraph. Additional complications will arise because buyers from different periods will have to be treated asymmetrically. In the third period, the mechanism designer has to design an optimal auction with three different bidders, two of which have type-dependent participation constraints. In the case of two periods and two bidders, the feasibility constraint could be used to eliminate the winning probability of one bidder (see Appendix 1.A). A generalization of this approach to three bidders is not obvious.

1.A. Proof of Theorem 1.6.5

It will be convenient to make the changes of variables $t_1 = F_1(v_1|2)$ and $t_2 = F_2(v_2)$. Defining $v_1(t_1) := F_1^{-1}(t_1|2)$ and $v_2(t_2 := F_2^{-1}(t_2)$, we have

$$\begin{aligned} t_i &\sim U[0,1] \quad \text{for } i = 1,2, \\ v_1'(t_1) &= \frac{1}{f_1(v_1(t_1)|2)}, \\ \text{and} \quad v_2'(t_2) &= \frac{1}{f_2(v_2(t_2))}, \end{aligned}$$

Furthermore, for $i = 1, 2$ we introduce

$$\begin{aligned} q_i(t) &= q_i(v_i(t), 2), \\ U(t) &= U_1(v_1(t), 2), \\ M_1(t) &= J_1(v_1(t)|2) = v_1(t) - (1-t)v_1'(t) \\ M_2(t) &= J_2(v_2(t)) = v_2(t) - (1-t)v_2'(t) \\ t_1^0 &= F_1(v_1^0|2\,|2). \\ \text{and} \qquad t_2^0 &= F_2(v_2^0). \end{aligned}$$

The objective of the seller becomes

$$R[q_1, q_2] := \int_0^1 q_1(t)M_1(t) + q_2(t)M_2(t)dt. \tag{1.A.1}$$

In order to employ control theory, we have to formulate the feasibility constraint in terms of q. Border (1991) provides a characterization of feasibility for symmetric reduced form allocation rules. The following Theorem generalizes the result to asymmetric allocation rules.

THEOREM 1.A.1 (Chapter 2). *For $i = 1, 2$, let $q_i : [0, 1] \to [0, 1]$ be nondecreasing. (q_1, q_2) is the reduced form of a feasible allocation rule if and only if for all $t_1, t_2 \in [0, 1]$,*

$$\int_{t_1}^{1} q_1(t)dt + \int_{t_2}^{1} q_2(t)dt \leq 1 - t_1 t_2.$$

Now we can restate $\mathcal{P}_2$ as $\mathcal{P}_2'$:

$$\pi_2(\bar{U}) = \sup_{(q_1,q_2)} R[q_1, q_2] \qquad (\mathcal{P}_2')$$

subject to

$$\forall t \in [0,1]: \qquad q_i(t) \in [0,1], \qquad (1.A.2)$$

$$\forall t > t', \qquad q_i(t) \geq q_i(t'), \qquad (1.A.3)$$

$$\forall t_1, t_2 \in [0,1]: \qquad \int_{t_1}^{1} q_1(\theta)d\theta + \int_{t_2}^{1} q_2(\theta)d\theta \leq 1 - t_1 t_2, \qquad (1.A.4)$$

$$\forall t \in [0,1]: \qquad U(t) = \int_0^t q_1(\theta)v_1'(\theta)d\theta, \qquad (1.A.5)$$

$$\text{and} \qquad U(1) \geq \bar{U}. \qquad (1.A.6)$$

Using $q_i(F_i(v_i|2)) = q_i(v_i, 2)$, a solution to $\mathcal{P}_2$ can be derived easily from a solution to $\mathcal{P}_2'$.

A direct solution of $\mathcal{P}_2'$ is difficult because (1.A.4) is not a standard constraint. Instead, we can use (1.A.4) to eliminate q_2 from the objective function. For $q_1 : [0, 1] \to [0, 1]$ non-decreasing, define the *inverse* as

$$q_1^{-1}(t) := \begin{cases} 1 & \text{if } q_1(1) < t, \\ \inf\{\theta \in [0,1] \mid q_1(\theta) \geq t\} & \text{otherwise.} \end{cases}$$

LEMMA 1.A.2. *Let $q_1 : [0, 1] \to [0, 1]$ be non-decreasing. Then an optimal solution to*

$$\sup_{q_2} \int_0^1 q_2(t)M_2(t)dt \qquad \text{subject to (1.A.2)–(1.A.4)},$$

is given by

$$q_2^*(t) = \begin{cases} q_1^{-1}(t) & \text{if } t \geq t_2^0, \\ 0 & \text{otherwise.} \end{cases}$$

The solution is unique for almost every t.

PROOF. (1.A.4) can be rewritten as

$$\forall t_2 \in [0,1]: \quad \int_{t_2}^{1} q_2(\theta)d\theta \leq \min_{t_1\in[0,1]} \left[1 - t_1 t_2 - \int_{t_1}^{1} q_1(\theta)d\theta\right].$$

On the right-hand side we minimize a convex function. Therefore, the first order condition is sufficient for a minimum and we have $t_2 \in [q_1(t_1^-), q_1(t_1^+)]$ for all $t_2 \in [q_1(0), q_1(1)]$, $t_1 = 0$ if $t_2 < q_1(0)$ and $t_1 = 1$ if $t_2 > q(1)$. Hence $t_1 = q_1^{-1}(t_2)$ is a minimizer for all t_2. Substituting this into (1.A.4) yields

$$\forall t_2 \in [0,1]: \quad \int_{t_2}^{1} q_2(\theta)d\theta \leq 1 - q_1^{-1}(t_2)t_2 - \int_{q_1^{-1}(t_2)}^{1} q_1(\theta)d\theta. \tag{1.A.7}$$

q_2^* fulfills this constraint with equality for all $t_2 \in [0,1]$.

Now consider an alternative solution $\tilde{q}_2$ that differs from q_2^* on a set of positive measure. If $\tilde{q}_2(t) > 0$ for some $t < t_2^0$, than it is not a maximizer. So suppose $\tilde{q}_2(t) = 0$ for $t < t_2^0$. By (1.A.7) we must have $\int_t^1 \tilde{q}_2(\theta)d\theta \leq \int_t^1 q_2^*(\theta)d\theta$ for all $t \in [0,1]$. Since $\tilde{q} \neq q^*$, on a set of positive measure, $\int_a^1 \tilde{q}_2(\theta)d\theta < \int_a^1 q_2^*(\theta)d\theta$ for some $a \in [t_2^0, 1]$. Let $Q(t)$ be the concave hull of

$$t \longmapsto \begin{cases} \int_t^1 \tilde{q}_2(\theta)d\theta, & \text{if } t \neq a, \\ \int_a^1 q_2^*(\theta)d\theta, & \text{if } t = a, \end{cases}$$

and define $\hat{q}_2(t) = -\frac{dQ(t)}{dt}$ for almost every t. By definition, $Q(t) = \int_t^1 \hat{q}_2(\theta)d\theta \leq \int_t^1 q_2^*(\theta)d\theta$. Hence $\hat{q}_2$ is a solution. Furthermore, there are $\underline{a}, \bar{a}$ such that

$$\hat{q}_2(t) = \begin{cases} \tilde{q}_2(t), & \text{if } t \notin [\underline{a}, \bar{a}], \\ \frac{\int_{\underline{a}}^1 \tilde{q}_2(\theta)d\theta - \int_a^1 q_2^*(\theta)d\theta}{a - \underline{a}}, & \text{if } t \in [\underline{a}, a), \\ \frac{\int_a^1 q_2^*(\theta)d\theta - \int_{\bar{a}}^1 \tilde{q}_2(\theta)d\theta}{\bar{a} - a}, & \text{if } t \in (a, \bar{a}]. \end{cases}$$

Hence $\hat{q}_2(t) < \tilde{q}_2(t)$ for $t \in (\underline{a}, a)$, $\hat{q}_2(t) > \tilde{q}_2(t)$ for $t \in (a, \bar{a})$ and $\hat{q}_2(t) = \tilde{q}_2(t)$ otherwise. Furthermore,

$$\int_{\underline{a}}^{a} \hat{q}_2(\theta) - \tilde{q}_2(\theta)d\theta = \int_a^{\bar{a}} \tilde{q}_2(\theta) - \hat{q}_2(\theta)d\theta.$$

This implies that we have constructed $\hat{q}_2$ from $\tilde{q}_2$ by shifting winning probability from types in $[\underline{a}, a]$ to types in $[a, \overline{a}]$. By Assumption 1.2.1, this increases the objective function. Hence $\tilde{q}_2$ cannot be optimal. □

Using Lemma 1.A.2, (1.A.1) becomes

$$\int_0^1 q_1(t)M_1(t)dt + \int_{t_2^0}^1 q_1^{-1}(t)M_2(t)dt. \tag{1.A.8}$$

If q_1 is absolutely continuous, substituting $s = q_1(t)$ in the second integral yields

$$\int_0^1 q_1(t)M_1(t) + tq_1'(t)\tilde{M}_2(q_1(t))dt + \int_{q(1)}^1 \tilde{M}_2(t)dt, \tag{1.A.9}$$

where we define $\tilde{M}_2(t) := \max\{0, M_2(t)\}$.[22]

Monotonicity implies some regularity of q_1. In particular $q_1 = q_1^C + q_1^J$ where q_1^C is a continuous function and q_1^J is a pure jump function. This leaves two problems unresolved. Firstly, we have to deal with jumps and secondly, absolute continuity of q_1^C is not guaranteed.[23]

These problems can be circumvented by solving the maximization problem under the restriction that q_1 be Lipschitz continuous with global Lipschitz constant K,

$$q_1 \in \mathcal{L}^K := \{q : [0,1] \to [0,1] \mid \forall t, t' \in [0,1] : \ |q(t) - q(t')| \leq K|t - t'|\}.$$

We define the maximization problem $\mathcal{P}_2^K$ as $\mathcal{P}_2'$ subject to the additional constraint $q_1 \in \mathcal{L}^K$. It will be shown that optimal solutions of $\mathcal{P}_2^K$ converge to the optimal solution of $\mathcal{P}_2'$ as $K \to \infty$. Using Lipschitz functions is convenient to show existence because $\mathcal{L}^K$ is sequentially compact.

THEOREM 1.A.3. *(a) An optimal solution of $\mathcal{P}_2'$ exists.*
(b) For every $K > 0$, an optimal solution of $\mathcal{P}_2^K$ exists.

PROOF. (i) Let $(q_1^n, q_2^n)_{n\in\mathbb{N}}$ be a sequence of solutions of $\mathcal{P}_2'$ such that $R[q_1^n, q_2^n] \to \pi_2(\bar{U})$ for $n \to \infty$. By Helly's Theorem, for $i = 1, 2$ there exists a subsequence $\left(q_i^{n_j}\right)_{j\in\mathbb{N}}$ and a non-decreasing function $q_i : [0,1] \to [0,1]$, such that $q_i^{n_j} \to q_i$ almost everywhere. If we consider the q_i as elements of $L_2([0,1])$, the set of winning probabilities that satisfy (1.A.4) is weakly-compact (cf. Lemma 2.3.5 in Chapter 2 and Lemma 5.4 in Border (1991)). Therefore, after taking subsequences again, $q_i^{n_j} \rightharpoonup q_i$ and q_i is feasible. As $M_i \in L_2([0,1])$ and $v_1' \in L_2([0,1])$, weak convergence of $q_i^{n_j}$ implies that q_1 fulfills (1.A.5)–(1.A.6), and $R[q_1, q_2] = \pi_2(\bar{U})$. Therefore (q_1, q_2) is an optimal solution.

(ii) Let $(q_1^n, q_2^n)_{n\in\mathbb{N}}$ be a sequence of solutions of $\mathcal{P}_2^K$ such that $R[q_1^n, q_2^n] \to \pi_2^K(\bar{U})$. After taking subsequences we can assume that this sequence converges to a solution satisfying (1.A.2)–(1.A.6) as in (i). Let q_1 be the limit of q_1^n. Since $q_i^n \in \mathcal{L}^K$, for all $s, t \in [0,1]$, $|q_1(t) - q_1(s)| = \lim_{n\to\infty} |q_1^n(t) - q_1^n(s)| \leq K|t - s|$. Hence $q_1 \in \mathcal{L}^K$.

□

The next step is to show that Lipschitz solutions converge to the general solution if K tends to infinity. The proof is based on Reid (1968).

LEMMA 1.A.4. *Let $(q_1^n, q_2^n)_{n\in\mathbb{N}}$ a sequence of optimal solutions of $\mathcal{P}_2^{K_n}$ where $K_n \to \infty$ as $n \to \infty$. Then, there exists a solution (q_1, q_2) of $\mathcal{P}_2'$ and a subsequence $(q_1^{n_j}, q_2^{n_j})_{j\in\mathbb{N}}$ such that $q_i^{n_j}(t) \overset{j\to\infty}{\longrightarrow} q_i(t)$ for almost every t and $R[q_1, q_2] = \pi_2(\bar{U})$.*

[22] If q is not absolutely continuous, the substitution yields $\int_0^1 q_1(t)M_1(t)dt + \int_0^1 t\tilde{M}_2(t)dq_1(t) + \int_{q(1)}^1 \tilde{M}_2(t)dt$. In the second integral, q_1 is interpreted as a measure that does not admit a density. This is not useful if we want to apply optimal control theory.

[23] For example, the Cantor function is non-decreasing and continuous but it cannot be described as the integral of a function. Hence it is not absolutely continuous.

PROOF. After taking a subsequence, we can assume that (q_1^n, q_2^n) converges a.e. to a solution $(\hat{q}_1, \hat{q}_2)$ of $\mathcal{P}_2'$ (see proof of Theorem 1.A.3). To show optimality of $(\hat{q}_1, \hat{q}_2)$, let (q_1, q_2) be an optimal solution of $\mathcal{P}_2'$. We can extend q_1 to $\mathbb{R}$ by setting $q_1(t) = 0$ if $t < 0$ and $q_1(t) = 1$ if $t > 1$. Define $q_{d,1} : \mathbb{R} \to [0, 1]$ as

$$q_{d,1}(t) := \frac{1}{2d} \int_{t-d}^{t+d} q_1(s)ds.$$

By the Lebesgue differentiation theorem $q_{d,1}(t) \to q_1(t)$ for almost every $t \in [0, 1]$ as $d \to 0$. Since q_1 is non-decreasing and $q_1(t) \in [0, 1]$, $q_{d,1}$ also has these properties. Furthermore $q_{d,1} \in \mathcal{L}^{\frac{1}{2d}}$:

$$\begin{aligned} \forall t > t' : \qquad 0 \le q_{d,1}(t) - q_{d,1}(t') &= \frac{1}{2d}\left(\int_{t-d}^{t+d} q_1(s)ds - \int_{t'-d}^{t'+d} q_1(s)ds\right) \\ &= \frac{1}{2d}\left(\int_{t'+d}^{t+d} q_1(s)ds - \int_{t'-d}^{t-d} q_1(s)ds\right) \\ &\le \frac{1}{2d}\int_{t'+d}^{t+d} q_1(s)ds \\ &\le \frac{1}{2d}(t - t') \end{aligned}$$

Since $q_{d,1}$ may violate $\int_0^1 q_{d,1}(t)v_1'(t)dt \ge \bar{U}$, we define $\tilde{q}_{d,1} := \lambda_d + (1 - \lambda_d)q_{d,1}$ and

$$\tilde{q}_{d,2}(t) := \begin{cases} \tilde{q}_{d,1}^{-1}(t), & \text{if } M_2(t) \ge 0, \\ 0, & \text{otherwise,} \end{cases}$$

where $\lambda_d := \max\left\{0, \frac{\bar{U} - \int_0^1 q_{d,1}(t)v_1'(t)dt}{\bar{v} - \int_0^1 q_{d,1}(t)v_1'(t)dt}\right\}$. For every d, $(\tilde{q}_{d,1}, \tilde{q}_{d,2})$ is a solution of $\mathcal{P}_2^{\frac{1}{2d}}$. λ_d converges to zero as $d \to 0$. By Lemma 1.A.2, $q_2(t) = q_1^{-1}(t)$ for a.e. t such that $M_2(t) \ge 0$ and $q_2(t) = 0$ otherwise. Hence, for $i = 1, 2$, $\tilde{q}_{d,i} \to q_i$ almost everywhere as $d \to 0$. By the dominated convergence theorem, $R[\tilde{q}_{d,1}, \tilde{q}_{d,2}] \to R[q_1, q_2]$ and $R[q_1^n, q_2^n] \to R[\hat{q}_1, \hat{q}_2]$. Define $d_n = \frac{1}{2K_n}$. Then, $R[\tilde{q}_{d_n,1}, \tilde{q}_{d_n,2}] \le R[q_1^n, q_2^n]$ and we have $R[q_1^n, q_2^n] \to R[q_1, q_2]$ and hence $R[\hat{q}_1, \hat{q}_2] = R[q_1, q_2]$. □

1.A.1. Solution on the class $\mathcal{L}^K$. Using Lemma 1.A.2, we rewrite $\mathcal{P}_2^K$ as a control problem. The state variables are the expected utility of bidder one, denoted $U(t)$, and the winning probability, denoted $q(t)$, (in the control problem we write q instead of q_1). As q is absolutely continuous, we can use $u(t) = q'(t)$ as a control variable. The objective is defined as

$$R_c[U, q, u] := \int_0^1 q(t)M_1(t) + tu(t)\tilde{M}_2(q(t))dt + \int_{q(1)}^1 \tilde{M}_2(t)dt.$$

where u is a measurable control

$$u : [0, 1] \to [0, K]. \tag{1.A.10}$$

The evolution of the state variables is governed by

$$U'(t) = q(t)v_1'(t), \tag{1.A.11}$$
$$q'(t) = u(t). \tag{1.A.12}$$

We impose the state constraint

$$\forall t \in [0,1]: \qquad q(t) \leq 1. \tag{1.A.13}$$

Furthermore, we impose the following constraints on the start- and endpoints:

$$U(0) = 0, \tag{1.A.14}$$
$$q(0) \geq 0, \tag{1.A.15}$$
$$(1) \geq \bar{U}, \tag{1.A.16}$$

To summarize, we have the following control problem:

$$\max_{(U,q,u)} R_c[U,q,u], \qquad \text{subject to (1.A.10)–(1.A.16).} \tag{$\mathcal{P}_C^K$}$$

(1.A.11) is (1.A.5) in differential form. (1.A.10) and (1.A.12) ensure that $q \in \mathcal{L}^K$ and non-decreasing. (1.A.10), (1.A.12) and (1.A.15) imply $q(t) \geq 0$ for all t. Hence, we can dispense with a second state constraint.

The Pontryagin maximum principle yields the following necessary conditions for an optimum.

Theorem 1.A.5 (Clarke (1983), pp. 210-212). *Let (U,q,u) be a solution of $\mathcal{P}_C^K$. If (U,q,u) is optimal, there exists $\omega \in \{0,1\}$, an absolutely continuous function $p:[0,1] \to \mathbb{R}^2$, the components of which we denote by (p_U, p_q), and a non-negative measure μ on $[0,1]$, such that the following conditions hold:*

(i) For almost every $t \in [0,1]$,

$$p_U'(t) = 0, \tag{1.A.17}$$
$$p_q'(t) = -\omega\left[M_1(t) + tu(t)\tilde{M}_2'(q(t))\right] - p_U v_1'(t). \tag{1.A.18}$$

(ii) For almost every $t \in [0,1]$, $u(t)$ maximizes

$$\left[\omega t\tilde{M}_2(q(t)) + p_q(t) + \mu[0,t)\right] u.$$

(iii) μ is supported on $\{q(t) = 1\}$,

(iv) p satisfies the transversality conditions

$$p_q(0) \leq 0, \qquad (\textit{with equality if } q(0) > 0,)$$
$$p_U(1) \geq 0, \qquad (\textit{with equality if } U(1) > \underline{U},)$$
$$p_q(1) = -\omega\tilde{M}_2(q(1)) - \mu[0,1].$$

(v) $\omega + \|p\| + \|\mu\| > 0$.

Note that (1.A.17) implies that p_U is constant. First, we show that trivial solutions do not occur.

LEMMA 1.A.6 (Non-triviality). *If* $\bar{U} < \overline{v}$, $\omega = 1$.

PROOF. Suppose that $\omega = 0$. By (1.A.18), $p_q'(t) = -p_U v_1'(t)$. By the transversality conditions, $p_U \geq 0$. $p_U = 0$ implies, $p_q'(t) = 0$ and $p_q(t) = p_q(0)$ for all t. $p_U > 0$ implies, $p_q'(t) < 0$ and $p_q(t) < 0$ for all $t > 0$.

Suppose $p_U > 0$. By, the transversality condition this implies $U(1) = \bar{U}$. By (ii), $u(t)$ maximizes $(p_q(t) + \mu[0,t))u$. If $q(0) < 1$, $\mu[0,t) = 0$ for t close to zero and hence $u(t) = 0$. As $\mu[0,t)$ cannot become positive we must have $q(t) = q(0) < 1$ for all t and consequently $\mu[0,1] = 0$. The transversality condition therefore requires $p_q(1) = 0$, a contradiction. If, however, $q(0) = 1$ we would have $U(1) = \overline{v} > \bar{U}$. Again a contradiction.

Now suppose that $p_U = 0$. If $q(1) < 1$, $\mu[0,1] = 0$ and by the transversality conditions, $p(t) = 0$ for all t. This implies $\omega + \|p\| + \|\mu\| = 0$, in contradiction to (v). Hence, $q(1) = 1$. Since $p_q(t) = p_q(1)$, we have $p_q(t) = -\mu[0,1]$. To fulfil (v) we must have $\mu[0,1] > 0$. $u(t)$ maximizes $(\mu[0,t) - \mu[0,1])u$. This implies that $u(t) = 0$ if $q(t) < 1$. Hence, we must have $q(t) = 1$ for all $t \in [0,1]$. This implies $U(1) = \overline{v}$ which cannot be optimal if $\bar{U} < \overline{v}$. □

Defining $M_1^{p_U}(t) := M_1(t) + p_U v_1'(t)$, we can rewrite (1.A.18) as

$$-p_q'(t) = M_1^{p_U}(t) + tu(t)\tilde{M}_2'(q(t)), \text{ for a. e.} \, t \in [0,1]. \tag{1.A.19}$$

Condition (ii) implies that for almost every $t \in [0,1]$,

$$u(t) = K \qquad \text{if } t\tilde{M}_2(q(t)) + p_q(t) > 0, \tag{1.A.20}$$

$$u(t) \in [0,K] \qquad \text{if } t\tilde{M}_2(q(t)) + p_q(t) + \mu[0,t) = 0, \tag{1.A.21}$$

$$u(t) = 0 \qquad \text{if } t\tilde{M}_2(q(t)) + p_q(t) + \mu[0,t) < 0. \tag{1.A.22}$$

In (1.A.20), $\mu[0,t)$ was omitted because $q(t) < 1$ if $u(t) = K$. Integrating (1.A.19) yields for $s, t \in [0,1]$:

$$\begin{aligned} p_q(t) &= p_q(s) - \int_s^t M_1^{p_U}(\theta) + \theta u(\theta)\tilde{M}_2'(q(\theta))d\theta \\ &= p_q(s) - \int_s^t M_1^{p_U}(\theta) - \tilde{M}_2(q(\theta))d\theta - t\tilde{M}_2(q(t)) + s\tilde{M}_2(q(s)). \end{aligned} \tag{1.A.23}$$

If we substitute (1.A.23) in (1.A.20)–(1.A.22) and define $H^{p_U}(t) = \int_0^t M_1^{p_U}(\theta)d\theta$ and $m_q(t) = \int_0^t \tilde{M}_2(q(\theta))d\theta$, we have that for almost every $t \in [0,1]$,

$$u(t) = K \qquad \text{if } p_q(0) + m_q(t) > H^{p_U}(t), \tag{1.A.24}$$

$$u(t) \in [0,K] \qquad \text{if } p_q(0) + m_q(t) + \mu[0,t) = H^{p_U}(t), \tag{1.A.25}$$

$$u(t) = 0 \qquad \text{if } p_q(0) + m_q(t) + \mu[0,t) < H^{p_U}(t). \tag{1.A.26}$$

LEMMA 1.A.7 (Reid (1968)). *Suppose* $p_q(0) + m_q(t) = H^{p_U}(t)$ *for* $t \in \{\underline{t}, \overline{t}\}$, $\underline{t} < \overline{t}$ *and* $q(t) < 1$ *for* $t < \overline{t}$. *Let* $\alpha, \beta \in \mathbb{R}$ *and* $l(t) = \alpha + \beta t$. *If* $l(t) \leq H^{p_U}(t)$ *for all* $t \in [\underline{t}, \overline{t}]$, *then* $p_q(0) + m_q(t) \geq l(t)$ *for all* $t \in [\underline{t}, \overline{t}]$.

