Factorable Monoids: Resolutions and Homology via Discrete Morse Theory

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Zusammenfassung

Diese Arbeit ist im Bereich der kombinatorische Gruppen- bzw. Monoidtheorie einzuordnen. Für Monoide X, die mit einer sogenannten Faktorabilitätsstruktur versehen sind, konstruieren wir freie Auflösungen von $\mathbb{Z}$ über dem Monoidring $\mathbb{Z}X$, die deutlich kleiner sind als die Barauflösung. Damit erweitern wir Resultate aus [Vis11] und [Wan11]. Wir geben außerdem einige Berechnungsbeispiele. Unser Hauptwerkzeug ist die diskrete Morsetheorie nach Forman [For98].

Faktorabilität hat ihren Ursprung in der Berechnung von Homologiegruppen von Modulräumen Riemannscher Flächen. Diese Modulräume lassen eine rein kombinatorische Beschreibung durch symmetrische Gruppen zu, siehe [Böd90], [Böd], [Ebe], [ABE08]. Dadurch ist es möglich, Homologiegruppen mit Hilfe von Computerprogrammen zu bestimmen, so geschehen bspw. in [ABE08], [Wan11], [Meh11].

Bei diesen Berechnungen tritt ein interessantes Phänomen zu Tage, nämlich kollabiert der Barkomplex der symmetrischen Gruppen zu einem deutlich kleineren Kettenkomplex. Um diese Beobachtung zu erklären, führt [Vis11] den Begriff der faktorablen Gruppe ein.

Eine faktorable Gruppe ist ein Tripel bestehend aus einer Gruppe G, einem Erzeugendensystem S sowie einer Faktorisierungsabbildung $\eta : G \to G \times S$. Das Bild das man hierbei vor Augen haben sollte ist, dass η von einem Gruppenelement einen Erzeuger abspaltet. In [Wan11] wird eine Verallgemeinerung dieses Begriffs auf Monoide vorgeschlagen. Dort werden, aufbauend auf [Vis11], kleine Kettenkomplexe zur Berechnung der Homologie rechtskürzbarer, faktorabler Monoide mit endlichem Erzeugendensystem konstruiert.

Der von uns gewählte Zugang zu Faktorabilität ist konzeptioneller. Dies erlaubt einerseits, die Faktorisierungsabbildung η als diskrete Morsefunktion auf der Barauflösung zu interpretieren. Daraus werden unter anderem geometrische und homologische Endlichkeitsaussagen gewonnen. Andererseits ermöglicht unsere Herangehensweise, faktorable Monoide in den Kontext von Termersetzungssystemen (siehe z.B. [Coh97]) einzuordnen.

Der Aufbau dieser Arbeit ist wie folgt. In Kapitel 1 rekapitulieren wir ausführlich diskrete Morsetheorie und einige ihrer Varianten. In Kapitel 2 geben wir unsere Definition von faktorablem Monoid. Kapitel 3 zeigt, wie man aus Faktorabilitätsstrukturen "kleine" Auflösungen gewinnt und wie dieses Resultat die Konstruktionen in [Vis11] und [Wan11] vereinheitlicht und verallgemeinert. In Kapitel 4 präsentieren wir eine Anwendung und berechnen Homologiegruppen von sogenannten Thompsonmonoiden. Dies beinhaltet insbesondere eine Neuberechnung der Homologie der Thompsongruppe F selbst (vgl. [CFP96], [BG84]) sowie verwandter Gruppen $F_{n,\infty}$ (vgl. [Bro92], [Ste92]).

Contents

Introduction **7**

Preliminaries on monoids **17**

1 Discrete Morse theory and rewriting systems **21**

1.1 Discrete Morse theory 21
1.1.1 Simplicial collapse 21
1.1.2 Algebraic collapse 23
1.1.3 Labelled graphs and matchings 26
1.1.4 Based chain complexes and incidence graphs 28
1.1.5 Discrete gradient flow and invariant chains 30
1.1.6 Reduced gradient flow and essential chains 33
1.1.7 Historical Remarks & References 37
1.2 Rewriting Systems 38
1.2.1 Reminder on bar constructions 38
1.2.2 Noetherian matchings on the normalized bar resolution 42
1.2.3 Rewriting Systems 43
1.2.4 Brown's proof of the Anick–Groves–Squier Theorem 44
1.3 A new proof of a Theorem by Charney–Meier–Whittlesey 47

2 Factorability **53**

2.1 Prerequisites 53
2.1.1 Generating sets and filtrations 53
2.1.2 Factorable groups 55
2.1.3 Weakly factorable monoids 59
2.1.4 The Visy complex 61
2.1.5 The recognition principle 63
2.2 Factorable monoids 65
2.2.1 Definition and classification 65
2.2.2 Products 69
2.2.3 Finite sequences and monoid actions 72
2.3 The monoids P_n and Q_n 75
2.3.1 The monoids P_n 75
2.3.2 The monoids Q_n 78
2.3.3 Smallness 85
2.3.4 A new description of κ_n 89

3 The Visy resolution **93**
3.1 A noetherian matching on $\overline{\mathbb{E}}_* X$ . . . 93
3.1.1 The matching μ and $\mathbb{Z}X$-compatibility . . . 93
3.1.2 Coherent sequences . . . 98
3.1.3 Noetherianity of μ . . . 101
3.2 Computing the differential $\theta^\infty \circ \overline{\partial}$ . . . 103
3.2.1 $\mathbf{\Theta}^\infty$ and coherent sequences . . . 103
3.2.2 $\mathbf{\Theta}^\infty$ and right-most, reduced, small sequences . . . 106
3.3 $\mathbb{Z}$-coefficients . . . 113
3.3.1 The E^1-page revisited . . . 113
3.3.2 κ is an isomorphism . . . 117
3.4 Connection to Rewriting Systems . . . 118

4 Applications to generalized Thompson groups and monoids **125**
4.1 Homology of groups via homology of monoids . . . 125
4.2 Braid factorability . . . 127
4.2.1 The monoids Z_n . . . 128
4.2.2 A class of braid factorable monoids . . . 133
4.3 Thompson monoids . . . 137
4.3.1 Factorability of $t_m(p,q)$ . . . 139
4.3.2 Embedding $t_m(p,q)$ into $\mathcal{T}_m(p,q)$ . . . 142
4.4 Reducing the Visy complexes $\mathbb{V}_* t_m(p,q)$ . . . 144
4.4.1 Notational conventions . . . 145
4.4.2 Describing the differential $\partial_*^{\mathbb{V}}$ . . . 146
4.4.3 A perfect matching on $\mathbb{V}_* t_m(p,q)$ . . . 147
4.4.4 Perfectness of μ . . . 153
4.5 Homological results on Thompson monoids . . . 155
4.5.1 Homological dimension . . . 155
4.5.2 The homology of Thompson monoids . . . 156
4.5.3 A remark on the monoids $t_m(1,q), \ldots, t_m(q,q)$ . . . 159

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Introduction

Group homology is an important invariant in many algebraically flavoured branches of mathematics. The homology of a group G with coefficients in a $\mathbb{Z}G$-module M is defined as

$$H_*(G; M) := \mathrm{Tor}_*^{\mathbb{Z}G}(\mathbb{Z}; M),$$

where $\mathbb{Z}$ has the trivial $\mathbb{Z}G$-module structure. To compute these Tor-groups, we need to find a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.

The normalized inhomogeneous bar resolution $\overline{\mathbb{E}}_*G$, for short bar resolution of G, serves as such. Tensoring with $\mathbb{Z}$, we obtain a well-known model for the homology of G, the bar complex $\overline{\mathbb{B}}_*G := \overline{\mathbb{E}}_*G \otimes_{\mathbb{Z}G} \mathbb{Z}$. As a $\mathbb{Z}$-module, each $\overline{\mathbb{B}}_nG$ is freely generated by tuples $[g_n|\dots|g_1]$ with $g_i \neq 1$ for all i.

In practice, the bar complex is too huge to do homology computations straight away. It is therefore convenient to take into account potential extra structure on G. For example, the existence of a finite complete rewriting system for G gives rise to a free resolution of $\mathbb{Z}$ of finite type, see e.g. [Ani86], [Squ87], [Gro90], [Bro92] and the survey articles [NR93], [Coh97]. More specifically, in [CMW04] Garside structures are used to construct finite free resolutions.

The notion of factorability should be considered as lying somewhere between Garside structures and complete rewriting systems. This will be made precise later. The upshot is that if G is equipped with a reasonable factorability structure, then one can explicitly write down a free resolution which is considerably smaller than the bar resolution and thus more amenable to computation.

Before going into details, we briefly discuss a collapsing phenomenon that has first been observed for symmetric groups and that motivated the notion of factorability.

Denote by $\mathcal{S}_k$ the k-th symmetric group. Let $T \subset \mathcal{S}_k$ be the generating set of *all* transpositions. The word length ℓ with respect to T gives rise to a filtration by subcomplexes of the bar complex $\overline{\mathbb{B}}_*\mathcal{S}_k$. More precisely, define $\mathcal{F}_h\overline{\mathbb{B}}_n\mathcal{S}_k$ to be the $\mathbb{Z}$-module freely generated by all tuples $[\sigma_n|\dots|\sigma_1]$ with $\sigma_i \neq \mathrm{id}$ and $\ell(\sigma_n) + \dots + \ell(\sigma_1) \leq h$. Since ℓ is subadditive, the filtration levels $\mathcal{F}_h\overline{\mathbb{B}}_*\mathcal{S}_k$ are indeed subcomplexes of $\overline{\mathbb{B}}_*\mathcal{S}_k$. Note that $\mathcal{F}_h\overline{\mathbb{B}}_*\mathcal{S}_k = 0$ for $* < 0$ or $* > h$.

The following observation is due to Visy [Vis11]. It marks the starting point of the study of factorability.

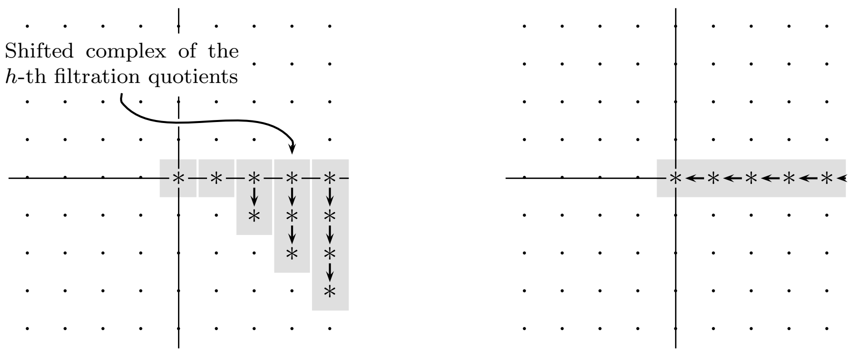
Proposition (Visy) *The complex of filtration quotients $\mathcal{F}_h\overline{\mathbb{B}}_*S_k/\mathcal{F}_{h-1}\overline{\mathbb{B}}_*S_k$,*

$$0 \longrightarrow \mathcal{F}_h\overline{\mathbb{B}}_h\mathcal{S}_k/\mathcal{F}_{h-1}\overline{\mathbb{B}}_h\mathcal{S}_k \longrightarrow \cdots \longrightarrow \mathcal{F}_h\overline{\mathbb{B}}_0\mathcal{S}_k/\mathcal{F}_{h-1}\overline{\mathbb{B}}_0\mathcal{S}_k \longrightarrow 0 \qquad (*)$$

has homology concentrated in top-degree $* = h$.

Let us reformulate this observation in terms of spectral sequences. $\mathcal{F}_\bullet\overline{\mathbb{B}}_*\mathcal{S}_k$ is an increasing filtration by chain complexes, and hence there is an associated homology spectral sequence, cf. Weibel [Wei94, §5.4]. More precisely, we have $\mathsf{E}^0_{p,q} = \mathcal{F}_p\overline{\mathbb{B}}_{p+q}\mathcal{S}_k/\mathcal{F}_{p-1}\overline{\mathbb{B}}_{p+q}\mathcal{S}_k$. Note that our spectral sequence lives in the fourth quadrant.

The chain complexes of filtration quotients $\mathcal{F}_h\overline{\mathbb{B}}_*\mathcal{S}_k/\mathcal{F}_{h-1}\overline{\mathbb{B}}_*\mathcal{S}_k$ in $(*)$ occur as (shifted) columns $(\mathsf{E}^0_{h,*-h}, \mathsf{d}^0_{h,*-h})$ of this spectral sequence. Visy's observation is therefore equivalent to saying that the homology of each column is concentrated in degree 0. In particular, the E^1-page consists of a single chain complex $(\mathsf{E}^1_{*,0}, \mathsf{d}^1_{*,0})$, and our spectral sequence collapses on the E^2-page.



The E^0-page. The E^1-page.

Clearly, the complex $(\mathsf{E}^1_{*,0}, \mathsf{d}^1_{*,0})$ is a model for the homology of G, cf. Weibel [Wei94, §5.5], and it is considerably smaller than the bar complex.

The notion of factorability has been introduced to explain this collapsing phenomenon and to give a description of the complex $(\mathsf{E}^1_{*,0}, \mathsf{d}^1_{*,0})$. We now outline the original definition by Visy [Vis11].

A factorable group consists of the following data. A discrete group G, a generating set $S \subset G$ which is closed under taking inverses, and a splitting map $\eta : G \to G \times G$, $g \mapsto (\overline{g}, g')$. The element g' is called the *prefix* of g and $\overline{g}$ is called the *remainder* of g. Denote by ℓ the word length with respect to S. We say that η is a *factorization map* for the pair (G, S) if it satisfies the following axioms:

(F1) $g = \overline{g} \cdot g'$.

(F2) $\ell(g) = \ell(\overline{g}) + \ell(g')$.

(F3) If $g \neq 1$ then $g' \in S$.

We remark that if η is a factorization map, then iteratively applying η to the remainder we obtain a normal form, i.e. a section $G \to S^*$ of the canonical projection $S^* \to G$, where S^* denotes the free group over S.

The triple (G, S, η) is called a *factorable group* if the factorization map η satisfies $(gt)' = (g't)'$ for all pairs $(g, t) \in G \times S$ with $\ell(gt) = \ell(g) + \ell(t)$. This latter property should be regarded as a compatibility condition about prefixes of products. The intuition behind it is that factorability assures the existence of particularly nice normal forms. For example, if (G, S, η) is factorable, then the image of the induced normal form $G \to S^*$ is closed under taking subwords. To get an idea of what this has to do with the spectral sequence situation, we remark that the factorization map η can be thought of as encoding a collapsing chain homotopy on each vertical complex $(\mathsf{E}^0_{p,*}, \mathsf{d}^0_{p,*})$.

Our prototypical example $(\mathcal{S}_k, T)$, the k-th symmetric group together with the generating set of all transpositions, can be endowed with a factorability structure as follows. By (F2) we must have $\eta(\mathrm{id}) = (\mathrm{id}, \mathrm{id})$. Let $\sigma \in \mathcal{S}_k$ be a non-trivial permutation and denote by m its largest non-fixed point. As prefix σ' we take the transposition $\tau = (m\ \ \sigma^{-1}(m))$. The remainder is then given by $\overline{\sigma} = \sigma \circ \tau$. For example, in cycle notation, $(1\,2\,4) \in \mathcal{S}_4$ is mapped to $((1\,2), (2\,4))$. Visy [Vis11] showed that $(\mathcal{S}_k, T, \eta)$ is a factorable group.

Further interesting examples including alternating groups, dihedral groups and certain Coxeter groups can be found in the works of Ozornova [Ozo], Rodenhausen [Rod], [Rod11] and Visy [Vis11]. Also, free and direct products (or, more generally, graph products) as well as semidirect products of factorable groups are again factorable, cf. [Rod] and [Vis11].

We now briefly survey the results of Visy [Vis11] and Wang [Wan11] in finding small homology models for factorable groups. Recall from page 8 the spectral sequence associated to the generating set S for the group G.

> **Theorem (Visy)** *If (G, S, η) is a factorable group then the homology of each vertical complex $(\mathsf{E}^0_{p,*}, \mathsf{d}^0_{p,*})$ is concentrated in degree 0.*

In other words, the E^1-page of the spectral sequence consists of a single chain complex $(\mathsf{E}^1_{*,0}, \mathsf{d}^1_{*,0})$. To describe this complex, Visy introduces the complex $(\mathbb{V}_*, \partial^{\mathbb{V}}_*)$ together with an embedding of chain complexes

$$\kappa : (\mathbb{V}_*, \partial^{\mathbb{V}}_*) \longrightarrow (\mathsf{E}^1_{*,0}, \mathsf{d}^1_{*,0}).$$

He also proves that κ is surjective for symmetric groups with the above factorability structure. Wang [Wan11] uses a clever counting argument to conclude the following:

> **Theorem (Wang)** *Let (G, S, η) be a factorable group. If S is finite then κ is an isomophism.*

There is an obvious generalization of the notion of factorability to monoids, and Wang in loc. cit. points out that her Theorem still holds when the group G is replaced by a right-cancellative monoid.

The works of Visy and Wang investigate the homological behavior mentioned above from the perspective of factorability of groups or monoids. One aim of this thesis is to study factorability in a far broader context and to provide a conceptual treatment for it, putting it in perspective with rewriting systems and discrete Morse theory. To this end, we first equivalently reformulate factorability of groups in terms of actions of monoids P_n and Q_n. This point of view suggests a notion of factorability for monoids which is slightly stronger than the one used by Wang.

The advantage of our approach is that now a factorability structure on a monoid gives rise to a discrete Morse function on its bar complex.

A discrete Morse function f on a cell complex K assigns to every (open) cell of K a natural number, providing a partition of the cells of K into regular and critical ones. As in classical Morse theory, a discrete Morse function induces a discrete gradient flow, and along this flow we can collapse K onto a complex that is built up from the critical cells. This latter complex is called the discrete Morse complex associated to f.

Here we are mostly interested in an algebraic version of discrete Morse theory, where the objects of study are based chain complexes.

Large parts of this thesis are devoted to studying the Morse complex associated to the bar resolution and the bar complex of a factorable monoid. For example, as an immediate consequence, we obtain geometric and homological finiteness properties of factorable monoids.

A more detailed analysis will reveal that our discrete Morse complex coincides with Visy's complex $(\mathbb{V}_*, \partial_*^{\mathbb{V}})$ and furthermore that, with our stronger notion of factorable monoid, the map κ is always an isomorphism. Moreover, we will find that κ has an interpretation in the world of discrete Morse theory.

Note that here we are only concerned with one factorable group at a time. Another interesting direction in the study of factorability takes into account families of factorable groups. This has applications in the computation of the homology of moduli spaces of Riemann surfaces. Before discussing the organization of this thesis, let us briefly report on this connection.

Let $F_{g,1}^m$ be a Riemann surface of genus g with one boundary curve and m marked points ("punctures"). Bödigheimer's *Hilbert Uniformization* provides a finite double complex $\mathbb{Q}_{\bullet,\bullet}$ that computes the homology of the moduli space $\mathfrak{M}_{g,1}^m$ of $F_{g,1}^m$. This double complex has a description in terms of the family of symmetric groups equipped with the aforementioned factorability structure. For further reading see [ABE08], [Böd90], [Böd], [Ebe], [Vis11], [Wan11].

Let $(G^{(k)}, S^{(k)}, \eta^{(k)})$ be a factorable group for every $k \geq 0$. Recall that for each of these factorable groups we have an associated spectral sequence, and we are now aiming for obtaining connecting homomorphisms between the respective E^0-pages. To this end, we arrange the sets underlying the groups $G^{(k)}$ in a semisimplicial set. That is, for every

$k \geq 1$ and all $i = 0, \ldots, n$ we are given face maps (not necessarily homomorphisms!)

$$D_i^{(k)} : G^{(k)} \to G^{(k-1)},$$

satisfying the simplicial identities $D_i D_j = D_{j-1} D_i$ for $i < j$. (For simplicity of notation we suppress the upper index k.) In order to make these face maps compatible with the respective generating sets S, we need to impose further conditions, namely that

- $D_i(1) = 1$ for all i, and
- $\ell(D_i(gh) \cdot (D_i(h))^{-1}) \leq \ell(g)$ for all $g, h \in G$, where the word length ℓ is taken with respect to the respective generating set.

Our standard example is the family of symmetric groups $G^{(k)} = \mathcal{S}_{k+1}$. As generating set $S^{(k)}$ we take all transpositions, and the factorization map is as defined on page 9. We now describe the face maps D_i. Let $\sigma \in G^{(k)} = \mathcal{S}_{k+1}$. Then, in the cycle notation of σ, D_i removes the entry $i+1$ and renormalizes all larger entries, meaning that every entry $j > i+1$ is replaced by $j-1$. For example, consider $\sigma = (1\,2\,4)\,(3\,6)\,(5) \in G^{(5)} = \mathcal{S}_6$. The $D_i(\sigma)$ then are as follows:

$$\begin{array}{lll} D_0(\sigma) = (1\,3)\,(2\,5)\,(4) & D_2(\sigma) = (1\,2\,3)\,(4)\,(5) & D_4(\sigma) = (1\,2\,4)\,(3\,5) \\ D_1(\sigma) = (1\,3)\,(2\,5)\,(4) & D_3(\sigma) = (1\,2)\,(3\,5)\,(4) & D_5(\sigma) = (1\,2\,4)\,(3)\,(5) \end{array}$$

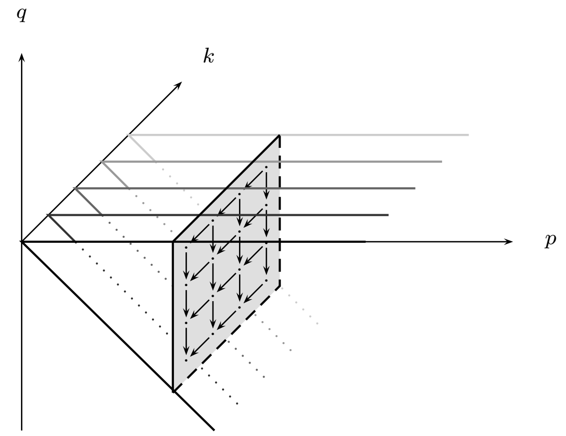
We remark that the maps $D_i : \mathcal{S}_{k+1} \to \mathcal{S}_k$ arise quite naturally in the study of flow lines of harmonic functions on Riemann surfaces. For details see [ABE08], [Böd90], [Böd], [Ebe].

Now, given a family of factorable groups arranged in a semisimplicial set, with the face maps satisfying the above compatibility conditions, we obtain maps $\delta_n^{(k)} : \overline{\mathbb{B}}_n G^{(k)} \to \overline{\mathbb{B}}_n G^{(k-1)}$ by assigning to a cell in homogeneous notation $[g_n : \ldots : g_1] \in \overline{\mathbb{B}}_n G^{(k)}$ the alternating sum $\sum_{i=0}^{k} (-1)^i [D_i(g_n) : \ldots : D_i(g_1)]$ and extending linearly. Observe that the $\delta_n^{(k)}$ commute with the face maps of the bar complex, which in homogeneous notation are given by deletion of entries.

Our assumptions guarantee that the $\delta_n^{(k)}$'s respect the filtration $\mathcal{F}_\bullet \overline{\mathbb{B}}_n G^{(k)}$ induced by $S^{(k)}$, hence descend to filtration quotients, and in this way yield the desired connecting homomorphisms between the respective E^0-pages,

$$\delta_{p,q}^{(k)} : \mathsf{E}^0_{p,q} G^{(k)} \longrightarrow \mathsf{E}^0_{p,q} G^{(k-1)}.$$

Fix $h \geq 0$ and simultaneously look at the h-th column of the E^0-pages of all groups $G^{(k)}$. We obtain a double complex $\mathsf{E}^0_{h,*} G^{(*)}$. The horizontal differential is $\delta_{h,*}^{(*)}$ and the vertical differential is just the differential of the respective E^0-page, see the figure below. Note that the k-th column of this double complex is the h-th column of the E^0-page of the factorable group $G^{(k)}$, and thus the homology of each column of this double complex is concentrated in top-degree.



The double complex $\mathsf{E}^0_{h,*}G^{(*)}$.

Taking $G^{(k)} = \mathcal{S}_{k+1}$ as above, this double complex computes the homology of a wedge sum of moduli spaces. More precisely, fix $h \geq 0$ and define the truncated double complex $\mathbb{T}_{\bullet,\bullet}$ by

$$\mathbb{T}_{i,j} = \begin{cases} \mathsf{E}^0_{h,j}\mathcal{S}_{i+1} & \text{if } 0 \leq i \leq 2h \text{ and } 0 \leq j \leq h, \\ 0 & \text{else.} \end{cases}$$

One version of Bödigheimer's result reads as follows, cf. [ABE08], [Böd].

Theorem (Bödigheimer). $H_*(\mathbb{T}_{\bullet,\bullet}) \cong \bigoplus_{2g+m=h} H_*(\mathfrak{M}^m_{g,1})$.

In [ABE08], a double complex $\mathbb{Q}_{\bullet,\bullet}$ is introduced for every choice of parameters $g, m \geq 0$, and the direct sum of all $\mathbb{Q}_{\bullet,\bullet}$'s with parameters $2g + m = h$ is isomorphic to $\mathbb{T}_{\bullet,\bullet}$. The main result of [Böd] is the following.

Theorem (Bödigheimer). $H_*(\mathbb{Q}_{\bullet,\bullet}) \cong H_*(\mathfrak{M}^m_{g,1})$.

The complexes $\mathbb{Q}_{\bullet,\bullet}$ allow a purely combinatorial description. Furthermore, all the respective vertical complexes have homology concentrated in top-degree, and the associated top-degree chain complexes can be described in terms of the factorability structure on symmetric groups. This combinatorial model has been used in the works of Mehner [Meh11] and Wang [Wan11] to do homology computations for moduli spaces with parameters $2g + m \leq 7$.

Considering families of factorable groupoids, [Wan11] is able to do similar calculations for moduli spaces of Kleinian surfaces.

We now discuss the organization of this thesis.

In **Chapter 0** we collect prerequisites on monoids and recall the definitions of geometric and homological finiteness properties.

Chapter 1 is concerned with various kinds and applications of discrete Morse theory. The chapter is separated into three parts. In the first part, we give a self-contained exposition of Morse theory for chain complexes. The second part starts with a reminder on several types of bar constructions on monoids. We then introduce rewriting systems and give a brief survey on Brown's proof of the Anick–Groves–Squier Theorem, stating that a complete rewriting system on a monoid X gives rise to small resolutions of X. This proof is based on discrete Morse theory. Section 1.3 is joint work with Ozornova. We give an alternative proof of a Theorem by Charney, Meier and Whittlesey on the existence of finite resolutions for Garside monoids, using discrete Morse theory only.

In **Chapter 2** we survey Visy's work on factorable groups and Wang's generalization to monoids. To the latter we will refer to as *weak* factorability. Wang proved the following, cf. Theorem 2.1.23:

> **Theorem (Wang)** *Let (X, S, η) be a weakly factorable monoid. Assume that X is right-cancellative and that S is finite. Then the map $\kappa : (\mathbb{V}_*, \partial^{\mathbb{V}}_*) \to (\mathsf{E}^1_{*,0}, \mathsf{d}^1_{*,0})$ is an isomorpism.*

Mehner pointed out that the above Theorem does not hold if the assumption of X being right-cancellative is dropped. To make this precise, we introduce a local-to-global condition for normal forms $X \to S^*$, called the *recognition principle*. In Section 2.2 we define factorability via actions of certain monoids P_n and Q_n. For groups (and, more generally, for right-cancellative monoids) the two notions coincide. For arbitrary monoids we have Theorem 2.2.6:

> **Theorem** *Let X be a monoid, S a generating set and $\eta : X \to X \times X$ a factorization map. Then the following are equivalent:*
>
> *(a) (X, S, η) is a factorable monoid.*
>
> *(b) (X, S, η) is a weakly factorable monoid and η satisfies the recognition principle.*

Indeed, if X is right-cancellative, then every factorization map $\eta : X \to X \times X$ satisfies the recognition principle.

The remainder of Chapter 2 is devoted to study the monoids P_n and Q_n in detail. For example, we show that every monoid Q_n admits an absorbing element. This allows to explicitly write down a normal form algorithm for factorable monoids, yielding the following result, which is Corollary 2.3.16.

Corollary *Let (X, S, η) be a factorable monoid. If S is finite then X has Dehn function of at most cubic growth, and in particular X has solvable word problem.*

We conclude Chapter 2 by investigating a connection between the monoids P_n, Q_n and Visy's map κ: The monoid Q_n is a quotient of P_n, and we say that an element in P_n is *small* if its fibre under the quotient map $P_n \twoheadrightarrow Q_n$ consists of exactly one element. Proposition 2.3.36 then states that κ has a universal description in terms of a sum indexed by the small elements of P_n.

Chapter 3 is the heart of this thesis. To begin, we show that the normalized bar resolution and complex of a factorable monoid are highly structured. More precisely, we have the following, cf. Theorem 3.1.8:

Theorem *A factorability structure on a monoid X naturally gives rise to a discrete Morse function on the normalized bar complex $(\overline{\mathbb{B}}_* X, \overline{\partial}_*)$.*

An analogous result holds for the classifying space BX, and we record the following immediate consequence, cf. Corollary 3.1.22 and Remark 3.1.23:

Corollary *Let (X, S, η) be a factorable monoid. If S is finite then X satisfies the geometric and homological finiteness properties F_∞ and FP_∞.*

Associated to our Morse function, discrete Morse theory provides two distinct but isomorphic chain complexes, which are chain homotopy equivalent to the bar complex. Namely, these are the discrete Morse complex, denoted by $((\overline{\mathbb{B}}_* X)^\theta, \overline{\partial}^\theta_*)$, and the complex of discrete harmonic forms, for which we write $((\overline{\mathbb{B}}_* X)^\Theta, \overline{\partial}_*)$.

The main result of this thesis is Theorem 3.3.8. It draws a connection between Visy's complex $(\mathbb{V}_*, \partial^{\mathbb{V}}_*)$, the E^1-page of the previously discussed spectral sequence, and the just-mentioned two complexes from discrete Morse theory. The following is a slightly weakened version of Theorem 3.3.8.

Theorem *For every factorable monoid (X, S, η) we have the following commutative diagram of chain complexes:*

$$\begin{array}{ccc}
(\mathbb{V}_*, \partial^{\mathbb{V}}_*) & \xrightarrow[\cong]{\kappa} & (\mathsf{E}^1_{*,0}, \mathsf{d}^1_{*,0}) \\
\| & & \| \\
((\overline{\mathbb{B}}_* X)^\theta, \overline{\partial}^\theta_*) & \xrightarrow[\cong]{\Theta^\infty} & ((\overline{\mathbb{B}}_* X)^\Theta, \overline{\partial}_*)
\end{array}$$

As an immediate consequence we obtain Corollary 3.3.9:

Corollary *κ is an isomorphism for every factorable monoid.*

Moreover, κ is not just *any* isomorphism, but can be identified with the stabilization isomorphism Θ^∞, which occurs naturally in discrete Morse theory, cf. Section 1.1.5.

We conclude Chapter 3 by pointing out that the notion of factorability fits into the framework of complete rewriting systems. Theorem 3.4.1 is joint work with Ozornova:

> **Theorem (H, Ozornova)** *If (X, S, η) is a factorable monoid then X possesses a complete rewriting system over the alphabet S.*

It is well-known that a complete rewriting system on a monoid X gives rise to a discrete Morse function on the bar complex BX, see e.g. Brown [Bro92] or Subsection 1.2.4. When proving the above theorem, we furthermore show that the discrete Morse function induced by this complete rewriting system coincides with our construction of a discrete Morse function from a factorability structure. The upshot is that a factorability structure is a special case of a complete rewriting system. The advantage we gain from this extra structure is that we have an explicit description of the differentials in the associated Morse complex.

In **Chapter 4** we use our previous results to compute homology groups of a 3-parameter family of monoids $t_m(p,q)$. These monoids occur as abstract generalizations of Thompson's group F, and we find F as the group of fractions of $t_\infty(1,2)$. We derive recursion formulas for the homology of $t_m(p,q)$ for all values of $m > 0$, $q > 0$ and $0 < p \leq q$, cf. Corollary 4.5.4. For $m = \infty$, Proposition 4.5.5 provides an explicit computation:

> **Proposition** *For $0 < p \leq q$ and $n > 0$ we have $H_n(t_\infty(p,q)) \cong \mathbb{Z}^{(q-1)^{n-1}\cdot q}$.*

Here we remark that the homology of the monoids $t_m(p,q)$ does in fact not depend on the particular choice of the parameter p.

Denote by $\mathcal{T}_m(p,q)$ the group of fractions of $t_m(p,q)$. Fixing $m = \infty$ and $q = p + 1$, we obtain a 1-parameter family of groups $\mathcal{T}_\infty(n-1,n)$; as mentioned, $\mathcal{T}_\infty(1,2)$ is Thompson's famous group F. This family also arises from geometric considerations in Brown [Bro87], and, using this geometric intuition, Stein [Ste92] computes the homology of each of these groups.

In Subsection 4.3.2 we show that for $q = p+1$ the monoids $t_\infty(p,q)$ are cancellative and satisfy the right Ore condition. It follows that the canonical map $i : t_\infty(p,q) \to \mathcal{T}_\infty(p,q)$ induces an isomorphism on homology, cf. Cartan–Eilenberg [CE99, Proposition 4.1]. Therefore, our above Proposition provides in particular a recomputation of the homology of each group $\mathcal{T}_\infty(n-1,n)$.

To conclude this introduction, let us remark that, rather than groups, it seems that monoids are the natural setting for factorability. The axioms of a factorable group effectively do not make use of the existence of invertible elements, because the requirement that the generating set S of a factorable group G is closed under taking inverses is merely needed to guarantee that the sub*monoid* in G generated by S is the whole group.

The existence of non-trivial invertible elements can sometimes even be obstructive. In [Bro92, p.157], Brown writes: *"When we are interested in a group G, however, we will often get results about G by studying a suitable submonoid $M \subset G$. (This idea is suggested by the work of Craig Squier)."* Indeed, for every non-trivial group, the homology model $\mathbb{V}_*$ is infinite dimensional. For monoids, however, it can happen that $\mathbb{V}_*$ is finite dimensional, and even if it is infinite dimensional, it is still considerably smaller than the model associated to its group of fractions. Examples are provided by the large class of generalized Thompson groups and monoids in Chapter 4.

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Preliminaries on monoids

A *monoid* is a triple $(X, \cdot, e)$, where X is a set, $\cdot : X \times X \to X$ is an associative operation, and e is a neutral element. We will usually suppress the multiplication symbol $\cdot$ and just write xy instead of $x \cdot y$.