PROOF. Suppose that $m_q(s) + p_q(0) < l(s)$ for some $s \in (\underline{t}, \overline{t})$. Then there exists $\varepsilon > 0$ and $\underline{t} < t_1 < t_2 < \overline{t}$ such that $m_q(t) + p_q(0) < l(t) - \varepsilon$ for $t \in (t_1, t_2)$, $m_q(t_1) + p_q(0) = l(t_1) - \varepsilon$, and $p_q(0) + m_q(t_2) = l(t_2) - \varepsilon$. This implies that $m_q'(t) = \tilde{M}_2(q(t))$ cannot be constant on (t_1, t_2). On the other hand, $m_q(t) + p_q(0) + \mu[0, t) = m_q(t) + p_q(0) < l(t) - \varepsilon < H(t)$ and hence $u(t) = 0$ for $t \in (t_1, t_2)$ which implies that $m_q'(t)$ is constant, a contradiction. □

An immediate implication of the Lemma is that $p_q(0) + m_q(t) \geq \bar{H}^{p_U}_{[\underline{t},\overline{t}]}(t)$, where $\bar{H}^{p_U}_{[\underline{t},\overline{t}]}(t)$ denotes the convex hull of H^{p_U} restricted to $[\underline{t}, \overline{t}]$, i.e. the greatest convex function $G : [\underline{t}, \overline{t}] \to \mathbb{R}$ such that $G(t) < H^{p_U}(t)$ for all $t \in [\underline{t}, \overline{t}]$. Furthermore, $p_q(0) + m_q(t)$ is convex because q and $\tilde{M}_2$ are non-decreasing. This yields the following

COROLLARY 1.A.8. *Suppose $p_q(0) + m_q(t) \leq H^{p_U}(t)$ for all $t \in [\underline{t}, \overline{t}]$, with equality at the endpoints of the interval and $q(t) < 1$ for $t < \overline{t}$. Then $p_q(0) + m_q(t) = \bar{H}^{p_U}_{[\underline{t},\overline{t}]}(t)$, for all $t \in [\underline{t}, \overline{t}]$.*

If $M_1^{p_U}$ is non-decreasing on $[\underline{t}, \overline{t}]$, then $H^{p_U}(t) = \bar{H}^{p_U}_{[\underline{t},\overline{t}]}(t)$. Differentiating $p_q(0) + m_q(t) = \bar{H}^{p_U}_{[\underline{t},\overline{t}]}$ yields $M_1^{p_U} = \tilde{M}_2(q(t))$ for $t \in [\underline{t}, \overline{t}]$.

If, however, $M_1^{p_U}$ is not monotonic on $[\underline{t}, \overline{t}]$, differentiating yields $\bar{M}^{p_U}_{[\underline{t},\overline{t}]}(t) = \tilde{M}_2(q(t))$, where $\bar{M}^{p_U}_{[\underline{t},\overline{t}]} = \frac{d\bar{H}^{p_U}_{[\underline{t},\overline{t}]}(t)}{dt}$ is non-decreasing. Hence, Reid's Lemma provides a control theoretic technique to show that Myerson's ironing procedure can be used to solve irregular instances of mechanism design problems.

Now we establish some properties of the optimal solution. Define

$$x_{p_U}(t) = \begin{cases} 0, & \text{if } M_1^{p_U}(t) < M_2(0), \\ M_2^{-1}(M_1^{p_U}(t)), & \text{if } M_1^{p_U}(t) \in [M_2(0), \overline{v}] \ , \\ 1, & \text{if } M_1^{p_U}(t) > \overline{v}, \end{cases}$$

and

$$x_{p_U}^{[\underline{t},\overline{t}]}(t) = \begin{cases} 0, & \text{if } \bar{M}^{p_U}_{[\underline{t},\overline{t}]}(t) < M_2(0), \\ M_2^{-1}(\bar{M}^{p_U}_{[\underline{t},\overline{t}]}(t)), & \text{if } \bar{M}^{p_U}_{[\underline{t},\overline{t}]}(t) \in [M_2(0), \overline{v}] \ , \\ 1, & \text{if } \bar{M}^{p_U}_{[\underline{t},\overline{t}]}(t) > \overline{v}. \end{cases}$$

The derivative of x_{p_U} is given by

$$x_{p_U}'(t) = \frac{M_1'(t) + p_U v''(t)}{M_2'(x_{p_U}(t))}.$$

The assumptions on f_i and F_i guarantee that $x_{p_U}'(t)$ is continuous on $[0, 1]$. Let $K^{p_U} := \max_{t \in [0,1]} x_{p_U}'(t)$. Then $x_{p_U} \in \mathcal{L}^{K^{p_U}}$. In what follows, we write $\bar{H}^{p_U}$ for $\bar{H}^{p_U}_{[0,1]}$ and $\bar{M}_1^{p_U}$ for $\bar{M}^{p_U}_{[0,1]}$.

LEMMA 1.A.9 (interior solution). *Suppose $u(t) \in (0, K)$ for a.e. $t \in [\underline{t}, \overline{t}]$, $\underline{t} < \overline{t}$. Then for all $t \in [\underline{t}, \overline{t}]$,*

(i) $q(t) = x_{p_U}(t)$ if $q(t) \geq t_2^0$,
(ii) $M_1^{p_U}(t) = 0$ if $q(t) < t_2^0$.

Proof. If $u(t) > 0$, we must have $\mu[0,t) = 0$. (1.A.24) – (1.A.26) imply that $p_q(0) + m_q(t) = H^{pu}(t)$ for all$t \in (\underline{t}, \overline{t})$. Differentiating this w.r.t. t yields

$$\tilde{M}_2(q(t)) = M_1^{pu}(t).$$

If $q(t) \geq t_2^0$, $\tilde{M}_2(q(t)) = M_2(q(t))$ and hence that $q(t) = x_{pu}(t)$. If $q(t) < t_2^0$, $\tilde{M}_2(q(t)) = 0$ and hence $M_1^{pu}(t) = 0$. By continuity, the results extend to $\underline{t}$ and $\overline{t}$. □

Next, we derive necessary conditions for intervals where $u(t)$ is in $\{0, K\}$.

Lemma 1.A.10 (constant q). *Suppose $q(t) = a \in [0,1]$ on $[\underline{t}, \overline{t}]$, $\underline{t} < \overline{t}$, and let $[\underline{t}, \overline{t}]$ be chosen maximally. Then*

$$p_q(t) + t\tilde{M}_2(q(t)) = 0,$$
$$p_q(0) + m_q(t) = H^{pu}(t),$$

for $t = \underline{t}$ if $\underline{t} > 0$ and for $t = \overline{t}$ if $\overline{t} < 1$, and furthermore

$$M_1^{pu}(\underline{t}) \geq \tilde{M}_2(a), \qquad \text{if } \underline{t} > 0, \tag{1.A.27}$$

$$\text{and} \quad M_1^{pu}(\overline{t}) \leq \tilde{M}_2(a), \qquad \text{if } \overline{t} < 1. \tag{1.A.28}$$

Proof. If $q(t)$ is constant, then for almost every $t \in (\underline{t}, \overline{t})$, $u(t) = 0$ and therefore $p_q(t) + t\tilde{M}_2(q(t)) + \mu[0,t) \leq 0$ and $p_q(0) + m_q(t) + \mu[0,t) \leq H^{pu}(t)$. As $\mu \geq 0$ and by continuity, $p_q(t) + t\tilde{M}_2(q(t)) \leq 0$ and $p_q(0) + m_q(t) \leq H^{pu}(t)$ for $t \in \{\underline{t}, \overline{t}\}$.

Suppose $\underline{t} > 0$ and let $S_- := \{0 < t < \underline{t} \mid u(t) > 0\}$. Since $q(t) < a$ for $t < \underline{t}$, and q is absolutely continuous, $S_- \cap [\underline{t} - \delta, \underline{t}]$ has positive measure for every $\delta > 0$. Hence, there exists a sequence $t_n \nearrow \underline{t}$ with $p_q(t_n) + t_n\tilde{M}_2(q(t_n)) \geq 0$ and $p_q(0) + m_p(t_n) \geq H^{pu}(t_n)$ for all n. By continuity, the first two equalities in the Lemma follow for $\underline{t} > 0$. For $\overline{t} < 1$ set $S_+ := \{\overline{t} < t < 1 \mid u(t) > 0\}$. $S_+ \cap [\overline{t}, \overline{t} + \delta]$ has positive measure for every $\delta > 0$. Hence, there exists a sequence $t_n \searrow \overline{t}$ with $p_q(t_n) + t_n\tilde{M}_2(q(t_n)) \geq 0$ and $p_q(0) + m_p(t_n) \geq H^{pu}(t_n)$ for all n. By continuity, the first two equations in the Lemma follow for $\overline{t} < 1$.

To show (1.A.27), note that for almost every $t \in S_-$, $p_q(t) + t\tilde{M}_2(q(t)) \geq 0$. (1.A.23) yields

$$p_q(\underline{t}) = p_q(t) - \int_t^{\underline{t}} M_1^{pu}(\theta) - \tilde{M}_2(q(\theta))d\theta - \underline{t}\tilde{M}_2(q(\underline{t})) + t\tilde{M}_2(q(t)).$$

With $p_q(\underline{t}) = -\underline{t}\tilde{M}_2(q(\underline{t}))$ and $p_q(t) + t\tilde{M}_2(q(t)) \geq 0$ this implies

$$\int_t^{\underline{t}} M_1^{pu}(\theta) - \tilde{M}_2(q(\theta))d\theta \geq 0,$$

for almost every $t \in S_-$. If this inequality is fulfilled, there must be a $t' \in [t, \underline{t}]$ with

$$M_1^{pu}(t') - \tilde{M}_2(q(t')) \geq 0.$$

As $S_- \cap [\underline{t} - \delta, \underline{t}]$ has positive measure for every $\delta > 0$, t and hence t' can be chosen arbitrarily close to $\underline{t}$. By continuity this implies

$$M_1^{p_U}(\underline{t}) - \tilde{M}_2(q(\underline{t})) \geq 0.$$

To show (1.A.28), note that for almost every $t \in S_+$, $p_q(t) + t\tilde{M}_2(q(t)) \geq 0$. (1.A.23) yields

$$p_q(t) = p_q(\bar{t}) - \int_{\bar{t}}^{t} M_1^{p_U}(\theta) - \tilde{M}_2(q(\theta))d\theta - t\tilde{M}_2(q(t)) + \bar{t}\tilde{M}_2(q(\bar{t})).$$

With $p_q(\bar{t}) = -\bar{t}\tilde{M}_2(q(\bar{t}))$ and $p_q(t) + t\tilde{M}_2(q(t)) \geq 0$ this implies

$$\int_{\bar{t}}^{t} M_1^{p_U}(\theta) - \tilde{M}_2(q(\theta))d\theta \leq 0,$$

for almost every $t \in S_+$. As above there exists $t' \in [\bar{t}, t]$ such that the integrand is non-positive at t'. t and t' can be chosen arbitrarily close to $\bar{t}$. Therefore, by continuity

$$M_1^{p_U}(\bar{t}) - \tilde{M}_2(q(\bar{t})) \leq 0.$$

□

Lemma 1.A.10 implies that there cannot be an interval where q is constant and $q \in (0, 1)$ if x_{p_U} is strictly increasing.

LEMMA 1.A.11. *Suppose $u(t) = K$ for almost every $t \in (\underline{t}, \bar{t})$, $\underline{t} < \bar{t}$. Let $(\underline{t}, \bar{t})$ be chosen maximally. Then for $t = \underline{t}$ and for $t = \bar{t}$ if $\bar{t} < 1$,*

$$p_q(t) + t\tilde{M}_2(q(t)) = 0,$$

for $t = \underline{t}$ if $\underline{t} > 0$ and for $t = \bar{t}$ if $\bar{t} < 1$

$$p_q(0) + m_q(t) = H^{p_U}(t).$$

Furthermore,

$$M_1^{p_U}(\underline{t}) \leq \tilde{M}_2(q(\underline{t})), \qquad \text{if } \underline{t} > 0, \tag{1.A.29}$$

$$\text{and} \quad M_1^{p_U}(\bar{t}) \geq \tilde{M}_2(q(\bar{t})), \qquad \text{if } \bar{t} \in [0, 1]. \tag{1.A.30}$$

PROOF. The proof is very similar to the proof of the preceding Lemma. To show the first equality for $\underline{t} = 0$, the transversality condition can be used to obtain $p_q(0) \leq 0$. For $\bar{t} = 1$, (1.A.30) follows from $M_1^{p_U}(1) \geq \bar{v}$ and $\tilde{M}_2(q(t)) \leq \bar{v}$. □

Setting $q(t) = x_0(t)$ for $t \geq t_1^0$ and $q(t) = 0$ otherwise, yields the optimal solution of Myerson (1981). This is not surprising because p_U would be zero if the incentive compatibility constraint for the deadline were ignored. The following Lemma, which does not depend on the maximum principle, excludes solutions that have lower winning probabilities than the undistorted solution x_0.

LEMMA 1.A.12. *For $K > K^0$, let $b \geq t_1^0$ be the unique solution to $(b - t_1^0)K = x_0(b)$. If $q(t) \leq x_0(t)$ for all $t \in [t_1^0, 1]$ and $q(t) < x_0(t)$ for some $t \in [b, 1]$, then q is not optimal.*

Proof. Suppose by contradiction that q is an optimal solution with the properties stated in the Lemma. Let $b' \in [0, b]$ be the unique solution to $q(t_1^0) + (b' - t_1^0)K = x_0(b')$. Define

$$\tilde{q}(t) = \begin{cases} q(t), & \text{if } t < t_1^0, \\ q(t_1^0) + (t - t_1^0)K, & \text{if } t \in [t_1^0, b'], \\ x_0(t), & \text{if } t > b'. \end{cases}$$

Obviously, $\tilde{q} \in \mathcal{L}^K$ and $\tilde{U}(1) \geq \bar{U}$. Since x_0 is the optimal solution absent constraints, $\tilde{q}$ yields higher revenue than q. This contradicts the optimality of q. □

Lemma 1.A.13. *If $\bar{U} < \overline{v}$, then $p_U \leq \bar{p}_U := 1 + \max_{t \in [0,1]} \frac{\overline{v} - v_1(t)}{v_1'(t)} < \infty$.*

Proof. Suppose to the contrary that, $p_U > \bar{p}_U$. Then $M_1^{p_U}(t) > \tilde{M}_2(1) = \overline{v}$ for all $t \in [0, 1]$. By Lemma 1.A.9.ii, this implies $q(\underline{t}) \geq t_2^0$ if $u(t) \in (0, K)$ on a maximal interval $[\underline{t}, \overline{t}]$. By Lemma 1.A.9.i, this implies $q(t) = x_{p_U}(t)$, for all $t \in [\underline{t}, \overline{t}]$, but this contradicts $u(t) > 0$ if $M_1^{p_U}(t) > \overline{v}$. Hence we have $u(t) \in \{0, K\}$ for all $t \in [0, 1]$.

Suppose $u(t) = 0$ on a maximal interval $[\underline{t}, \overline{t}]$. By Lemma 1.A.10, this implies $\overline{t} = 1$. If $u(t) = K$ on a maximal interval $[\underline{t}, \overline{t}]$, Lemma 1.A.11 implies $\underline{t} = 0$. Therefore, there exists $a \in [0, 1]$ such that $u(t) = K$ for $t < a$ and $u(t) = 0$ for $t > a$. Suppose $a > 0$. Lemma 1.A.11 implies $p_q(0) = 0$ if $a > 0$. As $M_1^{p_U}(t) > \tilde{M}_2(q(t))$ for all t, we have $p_q(t) + m_q(t) < H^{p_U}(t)$ for all $t > 0$. Hence, $u(t) = 0$ for all $t > 0$ and $a = 0$.

If $q(t) = q$ is constant, Lemma 1.A.12 implies that $q > t_2^0$. Therefore, $p_q(0) = 0$ by the transversality condition. Using (1.A.23), we get $p_q(1) = -\int_0^1 M_1^{p_U}(t)dt < 0$. The transversality condition and $p_U > 0$ imply $U(1) = \bar{U}$. This yields $q = \frac{\bar{U}}{\overline{v}}$. If $q < 1$, then $\mu[0, 1] = 0$, and hence, $p_q(1) = -\tilde{M}_2(q(1)) > -\int_0^1 M_1^{p_U}(t)dt$ by the transversality condition. So we must have $q = 1$ and hence $\bar{U} = \overline{v}$ which is ruled out by assumption. □

Note that $|x_{p_U}'(t)| \leq \frac{M_1'(t) + p_U |v_1'(t)|}{\min_{x \in [0,1]} |M_2'(x)|}$. Defining $\overline{K} := \max_{t \in [0,1]} \frac{M_1'(t) + \bar{p}_U |v_1'(t)|}{\min_{x \in [0,1]} |M_2'(x)|}$ we have $x_{p_U} \in \mathcal{L}^{\overline{K}}$ for all $p_U \leq \bar{p}_U$.

Lemma 1.A.14. *Let $(\underline{t}, \overline{t})$ be a maximal interval such that $u(t) = K$ for all $t \in (\underline{t}, \overline{t})$ and $K > \overline{K}$. Then $q(t) < \max\{t_2^0, x_{p_U}(t)\}$ for all $t \in [\underline{t}, \overline{t})$. If $\underline{t} > 0$, then $q(\underline{t}) < t_2^0$. Furthermore $\overline{t} < 1$.*

Proof. If $q(t) \geq \max\{t_2^0, x_{p_U}(t)\}$, then $q(\overline{t}) > \max\{t_2^0, x_{p_U}(\overline{t})\}$ because $K > \overline{K}$. Hence $\tilde{M}_2(q(\overline{t})) > M_1^{p_U}(\overline{t})$, a contradiction by Lemma 1.A.11. If $\underline{t} > 0$, $q(\underline{t}) < t_2^0$ because otherwise (1.A.29) and $K > \overline{K}$ would imply $q(\overline{t}) \geq \max\{t_2^0, x_{p_U}(\overline{t})\}$, which is a contradiction. Finally, $\overline{t} = 1$ would imply $q(t) < x_0(t)$ for all $t \in [t_1^0, 1)$. This is also a contradiction by Lemma 1.A.12. □

Lemma 1.A.15. *For $K > K^0$, $q(1) = 1$.*

Proof. Suppose $q(1) < 1$. By Lemma 1.A.12, $q(1) > t_2^0$. By the transversality condition, $p_q(1) = -\tilde{M}_2(q(1))$. Differentiating $p_q(t) + t\tilde{M}_2(q(t))$, we get $\frac{d}{dt}(p_q(t) + t\tilde{M}_2(q(t))) = p_q'(t)+\tilde{M}_2(q(t))+t\tilde{M}_2'(q(t))q'(t) = \tilde{M}_2(q(t))-M_1^{p_U}(t)$. As $q(1) < x_{p_U}(1)$ we have $\frac{d}{dt}(p_q(t)+t\tilde{M}_2(q(t))) < 0$, and thus $p(t)+t\tilde{M}_2(q(t)) > 0$ for t sufficiently close to one. Hence $u(t) = K$ on a maximal interval $[\underline{t}, 1]$. As $K > K^0$, $\underline{t} > 0$ and hence $q(\underline{t}) < t_2^0$ by Lemma 1.A.14. This contradicts optimality by Lemma 1.A.12. □

Define $c := \min\{t \mid q(t) = 1\}$. By the preceding Lemma, this is well defined for $K > K^0$.

Lemma 1.A.16. *For $K > \overline{K}$,*

$$p_q(0) + m_q(c) = H^{p_U}(c),$$
$$p_q(c) + c\tilde{M}_2(q(c)) = 0,$$
$$M_1^{p_U}(c) = \tilde{M}_2(1).$$

Proof. If $c < 1$ the first two equations are implied by Lemma 1.A.10. If $c = 1$, $u(t) \notin \{0, K\}$ for a set of types with positive measure, arbitrarily close to one. ($u(t) = 0$ is ruled out by $c = 1$, $u(t) \neq K$ follows from the same argument as in the proof of Lemma 1.A.15). Hence, the first two equalities hold for t close to c and by Lemma 1.A.9 also the third equality holds for t close to c. By continuity the equalities also hold for c. If $c < 1$, $M_1^{p_U}(c) \geq \tilde{M}_2(q(c))$ by Lemma 1.A.10. For $K > \overline{K}$, $u(t) = K$ for a maximal interval $[\underline{t}, c]$ is not possible as Lemma 1.A.11 requires $M_1^{p_U}(\underline{t}) \leq \tilde{M}_2(q(\underline{t}))$. Hence $u(t) \notin \{0, K\}$ for a set of types with positive measure, arbitrarily close to c. By Lemma 1.A.9 and continuity, the third equality follows for c. □

Lemma 1.A.17. *Let (U, q, u) be an optimal solution to $\mathcal{P}_C^K$ for $K > \overline{K}$.*

(i) Let $\underline{b} = \min\{q(t) \geq t_2^0\}$. Then there exists $\overline{b} \in [\underline{b}, c]$ such that $u(t) = K$ for $t \in [\underline{b}, \overline{b}]$, and $\tilde{M}_2(q(t)) = \bar{M}^{p_U}_{[\overline{b},1]}(t)$ for $t \in [\overline{b}, c]$. Furthermore, $c = \min\{t \mid \bar{M}^{p_U}_{[\overline{b},1]}(t) = \overline{v}\}$.