In fancy language, a monoid is a category with exactly one object. This point of view allows a comfortable definition of monoid action: Let $\mathcal{O}$ be an object in an arbitrary category $\mathscr{C}$ and denote by $\mathrm{End}(\mathcal{O}) = \mathrm{Mor}_{\mathscr{C}}(\mathcal{O}, \mathcal{O})$ the endomorphism monoid of $\mathcal{O}$. An *action* of a monoid X on the object $\mathcal{O}$ is nothing but a morphism of monoids $X \to \mathrm{End}(\mathcal{O})$.

Congruence relations

If S is a set then we write S^* for the free monoid over S. Elements of S^* are called *words* over S, and S is sometimes referred to as a (formal) *alphabet*. Words $w \in S^*$ will be written as sequences $w = (s_n, \dots, s_1)$ with numbering from right to left. Multiplication in S^* is given by concatenation of words, and the empty word $\epsilon = (\,)$ is a neutral element. Equivalently speaking, S^* is the monoid of finite sequences in S.

Convention. This work is mostly about monoids, and it is for this reason that for us 0 is a natural number. This way, $(\mathbb{N}, +, 0)$ becomes a free monoid on one generator.

Recall that a congruence relation on a monoid is an equivalence relation that is compatible with the monoid multiplication map. More precisely, an equivalence relation $R \subseteq X \times X$ is a congruence relation if and only if for all elements $x, x', y, z \in X$ we have that $x \, R \, x'$ implies $yxz \, R \, yx'z$. In other words, an equivalence relation R on a monoid X is a congruence relation if and only if R is a submonoid of the direct product $X \times X$.

Given an arbitrary relation $R \subseteq X \times X$, we denote by $\langle R \rangle$ the congruence relation induced by R, that is, $\langle R \rangle$ is the smallest submonoid of $X \times X$ containing the reflexive, symmetric and transitive closure of R.

If R is a congruence relation on X, then multiplication descends to congruence classes, and thus the quotient X/R inherits the structure of a monoid.

Presentations

Let S be a formal alphabet and $R \subseteq S^* \times S^*$ a relation. We then abbreviate

$$\langle S \mid R \rangle := S^*/\langle R \rangle,$$

and we call $\langle S \mid R \rangle$ a *presentation* for the monoid $X := S^*/\langle R \rangle$. The canonical quotient map $\mathrm{ev} : S^* \twoheadrightarrow X$ will sometimes be referred to as *evaluation*.

Let $X = \langle S \mid R \rangle$. A *normal form* (with respect to the generating set S) is a section $X \to S^*$ for the evaluation map $S^* \to X$. In other words, a normal form is the same as for every element $x \in X$ choosing a preferred way to write x as a product of generators.

Convention. If X is an "abstract" monoid, i.e. if X is defined in terms of a presentation, then we will usually denote its neutral element by ϵ_X or simply ϵ. (This is motivated by the fact that the empty word $\epsilon \in S^*$ is a representative for the neutral element in X.)

In contrast, we stick to writing 0 for the neutral element of $\mathbb{N}$, and we write 1 for the mulitplicative neutral element of $\mathbb{Z}$.

Finiteness properties

Let X be a monoid and denote by $\mathbb{Z}X$ its monoid ring. A *projective resolution* of X by right $\mathbb{Z}X$-modules is a resolution of the form

$$\cdots \xrightarrow{\partial} F_1 \xrightarrow{\partial} F_0 \xrightarrow{\partial} \mathbb{Z} \longrightarrow 0,$$

with X acting trivially on $\mathbb{Z}$, and each F_i being a projective right $\mathbb{Z}X$-module.

Definition (Homological finiteness properties). Let X be a monoid.

(a) We say that X is of type *right*-FP if it possesses a finite projective resolution by right $\mathbb{Z}X$-modules. (The term "finite" refers to F_* being finitely generated over $\mathbb{Z}X$.)

(b) X is of type *right*-FP_∞ if it possesses a projective resolution $F_* \to \mathbb{Z}$ by right $\mathbb{Z}X$-modules which is of finite type, that is, every F_i is finitely generated (as a $\mathbb{Z}X$-module).

(c) X is of type *right*-FP_n if the above holds for all $i \le n$, i.e. each module $F_1, \ldots, F_n$ is finitely generated.

The properties left-FP, left-FP_∞ and left-FP_n are defined analogously. We say that a monoid is FP_∞ if it is right-FP_∞ and left-FP_∞.

Remark. Note that if G is a group then every right $\mathbb{Z}G$-module can be made into a left $\mathbb{Z}G$-module and vice versa. In particular, for groups, the notions of left-FP and right-FP etc. are equivalent. For monoids we can no longer identify right $\mathbb{Z}X$-modules and left $\mathbb{Z}X$-modules. Indeed, in Cohen [Coh92] a monoid is presented that is right-FP_∞ but not left-FP_1.

Cohen [Coh97, §6] provides lots of examples and counterexamples of monoids satisfying respectively failing the aforementioned homological finiteness properties. Further finiteness properties (for groups) may be found in Bestvina–Brady [BB97, §3].

Definition (Geometric finiteness properties). Let X be a monoid.

(a) We say that X satisfies the geometric finiteness property F, if its classifying space BX is homotopy equivalent to a finite CW complex.

(b) X is of type F_∞ if it admits a classifying space of finite type, meaning that BX is homotopy equivalent to a CW complex with only finitely many cells in each dimension.

It is well-known that every group of type F_∞ (resp. F) is of type FP_∞ (resp. FP), simply by considering universal covers.

1 Discrete Morse theory and rewriting systems

1.1 Discrete Morse theory

Discrete Morse theory comes in two flavours, either topological (see e.g. [Bro92], [For98]) or algebraic (see e.g. [Coh97], [Koz08]). In the former case one studies CW complexes, in the latter case based chain complexes. The main idea is to simplify a given complex by projecting onto a homotopy equivalent quotient complex. This chapter covers only the algebraic version. Yet, to make it more accessible, we first briefly discuss the concept of simplicial collapse, which is the geometric intuition behind discrete Morse theory.

1.1.1 Simplicial collapse

Let K be a simplicial complex. Choose a maximal cell x and a free codimension 1-face y of x (that is, y is not the face of any other cell). We can then deformation retract the interiors of x and y onto the complementary boundary $\partial x \setminus y$. This is referred to as an elementary collapse (of x from y away). The cell x is called *collapsible* and y is called *redundant*. The idea behind this nomenclature is that we think of y as being redundant in the sense that we can remove its interior from K without changing the homotopy type of K.

In Figure 1.1 we start with the simplicial 2-disk and iteratively perform elementary collapses, ending up with a one-point space. The aforementioned deformation retractions are indicated by gray shaded arrows.

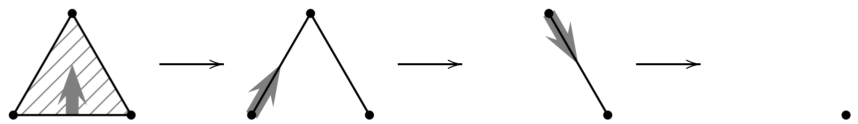


Figure 1.1: Collapsing the simplicial 2-disk onto one of its vertices.

The notion of elementary collapse of free faces in simplicial complexes is just a special case of the concept of simplicial collapse of regular faces in CW complexes:

Definition 1.1.1 Let K be a CW complex. Let x be an n-cell in K and denote by

$h : D^n \to K$ its characteristic map. Let y be a face of x. We say that y is a *regular* face of x if the following conditions hold:

(a) The restriction $h|_{h^{-1}(y)} : h^{-1}(y) \to y$ is a homeomorphism, and

(b) $\overline{h^{-1}(y)} \cong D^{n-1}$.

Example 1.1.2 In Figure 1.2.(a), y is a regular face of x. In Figure 1.2.(b) y is not a regular face of x, because $\overline{h^{-1}(y)} \cong S^1$.

Figure 1.2: Regular and irregular faces.

Remark 1.1.3 (a) In a simplicial complex every face is regular.

(b) We warn the reader that in a CW complex a free face need not be regular: In Example 1.1.2.(b) y is a free face of x, but not a regular one.

Let K be a CW complex, x an n-cell in K and y a regular face of x. We can then modify K in a way very similar to the elementary collapse of free faces. We denote the resulting CW complex by K'. Note that y might have more cofaces than just x. In K' these cofaces are glued along y's complementary boundary $\partial x \setminus y$. Figure 1.3 depicts the simplicial collapse of $x = ABC$ away from $y = AB$. Note how in K' the cell ADB is glued along $\partial(ABC) \setminus AB = BC \cup CA$.

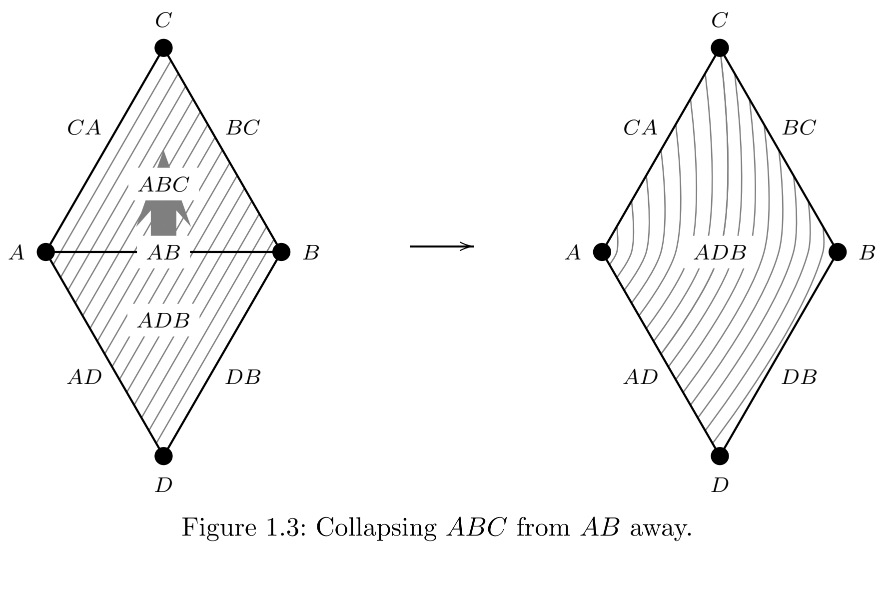


Figure 1.3: Collapsing ABC from AB away.

If K' is obtained from K by a simplicial collapse then $K' \simeq K$. Furthermore, if K was finite, then K' consists of two cells less than K. It is in this sense that we think of K' as a "simpler" model of K. The price to pay is that the boundaries in K' might be more complicated. For example, considering the complex K from Figure 1.3, we have $\partial(ADB) = AD \cup DB \cup AB$, whereas in K' we have $\partial(ADB) = AD \cup DB \cup BC \cup CA$.

Forman's discrete Morse theory (for cell complexes) provides an efficient way of encoding series of simplicial collapses and to describe the resulting quotient complex. Intuitively speaking, algebraic Morse theory discards geometric aspects and studies the impact of simplicial collapses on the underlying cellular chain complexes.

1.1.2 Algebraic collapse

In the previous subsection we discussed the geometric origins of discrete Morse theory. We now try to motivate the algebraic analogon, discrete Morse theory for based chain complexes, from an algebraic point of view. For the sake of simplicity we will not give rigorous proofs here and all complexes are assumed to be finite. Sections 1.1.3 to 1.1.6 are then devoted to a more thorough treatment.

Assume we are interested in the homology of the following elementary free chain complex:

$$\mathsf{C}: \qquad 0 \longrightarrow \mathbb{Z}^m \xrightarrow{\partial} \mathbb{Z}^n \longrightarrow 0$$

Let us fix $\mathbb{Z}$-bases $\{x_1, \ldots, x_m\}$ for $\mathbb{Z}^m$ and $\{y_1, \ldots, y_n\}$ for $\mathbb{Z}^n$, respectively. We implicitly understand $\mathbb{Z}^n$ to be equipped with the inner product $[\_ : \_] : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ that is obtained by regarding $\{y_1, \ldots, y_n\}$ as an orthonormal basis. We can then associate a matrix $A \in \mathbb{Z}^{n\times m}$, $A = (a_{j,i})$, to $\partial : \mathbb{Z}^m \to \mathbb{Z}^n$ as follows,

$$a_{j,i} = [\partial x_i : y_j].$$

Let us further assume that some entry of A is invertible, say $a_{j,i}$. (We consider i and j to be fixed from now on.) We will refer to $a_{j,i}$ as our *pivot element*. One says that x_i is *collapsible*, y_j is *redundant*, and all the other basis elements are called *essential*. Geometrically, one should think of y_j being a face of x_i, and in this context invertibility of $a_{j,i}$ corresponds to regularity of y_j in x_i.

Define matrices $M \in \mathbb{Z}^{m\times m}$ and $N \in \mathbb{Z}^{n\times n}$ as follows:

$$\begin{aligned} M &= \left(x_i \,\middle|\, x_1 - \frac{a_{j,1}}{a_{j,i}} x_i \,\middle|\, \ldots \,\middle|\, \widehat{0} \,\middle|\, \ldots \,\middle|\, x_m - \frac{a_{j,m}}{a_{j,i}} x_i \right), \\ N &= (Ax_i \,|\, y_1 \,|\, \ldots \,|\, \widehat{y_j} \,|\, \ldots \,|\, y_n). \end{aligned}$$

Observe that $\det(M) = (-1)^{i+1} \cdot \det(x_1|\ldots|x_m)$ and $\det(N) = [\partial x_i : y_j] \cdot (-1)^{j+1} \cdot \det(y_1|\ldots|y_n)$, and therefore both matrices are invertible. An easy computation shows that $N^{-1}AM \in \mathbb{Z}^{n\times m}$ takes the following form,

$$N^{-1}AM = \begin{pmatrix} 1 & 0 \\ 0 & A' \end{pmatrix},$$

for some matrix $A' \in \mathbb{Z}^{(n-1)\times(m-1)}$. In other words, the following diagram commutes

$$\begin{tikzcd}
\mathbb{Z}^m \arrow[rr, "A"] & & \mathbb{Z}^n \\
\mathbb{Z} \oplus \mathbb{Z}^{m-1} \arrow[u, "M"', "\cong"] \arrow[rr, "\mathrm{id} \oplus A'"'] & & \mathbb{Z} \oplus \mathbb{Z}^{n-1} \arrow[u, "\cong", "N"']
\end{tikzcd}$$

and thus the homology of the complex C is isomorphic to the homology of the following smaller complex

$$\mathsf{C}' : \qquad 0 \longrightarrow \mathbb{Z}^{m-1} \xrightarrow{A'} \mathbb{Z}^{n-1} \longrightarrow 0.$$

The complex C' is freely generated by the essential basis elements of C. It is called the *Morse complex* with respect to the matching $\{(x_i, y_j)\}$. Geometrically, one should think of C' as being the cellular chain complex of the CW complex obtained from collapsing x_i from y_j away.

Algebraically, the way we obtained C' from C can be thought of as a first step of a two-sided Gauß elimination. As long as we find pivot elements, we can iterate this procedure and go over to smaller and smaller complexes with isomorphic homology. For later reference, we discuss one example explicitly.

Example 1.1.4 We take $m = 3$ and $n = 2$ and consider $\mathbb{Z}^m$ and $\mathbb{Z}^n$ to be equipped with the standard basis. Consider the following chain complex

$$\mathsf{C} : \qquad 0 \longrightarrow \mathbb{Z}^{3} \xrightarrow{A} \mathbb{Z}^{2} \longrightarrow 0$$

with

$$A = \begin{pmatrix} 3 & 2 & -1 \\ 0 & 1 & 4 \end{pmatrix}.$$

As pivot element we take $a_{2,2} = 1$. Calculating M and N gives

$$\begin{aligned} M &= (x_2 \,|\, x_1 \,|\, x_3 - 4x_2) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -4 \\ 0 & 0 & 1 \end{pmatrix}, \\ N &= (Ax_2 \,|\, y_1) = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}, \end{aligned}$$

and we obtain

$$N^{-1}AM = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & -9 \end{pmatrix}.$$

Therefore the homology of the complex C is isomorphic to the homology of the complex

$$\mathsf{C}' : \qquad 0 \longrightarrow \mathbb{Z}^{2} \xrightarrow{(3 \quad -9)} \mathbb{Z} \longrightarrow 0.$$

We now investigate the question how the above construction can be expressed in terms of the associated incidence graphs of C and C'.

Recall that the *incidence graph* of a based chain complex has one vertex for each basis element, and there is a labelled edge $v \to w$ if $\dim(v) = \dim(w) + 1$ and $[\partial v : w] \neq 0$. In this case, the label of $v \to w$ is the incidence number $[\partial v : w]$. For convenience one might want to compare Example 1.1.4 and Figure 1.4. A precise description is given in Definition 1.1.14.

Denote by Γ the incidence graph of C and denote by Γ' the incidence graph of C'. How does Γ' arise from Γ?

Clearly, the vertices of Γ' are the essential vertices of Γ, i.e. all vertices of Γ except for x_i and y_j. To elaborate the relation between the edges in Γ and Γ' we need an intermediate step. Denote by e the edge in Γ that corresponds to the chosen pivot element $a_{j,i}$, i.e. e points from x_i to y_j, and the label of e is $a_{j,i}$. We now invert e. By this we mean two things: Firstly, inverting the direction of the arrow and secondly changing the label to $-a_{j,i}^{-1}$. We write $\check{\Gamma}$ for the labelled graph obtained this way. The graph Γ' is now the full subgraph of the total flow of $\check{\Gamma}$, that is, the label of an edge $v \to w$ in Γ' is the sum over all labels of paths from v to w in $\check{\Gamma}$, where the label of a path is the product of the labels of its edges.

Example 1.1.5 Consider the based chain complex C from Example 1.1.4. Its incidence graph Γ is depicted in Figure 1.4.

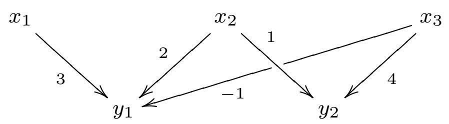


Figure 1.4: The incidence graph Γ.

Inverting the labelled edge $x_2 \to y_2$ yields the following graph:

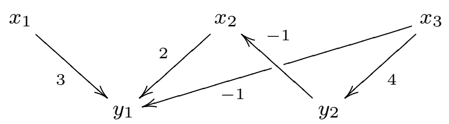


Figure 1.5: The graph $\check{\Gamma}$.

The total flow is now easily read off. Clearly, the total flow between x_1 and y_1 is just 3 and the total flow from x_3 to y_1 is $-1 + 4 \cdot (-1) \cdot 2 = -9$. We therefore obtain the graph depicted in Figure 1.6 which is indeed the incidence graph of C', compare Example 1.1.4.

Note that there is no loss of information when passing from a based chain complex C to its incidence graph Γ, for we can completely recover C from Γ. We can therefore go back

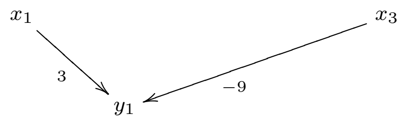


Figure 1.6: The graph Γ'.

and forth between based chain complexes and their incidence graphs. This observation allows a purely graph-theoretical formulation of the two-sided Gauß elimination discussed on page 23.

For efficiency reasons it would be nice if we could invert several edges simultaneously, without determining the total flow of $\check{\Gamma}$ after every single inversion. For this we need some criterion that tells us whether a given set of edges may or may not be inverted simultaneously. One condition is that this set of edges constitutes a matching on Γ, meaning that no two edges of the set share a vertex. Of course, this condition alone is not sufficient: For example, consider the graph Γ in Figure 1.4. The edges $x_2 \to y_2$ and $x_3 \to y_1$ must not be inverted simultaneously, because then x_1 would be the only essential vertex, and thus the associated chain complex would be given by $0 \to \mathbb{Z} \to 0 \to 0$, which does not compute the right homology. A plausibility argument for this fact is that after inverting $x_2 \to y_2$ and $x_3 \to y_1$ in Γ, the graph $\check{\Gamma}$ contains a cycle, and thus our naive notion of "total flow" does not make sense here. Conversely, assume that we chose our matching in such a way that the resulting graph $\check{\Gamma}$ is acyclic. Then our notion of "total flow" makes sense, and it turns out that this is actually sufficient for our purpose of simultaneously inverting edges.

The aim of this chapter is to prove the Main Theorem of Morse theory for based chain complexes (Theorem 1.1.29), which, in the finitely generated case, can be summarized as follows: Let C be a based chain complex and consider a matching on its incidence graph. If the associated graph $\check{\Gamma}$ is acyclic, then there is a complex C', freely generated by the essential basis elements of C, and a chain homotopy equivalence $\mathsf{C} \to \mathsf{C}'$. The theorem also provides formulas for the projection map $\mathsf{C} \to \mathsf{C}'$ as well as for the differential of C'.

1.1.3 Labelled graphs and matchings

We now introduce the graph-theoretical vocabulary that we need for discrete Morse theory. Most of this material has already been addressed in the motivation. The only new player will be the notion of noetherianity which replaces acyclicity, for the latter is too weak in case of infinite graphs.

Recall that a *directed graph* is a pair $\Gamma = (V, E)$, where V is the set of vertices and $E \subseteq V \times V$ is the set of oriented edges. All graphs in this chapter are directed, and we will refer to them simply as *graphs*.

We are only interested in graphs as combinatorial objects, meaning that we do not care about topological properties that occur when considering (undirected) graphs as

1-dimensional CW complexes.

Throughout, let R be an associative ring with unity 1_R, and denote by $R^\times$ its group of units, i.e. $R^\times$ contains all elements that possess a left and right inverse with respect to the ring multiplication.

Definition 1.1.6 (a) A *graph with labels in* R, or R*-graph*, is a pair (V, λ), where V is the set of vertices and $\lambda : V \times V \to R$ is a so-called *labelling*.

(b) A *path* in (V, λ) of length n is an $(n+1)$-tuple $(v_0, \ldots, v_n) \in V^{n+1}$ such that for all i, $1 \le i \le n$, we have $\lambda(v_{i-1}, v_i) \neq 0$.

(c) Define the *height* of a vertex $v \in V$ as follows,

$$\mathrm{ht}(v) = \sup\{n : \text{there is a path } (v_0, \ldots, v_n) \text{ with } v_0 = v\}.$$

(d) We say that (V, λ) is *noetherian* if the height of every vertex is finite.[1]

Remark 1.1.7 The intuition behind labelled graphs is that "there is no edge $v \to w$" if $\lambda(v, w) = 0$ and "there is an edge $v \to w$ with label $\lambda(v, w)$" else. This way, the concept of labelled graphs generalizes usual directed graphs; there is an obvious identification between directed graphs and graphs with labels in $\mathbb{Z}/2$.

Definition 1.1.8 An R-graph (V, λ) is called *thin* if for all $(v, w) \in V^2$ at least one of $\lambda(v, w)$ and $\lambda(w, v)$ is zero. In particular, $\lambda(v, v) = 0$ for all $v \in V$.

Note that every noetherian graph is thin.

Definition 1.1.9 Let $\Gamma = (V, \lambda)$ be a thin R-graph. An R*-compatible matching*, for short *matching*, on Γ is an involution $\mu : V \to V$ satisfying the following property: If $v \in V$ is not a fixed point of μ then one of $\lambda(v, \mu(v))$, $\lambda(\mu(v), v)$ is invertible in R (and the other one is necessarily 0).

Let (V, λ) be a thin R-graph and let μ be a matching on it. We define a new graph $\check{\Gamma} = (\check{V}, \check{\lambda})$ by "inverting edges between matched vertices". More precisely, $\check{V} = V$ and the labelling $\check{\lambda}$ is given as follows:

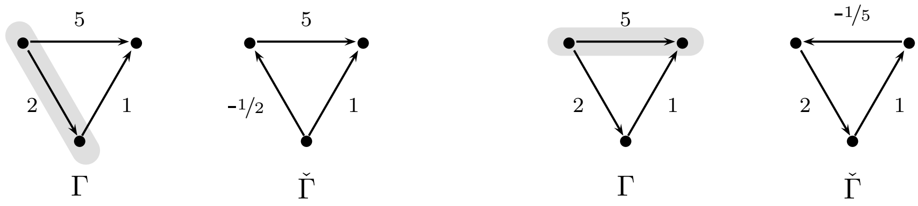
$$\check{\lambda}(v, w) = \begin{cases} \lambda(v, w) & \text{if } \mu(v) \neq w \\ 0 & \text{if } \mu(v) = w \text{ and } \lambda(w, v) = 0 \\ -\lambda(w, v)^{-1} & \text{if } \mu(v) = w \text{ and } \lambda(w, v) \in R^\times \end{cases}$$

It is easily seen that $\check{\Gamma}$ is thin. Furthermore, μ is a matching on $\check{\Gamma}$, and the associated graph $\check{\check{\Gamma}}$ is just Γ.

[1] Note that there are two reasons why the height of a vertex could be infinite: Firstly, there could be an infinite path $(v_0, v_1, v_2, \ldots)$. Secondly, there could exist a sequence of paths of increasing lengths. When speaking of noetherianity, one usually only excludes the first. However, in our situation of incidence graphs of chain complexes both cases are equivalent. This follows from König's Lemma, see e.g. Cohen [Coh97, Lemma 1].

Definition 1.1.10 A matching μ on a labelled graph Γ is called *noetherian* if the associated graph $\check{\Gamma}$ is noetherian.

Example 1.1.11 In Figure 1.7 we consider twice the same graph Γ with labels in $R = \mathbb{Q}$, but with different matchings, indicated by the gray shaded boxes. The first matching is noetherian whereas the second is not. Note that if we considered the labels to live in $R = \mathbb{Z}$ then none of them would be a matching.



(a) A noetherian matching (b) Not notherian

Figure 1.7: Different matchings on a graph.

1.1.4 Based chain complexes and incidence graphs

Definition 1.1.12 A *based chain complex of right R-modules*, for short *based chain complex*, is a non-negatively graded chain complex (C_*, ∂_*), where each C_n is a free right R-module, together with a choice of basis Ω_n for each C_n. The elements of Ω_n are called *n-cells*. Based chain complexes will be denoted by $(C_*, \Omega_*, \partial_*)$.

Remark 1.1.13 The cellular complex of any CW complex naturally carries the structure of a based chain complex: As basis Ω_n one takes all (geometric) n-cells of the CW complex.

For us, the most important example of based chain complexes are the various kinds of bar constructions on a monoid X. Any of these admits a canonical basis consisting of certain tuples $\underline{x} = (x_n, \dots, x_1) \in X^n$. It is for this reason that we will usually decorate elements of Ω_* with an underbar.

Definition 1.1.14 Let $(C_*, \Omega_*, \partial_*)$ be a based chain complex.

(a) Every chain $c \in C_*$ can uniquely by written as a finite sum of the form

$$c = \sum_{\underline{x} \in \Omega_*} \underline{x} \cdot \alpha_{\underline{x}},$$

and for every $\underline{x} \in \Omega_*$ we define the *incidence number* of $\underline{x}$ in c as $[c : \underline{x}] := \alpha_{\underline{x}}$.

(b) The *incidence graph* associated to $(C_*, \Omega_*, \partial_*)$ is the R-graph (V, λ) with vertex set $V = \Omega_*$ and labelling $\lambda(\underline{x}, \underline{y}) = [\partial \underline{x} : \underline{y}]$.

We see that if in an incidence graph we have $\lambda(\underline{x}, \underline{y}) \neq 0$ then $\dim(\underline{y}) = \dim(\underline{x}) - 1$ (as elements of the chain complex), implying that incidence graphs are noetherian and in particular thin.

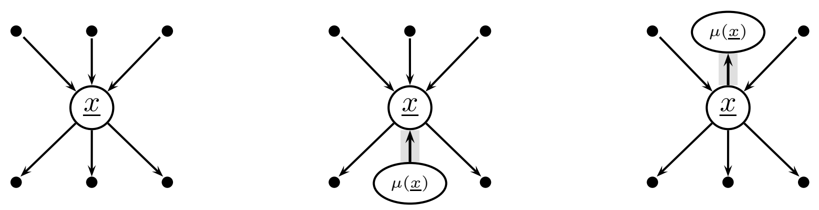
Definition 1.1.15 (a) Consider a based chain complex $(C_*, \Omega_*, \partial_*)$. A *noetherian matching* on $(C_*, \Omega_*, \partial_*)$ is a noetherian, R-compatible matching $\mu : \Omega_* \to \Omega_*$ on the incidence graph $\Gamma = (\Omega_*, \lambda)$.

(b) Given a noetherian matching, the fixed points of $\mu : \Omega_* \to \Omega_*$ are called *essential*. If $\underline{x} \in \Omega_n$ is not a fixed point then, geometrically speaking, $\mu(\underline{x})$ is a face or coface of $\underline{x}$ and thus $\mu(\underline{x}) \in \Omega_{n-1} \cup \Omega_{n+1}$. We say that $\underline{x}$ is *collapsible* if $\mu(\underline{x}) \in \Omega_{n-1}$, and it is called *redundant* if $\mu(\underline{x}) \in \Omega_{n+1}$.

(c) A chain $c \in C_*$ is called *essential*, *collapsible*, *redundant*, respectively, if every $\underline{x} \in \Omega_*$ with $[c : \underline{x}] \neq 0$ is essential, collapsible, redundant, respectively.

(d) A chain $c \in C_*$ is called *essential-collapsible* if for every redundant cell $\underline{x} \in \Omega_*$ we have $[c : \underline{x}] = 0$.

Remark 1.1.16 Let $(C_*, \Omega_*, \partial_*)$ be a based chain complex with incidence graph Γ. Consider a noetherian matching μ on $(C_*, \Omega_*, \partial_*)$. Let $\check{\Gamma} = (\Omega_*, \check{\lambda})$ be defined as above. In Figure 1.8 we indicate how $\check{\Gamma}$ looks locally around an essential, a collapsible, and a redundant vertex $\underline{x} \in \Omega_*$.



(a) essential (b) collapsible (c) redundant

Figure 1.8: How the graph $\check{\Gamma}$ looks locally.

For simplicity we surpressed labels and we drew an arrow $v \to w$ if and only if $\check{\lambda}(v, w) \neq 0$. An arrow is shaded gray if its orientation is reversed when passing from Γ to $\check{\Gamma}$. So, roughly speaking, gray shaded arrows indicate the matching μ.

We will be very much concerned with the heights of vertices in $\check{\Gamma}$.

Definition 1.1.17 (a) Define the *μ-height* $\check{\mathrm{ht}}(\underline{x})$ of a basis element $\underline{x} \in \Omega_*$ as the height of $\underline{x}$ when considered as a vertex in $\check{\Gamma}$, i.e. with respect to the labelling $\check{\lambda}$.

(b) For the trivial chain $c = 0$ we set $\check{\mathrm{ht}}(c) := -\infty$. For a non-trivial chain $c \in C_*$ we define its μ-height as follows:

$$\check{\mathrm{ht}}(c) = \max\{\check{\mathrm{ht}}(\underline{x}) \mid \underline{x} \in \Omega_* \text{ with } [c : \underline{x}] \neq 0\}.$$

1.1.5 Discrete gradient flow and invariant chains

In what follows we fix some based chain complex $(C_*, \Omega_*, \partial_*)$ of right R-modules and a noetherian matching μ. The associated *discrete gradient vector field* $V_* : C_* \to C_{*+1}$ is defined as follows. Let $\underline{x} \in \Omega_*$. If $\underline{x}$ is essential or collapsible then we set $V(\underline{x}) = 0$. Otherwise, i.e. if $\underline{x}$ is redundant, then $\lambda(\mu(\underline{x}), \underline{x}) = [\partial\mu(\underline{x}) : \underline{x}]$ is invertible in R, and we set

$$V(\underline{x}) := -\mu(\underline{x}) \cdot [\partial\mu(\underline{x}) : \underline{x}]^{-1}. \tag{1.1}$$

We obtain $V : C_* \to C_{*+1}$ by linear extension, i.e.

$$V\left(\sum_{\underline{x}\in\Omega_*} \underline{x} \cdot \alpha_{\underline{x}}\right) = -\sum_{\substack{\underline{x}\in\Omega_* \\ \underline{x} \text{ red.}}} \mu(\underline{x}) \cdot [\partial\mu(\underline{x}) : \underline{x}]^{-1} \cdot \alpha_{\underline{x}}.$$

Observe that for every chain $c \in C_*$ its image $V(c)$ is a collapsible chain, implying that

$$V^2 = 0. \tag{1.2}$$

Vice versa, if $\underline{x}$ is a collapsible cell then $V(\mu(\underline{x})) = r \cdot \underline{x}$ for some unit $r \in R^\times$. In particular, every collapsible chain lies in the image of V.

Remark 1.1.18 The definition of V is made such that for a redundant basis element $\underline{x} \in \Omega_*$ we have $[\partial V(\underline{x}) : \underline{x}] = -1$ and thus $[\underline{x} + \partial V(\underline{x}) : \underline{x}] = 0$. Similarly one shows that for a collapsible cell $\underline{x} \in \Omega_*$ it holds $[\underline{x} + V\partial(\underline{x}) : \underline{x}] = 0$.

We define the *discrete gradient flow* $\Theta_* : C_* \to C_*$ associated to the discrete gradient vector field as follows,

$$\Theta := \mathrm{id} + \partial V + V\partial. \tag{1.3}$$

Observe that, since $\partial^2 = 0$ and $V^2 = 0$, we have

$$V\Theta = \Theta V, \tag{1.4}$$
$$\partial\Theta = \Theta\partial. \tag{1.5}$$

Geometrially one could think of Θ as mapping an n-cell $\underline{x} \in \Omega_n$ to a sum of n-cells $c \in C_n$ with the property that each cell occuring in c has a common face or coface with $\underline{x}$.

Remark 1.1.19 Note that if $c \in C_*$ is essential-collapsible then $V(c) = 0$ and thus $\Theta(c) := c + V\partial(c)$ is again essential-collapsible.

Our next aim is to prove a stabilization result for Θ. We will need the following technical lemma, which tells us that "applying Θ decreases heights".

Lemma 1.1.20 *If $\underline{x} \in \Omega_*$ is collapsible or redundant then* $\check{\mathrm{ht}}(\Theta(\underline{x})) \leq \check{\mathrm{ht}}(\underline{x}) - 2$.

PROOF. We prove the Lemma only for collapsible cells. For redundant cells the proof is very similar. Observe that if in the graph $\check{\Gamma}$ there is an arrow $\underline{x} \to \underline{y}$, i.e. $\lambda(\underline{x}, \underline{y}) \neq 0$, then $\check{\mathrm{ht}}(\underline{x}) \geq \check{\mathrm{ht}}(\underline{y}) + 1$. With Figure 1.8 in mind, the following arguments are quite obvious.