(ii) Let $\overline{t}_1^0 := \max\{t \mid \bar{M}_1^{p_U}(t) \leq 0\}$ and $\overline{t}_1^0 = 0$ if $\bar{M}_1^{p_U}(0) > 0$. Then $\underline{b} \to \overline{t}_1^0$ and $\overline{b} \to \overline{t}_1^0$ as $K \to \infty$.

(iii) For almost every $t < \underline{b}$,

$$u(t) \begin{cases} = 0, & \text{if } p_q(0) < H^{p_U}(t), \\ \in [0, K], & \text{if } p_q(0) = H^{p_U}(t), \\ = K, & \text{if } p_q(0) > H^{p_U}(t). \end{cases}$$

Proof. (iii) follows directly from (1.A.24)–(1.A.26) as $q(t) \leq t_2^0$ for $t < \underline{b}$ and hence $m_q(t) = 0$.

If $p_q(0) < H^{p_U}(t)$ for all $t \in [0, 1]$, then $p_q(0) < 0$ and therefore $q(0) = 0$ by the transversality condition. Hence $p_q(0) + m_q(t) < H^{p_U}(t)$, and $q(t) = 0$ for all t, contradicting Lemma 1.A.12. Therefore $p_q(0) \geq \min_t H^{p_U}(t)$.

To show (i), we first show that $\tilde{M}_2(q(t)) = \bar{M}^{p_U}_{[\bar{b},c]}(t)$ for all $t \in [\bar{b}, c]$. Three cases have to be considered. To do this we need the following definitions:

$$\underline{p}_q := \begin{cases} \min\{p_q \mid \lambda\{H^{p_U}(t) \leq p_q\}K \geq t_2^0\}, & \text{if } \lambda\{H^{p_U}(t) \leq 0\}K \geq t_2^0, \\ 0, & \text{otherwise,} \end{cases}$$

$$b^{\max} := \max\{b \mid \underline{p}_q \geq H^{p_U}(b)\},$$

where λ denotes the Lebesgue measure on $[0, 1]$.

Case 1: $H^{p_U}(t) > 0$ for all $t > 0$. ($\Rightarrow \underline{p}_q = 0$, $b^{\max} = 0$)
In this case, $q(0) \geq t_2^0$. Otherwise $p_q(0) + m_q(t) < H^{p_U}(t)$ for all $t > 0$. This would imply $q(1) = q(0) < 1$, a contradiction. Suppose $u(t) = K$ for a maximal interval $[\underline{t}, \bar{t}]$. By Lemma 1.A.14, $\underline{t} > 0$ would imply $q(\underline{t}) < t_2^0$. Hence $\underline{t} = 0$. Also by Lemma 1.A.14, $q(t) \leq x_{p_U}(t)$ for all $t \in [\underline{t}, \bar{t}]$ and hence $q(0) < x_{p_U}(0)$. This implies $p_q(0) + m_q(t) < H^{p_U}(t)$ for t close to zero, contradicting $u(t) = K$. Hence $u(t) < K$ for all $t \in [0, 1]$. This requires $p_q(0) + m_q(t) \leq H(t)$ for all t by (1.A.24)–(1.A.26), and by Reid's Lemma, we have $\tilde{M}_2(q(t)) = M^{p_U}_{[0,c]}(t)$ for all $t \in [0, c]$. With $\underline{b} = \bar{b} = 0$, this shows $\tilde{M}_2(q(t)) = \bar{M}^{p_U}_{[\bar{b},c]}(t)$ for all $t \in [\bar{b}, c]$ in case 1.

Case 2: $H^{p_U}(t) \leq 0$ for some $t > 0$ and $M_1^{p_U}(b^{\max}) = 0$.
In this case, $q(b^{\max}) = t_2^0$. Suppose to the contrary that $q(b^{\max}) < t_2^0$. This implies $p_q(0) \leq \underline{p}_q$. Hence $p_q(0) + m_q(t) \leq \underline{p}_q < H^{p_U}(t)$ for all $t > b^{\max}$. This is a contradiction to optimality. Next, suppose that $q(b^{\max}) > t_2^0$. This implies $p_q(0) \geq \underline{p}_q$ and therefore $p_q(0) + m_q(b^{\max}) > H^{p_U}(b^{\max})$. Therefore $b^{\max}$ is contained in an interval $[\underline{t}, \bar{t}]$ where $u(t) = K$. By Lemma 1.A.14, this is a contradiction. Therefore $q(b^{\max}) = t_2^0$. By (iii) we must have $p_q(0) = \underline{p}_q$ and hence $p_q(0) + m_q(b^{\max}) = \underline{p}_q = H^{p_U}(b^{\max})$. Set $\underline{b} = \bar{b} = b^{\max}$. Lemma 1.A.14 also implies that $p_q(0) + m_q(t) \leq H^{p_U}(t)$ for all $t \in [b^{\max}, c]$. Reid's Lemma then implies that $\tilde{M}_2(q(t)) = \bar{M}^{p_U}_{[\bar{b},c]}(t)$ for all $t \in [\underline{b}, c]$ for case 2.

Case 3: $H^{p_U}(t) \leq 0$ for some $t > 0$ and $M_1^{p_U}(b^{\max}) > 0$.
In this case, $q(b^{\max}) > t_2^0$ because otherwise $q(1) = q(b^{\max}) < 1$, which is a contradiction. This implies $\underline{b} < b^{\max}$ and $p_q(0) \geq \underline{p}_q$. Since $p_q(0) \geq \underline{p}_q$, $p_q(0) + m_q(b^{\max}) > H(b^{\max}) = \underline{p}_q$. Hence $b^{\max}$ is in the interior of a maximal interval $[\underline{t}, \bar{t}]$ such that $u(t) = K$ for all $t \in [\underline{t}, \bar{t}]$. By Lemma 1.A.14, $q(\underline{t}) < t_2^0$. This implies that $\underline{b} \in (\underline{t}, b^{\max})$. By Lemma 1.A.11, $p_q(0) + m_q(\bar{t}) = H(\bar{t})$ and by Lemma 1.A.14, $p_q(0) + m_q(t) \leq H(t)$, for $t \in [\bar{t}, c]$. Hence, we can set $\bar{b} = \bar{t}$ and have thus shown $\tilde{M}_2(q(t)) = \bar{M}^{p_U}_{[\bar{b},c]}(t)$ for all $t \in [\bar{b}, c]$ for case 3.

Claim: $\tilde{M}_2(q(t)) = \bar{M}^{p_U}_{[\bar{b},1]}(t)$ for all $t \in [\bar{b}, c]$.
Note that $\bar{M}^{p_U}_{[\bar{b},1]}(c) \leq \bar{M}^{p_U}_{[\bar{b},c]}(c)$. To show the converse, note that as q is constant on

$[c, 1]$, $p_q(0) + m_q(t) + \mu[0, t) \leq H^{p_U}(t)$ for a.e. $t \geq c$. This implies

$$p_q(0) + m_q(c) + (t - c)\overline{v} + \mu[c, t) \leq H^{p_U}(c) + \int_c^t M_1^{p_U}(s)ds,$$

$$\Leftrightarrow \qquad \int_c^t M_1^{p_U}(s)ds \geq \overline{v}(t - c) + \mu[c, t). \tag{1.A.31}$$

If $\bar{M}^{p_U}_{[\bar{b},1]}(c) < \overline{v}$, then $\int_c^t M_1^{p_U}(s)ds = H^{p_U}(t) - H^{p_U}(c) \leq H^{p_U}(t) - \bar{H}^{p_U}(c) < (t - c)\overline{v}$ for some $t > c$. This would contradict (1.A.31) so we must have $\bar{M}^{p_U}_{[\bar{b},1]}(c) \geq \overline{v} = \bar{M}^{p_U}_{[\bar{b},c]}(c)$. If $\bar{M}^{p_U}_{[\bar{b},c]}(c) = \bar{M}^{p_U}_{[\bar{b},1]}(c)$ we must have $\bar{M}^{p_U}_{[\bar{b},c]}(t) = \bar{M}^{p_U}_{[\bar{b},1]}(t)$ for all $t \in [\bar{b}, c]$. This proves the claim and $c = \min\{t \mid \bar{M}^{p_U}_{[\bar{b},1]}(t) = \overline{v}\}$ follows immediately. Hence we have shown (i).

It remains to show (ii): $\underline{p}_q \to \min_{t\in[0,1]} H^{p_U}(t)$ as $K \to \infty$. This implies that $b^{\max} \to \overline{t}_1^0$. Furthermore $\overline{b} \geq b^{\max} \geq \underline{b}$ and $\underline{b} - \overline{b} < \frac{1}{K}$. Hence $\underline{b} \to \overline{t}_1^0$ and $\overline{b} \to \overline{t}_1^0$ as $K \to \infty$. □

Now we can turn to the limiting solution as $K \to \infty$.

Proof of Theorem 1.6.5. The reduced form of $\bar{x}_i$ as defined in (1.6.3) is

$$\bar{q}_1(v_1, 2) = \begin{cases} 0, & \text{if } \bar{J}_1^{p_U}(v_1) < 0 \\ \underline{x}_1^0 F_2(v_2^0), & \text{if } \bar{J}_1^{p_U}(v_1) = 0 \\ F_2(J_2^{-1}(\bar{J}_1^{p_U}(v_1)), & \text{if } 0 < \bar{J}_1^{p_U}(v_1) \leq \overline{v}, \\ 1, & \text{otherwise,} \end{cases}$$

$$\bar{q}_2(v_2, 1) = \begin{cases} 0, & \text{if } J_2(v_2) < 0, \\ F_1((\bar{J}_1^{p_U})^{-1}(J_2(v_2))), & \text{otherwise.} \end{cases}$$

Changing variables, we have

$$\bar{q}_1(t) = \begin{cases} 0, & \text{if } t < \underline{t}_1^0, \\ \underline{x}_1^0 t_2^0, & \text{if } t \in [\underline{t}_{1,}^0 \overline{t}_1^0], \\ M_2^{-1}(\bar{M}_1^{p_U}(t)), & \text{if } 0 < \bar{M}_1^{p_U}(t) \leq \overline{v}, \\ 1, & \text{otherwise,} \end{cases}$$

$$\bar{q}_2(t) = \begin{cases} 0, & \text{if } M_2(t) < 0, \\ (\bar{M}_1^{p_U})^{-1}(M_2(t)), & \text{otherwise,} \end{cases}$$

where $\underline{t}_1^0 = \min\{t | \bar{M}_1^{p_U}(t) \geq 0\}$.

Obviously, $\bar{q}_2(t) = \bar{q}_1^{-1}(t)$ if $t \geq t_2^0$ and $\bar{q}_2(t) = 0$ otherwise. Therefore, by Lemma 1.A.2, we only have to show optimality of $\bar{q}_1$. Let (q_1^n, q_2^n) be a sequence of optimal solutions of $\mathcal{P}_2^{K_n}$ where $\overline{K} < K_n \to \infty$ as $n \to \infty$. Denote the adjoint variables in these solutions by p_U^n and p_q^n, respectively, and let (q_1, q_2) be the a.e.-limit of the sequence. By Theorem 1.A.4, (q_1, q_2) is an optimal solution. We will show that $(\bar{q}_1, \bar{q}_2)$ yields the same expected revenue as the limit of any such sequence. Since

$\bar{M}^{p_U}_{[\bar{t}^0_1,1]}(t) = \bar{M}^{p_U}_1(t)$ for all $t \in [\bar{t}^0_1, 1]$, Lemma 1.A.17 implies that $q_1(t) = \bar{q}_1(t)$ for $t > \bar{t}^0_1$ where $p_U = \lim_{n\to\infty} p^n_U$.

Next we consider the limiting solution for $t < \bar{t}^0_1$. If $\underline{t}^1_0 > 0$, then $q_1(0) = 0$ and $u(t) = 0$ for $t \leq \underline{t}^0_1$ as for $\bar{q}_1$. Now suppose that $\underline{t}^0_1 < \bar{t}^0_1$.

Claim: If $q_1(t)$ is not constant at $t \in [\underline{t}^0_1, \underline{t}^0_1]$, then $H^{p_U}(t) = \min_\theta H^{p_U}(\theta)$. Suppose to the contrary that $H^{p_U}(t) > \min_\theta H^{p_U}(\theta)$. Then there exist $\varepsilon > 0$ and $\delta > 0$ such that $H^{p_U}(\tau) > \min_\theta H^{p_U}(\theta) + \delta$ for all $\tau \in (t-\varepsilon, t+\varepsilon)$. Since $p^n_q(0) \to \min_\theta H^{p_U}(\theta)$ for $n \to \infty$, there exists $N > 0$ and $\varepsilon' \in (0, \varepsilon)$ such that for all $n > N$, $p^n_q(0) < H^{p^n_U}(\tau)$ for all $\tau \in (t-\varepsilon', t+\varepsilon')$. This implies that q^n_1 is constant on $(t-\varepsilon', t+\varepsilon')$ for $n > N$ and hence q_1 is constant on $(t-\varepsilon', t+\varepsilon')$ which is a contradiction. This proves the claim.

Now set $\underline{q}^0_1 = \left[(v_1(\bar{t}^0_1) - v_1(\underline{t}^0_1))\right]^{-1} \int_{\underline{t}^0_1}^{\bar{t}^0_1} q_1(s) v_1'(s) ds$ and let $[\underline{t}, \bar{t}]$ be the interval where $q_1(t) = \underline{q}^0_1$ (if $q_1(t) \neq \underline{q}^0_1$ for all t, set $\underline{t} = \bar{t}$ such that $q_1(t) < \underline{q}^0_1$ if $t < \underline{t}$ and $q_1(t) > \underline{q}^0_1$ if $t > \underline{t}$). With this definition, $q_1(t) < \underline{q}^0_1$ for $t < \underline{t}$ and $q_1(t) > \underline{q}^0_1$ for $t > \underline{t}$, and q_1 is not constant at $\underline{t}$ and $\bar{t}$. The claim implies that $[\underline{t}^0_1, \underline{t}]$ and $[\bar{t}, \bar{t}^0_1]$ are unions of intervals $[a, b]$ such that either $M^{p_U}_1(t) = 0$ for all $t \in [a, b]$, or q_1 is constant on $[a, b]$ and $\int_a^b M^{p_U}_1(t) dt = 0$. Hence, setting $q_1(t) = \underline{q}^0_1$ does not change the value of the objective and by definition of $\underline{q}^0_1$, $U_1(1)$ is left unchanged. Since, $\underline{q}^0_1 = \underline{x}^0_1 t^0_2$, the (q_1, q_2) yields the same expected revenue as $(\bar{q}_1, \bar{q}_2)$. Uniqueness of p_U and $\underline{x}^0_1$ are obvious.

For the proof of (ii) and (iii) note that π_2 can be written as

$$\pi_2(\bar{U}) = \int_0^1 \left[\bar{x}_{p_{\bar{U}}}(t) M_1(t) + \int_{\bar{x}_{p_{\bar{U}}}(t)}^1 \tilde{M}_2(q) dq \right] dt.$$

We first show that $\pi_2(\bar{U})$ is Lipschitz. For $\bar{U}' > \bar{U}$,

$$\begin{aligned}
\left|\pi_2(\bar{U}') - \pi_2(\bar{U})\right| &= \left| \int_0^1 \int_{\bar{x}_{p_{\bar{U}}}(t)}^{\bar{x}_{p_{\bar{U}'}}(t)} M_1(t) - \tilde{M}_2(q) dq dt \right|, \\
&\leq \int_0^1 \left| \int_{\bar{x}_{p_{\bar{U}}}(t)}^{\bar{x}_{p_{\bar{U}'}}(t)} \underbrace{M_1(t) - \tilde{M}_2(q)}_{|.| \leq M < \infty} dq \right| dt, \\
&\leq M \int_0^1 \bar{x}_{p_{\bar{U}'}}(t) - \bar{x}_{p_{\bar{U}}}(t) dt, \\
&\leq \frac{M}{\underline{v}_1'} (\bar{U}' - \bar{U}),
\end{aligned}$$

where $\underline{v}_1' = \min_{t\in[0,1]} v_1'(t) > 0$ by our assumptions on the type distributions. Next we show that $\pi_2'(\bar{U}) = -p_U$. for almost every $\bar{U}$. For $h > 0$,

$$\begin{aligned}
\frac{1}{h}(\pi_2(\bar{U}+h) - \pi_2(\bar{U})) &= \frac{1}{h}\int_0^1 \int_{\bar{x}_{p_{\bar{U}}}(t)}^{\bar{x}_{p_{\bar{U}+h}}(t)} M_1(t) - \tilde{M}_2(q)dqdt, \\
&= \frac{1}{h}\int_{\underline{t}_1^0(\bar{U}+h)}^{c(\bar{U})} \int_{\bar{x}_{p_{\bar{U}}}(t)}^{\bar{x}_{p_{\bar{U}+h}}(t)} M_1(t) - \tilde{M}_2(q)dqdt, \\
&= -p_{\bar{U}}\frac{1}{h}\underbrace{\int_{\underline{t}_1^0(\bar{U}+h)}^{c(\bar{U})} \int_{\bar{x}_{p_{\bar{U}}}(t)}^{\bar{x}_{p_{\bar{U}+h}}(t)} v_1'(t)dqdt}_{=h} + \\
&\quad + \int_{\underline{t}_1^0(\bar{U}+h)}^{c(\bar{U})} \frac{1}{h}\int_{\bar{x}_{p_{\bar{U}}}(t)}^{\bar{x}_{p_{\bar{U}+h}}(t)} M_1^{p_U}(t) - \tilde{M}_2(q)dqdt.
\end{aligned}$$

A similar expression can be derived for $h < 0$. $\underline{t}_1^0$ and c are are continuous in $\bar{U}$ for almost every $\bar{U}$ (for all $\bar{U}$ if $M_1^{p_U}$ is strictly increasing). Hence, by the Lebesgue differentiation theorem and dominated convergence, for almost every $\bar{U}$ (every $\bar{U}$ if $M_1^{p_U}$ is strictly increasing),

$$\begin{aligned}
\pi_2'(\bar{U}) = \lim_{h\to 0}\frac{1}{h}(\pi_2(\bar{U}+h) - \pi_2(\bar{U})) &= -p_{\bar{U}} + \int_{\underline{t}_1^0}^{c} M_1^{p_U}(t) - \tilde{M}_2(\bar{x}_{p_{\bar{U}}}(t))dt, \\
&= -p_{\bar{U}} + \int_{\underline{t}_1^0}^{c} \bar{M}_1^{p_U}(t) - \tilde{M}_2(\bar{x}_{p_{\bar{U}}}(t))dt, \\
&= -p_{\bar{U}}.
\end{aligned}$$

Since $\pi_2(\bar{U})$ is Lipschitz continuous it is absolutely continuous and $\pi_2(\bar{U}) = \pi_2(0) - \int_0^{\bar{U}} p_U(s)ds$. Therefore, as $p_U(\bar{U})$ is non-decreasing, π_2 is weakly concave. If $\{t|\bar{M}_1^{p_U}(t) = 0\}$ is a singleton $p_U(\bar{U})$ is strictly increasing an hence π_2 strictly concave. □

1.B. Other Proofs

Proof of Lemma 1.3.1. $\hat{x}$ is derived from x as follows. Whenever a buyer is alloted a unit before his deadline is reached, this allotment is postponed to the deadline. Whenever a buyer is alloted unit after his deadline has elapsed, the unit is withheld under the new allocation rule. In all other cases, the new allocation rule is the same as the old one.

This implies that buyers who report their deadline truthfully enjoy the same expected payoff in both mechanisms:

$$\forall a \in \{1, \ldots T\}, d \in \{a, \ldots T\}, \forall v \in [0, \overline{v}] : \quad \sum_{\tau=a}^{d} \hat{q}_a(v,d) = \sum_{\tau=a}^{d} q_a(v,d).$$

On the other hand, for $d' \neq d$, we have

$$\sum_{\tau=a}^{d} \hat{q}_a(v, d') \leq \sum_{\tau=a}^{d} q_a(v, d').$$

Hence,

$$\hat{U}_a(v, d) = U_a(v, d) \geq U_a(v, d, v', d') \geq \hat{U}_a(v, d, v', d').$$

□

Proof of Lemma 1.5.1. The result will be shown separately for the two cases $K = 1$ and $T = 2$.

Case 1 ($K = 1$)*:* To simplify notation, define $c_a^\tau := \max_{j \in \{i \in I_a | d_i = \tau\}} J_a(v_j|\tau)$ and $c_{\leq a}^\tau := \max\{c_1^\tau, \ldots, c_a^\tau\}$. For fixed $i \in I_{\leq \tau}$ define $c_a^{\tau,-i} := \max_{j \in \{l \in I_a \setminus \{i\} | d_l = \tau\}} J_a(v_j|\tau)$ and $c_{\leq a}^{\tau,-i} := \max\{c_1^{\tau,-i}, \ldots, c_a^{\tau,-i}\}$. The results from Chapter 3 (see the remark at the end of Section 3.3) imply that for each state s_t in which the object is still available, there is a unique period $\theta_t \geq t$, in which the object will be allocated if it is allocated to a buyer $i \in I_{\leq t}$. θ_t is determined by

$$\begin{aligned} & c_{\leq t}^\tau \leq E_{s_{\tau+1}}\left[V_{\tau+1}(s_{\tau+1}) | s_\tau = s_t, k_{\tau+1} = 1\right] \qquad \forall \tau < \theta_t, \\ \text{and} \qquad & c_{\leq t}^{\theta_t} > E_{s_{\theta_t+1}}\left[V_{\theta_t+1}(s_{\theta_t+1}) | s_{\theta_t} = s_t, k_{\theta_t+1} = 1\right]. \end{aligned}$$

Furthermore, there is a unique tentative winner $i_t^* \in I_{\leq t}$ in state s_t. i_t^* has deadline $d_{i_t^*} = \theta_t$, and virtual valuation $J_{a_{i_t^*}}(v_{i_t^*}|d_{i_t^*}) = c_{\leq t}^{\theta_t}$. For all other buyers in $I_{\leq t}$, the winning probability conditional on s_t is zero. Hence, in order to compute the value function in state s_t, H_t can be replaced $(\theta_t, c_{\leq t}^{\theta_t})$:

$$\begin{aligned} V_T(s_T) &= \mathbf{1}_{\{k_T=1\}} \max\{0, J_{a_{i_T^*}}(v_{i_t^*}|T)\} \\ &= \mathbf{1}_{\{k_T=1\}} \max\{0, c_{\leq T}^T\} =: V_T((\theta_T, c_{\leq T}^T)), \\ \text{and} \qquad V_t(s_t) &= \begin{cases} J_{a_{i_t^*}}(v_{i_t^*}|t), & \text{if } d_{i_t^*} = t \text{ and } k_t = 1, \\ E_{s_{t+1}}\left[V_{t+1}(s_{t+1}) | s_t\right], & \text{otherwise.} \end{cases} \\ &= \begin{cases} c_{\leq t}^{\theta_t} & \text{if } \theta_i = t \text{ and } k_t = 1, \\ E_{s_{t+1}}\left[V_{t+1}(s_{t+1}) | (\theta_t, c_{\leq t}^{\theta_t}), k_{t+1} = 1\right], & \text{otherwise.} \end{cases} \\ &=: V_t((\theta_t, c_{\leq t}^{\theta_t})), \end{aligned}$$

In order to compute the critical virtual valuation of the winning buyer, of course, more information is needed. Suppose buyer i arrives in period a_i and $k_{a_i} = 1$. Then, he wins in period d_i, if and only if

$$\begin{aligned} \forall t \in \{a_i, \ldots, d_i\} \qquad & c_{\leq t}^t \leq V_t((d_i, J_{a_i}(v_i|d_i)), k_t = 1), \\ \text{and} \qquad & J_{a_i}(v_i|d_i) > E_{s_{d_i+1}}\left[V_{d_i+1}(s_{d_i+1}) | s_{d_i}, k_{d_i+1} = 1\right], \end{aligned}$$

where we define the expected value in the last line as zero if $d_i = T$. To give an expression for the critical virtual valuation, we define

$$z_t^d(c_{\leq t}^t) = \min\left\{z \geq 0 \middle| c_{\leq t}^t = E_{s_{t+1}}\left[V_{t+1}(s_{t+1}) | (d, z), k_{t+1} = 1\right]\right\}$$

where here and in the following, $E_{s_{t+1}}[\dots|(d,z),k_{t+1}=1] = E_{s_{t+1}}[\dots|(\theta_t, c^{\theta_t}_{\leq\theta_t}) = (d,z), k_{t+1}=1]$. With this definition we have

$$\zeta^i_{a_i,d_i}(H_{d_i},1) = \max\Big\{ z^{d_i}_{a_i}(c^{a_i,-i}_{\leq a_i}), \dots, z^{d_i}_{d_i-1}(c^{d_i-1,-i}_{\leq d_i-1}), c^{d_i,-i}_{\leq d_i}, \\ E_{s_{d_i+1}}\left[V_{d_i+1}(s_{d_i+1})|H_{d_i,-i}, k_{d_i+1}=1\right]\Big\}.$$

CLAIM 1.B.1. $z^d_t(c^t_{\leq t}) = z^d_{d-1}(z^{d-1}_t(c^t_{\leq t}))$.

PROOF OF CLAIM 1.B.1. If $z^d_t(c^t_{\leq t}) = 0$ then also $z^{d-1}_t(c^t_{\leq t}) = 0$ and therefore $z^d_{d-1}(z^{d-1}_t(c^t_{\leq t})) = 0$. Suppose $z^d_t(c^t_{\leq t}) > 0$ and $z^{d-1}_t(c^t_{\leq t}) > 0$. This implies

$$\begin{aligned} c^t_{\leq t} &= E_{s_{t+1}}\left[V_{t+1}(s_{t+1})\big|(d-1, z^{d-1}_t(c^t_{\leq t})), k_{t+1}=1\right] \\ &= E_{s_{t+1}}\left[V_{t+1}(s_{t+1})\big|(d, z^d_t(c^t_{\leq t})), k_{t+1}=1\right] \end{aligned}$$

The second equation is equivalent to

$$\begin{aligned} &E_{s_{t+1}}\Big[\max\Big\{c^{t+1}_{t+1}, E_{s_{t+2}}\Big[\dots \\ &\quad E_{s_{d-1}}\left[\mathbf{1}_{\{k_{d-1}=1\}}\max\left\{c^{d-1}_{\leq d-1}, z^{d-1}_t(c^t_{\leq t}), E_{s_d}\left[V_s(s_d)|H_{d-1},k_d=1\right]\right\}\big|s_{d-2}\right] \\ &\quad\quad \dots\big|s_{t+1}\Big]\Big\}\big|I_{\leq t}=\emptyset, k_{t+1}=1\Big] \\ =&E_{s_{t+1}}\Big[\max\Big\{c^{t+1}_{t+1}, E_{s_{t+2}}\Big[\dots \\ &\quad E_{s-1}\Big[\mathbf{1}_{\{k_{d-1}=1\}}\max\Big\{c^{d-1}_{\leq d-1}, E_{s_d}\left[V_s(s_d)\big|(d, z^d_t(c^t_{\leq t})), k_d=1\right], \\ &\quad\quad E_{s_d}\left[V_s(s_d)|H_{d-1},k_d=1\right]\Big\}\big|s_{d-2}\Big] \quad \dots\big|s_{t+1}\Big]\Big\}\big|I_{\leq t}=\emptyset, k_{t+1}=1\Big] \end{aligned} \tag{1.B.1}$$

Now suppose by contradiction that $z^d_t(c^t_{\leq t}) > z^d_{d-1}(z^{d-1}_t(c^t_{\leq t}))$. This implies

$$E_{s_d}\left[V_d(s_d)\big|(d, z^d_t(c^t_{\leq t})), k_d=1\right] > z^{d-1}_t(c^t_{\leq t}).$$

Conditional on $I_{\leq t}=\emptyset$, with positive probability the realization of s_{d-1} is such that $k_{d-1}=1$ and

$$\begin{aligned} E_{s_d}\left[V_d(s_d)\big|(d, z^d_t(c^t_{\leq t})), k_d=1\right] &> \max\left\{c^{d-1}_{\leq d-1}, E_{s_d}\left[V_d(s_d)|H_{d-1},k_d=1\right]\right\} \\ &> z^{d-1}_t(c^t_{\leq t}). \end{aligned}$$

But this contradicts (1.B.1). Similarly, $z^d_t(c^t_{\leq t}) < z^d_{d-1}(z^{d-1}_t(c^t_{\leq t}))$ leads to a contradiction. This proves the claim. □

Now, consider the critical virtual valuation for deadline $d_i - 1$:

$$\zeta^i_{a_i,d_i-1}(H_{d_i-1},1) = \max\Big\{ z^{d_i-1}_{a_i}(c^{a_i,-i}_{\leq a_i}), \dots, z^{d_i-1}_{d_i-2}(c^{d_i-2,-i}_{\leq d_i-2}), c^{d_i-1,-i}_{\leq d_i-1}, \\ E_{s_{d_i}}\left[V_{d_i}(s_{d_i})|H_{d_i-1,-i}, k_{d_i}=1\right]\Big\}.$$

Claim 1.B.1 allows us to replace the cutoff values $z^{d_i-1}_\tau$. $\forall\tau\in\{a_i,\dots,d_i-2\}$:

$$\begin{aligned} x^{d_i-1}_\tau(c^{\tau,-i}_{\leq\tau}) &= E_{s_{d_i}}\left[V_{d_i}(s_{d_i})\big|(d_i, z^{d_i}_{d_i-1}(z^{d_i-1}_\tau(c^{\tau,-i}_{\leq\tau}))), k_{d_i}=1\right], \\ &= E_{s_{d_i}}\left[V_{d_i}(s_{d_i})\big|(d_i, z^{d_i}_\tau(c^{\tau,-i}_{\leq\tau})), k_{d_i}=1\right]. \end{aligned}$$

Hence,

$$\begin{aligned}
&\zeta^i_{a_i,d_i-1}(H_{d_i-1},1)=\\
&=\max\Big\{E_{s_{d_i}}\left[V_{d_i}(s_{d_i})\big|(d_i,z^{d_i}_{a_i}(c^{a_i,-i}_{\leq a_i})),k_{d_i}=1\right],\ldots\\
&\qquad\ldots,E_{s_{d_i}}\left[V_{d_i}(s_{d_i})\Big|(d_i,z^{d_i}_{d_i-1}(c^{d_i-1,-i}_{\leq d_i-1})),k_{d_i}=1\right],E_{s_{d_i}}\left[V_{d_i}(s_{d_i})|H_{d_i-1,-i},k_{d_i}=1\right]\Big\}\\
&=E_{s_{d_i}}\Big[\max\Big\{z^{d_i}_{a_i}(c^{a_i,-i}_{\leq a_i}),\ldots,z^{d_i}_{d_i-1}(c^{d_i-1,-i}_{\leq d_i-1}),\\
&\qquad\qquad c^{d_i}_{\leq d_i},E_{s_{d_i+1}}\left[V_{d_i+1}(s_{d_i+1})|H_{d_i,-i},k_{d_i+1}=1\right]\Big\}\Big|H_{d_i-1}\Big]\\
&=E_{H_{d_i}}[\zeta^i_{a_i,d_i}(H_{d_i},1)|H_{d_i-1}].
\end{aligned}$$

As $\zeta^i_{a_i,d_i-1}(H_{d_i-1},1)|H_{d_i-1}$ is deterministic for each H_{d_i-1},

$$\zeta^i_{a_i,d_i-1}(H_{d_i-1},1)|H_{d_i-1}\succ_{\text{SSD}}\zeta^i_{a_i,d_i}(H_{d_i},1)|H_{d_i-1}$$

and the lemma follows.

Case 2 ($T=2$)*:* Now we revert to the notation from the main text and use $c^t_{(K)}$. Let $c^{t,-i}_{(k)}$ denote the k$^{\text{th}}$ highest virtual valuation among the buyers with deadline t in $I_{\leq t}\backslash\{i\}$. Fix any state $s_1=(H_1,K)$. Let K_1 denote the number of units that are allocated in period one in state $(H_{1,-i},K-1)$. We distinguish two sub-cases.

Case A—In state $(H_{1,-i},K)$*,* K_1 *units are allocated in the first period:* If, in state $((H_{1,-i},(1,v_i,1)),K)$, buyer i gets a unit in the first period, then the remaining $K-1$ units are allocated as in state $(H_{1,-i},K-1)$. This means that K_1 units are allocated to buyers other than i in period one and $K-K_1-1$ units are retained. Hence, i's virtual valuation must exceed the option value of retaining the $K-K_1^{\text{st}}$ unit. We have

$$\begin{aligned}
\zeta^i_{1,1}(H_1,K)&=E_{s_2}\left[V_2(s_2)|H_{1,-i},k_2=K-K_1\right]-E_{s_2}\left[V_2(s_2)|H_{1,-i},k_2=K-K_1-1\right],\\
&=E_{s_2}\left[\max\left\{0,c^{2,-i}_{(K-K_1)}\right\}\Big|H_1\right].
\end{aligned}$$

In state $((H_{1,-i},(1,v_i,2)),K)$, the number of units that are allocated in the first period must also be K_1. It is obvious that the arrival of buyer i with $d_i=2$ cannot increase the number of units allocated in the first period. On the other hand, suppose that in state $((H_{1,-i},(1,v_i,2)),K)$, only K_1-1 units are allocated in the first period. Then

$$\begin{aligned}
c^{1,-i}_{(K_1)}&\leq E_{s_2}\left[\max\left\{0,c^2_{(K-K_1+1)}\right\}\big|(H_{1,-i},(1,v_i,2))\right],\\
&\leq E_{s_2}\left[\max\left\{0,c^2_{(K-K_1+1)}\right\}\big|(H_{1,-i},(1,\overline{v},2))\right],\\
&=E_{s_2}\left[\max\left\{0,c^2_{(K-K_1)}\right\}\big|H_{1,-i}\right],\\
&<c^{1,-i}_{(K_1)},
\end{aligned}$$

where the last inequality follows from our assumption that in state $(H_{1,-i},K-1)$, K_1 units are allocated in the first period. This is a contradiction. But if K_1 objects are allocated in the first period, then

$$\zeta^i_{1,2}(H_2, K) = \max\left\{0, c^{2,-i}_{(K-K_1)}\right\}.$$

Hence, in case A, $E_{s_2}[\zeta^i_{1,2}(H_2, K)|H_1] = \zeta^i_{1,1}(H_1, K)$ and $\zeta^i_{1,1}(H_1, K)|H_1 \succ_{\text{SSD}} \zeta^i_{1,2}(H_2, K)|H_1$.

Case B—In state $(H_{1,-i}, K)$, $K_1 + 1$ objects are allocated in the first period: Again, if in state $((H_{1,-i}, (1, v_i, 1)), K)$, buyer i gets an object in the first period, then the remaining $K - 1$ objects are allocated as in state $(H_{1,-i}, K - 1)$. Hence, in case B we have

$$\zeta^i_{1,1}(H_1, K) = c^{1,-i}_{(K_1+1)}.$$

In state $((H_{1,-i}, (1, v_i, 2)), K)$, it depends on v_i, how many objects are retained for the second period. Define z by

$$c^{1,-i}_{(K_1+1)} = E_{s_2}\left[\max\left\{0, c^2_{(K-K_1)}\right\}\big|(H_{1,-i}, (1, J_1^{-1}(z|2), 2))\right].$$

If $J_1(v_i|2) \geq z$, then $K - K_1$ objects are retained, otherwise only $K - K_1 - 1$ objects are retained. Hence, we have

$$\zeta^i_{1,2}(H_2, K) = \begin{cases} c^{2,-i}_{(K-K_1)}, & \text{if } z < c^{2,-i}_{(K-K_1)}, \\ z & \text{if } c^{2,-i}_{(K-K_1)} \leq z < c^{2,-i}_{(K-K_1-1)}, \\ c^{2,-i}_{(K-K_1-1)} & \text{if } c^{2,-i}_{(K-K_1-1)} \leq z. \end{cases}$$

Note that for $H_1 = (H_{1,-i}, (1, J_1^{-1}(z|2), 2))$ this equals $\max\left\{0, c^2_{(K-K_1)}\right\}$. Therefore, also in case B, $E_{s_2}[\zeta^i_{1,2}(H_2, K)|H_1] = \zeta^i_{1,1}(H_1, K)$ and $\zeta^i_{1,1}(H_1, K)|H_1 \succ_{\text{SSD}} \zeta^i_{1,2}(H_2, K)|H_1$. □

Proof of Theorem 1.5.2. Consider a buyer i with type (a, v, d), where $a < d \leq T$ and let $d' \in \{1, \ldots, d - 1\}$. Fix the state in the arrival period s_a, and let

$$G(\zeta) = \text{Prob}\left\{\zeta^i_{a,d}(H_d, k_a) \leq \zeta\big|s_a\right\},$$
$$\text{and} \qquad G'(\zeta) = \text{Prob}\left\{\zeta^i_{a,d'}(H_{d'}, k_a) \leq \zeta\big|s_a\right\}.$$

Lemma 1.5.1 implies that G and G' have the same mean and $G' \succ_{\text{SSD}} G$.

(i) Suppose $v = \overline{v}$, $J_a(v|d')$ is strictly concave and $J_a(v|d') \geq J_a(v|d)$ for all $v \in [v^0_a|d', \overline{v}]$. Conditional on s_a the expected payoff of i is given by

$$\begin{aligned} U_a(\overline{v}, d) &= \int_0^{\overline{v}} (\overline{v} - J_a^{-1}(\zeta|d))dG(\zeta), \\ &\leq \int_0^{\overline{v}} (\overline{v} - J_a^{-1}(\zeta|d'))dG(\zeta), \\ &< \int_0^{\overline{v}} (\overline{v} - J_a^{-1}(\zeta|d')dG'(\zeta) = U_a(\overline{v}, d'). \end{aligned}$$

In the second line we have used that $J_a^{-1}(\zeta|d') \leq J_a^{-1}(\zeta|d)$ for $\zeta > 0$. In the third line we have used strict convexity of $J_a^{-1}(\zeta|d')$ as a function of ζ. A similar argument can be made if $J_a(v|d)$ is strictly concave. If $J_a(v|d') > J_a(v|d)$ for all $v < \overline{v}$, the first inequality becomes strict and strict concavity can be replaced by weak concavity.

(ii) Suppose $J_a(v|d')$ is convex and $J_a(v|d) \geq J_a(v|d')$ for all $v \in [v_a^0|d, \overline{v}]$. Conditional on s_a we have

$$\begin{aligned} U_a(v,d) &= \int_0^{J_a(v|d)} (v - J_a^{-1}(\zeta|d))dG(\zeta), \\ &\geq \int_0^{J_a(v|d')} (v - J_a^{-1}(\zeta|d'))dG(\zeta), \\ &= J_a^{-1}(0|d')G(0) + \int_0^{J_a(v|d')} \frac{d}{d\zeta} J_a^{-1}(v|d')\, G(\zeta)d\zeta, \\ &\geq J_a^{-1}(0|d')G'(0) + \int_0^{J_a(v|d')} \frac{d}{d\zeta} J_a^{-1}(v|d')\, G'(\zeta)d\zeta = U_a(v,d') \end{aligned}$$

The last line follows because $\frac{d}{d\zeta} J_a^{-1}(v|d')$ is non-negative and non-increasing and for all non-negative and non-increasing functions $\phi : [0,\overline{v}] \to [0,\overline{v}]$, we have

$$\forall x \in [0,\overline{v}] : \qquad \int_0^x \phi(s)G'(s)ds \leq \int_0^x \phi(s)G(s)ds.$$

For $\phi(s) = \mathbf{1}_{\{s \leq x\}}$ this follows directly from SSD and since any non-increasing function $\phi : [0,\overline{v}] \to [0,\overline{v}]$ can be uniformly approximated by non-increasing step functions the result follows.

A similar argument can be made if $J_a(v|d)$ is convex.

□

Proof of Theorem 1.6.2. Substituting V_2^{opt} into the objective function we get

$$\pi_1(U) = \max_{q_1(.,1)} V_2^{\text{opt}} + \int_0^{\overline{v}} q_1(v,1) \left(J_1(v|1) - V_2^{\text{opt}}\right) f_1(v|1)dv, \tag{1.B.2}$$

Subject to $q_1(v,1) \in [0,1]$, $\forall v \in [0,\overline{v}]$. ($M_1$), ($PE_1$) and ($ICD_U^d$). This is a control problem with state $U_1(v) = U_1(v,1)$ and measurable control $q_1(.) = q_1(v,1)$. The law of motion for the state is $U_1'(v) = q_1(v)$. We account for $q_1(v,1) \in [0,1]$ and (ICD_U^d) by imposing the state constraint $U_1(v) \leq U(v)$, requiring the state to start at zero, $U_1(0) = 0$, and the control to take values between zero and one, $q_1(v) \in [0,1]$. (M) will be neglected for the moment.

The Hamiltonian of this problem is

$$\mathcal{H}(U_1, q_1, p, v) = pq_1 + q_1 \left(J_1(v|1) - V_2^{\text{opt}}\right) f_1(v|1)$$

where p is the adjoint variable of the state U_1. Let (U_1, q_1) be an optimal solution. By the Pontryagin maximum principle (c.f. Clarke (1983, pp. 211-212)) we have that $p(v) = p$ is constant and $p + \mu[0,\overline{v}] = 0$, where μ is a non-negative measure supported on the set $\{v \mid U_1(v) = U(v)\}$. Furthermore, for almost every v, $q_1(v)$

maximizes $\mathcal{H}(U_1(t), q_1, p + \mu[0,v), v)$. This implies that for almost every v,

$$\begin{aligned} & q_1(v) = 1, && \text{if } p + \mu[0,v) + (J_1(v|1) - V_2^{\text{opt}})f_1(v|1) > 0, \\ & q_1(v) \in [0,1], && \text{if } p + \mu[0,v) + (J_1(v|1) - V_2^{\text{opt}})f_1(v|1) = 0, \\ \text{and} \quad & q_1(v) = 0, && \text{if } p + \mu[0,v) + (J_1(v|1) - V_2^{\text{opt}})f_1(v|1) < 0. \end{aligned}$$

Since $p + \mu[0,v) \leq 0$, $q_1(v) = 0$ if $J_1(v|1) < V_2^{\text{opt}}$. But if $J_1(v|1) \geq V_2^{\text{opt}}$, 1.6.1 implies that $(J_1(v|1) - V_2^{\text{opt}})f_1(v|1)$ is strictly increasing. Since $\mu[0,v)$ is non-decreasing, $p + \mu[0,v) + (J_1(v|1) - V_2^{\text{opt}})f_1(v|1) = 0$ implies $p + \mu[0,v') + (J_1(v'|1) - V_2^{\text{opt}})f_1(v'|1) > 0$ for all $v' > v$. Therefore there is a unique value r_1 such that

$$q_1(v_1) = \begin{cases} 0, & \text{if } v_1 < r_1 \\ 1, & \text{if } v_1 > r_1. \end{cases}$$

Obviously, any such solution satisfies (M). r_1 can be determined without resorting to optimal control theory. As the mechanism is deterministic, it is the lowest value such that $J_1(r_1) \geq V_2^{\text{opt}}$ and $U_1(\overline{v}, 1) = \overline{v} - r_1 \leq U(\overline{v})$. This yields the solution stated in the Theorem.

If we set $r_1 = \max\{J_1^{-1}(V_2^{\text{opt}}|1), \overline{v} - \bar{U}\}$ and insert the optimal solution in the objective function we obtain

$$\pi_1(\bar{U}) = \int_{r_1}^{\overline{v}} J_1(v|1)f_1(v|1)dv + V_2^{\text{opt}}F_1(r_1|1).$$

$$\pi_1'(\bar{U}) = \begin{cases} (J_1(\overline{v} - \bar{U}|1) - V_2^{\text{opt}})f_1(\overline{v} - \bar{U}|1), & \text{if } J_1(\overline{v} - \bar{U}|1) > V_2^{\text{opt}}, \\ 0 & \text{otherwise.} \end{cases}$$

For $\bar{U} \to \overline{v} - J_1^{-1}(V_2^{\text{opt}}|1)$ we have $\pi_1'(\bar{U}) \to 0$ since f_1 is bounded. Hence, $\pi_1'(\bar{U})$ is continuous. Using Assumption 1.6.1, we conclude that $\pi_1'(\bar{U})$ is strictly decreasing if $J_1(\overline{v} - \bar{U}|1) > V_2^{\text{opt}}$ and hence π_1 is strictly concave. □

1.C. Reduced Forms

The interim winning probability for period t of a buyer with type (a_i, v_i, d_i) is given by:

$$q_{a_i}^t(v_i, d_i) = \sum_{(N_1,\dots,N_t)\in\{0,\dots,\bar{N}\}^t} \left(\frac{N_{a_i}\nu_{a_i,N_{a_i}}}{\sum_{r=1}^{\bar{N}} r\nu_{a_i,r}} \prod_{a\in\{1,\dots,t\}\setminus a_i} \nu_{a,N_a} \right) \left[\sum_{d_1=a_1}^{T} \cdots \sum_{d_{i-1}=a_{i-1}}^{T} \right.$$
$$\sum_{d_{i+1}=a_{i+1}}^{T} \cdots \sum_{d_{N_{\leq t}}=a_{N_{\leq t}}}^{T} \left(\prod_{j\in I_{\leq t}\setminus i} \rho_{a_j,d_j} \right) \int\!\!\cdots\!\!\int_{v_1\dots v_{i-1}} \int\!\!\cdots\!\!\int_{v_{i+1}\dots v_{N_{\leq t}}} \left(\sum_{\xi_1\in\Phi_1(s_1)} x_1(\xi_1|s_1) \right.$$
$$\left. \left. \dots \sum_{\xi_t\in\Phi_t(s_t)} x_t(\xi_t|s_t)\xi_{t,i} \right) \prod_{j\in I_{\leq t}\setminus i} f_{a_j}(v_j|d_j)dv_j \right].$$

The interim expected payment of a buyer with type (a_i, v_i, d_i) is given by:

$$p_{a_i}(v_i, d_i) = \sum_{(N_1,\dots,N_t)\in\{0,\dots,\bar{N}\}^t} \left(\frac{N_{a_i}\nu_{a_i,N_{a_i}}}{\sum_{r=1}^{\bar{N}} r\nu_{a_i,r}} \prod_{a\in\{1,\dots,t\}\setminus a_i} \nu_{a,N_a} \right) \left[\sum_{d_1=a_1}^{T} \cdots \sum_{d_{i-1}=a_{i-1}}^{T} \sum_{d_{i+1}=a_{i+1}}^{T} \cdots \sum_{d_{N_{\leq t}}=a_{N_{\leq t}}}^{T} \left(\prod_{j\in I_{\leq t}\setminus i} \rho_{a_j,d_j} \right) \int\!\!\cdots\!\!\int_{v_1\dots v_{i-1}} \int\!\!\cdots\!\!\int_{v_{i+1}\dots v_{N_{\leq t}}} \left(\sum_{\xi_1\in\Phi_1(s_1)} x_1(\xi_1|s_1) \right. \right.$$

$$\dots \sum_{\xi_{a_i}\in\Phi_{a_i}(s_{a_i})} x_{a_i}(\xi_{a_i}|s_{a_i}) \left(y_{a_i,i}(s_{a_i},\xi_{a_i}) + \sum_{\xi_{a_i+1}\in\Phi_{a_i+1}(s_{a_i+1})} x_{a_i+1}(\xi_{a_i+1}|s_{a_i+1}) \left(\dots \right.\right.$$

$$\left.\left.\left.\left. \sum_{\xi_T\in\Phi_T(s_T)} x_T(\xi_T|s_T) y_{T,i}(s_T,\xi_T) \right)\right)\right) \prod_{j\in I_{\leq t}\setminus i} f_{a_j}(v_j|d_j)dv_j \right].$$

CHAPTER 2

Asymmetric Reduced Form Auctions

2.1. Introduction

An auction is a mechanism to sell (or buy) an object to (from) one of several bidders. An asymmetric auction has the special feature that its rules treat different bidders or groups of bidders differently. For example in a procurement auction the buyer may want to grant bidders different bonuses according to his monetary assessment of non-price attributes of the bidders' products. In dynamic settings, the auctioneer may want to treat bidders differently if they arrive in different periods. If the auctioneer knows that different (groups of) bidders have different outside options, or budget constraints, he may also want to treat them differently.