If $\underline{x}$ is essential or collapsible then $V(\underline{x}) = 0$ and hence $\check{\mathrm{ht}}(V(\underline{x})) = -\infty$. If $\underline{x}$ is redundant then $V(\underline{x}) = r \cdot \mu(\underline{x})$ for some $r \in R^\times$, and there is an arrow $\underline{x} \to \mu(\underline{x})$ in $\check{\Gamma}$. Altogether we have shown that for every cell $\underline{x} \in \Omega_*$ we have $\check{\mathrm{ht}}(V(\underline{x})) \leq \check{\mathrm{ht}}(\underline{x}) - 1$.

If $\underline{x}$ is essential or redundant then every face of $\underline{x}$ has smaller height than $\underline{x}$. If $\underline{x}$ is collapsible then there is face of $\underline{x}$ having larger height than $\underline{x}$, namely its redundant partner $\mu(\underline{x})$. All other faces have smaller heights.

We conclude that if $\underline{x}$ is collapsible and $[V\partial(\underline{x}) : \underline{x}'] \neq 0$ and $\underline{x}' \neq \underline{x}$ then $\check{\mathrm{ht}}(\underline{x}') \leq \check{\mathrm{ht}}(\underline{x}) - 2$, cf. Figure 1.9. This clearly remains true when replacing the chain $V\partial(\underline{x})$ by

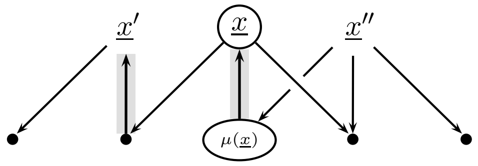


Figure 1.9: $\check{\Gamma}$ around $\underline{x}$ and $[V\partial(\underline{x} : \underline{x}')] \neq 0$.

$\underline{x} + V\partial(\underline{x})$. Recall that $\underline{x} + V\partial(\underline{x}) = \Theta(\underline{x})$ (because $\underline{x}$ is collapsible) and thus, by Remark 1.1.18, $[\Theta(\underline{x}) : \underline{x}] = 0$. It follows that if $[\Theta(\underline{x}) : \underline{x}'] \neq 0$ then $\check{\mathrm{ht}}(\underline{x}') \leq \check{\mathrm{ht}}(\underline{x}) - 2$ and thus $\check{\mathrm{ht}}(\Theta(\underline{x})) \leq \check{\mathrm{ht}}(\underline{x}) - 2$. □

Proposition 1.1.21 *For every $\underline{x} \in \Omega_*$ the sequence $\Theta(\underline{x}), \Theta^2(\underline{x}), \Theta^3(\underline{x}), \ldots$ stabilizes.*

PROOF. The proof will be done in three steps, according to whether $\underline{x}$ is collapsible, essential or redundant.

- Assume that $\underline{x}$ is collapsible. Then $\Theta(\underline{x})$ is a collapsible chain and by Lemma 1.1.20 we have $\check{\mathrm{ht}}(\Theta(\underline{x})) \leq \check{\mathrm{ht}}(\underline{x}) - 2$. Therefore, for N sufficiently large, $\check{\mathrm{ht}}(\Theta^N(\underline{x})) < 0$ and hence $\Theta^N(\underline{x}) = 0$.
- Assume that $\underline{x}$ is essential. Set $c := V\partial(\underline{x})$, which is a collapsible chain. Since $\partial V(\underline{x}) = 0$ we have $\Theta(\underline{x}) = \underline{x} + c$, yielding $\Theta^m(\underline{x}) = \underline{x} + c + \Theta(c) + \ldots + \Theta^{m-1}(c)$. Now, for N sufficiently large, $\Theta^N(c) = 0$ and thus we have $\Theta^N(\underline{x}) = \Theta^{N+1}(\underline{x})$.

From what we have shown so far, it directly follows that the sequence $\Theta(c), \Theta^2(c), \ldots$ stabilizes for every essential-collapsible chain c.

- Assume that $\underline{x}$ is redundant. (Note that $\Theta(\underline{x})$ will in general not be redundant.) By Lemma 1.1.20 we have $\check{\mathrm{ht}}(\Theta(\underline{x})) \leq \check{\mathrm{ht}}(\underline{x}) - 2$. Thus, if $\check{\mathrm{ht}}(\underline{x}) \leq 1$ then $\Theta(\underline{x}) = 0$.

  Assume now that for every redundant basis element $\underline{x} \in \Omega_*$ with $\check{\mathrm{ht}}(\underline{x}) < h$ we have that the sequence $\Theta(\underline{x}), \Theta^2(\underline{x}), \ldots$ stabilizes. By our previous discussion this then holds for every chain $c \in C_*$ satisfying $\check{\mathrm{ht}}(c) < h$. Now, if $\underline{x} \in \Omega_*$ is redundant and $\check{\mathrm{ht}}(\underline{x}) = h$ then $\check{\mathrm{ht}}(\Theta(\underline{x})) \leq h - 2$, and hence the sequence $\Theta(\underline{x}), \Theta^2(\underline{x}), \ldots$ stabilizes.

The Proposition if proven. □

As an immediate consequence of Proposition 1.1.21 we obtain that for every chain $c \in C_*$ there is an N such that $\Theta^{N+1}(c) = \Theta^N(c)$, and we set $\Theta^\infty(x) := \Theta^N(c)$. Denote by C_*^Θ the submodule of Θ-invariant chains. Clearly, Θ^∞ projects onto C_*^Θ. In what follows we consider Θ^∞ as a map $\Theta^\infty : C_* \to C_*^\Theta$. The subsequent proposition partially answers the question which chains are Θ-invariant.

Lemma 1.1.22 *For every chain $c \in C_*$ its image $\Theta^\infty(c)$ is essential-collapsible.*

Proof. This can be seen as follows. Define $\rho(c) = \max\{\check{\mathrm{ht}}(\underline{x})\}$, where the maximum is taken over all redundant $\underline{x} \in \Omega_*$ satisfying $[c : \underline{x}] \neq 0$. The proof of Proposition 1.1.21 showed that if $\rho(c) \geq 0$ then $\rho(\Theta(c)) \leq \rho(c) - 2$, and therefore $\rho(\Theta^\infty(c)) < 0$. □

Recall from (1.5) that $\partial\Theta = \Theta\partial$. Hence, ∂ restricts and corestricts[2] to Θ-invariant chains, implying that (C_*^Θ, ∂_*) is a sub-chain complex of (C_*, ∂_*). Moreover, we see that Θ^∞ is a chain map:

$$
\begin{tikzcd}
\cdots \arrow[r, "\partial"] & C_n \arrow[r, "\partial"] \arrow[d, "\Theta^\infty"] & C_{n-1} \arrow[r, "\partial"] \arrow[d, "\Theta^\infty"] & \cdots \\
\cdots \arrow[r, "\partial"] & C_n^\Theta \arrow[r, "\partial"] & C_{n-1}^\Theta \arrow[r, "\partial"] & \cdots
\end{tikzcd}
$$

The following is Theorem 7.3 in Forman [For98].

Theorem 1.1.23 (Forman) *Let $(C_*, \Omega_*, \partial_*)$ be a based chain complex over R and let μ be a noetherian matching on it. Then the map*

$$\Theta^\infty : (C_*, \partial_*) \longrightarrow (C_*^\Theta, \partial_*)$$

is a chain homotopy equivalence.

[2] By this we mean that $\partial(C_*^\Theta) \subseteq C_*^\Theta$. The map ∂ therefore naturally induces a map $C_*^\Theta \to C_*^\Theta$ which we again denote by ∂.

Proof. We are going to show that $i : C_*^\Theta \hookrightarrow C_*$ is a homotopy inverse for Θ^∞. Clearly, $\Theta^\infty \circ i = \mathrm{id} : C_*^\Theta \to C_*^\Theta$. It remains to show that $i \circ \Theta^\infty \simeq \mathrm{id} : C_* \to C_*$. For this, we define a chain homotopy $H : C_* \to C_{*+1}$ as follows. Let $c \in C_*$ and let N be sufficiently large, meaning that $\Theta^\infty(c) = \Theta^N(c)$. We then set

$$H(c) := -\sum_{i=0}^{N-1} V \circ \Theta^i(c).$$

Clearly, H is R-linear, and Lemma 1.1.22 guarantees that H is well-defined, because $\Theta^N(c) = \Theta^\infty(c)$ is essential-collapsible and hence lies in the kernel of V. Calculating

$$\begin{aligned}(\partial H + H\partial)(c) &= -[(V\partial + \partial V) \circ (\mathrm{id} + \Theta + \ldots + \Theta^{N-1})](c) \\ &= [(\mathrm{id} - \Theta) \circ (\mathrm{id} + \Theta + \ldots + \Theta^{N-1})](c) \\ &= c - \Theta^\infty(c)\end{aligned}$$

we see that $i \circ \Theta^\infty \simeq \mathrm{id}$, as desired. (The first equality holds because Θ and ∂ commute, cf. (1.5) on page 30.) $\square$

Remark 1.1.24 Theorem 1.1.23 should be regarded as a kind of weak discrete Hodge Theorem, and we are now going to justify this point of view. Let M be a compact, orientable smooth Riemannian manifold. The exterior derivative d on M admits an adjoint operator which we denote by δ. The Laplace operator is defined as $\Delta = d\delta + \delta d$, and a k-form is called *harmonic* if it lies in the kernel of Δ. Every harmonic form is a cycle, and hence the harmonic forms constitute a subcomplex (with trivial differential) of the de Rham complex. The first part of Hodge's Theorem states that the space of harmonic k-forms is isomorphic to the k-th de Rham cohomology.

In our setting, a noetherian matching μ on a based chain complex $(C_*, \Omega_*, \partial_*)$ induces a codifferential $V_* : C_* \to C_{*+1}$, cf. (1.1) and (1.2). We define the associated (abstract) Laplace operator as $\Delta := \partial V + V\partial : C_* \to C_*$. Call a chain *$\mu$-harmonic* if it lies in the kernel of Δ. Since $\Delta = \Theta - \mathrm{id}$, a chain is μ-harmonic if and only if it is Θ-invariant. Of course, we cannot expect C_k^Θ to be isomorphic to the k-th homology group $H_k(C_*)$, e.g. we have $C_*^\Theta = C_*$ if μ is the identity on Ω_*. But, by Theorem 1.1.23, the complexes (C_*^Θ, ∂_*) and (C_*, ∂_*) have naturally isomorphic homology groups.

1.1.6 Reduced gradient flow and essential chains

We now define a map $\theta_* : C_* \to C_*$ that should be thought of as a reduced version of the discrete gradient flow Θ. On basis elements θ is given as follows:

$$\theta(\underline{x}) = \begin{cases} \underline{x} & \text{if } \underline{x} \text{ is essential} \\ 0 & \text{if } \underline{x} \text{ is collapsible} \\ \underline{x} + \partial V(\underline{x}) & \text{if } \underline{x} \text{ is redundant} \end{cases}$$

We obtain $\theta : C_* \to C_*$ by linear extension.

Remark 1.1.25 The map θ has a nice geometric interpretation. Let $\underline{x}$ be a redundant cell. Then, up to a unit, $\partial V(\underline{x})$ is the boundary of the collapsible partner $\mu(\underline{x})$ of $\underline{x}$. Recall from Remark 1.1.18 that $[\underline{x} + \partial V(\underline{x}) : \underline{x}] = 0$. So, geometrically speaking, $\theta(\underline{x})$ is the complementary boundary part of $\underline{x}$ (with respect to $\mu(\underline{x})$), see Figure 1.10.

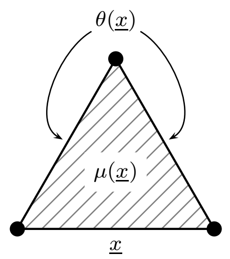


Figure 1.10: Geometric interpretation of θ.

Note that if $\underline{x}$ is collapsible then $0 = \theta(\underline{x}) = \theta^2(\underline{x})$, and if $\underline{x}$ is essential then $\underline{x} = \theta(\underline{x})$. The same arguments as in the proof of Proposition 1.1.21 yield the following:

Proposition 1.1.26 *For every $\underline{x} \in \Omega_*$ the sequence $\theta(\underline{x}), \theta^2(\underline{x}), \theta^3(\underline{x}), \dots$ stabilizes.*

For a chain $c \in C_*$ we set $\theta^\infty(c) = \theta^N(c)$ for N sufficiently large. Denote by C_*^θ the submodule of θ-invariant chains. Clearly, θ^∞ projects onto C_*^θ, and we consider θ^∞ as a map $\theta^\infty : C_* \to C_*^\theta$. The subsequent Proposition classifies θ-invariant chains.

Lemma 1.1.27 *A chain is θ-invariant if and only if it is essential.*

Proof. Obviously, every essential chain is θ-invariant. To show that every θ-invariant chain is essential we apply an argument very similar to the one used in the proof of Lemma 1.1.22. □

We are now going to make precise the relationship between the maps Θ^∞ and θ^∞. Denote by $\pi : C_* \to C_*^\theta$ the "orthogonal projection" onto essential chains,

$$\pi(c) := \sum_{\substack{\underline{x} \in \Omega_* \\ \underline{x} \text{ ess.}}} \underline{x} \cdot [c : \underline{x}].$$

Lemma 1.1.28 *(a) For every $n \geq 0$ and every $c \in C_*$ we have $\Theta^n(c) - \theta^n(c) \in \mathrm{im}(V)$.*

(b) $\theta^\infty = \pi\Theta^\infty : C_* \to C_*^\theta$.

(c) $\Theta^\infty\pi\Theta^\infty = \Theta^\infty : C_* \to C_*^\Theta$.

Proof. (a) It is easily checked that for every $\underline{x} \in \Omega_*$ we have that $\Theta(\underline{x}) - \theta(\underline{x})$ is a collapsible chain, i.e. lies in the image of V. Let $n \geq 2$ and assume now that the claim holds true for all powers $< n$. Let $c \in C_*$ and take d such that $\Theta(c) = \theta(c) + V(d)$.

$$\begin{aligned}\Theta^n(c) - \theta^n(c) &= \Theta^{n-1}(\theta(c) + V(d)) - \theta^{n-1}(\theta(c)) \\ &= (\Theta^{n-1} - \theta^{n-1})(\theta(c)) + \Theta^{n-1}(V(d)).\end{aligned}$$

Both summands lie in the image of V, the first one by the induction hypothesis and the second one due to the fact that $\Theta V = V\Theta$.

(b) We have $\mathrm{im}(V) \subseteq \ker(\pi)$ and therefore part (a) yields $\pi\Theta^\infty = \pi\theta^\infty = \theta^\infty$. (The last equality follows from the fact that π and θ^∞ both project onto C_*^θ.)

(c) We first show that for an essential-collapsible chain c we have $\Theta^\infty(\pi(c)) = \Theta^\infty(c)$. Indeed, if $\underline{x}$ is essential then $\pi(\underline{x}) = \underline{x}$, hence $\Theta^\infty(\pi(\underline{x})) = \Theta^\infty(\underline{x})$, and if $\underline{x}$ is collapsible then $\pi(\underline{x}) = 0$ and $\Theta^\infty(\underline{x}) = 0$, as seen in the first part of the proof of Proposition 1.1.21. It follows that if $c \in C_*$ is an essential-collapsible chain then $\Theta^\infty(\pi(c)) = \Theta^\infty(c)$.

If $c \in C_*$ is an arbitrary chain then by Lemma 1.1.22 the chain $\Theta^\infty(c)$ is essential-collapsible and thus $\Theta^\infty(\pi(\Theta^\infty(c))) = \Theta^\infty(\Theta^\infty(c)) = \Theta^\infty(c)$. □

We can now state and prove the Main Theorem of Morse theory for based chain complexes, cf. [Bro92, Proposition 1], [Coh97, Theorem 2], [For98, §8].

Theorem 1.1.29 (Brown, Cohen, Forman) *Let $(C_*, \Omega_*, \partial_*)$ be a based chain complex of right R-modules and let μ be a noetherian matching on it. Then $\partial^\theta := \theta^\infty \circ \partial$ is a differential for C_*^θ and the map*

$$\theta^\infty : (C_*, \partial_*) \longrightarrow (C_*^\theta, \partial_*^\theta)$$

is a chain homotopy equivalence.

Proof. Recall that Θ^∞ and ∂ commute. Using Lemma 1.1.28 we obtain

$$\partial^\theta\partial^\theta = \theta^\infty\partial\theta^\infty\partial = \pi\Theta^\infty\partial\pi\Theta^\infty\partial = \pi\partial\Theta^\infty\pi\Theta^\infty\partial = \pi\partial\Theta^\infty\partial = \pi\Theta^\infty\partial\partial = 0.$$

We will now show that $\pi : (C_*^\Theta, \partial_*) \to (C_*^\theta, \partial_*^\theta)$ is an isomorphism of chain complexes. The above Theorem then follows from Theorem 1.1.23. Consider the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r, "\partial"] & C_n^\Theta \arrow[r, "\partial"] \arrow[d, "\pi"] & C_{n-1}^\Theta \arrow[r, "\partial"] \arrow[d, "\pi"] & \cdots \\
\cdots \arrow[r, "\partial^\theta"] & C_n^\theta \arrow[r, "\partial^\theta"] \arrow[d, "\Theta^\infty"] & C_{n-1}^\theta \arrow[r, "\partial^\theta"] \arrow[d, "\Theta^\infty"] & \cdots \\
\cdots \arrow[r, "\partial"] & C_n^\Theta \arrow[r, "\partial"] & C_{n-1}^\Theta \arrow[r, "\partial"] & \cdots
\end{tikzcd}$$

Part (b) of Lemma 1.1.28 implies $\pi \circ \Theta^\infty = \mathrm{id} : C_*^\theta \to C_*^\theta$, and from part (c) we conclude that $\Theta^\infty \circ \pi = \mathrm{id} : C_*^\Theta \to C_*^\Theta$. Therefore the restrictions $\pi : C_*^\Theta \to C_*^\theta$ and $\Theta^\infty : C_*^\theta \to$

C_*^Θ are mutually inverse isomorphisms (of modules, a priori). We conclude that the above diagram commutes, for $\partial^\theta \pi = \theta^\infty \partial \pi = \pi \Theta^\infty \partial \pi = \pi \partial \Theta^\infty \pi = \pi \partial : C_*^\Theta \to C_{*-1}^\theta$ and $\Theta^\infty \partial^\theta = \Theta^\infty \theta^\infty \partial = \Theta^\infty \pi \Theta^\infty \partial = \Theta^\infty \partial = \partial \Theta^\infty : C_*^\theta \to C_{*-1}^\Theta$. The Theorem is proven. □

Definition 1.1.30 The complex $(C_*^\theta, \partial_*^\theta)$ is called the *Morse complex* of $(C_*, \Omega_*, \partial_*)$ with respect to the matching μ.

Note that $(C_*^\theta, \Omega_*^\theta, \partial_*^\theta)$ is a based chain complex, where $\Omega_*^\theta \subseteq \Omega_*$ is the set of essential cells. This observation allows to inductively simplify a based chain complex by iterating the procedure of finding a noetherian matching and going over to its associated Morse complex.

Remark 1.1.31 There are various topological versions of Theorem 1.1.29, differing in which kind of cell complexes one considers. In each of these settings one starts with a certain cell complex K and an "admissible" matching on the cells of K. As before, one uses this matching to classify the cells of K into essential, collapsible and redundant ones. The statement is then that K is homotopy equivalent to a CW complex K' built up from the essential cells.

The connection to discrete Morse theory of chain complexes is the following. Write Ω_n for the set of n-cells in K and denote by $(C_*(K), \partial_*)$ the cellular chain complex of K. The admissible matching from above naturally induces a noetherian, $\mathbb{Z}$-compatible matching on the based chain complex $(C_*(K), \Omega_*, \partial_*)$, and the associated Morse complex is isomorphic to the cellular chain complex of K'.

For a precise exposition we refer the reader to Brown [Bro92, Proposition 1, p.140], which is concerned with realizations of simplicial sets, and to Forman [For98, Corollary 3.5, p.107], where regular CW complexes are studied. Theorem 10.2 in [For98] provides a version for arbitrary CW complexes.

Example 1.1.32 For convenience we explicitly compute V, θ and ∂^θ for our complex C from Example 1.1.4. The matching function μ is given by $\mu(x_2) = y_2$, $\mu(y_2) = x_2$ and the identity else. We see that y_2 is the only redundant cell. The discrete gradient vector field V is thus given by $V(y_2) = -\mu(y_2) \cdot [\partial \mu(y_2) : y_2]^{-1} = -x_2$ and zero else. From this we obtain $\theta(y_2) = y_2 + \partial(-x_2) = -2y_1$. The differentials in the associated Morse complex therefore compute to $\theta^\infty \partial(x_1) = 3y_1$ and $\theta^\infty \partial(x_3) = -y_1 + 4\theta^\infty(y_2) = -9y_1$.

Remark 1.1.33 There is also a more explicit formula for the differential $\partial_*^\theta : C_*^\theta \to C_{*-1}^\theta$. For $\dim(\underline{y}) = \dim(\underline{x}) - 1$ we have that $[\partial^\theta(\underline{x}) : \underline{y}]$ is the total flow from $\underline{x}$ to $\underline{y}$ in the graph $\check{\Gamma}$. More precisely,

$$\partial^\theta(\underline{x}) = \sum_{\underline{y}} \underline{y} \cdot \sum_{(\underline{x_0}, \ldots, \underline{x_n})} \lambda(\underline{x_{n-1}}, \underline{x_n}) \cdot \ldots \cdot \lambda(\underline{x_0}, \underline{x_1}). \tag{1.6}$$

where the first sum is taken over all $\underline{y}$ with $\dim(\underline{y}) = \dim(\underline{x}) - 1$, and the second sum runs over all paths $(\underline{x_0}, \ldots, \underline{x_n})$ of arbitrary length, satisfying $\underline{x_0} = \underline{x}$ and $\underline{x_n} = \underline{y}$, cf.

e.g. Kozlov [Koz08, p.202ff] or Sköldberg [Skö06, Lemma 5]. However, for our purposes it will be more convenient to compute the differential ∂^θ by iteratively applying θ.

Remark 1.1.34 It can be quite tedious to check that a certain matching is noetherian. We now present an equivalent condition which is sometimes more convenient to verify, because it only takes into account redundant cells. For redundant cells $\underline{x}$, $\underline{x}' \in \Omega_*$ we define

$$\underline{x} \succ \underline{x}' :\Longleftrightarrow [\partial\mu(\underline{x}) : \underline{x}'] \neq 0.$$

It is easy to see that noetherianity of μ is equivalent to saying that there is no infinite strictly descending chain $\underline{x_1} \succ \underline{x_2} \succ \dots$ of redundant cells, where "strict" refers to $\underline{x_i} \neq \underline{x_{i+1}}$ for all i.

1.1.7 Historical Remarks & References

A first hint towards discrete Morse theory appears in Brown–Geoghegan [BG84]. They prove that the Thompson group $\mathcal{T}$ satisfies the homological finiteness condition FP_∞. To do so, they start with a rather huge Eilenberg–MacLane complex $K(\mathcal{T}, 1)$ and then collapse away pairs of cells in adjacent dimensions. This way, they end up with a $K(\mathcal{T}, 1)$ that has only two cells in each positive dimension.

In [Bro92], Brown remarks that "*the method seemed ad hoc at the time, but it turns out to have much wider applicability [...]*". Loc. cit. introduces what today is called discrete Morse theory for CW complexes: What he calls *collapsing scheme* corresponds to a noetherian matching in our setting.

Later on, Forman [For98] independently discovered the concept of simplifying CW complexes by pairwise collapsing cells and how to efficiently encode these collapses in a so-called *discrete Morse function*. Forman's discrete Morse theory is very similar to Brown's theory of collapsing schemes. However, today it seems that to most authors only Forman's work is known.

The term discrete Morse theory can be justified as follows: Depending on the values on faces and cofaces, a discrete Morse function provides a partition of the cells into collapsible, redundant and essential ones. ([For98] uses the term "critical" instead of "essential".) Forman proves that the original CW complex is homotopy equivalent to a CW complex built up from essential cells, and that the attaching maps are completely determined by the induced discrete gradient flow, compare (1.6).

Discrete Morse theory easily carries over to based chain complexes, and it is then sometimes referred to as *algebraic Morse theory*, cf. Jöllenbeck–Welker [JW09] and Sköldberg [Skö06]. Cohen's survey article [Coh97] offers an algebraic approach to Brown's theory of collapsing schemes.

Note that Forman's machinery requires as input a discrete Morse function. Such a function naturally induces a noetherian matching on the incidence graph, see e.g. Forman

[For98, p.102]. Vice versa, Chari [Cha00, p.7] pointed out that every noetherian matching can be extended to a discrete Morse function. Indeed, the height function $\check{\mathrm{ht}} : \Omega_* \to \mathbb{N}$ on the modified incidence graph $\check{\Gamma}$ is a discrete Morse function.

To cut a long story short, the concepts of collapsing scheme, discrete Morse function, and noetherian matching are very strongly related, and this introductory chapter incorporates notions from all three of them, depending on which one seemed most convenient. For example, the definition of the discrete gradient vector field V and the discrete gradient flow Θ originate from Forman [For98]. In contrast, the definition of the reduced gradient flow θ is taken from Cohen [Coh97].

Note that we do not claim any originality here. Indeed, large parts of the material in this section, and especially most proofs, are gathered from Forman's article [For98], which we highly recommend as an easily accessible introduction to discrete Morse theory. A slightly more modern treatment of the topic can be found in Kozlov's book [Koz08]. We remark that the aforementioned references work with commutative ground rings, which simplifies the exposition. Sköldberg [Skö06] provides an introduction to algebraic Morse theory for arbitrary ground rings.

1.2 Rewriting Systems

1.2.1 Reminder on bar constructions

Let X be a monoid and let M be a left $\mathbb{Z}X$-module. The homology of X with coefficients in M is defined as

$$H_*(X; M) := \mathrm{Tor}_*^{\mathbb{Z}X}(\mathbb{Z}; M),$$

where X acts trivially on $\mathbb{Z}$. If $M = \mathbb{Z}$ then we will often simply write $H_*(X)$ instead of $H_*(X;\mathbb{Z})$. To compute these Tor-groups we need a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}X$. In this section we recall the normalized and unnormalized bar construction on monoids. They are important examples of free (hence projective) resolutions and fit into the setting of discrete Morse theory.

As a sample application of discrete Morse theory we show how it can be used to obtain some well-known contractibility results of bar constructions. One might find these alternative proofs intricate at first glance, especially when compared to the usual ones. We included them for two reasons. Firstly, to give concrete examples of how to apply discrete Morse theory. Secondly, in the course of this thesis, we will do several similar but more complicated constructions. We hope that our future constructions become more accessible when having the examples of the present section in mind.

We remind the reader that we number tuples from right to left.

Let X be a monoid and denote by $\mathbb{E}_n X$ the right $\mathbb{Z}X$-module freely generated by all

n-tuples $(x_n|\dots|x_1)$ with $x_i \in X$ for all i. For $0 \le i \le n$ define

$$d_i(x_n \mid \dots \mid x_1) = \begin{cases} (x_n \mid \dots \mid x_2)x_1 & \text{if } i = 0, \\ (x_n \mid \dots \mid x_{i+1}x_i \mid \dots \mid x_1) & \text{if } n > i > 0, \\ (x_{n-1} \mid \dots \mid x_1) & \text{if } i = n. \end{cases} \tag{1.7}$$

We obtain $d_i : \mathbb{E}_n X \to \mathbb{E}_{n-1} X$ by linear extension. The *unnormalized inhomogeneous bar resolution* of X is the chain complex $(\mathbb{E}_* X, \partial_*)$ where $\partial_n : \mathbb{E}_n X \to \mathbb{E}_{n-1} X$ is given by the alternating sum $\partial_n = \sum_{i=0}^{n} (-1)^i d_i$. It is well-known that this chain complex is acyclic. A contracting homotopy is given by $\mathbb{Z}$-linear extension of the map $(x_n|\dots|x_1)x_0 \mapsto (x_n|\dots|x_1|x_0)$. Below we give an alternative proof of this fact, using discrete Morse theory only.

Proposition 1.2.1 *The complex* $(\mathbb{E}_* X, \partial_*)$ *is acyclic.*

Proof. Acyclicity of a complex does not depend on the ground ring. In this proof we consider $(\mathbb{E}_* X, \partial_*)$ as chain complex over $\mathbb{Z}$. For $n \ge 0$ denote by Ω_n the set consisting of all tuples of the form $(x_n|\dots|x_1)x_0$ with $x_i \in X$ for all i. Clearly, Ω_* is a $\mathbb{Z}$-basis for $\mathbb{E}_* X$. We are now going to construct a noetherian matching μ on $(\mathbb{E}_* X, \Omega_*, \partial_*)$. This matching will have exactly one fixed point, namely $(\,)\epsilon \in \Omega_0$. (Recall that write ϵ for the neutral element in X.) Let $(x_n|\dots|x_1)x_0 \in \Omega_*$ be a generator different from $(\,)\epsilon$. Let k be maximal subject to $x_{k-1} = \dots = x_1 = x_0 = \epsilon$, and define $\mu((x_n|\dots|x_1)x_0)$ as follows,

$$\mu((x_n|\dots|x_1)x_0) = \begin{cases} (x_n|\dots|x_1|x_0)\epsilon & \text{if } k \text{ is even} \\ (x_n|\dots|x_2)x_1x_0 & \text{if } k \text{ is odd.} \end{cases}$$

Note that if k is odd then $x_0 = \epsilon$ and thus $x_1x_0 = x_1$. Futhermore, a cell $(x_n|\dots|x_1)x_0$ is redundant if and only if k is even.

Obviously, μ is an involution. We need to show that μ is $\mathbb{Z}$-compatible and noetherian. For this, let $(x_n|\dots|x_1)x_0$ be redundant and consider

$$\begin{aligned} &[\partial\mu(x_n|\dots|x_1)x_0 : (x_n|\dots|x_1)x_0] \\ &= [\partial(x_n|\dots|x_1|x_0)\epsilon : (x_n|\dots|x_1)x_0] \\ &= \sum_{i=0}^{n+1} (-1)^i \cdot [d_i(x_n|\dots|x_0)\epsilon : (x_n|\dots|x_1)x_0]. \end{aligned} \tag{1.8}$$

To compute (1.8) we distinguish three cases.

- For $k = 0$ we have $d_0(x_n|\dots|x_0)\epsilon = (x_n|\dots|x_1)x_0$.
- For $k \ge i \ge 1$ we have $d_i(x_n|\dots|x_0)\epsilon = (x_n|\dots|x_ix_{i-1}|\dots|x_0)\epsilon$. Recall that $x_{k-1} = \dots = x_0 = \epsilon$. It follows that $x_{i-1} = \epsilon$ and hence $(x_n|\dots|x_ix_{i-1}|\dots|x_0)\epsilon = (x_n|\dots|x_i|x_{i-2}|\dots|x_0)\epsilon = (x_n|\dots|x_1)x_0$.

- For $i \geq k+1$ consider $(y_n|\dots|y_1)y_0 := d_i(x_n|\dots|x_0)\epsilon = (x_n|\dots|x_i x_{i-1}|\dots|x_0)\epsilon$. Since $k < i$ we have $y_k = x_{k-1} = \epsilon$. But $x_k \neq \epsilon$ and hence $(y_n|\dots|y_1)y_0 \neq (x_n|\dots|x_1)x_0$.

  For proving noetherianity of our matching, the following further observations will be important. If $i \geq k+2$ then $y_{k+1} = x_k \neq \epsilon$, implying that $(y_n|\dots|y_1)y_0$ is collapsible. If $i = k+1$ then $y_{k+1} = x_{k+1}x_k$, and if $(y_n|\dots|y_1)y_0$ is redundant then we necessarily have $x_{k+1}x_k = \epsilon$. In this case, $(y_n|\dots|y_1)y_0$ has strictly less non-trivial entries than $(x_n|\dots|x_1)x_0$.

Putting everything together, we see that (1.8) computes to $\sum_{i=0}^{k}(-1)^i$. Since k is even, we obtain 1, proving that μ is $\mathbb{Z}$-compatible. The above calculation also shows that if $\underline{x}$ and $\underline{x}'$ are redundant and $\underline{x} \nrightarrow \underline{x}'$ then either $\underline{x} = \underline{x}'$ or $\underline{x}'$ has fewer non-trivial entries than $\underline{x}$. Therefore μ is noetherian.

Theorem 1.1.29 now tells us that the map $\theta^\infty : (\mathbb{E}_*X, \partial_*) \to (\mathbb{E}^\theta_*X, \partial^\theta_*)$ is a chain homotopy equivalence. The Morse complex $(\mathbb{E}^\theta_*X, \partial^\theta_*)$ is freely generated by the element $(\,)\epsilon$ sitting in degree 0. This proves that $(\mathbb{E}_*X, \partial_*)$ is indeed acyclic. □

Remark 1.2.2 (Scanning) Note that the map μ only takes into consideration the first k entries of a cell. In other words, for a cell being collapsible or redundant, respectively, does not, in general, depend on all its entries x_i. Here we were just interested in how many entries were ϵ until the first non-trivial entry appeared. For this, one can think of successively checking the entries from right to left until we have all the information we need. The remaing x_i's are then discarded. We will refer to this procedure as *scanning*. The idea of scanning tuples will be used at several places in this work.

Denote by D_* the submodule of $\mathbb{E}_*X$ generated by n-tuples $(x_n|\dots|x_1)$ for which $x_i = \epsilon$ for at least one i. It is easily seen that the differential ∂_* restricts and corestricts to D_*. The quotient

$$\overline{\mathbb{E}}_*X := \mathbb{E}_*X/D_*$$

is called the *normalized inhomogeneous bar resolution* of X. We write $[x_n|\dots|x_1]$ for the equivalence class of $(x_n|\dots|x_1)$. The differential of $\overline{\mathbb{E}}_*X$ will be denoted by $\overline{\partial}_*$. The following is well-known.

Proposition 1.2.3 *The projection map*

$$\begin{aligned} \psi : \mathbb{E}_*X &\longrightarrow \overline{\mathbb{E}}_*X, \\ (x_n|\dots|x_1) &\longmapsto [x_n|\dots|x_1] \end{aligned}$$

is a chain homotopy equivalence.

Below we give a proof that uses discrete Morse theory only. We warn the reader that the notion of height appearing in the proof is different from the one introduced in Definition 1.1.6. In the remainder of this work, the height of a vertex in the sense of Definition

1.1.6 will not again occur explicitly. Henceforth, the term "height" will be reserved for measuring "to what extent a cell is essential".

PROOF. As a $\mathbb{Z}X$-module, D_* is a direct summand of $\mathbb{E}_*X$ and thus $\overline{\mathbb{E}}_*X$ is canonically isomorphic to the direct summand freely generated by all tuples $(x_n|\dots|x_1)$ satisfying $x_i \neq \epsilon$ for all i. We will show that the orthogonal projection onto this direct summand induces a chain homotopy equivalence.