Formally, an auction consists of a set of admissible bids for each bidder, an allocation rule and a payment rule. Depending on the profile of submitted bids, the allocation and payment rules determine who will get the object and the payments of each participant. An auction implements a certain social choice function (i.e. allocation and payment rule) if there is an equilibrium in which the outcome coincides with the outcome of the social choice function. By the revelation principle, a social choice function is implementable if and only if the direct mechanism defined by the social choice function has a truth-telling equilibrium. The design problem of finding an optimal auction therefore amounts to the optimal choice of a social choice function subject to (a) incentive compatibility constraints and (b) a feasibility constraint that requires that for each profile of types, the object is allocated only once.

The reduced form of an allocation rule is given by the interim winning probabilities, i.e. the probabilities that an agent wins, conditional on his own type, provided that all other bidders tell the truth. If the desired solution concept is Bayes-Nash-Equilibrium, standard payoff equivalence results characterize incentive compatibility as a condition on the reduced form allocation and payment rules. The feasibility constraint, on the other hand, is a condition on the allocation rule (rather than the reduced form). In order to incorporate both constraints in the mechanism design problem, a condition is needed, that characterizes reduced forms which are implementable by a feasible allocation rule.[1]

[1] In the classic optimal auction problem, this problem does not arise. Incentive compatibility constraints can be used to attach a virtual valuation to each type of a buyer. The optimal mechanism can then be derived by point-wise maximization. This admits a direct application of the feasibility constraint (Myerson, 1981).

For the case of symmetric allocation rules, Border (1991) gives a characterization of implementable reduced forms.[2] A reduced form is shown to be implementable if and only if for all measurable subsets A of the type-space,[3] the probability that a bidder with type in A wins the object is less than or equal to the probability that there is a bidder with type in A. Following Border (2007), this condition shall be called the Maskin-Riley-Matthews (MRM) condition. Border also shows that it suffices to check the MRM condition for a one-dimensional family of subsets, viz. the upper contour sets of the reduced form allocation rule. Bayes-Nash incentive compatibility requires monotonicity of the reduced form, and therefore the upper contour sets take the form of intervals if types are one-dimensional. Border (2007) generalizes the first result to asymmetric auctions for the case of finite type-spaces. In this characterization, the MRM condition must be satisfied for all sets of type profiles. The second result, i.e. a reduction of the condition to a low-dimensional family of sets has not been generalized to the asymmetric case, so far.

In this note, asymmetric reduced form auctions are studied with general type-spaces and the second result is generalized.[4] In the asymmetric case, bidders may have different type spaces and their types need not be identically distributed. There may be groups of bidders that are treated identically by the allocation rule. Bidders in the same group are assumed to be symmetric, i.e. all bidders in the same group have identical type spaces and identically distributed types. The number of groups is denoted by L and hence there are L (possibly) different type spaces.

We show that the MRM condition is necessary and sufficient for feasibility if it is imposed for sets of the form $A^1 \times \ldots \times A^L$, where each A^l is a measurable subset of the type-space of group l. This family of sets is a proper subset of the family considered in Border (2007). Second, we show that for each group l the family of sets A^l can be further restricted to the upper contour sets of group l's reduced form winning probability. For each component of $A^1 \times \ldots \times A^l$, there is only a one-dimensional family of sets. Hence, it is necessary and sufficient that the MRM condition is satisfied for an L-dimensional family of sets. Again, together with incentive compatibility, this yields a tractable characterization of feasibility.[5]

In the next section, the formal model is introduced and the results are stated. Section 2.3 contains the proofs. The general approach to prove the results is the same as in Border (1991). Some generalizations are necessary to incorporate the

[2]Maskin and Riley (1984) use a special case of this result to study the optimal auction problem with risk-aversion. Matthews (1984) conjectured the result proven by Border and proved a partial result. See also (Chen, 1986).

[3]In the symmetric case, all bidders have the same type space and their types are identically distributed.

[4]Daniele Condorelli, Yoen-Koo Che and Jinwoo Kim (private communication) independently derived a similar result.

[5]In Chapter 1, we use this characterization to solve a dynamic mechanism design problem. Pai and Vohra (2008a) use Border's symmetric characterization to analyze optimal auctions with budget constraints.

more complicated asymmetric structure (Lemmas 2.3.4 and 2.3.6). Furthermore some parts are (slightly) simplified by treating the set of feasible allocation rules and the set of feasible reduced forms as subsets of L_2 rather than L_∞.

2.2. Definitions and Results

There are N bidders. Each bidder belongs to one of $L \leq N$ groups. The function $\gamma : \{1, \dots, N\} \to \{1, \dots, L\}$ associates with each bidder i his group $\gamma(i)$. $\gamma^{-1}(l)$ denotes the set of bidders in group l and $N^l = |\gamma^{-1}(l)|$ denotes the number of bidders in group l. (The sets $\gamma^{-1}(1), \dots, \gamma^{-1}(L)$ partition the set of bidders $\{1, \dots, N\}$. Hence $\sum_{l=1}^{L} N^l = N$.) For each group l, there is a probability space $(T^l, \mathcal{T}^l, \mu^l)$. The type of a bidder i is denoted $t^i \in T^l$ where $l = \gamma(i)$. The types of all bidders that belong to group l are identically distributed according to the probability measure μ^l. The space of all type profiles $t = (t^1, \dots, t^N)$ is the product space of all type-spaces and is denoted by $(T, \mathcal{T}, \mu) = (T^{\gamma(1)} \times T^{\gamma(2)} \times \dots \times T^{\gamma(N)}, \mathcal{T}^{\gamma(1)} \otimes \mathcal{T}^{\gamma(2)} \otimes \dots \otimes \mathcal{T}^{\gamma(N)}, \mu = \prod_{i=1}^{N} \mu^{\gamma(i)})$. As usual, a type profile where the type of bidder i is excluded, is denoted by t^{-i} with probability space $(T^{-i}, \mathcal{T}^{-i}, \mu^{-i})$. $\theta = (\theta^1, \dots, \theta^L)$ denotes a profile of L types, one for each group. The associated probability space is $(\hat{T}, \hat{\mathcal{T}}, \hat{\mu}) = (T^1 \times \dots \times T^L, \mathcal{T}^1 \otimes \dots \otimes \mathcal{T}^L, \prod_{l=1}^{L} \mu^l)$.

DEFINITION 2.2.1. (a) An *allocation rule* is a measurable function $q : T \to [0,1]^N$ that satisfies the following feasibility condition for all $t \in T$.

$$\sum_{i=1}^{N} q^i(t) \leq 1. \tag{F}$$

(b) An allocation rule is *group symmetric* if for all $l \in \{1, \dots, L\}$, $i, j \in \gamma^{-1}(l)$ and $t \in T$

$$q^i(t) = q^j(\sigma_{i,j}(t)),$$

where $\sigma_{i,j}$ interchanges the ith with the jth component of its argument.

$q^i(t)$ is the probability that bidder i gets the object if the profile of types is t. The feasibility condition (F) ensures that the object is allocated at most once. The set of all group symmetric allocation rules is denoted by $\mathcal{Q}_0$. The *(group) reduced form* $\hat{Q} : \hat{T} \to [0,1]^L$ of a group symmetric allocation rule q is given by the interim winning probabilities

$$\hat{Q}^l(\theta^l) := \int_{T^{-i}} q^i(\theta^l, t^{-i}) d\mu^{-i}(t^{-i}), \qquad \text{where } \gamma(i) = l.$$

The l^{th} component of $\hat{Q}(\theta)$ is the probability that a bidder from group l gets the object when his type is θ^l. Note that $\hat{Q}^l$ is a function of θ^l rather than θ. In what follows, a function with this property shall be called *diagonal*. A measurable diagonal function $\hat{Q} : \hat{T} \to [0,1]^L$ is *implementable by a group symmetric allocation* *rule* if it is the group reduced form of some $q \in \mathcal{Q}_0$.[6] The set of all such functions is denoted by $\hat{\mathcal{Q}}$.

Let $\mathcal{F}$ denote the set of L-tuples of measurable subsets of the individual type-spaces

$$\mathcal{F} := \left\{(A^1, \dots, A^L) \mid \forall l : A^l \in \mathcal{T}^l\right\}.$$

For each $A \in \mathcal{F}$, $A^1 \times \ldots \times A^L$ is a subset of $\hat{T}$. The converse, however, is not true. This is the difference to the approach taken by Border (2007) who considers general subsets of $\hat{T}$.[7]

$A \in \mathcal{F}$ is called empty if for all $l = 1, \dots, L$, $A^l = \emptyset$. Two elements $A, B \in \mathcal{F}$ are called $\mathcal{F}$-disjoint (denoted $A \cap_{\mathcal{F}} B = \emptyset$) if for all $l = 1, \dots, L$: $A^l \cap B^l = \emptyset$.

We can now state the generalization of Theorem 3.1 in Border (1991) for asymmetric allocation rules.

THEOREM 2.2.2. *Let $\hat{Q} : \hat{T} \to [0,1]^L$ be measurable and diagonal. Then $\hat{Q} \in \hat{\mathcal{Q}}$ if and only if for each $A \in \mathcal{F}$,*

$$\sum_{l=1}^{L} N^l \int_{A^l} \hat{Q}^l(t^l) d\mu^l(t^l) \leq 1 - \prod_{l=1}^{L} \left(1 - \mu^l(A^l)\right)^{N^l}. \tag{2.2.1}$$

As in the symmetric case, the family of sets $A \in \mathcal{F}$ for which (2.2.1) must be checked can be reduced. In the symmetric case it suffices to check a one dimensional family. In the group symmetric case, this family has dimension L.

THEOREM 2.2.3. *Let $\hat{Q} : \hat{T} \to [0,1]^L$ be measurable and diagonal. For each $\alpha \in [0,1]^L$ define $E_\alpha = (E_\alpha^1, \dots, E_\alpha^L)$ by $E_\alpha^l := \{t^l \in T^l \mid \hat{Q}^l(t^l) \geq \alpha^l\}$.*
Then $\hat{Q} \in \hat{\mathcal{Q}}$ if and only if for each $\alpha \in [0,1]^L$,

$$\sum_{l=1}^{L} N^l \int_{E_\alpha^l} \hat{Q}^l(t^l) d\mu^l(t^l) \leq 1 - \prod_{i=1}^{L} \left(1 - \mu^l(E_\alpha^l)\right)^{N^l}.$$

2.3. Proofs

For applications it is convenient to work with group reduced forms but for the proofs, the *bidder* reduced form is more convenient. The bidder reduced form $Q : T \to [0,1]^N$ of a feasible group-symmetric allocation rule q is defined as

$$Q^i(t^i) = \int_{T^{-i}} q^i(t^i, t^{-i}) d\mu^{-i}(t^{-i}).$$

The set of bidder reduced forms of feasible and group symmetric allocation rules is denoted by $\mathcal{Q}$. Note that each $Q \in \mathcal{Q}$ is diagonal and group-symmetric ($Q^i(t) = Q^j(t)$ for all $t \in T$ if $i \in \gamma(j)$). Hence, each $Q \in \mathcal{Q}$ has a representation $\hat{Q} \in \hat{\mathcal{Q}}$ that satisfies $\hat{Q}^l = Q^i$ if $l = \gamma(i)$.

[6]Implementability is not to be confused with usual meaning in the mechanism design literature.
[7]In that paper $L = N$ and therefore $\hat{T} = T$.

As in Border (1991), *hierarchical allocation rules* are an important tool in the proofs. This notion has to be generalized to fit the asymmetric case.

DEFINITION 2.3.1. Let $A_1, \dots, A_K \in \mathcal{F}$ be a family of pairwise $\mathcal{F}$-disjoint sets. The *hierarchical allocation rule* $q_{A_1,\dots,A_K}$ generated by $A_1, \dots, A_K$ is defined as

$$q^i_{A_1,\dots,A_K}(t) := \begin{cases} \frac{1}{|\{j:\, t^j \in A_k^{\gamma(j)}\}|} & \text{if } t^i \in A_k^{\gamma(i)} \text{ and } \nexists j:\ t^j \in A_1^{\gamma(j)} \cup \ldots \cup A_{k-1}^{\gamma(j)} \\ 0 & \text{otherwise.} \end{cases}$$

The sets $A_1, \dots, A_K$ define a hierarchy of types. A^l_k is the set of types of bidders in group l that are at the kth level of the hierarchy. The hierarchical allocation rule $q_{A_1,\dots,A_K}$ works as follows. If there are bidders with types at the first level, the object is given to one of these bidders with equal probability. Otherwise the auctioneer checks whether there are bidders with types at the second level, and in this case, allocates the object with equal probability to one of them. The auctioneer continues until either he has allocated the object or he has checked for bidders at all levels of the hierarchy. In the latter case the object is not sold.

2.3.1. Proof of Theorem 2.2.2. The general approach is the same as in Border (1991). Since μ is a finite measure, $\mathcal{Q}_0$ and $\mathcal{Q}$ are subsets of the Hilbert space $L_2(T, \mu, \mathbb{R}^N)$. For $Q, f \in L_2(T, \mu, \mathbb{R}^N)$ the scalar product is given by $\langle Q, f\rangle = \int_T (Q(t), f(t)) d\mu(t)$, where $(.,.)$ is the Euclidean scalar product in $\mathbb{R}^N$. To simplify notation $L_2(T, \mu, \mathbb{R}^N)$ will be abbreviated as L_2^N, and $L_2(T, \mu, \mathbb{R})$ as L_2.

The feasibility condition (2.2.1) can be written as a condition on $\langle Q, f\rangle$ for certain functions $f \in L_2^N$. To do this write the vector of indicator functions for $(A^1, \dots, A^L) = A \in \mathcal{F}$, as $\chi_A(t) := (\chi_{A^{\gamma(1)}}(t^1), \dots, \chi_{A^{\gamma(N)}}(t^N))$ so that $\chi_A : T \to \{0,1\}^N$. Clearly, $\chi_A \in L_2^N$ if $A \in \mathcal{F}$. Furthermore define $B(A) := 1 - \prod_{l=1}^L \left(1 - \mu^l(A^l)\right)^{N^l}$. If $\hat{Q}$ is diagonal and a representation of Q, (2.2.1) can be rewritten as

$$\langle Q, \chi_A\rangle \le B(A).$$

With this notation, Lemmas 5.1 and 5.3 from Border (1991) can be reproduced for the asymmetric case.

LEMMA 2.3.2 (cf. Lemma 5.1, Border (1991)). *For all $A \in \mathcal{F}$ and all $Q \in \mathcal{Q}$,*

$$\langle Q, \chi_A\rangle \le B(A).$$

LEMMA 2.3.3 (cf. Lemma 5.3, Border (1991)). *Let $Q : T \to [0,1]^N$ be measurable and suppose that the function $f = \sum_{j=1}^M \alpha_j \chi_{A_j}$ with $\alpha_1, \dots, \alpha_M \in \mathbb{R}$ and $A_1, \dots, A_M \in \mathcal{F}$ separates Q from $\mathcal{Q}$. That is, for all $\tilde{Q} \in \mathcal{Q}$:*

$$\left\langle Q, \sum_{j=1}^M \alpha_j \chi_{A_j} \right\rangle > \left\langle \tilde{Q}, \sum_{j=1}^M \alpha_j \chi_{A_j} \right\rangle.$$

Then for some set $A \in \mathcal{F}$, $\langle Q, \chi_A\rangle > B(A)$.

In Lemma 2.3.3, the simple function $f = \sum_{j=1}^{M} \alpha_j \chi_{A_j}$ is *diagonal* and *group symmetric*. The following Lemma implies that whenever a function $\tilde{f} \in L_2^N$ separates Q from $\mathcal{Q}$ and Q is diagonal and group symmetric, then there exists a diagonal group symmetric function $f \in L_2^N$ that separates $\overline{Q}$ from $\mathcal{Q}$.

LEMMA 2.3.4. *For every $\tilde{f} \in L_2^N$, there exist a diagonal and group symmetric $f \in L_2^N$ such that $\left\langle Q, \tilde{f} \right\rangle = \langle Q, f \rangle$ for all diagonal and group symmetric $Q \in L_2^N$.*

PROOF. Let $Q \in L_2^N$ be diagonal and group symmetric with representation $\hat{Q} : \hat{T} \to [0,1]^L$. Then,

$$\begin{aligned}
\left\langle Q, \tilde{f} \right\rangle &= \int_T \sum_{i=1}^{N} \tilde{f}^i(t) Q^i(t) \mu(t), \\
&= \sum_{i=1}^{N} \int_{T^i} \left(\int_{T^{-i}} \tilde{f}^i(t^i, t^{-i}) Q^i(t^i, t^{-i}) d\mu^{-i}(t^{-i}) \right) d\mu^{\gamma(i)}(t^i), \\
&= \sum_{i=1}^{N} \int_{T^i} \underbrace{\left(\int_{T^{-i}} \tilde{f}^i(t^i, t^{-i}) d\mu^{-i}(t^{-i}) \right)}_{=:\xi^i(t^i)} Q^i(t^i) d\mu^{\gamma(i)}(t^i), \\
&= \sum_{i=1}^{N} \int_{T^i} \xi^i(t^i) Q^i(t^i) d\mu^{\gamma(i)}(t^i), \\
&= \sum_{l=1}^{L} \sum_{i \in \gamma^{-1}(l)} \int_{T^l} \xi^i(t^l) \hat{Q}^l(t^l) d\mu^l(t^l), \\
&= \sum_{l=1}^{L} \int_{T^l} \underbrace{\left(\sum_{i \in \gamma^{-1}(l)} \xi^i(t^l) \right)}_{=:N^l \hat{f}^l(t^l)} \hat{Q}^l(t^l) d\mu^l(t^l), \\
&= \sum_{i=1}^{N} \int_{T^i} f^i(t^i) Q^i(t^i) d\mu^{\gamma(i)}(t^i) = \langle Q, f \rangle.
\end{aligned}$$

$\xi : T \to [0,1]^N$ is diagonal by definition and therefore $\hat{f} : \hat{T} \to [0,1]^L$ is also diagonal. With $f : T \to [0,1]^N$ defined as $f^i(t) = \hat{f}^{\gamma(i)}(t^i) = \frac{1}{N^{\gamma(i)}} \sum_{j:\gamma(j)=\gamma(i)} \xi^j(t^i)$, the desired diagonal and group symmetric function is obtained. □

LEMMA 2.3.5. *$\mathcal{Q}_0$ and $\mathcal{Q}$ are weakly compact subsets of L_2^N.*

PROOF. The proof is very similar to the proof of Lemma 5.4 in Border (1991). Since we work with the Hilbert-Space L_2^N and $\mathcal{Q}_0$ is bounded, we have that every sequence (q_n) in $\mathcal{Q}_0$ has a weakly convergent subsequence (with limit in L_2^N). Following Border, it can be shown that $\mathcal{Q}_0$ is weakly closed and hence weakly compact. Furthermore, the mapping Λ, that associates an allocation rule with its reduced form is weakly continuous. As $\mathcal{Q}$ is the image of a compact set under Λ, it is also weakly compact. $\square$

Proof of Theorem 2.2.2. Let $\hat{Q} : \hat{T} \to [0,1]^L$ be diagonal. Then it is the representation of a diagonal and group symmetric function $Q : T \to [0,1]^N$.

Lemma 2.3.2 shows that condition (2.2.1) is necessary for feasibility. Conversely suppose $Q \notin \mathcal{Q}$. $\mathcal{Q}$ is a convex and weakly compact subset of L_2^N. By a standard separation theorem,[8] there exists a function $f \in L_2^N$ such that $\langle Q, f \rangle > \max\left\{\left\langle \tilde{Q}, f \right\rangle \middle| \tilde{Q} \in \mathcal{Q}\right\}$. By Lemma 2.3.4, f can be chosen to be diagonal and group symmetric. Furthermore, as the simple functions are dense in L_2, we can take each component f^i to be a simple function. Hence f satisfies the conditions of Lemma 2.3.3 and there exists $A \in F$ such that (2.2.1) is violated. It remains to be shown that for every $Q \in \mathcal{Q}$ there exists a $q^* \in \mathcal{Q}_0$ such that $\Lambda(q^*)(t) = Q(t)$ for *every* $t \in T$ (so far this has been shown for almost every $t \in T$). The proof can be found in Border (1991) and is omitted here. $\square$

2.3.2. Proof of Theorem 2.2.3. As in the symmetric case, the proof of Theorem 2.2.3 starts by showing the result for simple functions which requires a bit more work than the proof for symmetric auctions. For $A \in \mathcal{F}$ and $\chi_A : T \to [0,1]^N$ as above, let $\hat{\chi}_A : \hat{T} \to [0,1]^L$ denote the representation of χ_A.