Denote by Ω_n the set consisting of all tuples $(x_n|\dots|x_1)$ with entries in X. Clearly, Ω_* is a $\mathbb{Z}X$-basis for $\mathbb{E}_*X$. We say that $(x_n|\dots|x_1) \in \Omega_n$ is essential if and only if $x_i \neq \epsilon$ for all i. The classification of the remaining cells into collapsible and redundant is again done by scanning: The *height* of a basis element $(x_n|\dots|x_1)$ is defined to be the maximum integer h subject to $(x_h|\dots|x_1)$ being essential. Clearly, $(x_n|\dots|x_1)$ is essential if and only if $h = n$, and if it is not essential then $x_{h+1} = \epsilon$. Let k be maximal subject to $x_{h+k} = \dots = x_{h+2} = x_{h+1} = \epsilon$. On non-essential cells the map μ is given as follows:

$$\mu(x_n|\dots|x_1) = \begin{cases} (x_n|\dots|\widehat{x_{h+1}}|\dots|x_1) & \text{if } k \text{ is even} \\ (x_n|\dots|x_{h+1}\,|\,\epsilon\,|\,x_h|\dots|x_1) & \text{if } k \text{ is odd} \end{cases}$$

In particular, $(x_n|\dots|x_1)$ is redundant if and only if k is odd.

Obviously, μ is an involution. We need to show that μ is $\mathbb{Z}X$-compatible and noetherian. For this, let $(x_n|\dots|x_1)$ be redundant and consider

$$\begin{aligned} &[\partial\mu(x_n|\dots|x_1) : (x_n|\dots|x_1)] \\ =\,&[\partial(x_n|\dots|x_{h+1}|\epsilon|x_h|\dots|x_1) : (x_n|\dots|x_1)] \\ =\,&\sum_{i=0}^{n+1}(-1)^i \cdot [d_i(x_n|\dots|x_{h+1}|\epsilon|x_h|\dots|x_1) : (x_n|\dots|x_1)]. \end{aligned} \tag{1.9}$$

To conclude that this term takes values in $(\mathbb{Z}X)^\times$ we again distinguish three cases.

- For $h-1 \geq i \geq 0$ the cell $d_i(x_n|\dots|x_{h+1}|\epsilon|x_h|\dots|x_1)$ is either collapsible or redundant, the latter being the case if and only if $x_{i+1}x_i = \epsilon$. Thus, if $i < h$ and $d_i(x_n|\dots|x_{h+1}|\epsilon|x_h|\dots|x_1)$ is redundant then its height is strictly smaller than h.
- For $h+k+1 \geq i \geq h$ it is easily seen that $d_i(x_n|\dots|x_{h+1}|\epsilon|x_h|\dots|x_1) = (x_n|\dots|x_1)$.
- For $i \geq h+k+2$ we have $d_i(x_n|\dots|x_{h+1}|\epsilon|x_h|\dots|x_1) \neq (x_n|\dots|x_1)$. This can be seen by comparing the $(h+k+1)$-st entries; in the former case we have $x_{h+k} = \epsilon$, in the latter we have $x_{h+k+1} \neq \epsilon$.

  Note that for $i \geq h+k+3$ the cell $d_i(x_n|\dots|x_{h+1}|\epsilon|x_h|\dots|x_1)$ is collapsible. If $i = h+k+2$ then $d_i(x_n|\dots|x_{h+1}|\epsilon|x_h|\dots|x_1)$ may or may not be redundant. If it is redundant then necessarily $x_{h+k+2}x_{h+k+1} = \epsilon$, and thus $d_i(x_n|\dots|x_{h+1}|\epsilon|x_h|\dots|x_1)$ has strictly less non-trivial entries than $(x_n|\dots|x_1)$. Yet they have the same height.

Altogether, the incidence number in (1.9) computes to $\sum_{i=h}^{h+k+1}(-1)^i = (-1)^h$, which is invertible. This proves that our matching is $\mathbb{Z}X$-compatible.

We now prove that the relation $\succ\!\!\!-$ is noetherian. Let $\underline{x}$, $\underline{x}'$ be redundant and assume that $\underline{x} \succ\!\!\!- \underline{x}'$. Our above analysis shows that $\underline{x} = \underline{x}'$, or the height of $\underline{x}'$ is strictly smaller than the height of $\underline{x}$, or $\underline{x}$ and $\underline{x}'$ have the same height and $\underline{x}'$ has strictly less non-trivial entries than $\underline{x}$. Therefore $\succ\!\!\!-$ is noetherian.

So far we have shown that μ is a noetherian matching on $(\mathbb{E}_*X, \Omega_*, \partial_*)$. Applying Theorem 1.1.29 we obtain a chain homotopy equivalence $\theta^\infty : (\mathbb{E}_*X, \partial_*) \to ((\mathbb{E}_*X)^\theta, \partial_*^\theta)$. We are now going to compute this map. Recall that a chain is θ-invariant if and only if it is essential. If $\underline{x}$ is collapsible then $\theta(\underline{x}) = 0$. Assume now that $\underline{x}$ is redundant. We then have $\theta(\underline{x}) = \underline{x} \pm \partial\mu(\underline{x})$, which, by our above analysis, is a sum of collapsible cells and redundant cells. It follows that $\theta^\infty(\underline{x})$ is a sum of collapsible and redundant cells. On the other hand, $\theta^\infty(\underline{x})$ is essential, and we conclude that $\theta^\infty(\underline{x}) = 0$. We therefore have $\theta^\infty = \pi : \mathbb{E}_*X \twoheadrightarrow (\mathbb{E}_*X)^\theta$, where π is the orthogonal projection onto essential cells. Thus $\partial_*^\theta = \pi \circ \partial$, as claimed. The Proposition is proven. $\square$

Take $M = \mathbb{Z}$ with trivial X-action. The chain complex

$$(\overline{\mathbb{B}}_*X, \overline{\partial}_*) := (\overline{\mathbb{E}}_*X \otimes_{\mathbb{Z}X} \mathbb{Z}, \overline{\partial}_* \otimes \mathrm{id})$$

is called the *normalized inhomogeneous bar complex* of X. (For simplicity of notation we denote the differential of $\overline{\mathbb{B}}_*X$ again by $\overline{\partial}_*$.) As a $\mathbb{Z}$-module it is freely generated by all tuples $[x_n|\dots|x_1]$ with $x_i \neq \epsilon$ for all i. The differential is given by $\overline{\partial}_n = \sum_{i=0}^n (-1)^i d_i$, where

$$d_i[x_n \,|\, \dots \,|\, x_1] = \begin{cases} [x_n \,|\, \dots \,|\, x_2] & \text{if } i = 0, \\ [x_n \,|\, \dots \,|\, x_{i+1}x_i \,|\, \dots \,|x_1] & \text{if } n > i > 0, \\ [x_{n-1} \,|\, \dots \,|\, x_1] & \text{if } i = n. \end{cases}$$

The normalized inhomogeneous bar complex $(\overline{\mathbb{B}}_*X, \overline{\partial}_*)$ is a model for $H_*(X)$.

1.2.2 Noetherian matchings on the normalized bar resolution

In general, it is a very hard task to find a noetherian matching on a based chain complex that has only "few" fixed points. For the case of finite dimensional complexes Mehner [Meh11] implemented an algorithm that often finds good matchings. This is done by iterating the steps of first looking for an arbitrary matching and then going over to the associated Morse complex. His algorithm turns out to work very well in practice.

In this work we are mostly concerned with noetherian matchings on the normalized bar resolution, which is never finite (unless X is trivial). We therefore need a more conceptual approach. It is worthwhile mentioning that noetherian matchings on the bar resolution are of a very special form. Clearly, every matching μ fixes essential cells. If $\underline{x}$ is a collapsible n-cell then its partner $\mu(\underline{x})$ must be a face of $\underline{x}$, i.e. $\mu(\underline{x}) = d_i(\underline{x})$ for some i. Let us assume that $i \neq 0$ and $i \neq n$, i.e. $\mu(\underline{x})$ arises from multiplying the entries x_{i+1} and x_i. Looking through the eyes of the redundant cell $\mu(\underline{x}) = (y_{n-1}, \dots, y_1)$, its partner $\underline{x} = (x_n, \dots, x_1)$ arises from splitting a certain entry y_i into a product of two factors $y_i = x_{i+1}x_i$.

Thus, to find a matching with relatively few essential cells it is convenient to have a general idea of how to split monoid elements in an appropriate way. Such splittings often arise from some additional structure on the monoid X. A common example is the following. Assume we have chosen a generating set S for X. Furthermore, assume that for every element $x \in X$ we chose a preferred way to write it as a product of elements in S. (That is, we chose a normal form $X \to S^*$.) If this normal form behaves sufficiently nice then we can use it to define a suitable splitting map $X \to X \times X$. In this work we present three types of extra structure on a monoid X, each of which gives rise to sufficiently nice normal forms and hence yielding a noetherian matching on $(\overline{\mathbb{E}}_* X, \Omega_*, \overline{\partial}_*)$. These are complete rewriting systems (cf. Section 1.2.3), Garside monoids (cf. Section 1.3) and factorable monoids (cf. Chapters 2 and 3).

1.2.3 Rewriting Systems

In this section we introduce rewriting systems. So-called complete rewriting systems provide an important example of extra structure on a monoid, naturally giving rise to a noetherian matching on its normalized bar resolution. This result is due to Brown [Bro92]. The material in this section is taken from Cohen's survey article [Coh97].

Informally speaking, a rewriting system for a monoid X can be thought of as a presentation $X = \langle S \mid \mathcal{R} \rangle$ in which the relations $\mathcal{R}$ can only be applied in one direction. We now give a detailed definition.

Definition 1.2.4 (Rewriting system) Let S be a formal alphabet and denote by S^* the free monoid over S. A set of *rewriting rules* $\mathcal{R}$ on S is a set of tuples $(l, r) \in S^* \times S^*$. l is called the *left side* and r is called the *right side* of the rewriting rule.

(a) We introduce a relation on S^* as follows: We say that w *rewrites* to z, denoted by $w \to_{\mathcal{R}} z$, if there exist $u, v \in S^*$ and some rewriting rule $(l, r) \in \mathcal{R}$ such that $w = ulv$ and $z = urv$.

(b) A word $w \in S^*$ is called *reducible* (with respect to $\mathcal{R}$) if there is some z such that $w \to_{\mathcal{R}} z$. Otherwise it is called *irreducible* (with respect to $\mathcal{R}$).

(c) Denote by $\leftrightarrow_{\mathcal{R}}$ the reflexive, symmetric, transitive closure of $\to_{\mathcal{R}}$. Two words w, z over S are called *equivalent* if $w \leftrightarrow_{\mathcal{R}} z$. Set $X = S/\langle \leftrightarrow_{\mathcal{R}} \rangle$. We then say that $(S, \mathcal{R})$ is a rewriting system for the monoid X.

Rewriting systems are a very general concept. For our purposes it will be convenient to study rewriting systems with further properties.

Definition 1.2.5 (Complete rewriting system) Let $(S, \mathcal{R})$ be a rewriting system.

(a) $(S, \mathcal{R})$ is called *minimal* if the right side r of every rewriting rule $(l, r) \in \mathcal{R}$ is irreducible and if the left side l of every rewriting rule $(l, r) \in \mathcal{R}$ is irreducible with respect to $\mathcal{R} \setminus \{(l, r)\}$.

(b) $(S, \mathcal{R})$ is called *strongly minimal* if it is minimal and if every element $s \in S$ is irreducible.

(c) $(S, \mathcal{R})$ is called *noetherian* if there is no infinite sequence $w_1 \to_{\mathcal{R}} w_2 \to_{\mathcal{R}} w_3 \to_{\mathcal{R}} \dots$ of rewritings. This implies that every sequence of rewritings eventually arrives at an irreducible word.

(d) $(S, \mathcal{R})$ is called *convergent* if it is noetherian and if in every equivalence class of $\leftrightarrow_{\mathcal{R}}$ there is only one irreducible element.

(e) A rewriting system is called *complete* if it is strongly minimal and convergent.

We say that a monoid X *possesses a complete rewriting system over the alphabet* S if there exists a set of rules $\mathcal{R}$ such that $(S, \mathcal{R})$ is a complete rewriting system for X.

Remark 1.2.6 Instead of convergence (cf. axiom (d)) one sometimes requires confluence: The rewriting system $(S, \mathcal{R})$ is called *confluent* if for every $u \in S^*$ and successors $u \to_{\mathcal{R}} v$, $u \to_{\mathcal{R}} w$ there exists $z \in S^*$ and chains of reductions $v = v_1 \to_{\mathcal{R}} \dots \to_{\mathcal{R}} v_n = z$ and $w = w_1 \to_{\mathcal{R}} \dots \to_{\mathcal{R}} w_m = z$. It is well-known that for a noetherian rewriting system the notions of convergence and confluence are equivalent.

A complete rewriting system $(S, \mathcal{R})$ gives rise to a normal form $X \to S^*$ by mapping an element $x \in X$ to its uniquely determined irreducible representative in S^*. Moreover, it provides an algorithmic way of computing this normal form: Let $x \in X$ and take any representative $w \in S^*$. If w is irreducible then we are done. Otherwise we can apply a rewriting rule $w \to_{\mathcal{R}} w'$. We take w' as new representative and restart our procedure. After finitely many steps we arrive at the unique irreducible representative, and we call it the *normal form* of x.

Example 1.2.7 (a) Denote by $X = \langle a, b \mid ab = ba \rangle$ the free abelian monoid of rank 2. It is easily checked that $S = \{a, b\}$ and $\mathcal{R} = \{(ab, ba)\}$ is a complete rewriting system for X. A word $w \in S^*$ is a normal form if and only if it is of the form $w = b^m a^n$ for some $m, n \geq 0$.

(b) The rewriting system $S = \{a, b\}$, $\mathcal{R} = \{(aba, bab)\}$ for the "braid monoid on three strands" is not complete, because $babba \;{}_{\mathcal{R}}\!\leftarrow ababa \to_{\mathcal{R}} abbab$ and $babba$ and $abbab$ are both irreducible. (In other words, this rewriting system is not confluent, cf. Remark 1.2.6.)

1.2.4 Brown's proof of the Anick–Groves–Squier Theorem

In this section we indicate Brown's proof of the Anick–Groves–Squier Theorem, which states that a monoid that possesses a finite complete rewriting system is of type FP_∞, cf. [Bro92], [Ani86], [Gro90], [Squ87]. This goes in two steps. First, constructing a noetherian matching out of a complete rewriting system and secondly concluding that the new resolution is of finite type. We follow Cohen's survey article [Coh97]. However, note that our exposition is "opposite" to his in the sense that we work from right to left. The reason for this will become clear in Chapter 2 when we introduce factorable monoids.

Let X be a monoid and let $(S, \mathcal{R})$ be a complete rewriting system for X. As a $\mathbb{Z}X$-module, the normalized inhomogeneous bar resolution is freely generated by tuples $[x_n|\dots|x_1] \in X^n$ with $x_i \neq \epsilon$ for all i. Denote by Ω_* the set of all such tuples. We are now going to define a noetherian matching on $(\overline{\mathbb{E}}_*X, \Omega_*, \overline{\partial}_*)$. This will be done in several steps. First we say what the essential cells should be. Secondly, we define the notion of height of an arbitrary cell. We then use scanning to define collapsible and redundant cells. Finally we give the matching function $\mu : \Omega_* \to \Omega_*$.

We begin by saying which cells are to be essential. Firstly, the 0-cell $[\,]$ is essential. For $n > 0$ consider an n-cell $[x_n|\dots|x_1] \in X^n$ and denote by $w_i \in S^*$ the normal form of x_i, i.e. the uniquely determined irreducible representative of x_i. We say that $[x_n|\dots|x_1]$ is essential if

(a) $x_1 \in S$.

(b) For all $n > i \geq 1$, the concatenation $w_{i+1}w_i \in S^*$ is reducible, i.e. there exist $u, v \in S^*$ and $(l, r) \in \mathcal{R}$ such that $w_{i+1}w_i = ulv$.

(c) For all $n > i \geq 1$, no proper prefix[3] of $w_{i+1}w_i$ is reducible, i.e. if $w_{i+1}w_i = ulv$ for some left side l then $u = \epsilon$.

Note that if $[x_{n+1}|\dots|x_1]$ is essential then so is $[x_n|\dots|x_1]$.

For an arbitrary n-cell $[x_n|\dots|x_1]$ we define its *height* to be the maximal number h subject to $[x_h|\dots|x_1]$ being essential. Clearly, an n-cell is essential if and only if its height is n. Assume now that $[x_n|\dots|x_1]$ is not essential, and denote by h its height.

- If $[x_{h+1}|\dots|x_1]$ does not satisfy (a), i.e. if $x_1 \notin S$, then we say that $[x_n|\dots|x_1]$ is redundant (necessarily of height 0).
- If $[x_{h+1}|\dots|x_1]$ satisfies (a) but fails to satisfy (b) then $w_{h+1}w_h$ is irreducible and the cell $[x_n|\dots|x_1]$ is called collapsible.
- If $[x_{h+1}|\dots|x_1]$ satisfies (a), (b) but fails to satisfy (c) then some proper prefix of $w_{h+1}w_h$ is reducible and $[x_n|\dots|x_1]$ is called redundant.

We will now define the matching $\mu : \Omega_* \to \Omega_*$. Clearly, μ will fix essential cells. On collapsible cells μ is easy to guess: If $\underline{x}$ is collapsible then $\mu(\underline{x})$ has to be a face of $\underline{x}$ and thus $\mu(\underline{x}) = d_i(\underline{x})$ for some i, and we take $\mu(\underline{x}) = d_h(\underline{x})$, where h denotes the height of $\underline{x}$. Note that $0 < h < n$.

On redundant cells μ is a little more complicated. We already know that μ will have to

[3] The term "prefix" might be a bit misleading here, because it stands on the right-hand side. However, recall that we number tuples in the opposite direction, and moreover, in the next chapter, when we define factorable monoids, prefixes will be split up to the right.

"split some entry into two non-trivial factors". We have to distinguish two cases: First, assume that $\underline{x} = [x_n|\dots|x_1]$ is redundant of height 0, i.e. $x_1 \notin S$. Let $(s_k, \dots, s_1) \in S^*$ be the normal form of x_1 and define $\overline{x_1} = s_k \dots s_2 \in X$ and $(x_1)' = s_1$. We then set $\mu(\underline{x}) = [x_n|\dots|x_2|\overline{x_1}|(x_1)']$. Assume now that $[x_n|\dots|x_1]$ is redundant of height $h > 0$. Denote by $w_i \in S^*$ the normal form of x_i. We know that some proper prefix of $w_{h+1}w_h$ is reducible. Write w_{h+1} as concatenation

$$w_{h+1} = ab \tag{1.10}$$

with b minimal subject to bw_h being reducible. Note that $a \neq \epsilon$ (by assumption) and $b \neq \epsilon$ (because w_h is irreducible). Denote by $\alpha, \beta \in X$ the respective classes of $a, b \in S^*$. μ will split the entry x_{h+1} into α and β.

Altogether, μ takes the following form,

$$\mu([x_n|\dots|x_1]) = \begin{cases} [x_n|\dots|x_1] & \text{if essential} \\ [x_n|\dots|x_{h+1}x_h|\dots|x_1] & \text{if collapsible of height } h \\ [x_n|\dots|x_2|\overline{x_1}|(x_1)'] & \text{if redundant of height 0,} \\ [x_n|\dots|x_{h+2}|\alpha|\beta|x_h|\dots|x_1] & \text{if redundant of height } h > 0, \end{cases} \tag{1.11}$$

where α, β are as above.

Theorem 1.2.8 (Brown [Bro92]) *The map $\mu : \Omega_* \to \Omega_*$ is a noetherian, $\mathbb{Z}X$-compatible matching.*

For a proof see e.g. Cohen [Coh97, §7.3]. The proof is not very difficult but quite tedious. This is due to the fact that the matching distinguishes between two kinds of splittings, depending on whether or not the height of a redundant cell is zero.

A rewriting system $(S, \mathcal{R})$ is called *finite* if S and $\mathcal{R}$ are finite.

We now use Theorem 1.2.8 to conclude that if a monoid X admits a finite complete rewriting system, then X satisfies the homological finiteness condition FP_∞, i.e. we need to show that X possesses projective resolutions of left resp. right $\mathbb{Z}X$-modules, both of finite type. The Morse complex associated to the matching μ is free, and we will show that in case of a finite complete rewriting system there are only finitely many essential cells in each dimension.

Let $(S, \mathcal{R})$ be a finite complete rewriting system. Obviously, the essential 1-cells are in one-to-one correspondance with the non-trivial elements of S, i.e. $[x]$ is an essential 1-cell if and only if $x \in S$ and $x \neq \epsilon$.

Recall that if $[x_{n+1}|\dots|x_1]$ is essential then so is $[x_n|\dots|x_1]$. We are now going to show that for every essential n-cell $[x_n|\dots|x_1]$ there are only finitely many ways to extend it to an essential $(n+1)$-cell $[x_{n+1}|x_n|\dots|x_1]$. From this it inductively follows that there are only finitely many essential cells in each dimension.

Denote by w_i the normal form of x_i. If $[x_{n+1}|x_n|\dots|x_1]$ is essential then $w_{n+1}w_n$ is reducible, i.e. $w_{n+1}w_n = ulv$ for some $u, v \in S^*$ and some $(l, r) \in \mathcal{R}$. Furthermore, no proper prefix of $w_{n+1}w_n$ is reducible and thus $u = \epsilon$ and $w_{n+1}w_n = lv$. In particular, one of w_{n+1}, l is a subword of the other. Since the w_i's are irreducible, w_{n+1} must be a (proper) subword of l. Spelled out, w_{n+1} must be a proper subword of the left side of some rewriting rule $(l, r) \in \mathcal{R}$. Since $\mathcal{R}$ is finite, there are only finitely many words satisfying this condition.

Using the Main Theorem of discrete Morse theory (cf. Theorem 1.1.29), we conclude that the Morse complex associated to the matching μ is a free resolution of finite type by right $\mathbb{Z}X$-modules. Therefore, X is of type right-FP_∞. Note that the matching function μ only takes into account d_i for $0 < i < n$. Considering the left bar resolution (i.e. the face map d_n now produces a coefficient x_n and d_0 simply kills entries, compare (1.7)), one analogously shows that X is of type left-FP_∞, cf. Cohen [NR93, p.42]. This finishes Brown's proof of the the following result, cf. [Ani86], [Gro90], [Squ87].

Theorem 1.2.9 (Anick, Groves, Squier) *A monoid admitting a finite complete rewiting system satisfies the homological finiteness condition* FP_∞.

Remark 1.2.10 Brown's original proof in [Bro92] is not purely algebraic but involves some topology. Indeed, he uses what today is called discrete Morse theory for cell complexes, compare Remark 1.1.31. This way he proves that a monoid that admits a finite complete rewriting system satisfies the geometric finiteness condition F_∞, i.e. the classifying space BX has the homotopy type of a CW complex with only finitely many cells in each dimension. He then points out that very similar methods may be used to show that X is of type FP_∞.

1.3 A new proof of a Theorem by Charney–Meier–Whittlesey

This section reports on joint work with Ozornova. It is mainly a copy of the unpublished article [HO]. We present a new proof of a finiteness result for Garside monoids by Charney, Meier, Whittlesey using discrete Morse theory.

Let X be a monoid. For $x \neq \epsilon$ denote by $\|x\|$ the maximum number of non-trivial factors x can be expanded into,

$$\|x\| = \sup\{n \geq 1 \mid \exists x_n, \dots, x_1 \in X \smallsetminus \{\epsilon\} : \ x = x_n \dots x_1\}.$$

For $x = \epsilon$ we set $\|\epsilon\| = 0$.

Definition 1.3.1 A monoid X is called *atomic* if $\|x\| < \infty$ for all $x \in X$.

We write $x \succeq y$ (resp. $y \preceq x$) if y is a right divisor (resp. left divisor) of x, i.e. if there exists $z \in X$ such that $x = zy$ (resp. $yz = x$). Note that the relations $\succeq$ and $\preceq$ both are reflexive and transitive, and if X is atomic then they are also antisymmetric (because there are no non-trivial invertible elements). Recall that $z \in X$ is called the right greatest common divisor of x and y if $x \succeq z$, $y \succeq z$, and for all z' satisfying $x \succeq z'$,

$y \succeq z'$ we have $z \succeq z'$. The notions of left greatest common divisor and right resp. left least common multiple are defined analogously.

The following definition is due to Dehornoy [Deh02].

Definition 1.3.2 A monoid X is called *Garside monoid* if it is atomic, cancellative and the following conditions hold.

(a) For any two elements x, y in X, their left and right least common multiple, as well as their left and right greatest common divisor do exist.

(b) There is an element $\Delta \in X$, called *Garside element*, with the property that its set of left divisors is finite, generates X, and coincides with the set of right divisors of Δ.

We need to fix some notation. Let X be a Garside monoid with Garside element Δ. For x, $y \in X$ we will write $x \tilde{\wedge} y$ for their right greatest common divisor. Denote by S the set of left divisors of Δ and set $S_+ = S \setminus \{\epsilon\}$. By Part (b) of Definition 1.3.2, S_+ is a generating set for X.

For $s \in S$ denote by s^* (resp. ${}^*s$) the element uniquely determined by $ss^* = \Delta$ (resp. ${}^*ss = \Delta$). Observe that $({}^*s)^* = s$, for $({}^*s)^*$ is the uniquely determined element such that ${}^*s({}^*s)^* = \Delta = {}^*ss$.

The following is Theorem 3.6 in [CMW04].

Theorem 1.3.3 (Charney–Meier–Whittlesey) *Let X be a Garside monoid. For $n \geq 0$ define the sets*

$$\Upsilon_n = \{\, [x_n|\dots|x_1] \mid x_i \in S_+ \text{ for all } i \text{ and } x_n \dots x_1 \in S_+\}.$$

Then

$$0 \longrightarrow \mathbb{Z}X[\Upsilon_{\|\Delta\|}] \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \mathbb{Z}X[\Upsilon_2] \xrightarrow{\overline{\partial}} \mathbb{Z}X[\Upsilon_1] \xrightarrow{\overline{\partial}} \mathbb{Z} \longrightarrow 0$$

is a free resolution of $\mathbb{Z}$ as a trivial $\mathbb{Z}X$-module. (The differentials are as in the bar resolution.)

As an immediate consequence of Theorem 1.3.3, Garside monoids satisfy the homological finiteness property FP, and in particular, every Garside monoid is of type FP_∞, cf. the discussion before Theorem 1.2.9.

We are going to give an alternative proof of Theorem 1.3.3, using discrete Morse theory.

Let X be a Garside monoid and denote by $(\overline{\mathbb{E}}_*X, \overline{\partial}_*)$ the normalized inhomogeneous bar resolution of X. $\overline{\mathbb{E}}_*X$ has a canonical $\mathbb{Z}X$-basis Ω_*, consisting of all tuples $[x_n|\dots|x_1]$ satisfying $x_i \neq \epsilon$ for all i. Obviously, $\Upsilon_n \subseteq \Omega_n$ and from property (b) of Definition 1.3.2 we conclude that $[x_n|\dots|x_1] \in \Upsilon_n$ if and only if $x_n \dots x_1 \in S_+$.

Proposition 1.3.4 *There exists a noetherian matching $\mu : \Omega_* \to \Omega_*$ on $(\overline{\mathbb{E}}_* X, \Omega_*, \overline{\partial}_*)$ with the property that $\underline{x} \in \Omega_n$ is a fixed point of μ if and only if $\underline{x} \in \Upsilon_n$.*

We will need the following easy lemma:

Lemma 1.3.5 *For elements x, y, $z \in X$ we have*

$$xz \mathbin{\tilde{\wedge}} yz = (x \mathbin{\tilde{\wedge}} y)\, z.$$

Proof. By definition of the right greatest common divisor we have $x \succeq (x \mathbin{\tilde{\wedge}} y)$ and $y \succeq (x \mathbin{\tilde{\wedge}} y)$, implying that $xz \succeq (x \mathbin{\tilde{\wedge}} y)z$ and $yz \succeq (x \mathbin{\tilde{\wedge}} y)z$. We therefore have $(xz \mathbin{\tilde{\wedge}} yz) \succeq (x \mathbin{\tilde{\wedge}} y)z$. Let $u \in X$ be such that $(xz \mathbin{\tilde{\wedge}} yz) = u(x \mathbin{\tilde{\wedge}} y)z$. We need to show that $u = \epsilon$. There are s, $t \in X$ such that $xz = su(x \mathbin{\tilde{\wedge}} y)z$ and $yz = tu(x \mathbin{\tilde{\wedge}} y)z$. Cancelling z yields $x = su(x \mathbin{\tilde{\wedge}} y)$ and $y = tu(x \mathbin{\tilde{\wedge}} y)$. This proves that $u(x \mathbin{\tilde{\wedge}} y)$ is a right common divisor of x and y, whence $u = \epsilon$. □

We now prove Proposition 1.3.4.

Proof. The classification into collapsible and redundant cells is again done by scanning. First, we define the *height* of a generator $[x_n|\ldots|x_1] \in \Omega_*$ to be the maximal integer $h \geq 0$ subject to $[x_h|\ldots|x_1] \in \Upsilon_h$. If $h = n$ then μ will fix this element. Otherwise, $x_{h+1}x_h \ldots x_1 \notin S_+$. (Note that $x_{h+1} \notin S_+$ implies $x_{h+1}x_h \ldots x_1 \notin S_+$.) Let $d = x_{h+1} \mathbin{\tilde{\wedge}} {}^*(x_h \ldots x_1)$.

(a) If $d = \epsilon$ then we set

$$\mu([\,x_n \,|\, \ldots \,|\, x_1\,]) = [\,x_n \,|\, \ldots \,|\, x_{h+2} \,|\, x_{h+1}x_h \,|\, x_{h-1} \,|\, \ldots \,|\, x_1\,].$$

(b) If $d \neq \epsilon$ then we set

$$\mu([\,x_n \,|\, \ldots \,|\, x_1\,]) = [\,x_n \,|\, \ldots \,|\, x_{h+2} \,|\, a \,|\, d \,|\, x_h \,|\, \ldots \,|\, x_1\,],$$

where a is the uniquely determined element satisfying $ad = x_{h+1}$. Note that $a \neq \epsilon$, because $dx_h \ldots x_1 \in S_+$ but $adx_h \ldots x_1 \notin S_+$.

Claim 1. μ is an involution.

Let $\underline{x} = [x_n|\ldots|x_1]$ be redundant of height h. We will first show that the cell $\mu(\underline{x}) = [x_n|\ldots|x_{h+2}|a|d|x_h|\ldots|x_1]$ is collapsible of height $h+1$. By definition, there exists $c \in M$ such that ${}^*(x_h \ldots x_1) = cd$, yielding

$$\Delta = {}^*(x_h \ldots x_1) \cdot (x_h \ldots x_1) = cd(x_h \ldots x_1),$$

and thus $dx_h \ldots x_1 \in S_+$. We have $adx_h \ldots x_1 = x_{h+1}x_h \ldots x_1 \notin S_+$, proving that $\mu(\underline{x})$ has height $h+1$. Set $b = a \mathbin{\tilde{\wedge}} {}^*(dx_h \ldots x_1)$. We have to show that $b = \epsilon$.

From $\Delta = {}^*(dx_h \dots x_1) \cdot (dx_h \dots x_1) = {}^*(x_h \dots x_1) \cdot (x_h \dots x_1)$ one concludes ${}^*(dx_h \dots x_1) \cdot d = {}^*(x_h \dots x_1)$. This gives

$$\begin{aligned} d &= x_{h+1} \,\tilde{\wedge}\, {}^*(x_h \dots x_1) \\ &= (ad) \,\tilde{\wedge}\, ({}^*(dx_h \dots x_1)d) \\ &\overset{1.3.5}{=} bd. \end{aligned}$$

So $b = \epsilon$ and thus $\mu(x)$ is collapsible of height $h+1$. Hence,

$$\mu^2(\underline{x}) = \mu([x_n|\dots|x_{h+2}|a|d|x_h|\dots|x_1]) = [x_n|\dots|x_{h+2}|ad|x_h|\dots|x_1] = \underline{x}.$$

Now let $\underline{x} = [x_n|\dots|x_1]$ be collapsible of height h. We will first show that $\mu(\underline{x}) = [x_n|\dots|x_{h+2}|x_{h+1}x_h|x_{h-1}|\dots|x_1]$ is redundant of height $h-1$. Since $\underline{x}$ has height h, we have $x_h \dots x_1 \in S_+$ and thus its right divisor $x_{h-1} \dots x_1$ also lies in S_+. On the other hand $x_{h+1}x_h \dots x_1 \notin S_+$. This proves that $\mu(\underline{x})$ has height $h-1$. Set $b = (x_{h+1}x_h) \,\tilde{\wedge}\, {}^*(x_{h-1} \dots x_1)$. We have to show that $b = x_h$.

Observe that ${}^*(x_{h-1} \dots x_1) = {}^*(x_h \dots x_1)x_h$, yielding

$$\begin{aligned} b &= (x_{h+1}x_h) \,\tilde{\wedge}\, {}^*(x_{h-1} \dots x_1) \\ &= (x_{h+1}x_h) \,\tilde{\wedge}\, ({}^*(x_h \dots x_1)x_h) \\ &\overset{1.3.5}{=} (x_{h+1} \,\tilde{\wedge}\, {}^*(x_h \dots x_1))x_h \\ &= x_h. \end{aligned}$$

Since $x_h \neq \epsilon$, this proves that $\mu(\underline{x})$ is redundant of height $h-1$. Hence,

$$\mu^2(\underline{x}) = \mu([x_n|\dots|x_{h+2}|x_{h+1}x_h|x_{h-1}|\dots|x_1]) = \underline{x}.$$

Claim 1 is proven.

Claim 2. If $\underline{x}$ is redundant then $[\partial\mu(\underline{x}) : \underline{x}] = \pm 1$.

Let $\underline{x} = [x_n|\dots|x_1]$ be redundant of height h. Then its partner $\mu(\underline{x})$ is collapsible of height $h+1$. Claim 2 follows from the observation that $d_{h+1}(\mu(\underline{x})) = \underline{x}$, and that $d_i(\mu(\underline{x})) \neq \underline{x}$ for $i \neq h+1$.

Claim 3. The matching is noetherian.

Let $\underline{x} = [x_n|\dots|x_1]$ be a redundant cell of height h, and let $\underline{z} \neq \underline{x}$ be redundant with $\underline{x} \succ \underline{z}$. We will now prove that $\underline{z}$ has height $h+1$. For this, let $\underline{y} = \mu(\underline{x}) = [x_n|\dots|x_{h+2}|a|d|x_h|\dots|x_1]$ and consider the boundaries $d_i\underline{y}$ for $i \neq h+1$. We distinguish several cases.