Lemma 2.3.6. *Let $\hat{Q} : \hat{T} \to [0,1]^L$ be a diagonal simple function with $\hat{Q} = \sum_{k=1}^K \alpha_k \hat{\chi}_{A_k}$ where $\alpha_1 > \alpha_2 > \ldots > \alpha_K \geq 0$, the $A_k \in \mathcal{F}$ are pairwise $\mathcal{F}$-disjoint and $A_1^l \cup \ldots \cup A_K^l = T^l$ for all l. For $l = 1, \ldots, L$ and $k = 1, \ldots, K$ set $E_k^l := A_1^l \cup \ldots \cup A_k^l$ and set $E_0^l := \emptyset$.*
If for each $(k_1, \ldots, k_L) \in \{0, 1, \ldots, K\}^L$:

$$\sum_{l=1}^L N^l \int_{E_{k_l}^l} \hat{Q}(t^l) d\mu^l(t^l) \leq B(E_{k_1}^1, \ldots, E_{k_L}^L), \tag{2.3.1}$$

then $\hat{Q} \in \hat{\mathcal{Q}}$.

Proof. Define $f : [0,1]^L \to [0,1]$ as $f(x) := 1 - \prod_{m=1}^L (1 - x^m)^{N^m}$. This implies

$$\partial_l^2 f(x) = -N^l(N^l - 1)(1 - x^l)^{N^l - 2} \prod_{m \neq l} (1 - x^m)^{N^m} \leq 0, \tag{2.3.2}$$

$$\text{and } B(A) = f(\mu^1(A^1), \ldots, \mu^L(A^L)), \text{ for } A \in \mathcal{F}. \tag{2.3.3}$$

To simplify notation define $c_k^l := \mu^l(E_k^l)$. In order to bound the left hand side of (2.2.1), define $g : [0,1]^L \to [0,1]$ as a continuous and piecewise linear function with $g(0) = 0$. For $x \in (c_{k_1-1}^1, c_{k_1}^1) \times \ldots \times (c_{k_L-1}^L, c_{k_L}^L)$, let the gradient of g be given by

$$\nabla g(x) = \begin{pmatrix} N^1 \alpha_{k_1} \\ \vdots \\ N^L \alpha_{k_L} \end{pmatrix}.$$

[8] cf. Theorem 3.4 in Rudin (1973)

With this definition, $g(x) \leq f(x)$ on the grid of points $G_0 := \{(c^1_{k_1}, \dots, c^L_{k_L}) \mid k_i \in \{0, 1, \dots, L\}\}$:

$$\begin{aligned} \forall x \in G_0: \quad g(x) &= \sum_{l=1}^{L} N^l \sum_{k=1}^{k_l} \alpha_k \mu^l(A^l_k) \\ &= \sum_{l=1}^{L} N^l \int_{E^l_{k_l}} Q^l(t^l)\mu^l(t^l) \\ &\leq f(\mu^l(E^1_{k_1}), \dots, \mu^L(E^L_{k_L})) = f(x). \end{aligned}$$

The second equality follows from the definition of Q and the inequality follows from (2.3.1) and (2.3.3).

Now it is shown inductively, that $g(x) \leq f(x)$ on the sets

$$G_n := \left\{ x \in [0,1]^L \,\middle|\, L - n \leq |\{j \mid \exists k_j : \, x^j = c^j_{k_j}\}| \right\},$$

for $n = 1, \dots, L$.[9] Observe that $G_L = [0,1]^L$. Suppose that $g(x) \leq f(x)$ for all $x \in G_{n-1}$. Let $x \in G_n$. Then there exist l and k_l such that $\underline{x} = (x^1, \dots, x^{l-1}, c^l_{k_l-1}, x^{l+1}, \dots, x^L)$ and $\overline{x} = (x^1, \dots, x^{l-1}, c^l_{k_l}, x^{l+1}, \dots, x^L)$ are in G_{n-1} and $x = x(\delta) = (1-\delta)\underline{x} + \delta\overline{x}$ for some $\delta \in [0,1]$. As $\underline{x}$ and $\overline{x}$ differ only in the l^{th} coordinate, $f(x(\delta))$ is weakly concave as a function of δ by (2.3.2). Furthermore, as the gradient of g is constant on sets of the form $(c^1_{k_1-1}, c^1_{k_1}) \times \dots \times (c^L_{k_L-1}, c^L_{k_L})$, $g(x(\delta))$ is linear as a function of δ. By the induction hypothesis, $g(\underline{x}) \leq f(\underline{x})$ and $g(\overline{x}) \leq f(\overline{x})$. Therefore also $g(x) \leq f(x)$.

Now, for $A \in \mathcal{F}$ define $h : [0, \mu^1(A^1)] \times \dots \times [0, \mu^L(A^L)] \to [0,1]$ as a continuous and piecewise linear function with $h(0) = 0$. For $x \in (\mu^1(A^1 \cap E^1_{k_1-1}), \mu^1(A^1 \cap E^1_{k_1})) \times \dots \times (\mu^L(A^L \cap E^L_{k_L-1}), \mu^L(A^L \cap E^L_{k_L}))$ let the gradient of h be given by

$$\nabla h(x) = \begin{pmatrix} N^1 \alpha_{k_1} \\ \vdots \\ N^L \alpha_{k_L} \end{pmatrix}.$$

With this definition,

$$h(A) = \sum_{l=1}^{L} N^l \int_{A^l} \hat{Q}^l(t^l) d\mu^l(t^l).$$

Furthermore for all x and all l: $\nabla_l h(x) \leq \nabla_l g(x)$. Therefore $h(x) \leq g(x) \leq f(x)$ which implies (2.2.1) for all sets $A \in \mathcal{F}$ and therefore $\hat{Q} \in \hat{\mathcal{Q}}$ by proposition 2.2.2. □

Proof of Theorem 2.2.3. The proof works as the proof of proposition 3.2 in Border (1991). For the asymmetric case $\hat{Q}$ is approximated by the sequence of simple functions $\hat{Q}_n : \hat{T} \to \mathbb{R}^L$ which is constructed such that $\hat{Q}^l_n(t) = \frac{k}{2^n}$ on $\{t \mid \frac{k}{2^n} \leq \hat{Q}^l(t) < \frac{k+1}{2^n}\}$. □

[9] In the case $L = 3$ and $K = 1$, G_0 are the vertices of the cuboid $[0,1]^3$, G_1 are the edges, G_2 are the surfaces and G_3 is the cuboid itself.

CHAPTER 3

The Dynamic Vickrey Auction

SUMMARY. We construct a simple payment rule that implements the efficient allocation rule for a single indivisible object over T time periods. Buyers arrive randomly over time. Private information is multidimensional because valuations depend on the time at which the object is sold. It is shown that each type has a unique potential winning period and only the valuation for this period is important for the allocation decision. Therefore, types can be reduced to essentially one dimension and there is a natural order on the type space by which buyers can be compared.
These properties allow to define a simple payment rule in which only the winner has non-zero transfers, transfers are ex-post individually rational and can be made online. The payment rule is a generalization of the static Vickrey auction in which the winner pays the lowest valuation for the winning period that would suffice to win. Losers pay nothing.
Furthermore, in each period, there is only one buyer who has a chance to win the object in the future, all other buyers can be dismissed and will never be recalled. This allows to define a generalized ascending auction that implements the efficient allocation rule and the same payment rule as the dynamic Vickrey auction. Both the dynamic Vickrey auction and the generalized ascending auction are periodic ex-post incentive compatible.

3.1. Introduction

Standard auction models usually assume that all potential buyers are available at the same time, and that the valuations of buyers do not depend on the time of the allocation. In many allocation problems, however, time is an important factor. In online auctions, buyers typically arrive over time and since auctions usually last several days, some buyers may not be willing to wait until the end of the auction (for example think of buying a last-minute birthday present). Internet platforms like eBay offer a feature that allows to end the auction immediately for a predetermined price. One explanation why this feature is used, is that buyers are impatient and willing to pay a high price for closing a deal immediately (Mathews, 2004; Gallien and Gupta, 2007).[1] Time preferences of buyers as well as dynamic arrival are important in many other markets. For example, in the housing market, and in the markets for airline tickets or hotel reservations, a long time elapses between the start and the

[1]Another explanation is risk-aversion (Budish and Takeyama, 2001; Hidvégi, Wang, and Whinston, 2006; Reynolds and Wooders, 2006).

end of the selling mechanism, potential buyers arrive over time, and they may have privately known preferences about the time of purchase.

In this chapter, we study the dynamic allocation of a single object over a finite time horizon. The model generalizes the standard independent private values framework. Potential buyers arrive randomly over time, they are long-lived, and the valuation they derive from getting the object may depend on the time of allocation in an arbitrary way. We show that the efficient allocation rule can be implemented by a mechanism with a *simple payment rule* that generalizes the static Vickrey auction (Vickrey, 1961).

The implementability of the efficient allocation rule has already been demonstrated in great generality by Parkes and Singh (2003), Bergemann and Välimäki (2010) and Athey and Segal (2007). To ensure incentive compatibility, expected payments of each buyer have to be equal to the expected change in the welfare enjoyed by the other agents due to the report of the buyer. This is an application of the famous Vickrey-Clarke-Groves (VCG) mechanism to the dynamic framework (Vickrey, 1961; Clarke, 1971; Groves, 1973). Thus, incentive compatibility pins down the *expected* payments conditional on all information available at the time when agents observe their private information (i.e. at their arrival time in this chapter). There are many ways, however, in which *ex-post* payments can be distributed over different states of the world, while maintaining the VCG-property of the expected payments.

The central question of this chapter is whether *simple payment rules* can be defined to implement the efficient allocation rule. By simplicity we mean that,

(A) *only the winner makes a payment,*
(B) payments are *ex-post individually rational*,
(C) the mechanism *never transfers money* to any buyer, and
(D) payments are made *online*, i.e. all information that is needed to determine the payment must be available at the time of allocation.

Note that properties (A)–(C) are fulfilled by standard static auction formats. Moreover, if we leave the ideal world of the abstract mechanism design model, properties (A)–(D) are obviously desirable. Property (A) minimizes the number of transactions. This is important if the buyers or the seller incur transaction costs for financial transactions. Property (B) is important because ex-post individually rational payments are easier to enforce. A winner who has to pay more than his value may feel a strong desire to renege on his bid. Property (C) is convenient because payments to buyers may encourage persons who are not interested in the object, to speculate on getting such subsidies and trying to renege on their bids in the case they are selected as the winner. While it may be possible to prevent such abuse by strict enforcement of the mechanism's rules, this will certainly involve additional costs. The last property (D) seems indispensable. If payments cannot be determined online, this means that additional information has to be collected from future buyers after the winner has already been determined. Incentives for reporting such information are weak. Furthermore, online payments allow to match payments with delivery, which makes it easier to enforce payments.

There are examples of mechanisms in the literature, which fulfill properties (A)–(D).[2] As systematic analysis of implementability by simple payments rules, however, seems to be missing. Only the requirement of online has been posited explicitly in the literature on dynamic mechanism design (Parkes and Singh, 2003).[3]

The Dynamic Vickrey Auction proposed in this chapter yields expected payments that ensure incentive compatibility. At the same time, payments are distributed over different states of the world such that properties (A)–(D) are satisfied. Consequently, the payment of an agent corresponds neither to the expected change in the other agents' welfare as in the *online VCG mechanisms* proposed by Parkes and Singh (2003), nor to the sum of *flow marginal contributions* as in the *dynamic pivot mechanism* proposed by Bergemann and Välimäki (2010). Instead, payments satisfy another property of the static Vickrey (or second-price) auction. The payment of the winner is equal to the lowest valuation for the winning period with which she could have won the object. This valuation is called the *critical type* of the winning buyer. In a static model, the critical type is equal to the second highest valuation. In the dynamic model, however, valuations for getting the object in one period (e.g. today) cannot be compared directly to a valuation for another period (e.g. tomorrow). Instead, valuations for tomorrow are compared to current valuations in terms of the option value of retaining the object until tomorrow. Loosely speaking, the critical type is determined by transforming all valuations using the option value function to make them comparable with valuations for the winning period. The second highest of these transformed valuations is the critical type of the winner. Because of the transformation, in general, the payment differs from the second highest (untransformed) valuation. Since the critical type determines the allocation decision, it can only depend on information that is available in the winning period. Therefore, the payment is not delayed beyond the time of allocation of the object; it can be made *online*.

There is difficulty in the definition of a critical type because the model has a multi-dimensional type-space. Types are multi-dimensional because buyers can have different valuations for different periods. The central result of this chapter is that the information about a buyer's type, that is relevant for determining the efficient allocation, is essentially one-dimensional. For each type, there is a unique period in which she can possibly win the object. Therefore, only the valuation for this period matters for the efficient allocation rule. This reduction to one dimension allows to

[2] For example, sequences of posted prices as in Gershkov and Moldovanu (2008); Gallien (2006) and many other papers lead to simple payment rules. The *dynamic pivot mechanism* proposed by Bergemann and Välimäki (2010) also yields a simple payment rule if it is applied to a scheduling problem.

[3] (Gershkov and Moldovanu, 2009a) demonstrate that the requirement of online payments can destroy the implementability of the efficient allocation rule if a buyer's type is informative about future buyers' types.

consider a *lowest type* in a well-defined way. As a corollary of this result, it is shown that at each point in time, there is only one bidder who has a positive chance of winning the object. All other bidders can be dismissed immediately and will never be recalled. In other words, the efficient allocation rule only needs a queue of bidders of length one. These properties also allow to design a generalized ascending auction that implements the efficient allocation rule in periodic ex-post equilibrium.

The chapter is structured as follows: In Section 3.2, the formal model is introduced, the efficient allocation rule is defined, and the mechanisms proposed by Parkes and Singh (2003) and Bergemann and Välimäki (2010) are discussed. In Section 3.3, it is proven that for each type, there is a unique potential winning period. In Section 3.4, the payment rule of the dynamic Vickrey auction is constructed. Section 3.5 describes the generalized ascending auction. Section 3.6 concludes with a discussion of possible generalizations and relationships of the results to revenue-maximizing auctions.

3.2. The Model

3.2.1. Setup and Notation. A seller wants to sell a single indivisible object within T time periods. In each period $t \in \{1, \dots, T\}$, a random number of buyers n_t arrives. The numbers n_t are independent random variables and the probability that $n_t = k$ is denoted by $\rho^t_k \geq 0$, with $\sum_{k=0}^{\infty} \rho^t_k = 1$. $N_t := \sum_{\tau=1}^{t} n_\tau$ denotes the number of buyers that have arrived in or before period t. Buyers are indexed in the order of arrival. Within periods, indexing is random. $I_t = \{1, \dots, N_t\}$ denotes the set of buyers that arrive in or before period t.

A typical buyer j who arrives in period t, attaches a monetary value of $v^\tau_j \in [0, \overline{v}]$ to the object if she gets it in period $\tau \geq t$, where $\overline{v} > 0$. Each buyer is completely characterized by her vector of valuations starting with her arrival period. Her type is thus $v_j = (v^t_j, \dots, v^T_j)$. Buyers' types are independent random variables with distribution functions $\Phi_t : [0, \overline{v}]^{T-t+1} \to [0, 1]$ and strictly positive densities on $[0, \overline{v}]^{T-t+1}$, where t is the arrival period of the respective buyer. We allow for dependencies between the components of a buyer's type. $(\rho^t_k)_{t=1,\dots T, k\in\mathbb{N}}$ and $(\Phi_t)_{t=1,\dots,T}$ are commonly known by the buyers and the seller. Realizations of type and arrival period are private information of each buyer, and we assume that she knows her complete type in the arrival period. The valuation for the object can depend on the time of allocation as described by the type-vector, but this relationship is completely determined when the buyer arrives.

Prior to the arrival period, the type of a buyer is not known to anybody. To emphasize this informational constraint, we distinguish the type of a buyer who arrives in period t: $(v^t_j, \dots, v^T_j) \in [0, \bar{v}]^{T-t+1}$ from the type of a buyer who arrives in period $\tau < t$ and has the same valuations for periods $t, \dots, T$ but valuations of zero for periods before t: $(0, \dots, 0, v^t_j, \dots, v^T_j) \in [0, \bar{v}]^{T-\tau+1}$.

Buyers are risk-neutral. If a buyer j has to make a total expected payment of p_j, and q_τ is the probability of getting the object in period τ, then her expected utility is given by $\sum_{\tau=t}^{T} v_j^\tau q_\tau - p_j$. The seller's valuation for the object is normalized to zero.

For $\tau \geq t$, let $\eta_\tau(I_t) := \max_{i \in I_t}\{v_i^\tau\}$ be the highest valuation for getting the object in period τ among all buyers in I_t and define $\eta(I_t) := (\eta_t(I_t), \ldots, \eta_T(I_t))$ and $\eta := (\eta(I_1), \ldots, \eta(I_T))$.

3.2.2. The Efficient Allocation Rule under Complete Information. In period t, allocation decisions can only depend on the types of buyers that have already arrived. The value of the object, on the other hand, depends on the time when a buyer gets it. Consequently, the ex-post efficient allocation rule is not feasible. It is not always possible to identify the identity of the buyer with the ex-post highest valuation with certainty, in the period where this valuation can be realized. Instead, we consider the ex-ante efficient allocation rule, i.e. the allocation rule that maximizes the expected value of the utility enjoyed by the buyers, subject to the informational constraint that information about future types cannot be used. This allocation rule is the optimal policy for the following dynamic program.

The *state* (h_t, a_t) at time t, consists of the history of types of all buyers in I_t, denoted h_t, and the availability of the object, denoted $a_t \in \{0, 1\}$. a_t equals zero if the object has already been allocated, and one if the object is still available in period t.

The *set of feasible decisions* x_t, in state (h_t, a_t) is

$$X(h_t, a_t) = \begin{cases} \{0, 1, \ldots, N_t(h_t)\}, & \text{if } a_t = 1, \\ \{0\}, & \text{if } a_t = 0, \end{cases}$$

where $x_t = 0$ means that the object is not allocated in period t and $x_t \in \{1, \ldots, N_t\}$ means that the object is allocated to buyer x_t.[4] A *policy* is a family of decision rules $x = (x_t(.,.))_{t=1}^T$ where $x_t(h_t, a_t) \in X(h_t, a_t)$. The total value (or welfare) in period T at history h_T after decisions $x = (x_1, \ldots, x_T)$ is given by $\sum_{t=1}^T v_{x_t}^t$ (define $v_0^t := 0$, $\forall t$). The *efficient allocation rule* is the optimal policy x^*, for the dynamic program $\mathcal{P}$:

$$\max_{(x_\tau(.,.))_{\tau=1}^T} E\left[\sum_{\tau=1}^T v_{x_t(h_t,a_t)}^\tau\right]. \qquad (\mathcal{P})$$

The value function is given by $V_{T+1}^*(h_{T+1}, a_{T+1}) = 0$ and

$$\begin{aligned} V_t^*(h_t, a_t) &= \max_{(x_\tau(.,.))_{\tau=t}^T} E\left[\sum_{\tau=t}^T v_{x_\tau(h_\tau,a_\tau)}^\tau\right] \\ &= v_{x_t^*(h_t,a_t)}^t + E\left[V_{t+1}^*(h_{t+1}, a_{t+1}) \mid h_t, a_t, x_t^*(h_t, a_t)\right]. \end{aligned}$$

The efficient allocation rule always allocates to a buyer who has the highest valuation for the selling period. Hence, the value function can be considered as a

[4]Without loss of efficiency we can restrict the decision space to deterministic decisions.

function of $\eta(I_t)$ instead of h_t.

$$V_t^*(h_t, a_t) = V_t^*(\eta(I_t), a_t).$$

The option value of retaining the object in period t is a function of the highest valuations for all future periods $\hat{v}_t : [0, \overline{v}]^{T-t+1} \to \mathbb{R}$, defined by[5]

$$\hat{v}_t(\eta(I_t)) := E\left[V_{t+1}^*(\eta(I_{t+1}), 1) \mid \eta(I_t)\right]. \tag{3.2.1}$$

It is efficient to allocate the object in period t, if the option value is below the highest valuation in period t. Hence, the object is allocated in the first period for which

$$\eta_t(I_t) > \hat{v}_t(\eta(I_t)). \tag{3.2.2}$$

In this period, the object is awarded to a buyer with valuation $\eta_t(I_t)$. If there is a tie, the buyer with the lowest index is chosen.[6]

3.2.3. Incentive Compatibility. A *direct mechanism* is a tuple (S, x, π). $S = (S_t)_{t\in\{1,\ldots,T\}}$ is the *sequence of signal spaces*, where $S_t = [0, \overline{v}]^{T-t+1}$ is the signal space for period t. In each period, buyers can only report a complete type of a buyer that arrived in the same period. A buyer can make a report in any period after her arrival. Without loss of generality we can assume that each buyer makes at most one report. The allocation rule x is a policy as defined in the last section. We will only use the efficient policy x^*. $\pi : h_T \mapsto (\pi_i)_{i=1,\ldots,N_T} \in \mathbb{R}^{N_t}$ is the payment rule. $\pi_j(h_T)$ specifies the payment of buyer j at the terminal history h_T. A mechanism does not explicitly specify the time at which payments have to be made. Payments may depend on all reports until the last period. If, however, in period t, and for some h_t, $\pi_j(h_T)$ is independent of all reports after period t, then the payments of buyer j can already be made in period t.

Now consider a buyer j, who arrives in period t and plans to make a report $v' \in S_r$ in period $r \geq t$. Denote the history of reports of all buyers in $I_t \setminus \{j\}$, by $h_{t,-j}$. For given $h_{t,-j}$ and assuming that all other buyers report truthfully, the *winning probability* of buyer j for period $\tau \geq r$ conditional on all information available in the arrival period is given by

$$q_\tau^r(v', h_{t,-j}) := \text{Prob}\left[x_\tau^*(h_\tau) = j \mid h_{t,-j}, v_j = v'\right].$$

We omit a_t as an argument of the winning probability because $q_\tau^r(h_{t,-j}, v')$ will only be used when $a_t = 1$. The *expected payment* is given by

$$p^r(v', h_{t,-j}) := E\left[\pi_j(h_T) \mid h_{t,-j}, v_j = v'\right].$$

[5]For simplicity, $\eta_t(I_t)$ is included in the arguments although $\hat{v}_t$ does not depend on $\eta_t(I_t)$.

[6]This implies a random selection among buyers with the same arrival period because indices are assigned randomly. Furthermore, there is preference for buyers that arrived earlier. In this chapter, the *efficient allocation rule* always refers to the allocation rule that has just been described. There are other allocation rules which achieve the same total expected welfare. For example, the allocation rule that allocates in the first period for which $\eta_t(I_t) \geq \hat{v}_t(\eta(I_t))$, is also efficient. Also, other tie-breaking rules could be used.

With these definitions, the *expected utility* from participating in the mechanism with a reported type $v' \in S_r$ and true type $v \in S_t$ is

$$U(v, v', h_{t,-j}) := \sum_{\tau=r}^{T} v^{\tau} q_{\tau}^{r}(v', h_{t,-j}) - p^{r}(v', h_{t,-j}).$$

The expected utility from truth-telling is abbreviated $U(v, h_{t,-j}) := U(v, v, h_{t,-j})$.