(a) $n \geq i \geq h+3$: We have $d_i(\underline{y}) = [x_n|\dots|x_ix_{i-1}|\dots|x_{h+2}|a|d|x_h|\dots|x_1]$, which has height $h+1$ since, as above, $dx_h \dots x_1 \in S_+$ and $adx_h \dots x_1 \notin S_+$. As computed above, $a \,\tilde{\wedge}\, {}^*(dx_h \dots x_1) = \epsilon$, so $d_i(\underline{y})$ is collapsible.

(b) $h \geq i \geq 1$: For $i \leq h-1$ we have $d_i(\underline{y}) = [x_n|\dots|x_{h+2}|a|d|x_h|\dots|x_{i+1}x_i|\dots|x_1]$, and for $i = h$ we have $d_i(\underline{y}) = [x_n|\dots|x_{h+2}|a|dx_h|x_{h-1}|\dots|x_1]$. In both cases $d_i(\underline{y})$ has height h, because the product of the first h entries from the right is $dx_h \dots x_1 \in S_+$, whereas the product of the first $h+1$ entries from the right gives $adx_h \dots x_1 \notin S_+$. Computing $a \tilde{\wedge}^*(dx_h \dots x_1) = \epsilon$, we see that $d_i(\underline{y})$ is again collapsible.

(c) $i = h+2$: Here, $d_i(\underline{y}) = [x_n|\dots|x_{h+3}|x_{h+2}a|d|x_h|\dots|x_1])$. This cell has height $h+1$, for $dx_h \dots x_1 \in S_+$, but $x_{h+2}adx_h \dots x_1 \notin S_+$. The latter follows from the fact that its right divisor $adx_h \dots x_1$ is not in S_+. The cell $d_i(\underline{y})$ may or may not be redundant.

Altogether we have shown that if $\underline{z} \neq \underline{x}$ and $\underline{x} \succ \underline{z}$ then $\underline{z}$ has strictly larger height than $\underline{x}$. Note that the height of a cell is bounded by $\|\Delta\|$, the maximal number of nontrivial factors Δ can be expanded into. It follows that every chain $x_1 \succ x_2 \succ \dots$ eventually stabilizes.

This finishes the proof of the Proposition. □

Applying Theorem 1.1.29 to the matching constructed in the proof of Proposition 1.3.4, we obtain that

$$\theta^\infty : (\mathbb{Z}X[\Omega_*], \overline{\partial}_*) \longrightarrow (\mathbb{Z}X[\Upsilon_*], \theta^\infty \circ \overline{\partial}_*)$$

is chain homotopy equivalence.

To conclude Theorem 1.3.3, we only have to show that for $\underline{x} \in \Upsilon_*$ we have $\theta^\infty(\overline{\partial}(\underline{x})) = \overline{\partial}(\underline{x})$. For this, consider $[x_n|\dots|x_1] \in \Upsilon_n$. So, in particular, $x_n \dots x_1 \in S_+$. As a consequence, $x_n \dots x_2 \in S_+$ as well as $x_k \dots x_1 \in S_+$ for all $k \geq 1$. It follows that for all $i = 0, \dots, n$ the face $d_i[x_n|\dots|x_1]$ is a multiple of an element in Υ_{n-1}. Therefore $\overline{\partial}([x_n|\dots|x_1])$ is θ-invariant. Theorem 1.3.3 is proven. □

Remark 1.3.6 (a) In [Ozo], Ozornova uses similar arguments to extend Theorem 1.3.3 to locally left Gaussian monoids.

(b) Besides Theorem 1.3.3, Charney–Meier–Whittlesey [CMW04, Theorem 3.1] also provides a geometric finiteness result, namely that every Garside group G is of type F, i.e. its classifying space BG has the homotopy type of a finite CW complex.

We can use the same strategy as above to give an alternative proof of this geometric statement: Our matching constructed in the proof of Proposition 1.3.4 carries over to the topological situation, meaning that we obtain an admissible matching on the (geometric) cells of the classifying space of X. Discrete Morse theory for cell complexes (compare Remark 1.1.31) then tells us that BX collapses onto a finite complex with one n-cell for each element $[x_n|\dots|x_1]$ in Υ_*. Since X is a Garside monoid (and hence satisfies the Ore condition), the spaces BX and BG are homotopy equivalent, cf. Fiedorowicz [Fie84, Proposition 4.4].

2 Factorability

The organization of this chapter is as follows. In Section 2.1 we briefly review Visy's and Wang's work on factorable groups and monoids. In Section 2.2 we introduce our notion of factorable monoid and compare it with the previous ones. Our notion of factorable monoid can be expressed via certain actions of monoids P_n and Q_n, and Section 2.3 is devoted to investigating these monoids. Finally, we draw a connection between the monoids P_n and Q_n and the so-called Visy complex.

We chose this approach for two reasons. First, we assume the reader to have basic familiarity with factorable groups as introduced by Visy [Vis11] and the generalization to categories by Wang [Wan11]. Thus, the first section may be seen as a warm-up. Secondly, Wang's definition of factorable monoid has the defect that her Main Theorem does not hold in full generality, cf. Theorem 2.1.23. We introduce the recognition principle to point out in detail why this issue occurs. Subsequently, we suggest a definition of factorable monoid that avoids this problem. Having this motivation in mind, Sections 2.2 and 2.3 should then be much more accessible.

2.1 Prerequisites

2.1.1 Generating sets and filtrations

Let X be a monoid and let S be a generating set for X. This means that every non-trivial element of X can be represented as a product of positive powers of elements of S. In particular, if X happens to be a group, we require S to generate X "as a monoid". This will later be guaranteed by requiring that generating sets of groups are closed under taking inverses.

Convention. For technical reasons, it will be convenient to implicitly understand the neutral element $\epsilon \in X$ as an element of our generating set S, even if it does not occur in the explicit description of S. In case we want to exclude ϵ we will write S_+.

In this section we introduce two filtrations associated to a the pair (X, S). Denote by $\ell_S : X \to \mathbb{N}$ the word length in X with respect to S. With the above convention we have $S = \{x \in X : \ell(x) \leq 1\}$ and $S_+ = \{x \in X : \ell(x) = 1\}$.

Clearly, ℓ_S will in general not be additive.

Definition 2.1.1 The pair (X, S) is called *balanced* if $\ell_S : X \to \mathbb{N}$ is a homomorphism (of monoids). It is called 2-*balanced* if the composite $p \circ \ell_S : X \to \mathbb{Z}/2\mathbb{Z}$ is a homomorphism, where $p : \mathbb{N} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z}$ is the homomorphism that sends $1 \in \mathbb{N}$ to the class $[1] \in \mathbb{Z}/2\mathbb{Z}$.

$$\begin{tikzcd} X \arrow[r, "\ell_S"] \arrow[dr, "p\circ\ell_S"'] & \mathbb{N} \arrow[d, "p"] \\ & \mathbb{Z}/2\mathbb{Z} \end{tikzcd}$$

Observe that if (X, S) is balanced then there are no non-trivial invertible elements in X.

Remark 2.1.2 In [Vis11], [Wan11] the term "atomic" is used instead of 2-balanced.

Definition 2.1.3 An n-tuple $(x_n, \dots, x_1) \in X^n$ is called *geodesic* (with respect to S) if $\ell(x_n \dots x_1) = \ell(x_n) + \dots + \ell(x_1)$. If (x_2, x_1) is a geodesic pair then we write $x_2 \| x_1$.

A generating set S for X induces a filtration (by sets) on every subset $Y \subseteq X$:

$$Y(h) := \{y \in Y \mid \ell_S(y) \leq h\}$$

Note that $Y(h) = \varnothing$ for $h < 0$. The associated filtration quotients are given by

$$Y[h] := Y(h) \,/\, Y(h-1).$$

Each quotient $Y[h]$ is a pointed set with base point the equivalence class corresponding to $Y(h-1)$. Since there is no topology involved, another way of describing the filtration quotients is to say that $Y[h]$ is the stratum $Y(h) \smallsetminus Y(h-1)$ with an additional basepoint.

If S_i is a generating set for X_i $(i = 1, 2)$ then $S_{12} = (S_1 \times \{\epsilon_{X_2}\}) \cup (\{\epsilon_{X_1}\} \times S_2)$ is a generating set for the direct product $X_1 \times X_2$, and the word length $\ell_{S_{12}} : X_1 \times X_2 \to \mathbb{N}$ satisfies

$$\ell_{S_{12}}(x_1, x_2) = \ell_{S_1}(x_1) + \ell_{S_2}(x_2).$$

Taking $X_1 = X_2$ and iterating the above procedure we obtain the following filtration of X^n by sets,

$$X^n(h) = \{(x_n, \dots, x_1) \in X^n \mid \ell_S(x_n) + \dots + \ell_S(x_1) \leq h\}.$$

Definition 2.1.4 (a) Let A, B be filtered sets. We say that $f : A \to B$ is a *graded map* if $f(A(h)) \subseteq B(h)$ for all $h \geq 0$. Graded maps descend to the filtration quotients, i.e. for every $h \geq 0$ we obtain an induced map $A[h] \to B[h]$ which will also be denoted by f.

(b) Two graded maps $f, g : A \to B$ are called *equal in the graded sense*, $f \equiv g$, if for every $h \geq 0$ the induced maps on the filtration quotients $f, g : A[h] \to B[h]$ are equal.

Remark 2.1.5 Observe that $\equiv$ is compatible with pre- and post-composition by graded maps.

By a very similar construction we obtain a filtration of the normalized inhomogeneous bar complex $(\overline{\mathbb{B}}_*X, \overline{\partial}_*)$. Denote by $\mathcal{F}_h\overline{\mathbb{B}}_nX$ the free $\mathbb{Z}$-module generated by all n-tuples $[x_n|\dots|x_1]$ satisfying $x_i \neq \epsilon$ for all i and $\ell_S(x_n) + \dots + \ell_S(x_1) \leq h$. The word length is subadditive, and thus the filtration levels $\mathcal{F}_h\overline{\mathbb{B}}_*X$ are indeed subcomplexes. The associated complexes of filtration quotients are given by

$$\mathcal{G}_h\overline{\mathbb{B}}_*X := \mathcal{F}_h\overline{\mathbb{B}}_*X/\mathcal{F}_{h-1}\overline{\mathbb{B}}_*X.$$

As a $\mathbb{Z}$-module, $\mathcal{G}_h\overline{\mathbb{B}}_*X$ is freely generated by all tuples $[x_n|\dots|x_1]$ satisfying $x_i \neq \epsilon$ for all i and $\ell_S(x_n) + \dots + \ell_S(x_1) = h$. In particular, a tuple in $\mathcal{G}_h\overline{\mathbb{B}}_*X$ has length at most h, and hence $\mathcal{G}_h\overline{\mathbb{B}}_*X = 0$ for $* < 0$ or $* > h$.

Remark 2.1.6 Visy [Vis11] writes $\mathcal{N}_*X[h]$ instead of $\mathcal{G}_h\overline{\mathbb{B}}_*X$ and calls it the *h-th norm complex* of X (with respect to the generating set S).

Associated to our increasing filtration $\mathcal{F}_\bullet\overline{\mathbb{B}}_*X$ there is a homology spectral sequence (see e.g. Weibel [Wei94, §5.4]), the E^0-page of which has as entries the filtration quotients

$$\mathsf{E}^0_{p,q} = \mathcal{G}_p\overline{\mathbb{B}}_{p+q}X.$$

We thus find the complexes of filtration quotients $\mathcal{G}_h\overline{\mathbb{B}}_*X$ as the vertical complexes $\mathsf{E}^0_{h,*-h}$. Note that $\mathsf{E}^*_{*,*}$ is a fourth quadrant spectral sequence. Indeed, the E^0-page is concentrated in the right upper triangle of the fourth quadrant, cf. Figure 2.1. The spectral sequence therefore converges to the homology of the monoid X (see e.g. [Wei94, §5.5]).

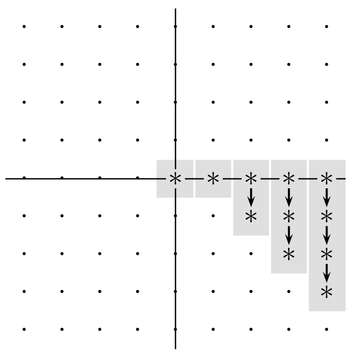


Figure 2.1: The E^0-page.

2.1.2 Factorable groups

For the sake of readability we henceforth denote the word length by ℓ instead of ℓ_S.

Definition 2.1.7 (Visy) A *factorization map* for the pair (X, S) is a map $\eta : X \to X \times X$, $x \mapsto (\overline{x}, x')$ satisfying the following axioms:

(F1) $\overline{x}\, x' = x$.

(F2) $\ell(\overline{x}) + \ell(x') = \ell(x)$.

(F3) $\ell(x') = 1$ if $\ell(x) \geq 1$.

We call x' the *prefix* and $\overline{x}$ the *remainder* of x.

Note that we do not require $\eta : X \to X \times X$ to be a morphism of monoids.

Some remarks are in order. (F1) states that η is a section for the monoid multiplication $d : X \times X \to X$. (F2) states that the factorization η is geodesic, i.e. $\overline{x} \| x'$. (F3) should be seen as "normalization axiom"; as long as $x \neq \epsilon$, the map η splits up a generator.

Remark 2.1.8 In terms of Cayley graphs, η does the following. Given any vertex $g \in G \smallsetminus \{e_G\}$, η picks an edge incident to g with the property that this edge can be extended to a path of minimal length joining g and e_G, see Figure 2.2.

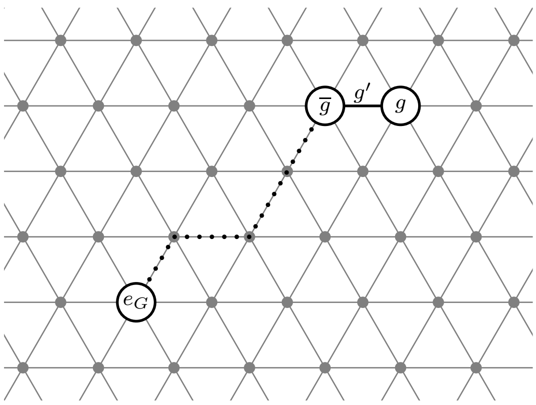


Figure 2.2: The factorization map η in terms of Cayley graphs.

A factorization map $\eta : X \to X \times X$ induces maps $\eta_i : X^n \to X^{n+1}$ by applying η to the i-th entry. More precisely, for $n \geq i \geq 1$ we set $\eta_i := \mathrm{id}^{n-i} \times \eta \times \mathrm{id}^{i-1} : X^n \to X^{n+1}$. Spelled out we have

$$\eta_i(x_n, \ldots, x_1) = (x_n, \ldots, x_{i+1}, \overline{x_i}, (x_i)', x_{i-1}, \ldots, x_1).$$

For $n > i \geq 1$ we define maps $d_i : X^n \to X^{n-1}$ by multiplying the entries x_{i+1}, x_i. Finally, for $n > i \geq 1$ we set $f_i := \eta_i \circ d_i : X^n \to X^n$. Spelled out we have

$$f_i(x_n, \ldots, x_1) = (x_n, \ldots, \overline{x_{i+1} x_i}, (x_{i+1} x_i)', \ldots, x_1). \tag{2.1}$$

Definition 2.1.9 A pair (x_2, x_1) is called *η-stable* (or just *stable* if η is understood), if $\eta(x_2x_1) = (x_2, x_1)$. A tuple $\underline{x} = (x_n, \ldots, x_1)$ is called *stable at position i* if $f_i(\underline{x}) = \underline{x}$. We say that a tuple is *totally stable* if it is stable at every position.

Definition 2.1.10 (Visy) Let G be a group and let S be a generating set that is closed under taking inverses. Let $\eta : G \to G \times G$ be a factorization map. We say that the triple (G, S, η) is a *factorable group* if the maps

$$\begin{aligned} d_2\eta_1 d_1 \eta_2 : G \times S &\longrightarrow G \times S, \\ \eta_1 d_1 : G \times S &\longrightarrow G \times S \end{aligned}$$

are equal in the graded sense, i.e. if the following diagram commutes for every $h \geq 0$:

$$\begin{tikzcd} (G \times S)[h] \arrow[r, "\eta_2"] \arrow[dd, "d_1"'] & (G \times S \times S)[h] \arrow[d, "d_1"] & \\ & (G \times G)[h] \arrow[r, "\eta_1"] & (G \times G \times S)[h] \arrow[d, "d_2"] \\ G[h] \arrow[rr, "\eta_1"'] & & (G \times S)[h] \end{tikzcd} \tag{2.2}$$

Note that the d_i's and η_i's are graded maps and therefore the above diagram makes sense. Furthermore, note that (2.2) always commutes for elements of the form (g, ϵ). Hence, for the definition of factorable group it does not matter whether $\epsilon \in S$ or not.

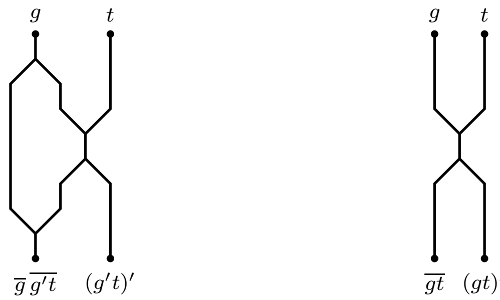
We now give a reformulation of (2.2). Set $\alpha_u = d_2\eta_1 d_1\eta_2 : G \times S \to G \times S$ and $\alpha_l = \eta_1 d_1 : G \times S \to G \times S$. Thus, when passing to filtration quotients, α_u is just the "upper" composition in (2.2), and α_l is the "lower" composition. Spelled out, we have

$$\begin{aligned} \alpha_u(g, t) &= (\overline{g}\,\overline{g't}, (g't)') \\ \alpha_l(g, t) &= (\overline{gt}, (gt)'). \end{aligned}$$

For convenience we show pictures for α_u and α_l in Figure 2.3. We draw a symbol Y to indicate multiplication of two elements and we draw a symbol ⅄ to indicate the map η that factorizes one input variable into its prefix and remainder.

Commutativitiy of (2.2) can now be restated as follows. For any two elements $g \in G$, $t \in S$ we require that $\ell(\overline{g}\,\overline{g't}) + \ell((g't)') = \ell(g) + \ell(t)$ if and only if $\ell(\overline{gt}) + \ell((gt)') = \ell(g) + \ell(t)$, and in this case we furthermore require that $\overline{g}\,\overline{g't} = \overline{gt}$ and $(g't)' = (gt)'$.

Example 2.1.11 (Trivial factorability structure) Every group G can be endowed with the so-called trivial factorability structure: As generating set we take the whole group $S = G$, and we define $\eta : G \to G \times G$ by $\eta(g) = (\epsilon, g)$, i.e. $g' = g$ and $\overline{g} = \epsilon$. Then (G, S, η) is a factorable group. Axioms (F1) - (F3) are obvious. Furthermore, the map η_i is just the simplicial degeneracy map $s_i : G^n \to G^{n+1}$ in the simplicial model of the classifying space BG. Commutativity of (2.2) now follows from the simplicial identities $d_2 s_1 = \mathrm{id}$ and $d_1 s_2 = s_1 d_1$.



(a) α_u (b) α_l

Figure 2.3: Visualization of the upper and lower composition of (2.2).

Remark 2.1.12 (a) For a factorable group (G, S, η) the maps η_i could vaguely be regarded as generalized degeneracies in the following sense. Call a tuple η-degenerate if it lies in the image of some $\eta_i : G^n \to G^{n+1}$. We will later see that, roughly speaking, the maps η_i give rise to a (complicated) collapsing map from the bar resolution of G to a certain quotient consisting of η-non-degenerate cells only.

(b) In general, $\eta : G \to G \times G$ will not be coassociative. Indeed, the only possible factorability structure with η coassociative is the trivial one. This can be seen as follows. η is coassociative if and only if for all $g \in G$ we have $\eta_1(\eta_1(g)) = \eta_2(\eta_1(g))$. This implies $\epsilon = \overline{g}'$, cf. Figure 2.4. Axiom (F3) yields $\overline{g} = \epsilon$ and hence, by axiom (F1), we have $g' = g$.

(a) $\eta_1\eta_1(g)$ (b) $\eta_2\eta_1(g)$

Figure 2.4: Why η is not coassociative in general.

Example 2.1.13 (The infinite cyclic group) The integers $\mathbb{Z}$ are factorable with respect to the generating set $S = \{\pm 1\}$. The factorization map is given by $z \mapsto (z - \operatorname{sign}(z), \operatorname{sign}(z))$, where $\operatorname{sign} : \mathbb{Z} \to \{-1, 0, +1\}$ denotes the sign-function. It is easily checked that $(\mathbb{Z}, S, \eta)$ is a factorable group.

In contrast, Visy [Vis11] showed that for $n > 3$ the finite cyclic groups $\mathbb{Z}/n\mathbb{Z}$ are not factorable with respect to the generating set consisting of the classes $[-1]$ and $[+1]$ only.

Example 2.1.14 (Symmetric groups) Fix some $n \geq 1$ and denote by $\mathcal{S}_n$ the n-th symmetric group. As generating set we take all transpositions, $S = \{(i\ j) \mid 1 \leq i <$

$j \leq n\}$. We define a factorization map as follows. For the identity we necessarily have $\eta(\mathrm{id}) = (\mathrm{id}, \mathrm{id})$. Now, let $\sigma \in \mathcal{S}_n$ be non-trivial. Denote by k the largest non-fixed point of σ and set $\eta(\sigma) = (\sigma\tau, \tau)$, where τ is the transposition $(k,\, \sigma^{-1}(k))$. Visy [Vis11] shows that this data endows $\mathcal{S}_n$ with the structure of a factorable group.

Similarly, the alternating group $\mathcal{A}_n \subset \mathcal{S}_n$ is factorable with respect to the generating set consisting of the products of any two transpositions. The factorization map in this case is roughly given by "applying η twice", cf. Visy [Vis11, §5.3, pp.46-49].

Example 2.1.15 (Products) Let (G, S, η) and (H, T, ν) be factorable groups.

(a) The free product $G * H$ is factorable with respect to the generating set $S \sqcup T$.

(b) Consider a group homomorphism $\varphi : H \to \mathrm{Aut}(G)$. If for every $h \in H$ the automorphism $\varphi(h) : G \to G$ is a graded map, then the semidirect product $G \rtimes_\varphi H$ is factorable with respect to the generating set $(S \times \{\epsilon_H\}) \cup (\{\epsilon_G\} \times T)$. In particular, the direct product $G \times H$ is factorable with respect to the aforementioned generating set. Note that the factorability structure on $G \rtimes_\varphi H$ is non-trivial (unless one of them is the trivial group). Semidirect products therefore provide a lot of interesting examples.

Example 2.1.16 (Dihedral groups) Consider $\mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ equipped with the trivial factorability structure. Taking their semidirect product, we obtain a non-trivial factorability structure on the n-th dihedral group $\mathcal{D}_{2n} = \mathbb{Z}/n\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ with n non-trivial generators. For $n \geq 3$ this factorability structure on $\mathcal{D}_{2n}$ is not 2-balanced.

We remark that there exists another factorability structure on $\mathcal{D}_{2n}$, found by Rodenhausen [Rod], which has again n non-trivial generators, but which is 2-balanced. More precisely, the generating set is given by the reflections in $\mathcal{D}_{2n}$. This factorability structure has been used by Rodenhausen to recompute $H_*(\mathcal{D}_{2n}; \mathbb{Z}[\frac{1}{n}])$ for odd n.

Remark 2.1.17 Factorability of a group G with respect to a "small" generating set S is a rather strong condition. Indeed, Corollary 3.1.22 will later tell us that if S is finite then G satisfies the homological finiteness property FP_∞.

2.1.3 Weakly factorable monoids

Note that the definition of factorable group at no time refers to inverses. We can therefore directly give a prototypical definition of factorable monoid. Indeed, following this idea, Wang [Wan11] introduced the notion of factorable category. The definition is made such that when we take a group and consider it as a category with one object $*$ and morphism set $\mathrm{Mor}(*, *) \cong G$ then it coincides with Visy's definition. For a monoid together with some generating set S this yields the following:

Definition 2.1.18 (Wang) Let X be a monoid, S a generating set, and $\eta : X \to X \times X$ a factorization map. We say that the triple (X, S, η) is a *weakly factorable monoid* if the maps

$$\begin{aligned} d_2\eta_1 d_1\eta_2 &: X \times S \longrightarrow X \times S, \\ \eta_1 d_1 &: X \times S \longrightarrow X \times S \end{aligned}$$

are equal in the graded sense, i.e. if for every $h \geq 0$ the diagram (2.2) commutes (with G replaced by X).

Clearly, every factorable group is a weakly factorable monoid.

Remark 2.1.19 We warn the reader that what we call weakly factorable monoid is called "factorable monoid" in Wang [Wan11]. Subsection 2.2.1 will justify our renaming.

For later use it will be convenient to not only have that $d_2\eta_1 d_1\eta_2$, $\eta_1 d_1$ are equal in the graded sense when considered as maps $X \times S \to X \times S$ but as maps $X^2 \to X \times S$. (For simplicity of notation we will write $X^2 \to X^2$.) The following result tells us that we can always assume this without loss of generality.

Proposition 2.1.20 *Let (X, S, η) be a weakly factorable monoid. Then the maps*

$$\begin{aligned} d_2\eta_1 d_1\eta_2 &: X^2 \longrightarrow X^2, \\ \eta_1 d_1 &: X^2 \longrightarrow X^2 \end{aligned}$$

are equal in the graded sense, i.e. the following diagram commutes for all $h \geq 0$:

$$\begin{tikzcd} X^2[h] \arrow[r, "\eta_2"] \arrow[dd, "d_1"'] & X^3[h] \arrow[d, "d_1"] & \\ & X^2[h] \arrow[r, "\eta_1"] & X^3[h] \arrow[d, "d_2"] \\ X[h] \arrow[rr, "\eta_1"'] & & X^2[h] \end{tikzcd} \tag{2.3}$$

Proof. Let $(x, y) \in X \times X$. The proof is by induction on the word length of (x, y), and we abbreviate $\ell(x, y) := \ell(x) + \ell(y)$. If $\ell(x, y) = 1$ then the claim is obviously true. Assume that the Proposition holds true for all pairs (x, y) satisfying $\ell(x, y) < N$ for some fixed N. Take $(x, y) \in X \times X$ with $\ell(x, y) = N$ and consider the diagram depicted in Figure 2.5.

In this diagram, all non-marked polygons already commute when η is a factorization map for the pair (X, S). We now check that the remaining three polygons commute in the graded sense. The polygon marked with 1 is applied to the triple $(\overline{x}, x'\overline{y}, y')$ and affects the pair $(x'\overline{y}, y')$. It commutes in the graded sense, because $y' \in S$ and by assumption (X, S, η) is weakly factorable. For the same reason, the polygon marked

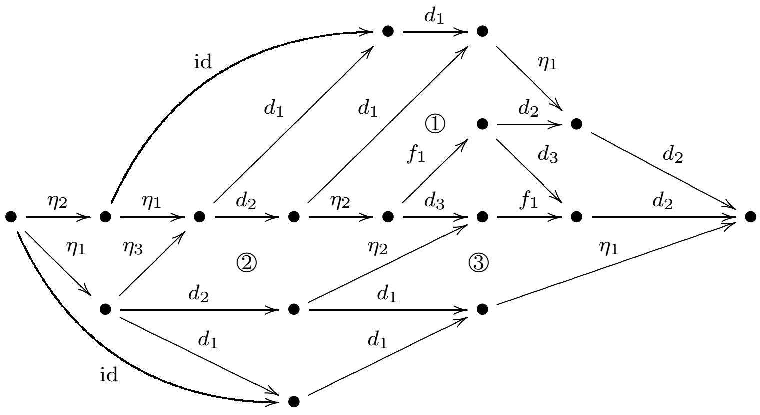


Figure 2.5: The induction step of the proof of Proposition 2.1.20.

with ③ commutes in the graded sense, for it is applied to the pair $(x\overline{y}, y')$. The polygon marked with ② is applied to the triple $(x, \overline{y}, y')$ and affects the pair $(x, \overline{y})$. If $y = \epsilon$ then ② always commutes. Otherwise $\ell(x, \overline{y}) = N - 1$ and the induction hypothesis tells us that ② commutes in the graded sense.

It follows that the top-most composition $d_2\eta_1 d_1\eta_2 : X^2 \to X^2$ and the bottom-most composition $\eta_1 d_1 : X^2 \to X^2$ are equal in the graded sense. $\square$

2.1.4 The Visy complex

We briefly recollect here Visy's main results about factorable groups and Wang's generalizations to weakly factorable monoids.

Theorem 2.1.21 (Visy, Wang) *If (X, S, η) is a weakly factorable monoid then the homology of the complexes $\mathcal{G}_h\overline{\mathbb{B}}_*X$ is concentrated in degree h.*

For a proof see e.g. [Vis11, pp.31-35] or [Wan11, pp.16-20]. We provide a sketch of proof of this result for factorable monoids in Section 3.3.

Equivalently speaking, the E^1-page of the associated spectral sequence consists of a single chain complex $0 \leftarrow \mathsf{E}^1_{0,0} \leftarrow \mathsf{E}^1_{1,0} \leftarrow \mathsf{E}^1_{2,0} \leftarrow \ldots$, cf. Figure 2.6. The spectral sequence therefore collapses on the E^2-page. The E^1-page is of particular interest, because it is a model for the homology of X.

Before we can investigate the complex $(\mathsf{E}^1_{*,0}, d^1_{*,0})$ we need to fix some notation. Recall the maps $f_i : X^n \to X^n$. They induce $\mathbb{Z}$-linear maps $f_i : \mathbb{Z}[X^n] \to \mathbb{Z}[X^n]$ in the obvious way. Note that, as a $\mathbb{Z}$-module, $\mathbb{Z}[X^n]$ is canonically isomorphic to the unnormalized "bar module" $\mathbb{B}_nX$. The maps $f_i : \mathbb{B}_nX \to \mathbb{B}_nX$ obtained this way descend to the normalized bar modules $\overline{\mathbb{B}}_nX$. For convenience we make this explicit. For every i,

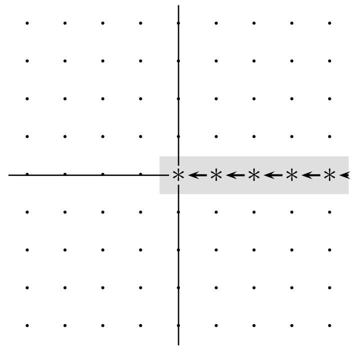


Figure 2.6: The E^1-page for a factorable group.

$n > i \geq 1$, we obtain a $\mathbb{Z}$-linear map $f_i : \overline{\mathbb{B}}_n X \to \overline{\mathbb{B}}_n X$, which, on basis elements, is given by

$$f_i[x_n|\dots|x_1] = [x_n|\dots|\overline{x_{i+1}x_i}|(x_{i+1}x_i)'|\dots|x_1]. \tag{2.4}$$

Note that $f_i[x_n|\dots|x_1] = 0$ if $\ell(x_{i+1}x_i) \leq 1$.

To describe the chain complex $(\mathsf{E}^1_{*,0}, d^1_{*,0})$, Visy introduced the following complex. Denote by $\mathbb{V}_n$ be free $\mathbb{Z}$-module generated by n-tuples $[x_n|\dots|x_1]$ satisfying $\ell(x_i) = 1$ for all i and which are unstable at every position. The differential $\partial^{\mathbb{V}}_n : \mathbb{V}_n \to \mathbb{V}_{n-1}$ is given by the following composition

$$\mathbb{V}_n \xrightarrow{i_n} \overline{\mathbb{B}}_n X \xrightarrow{\kappa_n} \overline{\mathbb{B}}_n X \xrightarrow{\overline{\partial}_n} \overline{\mathbb{B}}_{n-1} X \xrightarrow{\pi_{n-1}} \mathbb{V}_{n-1}. \tag{2.5}$$

We now explain the maps showing up in (2.5). $i_n : \mathbb{V}_n \to \overline{\mathbb{B}}_n X$ is the inclusion of $\mathbb{V}_n$ into the normalized bar complex. The map $\kappa_n : \overline{\mathbb{B}}_n X \to \overline{\mathbb{B}}_n X$ is quite complicated. On generators it is defined as $\kappa_n = K_n \circ \dots \circ K_1$, where

$$K_q = \sum_{i=1}^{q} (-1)^{q-i} \Phi^q_i$$

and $\Phi^q_i = f_i \circ f_{i+1} \circ \dots \circ f_{q-1}$. Spelled out, we have $K_1 = \mathrm{id}$, $K_2 = \mathrm{id} - f_1$, $K_3 = \mathrm{id} - f_2 + f_1 f_2$, and we obtain $\kappa_1 = \mathrm{id}$, $\kappa_2 = \mathrm{id} - f_1$, $\kappa_3 = \mathrm{id} - f_1 - f_2 + f_1 f_2 + f_2 f_1 - f_1 f_2 f_1$, and so on. Note that the number of summands occuring in κ_n grows factorially. The map $\overline{\partial}_n : \overline{\mathbb{B}}_n X \to \overline{\mathbb{B}}_{n-1} X$ is just the differential in the normalized bar complex and $\pi_{n-1} : \overline{\mathbb{B}}_{n-1} X \to \mathbb{V}_{n-1}$ is the "orthogonal" projection onto the direct summand.

The following result is due to Visy.

Theorem 2.1.22 (Visy) *Let (G, S, η) be a factorable group. Then the map*

$$\kappa : (\mathbb{V}_*, \partial^{\mathbb{V}}_*) \longrightarrow (\mathsf{E}^1_{*,0}, d^1_{*,0}) \tag{2.6}$$

is an embedding of chain complexes.

This theorem has been generalized as follows.

Theorem 2.1.23 (Wang) *Let (X, S, η) be a weakly factorable monoid. Assume that X is right-cancellative and that S is finite. Then the map κ in (2.6) is an isomorpism, and thus $(\mathbb{V}_*, \partial_*^{\mathbb{V}})$ is a model for the homology of X.*

For a proof of Theorem 2.1.22 see e.g. [Vis11, Proposition 4.3.5]. For a proof of Theorem 2.1.23 see [Wan11, Theorem 1.3.3]. In this thesis we are going to prove a stronger statement, from which Theorem 2.1.23 immediately follows. In particular, we show that the assumption of S being finite is not necessary, cf. Corollary 3.3.9 and Remark 3.3.10.

The following subsection provides an example why Theorem 2.1.23 does not hold when we drop the assumption of X being right-cancellative.