A mechanism is *periodic ex-post incentive compatible* if for all $t, r \in \{1, \ldots, T\}$ with $r \geq t$, all $v \in S_t$, $v' \in S_r$, and all possible histories of reports $h_{t,-j}$,

$$U(v, h_{t,-j}) \geq U(v, v', h_{t,-j}). \tag{3.2.3}$$

Periodic ex-post incentive compatibility requires that a truthful report at the arrival period must be optimal for all possible histories, under the assumption that all buyers (past, current and future arrivals) report their types truthfully at their respective arrival periods. Periodic ex-post incentive compatibility is a hybrid concept that reflects the informational constraint of the dynamic model. Expectations are taken with respect to the types of future buyers. In this sense, it resembles Bayes-Nash incentive compatibility. With respect to past and current buyers, incentive compatibility constraints must hold for every profile of types. Therefore, ex-post incentive compatibility is required only for information that is already realized at the time when a buyer makes a report.[7]

Incentive compatibility of the efficient allocation rule has been shown by Parkes and Singh (2003) for discrete type-spaces[8] and by Bergemann and Välimäki (2010) for continuous type-spaces.[9]

Adapted to the model of this chapter, the *online VCG mechanism* of Parkes and Singh (2003) uses the following payment rule. The payment of j, when she makes a report $v' \in S_r$, is defined as

$$\pi_j^{\text{oVCG}}(h_{T,-j}, v') = \sum_{\tau=r}^{T} v'^{\tau} \mathbf{1}_{\{x_{\tau}^{*}(h_{\tau,-j}, v')=j\}} - \left[V_r^{*}((h_{r,-j}, v'), a_r) - V_r^{*}(h_{r,-j}, a_r)\right]. \tag{3.2.4}$$

[7]Note that (3.2.3) also rules out profitable deviations in which a buyer delays her report and reports different types in later periods, conditional on the valuations of buyers who have arrived in the meantime. For period T, (3.2.3) ensures that it is optimal to report v_j^T truthfully, for every history $h_{T,-j}$. This applies to buyers who arrived in period T as well as to buyers who delayed their report because the mechanism cannot distinguish between them. In period $T-1$, (3.2.3) rules out that a delayed but truthful report of v_j^T is a profitable deviation. Therefore in period $T-1$, it is optimal to report (v_j^{T-1}, v_j^T) truthfully and without delay. Working backwards in time it follows inductively, that (3.2.3) rules out all feasible reporting strategies except a truthful report in the arrival period.

[8]These authors use a very similar equilibrium concept. In their concept, ex-post incentive compatibility is required with respect to information of buyers with lower index. This excludes the types of buyers who arrive simultaneously but were assigned a higher index.

[9]Athey and Segal (2007) also show implementability of the efficient allocation rule, but the proposed mechanism requires all agents to be available in all periods.

The first term is equal to the private utility j enjoys according to her reported type. If j wins the object in period τ, this is equal to v'^{τ}. If she does not win the object in any period, it is zero. The term in parentheses is the change in expected total welfare due to the report of buyer j given the information available in period r. As in the standard static VCG mechanism, the payment replaces the private surplus of each buyer by the (expected) change in total welfare due to her report. The allocation rule maximizes welfare subject to the informational constraint that future types are not known. Therefore, it is optimal for j to report her true type, because she faces the same informational constraint.

THEOREM 3.2.1 (Parkes and Singh (2003)). *The mechanism* (S, x^*, π^{oVCG}) *is periodic ex-post incentive compatible.*

π^{pVCG} is not the only payment rule that implements the efficient allocation. The *dynamic pivot mechanism* of Bergemann and Välimäki (2010) does not aggregate payments over periods. The payment of a buyer j in period $s \geq r$ is defined as

$$\begin{aligned} \pi_j^{DP}(h_{s,-j}, v') = {} & v_{x_s^*(h_{s,-j})} + E\left[V_{s+1}^*(h_{s+1,-j}) \middle| h_{s,-j}, x_s^*(h_{s,-j})\right] \\ & - \left(v_{x_s^*(h_{s,-j},v')} + E\left[V_{s+1}^*(h_{s+1,-j}) \middle| h_{s,-j}, x_s^*(h_{s,-j}, v')\right]\right) \end{aligned}$$

None of the mechanisms fulfills all properties (A)–(C). In the truth-telling equilibrium of the online VCG mechanism, j always receives a payoff given by

$$V_t^*(h_t, a_t) - V_t^*(h_{t,-j}, a_t). \tag{3.2.5}$$

The payoff is independent of the event that she wins the object. In particular, this implies that the mechanism must transfer a positive amount of money to every buyer who has a positive chance of winning at the time of arrival, if that buyer does not win the object. This violates property (C).

The dynamic pivot mechanism requires payments from buyers who are pivotal for postponing the allocation even if they do not win the object. To see this consider the following example. Let $T = 2$, $I_1 = \{1, 2\}$ with $v_1 = (v_1^1, 0)$ and $v_2 = (0, v_2^2)$, and assume that $\rho_1^2 = 1$ so that $I_2 = \{1, 2, 3\}$. Furthermore, assume that $v_1^1 = \frac{5}{8}$, $v_2^2 = \frac{3}{4}$ and $v_3^2 \sim U[0, 1]$. In this case, it is efficient not to allocate in the first period because $v_1^1 = \frac{5}{8} < E\left[\max\left\{\frac{3}{4}, v_3^2\right\}\right] = \frac{25}{32}$. Buyer two is pivotal for the allocation decision; without her, the object would be allocated to buyer one in the first period. Her payment in period one is therefore given by $\pi_2^1 = v_1 - E[v_3] = \frac{1}{8}$. If the realized valuation v_3^2 exceeds v_2^2, then buyer two does not receive the object and his payment in the second period is $\pi_2^2 = 0$. Hence, her total payment is $\frac{1}{8}$ which violates property (A) and ex-post individual rationality.[10]

[10]This does not contradict the individual rationality result of Bergemann and Välimäki (2010) because the authors consider *periodic* ex-post individual rationality. Indeed, the expected payoff of buyer two in the first period is $\frac{3}{4}(\frac{3}{4} - \frac{3}{8}) - \frac{1}{8} = \frac{5}{32} > 0$.

3.3. Properties of the Efficient Allocation Rule

The efficient allocation rule allocates the object in the first period where $\eta_t(I_t) > \hat{v}_t(\eta(I_t))$. We show below that if we apply this condition to the type v_j of buyer j, then we get the unique period in which this type can win.

DEFINITION 3.3.1. For $t \in \{1, \ldots, T\}$, the *potential winning period* θ_j of buyer j with type $v_j \in S_t$, is the earliest period $\theta \geq t$ for which $v_j^\theta > \hat{v}_\theta(v_j)$, i.e.,

$$\begin{aligned} & v_j^\tau \leq \hat{v}_\tau(v_j), \qquad \text{for } \tau \in \{t, \ldots, \theta_j - 1\}, \\ \text{and} \qquad & v_j^{\theta_j} > \hat{v}_{\theta_j}(v_j). \end{aligned} \tag{3.3.1}$$

Definition 3.3.1 partitions the type-space. Hence, there is a unique potential winning period for each type. The potential winning period θ_j only depends on the type of buyer j and the structure of the allocation problem, i.e. the arrival process and the distributions from which valuations are drawn. It does not depend on the realized types of the other buyers or the realized numbers of buyers.

Examples:

(1) A buyer with constant valuation $v_j = (v, v, \ldots, v)$ has potential winning period T.
(2) A buyer with $v_j^\tau = v$ for some τ and $v_j^t = 0$ for $t \neq \tau$, has potential winning period T if $v_j^\tau \leq \hat{v}_\tau(0, \ldots, 0)$, otherwise she has potential winning period τ.

The following theorem states that under the efficient allocation rule, a buyer can win the object only in her potential winning period.

THEOREM 3.3.2. *Fix a buyer j with type $v_j \in S_t$. If $x_s^*(h_s, a_s) = j$ for some $s \geq t$, and some h_s, then $s = \theta_j$.*

PROOF. See Appendix 3.A. □

To get an intuition for the result, consider the case $T = 2$. Suppose buyer j arrives in period one and has type $v_j = (v_j^1, v_j^2)$. The theorem states that she can either win in period one or in period two, but not in both periods. Of course, it depends on the types of the other buyers whether she wins at all.

First, suppose that for some profile of types, it is efficient that j gets the object in period two. In this case, the highest valuation for the first period $\eta_1(I_1)$, must not be greater than the option value of retaining the object:

$$\eta_1(I_1) \leq \hat{v}_1(\eta(I_1)).$$

If j wins in the second period, she must have the highest valuation for period two among the buyers from period one: $v_j^2 = \eta_2(I_1)$. Hence, the option value of retaining the object only depends on her valuation: $\hat{v}_1(\eta(I_1)) = \hat{v}_1(v_j^2)$. On the other hand, her valuation for the first period cannot be greater than $\eta_1(I_1)$. We conclude that

$$v_j^1 \leq \hat{v}_1(v_j^2).$$

Loosely speaking, if j had a higher valuation for period one, she would overbid the option value defined by her own valuation for period two. But this must not be the case if j wins in the second period.

Second, suppose that for some other profile of types it is efficient to allocate the object to j in period one. Then, $v_j^1 = \eta_1(I_1)$ and hence

$$v_j^1 > \hat{v}_1(\eta(I_1)) \geq \hat{v}_1(v_j^2).$$

Loosely speaking, j's valuation for period one must overbid the option value of retaining the object. Especially, j must overbid her own valuation for period two (transformed by the option value function). The conditions on v_j for winning in the first and in the second period, cannot be fulfilled simultaneously. Hence, it is not possible that j wins in different periods for different profiles of the other buyers' types.

Theorem 3.3.2 greatly reduces the dimension of the signal space that is necessary to implement the value-maximizing allocation rule. The type of each buyer j can be summarized by θ_j and $v_j^{\theta_j}$. In addition, once the winning period θ_j is fixed, the notion of the *lowest type* that can win the auction for a particular state of the world becomes well defined. This property will be used to define the dynamic Vickrey auction.

Theorem 3.3.2 has two important implications that are useful to define auction rules. First, in each period t, there is a unique buyer j_t^* among those who have already arrived, who has a chance of winning. This buyer is called the *tentative winner in period t*. Furthermore, if we partition the set of buyers in period t into two subsets A and B, two tentative winners can be determined under the assumption that only the buyers in A or B, respectively, have arrived. The tentative winner for the set of all buyers ($A \cup B$), must be one of the two tentative winners determined for the subsets A and B. Formally, we have:

COROLLARY 3.3.3. *(i) For each period t and every state (h_t, a_t) with $a_t = 1$, there exists a unique buyer $j_t^* \in I_t$ such that $x_\tau^*(h_\tau, a_\tau) \notin I_t \backslash j_t^*$ for all $\tau \geq t$ and all future states (h_τ, a_τ) that can occur after (h_t, a_t).*

(ii) Suppose $I_t = A \cup B$ with $A, B \neq \emptyset$. Let $a \in A$ be the tentative winner if the set of buyers is $I_t' = A$, and let $b \in B$ be the tentative winner if $I_t' = B$. Then $j_t^* \in \{a, b\}$.

PROOF. (i) For $t \in \{1, \ldots, T\}$ consider a hypothetical buyer k with valuations $v_k = \eta(I_t)$. By Theorem 3.3.2, this buyer has a unique potential winning period θ_k. The stopping rule of the efficient allocation only depends on the highest valuation for each period (cf. condition (3.2.2)). Therefore, the time at which the object is allocated with I_t, is the same as with $I_t' = \{k\}$ and $v_k = \eta(I_t)$. Hence, buyers in I_t can only win in period θ_k. If a buyer from I_t wins, it must be $j_t^* = \min\left(\arg\max_{i \in I_t}\{v_i^{\theta_k}\}\right)$. (If there are ties, the tie-breaking rule described in Section 3.2.2 eventually selects the buyer with the lowest index. As the indices of the buyers in I_t are already known in period t, the identity of the tentative winner is unique in period t.)

(ii) Without loss of generality we can assume that $j_t^* \in A$. But then j_t^* must also be the tentative winner for $I_t' = A$ because with any other tentative winner, the expected value of the allocation for $I_t' = A$ must be weakly smaller than the expected value with j_t^* as tentative winner.

□

An immediate implication of Corollary 3.3.3 is that a buyer who was not tentative winner in period t, cannot become tentative winner in period $t+1$.

COROLLARY 3.3.4. *For $t \in \{1, \ldots, T-1\}$, let j_t^* and j_{t+1}^* be the tentative winners in t and $t+1$, respectively. Then $j_{t+1}^* \in (I_{t+1} \backslash I_t) \cup \{j_t^*\}$.*

These properties of the efficient allocation rule imply that it can be implemented as follows: In each period t, new buyers are asked to report complete types to the auctioneer.[11] If it is efficient to allocate immediately, the object is sold and the auction ends. If it is efficient to retain the object for the next period, the auctioneer declares a tentative winner which is either the tentative winner from the previous period or a new buyer who has made a report in period t. All other buyers are informed that they cannot win the auction and will never be recalled.

Remarks:

(1) The properties of the efficient allocation rule carry over to optimal policies of any dynamic program that has a similar structure as $\mathcal{P}$. For example, one could consider quasi-efficient allocation rules that maximize excepted welfare after valuations have been transformed by strictly increasing functions:

$$J_t(v_j) = (J_t^t(v_j^t), \ldots, J_t^T(v_j^T)),$$

where each J_t^T is strictly increasing.

(2) Note also that the assumption of full support of the type distribution has not been used in this section. Therefore, the results carry over to a model with constant valuations and deadlines. In this model, the types of all buyer have the form $(v_i, \ldots, v_i, 0, \ldots, 0)$ where the valuation v_i is repeated from the arrival time to the deadline d_i. Theorem 3.3.2 is trivial in this case because $\theta_i = d_i$ for all types, but the less obvious Corollary 3.3.3 also carries over to the model with deadlines.

3.4. Payments

In this section, a simple payment rule is constructed, that generalizes the static Vickrey auction. To highlight the similarity, we briefly review the payment rule of the static Vickrey auction.

[11] Alternatively, they could be asked to report their potential winning period and the valuation for that period.

3.4.1. The Static Vickrey Auction. Consider the standard independent private values model with N bidders. Valuations are drawn from $\Theta = [a, b]$ with distribution function F. Let q be the probability of winning the object in the second-price auction without reserve price.

By payoff-equivalence, the expected payoff of bidder 1 with valuation $v_1 \in \Theta$ in the second-price auction is given by

$$U(v_1) = \int_a^{v_1} q(v)dv. \tag{3.4.1}$$

Writing the winning probability explicitly,

$$U(v_1) = \int_a^{v_1} \int_{\Theta^{N-1}} \mathbf{1}_{\{v \geq \max\{v_2,\ldots,v_N\}\}} dF(v_2)\ldots dF(v_N)dv.$$

Writing $v_{(1)} = \max\{v_2, \ldots, v_N\}$ and changing the order of integration yields:

$$\begin{aligned} U(v_1) &= \int_\Theta \int_a^{v_1} \mathbf{1}_{\{v \geq v_{(1)}\}} dv dF^{N-1}(v_{(1)}) \\ &= \int_\Theta (v_1 - v_{(1)}) \mathbf{1}_{\{v_1 \geq v_{(1)}\}} dF^{N-1}(v_{(1)}) \\ &= \underbrace{\int_\Theta v_1 \mathbf{1}_{\{v_1 \geq v_{(1)}\}} dF^{N-1}(v_{(1)})}_{v_1 \cdot q(v_1)} - \underbrace{\int_\Theta v_{(1)} \mathbf{1}_{\{v_1 \geq v_{(1)}\}} dF^{N-1}(v_{(1)})}_{p(v_1)} \end{aligned}$$

This shows, that the payment of any bidder can be defined as zero if she does not win and as the highest valuation of the other bidders if she wins.

3.4.2. Construction of Payments for the Dynamic Vickrey Auction. The construction of payments in the dynamic setting follows the same logic. First, the dimension reduction of the type-space, incentive compatibility and payoff-equivalence for multi-dimensional mechanisms are used to derive a formula similar to (3.4.1). The result is (3.4.3) below. Second, as in the one-dimensional case, the winning probability is written explicitly and the order of integration is changed. Instead of the second-highest valuation, we will then obtain a *critical type* $\underline{v}^t(\theta_j, \eta_{-j})$ that depends on the arrival period t, the potential winning period θ_i, and the profile of the other bidders' highest valuations $\eta_{-j} = (\eta_s(I_\tau \backslash \{j\}))_{\tau=1,\ldots,T, s=\tau,\ldots,T}$. This is used to define payments for the winning bidder.

Consider a bidder j with type v_j who arrives in period $t \in \{1, \ldots, T\}$. Proposition 1 in Jehiel, Moldovanu, and Stacchetti (1999) implies that the expected payoff for j, from participating in an incentive compatible mechanism which implements the efficient allocation rule, is given by

$$U^t(v_j) = U^t(0) + \int_0^1 \left\langle q^t(\gamma(s)), \gamma'(s) \right\rangle ds \tag{3.4.2}$$

where $q^t(v) := (q_t^t(v, h_{t,-j}), \ldots, q_T^t(v, h_{t,-j}))$, $\gamma : [0, 1] \to [0, \overline{v}]^{T-t+1}$ parameterizes a piecewise smooth curve that connects the origin with v_j, and $\langle ., . \rangle$ denotes the standard scalar product on $\mathbb{R}^{T-t+1}$. The argument $h_{t,-j}$ is suppressed in the functions U and q, in order to simplify notation. Incentive compatibility of the efficient allocation rule implies that q^t is a conservative vector field (Jehiel, Moldovanu, and Stacchetti, 1999). Therefore, γ can be chosen such that it is composed of three straight lines. Let the first line connect the origin with $w_j^1 := (0, \ldots, \hat{v}_{\theta_j}(0), 0, \ldots)$. (This implies that the first line reduces to a point for $\theta_j = T$.) Let the second line connect w_j^1 with $w_j^2 := (0, \ldots, v_j^{\theta_j}, 0, \ldots)$. Let the third line connect w_j^2 with v_j.

If $\theta_j \neq T$, for each $v = \gamma(s)$ on the first line, $q^t_\tau(v) = 0$ for $\tau \neq T$ by Theorem 3.3.2 (see the example after Definition 3.3.1), and $\gamma'_T(s) = 0$ by the choice of γ. Hence, the path of integration is perpendicular to the vector-field and the integrand in (3.4.2) vanishes. For each $v = \gamma(s)$ on the third line segment, $q^t_\tau(v) = 0$ for $\tau \neq \theta_j$, and $\gamma'_{\theta_j}(s) = 0$. Hence, the integrand vanishes as well. On the second line segment, only the θ_j^{th} components of q^t and γ' are non-zero. Therefore, with a simple change of variables, (3.4.2) can be simplified to

$$U^t(v_j) = U^t(0) + \int_{\hat{v}_{\theta_j}(0)}^{v_j^{\theta_j}} q^t_{\theta_j}(0, \ldots, v, 0, \ldots) dv. \tag{3.4.3}$$

For given $v \in [0, \bar{v}]$, $q^t_{\theta_j}(0, \ldots, v, 0, \ldots)$ is equal to the probability conditional on $h_{t,-j}$, that η_{-j} belongs to the set

$$\begin{aligned}
\Omega^t_{\theta_j}(v) := \big\{ \tilde{\eta}_{-j} \;\big|\; & \tilde{\eta}_t(I_{t,-j}) \leq \hat{v}_t(\tilde{\eta}_t(I_{t,-j}), \ldots, \max\{\tilde{\eta}_{\theta_j}(I_{t,-j}), v\}, \ldots, \tilde{\eta}_T(I_{t,-j})), \\
& \ldots \\
& \tilde{\eta}_{\theta_j-1}(I_{\theta_j-1,-j}) \leq \hat{v}_{\theta_j-1}(\tilde{\eta}_{\theta_j-1}(I_{\theta_j-1,-j}), \max\{\tilde{\eta}_{\theta_j}(I_{\theta_j-1,-j}), v\}, \ldots \\
& \qquad \ldots, \tilde{\eta}_T(I_{\theta_j-1,-j})), \\
& v > \tilde{\eta}_{\theta_j}(I_{\theta_j,-j}), \\
& v > \hat{v}_{\theta_j}(\tilde{\eta}(I_{\theta_j,-j})) \big\}, \\
= \big\{ \tilde{\eta}_{-j} \mid & \tilde{\eta}_t(I_{t,-j}) \leq \hat{v}_t(0, \ldots, v, \ldots, 0), \\
& \ldots \\
& \tilde{\eta}_{\theta_j-1}(I_{\theta_j-1,-j}) \leq \hat{v}_{\theta_j-1}(0, v, 0 \ldots, 0), \\
& v > \tilde{\eta}_{\theta_j}(I_{\theta_j,-j}), \\
& v > \hat{v}_{\theta_j}(\tilde{\eta}(I_{\theta_j,-j})) \big\},
\end{aligned} \tag{3.4.4}$$

where $I_{\tau,-j} = I_\tau \backslash \{j\}$. The second equality follows because j must be the tentative winner in all periods $t, \ldots, \theta_j$, if he gets the object in period θ_j. In (3.4.4), the conditions on $\tilde{\eta}(I_{t,-j}), \ldots, \tilde{\eta}(I_{\theta_j-1,-j})$, ensure that it is efficient to retain the object until period θ_j. The second last line ensures that j has the highest valuation in period θ_j among all bidders. The last condition ensures that it is not efficient to retain the object in period θ_j. With this definition, (3.4.3) becomes

$$U^t(v_j) = U^t(0) + \int_{\hat{v}_{\theta_j}(0)}^{v_j^{\theta_j}} \int_{\Omega^t_{\theta_j}(v)} dG^t(\eta_{-j} \mid h_{t,-j}) dv,$$

where $G^t(\,.\,| \, h_{t,-j})$ shall denote the distribution function of η_{-j}, conditional on $h_{t,-j}$.