2.1.5 The recognition principle

Let X be a monoid. Let S be a generating set for X and denote by S^* the free monoid over S. A factorization map η gives rise to a normal form $\mathrm{NF} : X \to S^*$ by sending an element $x \in X$ to

$$\mathrm{NF}(x) := \eta_{\ell(x)-1} \dots \eta_2 \eta_1(x) \in S^*,$$

cf. Bödigheimer, Visy [BV, §5.1].

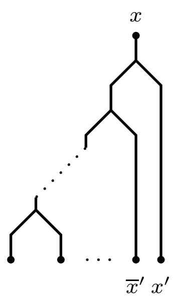


Figure 2.7: Visualization of $\mathrm{NF} : X \to S^*$.

Definition 2.1.24 A word $(s_n, \dots, s_1)$ in S^* is an *η-normal form* if $\mathrm{NF}(s_n \dots s_1) = (s_n, \dots, s_1)$.

Equivalently we can define the following: The empty word $() \in S^*$ is an η-normal form; and $(s_n, \dots, s_1) \in S^*$ is an η-normal form if and only if $(s_n \dots s_1)' = s_1$ and $\overline{s_n \dots s_1} = s_n \dots s_2$ and the tail $(s_n, \dots, s_2)$ is an η-normal form.

Remark 2.1.25 Our definition of η-normal form looks slightly stronger than the definitions by Bödigheimer, Visy [BV] and Rodenhausen [Rod], which do not require that $\overline{s_n \dots s_1} = s_n \dots s_2$. However, since they work in groups, this condition is automatically fulfilled.

Note that being an η-normal form is a "global" condition: If we want to check whether a given word $(s_n, \dots, s_1) \in S^*$ is an η-normal form we in particular have to compute the product $s_n \dots s_1 \in X$.

Definition 2.1.26 We say that a factorization map $\eta : X \to X \times X$ satisfies the *recognition principle* if for all $x \in X$, $t \in S$ we have that (x, t) is η-stable if and only if (x', t) is η-stable.

Remark 2.1.27 The impact of this definition is as follows. Assume we are given a word $(s_n, \dots, s_1) \in S^*$. This is an η-normal form if and only if the pair $(s_n \dots s_{i+1}, s_i)$ is stable for every i. If η satisfies the recognition principle then the latter is equivalent to (s_{i+1}, s_i) being stable for every i. In other words, $(s_n, \dots, s_1)$ is an η-normal form if and only if it is totally stable. Summarizing, the recognition principle guarantees that η-normal forms can be detected locally. This is a crucial property that will later allow us to classify generators $[x_n|\dots|x_1] \in \overline{\mathbb{B}}_n X$ into essential, collapsible and redundant by a scanning algorithm.

The following result has first been discovered by Rodenhausen.

Proposition 2.1.28 *Let (X, S, η) be a weakly factorable monoid. If X is right-cancellative then η satisfies the recognition principle.*

Proof. Assume that (x, t) is stable, i.e. $\alpha_l(x, t) = (x, t)$. In particular, α_l is norm-preserving and hence $\alpha_u(x, t) = (\overline{x}\,\overline{x't}, (x't)') = (x, t)$. From this we see $(x't)' = t$, and the equality $\overline{x't} = x'$ follows from axiom (F1) together with the fact that X is right-cancellative. Assume now that (x', t) is stable. Then $\alpha_u(x, t) = (x, t)$ and thus $\alpha_l(x, t) = (x, t)$, proving that (x, t) is a stable pair. (Note that this implication holds without assuming that X is right-cancellative.) □

Example 2.1.29 shows that Proposition 2.1.28 does not hold for all weakly factorable monoids. Intuitively speaking, the absence of a recognition principle for arbitrary weakly factorable monoids is the reason why in Theorem 2.1.23 the requirement of X being right-cancellative cannot be dropped.

Example 2.1.29 This counterexample has been communicated by Mehner. Set $S = \{a, b\}$ and define $\Xi = \langle a, b \mid a^2 = ab = ba = b^2 \rangle$. Obviously, Ξ is not right-cancellative. Ξ can be thought of as the natural numbers $\mathbb{N}$ with two different 1's, see Figure 2.8.

We define a factorization map as follows: $\eta(\epsilon) = (\epsilon, \epsilon)$, $\eta(a) = (\epsilon, a)$, $\eta(b) = (\epsilon, b)$ and for $k > 1$ we set $\eta(a^k) = (a^{k-1}, b)$. It is easily checked that (Ξ, S, η) is a weakly factorable monoid. However, Ξ does not satisfy the recognition principle: The pair (a^2, b) is stable, whereas the pair $((a^2)', b) = (b, b)$ is not stable. Indeed, Mehner pointed out that the homology of the complex $(\mathbb{V}_*, \partial^{\mathbb{V}}_*)$ is not isomorphic to the homology of X. We omit the details.

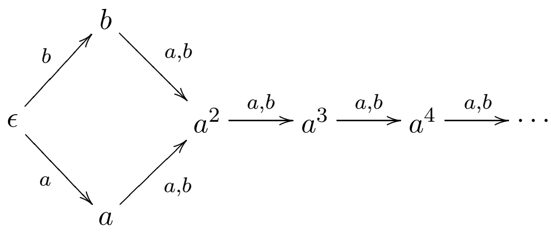


Figure 2.8: Cayley graph of the monoid Ξ.

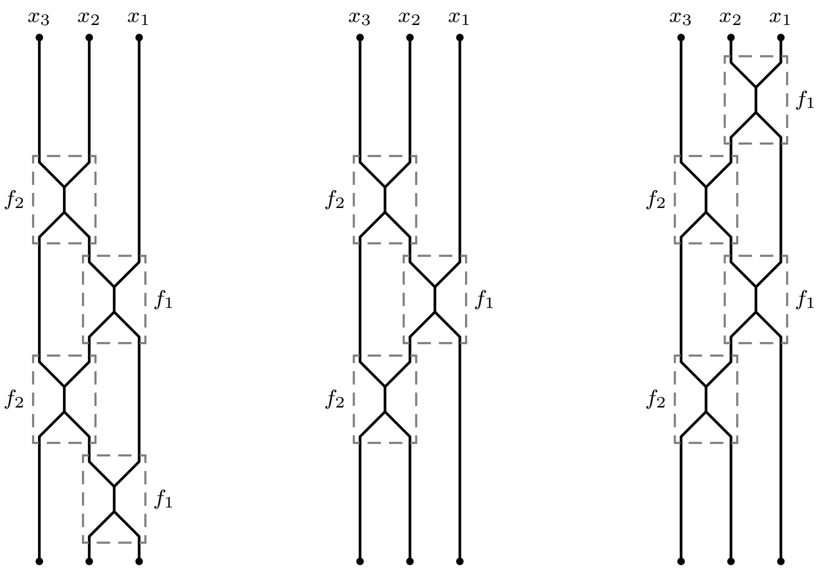
2.2 Factorable monoids

2.2.1 Definition and classification

Definition 2.2.1 (Factorable monoid) Let X be a monoid, S a generating set and $\eta : X \to X \times X$ a factorization map. We say that the triple (X, S, η) is a *factorable monoid* if the maps $f_1f_2f_1f_2$, $f_2f_1f_2$, $f_2f_1f_2f_1 : X^3 \to X^3$ are equal in the graded sense,

$$f_1f_2f_1f_2 \equiv f_2f_1f_2 \equiv f_2f_1f_2f_1 : X^3 \to X^3. \tag{2.7}$$

In Figure 2.9 we visualize the three compositions occuring in (2.7).



(a) $f_1f_2f_1f_2$ (b) $f_2f_1f_2$ (c) $f_2f_1f_2f_1$

Figure 2.9: Visualization of the three compositions occuring in (2.7).

Before continuing, let us briefly motivate the above definition. Let $\underline{x} \in X^3$ and assume that $\ell(f_2f_1f_2(\underline{x})) = \ell(\underline{x})$. We thus have $f_1f_2f_1f_2(\underline{x}) = f_2f_1f_2(\underline{x})$. This can be reformulated by saying that the triple $f_2f_1f_2(\underline{x})$ is everywhere stable (because $f_1 \circ f_2f_1f_2(\underline{x}) =$

$f_2f_1f_2(\underline{x})$ and $f_2 \circ f_2f_1f_2(\underline{x}) = f_2f_1f_2(\underline{x})$, since the f_i's are idempotent). For $\ell(\underline{x}) \leq 3$ this implies that $f_2f_1f_2(\underline{x})$ is an η-normal form. This observation gives rise to a normal form algorithm for factorable monoids. We will discuss this in more detail in Section 2.3.2.

Examples of factorable monoids will be given later. Indeed, Theorem 2.2.6 and in particular Corollary 2.2.7 will at once give a whole bunch of examples.

We now investigate the relation between factorable monoids and weakly factorable monoids.

Lemma 2.2.2 *If (X, S, η) is a factorable monoid then η satisfies the recognition principle.*

Proof. Let $x \in X$ and $t \in S$. We have to show that (x,t) is stable if and only if (x', t) is stable. Let us first assume that (x,t) is stable and consider $(\epsilon, \epsilon, xt) \in X^3$. We have

$$f_2f_1f_2(\epsilon, \epsilon, xt) = f_2f_1(\epsilon, \epsilon, xt) = f_2(\epsilon, x, t) = (\overline{x}, x', t),$$

and obviously $\ell(\epsilon, \epsilon, xt) = \ell(\overline{x}, x', t)$. It follows that $f_2f_1f_2(\epsilon, \epsilon, xt) = f_1f_2f_1f_2(\epsilon, \epsilon, xt)$ and thus (x', t) is a stable pair.

Assume now that (x', t) is stable and consider $(x, t, \epsilon) \in X^3$. We have

$$f_2f_1f_2f_1(x, t, \epsilon) = (\overline{x}, x', t),$$

and obviously $\ell(x, t, \epsilon) = \ell(\overline{x}, x', t)$. We conclude that

$$(\overline{\overline{xt}}, \overline{xt}', (xt)') = f_2f_1f_2(x, t, \epsilon) = f_2f_1f_2f_1(x, t, \epsilon) = (\overline{x}, x', t).$$

From this we see $(xt)' = t$ and $\overline{xt} = \overline{\overline{xt}}\,\overline{xt}' = \overline{x}\,x' = x$, proving that (x,t) is stable. $\square$

Lemma 2.2.3 *Every factorable monoid is weakly factorable.*

Coincidentally, Rodenhausen [Rod] almost gave a proof for this statement when looking for equivalent characterizations of factorability structures. We present here his original proof with some minor changes.

Proof. We have to prove that the diagram (2.2) on page 57 commutes with X replacing G. More precisely, we are going to show that for $x \in X$ and $t_0 \in S$ we have $x \| t$ if and only if $x' \| t$ and $\overline{x} \| \overline{x't}$, and in this case we have $(\overline{xt}, (xt)') = (\overline{x}\,\overline{x't}, (x't)')$. Let $x \in X$, $t_0 \in S$ and set $n = \ell(x)$.

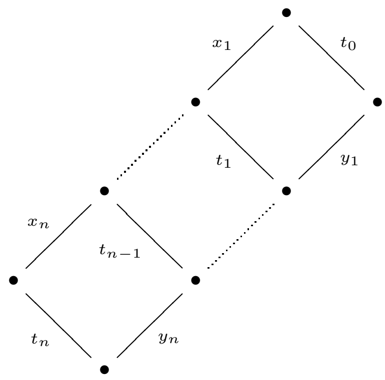
Assume that $x \| t_0$. We then have

$$n + 1 = \ell(x) + \ell(t_0) = \ell(xt_0) \leq \ell(\overline{x}) + \ell(x't_0) \leq \ell(\overline{x}) + \ell(x') + \ell(t_0) = n + 1,$$

proving that $x' \| t_0$. (Note that this argument only requires η to be a factorization map for the pair (X, S).) One similarly shows that $\overline{x} \| \overline{x't_0}$.

Assume now that $x' \| t_0$ and $\overline{x} \| \overline{x' t_0}$. Let $(x_n, \ldots, x_1) := \mathrm{NF}(x)$ and for $1 \leq i \leq n$ define

$$(t_i, y_i) := \eta(x_i t_{i-1}).$$



Claim. $\mathrm{NF}(xt_0) = (t_n, y_n, \ldots, y_1)$.

We first show that for all i, $n \geq i \geq 1$, we have $x_i \| t_{i-1}$. Indeed for $i = 1$ this has been shown above, and for $i \geq 2$ we use

$$\overline{x}\, \overline{x' t_0} = x_n \ldots x_2 t_1 = x_n \ldots x_i t_{i-1} y_{i-1} \ldots y_2$$

to conclude that

$$\ell(x_i t_{i-1}) \geq \ell(\overline{x}\, \overline{x' t_0}) - \ell(x_n \ldots x_{i+1}) - \ell(y_{i-1} \ldots y_2) \geq n - (n-i) - (i-2) = 2.$$

It follows that $x_i \| t_{i-1}$. For $i \geq 1$ consider the triple (x_{i+1}, x_i, t_{i-1}). By Lemma 2.2.2, η satisfies the recognition principle, and thus, by Remark 2.1.27, the pair (x_{i+1}, x_i) is stable. This yields

$$f_2 f_1 f_2(x_{i+1}, x_i, t_{i-1}) = f_2 f_1(x_{i+1}, x_i, t_{i-1}) = f_2(x_{i+1}, t_i, y_i) = (t_{i+1}, y_{i+1}, y_i).$$

Recall that for all i we have $x_i \| t_{i-1}$. Hence $f_2 f_1 f_2$ is norm-preserving for (x_{i+1}, x_i, t_{i-1}). It follows that the triple (t_{i+1}, y_{i+1}, y_i) is everywhere stable and thus $(t_n, y_n, \ldots, y_1)$ is everywhere stable. Using Remark 2.1.27, we conclude $(t_n, y_n, \ldots, y_1) = \mathrm{NF}(t_n y_n \ldots y_1) = \mathrm{NF}(x_n \ldots x_1 t_0) = \mathrm{NF}(xt_0)$, as claimed. It follows that $x \| t_0$.

So far we have shown that for any pair (x, t) the map $\alpha_u : X^2 \to X^2$ is norm-preserving if and only if $\alpha_l : X^2 \to X^2$ is norm-preserving. Assume now that they are both norm-preserving. We then have

$$\begin{aligned} \alpha_u(x, t_0) &= d_2 \eta_1 d_1 \eta_2(x, t_0) \\ &= d_2 \eta_1 d_1(x_n \ldots x_2, x_1, t_0) \\ &= d_2(x_n \ldots x_2, t_1, y_1) \\ &= (t_n y_n \ldots y_2, y_1), \end{aligned}$$

and on the other hand

$$\begin{aligned}\alpha_l(x,t_0) &= \eta_1 d_1(x_n \dots x_1, t_0)\\ &= \eta_1(t_n y_n \dots y_1)\\ &= (t_n y_n \dots y_2, y_1).\end{aligned}$$

Therefore, diagram (2.2) commutes (with G replaced by X). □

Remark 2.2.4 Note that in the above proof we used (2.7) only for triples (x_3, x_2, x_1) satisfying $\ell(x_i) = 1$ for all i.

Lemma 2.2.5 *Let (X, S, η) be a weakly factorable monoid. If η satisfies the recognition principle then (X, S, η) is factorable.*

Proof. Let (X, S, η) be weakly factorable. We thus have $d_2\eta_1 d_1\eta_2 \equiv \eta_1 d_1 : X^2 \to X^2$. We first prove $f_2 f_1 f_2 f_1 \equiv f_2 f_1 f_2 : X^3 \to X^3$. Note that $d_1 d_2 = d_1 d_1 : X^3 \to X$ and thus $d_1 d_2 f_1 = d_1 d_1 \eta_1 d_1 = d_1 d_1 = d_1 d_2 : X^3 \to X$. This way we obtain

$$f_2 f_1 f_2 f_1 = \eta_2 d_2 \eta_1 d_1 \eta_2 d_2 f_1 \equiv \eta_2 \eta_1 d_1 d_2 f_1 = \eta_2 \eta_1 d_1 d_2 = \eta_2 d_2 \eta_1 d_1 \eta_2 d_2 = f_2 f_1 f_2.$$

We now show $f_2 f_1 f_2 \equiv f_1 f_2 f_1 f_2 : X^3 \to X^3$. Consider $(x_3, x_2, x_1) \in X^3$. Clearly, if $f_2 f_1 f_2$ drops the norm of (x_3, x_2, x_1) then so does $f_1 f_2 f_1 f_2$. Assume that $f_2 f_1 f_2 = \eta_2 d_2 \eta_1 d_1 \eta_2 d_2$ is norm preserving for (x_3, x_2, x_1). So, in particular, $d_2\eta_1 d_1 \eta_2$ is norm-preserving for the pair $d_2(x_3, x_2, x_1) = (x_3 x_2, x_1)$. This yields

$$f_2 f_1 f_2(x_3, x_2, x_1) = \eta_2 \eta_1 d_1 d_2(x_3, x_2, x_1) = (\overline{\overline{x_3 x_2 x_1}}, \overline{x_3 x_2 x_1}', (x_3 x_2 x_1)').$$

If η satisfies the recognition principle then the latter is stable at position 1. □

Tacking together Lemmas 2.2.2, 2.2.3 and 2.2.5, we obtain the following.

Theorem 2.2.6 *Let X be a monoid, S a generating set and $\eta : X \to X \times X$ a factorization map. Then the following are equivalent:*

(a) (X, S, η) is a factorable monoid.

(b) (X, S, η) is a weakly factorable monoid and η satisfies the recognition principle.

In particular (cf. Proposition 2.1.28), if X is right-cancellative, then the notions of factorable monoid and weakly factorable monoid coincide.

Corollary 2.2.7 *Every factorable group (in the sense of Visy) is a factorable monoid.*

Example 2.2.8 (Garside monoids) Let (X, Δ) be a Garside monoid and denote by S the set of all left-divisors of the Garside element Δ. For an element $x \in X$ we set $x' = x \mathbin{\tilde{\wedge}} \Delta$ (cf. the notation introduced after Definition 1.3.2), and we denote by $\overline{x}$ the uniquely determined element satisfying $\overline{x}\, x' = x$. Then the map $\eta : X \to X \times X$, $x \mapsto (\overline{x}, x')$ endows (X, S) with the structure of a factorable monoid, cf. Ozornova [Ozo].

Indeed, loc. cit. defines a factorability structure on the larger class of locally left Gaussian monoids. In the special case of Garside monoids her factorability structure simplifies to the one above.

Example 2.2.9 (The natural numbers with funny generating sets) Take $X = \mathbb{N}$ and consider $S = \{0, \ldots, m\}$ for some fixed number $m \geq 1$. We claim that $\mathbb{N}$ is factorable with respect to S. For $n \geq 0$ set

$$\eta(n) = (n - \min\{n, m\},\, \min\{n, m\}),$$

i.e. $n' = \min\{n, m\}$. It is easily seen that η is a factorization map. We need to show that (X, S, η) is factorable. Since X is right-cancellative, it suffices to show that for every $x \in X$ and $t \in S$ we have $(x + t)' = (x' + t)'$. Indeed,

$$\begin{aligned}(x' + t)' &= \min\{x' + t,\, m\} \\ &= \min\{\min\{x, m\} + t,\, m\} \\ &= \min\{x + t,\, m + t,\, m\} \\ &= \min\{x + t,\, m\} \\ &= (x + t)'.\end{aligned}$$

2.2.2 Products

The results of this section will not be used elsewhere in this work. It is for this reason that we do not give the most rigorous proofs here. Instead, we try to keep the exposition as simple as possible.

The aim of this section is to show that "factorable monoids are closed under free, direct, and semidirect products". The strategy to prove this is as follows. Assume we are given two factorable monoids X and Y. (For the moment we suppress generating sets.) In particular, X and Y are weakly factorable. Visy [Vis11, §3.3] proves that free, direct, and semidirect products of factorable groups are again factorable. His proofs almost literally carry over to weakly factorable monoids, and we conclude that the product $Z = X \times Y$ is weakly factorable. By Theorem 2.2.6, it now suffices to show that the factorization map $\eta_Z : Z \to Z \times Z$ satisfies the recognition principle. Note that if Z is weakly factorable and (z', t) is stable, then (z, t) is stable, cf. the proof of Proposition 2.1.28. Thus, to show factorability of Z, the only thing we have to do is to verify the other implication of the recognition principle, that is, if (z, t) is a stable pair then so is (z', t).

Throughout this subsection let (X, S, η_X) and (Y, T, η_Y) be factorable monoids.

In order to avoid confusion we agree on the following convention: We write $\overline{x}$, x' and $\overline{y}$, y' for remainder and prefix in X and Y, respectively. In contrast, when taking remainder and prefix in the product $Z = X * Y$ or $Z = X \ltimes Y$, we will write $\overline{\eta}_Z(z)$ and $\eta'_Z(z)$ instead of $\overline{z}$ and z'.

Proposition 2.2.10 *The free product $Z = X * Y$ is factorable with respect to the generating set $S \sqcup T$.*

Proof. The factorization map $\eta_Z : Z \to Z \times Z$ is defined as follows. Every element $z \in Z$ can be uniquely written as a product $x_{n+1}y_nx_n \dots y_1x_1y_0$ with $x_i \in X$, $y_i \in Y$ and $x_i, y_i \neq \epsilon$ for all $1 \leq i \leq n$. (Note that $y_0 = \epsilon_Y$ and $x_{n+1} = \epsilon_X$ is allowed.) We then define

$$\eta_Z(z) = \begin{cases} (x_{n+1}y_nx_n \dots y_1x_1\,\overline{y_0},\ (y_0)') & \text{if } y_0 \neq \epsilon_Y, \\ (x_{n+1}y_nx_n \dots y_1\,\overline{x_1},\ (x_1)') & \text{if } y_0 = \epsilon_Y. \end{cases} \tag{2.8}$$

The proof of Proposition 3.3.3 in [Vis11] easily carries over to weakly factorable monoids and thus $(Z, S \sqcup T, \eta_Z)$ is a weakly factorable monoid. It remains to show that if (z, u) is an η_Z-stable pair then so is $(\eta'_Z(z), u)$. For $u = \epsilon$ this is trivial and in what follows we assume that $u \neq \epsilon$.

Clearly, $z', u \in S \sqcup T$. If $z' \in S$ and $u \in T$ (or vice versa $z' \in T$ and $u \in S$), then (2.8) gives $\eta_Z(\eta'_Z(z) \cdot u) = (\eta'_Z(z), u)$ and thus $(\eta'_Z(z), u)$ is stable. Otherwise, $z', u \in S$ or $z', u \in T$, and we assume that $z', u \in S$. (The case $z', u \in T$ is treated analogously.) As above, z can uniquely be written as $z = x_{n+1}y_nx_n \dots y_1x_1$. (Note that $y_0 = \epsilon_Y$ for $\eta'_Z(z) \in S$.) In particular, $\eta'_Z(z) = x'_1$. Since (z, u) is stable, we have $\overline{\eta}_Z(zu) = z$ and $\eta'_Z(zu) = u$. Furthermore, $x_1u \in X$, yielding

$$\eta_Z(zu) = \eta_Z(x_{n+1}y_nx_n \dots y_1x_1u) = (x_{n+1}y_nx_n \dots y_1\overline{x_1u}, (x_1u)') = (z, u).$$

We thus have $z = x_{n+1}y_nx_n \dots y_1x_1 = x_{n+1}y_nx_n \dots y_1\overline{x_1u}$. We use the uniqueness property to conclude that $x_1 = \overline{x_1u}$, and we see that (x_1, u) is stable. The recognition principle for (X, S, η) tells us that $(x'_1, u) = (\eta'_Z(z), u)$ is stable. Therefore, $(Z, S \sqcup T, \eta_Z)$ is factorable. □

Proposition 2.2.11 *Let $\varphi : X \to \mathrm{End}(Y)$ be a morphism of monoids with the property that for every $x \in X$ the morphism $\varphi(x) : Y \to Y$ is a graded map. Then the semidirect product $Z = X \ltimes_\varphi Y$ is factorable with respect to the generating set $U = (S \times \{\epsilon_Y\}) \cup (\{\epsilon_X\} \times T)$.*

Recall that multiplication in Z is given by

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1x_2, (\varphi(x_2).(y_1))y_2).$$

Proof. The factorization map is defined as follows. Consider $z = (x, y) \in Z$ and set

$$\eta_Z(z) = \begin{cases} ((x, \overline{y}),\ (\epsilon_X, y')) & \text{if } y \neq \epsilon_Y, \\ ((\overline{x}, \epsilon_Y),\ (x', \epsilon_Y)) & \text{if } y = \epsilon_Y. \end{cases} \tag{2.9}$$

It is easily seen that $\eta_Z : Z \to Z \times Z$ is indeed a factorization map. The proof of Proposition 3.3.1 in [Vis11] carries over to weakly factorable monoids and thus (Z, U, η_Z)

is a weakly factorable monoid.[1] It remains to show that if (z, u) is an η_Z-stable pair in Z then so is $(\eta'_Z(z), u)$. So let $z = (x, y) \in Z$, $u \in U$, and let (z, u) be η_Z-stable. We may assume that $u \neq (\epsilon_X, \epsilon_Y)$, for in this case η_Z-stability of (z, u) forces $z = (\epsilon_X, \epsilon_Y)$ and η_Z-stability of $(\eta'_Z(z), u)$ is trivial. We distinguish two cases:

- Assume that $u = (s, \epsilon_Y)$ for some $s \in S$. We claim that $y = \epsilon_Y$. We have

  $$(x, y) \cdot (s, \epsilon_Y) = (xs, \varphi(s)(y)).$$

  Since $((x, y), (s, \epsilon_Y))$ is η_Z-stable, we in particular have $\eta'_Z(xs, \varphi(s)(y)) = (s, \epsilon_Y)$, implying that $\varphi(s)(y) = \epsilon_Y$, cf. (2.9). This yields

  $$\eta_Z((xs, \varphi(s)(y))) = ((\overline{xs}, \epsilon_Y), ((xs)', \epsilon_Y)).$$

  Now, η_Z-stability of (z, u) implies that $z = (\overline{xs}, \epsilon_Y)$, and thus $y = \epsilon_Y$, as claimed. We therefore have $z \cdot u = (xs, \epsilon_Y)$ and $\eta'_Z(z) = (\overline{xs}', \epsilon_Y)$. We conclude that $(\eta'_Z(z), u) = ((\overline{xs}', \epsilon_Y), (s, \epsilon_Y))$ is η_Z-stable, because $s = (xs)'$ and $(\overline{xs}', (xs)')$ is η_X-stable by the recognition principle.

- Assume that $u = (\epsilon_X, t)$ for some $t \in T$. Then $z \cdot u$ reads as

  $$(x, y) \cdot (\epsilon_X, t) = (x, yt).$$

  Since (z, u) is η_Z-stable, we must have $yt \neq \epsilon_Y$, cf. (2.9). We obtain

  $$\eta_Z((x, yt)) = ((x, \overline{yt}), (\epsilon_X, (yt)')),$$

  and by η_Z-stability of (z, u) we have $(yt)' = t$ and $\overline{yt} = y$. In particular, the pair (y, t) is η_Y-stable.

  Now if $y = \epsilon_Y$ then $\eta_Z(z) = ((\overline{x}, \epsilon_Y), (x', \epsilon_Y))$. In this case we have $\eta'_Z(z) \cdot u = (x', \epsilon_Y) \cdot (\epsilon_X, t) = (x', t)$. Using (2.9), we compute $\eta_Z(x', t) = ((x', \epsilon_Y), (\epsilon_X, t)) = (\eta'_Z(z), u)$. Therefore the pair $(\eta'_Z(z), u)$ is η_Z-stable.

  If $y \neq \epsilon_Y$ then $\eta_Z(z) = ((x, \overline{y}), (\epsilon_X, y'))$ and $\eta'_Z(z) = (\epsilon_X, y')$. This yields $\eta'_Z(z) \cdot u = (\epsilon_X, y't)$. Recall that (y, t) is η_Y-stable, and thus by the recognition principle, the pair (y', t) is η_Y-stable. It follows that $\eta_Z(\eta'_Z(z) \cdot u) = ((\epsilon_X, \overline{y't}), (\epsilon_X, (y't)')) = ((\epsilon_X, y'), (\epsilon_X, t)) = (\eta'_Z(z), u)$.

The Proposition is proven. □

Taking for φ the constant map $\varphi(x) = \mathrm{id} : Y \to Y$ we obtain the following.

Corollary 2.2.12 *The direct product $X \times Y$ is factorable with respect to the generating set U.*

[1] Saying that $\varphi(x) : Y \to Y$ is graded is equivalent to saying that $\ell(\varphi(x)(y)) \leq \ell(y)$ for all $x \in X$. Note that Visy even requires $\ell(\varphi(x)(y)) = \ell(y)$ for all $x \in X$, $y \in Y$. However, this stronger condition is not really used in the proof. It is used implicitly, in the sense that it guarantees that his norm function $N : X \ltimes_\varphi Y \to \mathbb{N}$ is symmetric, meaning that the norm is invariant under taking inverses. For our purposes this requirement is redundant.

Remark 2.2.13 (Graph products) Consider a finite family of factorable monoids $X_1, \dots, X_k$. Given an undirected graph Γ with vertices $1, \dots, k$, one can define the *graph product* of $X_1, \dots, X_k$. The idea is that we start with the free product $X_1 * \dots * X_k$, and we add a commutativity relation $x_i x_j = x_j x_i$, where $x_i \in X_i$ and $x_j \in X_j$, whenever in Γ there is an edge between i and j. Graph products in particular include free and direct products. More precisely, the free product $X_1 * \dots * X_k$ corresponds to the graph product with respect to the graph containing no edges at all, and the direct product $X_1 \times \dots \times X_k$ corresponds to the graph product with respect to the complete graph on k vertices. Rodenhausen [Rod] pointed out that the graph product of factorable monoids is again factorable if the graph Γ is transitive.

2.2.3 Finite sequences and monoid actions

We now interprete factorability in terms of actions of monoids P_2 and Q_2 on the direct product X^3. As a consequence, we obtain actions of monoids P_n and Q_n on the filtered bar complex $\mathcal{F}_\bullet \overline{\mathbb{B}}_* X$. We need to fix some notation first.

Definition 2.2.14 Denote by F_n the free monoid over the formal alphabet $\{1, \dots, n\}$. Elements of F_n will be called *finite sequences of height* $\leq n$ and will be written as $(i_s, \dots, i_1)$. We also introduce $F_0 = \{()\}$. Concatenation of finite sequences will be denoted by a lower dot. The *length* of a finite sequence $I = (i_s, \dots, i_1)$ is defined as $\#I = s$.

We prefer to think of the elements of F_n as sequences rather than words. The reason for this is the notion of subsequence, which, in the language of words, would correspond to "non-connected subwords". We write $J \subset I$ if J is a subsequence of I. A subsequence is called *connected* if it is a subword in the usual sense.

For $1 \leq a, b \leq n$ and $1 \leq c < n$ we introduce the following relations in F_n:

$$
\begin{aligned}
(a,b) &\sim_{\text{dist}} (b,a) \text{ for } |a-b| \geq 2 && (\text{“distant commutativity”})\\
(a,a) &\sim_{\text{idem}} (a) && (\text{“idempotence”})\\
(c{+}1,c,c{+}1) &\sim_{\text{left}} (c).(c{+}1,c,c{+}1) && (\text{“left absorption”})\\
(c{+}1,c,c{+}1) &\sim_{\text{right}} (c{+}1,c,c{+}1).(c) && (\text{“right absorption”})
\end{aligned}
$$

Definition 2.2.15 We then define the monoids P_n and Q_n as follows,

$$
\begin{aligned}
P_n &= F_n / \langle \sim_{\text{dist}}, \sim_{\text{idem}} \rangle,\\
Q_n &= F_n / \langle \sim_{\text{dist}}, \sim_{\text{idem}}, \sim_{\text{left}}, \sim_{\text{right}} \rangle,
\end{aligned}
$$

where $\langle \sim_{\text{dist}}, \sim_{\text{idem}} \rangle$ denotes the congruence relation generated by $\sim_{\text{dist}}$ and $\sim_{\text{idem}}$, cf. Chapter 0 and analogously for the other quotient.

We write $p_n : F_n \twoheadrightarrow P_n$, $q_n : F_n \twoheadrightarrow Q_n$ for the respective quotient maps. When no confusion is possible we will often suppress the index n. Multiplication in P_n and Q_n will be denoted by a centered dot to distinguish it from multiplication in F_n. Elements of P_n and Q_n are equivalence classes of finite sequences, and we denote the class corresponding to the element $(i_s, \dots, i_1)$ by $[i_s, \dots, i_1]_P$ and $[i_s, \dots, i_1]_Q$, respectively.

For finite sequences $I, J \in F_n$ we set $I \sim_P J$ (resp. $I \sim_Q J$) if I and J lie in the same p-fibre (resp. q-fibre), i.e. if and only if $p(I) = p(J)$ (resp. $q(I) = q(J)$).

Convention. Obviously, $T = \{[1]_Q, \dots, [n]_Q\}$ is a generating set for Q_n. Conversely, every generating set for Q_n contains T. This can be seen as follows. For $1 \le k \le n$ we have

$$[k]_Q = \{(k),\ (k,k),\ (k,k,k), \dots\},$$

and thus if for non-empty finite sequences $I, J \in F_n$ we have $[I.J]_Q = [k]_Q$ then $[I]_Q = [J]_Q = [k]_Q$. Therefore T must contain $[k]_Q$. In particular, T is a minimal generating set for Q_n. When speaking of "word length" of elements of Q_n, we will always refer to the word length with respect to T. For the monoids P_n we stick to the analogous convention.

We now explain what the monoids P_n and Q_n have to do with factorability.

Let X be a monoid and let S be a generating set. Let $\eta : X \to X \times X$ be a factorization map and recall the maps $f_i : X^n \to X^n$,

$$f_i(x_n, \dots, x_1) = (x_n, \dots, \overline{x_{i+1}x_i}, (x_{i+1}x_i)', \dots, x_1). \tag{2.10}$$

We have an action

$$F_2 \longrightarrow \mathrm{Map}(X^3, X^3)$$

by sending a finite sequence $(i_s, \dots, i_1) \in F_2$ to the composition $f_{i_s} \circ \dots \circ f_{i_1} : X^3 \to X^3$. We remark that $\mathrm{Map}(X^3, X^3)$ denotes all maps from X^3 to X^3 and not only morphisms. By axiom (F1), we have $f_i^2 = f_i$. This can be expressed by saying that the above action factors through P_2:

$$\begin{tikzcd} F_2 \arrow[r, "f"] \arrow[d, "p"'] & \mathrm{Map}(X^3, X^3) \\ P_2 \arrow[ur] & \end{tikzcd}$$

It is convenient to denote the action $P_2 \to \mathrm{Map}(X^3, X^3)$ again by f. Note that the $f_i : X^3 \to X^3$ are graded maps. The action hence descends to the filtration quotients, i.e. for every $h \ge 0$ we have an action

$$f : P_2 \longrightarrow \mathrm{Map}(X^3[h], X^3[h]).$$

Assume now that (X, S, η) is a factorable monoid. Then on the filtration quotients we have $f_1 f_2 f_1 f_2 = f_2 f_1 f_2 = f_2 f_1 f_2 f_1 : X^3[h] \to X^3[h]$ for every $h \ge 0$, cf. Definitions 2.1.4 and 2.2.1. We record the following immediate reformulation of factorability.

Proposition 2.2.16 *Let X be a monoid, S a generating set and $\eta : X \to X \times X$ a factorization map. Then (X, S, η) is factorable monoid if and only if for every $h \geq 0$ the action $f : P_2 \to \mathrm{Map}(X^3[h], X^3[h])$ factors through Q_2.*

We now consider X^n for arbitrary n. From (2.10) it is obvious that f_i and f_j commute for $|i - j| \geq 2$, and therefore the action $f : F_{n-1} \to \mathrm{Map}(X^n, X^n)$ factors through P_{n-1}. Going over to filtration quotients, we obtain actions $f : P_{n-1} \to \mathrm{Map}(X^n[h], X^n[h])$. Again, we have $f_1 f_2 f_1 f_2 = f_2 f_1 f_2 = f_2 f_1 f_2 f_1 : X^3[h] \to X^3[h]$, and this statement clearly remains true when we replace the indices 1, 2 by i, $i{+}1$ for $1 \leq i < n$. (Because we required $f_1 f_2 f_1 f_2 \equiv f_2 f_1 f_2 \equiv f_2 f_1 f_2 f_1$ to hold for all triples, and thus in particular for (x_4, x_3, x_2) and so on.)

What we have shown is that if (X, S, η) is a factorable monoid then for every $n \geq 0$ and all $h \geq 0$ the action $f : P_{n-1} \to \mathrm{Map}(X^n[h], X^n[h])$ factors through Q_{n-1} (and by Proposition 2.2.16 the converse is also true).

$$
\begin{tikzcd}
P_{n-1} \arrow[r, "f"] \arrow[d] & \mathrm{Map}(X^n[h], X^n[h]) \\
Q_{n-1} \arrow[ur] &
\end{tikzcd}
$$

A very similar statement holds for the filtered normalized bar complex $\mathcal{F}_\bullet \overline{\mathbb{B}}_* X$: Here, the maps f_i induce an action $F_{n-1} \to \mathrm{End}(\overline{\mathbb{B}}_n X)$. (Recall from (2.4) on page 62 that in this setting the f_i's are $\mathbb{Z}$-linear.) As before, this action factors through P_{n-1}. Going over to filtration quotients we obtain an action $f : P_{n-1} \to \mathrm{End}(\mathcal{G}_h \overline{\mathbb{B}}_n X)$. The following is immediate.

Proposition 2.2.17 *Let X be a monoid, S a generating set and $\eta : X \to X \times X$ a factorization map. If (X, S, η) is a factorable monoid then for every $n \geq 0$ and all $h \geq 0$ the action $f : P_{n-1} \to \mathrm{End}(\mathcal{G}_h \overline{\mathbb{B}}_n X)$ factors through Q_{n-1}.*

Remark 2.2.18 The converse of Proposition 2.2.17 does not hold: Consider $\mathbb{Z}/3\mathbb{Z}$ with presentation $X = \langle a \mid a^3 = \epsilon \rangle$. As generating set we take $S = \{a\}$. Define a factorization map by $\eta(\epsilon) = (\epsilon, \epsilon)$, $\eta(a) = (\epsilon, a)$ and $\eta(a^2) = (a, a)$. Observe that (X, S, η) is not a factorable monoid. (The triple (a, a, a) is everywhere stable, but its η-normal form is ϵ.) However, the action $f : P_{n-1} \to \mathrm{End}(\mathcal{G}_h \overline{\mathbb{B}}_n X)$ factors through Q_{n-1}. This is because a cell $[x_n | \ldots | x_1] \in \overline{\mathbb{B}}_n X$ is either stable at position i (in which case we have $(x_{i+1}, x_i) = (a, a)$) or applying f_i strictly decreases the norm. In other words, in $X \smallsetminus \{\epsilon\}$ there are no unstable geodesic pairs, and therefore the failure of the recognition principle is not detected when working in the *normalized* bar complex.

Propositions 2.2.16 and 2.2.17 offer a rather conceptual approach to factorability via actions of monoids. Indeed, many results will later follow from abstract properties of the monoids P_n and Q_n, rather than from explicit calculations. In the following sections we investigate these monoids. For this purpose it will be helpful to have at hand a visualization for finite sequences. We draw in a (mirrored) coordinate chart the values (k, i_k) and we draw a line from (k, i_k) to $(k+1, i_{k+1})$ if $|i_k - i_{k+1}| \leq 1$, i.e. we draw a line if the two entries do not commute in the sense of $\sim_{\text{dist}}$. As an example, Figure 2.10 visualizes the sequence $(i_{10}, \ldots, i_1) = (4, 3, 4, 2, 3, 4, 1, 2, 3, 4)$.

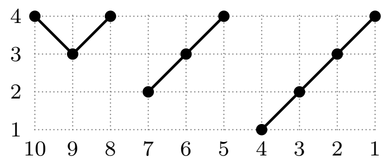


Figure 2.10: Visualization of $(4, 3, 4, 2, 3, 4, 1, 2, 3, 4)$.

In Figure 2.11 we express the relations $\sim_{\text{left}}$ and $\sim_{\text{right}}$ in pictures.

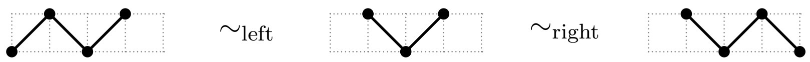


Figure 2.11: Visualization of $\sim_{\text{left}}$ and $\sim_{\text{right}}$.

Remark 2.2.19 (Evaluation Lemma) We state this remark for later reference. Let (X, S, η) be a factorable monoid. For the empty sequence $()$ we define $f_{()} := \text{id}$, and for a non-empty finite sequence $I = (i_s, \ldots, i_1)$ we set $f_I = f_{i_s} \circ \ldots \circ f_{i_1}$. Let $I, J \in F_n$ be finite sequences. Our above observations immediately yield the following.

(a) If $I \sim_P J$ then $f_I = f_J : \overline{\mathbb{B}}_{n+1}X \to \overline{\mathbb{B}}_{n+1}X$, and

(b) if $I \sim_Q J$ then $f_I \equiv f_J : \overline{\mathbb{B}}_{n+1}X \to \overline{\mathbb{B}}_{n+1}X$.

We will refer to this fact as *Evaluation Lemma*.

2.3 The monoids P_n and Q_n

2.3.1 The monoids P_n

Throughout this section fix some $n \in \mathbb{N}$. We will investigate the monoid P_n and show that it possesses a strongly minimal convergent rewriting system. We introduce the notions of being "right-most" and "left-most", which are opposite to each other in the sense that a finite sequence $(i_s, \ldots, i_1)$ is right-most if and only if its opposite $(i_1, \ldots, i_s)$ is left-most. In particular, for every statement about right-most finite sequences there is an equivalent formulation for left-most finite sequences. For example, in Lemma 2.3.3.(b), to get the opposite result for left-most finite sequences, one only has to replace $i_{t+1} \leq i_t + 1$ by $i_{t+1} \geq i_t - 1$. We leave these minor changes to the reader and concentrate on right-most finite sequences only.

Definition 2.3.1 (a) A finite sequence $(i_s, \ldots, i_1)$ is called *right-most* (resp. *left-most*), if for every $s > t \geq 1$ the following holds: If $|i_{t+1} - i_t| \geq 2$ then $i_{t+1} < i_t$ (resp. $i_{t+1} > i_t$). In other words, large entries stand right-most (resp. left-most).

(b) A finite sequence $(i_s, \ldots, i_1)$ is called *reduced* if for every t, $s > t \geq 1$, we have $i_{t+1} \neq i_t$.

Clearly, being right-most (resp. left-most) is a local condition. In Figure 2.12 we give pictures how right-most sequences look like or do not look like locally. Also observe that (a)-(c) are left-most, whereas (d) is not left-most.

(a) Not right-most (b) Right-most (c) Right-most (d) Right-most

Figure 2.12: Visualization of right-most.

Example 2.3.2 The finite sequence in Figure 2.10 is reduced and left-most. The first is obvious and the latter can be seen by checking adjacent entries.

The following lemma offers some immediate characterizations of the property of being right-most. The proof is very easy and should be regarded as a warm-up for the sections to come, where we have to do calculations in Q_n, which will be more complicated.

Lemma 2.3.3 *For a finite sequence $I := (i_s, \ldots, i_1)$ the following statements are equivalent:*

(a) $(i_s, \ldots, i_1)$ is right-most.

(b) For all t, $s > t \geq 1$, we have $i_{t+1} \leq i_t + 1$.

(c) For every connected subsequence J of I the following holds: If for $a < b$ we have $(b, a) \subset J$ then we also have $(b, \ldots, a+1, a) \subset J$.

Proof. (a) $\Rightarrow$ (b): Assume that $(i_s, \ldots, i_1)$ fails to satisfy (b), i.e. there exists t, $s > t \geq 1$, such that $i_{t+1} > i_t + 1$, hence $i_{t+1} \geq i_t + 2$. In particular, we have $|i_{t+1} - i_t| \geq 2$ and $i_{t+1} > i_t$. Therefore $(i_s, \ldots, i_1)$ is not right-most.

(b) $\Rightarrow$ (c): Assume that I satisfies (b) and let J be a connected subsequence of I. Note that J also satisfies (b). Consider $a < b$ such that $(b, a) \subset J$. Condition (b) states that the entries of J "increase by at most 1". Therefore, to get from a to b we will come across $a+1$, $a+2$ and so on. Hence $(b, \ldots, a+1, a) \subset J$.

(c) $\Rightarrow$ (a): Assume that $(i_s, \ldots, i_1)$ is not right-most. Hence there exists t, $s > t \geq 1$, such that $|i_{t+1} - i_t| \geq 2$ and $i_{t+1} \geq i_t$, i.e. $i_{t+1} \geq i_t + 2$. Set $J := (i_{t+1}, i_t)$. This is a connected subsequence of length 2 of I. Since $i_{t+1} \geq i_t + 2$, the sequence $(i_{t+1}, \ldots, i_t + 1, i_t)$ has length at least 3, therefore it cannot be a subsequence of J. $\square$

Proposition 2.3.4 *The monoid P_n admits a strongly minimal convergent rewriting system over the alphabet $\{[1]_P, \dots, [n]_P\}$.*

PROOF. Set $S = \{1, \dots, n\}$. (For simplicity of notation we write i instead of $[i]_P$.) The set of rewriting rules R consists of the following rules:

(a) $(j, i) \to (i, j)$ for all $1 \leq i < j \leq n$ and $j - i \geq 2$,

(b) $(i, i) \to (i)$ for all $1 \leq i \leq n$.

It is obvious that this is a rewriting system for P_n, i.e. $\langle S \,|\, R \rangle$ is a presentation of P_n. An isomorphism is given by the quotient map $p_n : (i_s, \dots, i_1) \mapsto [i_s, \dots, i_1]_P$. It is also easily seen that this rewriting system is strongly minimal: Every left side of a rewriting rule shows up only once. Every right side is irreducible. Every finite sequence of length 1 is irreducible.

We now prove convergence. Observe that there is no infinite chain of reductions: To see this, we introduce the *value* of a finite sequence:

$$\text{value}(i_s, \dots, i_1) := \sum_{k=1}^{s} 2^k \cdot i_k.$$

The value of a finite sequence is a natural number and applying a rewriting rule to a finite sequence strictly lowers its value. Our rewriting system is therefore noetherian.

It remains to show that there is only one irreducible representative in each equivalence class. This is most easily seen by showing that $\langle S \mid \mathcal{R} \rangle$ is confluent. (Recall from Remark 1.2.6 that for a noetherian rewriting system the notions of convergence and confluence are equivalent.) Formally, this would require several case distinctions, which are all straightforward. We omit the details. Instead, we depict one case in Figure 2.13. Arrows indicate the application of a rewriting rule. $\square$

$$\begin{tikzcd}
 & (5).(3,1) \arrow[r, equal] \arrow[d] & (5,3,1) \arrow[r, equal] & (5,3).(1) \arrow[d] & \\
(5,1).(3) \arrow[r, equal] \arrow[d] & (5).(1,3) & & (3,5).(1) \arrow[r, equal] & (3).(5,1) \arrow[d] \\
(1,5).(3) \arrow[r, equal] & (1).(5,3) \arrow[d] & & (3,1).(5) \arrow[r, equal] \arrow[d] & (3).(1,5) \\
 & (1).(3,5) \arrow[r, equal] & (1,3,5) \arrow[r, equal] & (1,3).(5) &
\end{tikzcd}$$

Figure 2.13: The rewriting system $\langle S \mid \mathcal{R} \rangle$ is confluent.

Looking at the rewriting rules in the proof of Proposition 2.3.4, we see that a word over the generating set S is irreducible if and only if it is right-most and reduced. Thus:

Corollary 2.3.5 *In every $\sim_P$-equivalence class of finite sequences there is exactly one representative that is right-most and reduced.*

For a finite sequence I we will sometimes denote its unique right-most and reduced representative within $[I]_P$ by $\overrightarrow{I}$. Accordingly, its unique left-most and reduced representative will be denoted by $\overleftarrow{I}$.

Remark 2.3.6 (a) The monoid P_n is factorable with respect to the generating set $T = \{[1]_P, \ldots, [n]_P\}$. We define a factorization map as follows. For $I \in F_n$ set $\eta([I]_P) = ([i_s, \ldots, i_2]_P, [i_1]_P)$, where $(i_s, \ldots, i_1) = \overrightarrow{I}$. Roughly speaking, $\eta : P_n \to P_n \times P_n$ splits up the largest generator that is a right-divisor of $[I]_P$. ("Large" refers to the canonical total order on T.) It is now easily checked that (P_n, T, η) is weakly factorable and that it satisfies the recognition principle.

(b) In contrast, for $n \geq 2$, the monoid Q_n is not even weakly factorable with respect to the generating set $\{[1]_Q, \ldots, [n]_Q\}$. This can be seen as follows. Assume that $\eta : Q_n \to Q_n \times Q_n$ is a factorization map. The element $[2,1,2]_Q$ admits a unique factorization into three generators, forcing $\eta([2,1,2]_Q) = ([2,1]_Q, [2]_Q)$. Similarly, we have $\eta([2,1]_Q) = ([2]_Q, [1]_Q)$. Considering the pair $([2,1,2]_Q, [1]_Q)$ and computing the upper and lower composition in (2.2) on page 57 gives

$$\alpha_u = d_2 \circ \eta_1 \circ d_1 \circ \eta_2([2,1,2]_Q, [1]_Q) = ([2,1,2]_Q, [1]_Q),$$
$$\alpha_l = \eta_1 \circ d_1([2,1,2]_Q, [1]_Q) = ([2,1]_Q, [2]_Q).$$

In particular, α_u is norm-preserving, but α_l is not.

2.3.2 The monoids Q_n

Throughout this chapter we will simultaneously work in F_n and its quotient Q_n. To avoid confusion, elements of Q_n will usually be denoted by small greek letters. We will investigate the monoids Q_n and show that each of them contains an absorbing element[2] Δ_n, that is, for all $\alpha \in Q_n$ we have $\alpha \cdot \Delta_n = \Delta_n = \Delta_n \cdot \alpha$.

For an element $\alpha \in Q_n$ denote by $\mathscr{L}(\alpha)$ (resp. $\mathscr{R}(\alpha)$) its set of left (resp. right) divisors.

Remark 2.3.7 Obviously, every left-divisor of α_2 is a left-divisor of $\alpha_2 \cdot \alpha_1$. Therefore $\mathscr{L}(\alpha_2 \cdot \alpha_1) \supseteq \mathscr{L}(\alpha_2)$. Similarly, $\mathscr{R}(\alpha_2 \cdot \alpha_1) \supseteq \mathscr{R}(\alpha_1)$. This elementary property will be used in several proofs.

Example 2.3.8 (a) Let us define some kind of (right) Cayley graph Γ for Q_n as follows. Vertices of Γ are the elements of Q_n, and we draw an edge $\alpha_1 \to \alpha_2$ with label $[k]_Q$ if there exists k, $1 \leq k \leq n$, such that $\alpha_1 \cdot [k]_Q = \alpha_2$. Figure 2.14 depicts the (right) Cayley graph of Q_2 without reflexive edges and labellings.

[2] There seems to be no consistent notation for absorbing elements. For the lack of a better symbol, we choose the letter Δ, because, apart from the finiteness condition, every absorbing element is a Garside element.

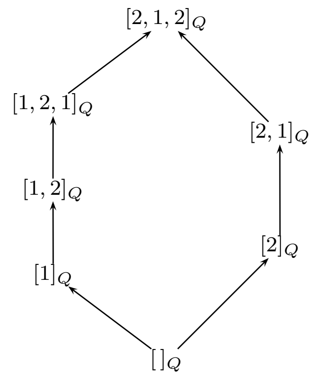


Figure 2.14: (Right) Cayley graph of Q_2 without reflexive edges and labellings.

From this graph we can easily read off left-divisors: We have $\alpha \in \mathscr{L}(\beta)$ if and only if in the associated right Cayley graph there is a directed path from α to β, possibly of length 0. For example

$$\begin{aligned} \mathscr{L}[\,]_Q &= \{[\,]\}, \\ \mathscr{L}[2]_Q &= \{[\,]_Q,\ [2]_Q\}, \\ \mathscr{L}[2,1]_Q &= \{[\,]_Q,\ [2]_Q,\ [2,1]_Q\}, \\ \mathscr{L}[2,1,2]_Q &= Q_2. \end{aligned}$$

(b) From Figure 2.14 we immediately see that $\#Q_2 = 7$. However, for $n \geq 3$, every Q_n is infinite. Indeed, the associated Cayley graph contains directed, cycle-free paths of infinite length, e.g.

$$[1]_Q \to [1,2]_Q \to [1,2,3]_Q \to [1,2,3,2]_Q \to [1,2,3,2,1]_Q \to [1,2,3,2,1,2]_Q \to \dots$$

Let $\alpha \in Q_n$. The set $\mathscr{L}(\alpha)$ will in general not be closed under multiplication. For example, we have $[2,1]_Q \in \mathscr{L}[2,1]_Q$, but $[2,1]_Q \cdot [2,1]_Q = [2,1,2]_Q \notin \mathscr{L}[2,1]_Q$. However, recall that every element in $T = \{[1]_Q, \dots, [n]_Q\}$ is an idempotent, and therefore

$$[k]_Q \cdot \alpha = \alpha$$

if $[k]_Q$ is a left-divisor of α. It follows that the submonoid generated by $T' := T \cap \mathscr{L}(\alpha)$ embeds (as a subset) into $\mathscr{L}(\alpha)$. We conclude that α is an absorbing element if and only if $T \subset \mathscr{L}(\alpha)$. Our next aim is to construct a finite sequence that is a representative for an absorbing element in Q_n. This requires some preparation.

For $b \geq a$ we introduce the following short hand notation,

$$I_a^b := (a, a+1, \dots, b-1, b) \in F_b.$$

For $a > b$ we set $I_a^b := (\,)$.

Lemma 2.3.9 *For all n we have $[I_2^{n-1}.I_1^{n-1}]_Q \in \mathscr{L}[I_2^n.I_1^n]_Q$.*

PROOF. For $n = 1$ the statement reads as $[\,]_Q \in \mathscr{L}[1]_Q$, which is obviously true. For $n \geq 2$ the proof is completely given in terms of Figure 2.15. (The reader should recall Figure 2.11 on page 75.) It shows that $I_2^{n-1}.I_1^{n-1}.(n, n-1, n) \sim_Q I_2^n.I_1^n$, and thus $[I_2^{n-1}.I_1^{n-1}]_Q$ is a left-divisor of $[I_2^n.I_1^n]_Q$. □

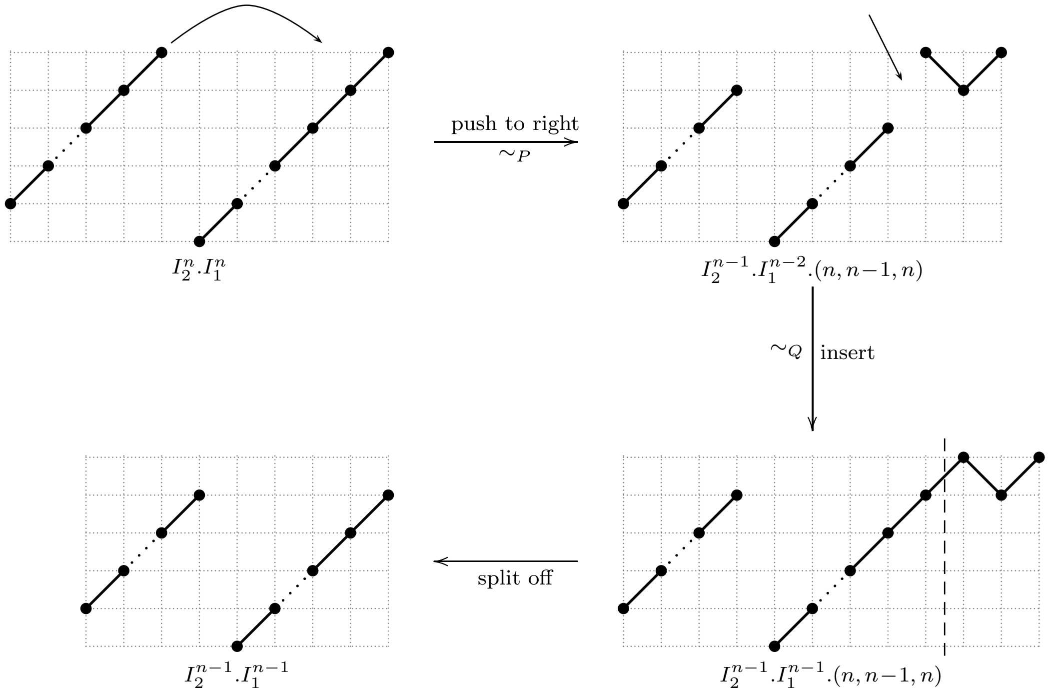


Figure 2.15: Proof of Lemma 2.3.9.

Corollary 2.3.10 *For every $n \geq 1$ we have $[1]_Q \in \mathscr{L}[I_2^n.I_1^n]_Q$.*

PROOF. Iterating Lemma 2.3.9 gives $[I_2^1.I_1^1]_Q \in \mathscr{L}[I_2^n.I_1^n]_Q$. Furthermore, $I_2^1.I_1^1 = (1)$, whence the claim. □

Recall that an *absorbing element* of a monoid X is an element Δ with the property that for all $\alpha \in X$ one has $\alpha \cdot \Delta = \Delta = \Delta \cdot \alpha$. In particular, absorbing elements are unique, if they exist.

Definition 2.3.11 For $k \geq 0$ and $n \geq 1$ define a map $\text{shift}_k : F_n \to F_{n+k}$,

$$\text{shift}_k(i_s, \ldots, i_1) := (i_s{+}k, \ldots, i_1{+}k).$$

Note that $\text{shift}_k : F_n \to F_{n+k}$ is an injective morphism of monoids. Futhermore, $\text{shift}_k : F_n \to F_{n+k}$ descends to a map $Q_n \to Q_{n+k}$ which we will also denote by shift_k.

We recursively define two candidates for a representative of an absorbing element:

$$\overleftarrow{D_1} := (1) \qquad \overrightarrow{D_1} := (1) \tag{2.11}$$
$$\overleftarrow{D_{n+1}} := \mathrm{shift}_1(\overleftarrow{D_n}).(1,\dots,n{+}1), \qquad \overrightarrow{D_{n+1}} := (n{+}1,\dots,1).\,\mathrm{shift}_1(\overrightarrow{D_n}).$$

Example 2.3.12 For convenience we explicitly list some of the $\overleftarrow{D_n}$ and $\overrightarrow{D_n}$ for small n. Note that the finite sequence illustrated in Figure 2.10 on page 75 is just $\overleftarrow{D_4}$.

$$\begin{aligned}
\overleftarrow{D_1} &= (1), & \overrightarrow{D_1} &= (1),\\
\overleftarrow{D_2} &= (2,1,2), & \overrightarrow{D_2} &= (2,1,2),\\
\overleftarrow{D_3} &= (3,2,3,1,2,3), & \overrightarrow{D_3} &= (3,2,1,3,2,3).\\
\overleftarrow{D_4} &= (4,3,4,2,3,4,1,2,3,4), & \overrightarrow{D_4} &= (4,3,2,1,4,3,2,4,3,4).
\end{aligned}$$

Lemma 2.3.13 *Set $(i_s,\dots,i_1) := \overleftarrow{D_n}$. Then $\overrightarrow{D_n} = (i_1,\dots,i_s)$. Furthermore, $\overleftarrow{D_n}$ is left-most and reduced, $\overrightarrow{D_n}$ is right-most and reduced, and $\overleftarrow{D_n} \sim_P \overrightarrow{D_n}$.*

Proof. The first statement is easily seen by closely looking at the definition of $\overleftarrow{D_n}$ and $\overrightarrow{D_n}$. It is also easily seen that $\overleftarrow{D_n}$ is left-most and reduced, implying that $\overrightarrow{D_n}$ is right-most and reduced. It remains to show that $\overleftarrow{D_n} \sim_P \overrightarrow{D_n}$. This will be done inductively. Example 2.3.12 gives $\overleftarrow{D_1} = \overrightarrow{D_1}$ and $\overleftarrow{D_2} = \overrightarrow{D_2}$. For $n \geq 2$ we have

$$\begin{aligned}
\overrightarrow{D_{n+1}} &= (n{+}1,\dots,1).\,\mathrm{shift}_1(\overrightarrow{D_n})\\
&\sim_P (n{+}1,\dots,1).\,\mathrm{shift}_1(\overleftarrow{D_n})
\end{aligned}$$

by the induction hypothesis. Plugging in the definition of $\overleftarrow{D_n}$ we obtain

$$\begin{aligned}
&= (n{+}1,\dots,1).\,\mathrm{shift}_1(\mathrm{shift}_1(\overleftarrow{D_{n-1}}).(1,\dots,n))\\
&= (n{+}1,\dots,1).\,\mathrm{shift}_2(\overleftarrow{D_{n-1}}).(2,\dots,n{+}1)
\end{aligned}$$

because $\mathrm{shift}_1 : F_n \to F_{n+1}$ is a morphism of monoids. Note that every entry of $\mathrm{shift}_2(\overleftarrow{D_{n-1}})$ is at least 3. We therefore have $(1).\,\mathrm{shift}_2(\overleftarrow{D_{n-1}}) \sim_P \mathrm{shift}_2(\overleftarrow{D_{n-1}}).(1)$. This yields

$$\begin{aligned}
&\sim_P (n{+}1,\dots,2).\,\mathrm{shift}_2(\overleftarrow{D_{n-1}}).(1,\dots,n{+}1)\\
&= \mathrm{shift}_1((n,\dots,1).\,\mathrm{shift}_1(\overleftarrow{D_{n-1}})).(1,\dots,n{+}1)\\
&\sim_P \mathrm{shift}_1((n,\dots,1).\,\mathrm{shift}_1(\overrightarrow{D_{n-1}})).(1,\dots,n{+}1)\\
&= \mathrm{shift}_1(\overrightarrow{D_n}).(1,\dots,n{+}1)\\
&\sim_P \mathrm{shift}_1(\overleftarrow{D_n}).(1,\dots,n{+}1)\\
&= \overleftarrow{D_{n+1}}.
\end{aligned}$$

The Lemma is proven. □

Proposition 2.3.14 *$\Delta_n := [\overleftarrow{D_n}]_Q$ is an absorbing element for Q_n.*

Proof. Recall that to prove the Proposition it suffices to show that every $[k]_Q$ ($1 \leq k \leq n$) is a left- and right-divisor of Δ_n. This will be done by induction on n. For $n = 1$ we have $\Delta_1 = [1]_Q$, and the statement is trivial. Assume now that for all k, $1 \leq k \leq n-1$, we have $[k]_Q \in \mathscr{L}(\Delta_{n-1})$.

Claim 1: $[k]_Q \in \mathscr{L}(\Delta_n)$ for all $1 \leq k \leq n$.

Claim 1 follows from Claim 1.A and Claim 1.B.

Claim 1.A: $[1]_Q \in \mathscr{L}(\Delta_n)$.

Since $\Delta_1 = [1]_Q$, the statement is obvious for $n = 1$. Corollary 2.3.10 for $n = 2$ gives $[1]_Q \in \mathscr{L}(\Delta_2)$. Consider $n \geq 3$. We have $\overleftarrow{D_n} = \operatorname{shift}_2(\overleftarrow{D_{n-2}}).I_2^n.I_1^n$. By Corollary 2.3.10 we know that $[1]_Q \in \mathscr{L}[I_2^n.I_1^n]_Q$, i.e. there exists a finite sequence $J \in F_n$ such that the concatenation $\operatorname{shift}_2(\overleftarrow{D_{n-2}}).(1).J$ is a representative for Δ_n. Furthermore, every entry of $\operatorname{shift}_2(\overleftarrow{D_{n-2}})$ is at least 3. Therefore we can push (1) to the very left, yielding that $(1).\operatorname{shift}_2(\overleftarrow{D_{n-2}}).J$ is a representative for Δ_n. Therefore $[1]_Q \in \mathscr{L}(\Delta_n)$ and Claim 1.A is proven.

Claim 1.B: $[k]_Q \in \mathscr{L}(\Delta_n)$ for all $1 < k \leq n$.

Recall that $\overleftarrow{D_n} = \operatorname{shift}_1(\overleftarrow{D_{n-1}}).I_1^n$ and hence $\mathscr{L}(\Delta_n) \supseteq \mathscr{L}[\operatorname{shift}_1(\overleftarrow{D_{n-1}})]_Q$. Claim 1.B now follows from the induction hypothesis together with the fact that $\overleftarrow{D_{n-1}}$ is a representative for Δ_{n-1}.

We have shown that every $[k]_Q$ is a left divisor of Δ_n. Therefore, Δ_n is left-absorbing: There is a finite sequence I such that $[k]_Q \cdot [I]_Q = \Delta_n$, and by idempotency of $[k]_Q$ we obtain $[k]_Q \cdot \Delta_n = [k]_Q \cdot [k]_Q \cdot [I]_Q = [k]_Q \cdot [I]_Q = \Delta_n$.

Claim 2: $[k]_Q \in \mathscr{R}(\Delta_n)$ for all $1 \leq k \leq n$.

This can be seen as follows: Set $(i_s, \ldots, i_1) := \overleftarrow{D_n}$. Then by Lemma 2.3.13 we have $[i_s, \ldots, i_1]_Q = [i_1, \ldots, i_s]_Q$. We therefore can copy Lemma 2.3.9, Corollary 2.3.10 and the above proof by "mirroring" our arguments. We omit the details. □

We conclude this section with deducing an efficient way of finding factorability structures on certain monoid presentations.

Corollary 2.3.15 *Let (X, S, η) be a factorable monoid. Let $x \in X$ and let $(s_n, \ldots, s_1) \in S^*$ be a representative of x of minimal length, i.e. $n = \ell(x)$. Then $f_{\overleftarrow{D_{n-1}}}(s_n, \ldots, s_1)$ is an η-normal form.*

Proof. First of all, note that if $(s_n, \ldots, s_1)$ is a geodesic tuple then for every i, $n > i \geq 1$, we have that f_i is norm-preserving for $(s_n, \ldots, s_1)$, and in particular $f_i(s_n, \ldots, s_1)$ is again geodesic.

Now, by assumption, n is minimal, so $(s_n, \ldots, s_1)$ is geodesic, and it follows that $f_{\overleftarrow{D_{n-1}}}$

is norm-preserving. We have $[\overleftarrow{D_{n-1}}]_Q = \Delta_{n-1}$ and hence $(k).\overleftarrow{D_{n-1}} \sim_Q \overleftarrow{D_{n-1}}$ for all $1 \le k < n$. The Evaluation Lemma yields

$$f_k \circ f_{\overleftarrow{D_{n-1}}} \equiv f_{\overleftarrow{D_{n-1}}} : X^n \longrightarrow X^n. \tag{2.12}$$

Since $f_{\overleftarrow{D_{n-1}}}$ is norm-preserving, one has equality in (2.12), proving that $f_{\overleftarrow{D_{n-1}}}(s_n, \ldots, s_1)$ is everywhere stable, and hence, by the recognition principle, it is an η-normal form. □

Assume that $(s_n, \ldots, s_1)$ is any representative of x, not necessarily of minimal length. Then applying $f_{\overleftarrow{D_{n-1}}}$ yields an η-normal form or drops the norm. In particular, applying $f_{\overleftarrow{D_{n-1}}}$ n-times and afterwards possibly removing a tail of trivial entries, we end up with an η-normal form. This algorithm has independently been found by Rodenhausen [Rod].

Observe that $\#\overleftarrow{D_{n-1}} = n \cdot (n+1)/2$, and hence the algorithm described above has complexity in $\mathcal{O}(n^3)$. Thus:

Corollary 2.3.16 *Let (X, S, η) be a factorable monoid. If S is finite then X has Dehn function of at most cubic growth, and in particular X has solvable word problem.*

Note that our normal form algorithm is "local" in the sense of Remark 2.1.27. The existence of such a local normal form algorithm implies that for a factorable monoid (X, S, η) the map $\eta : X \to X \times X$ can completely be recovered by its values on $X(2)$. More precisely, for $x \in X$ take any representative $(s_n, \ldots, s_1) \in S^*$. It follows from the recognition principle that

$$\eta(x) = d_2 \circ \ldots \circ d_{n-1} \circ (f_{\overleftarrow{D_{n-1}}})^n(s_n, \ldots, s_1), \tag{2.13}$$

and to compute the right-hand side of (2.13) we only need to know the values of η on products of generators. This observation gives rise to the following description of factorability, which, for factorable groups, is due to Rodenhausen [Rod]. His proof carries over to weakly factorable monoids. We only give a sketch of proof and point out that the recognition principle is satisfied.

Proposition 2.3.17 *Let S be a formal generating set that contains the empty word ϵ. Assume we are given a map $\phi : S \times S \to S \times S$ satisfying the following conditions:*

(a) $\phi \circ \phi = \phi$.

(b) $\phi(s, \epsilon_X) = (\epsilon_X, s)$ *for all* $s \in S$.