The inequalities defining $\Omega^t_{\theta_j}(v)$ in (3.4.4) are lower bounds for v. Therefore, $\Omega^t_{\theta_j}(v') \supseteq \Omega^t_{\theta_j}(v)$ if $v' \geq v$. We can rewrite the expected payoff and use Fubini as follows:

$$\begin{aligned} U^t(v_j) &= U^t(0) + \int_{\hat{v}_{\theta_j}(0)}^{v_j^{\theta_j}} \int_{\Omega^t_{\theta_j}(v_j^{\theta_j})} \mathbf{1}_{\{\eta_{-j} \in \Omega^t_{\theta_j}(v)\}} \, dG^t(\eta_{-j} \mid h_{t,-j}) \, dv, \\ &= U^t(0) + \int_{\Omega^t_{\theta_j}(v_j^{\theta_j})} \int_{\hat{v}_{\theta_j}(0)}^{v_j^{\theta_j}} \mathbf{1}_{\{\eta_{-j} \in \Omega^t_{\theta_j}(v)\}} \, dv \, dG^t(\eta_{-j} \mid h_{t,-j}). \end{aligned}$$

Finally, we define

$$\underline{v}^\tau(\theta_j, \eta_{-j}) := \inf \left\{ v \,\middle|\, \eta_{-j} \in \Omega^\tau_{\theta_j}(v) \right\}. \tag{3.4.5}$$

$\underline{v}^\tau(\theta_j, \eta_{-j})$ is the *critical type* of the winning bidder, i.e. the valuation $v_j^{\theta_j}$, for which j ties with η_{-j}. Using this, we can rewrite the expected payoff to get

$$\begin{aligned} U^t(v_j) &= U^t(0) + \int_{\Omega^t_{\theta_j}(v_j^{\theta_j})} v_j^{\theta_j} - \underline{v}^t(\theta_j, \eta_{-j}) \, dG^t(\eta_{-j} \mid h_{t,-j}) \qquad (3.4.6) \\ &= U^t(0) + \underbrace{\int_{\Omega^t_{\theta_j}(v_j^{\theta_j})} v_j^{\theta_j} \, dG^t(\eta_{-j} \mid h_{t,-j})}_{v_j^{\theta_j} \cdot q_{\theta_j}(v_j)} - \underbrace{\int_{\Omega^t_{\theta_j}(v_j^{\theta_j})} \underline{v}^t(\theta_j, \eta_{-j}) \, dG^t(\eta_{-j} \mid h_{t,-j})}_{p^t(v_j)} \end{aligned}$$

The last line shows that the expected payment is the integral over the critical type. The domain of integration is restricted to the set of profiles for which j wins the object. Therefore, a payment rule that requires no payment from losing bidders and a payment equal to the critical type from the winner, implements the efficient allocation rule in periodic ex-post equilibrium. Obviously, with this definition, it is ensured that buyer j has to pay a positive amount only if she gets the object. Furthermore, the payoff of bidder j is non-negative because her valuation $v_j^{\theta_j}$ is greater than or equal to the critical value if she wins the auction. As the inequalities in (3.4.4) only depend on information available in period θ_j, the payment can be determined at the same time as the allocation. To summarize, we can state

THEOREM 3.4.1. *Let π be a payment rule that defines the payment for bidder j, when she reports type v_j in period t, and the history of reports is h_T, as*

$$\pi_j(h_T) := \begin{cases} \underline{v}^t(\theta_j, \eta_{-j}) & \text{if } \eta_{-j} \in \Omega^t_{\theta_j}(v_j^{\theta_j}) \\ 0 & \text{if } \eta_{-j} \notin \Omega^t_{\theta_j}(v_j^{\theta_j}) \end{cases}. \tag{3.4.7}$$

Then,

(i) the mechanism (S, x^*, π) is periodic ex-post incentive compatible,
(ii) payments are non-negative for the winning bidder and zero for all other bidders,
(iii) payments are completely determined in the period when the object is allocated,
(iv) and the mechanism is ex-post individually rational.

Remarks:

(1) With these payments, $U^t(0) = 0$.
(2) The critical type $\underline{v}^t(\theta_j, \eta_{-j})$, can be computed by setting the inequalities in the definition of $\Omega^t_\theta(v)$ equal (see (3.4.4)) and solving each equality for v. The critical value is the maximum of these solutions. There is at least one equality that is fulfilled at this maximum. If the last equality is fulfilled we have $\underline{v}^t(\theta_j, \eta_{-j}) = \hat{v}_{\theta_j}(\eta(I_{\theta_j,-j}))$: j has to pay the option value of retaining the object to period $\theta_j + 1$. If the penultimate equality is fulfilled we have $\underline{v}^t(\theta_j, \eta_{-j}) = \eta_{\theta_j}(I_{\theta_j,-j})$: j has to pay the second highest valuation for period θ_j. If one of the other equalities is fulfilled, the auction would have ended earlier than θ_j if j had not made her report. Suppose for example, that the equality for $\tilde{\eta}_{t'}$ for some $t' \in \{t, \ldots, \theta_j - 1\}$ is fulfilled. Then, without j's report, the auction would have stopped in period t'. Solving $\tilde{\eta}_{t'}(I_{t',-j}) = \hat{v}_{t'}(0, \ldots, v, 0, \ldots, 0)$ for v yields $\underline{v}^t(\theta_j, \eta_{-j})$. The solution is lower than the highest valuation in period t'. Hence, in this case, the winner has to pay less that the valuation of the bidder who would have won without j.
(3) As in the static Vickrey auction, payments of bidder j do not depend directly on her report.
(4) Payments are defined as a function of the history h_T in the final period. We know from Corollary 3.3.3, that in each period only the tentative winner has a positive probability to win the object in the future. This implies, that for all bidders except the tentative winner, payments are determined as zero immediately after they have made their reports. For a tentative winner, the payment is determined if she wins the object or if another bidder becomes tentative winner. In the latter case, the payment is zero, in the former case, it can be made at the same time as the allocation of the object. Therefore, all payments can be made online.

3.5. A Generalized Ascending Auction

Theorem 3.4.1 defines a direct mechanism that generalizes the static second price auction. The ascending auction can also be generalized to the dynamic setting. In this construction, one property of the efficient allocation rule is crucial. We can define an order on the type-space such that efficient allocation rule always selects the bidder that ranks highest in this order as tentative winner, and allocates to her if her potential winning period is reached.

For bidder j with arrival period t, type $v_j \in S_t$ and potential winning period θ_j, we define the *comparison price* π_j^T by

$$\pi_j^T := \min \left\{ \pi \geq 0 \middle| v_j^{\theta_j} \leq \hat{v}_{\theta_j}(0, \ldots, 0, \pi) \right\}.$$

Since dismissed buyers are never recalled, we have that $\hat{v}_\tau(0, \ldots, 0, v_j^{\theta_j}, 0, \ldots, 0) = \hat{v}_\tau(0, \ldots, 0, \pi_j^T)$ for all v_j such that $\pi_j^T > 0$ and $\tau < \theta_j$. This implies that j wins against another bidder i with potential winning period $\theta_i = T$, if and only if $v_i^T < \pi_j^T$. More generally, if bidder i has type v_i, potential winning period θ_i and comparison price π_i^T, then j wins against i only if $\pi_j^T \geq \pi_i^T$.[12] If $\theta_i = \theta_j$ this is obvious. Otherwise, suppose without loss of generality that $\theta_j < \theta_i$. Then we have $\pi_j^T \geq \pi_i^T$ if and only if

$$v_j^{\theta_j} = \hat{v}_{\theta_j}(0, \ldots, 0, \pi_j^T) \geq \hat{v}_{\theta_j}(0, \ldots, 0, \pi_i^T) = \hat{v}_{\theta_j}(0, \ldots, 0, v_i^{\theta_i}, 0, \ldots, 0).$$

We will now use the order of types defined by their comparison price, to define a dynamic ascending auction. In each period t, there are $T - t + 1$ price clocks that show prices $\pi^t, \ldots, \pi^T$ for buying the object in periods $t, \ldots, T$. All prices π^τ, $\tau < T$ are linked to π^T by

$$\pi^\tau = \hat{v}_\tau(0, \ldots, 0, \pi^T). \tag{3.5.1}$$

In the first period, π^T starts at zero. In all periods $t > 1$, π^T starts at the value where it stopped in period $t - 1$. The other prices are set such that they satisfy (3.5.1). In each period, the auction has two phases, the clock phase and the buying phase. Before the clock phase, buyers can chose to become active. In the clock phase, π^T is raised continuously and the other prices are updated such that (3.5.1) is satisfied. Bidders are free to drop out at any time. A bidder who has dropped out cannot become active again. If all bidders but one have dropped out, or if all remaining bidders decide to drop out at the same time, the clock stops immediately. The remaining bidder, or a random bidder from the group of drop-outs if all bidders dropped out simultaneously, enters the buying phase.[13] In the buying phase, she can either buy immediately for the current price π^t, or she can wait. In the former case, the auction ends with a sale, in the latter case, the auction proceeds to the next period and she remains active.

A bidder j is said to bid *truthfully* in the dynamic ascending auction if she uses the following strategy:

- Before the clock phase of any period τ: become active if and only if $\exists s \geq \tau : v_j^s > \pi^s$.
- In the clock phase of any period τ: drop out if and only if $v_j^s \leq \pi^s$ for all $s \geq \tau$.
- In the buying phase of any period τ: Buy if and only if $\tau = \theta_j$.

[12] We ignore ties in this discussion.

[13] Here, for simplicity, we use a different tie-breaking rule than in the dynamic Vickrey auction. The probability that this affects the outcome is zero.

Theorem 3.5.1. *(i) If all bidders bid truthfully, the outcome of the dynamic ascending auction coincides with the outcome of the dynamic Vickrey auction with probability one.*

(ii) Truthful bidding is a periodic ex-post equilibrium in the dynamic ascending auction.

Proof. See Appendix 3.A. □

The main steps of the proof are as follows. Under truthful bidding, the ascending auction selects the buyer with the highest comparison price and allocates the object to him in his potential winning period. Hence, ignoring ties, the allocation rule is the same as in the dynamic Vickrey auction. Next, we show that the price for the winning period, at which the last competing bidder drops out is equal to the winner's critical type. Therefore, the payment rule implemented under truthful bidding is also identical to the payment rule in the dynamic Vickrey auction.

To show that truth-telling is an equilibrium we rule out several deviations that lead to non-positive expected payoffs and show that the remaining strategies yield the same expected payoffs as certain reports in the dynamic Vickrey auction. Truthful bidding corresponds to a truthful report in the dynamic Vickrey auction. Incentive compatibility of the latter therefore implies that truthful bidding is a periodic ex-post equilibrium of the dynamic ascending auction.

3.6. Conclusion

This chapter shows how the payments in a dynamic mechanism can be distributed over different states of the world such that (i) expected payments ensure incentive compatibility and (ii) the payment rule is *simple* in the sense that non-winning bidders do not make or receive a transaction, ex-post participation constraints are satisfied and payments can be made online. The result is the dynamic Vickrey auction in which the winning bidder pays her critical type, i.e., she pays the lowest valuation for the winning period that would suffice to win against the other bids. The crucial step in the construction of the payment rule was to show that for each type, there is a unique potential winning period. This reduces types to essentially one dimension. Furthermore, it was shown that the efficient allocation rule allows to define a tentative winner in each period. There is an order of the type-space and the tentative winner is the highest bidder in that order. The results have been used to generalize the ascending clock auction to the dynamic framework.

The model is restrictive in at least two ways. Firstly, the allocation of a single object is studied. The case of multiple objects is left for future research. However, if more than one object is at sale, it is possible to construct simple examples where bidders can win in different periods for different profiles of competing bidders' types. Therefore, future research will have to concentrate on the generalization of the weaker result of corollaries 3.3.3 and 3.3.4.

Secondly, more general allocation problems could be studied. In this case, as in the case of multiple objects, a reduction of types to essentially one dimension may not be possible. It should be noted, however, that Theorem 3.3.2 provides much more structure than is needed for the construction of payments in Section 3.4. It would suffice that for each type v_j, there exists a path from the origin to v_j such that the allocation to bidder j is monotonic along this path for all profiles of the other bidder's types. Depending on the choice of these paths, the payment rule may look significantly different from Vickrey payments, but nevertheless it would be possible to define payment rules that require transfers only from winning bidders.

Dynamic revenue maximization is an important question that has not been studied extensively in models with private information about time preferences.[14] The results of the present chapter do not characterize the allocation rule that maximizes revenue. They can, however, be generalized to other allocation rules that are the solution to a recursive dynamic program in which valuations are replaced by some increasing function of valuations. Given the dynamic structure of the model, it is possible that the revenue-maximizing allocation rule belongs to this class. In this case, the expected payments fixed by the allocation rule (via payoff equivalence), could be distributed over different states in the same way as in this chapter.

3.A. Omitted Proofs

Proof of Theorem 3.3.2. The result is proven by induction. For $T = 1$ the result is trivial. Assume that the theorem is true for allocation problems with $T-1$ periods. The statement for T is shown in four steps.

Step 1: If a buyer $j \in I_1$ gets the object in period one, $v_j^1 > \hat{v}_1(\eta(I_1)) \geq \hat{v}_1(v_j)$. Therefore $\theta_j = 1$.

Step 2: If it is not efficient to allocate in period one, we can consider the allocation of the retained object in periods $\{2, \ldots, T\}$ as a new allocation problem with $T-1$ periods. We only have to relabel $\tilde{I}_1 = I_2, \ldots, \tilde{I}_{T-1} = I_T$ and delete the first elements from the type vectors of bidders in I_1. The decisions in periods $\{2, \ldots, T\}$ of the original problem only depend on the buyers that are present in these periods and their valuations. Therefore the identity of the winning buyer is the same in the new problem and the original problem. The time of allocation is shifted by one. This implies, that buyer j with potential winning period $\theta_j^{\text{new}} \in \{1, \ldots, T-1\}$ in the new problem, can only win in period one or period $\theta_j' = \theta_j^{\text{new}} + 1$ of the original problem. Furthermore, as θ_j^{new} is characterized by condition (3.3.1) with $\tau \in \{1, \ldots, \theta_j^{\text{new}} - 1\}$, θ_j' is characterized by (3.3.1) with $\tau \in \{2, \ldots, \theta_j' - 1\}$. It therefore remains to show that $v_j^1 \leq \hat{v}_1(v_j)$, if j wins in θ_j' in the original problem.

Step 3: There is only one buyer from I_1 that can win the object in the original problem if it is retained for period two. To see this, consider again the new problem with $T-1$ periods. Define $A = I_1$ and $B = \tilde{I}_1 \backslash I_1$. Then $\tilde{I}_1 = A \cup B$. Assume $B \neq \emptyset$.

[14]See Pai and Vohra (2008b) and Chapter 1 for exceptions. In most other papers, buyers are either short-lived, or long-lived with a common, public discount factor and a fixed valuation.

By Corollary 3.3.3.ii, there are elements $a \in A$ and $b \in B$ such that the tentative winner in period one of the new problem is in $\{a, b\}$. Hence, buyer $a \in A = I_1$ is the only buyer in I_1 that can win the object in the original problem if it is retained for period two. If $B = \emptyset$ the argument is trivial.

Step 4: If $j \in I_1$ gets the object in $\theta'_j \neq 1$, we must have $\eta_1(I_1) \leq \hat{v}_1(\eta(I_1))$. By step 3 we know that j is the only bidder in I_1 that can win in periods $\{2, \ldots, T\}$. Therefore the option value of retaining the object only depends on her valuation: $\hat{v}_1(\eta(I_1)) = \hat{v}_1(v_j)$. As $v_j^1 \leq \eta_1(I_1)$ we have $v_j^1 \leq \hat{v}_1(v_j)$ as desired. □

Proof of Theorem 3.5.1. (i) If all bidders bid truthfully, a bidder j drops out if $\pi^T = \pi_j^T$. Therefore, in each period, a buyer with the highest comparison price enters the buying phase. As buyers buy in their potential winning periods, and only in this period, if they bid truthfully, the allocation coincides with the efficient allocation rule of the dynamic Vickrey auction, except for the case of ties, that occur with zero probability.

Now suppose that bidder j arrives in period t and wins the object in period θ_j. We show that the price π^{θ_j} at which the last competing bidder dropped out equals the critical type of bidder j. For each $i \in I_{\theta_j}$, define $\pi_i^{\theta_j} := \hat{v}_{\theta_j}(0, \ldots, 0, \pi_i^T)$. Then, j has to pay

$$\max_{i \in I_{\theta_j}} \pi_i^{\theta_j} = \max \left\{ \max_{i \in I_{\theta_j}, \theta_i < \theta_j} \pi_i^{\theta_j}, \max_{i \in I_{\theta_j}, \theta_i = \theta_j} \pi_i^{\theta_j}, \max_{i \in I_{\theta_j}, \theta_i > \theta_j} \pi_i^{\theta_j} \right\}.$$

If $\theta_i < \theta_j$, then for all $\tau < \theta_j$, $\hat{v}_\tau(0, \ldots, 0, \pi_i^{\theta_j}, 0, \ldots, 0) = \hat{v}_\tau(0, \ldots, 0, \pi_i^T) \geq v_i^\tau$. Hence,

$$\max_{i \in I_{\theta_j}, \theta_i < \theta_j} \pi_i^{\theta_j} = \max \left\{ v^{\theta_j} \middle| v^{\theta_j} = 0 \text{ or } \exists \tau \in \{t, \ldots, \theta_j - 1\}, i \in I_\tau : \theta_i < \theta_j \right. \tag{3.A.1}$$
$$\left. \text{and } v_i^\tau = \hat{v}_\tau(0, \ldots, 0, v^{\theta_j}, 0, \ldots, 0) \right\}.$$

If $\theta_j = \theta_i$, then $\pi_i^{\theta_j} = \hat{v}_{\theta_j}(0, \ldots, 0, \pi_i^T) = \hat{v}_{\theta_i}(0, \ldots, 0, \pi_i^T) = v_i^{\theta_i} = v_i^{\theta_j}$. Hence

$$\max_{i \in I_{\theta_j}, \theta_i = \theta_j} \pi_i^{\theta_j} = \max_{i \in I_{\theta_j}, \theta_i = \theta_j} v_i^{\theta_j}. \tag{3.A.2}$$

Finally if $\theta_i > \theta_j$, then $\pi_i^{\theta_j} = \hat{v}_{\theta_j}(0, \ldots, 0, \pi_i^T) \geq \hat{v}_{\theta_j}(0, \ldots, 0, v_i^{\theta_i}, 0, \ldots, 0)$. Hence

$$\max_{i \in I_{\theta_j}, \theta_i > \theta_j} \pi_i^{\theta_j} = \max_{i \in I_{\theta_j}, \theta_i > \theta_j} \hat{v}_{\theta_j}(v_i^{\theta_j}, \ldots, v_i^T). \tag{3.A.3}$$

We now compare (3.A.1)–(3.A.3) to the values defining the critical type of bidder j from (3.4.4). Define ξ_1 as the minimal value of v that satisfies all but the last two inequalities in (3.4.4), ξ_2 as the infimal value of v that satisfies the second-last inequality in (3.4.4), and ξ_3 as the infimal value of v that satisfies the last inequality in (3.4.4). In general ξ_1 is greater or equal than (3.A.1), ξ_2 is greater or equal than (3.A.2) and ξ_3 is greater or equal than (3.A.3) as the maximizations in (3.A.1)–(3.A.3) are restricted to the sets of bidders with $\theta_i < \theta_j$, $\theta_i = \theta_j$ and $\theta_i > \theta_j$, respectively. Now suppose that the last bidder i who dropped out before j won the auction has $\theta_i < \theta_j$. Then the price clock for period θ_i stopped at $\pi^{\theta_i} = v_i^{\theta_i}$. Let π^{θ_j} be the value at which the price clock for period θ_j stopped. As i is the last drop-out, $v_i^{\theta_i} = \eta_{\theta_i}(I_{\theta_i,-j}) = \hat{v}_{\theta_i}(0,\dots,0,\pi^{\theta_j},0,\dots,0)$ and $\eta_\tau(I_{\tau,-j}) \leq \hat{v}_\tau(0,\dots,0,\pi^{\theta_i},0,\dots 0)$ for all $\tau = t,\dots,\theta_j - 1$. Therefore, ξ_1 satisfies

$$v_i^{\theta_i} = \hat{v}_{\theta_i}(\underbrace{0,\dots,0}_{t,\dots,\theta_j-1},\xi_1,\underbrace{0,\dots,0}_{\theta_j+1,\dots,T}) = \hat{v}_{\theta_i}(0,\dots,0,\pi^{\theta_i},0,\dots,0).$$

This implies $\xi_1 = \pi_i^{\theta_j}$. If $\theta_i = \theta_j$, then i dropped out at $\pi^{\theta_j} = v_i^{\theta_j}$. This implies $\xi_2 = \pi_i^{\theta_j}$. If $\theta_i > \theta_j$, then i dropped out at $\pi^{\theta_i} = v_i^{\theta_i}$. Therefore $\pi_i^{\theta_j} = \hat{v}_{\theta_j}(0,\dots,0,v_i^{\theta_i},0,\dots,0) = \xi_3$. Finally, if no buyer except j arrives, the clock for period θ_j remains at its initial value $\pi_j = \hat{v}_{\theta_j}(0)$ and we have $\xi_3 = \hat{v}_{\theta_j}(0)$, $\xi_1 = 0$ and $\xi_2 = 0$. In summary this implies $\max_{i\in I_{\theta_j}} \pi_i^{\theta_i} = \underline{v}^t(\theta_j,\eta_{-j})$.

(ii) Suppose that for a dynamic ascending auction of length $T-1$, truthful bidding is an ex-post equilibrium. For length one this is trivial. We show by induction that the claim is also true for T periods.

Consider bidder $j \in I_1$ and suppose that all other bidders bid truthfully. If the auction reaches period two, and bidder j has not dropped out in the first period, truthful bidding is optimal for j by hypothesis.

If j enters the buying phase in period one, we have to distinguish two cases.
Case 1: $\pi^T \geq \pi_j^T$. In this case, $v_j^t \leq \pi^t$ for all $t = 1,\dots,T$. Therefore j's expected utility is non-positive regardless of the continuation strategy.
Case 2: $\pi^T < \pi_j^T$. In this case, (i) implies that buying immediately yields a payoff equal to $U(v_j,(v_j^1,0,\dots,0),\tilde{h}_{1,-j})$ and not buying followed by truthful bidding yields a payoff equal to $U(v_j,(0,v_j^2,\dots,v_j^T),\tilde{h}_{1,-j})$. $\tilde{h}_{1,-j} = h_{1,-j} \cup \{(\pi^1,\dots,\pi^T)\}$ denotes the history of types of the other bidders with the addition of an artificial bidder that has valuations equal to the prices at which the clock stopped when j entered the buying phase. $U(v,v',h)$ denotes the expected payoff from participating in the dynamic Vickrey auction. If $\theta_j = 1$

$$U(v_j,(v_j^1,0,\dots,0),\tilde{h}_{1,-j}) = U(v_j,\tilde{h}_{1,-j}) \geq U(v_j,(0,v_j^2,\dots,v_j^T),\tilde{h}_{1,-j}),$$

where the inequality follows from periodic ex-post incentive compatibility of the dynamic Vickrey auction. Similarly, if $\theta_j > 1$

$$U(v_j,(0,v_j^2,\dots,v_j^T),\tilde{h}_{1,-j}) = U(v_j,\tilde{h}_{1,-j}) \geq U(v_j,(v_j^1,0,\dots,0),\tilde{h}_{1,-j}).$$

This show that it is optimal to buy according to the truthful bidding strategy.

Finally, consider the clock phase. If $\pi^T \geq \pi_j^T$, remaining active yields a payoff of at most zero as shown before, therefore it is optimal to drop out immediately. If $\pi^T < \pi_j^T$, continuing by truthful bidding yields $U(v,\tilde{h}_{1,-j}) \geq 0$. Hence it is optimal to bid truthfully in the first period. $\square$

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