(c) $\phi_2\phi_1\phi_2\phi_1 \equiv \phi_2\phi_1\phi_2 \equiv \phi_1\phi_2\phi_1\phi_2 : S^3 \to S^3$, *where ϕ_i is defined in the obvious way.*

Then ϕ gives rise to a factorability structure on (X, S), where

$$X = \langle s \in S \mid ab = cd \text{ if } \phi(a, b) = \phi(c, d) \rangle. \tag{2.14}$$

Proof. We use (2.13) to extend ϕ to a well-defined factorization map $\eta : X \to X \times X$. (Well-definition follows from the fact that X is the defined as the quotient of the free monoid S^* "modulo the relations induced by ϕ" and the absorbing property of $f_{\overleftarrow{D_{n-1}}}$.) The map η gives rise to maps η_i and f_i on tuples, and by construction the restrictions $f_i| : S^3 \to S^3$ coincide with the maps $\phi_i : S^3 \to S^3$ (for $i = 1, 2$). In particular, condition (2.7) on page 65 holds for tuples of norm ≤ 3. Remark 2.2.4 now enables us to use the proof of Lemma 2.2.3 to show weak factorability of (X, S, η). As a consequence, a pair (x, t) is stable if (x', t) is stable.

To conclude the Proposition it only remains to show the "only if"-part. For convenience we do this in detail. Let (x, t) be a stable pair. Set $n := \ell(x)$ and $(x_n, \ldots, x_1) := \eta_{n-1} \ldots \eta_1(x)$. Furthermore we set

$$(a_n, \ldots, a_1, b) := f_n \circ \ldots \circ f_1(x_n, \ldots, x_1, t).$$

Note that $(a_n, \ldots, a_1, b)$ is geodesic. We have

$$\begin{aligned}(x, t) = \eta(xt) &= d_2 \ldots d_n f_{\overleftarrow{D_n}}(x_n, \ldots, x_1, t) \\ &= d_2 \ldots d_n f_{\overrightarrow{D_n}}(x_n, \ldots, x_1, t) \\ &= d_2 \ldots d_n \circ f_n \ldots f_1 \circ f_{\mathrm{shift}_1(\overrightarrow{D_{n-1}})}(x_n, \ldots, x_1, t).\end{aligned}$$

The tuple $(x_n, \ldots, x_1)$ is the normal form of x and thus everywhere stable. In particular, $f_{\mathrm{shift}_1(\overrightarrow{D_{n-1}})}$ fixes this tuple and we obtain

$$\begin{aligned}&= d_2 \ldots d_n f_n \ldots f_1(x_n, \ldots, x_1, t) \\ &= d_2 \ldots d_n(a_n, \ldots, a_1, b).\end{aligned}$$

Note that $(a_n, \ldots, a_1, b) = f_{\overleftarrow{D_n}}(x_n, \ldots, x_1, t)$ is everywhere stable. (Because (x, t) is stable, and hence $(x_n, \ldots, x_1, t)$ is geodesic.) From this it follows that $t = b = (x_1 t)'$. On the other hand, $a_n \ldots a_1 = x$ and the tuple $(a_n, \ldots, a_1)$ is everywhere stable. This implies $(x_n, \ldots, x_1) = (a_n, \ldots, a_1)$, yielding $\overline{x_1 t} = \overline{a_1 b} = a_1 = x_1$. So $(x_1, t) = (x', t)$ is indeed stable. □

Remark 2.3.18 The standard situation to apply Proposition 2.3.17 to is the following. Assume we are given a monoid X in terms of a presentation $X = \langle S \mid R \rangle$. Furthermore, assume that every relation in R is of the form $ab = cd$ for some $a, b, c, d \in S$. For each such relation $ab = cd$ specify one side as "stable", say (a, b), and set $\phi(c, d) = (a, b)$. If a pair of non-trivial entries does not occur in any of the relations then ϕ will fix this pair. Proposition 2.3.17 provides an efficient sufficient condition for $\phi : S \times S \to S \times S$ giving rise to a factorability structure on X with respect to the generating set S. We will see several Examples in Chapter 4.

Remark 2.3.19 There is a nice result by Rodenhausen [Rod], which states that every factorable monoid can be written as in (2.14). More precisely, let X be a monoid and

S a generating set. Rodenhausen proved that if (X, S, η) is a factorable monoid, then X admits a presentation $X = \langle S \mid R \rangle$ with the property that every relation $w_1 = w_2$ in R satisfies $\ell(w_1), \ell(w_2) \leq 2$. This result can be quite time-saving when looking for new factorability structures. For example, it tells us that for $n \geq 3$ the symmetric groups $\mathcal{S}_n$ are not factorable with respect to the generating set $S = \{(1\ 2),\ (2\ 3),\ \ldots,\ (n{-}1\ n)\}$ of elementary transpositions.

2.3.3 Smallness

In the previous sections we studied the monoids P_n and Q_n. Since Q_n is a quotient of P_n, the p-fibre $p^{-1}\{p(I)\} \subset F_n$ of a finite sequence I is always contained in its q-fibre $q^{-1}\{q(I)\}$. In this section we investigate finite sequences for which these fibres actually coincide. Finite sequences of this kind play a central role throughout this exposition. We give several characterizations of this property and derive an upper bound for word lengths of such sequences.

Definition 2.3.20 We say that a finite sequence $I \in F_n$ is *small* if the fibers $p^{-1}\{p(I)\}$ and $q^{-1}\{q(I)\}$ in F_n coincide.

Remark 2.3.21 (a) Clearly, being small descends to equivalence classes $[\,]_P$ and $[\,]_Q$, yielding a notion of smallness on P_n and Q_n.

(b) Observe that a finite sequence $(i_s, \ldots, i_1)$ is small if and only if the "opposite" sequence $(i_1, \ldots, i_s)$ is.

Example 2.3.22 (a) The finite sequence (1) is small, for the p-fibre of (1) consists of the elements (1), $(1,1)$, $(1,1,1)$, and so on. To none of those the relations $\sim_{\text{left}}$ and $\sim_{\text{right}}$ are applicable.

(b) The finite sequence $(2,1,2)$ is not small. This can be seen as follows: First, $(1,2,1,2) \sim_{\text{left}} (2,1,2)$, hence $(1,2,1,2) \sim_Q (2,1,2)$. Secondly, if $(i_s, \ldots, i_1) \sim_P (2,1,2)$ then $i_s = 2$, which immediately follows from looking at the relations $\sim_{\text{idem}}$ and $\sim_{\text{dist}}$. Therefore, $(1,2,1,2) \in q^{-1}\{q(2,1,2)\} \smallsetminus p^{-1}\{p(2,1,2)\}$.

The following Proposition shows that Example 2.3.22.(b) is exhaustive in the sense that if $I \in F_n$ is not small, then there exist m, $0 \leq m \leq n-2$, and L, $R \in F_n$ such that $I \sim_Q L.\operatorname{shift}_m(2,1,2).R$. We need to fix some notation first.

For an element $\alpha \in Q_n$ define the two-sided ideal generated by α as follows:

$$\langle \alpha \rangle := \{\beta \cdot \alpha \cdot \beta' : \beta, \beta' \in Q_n\}.$$

Recall that the maps $\operatorname{shift}_0, \ldots, \operatorname{shift}_{n-2}$ provide monomorphisms $Q_2 \hookrightarrow Q_n$. This way we can define the "two-sided ideal in Q_n generated by shifted absorbing elements",

$$\square_n := \langle \operatorname{shift}_0(\Delta_2) \rangle \cup \ldots \cup \langle \operatorname{shift}_{n-2}(\Delta_2) \rangle.$$

Note that $\square_n \subset Q_n$ is indeed a two-sided ideal in the sense that for all α, $\beta \in Q_n$ we have $\alpha \cdot \square_n \cdot \beta \subseteq \square_n$.

Proposition 2.3.23 *A finite sequence $(i_s, \dots, i_1) \in F_n$ is small if and only if its class $[i_s, \dots, i_1]_Q$ lies in the complement $Q_n \smallsetminus \square_n$.*

PROOF. "$\Rightarrow$": We have to show that for arbitrary $L, R \in F_n$ and $0 \le m \le n-2$ the finite sequence $L.\operatorname{shift}_m(2,1,2).R$ is not small. We have $(2,1,2) \sim_Q (2,1,2,2)$ (using $\sim_{\text{idem}}$) and $(2,1,2,2) \sim_Q (2,1,2,1,2)$ (using $\sim_{\text{right}}$). It follows that for every $k \ge 0$ we have $(2,1,2) \sim_Q (2,1,2).(1,2)^k$ and therefore

$$L.\operatorname{shift}_m(2,1,2).R \sim_Q L.\operatorname{shift}_m\left((2,1,2).(1,2)^k\right).R.$$

It remains to show that $L.\operatorname{shift}_m(2,1,2).R \not\sim_P L.\operatorname{shift}_m(2,1,2).(1,2)^k.R$ for some k. (Indeed, this holds for all $k \ge 1$.) Recall the word length $\ell : P_n \to \mathbb{N}$ with respect to the generating set $\{[1]_P, \dots, [n]_P\}$, and observe that for finite sequences $I, J \in F_n$ we have

$$\ell([I.J]_P) \ge \max\{\ell([I]_P), \ell([J]_P)\}.$$

Since $\ell([(1,2)^k]_P) = 2k$ we see that the word length of $[L.\operatorname{shift}_m(2,1,2).(1,2)^k.R]_P$ gets arbitrary large (as $k \to \infty$). Thus, for k sufficiently large, the $[\,]_P$-classes of $L.\operatorname{shift}_m(2,1,2).R$ and $L.\operatorname{shift}_m(2,1,2).(1,2)^k.R$ have different word length.

"$\Leftarrow$": Let $[i_s, \dots, i_1]_Q \in Q_n \smallsetminus \square_n$. To show that $(i_s, \dots, i_1)$ is small we have to show that if for some finite sequence J we have $J \sim_Q (i_s, \dots, i_1)$ then we already have $J \sim_P (i_s, \dots, i_1)$. Assume this was wrong. Take any chain

$$J = J_0 \mapsto J_1 \mapsto \dots \mapsto J_{r-1} \mapsto J_r = (i_s, \dots, i_1),$$

where every J_k, $1 \le k \le r$, is obtained from its predecessor J_{k-1} by applying one of the relations $\sim_{\text{idem}}$, $\sim_{\text{dist}}$, $\sim_{\text{left}}$, $\sim_{\text{right}}$ to some connected subsequence. Choose k minimal with the property that $J \not\sim_P J_k$. Clearly, $J \sim_P J_{k-1}$, and furthermore J_k arises from J_{k-1} by applying $\sim_{\text{left}}$ or $\sim_{\text{right}}$. This implies that there exist m, $0 \le m \le n-2$, and $L, R \in F_n$ such that J_{k-1} can be factorized as $J_{k-1} = L.\operatorname{shift}_m(2,1,2).R$. In particular, we have $J \sim_P L.\operatorname{shift}_m(2,1,2).R$, and hence $q(J) = [i_s, \dots, i_1]_Q \in \square_n$, contradicting $[i_s, \dots, i_1]_Q \in Q_n \smallsetminus \square_n$. $\square$

Remark 2.3.24 Before drawing some immediate corollaries from Proposition 2.3.23, we briefly give another justification of the name "smallness": Clearly, a class $[I]_P \in P_n$ is small if and only if its fibre under the canonical quotient map $P_n \twoheadrightarrow Q_n$ is trivial. If $[I]_P$ is not small then we can iterate the construction in the first part ("$\Rightarrow$") of the proof of by Proposition 2.3.23, to show that its fibre contains infinitely many elements.

Corollary 2.3.25 *A finite sequence is small if and only if every of its connected subsequences is.*

Corollary 2.3.26 *Consider a finite sequence $(i_s, \dots, i_1) \in F_n$. If there exist t, $s \geq t \geq 1$, and k, $1 \leq k \leq n-1$, such that $[k]_Q$, $[k+1]_Q \in \mathscr{L}[i_t, \dots, i_1]_Q$ or $[k]_Q$, $[k+1]_Q \in \mathscr{R}[i_s, \dots, i_t]_Q$ then $(i_s, \dots, i_1)$ is not small.*

PROOF. We prove the Corollary for $\mathscr{L}$ only. Assume $[k]_Q$, $[k+1]_Q \in \mathscr{L}[i_t, \dots, i_1]_Q$. By idempotency of the generators $[i]_Q$ in Q_n we then have

$$(i_s, \dots, i_1) \sim_Q (i_s, \dots i_{t+1}).(k{+}1, k, k{+}1).(i_t, \dots, i_1),$$

and by Proposition 2.3.23 the latter is not small. Hence, by Remark 2.3.21.(a), the former is neither. □

Recall that the notion of smallness is invariant under $\sim_Q$. In contrast, the formulation of the second condition of Corollary 2.3.26 depends on the particular choice of the representative $(i_s, \dots, i_1)$, and it is therefore not surprising that there is no converse to Corollary 2.3.26 in this generality. Indeed, the finite sequence $(3, 1, 2, 1, 3)$ provides a counterexample. However, compare Remark 2.3.31.(a).

The following important result gives a necessary and sufficient condition for when a right-most, reduced sequence is small.

Proposition 2.3.27 *A right-most, reduced finite sequence $(i_s, \dots, i_1)$ is small if and only if the following holds: For all t, $s > t \geq 1$, satisfying $i_{t+1} < i_t$ we have that for all r, $s \geq r > t$, it holds $i_r < i_t$.*

PROOF. "$\Rightarrow$": Let $(i_s, \dots, i_1)$ be right-most, reduced and assume there exist r and t, $s \geq r > t \geq 1$, such that $i_{t+1} < i_t$ and $i_r \geq i_t$. Since $(i_s, \dots, i_1)$ is right-most we may assume that $i_r = i_t$. Furthermore, without loss of generality we may assume that $r = s$ is minimal subject to $r > t$ and $i_r = i_t$. (Otherwise we truncate $(i_s, \dots, i_1)$ after the r-th entry.) These further assumptions imply that for all k, $s > k > t$, we have $i_k < i_t$. Note that by Lemma 2.3.3 there is at least one index k_0 such that $i_{k_0} = i_t - 1$.

- Case 1: Assume that there is only one such index k_0 satisfying $i_{k_0} = i_t - 1$. Since $(i_s, \dots, i_1)$ is right-most, reduced, we necessarily have $k_0 = s - 1$. In particular, for all k, $s > k > t$, $k \neq k_0$, we have $i_k \leq i_t - 2$. Hence, using the relation $\sim_{\text{dist}}$, we can push the entry i_t to the left (towards $i_{k_0} = i_{s-1}$), see Figure 2.16.

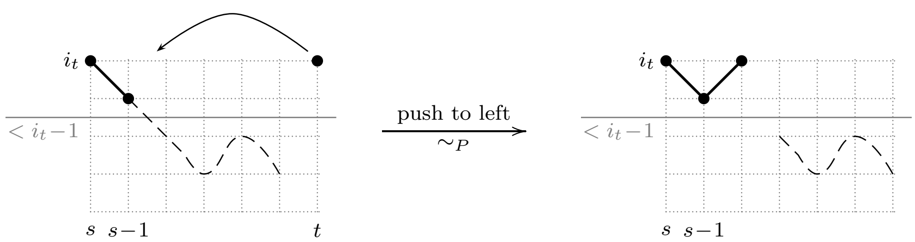


Figure 2.16: Pushing the entry i_t to the left.

This way we see that $(i_s, \dots, i_1)$ lies in the same $[\,]_P$-class as a sequence containing $(i_t, i_t{-}1, i_t)$ as a connected subsequence. Proposition 2.3.23 tells us that $(i_s, \dots, i_1)$ is not small.

- Case 2: If the entry $i_t - 1$ occurs at least two times in $(i_s, \dots, i_t)$, then let $t' > t$ be the smallest index such that $i_{t'} = i_t - 1$ and let s' be the second-smallest index. Note that $i_{t'+1} < i_{t'}$ (because otherwise we would have $i_{t'+1} = i_{t'} + 1 = i_r$, contradicting minimality of r). Since $(i_s, \dots, i_1)$ is right-most and reduced, the sequence $(i_{s'}, \dots, i_{t'})$ contains at least one entry with value $i_{t'} - 1$. We now continue as above, with t, s replaced by t' and s', respectively.

After sufficiently many applications of the construction of a connected subsequence in Case 2, we will finally end up in Case 1 and see that the original sequence $(i_s, \dots, i_1)$ is not small.

"$\Leftarrow$": Assume that $(i_s, \dots, i_1)$ is right-most, reduced and for all t satisfying $i_{t+1} < i_t$ we have that $i_r < i_t$ for all $r > t$. In other words, if for some indices r, t with $r > t$ we have $i_r = i_t$ then $i_{t+1} = i_t + 1$. In particular, $(i_s, \dots, i_1)$ satisfies the following:

($\star$) If for some indices r, t with $r > t$ we have $i_r = i_t$ then either $i_r = i_{r-1} = \dots = i_{t+1} = i_t$ or there exists some index k, $r > k > t$, such that $i_k = i_t + 1$.

Observe that ($\star$) is preserved when inserting or deleting entries in the sense of $\sim_{\text{idem}}$, or when one commutes entries in the sense of $\sim_{\text{dist}}$. Therefore, every representative of the class $[i_s, \dots, i_1]_P$ satisfies ($\star$). In particular, no such representative can possibly contain $(v{+}1, v, v{+}1)$ as a connected subsequence, for any v. Therefore $(i_s, \dots, i_1)$ is small. $\square$

The statement of Proposition 2.3.27 can be memorized by the slogan "*once you descend, you never return*": Whenever one descends, i.e. there is t such that $i_{t+1} < i_t$ then for all $r > t$ we have $i_r < i_t$, i.e. one never again sees an entry of value $\geq i_t$.

Corollary 2.3.28 *For every right-most, reduced, small finite sequence $(i_s, \dots, i_1)$ there is a unique index k such that $i_k = \max\{i_s, \dots, i_1\}$.*

Proof. Clearly, such k exists. Now assume there were more than one, say j and k. W.l.o.g. $k > j$. Since the sequence is reduced, $i_{j+1} \neq i_j$. Since i_j is a global maximum, $i_{j+1} < i_j$. But $k > j + 1$ and $i_k = i_j$. Therefore, by Proposition 2.3.27, $(i_s, \dots, i_1)$ is not small. $\square$

Corollary 2.3.29 *For every $n \in \mathbb{N}$ there is a uniform bound on the lengths of right-most, reduced, small finite sequences in F_n.*

Proof. For $n \in \mathbb{N}$ set

$$B(n) := \sup\{s \geq 0 : \exists (i_s, \dots, i_1) \in F_n \text{ right-most, reduced, small}\}.$$

First, observe that $B(1) = 1$: Since the finite sequence $(1) \in F_1$ is small, we have $B(1) \geq 1$. On the other hand we have $B(1) \leq 1$, since every finite sequence in F_1 of length strictly larger than 1 is not reduced. Corollary 2.3.28 together with Corollary 2.3.25 provides the recursion formula $B(n+1) \leq 2B(n)+1$, whence the claim. □

Remark 2.3.30 See Corollary 2.3.34 for an explicit computation of the $B(n)$'s.

We conclude this section with several remarks concerning Proposition 2.3.27.

Remark 2.3.31 (a) Similar arguments as in the proof of Proposition 2.3.27 give a partial converse to Corollary 2.3.26: A right-most and reduced finite sequence $(i_s, \dots, i_1)$ is small if and only if for all t, $s \geq t \geq 1$, and all k, $1 \leq k \leq n-1$, the elements $[k]_Q$, $[k+1]_Q$ do not both lie in $\mathscr{L}[i_t, \dots, i_1]_Q$.

(b) Let $(i_s, \dots, i_1)$ be a right-most, reduced, small finite sequence and consider $j < i_s$. Then the sequence $(j, i_s, \dots, i_1)$ is obviously right-most and reduced, and we use Proposition 2.3.27 to see that it is also small.

(c) Assume that $(i_s, \dots, i_1)$ is right-most, reduced but not small. Then there exist indices t and r, $s \geq r > t \geq 1$, such that $i_{t+1} < i_t$ and $i_r \geq i_t$. Note that, since $(i_s, \dots, i_1)$ is right-most, we can always assume that $i_r = i_t$. (This is because $(i_s, \dots, i_{t+1})$ contains (i_r, i_{t+1}) as a subsequence, and thus, by Lemma 2.3.3, it also contains the subsequence $(i_r, i_r - 1, \dots, i_{t+1}+1, i_{t+1})$, and this latter subsequence contains the entry i_t.) This observation will be made us of several times.

(d) Combining Proposition 2.3.27 and Remark 2.3.21.(b) we obtain a characterization of smallness for left-most, reduced finite sequences: A left-most, reduced finite sequence $(i_s, \dots, i_1)$ is small if and only if the following holds: For all t, $s \geq t > 1$, satisfying $i_t > i_{t-1}$ we have that $i_t > \max\{i_{t-1}, \dots, i_1\}$.

(e) Here's another characterization of smallness for right- or left-most, reduced finite sequences. Recursively define the property $(\wedge)$ as follows. The empty sequence $()$ has the property $(\wedge)$. A finite sequence $(i_s, \dots, i_1)$ has the property $(\wedge)$ if there is a unique index k, $s \geq k \geq 1$, with $i_k = \max\{i_s, \dots, i_1\}$ and $(i_s, \dots, i_{k+1})$ and $(i_{k-1}, \dots, i_1)$ both have the property $(\wedge)$.

From Proposition 2.3.27 and the aforementioned opposite statement for left-most sequences we conclude that a right-most (or left-most), reduced finite sequence is small if and only if it satisfies the property $(\wedge)$.

2.3.4 A new description of κ_n

In the preceding subsection we gave several characterizations for when a right-most, reduced finite sequence is small. We will now relate small sequences to Visy's map κ. The original definition of κ in [Vis11] uses left-most sequences, in a sense made precise in Proposition 2.3.36.

Definition 2.3.32 For $n \geq 0$ denote by $\overleftarrow{\Lambda}_n \subseteq F_n$ the set of left-most, reduced, small finite sequences. Accordingly, we denote by $\overrightarrow{\Lambda}_n \subset F_n$ the set of right-most, reduced, small finite sequences.

We reamrk that $\overleftarrow{\Lambda}_n$ and $\overrightarrow{\Lambda}_n$ have the same cardinality. Indeed, reversing $(i_s, \ldots, i_1) \mapsto (i_1, \ldots, i_s)$ defines a self-inverse bijection.

Recall that $I_a^b := (a, a{+}1, \ldots, b{-}1, b)$. In particular, $I_a^b = (\,)$ for $a > b$. Evaluation yields $f_{I_a^b} = \Phi_a^{b+1}$, cf. page 62.

Lemma 2.3.33 *For every $n \geq 1$, the set $\overleftarrow{\Lambda}_n$ splits as follows:*

$$\overleftarrow{\Lambda}_n = \{I_1^n, \ldots, I_n^n, I_{n+1}^n\}.\overleftarrow{\Lambda}_{n-1}$$

More precisely, every finite sequence in $\overleftarrow{\Lambda}_n$ can uniquely be written as concatenation $I.J$, where $I = I_a^n$ for some a, $1 \leq a \leq n+1$, and $J \in \overleftarrow{\Lambda}_{n-1}$.

Proof. Let $(i_s, \ldots, i_1) \in \overleftarrow{\Lambda}_n$. If $\max\{i_s, \ldots, i_1\} < n$ then we already have $(i_s, \ldots, i_1) \in \overleftarrow{\Lambda}_{n-1}$ and we take $a = n+1$, i.e. $I = (\,)$. Otherwise, by Corollary 2.3.28, there is a unique index t such that $i_t = n$. Clearly, $(i_{t-1}, \ldots, i_1) \in \overleftarrow{\Lambda}_{n-1}$. It remains to show that $(i_s, \ldots, i_t) = I_a^n$ for some a. For this we have to show that $i_{r+1} = i_r - 1$ for all r, $s > r \geq t$. Since $(i_s, \ldots, i_1)$ is left-most we have $i_{r+1} \geq i_r - 1$. The case $i_{r+1} = i_r$ does not occur, for $(i_s, \ldots, i_1)$ is reduced, and $i_{r+1} > i_r$ can be ruled out by Remark 2.3.31.(d), because $i_{r+1} < n = i_t$ and $r+1 > t$. $\square$

Using Lemma 2.3.33, we can give a sharp upper bound on the length of right- or left-most, reduced finite sequences:

Corollary 2.3.34 *For every n we have $B(n) = n + B(n-1)$ and thus*

$$B(n) = \frac{1}{2} \cdot n \cdot (n+1).$$

The maximum value $B(n)$ is attained for the left-most sequence $I_1^n.I_1^{n-1}.\ldots.I_1^2.I_1^1 \in \overleftarrow{\Lambda}_n$, cf. Figure 2.17.

Example 2.3.35 From Lemma 2.3.33 we obtain $\#\overleftarrow{\Lambda}_n = (n+1)!$, offering yet another proof of finiteness of each $B(n)$. Below we list some explicit examples.

(a) $\overleftarrow{\Lambda}_0 = \{(\,)\}$

(b) $\overleftarrow{\Lambda}_1 = \{(\,), (1)\}$.

(c) $\overleftarrow{\Lambda}_2 = \{(\,), (2), (1,2), (1), (2,1), (1,2,1)\}$.

(d) We can define some kind of (left) Cayley graph Γ for F_n as follows. Vertices of Γ are the elements of F_n, and we draw an edge $I_1 \to I_2$ with label k if $(k).I_1 = I_2$. Given any subset of F_n we can consider the subgraph of Γ generated by this subset. Figure 2.17 depicts this subgraph for $\overleftarrow{\Lambda}_3$ without labellings.

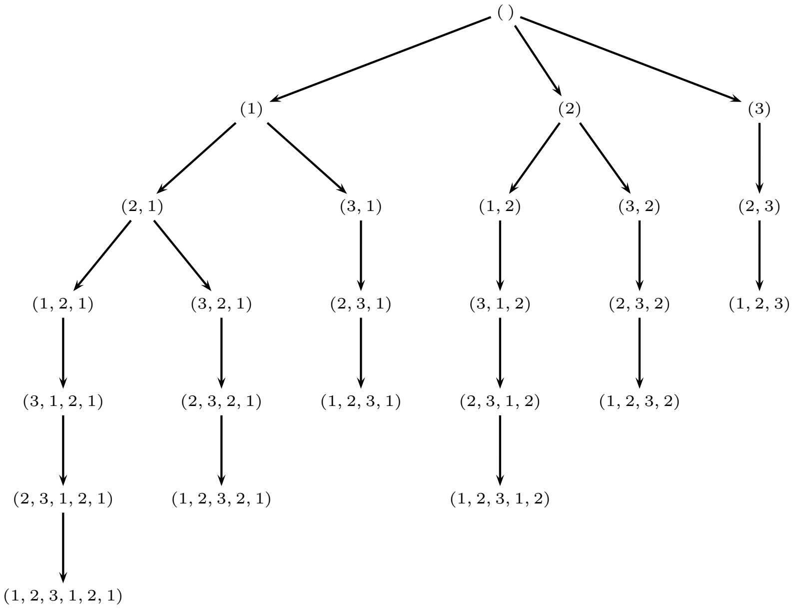


Figure 2.17: A kind of Cayley graph for $\overleftarrow{\Lambda}_3$.

Proposition 2.3.36 *Let (G, S, η) be a factorable group. Then for every $n \geq 1$ the maps $\kappa_n : \overline{\mathbb{B}}_n G \to \overline{\mathbb{B}_n} G$ introduced on page 62 can be written as follows,*

$$\kappa_n = \sum_{(i_s,\dots,i_1)\in\overleftarrow{\Lambda}_{n-1}} (-1)^s f_{i_s} \circ \ldots \circ f_{i_1}.$$

Proof. Recall that $\kappa_n = K_n \circ \ldots \circ K_1$ and $K_q = \sum_{i=1}^{q}(-1)^{q-i}\Phi_i^q$. The proof is by induction on n. For $n = 1$ we obtain $\kappa_1 = K_1 = \Phi_1^1 = \mathrm{id} : \overline{\mathbb{B}}_1 G \to \overline{\mathbb{B}_1} G$. On the other hand, $\overleftarrow{\Lambda}_0 = \{(\,)\}$ and therefore the right-hand side also simplifies to the identity. The induction step is immediate from Lemma 2.3.33; we have

$$\begin{aligned}
\sum_{(i_s,\dots,i_1)\in\overleftarrow{\Lambda}_n} (-1)^s f_{i_s} \circ \ldots \circ f_{i_1} &= \sum_{a=1}^{n+1} \sum_{J\in\overleftarrow{\Lambda}_{n-1}} (-1)^{n-a+1} f_{I_a^n} \circ (-1)^{\#J} f_J \\
&= \sum_{a=1}^{n+1} (-1)^{n+1-a} \Phi_a^{n+1} \circ \kappa_n \\
&= K_{n+1} \circ \kappa_n = \kappa_{n+1}.
\end{aligned}$$

The Proposition is proven. $\square$

Corollary 2.3.37 *For every $n \geq 1$ the differential in the Visy complex $\partial_n^{\mathbb{V}} : \mathbb{V}_n \to \mathbb{V}_{n-1}$ can be written as follows,*

$$\partial_n^{\mathbb{V}} = \pi_{n-1} \circ \overline{\partial}_n \circ \sum_{\alpha\in Q_{n-1}\smallsetminus\square_{n-1}} (-1)^{\ell(\alpha)} f_\alpha \circ i_n, \tag{2.15}$$

where ℓ denotes the word length with respect to the generating set $\{[1]_Q, \dots, [n-1]_Q\}$.

Remark 2.3.38 Note that the right-hand side of (2.15) is indeed well-defined. This follows from the Evaluation Lemma together with the fact that an element $\alpha \in Q_{n-1}$ lies in the complement of $\square_{n-1}$ if and only if α is small.

3 The Visy resolution

This chapter is the heart of the present thesis. We show that a factorability structure on a monoid X gives rise to a noetherian matching on the normalized bar resolution of X. This construction will occupy Section 3.1. The associated Morse complex provides a "new" free resolution of X which is often considerably smaller than the bar resolution, allowing us to derive an immediate homological finiteness result for factorable monoids. In Section 3.2 we investigate an explicit formula for the differential of the associated Morse complex. Section 3.3 is devoted to relating our work with the results of Visy and Wang. Most notably, we show that Visy's map κ (cf. page 62) is always an isomorphism. Indeed, it is not just *any* isomorphism but arises quite naturally in our approach. Finally, in Section 3.4 we show how the notion of factorability fits into the framework of complete rewriting systems.

3.1 A noetherian matching on $\overline{\mathbb{E}}_*X$

3.1.1 The matching μ and $\mathbb{Z}X$-compatibility

Throughout this chapter we fix a factorable monoid (X, S, η). We suppress the index S and write ℓ for the word length with respect to S. Set

$$\Omega_n = \{[x_n|\dots|x_1] \mid x_i \neq \epsilon_X \text{ for all } i\}.$$

Obviously, Ω_n is a $\mathbb{Z}X$-basis for $\overline{\mathbb{E}}_nX$. We are going to define a noetherian matching $\mu : \Omega_* \to \Omega_*$ on the based chain complex $(\overline{\mathbb{E}}_*X, \Omega_*, \overline{\partial}_*)$.

Definition 3.1.1 (a) The *partition type* of an n-cell $[x_n|\dots|x_1] \in \Omega_n$ is defined as follows,

$$\operatorname{pt}([x_n|\dots|x_1]) := (\ell(x_n), \dots, \ell(x_1)) \in \mathbb{N}^n.$$

(b) An n-cell $[x_n|\dots|x_1]$ has *elementary partition type* if $\operatorname{pt}([x_n|\dots|x_1]) = (1, \dots, 1)$, i.e. if every x_i lies in $S_+ = S \setminus \{\epsilon_X\}$.

(c) We say that a chain $c \in \overline{\mathbb{E}}_*X$ has elementary partition type if every $\underline{x} \in \Omega_*$ with $[c : \underline{x}] \neq 0$ has elementary partition type.

(d) The *norm* of an n-cell is defined as $\operatorname{norm}([x_n|\dots|x_1]) = \ell(x_n) + \dots + \ell(x_1)$. The norm of a non-trivial chain is then defined as $\operatorname{norm}(c) = \max\{\operatorname{norm}(\underline{x}) \mid [c : \underline{x}] \neq 0\}$. For $c = 0$ we set $\operatorname{norm}(c) := -\infty$.

Definition 3.1.2 A cell $\underline{x} = [x_n|\dots|x_1] \in \Omega_n$ is defined to be *essential*[1] if it has elementary partition type and if the stabilizer of $\underline{x}$ under the action $Q_{n-1} \curvearrowright X^n[n]$ is trivial, i.e. if for every $1 \leq i < n$ the pair (x_{i+1}, x_i) is unstable.

Remark 3.1.3 Note that if $[x_n|\dots|x_1]$ is an essential cell then so are $[x_{n-1}|\dots|x_1]$ and $[x_n|\dots|x_2]$.

Definition 3.1.4 The *height* of an n-cell $[x_n|\dots|x_1]$ is defined as follows,

$$\mathrm{ht}[x_n|\dots|x_1] := \max\{h \leq n \mid [x_h|\dots|x_1] \text{ is essential}\}.$$

In particular, an n-cell $[x_n|\dots|x_1]$ is essential if and only if its height equals n.

The classification of cells into collapsible and redundant ones is done by a scanning algorithm very similar to the one on page 45.

Definition 3.1.5 Consider an n-cell $\underline{x} = [x_n|\dots|x_1]$ of height $h < n$.

(a) If $x_1 \notin S_+$, i.e. if $\ell(x_1) > 1$, then $\underline{x}$ has height 0 and we call this cell *redundant*.

(b) If $x_1 \in S_+$ and (x_{h+1}, x_h) is a stable pair then we call $\underline{x}$ *collapsible*.

(c) Otherwise, i.e. if $x_1 \in S_+$ and (x_{h+1}, x_h) is unstable (and hence $x_{h+1} \notin S_+$), we call $\underline{x}$ *redundant*.

The following characterization of cells is an easy observation:

Remark 3.1.6 Consider an n-cell $\underline{x} = [x_n|\dots|x_1] \in \Omega_n$.

(a) $\underline{x}$ is essential (necessarily of height n) if and only if
   - for all i, $n \geq i \geq 1$, we have $\ell(x_i) = 1$ and
   - for all i, $n-1 \geq i \geq 1$, the pair (x_{i+1}, x_i) is unstable.

(b) $\underline{x}$ is collapsible of height $h < n$ if and only if
   - for all i, $h \geq i \geq 1$, we have $\ell(x_i) = 1$ and
   - for all i, $h-1 \geq i \geq 1$, the pair (x_{i+1}, x_i) is unstable and
   - the pair (x_{h+1}, x_h) is stable.

(c) $\underline{x}$ is redundant of height $h < n$ if and only if
   - for all i, $h \geq i \geq 1$, we have $\ell(x_i) = 1$ and
   - for all i, $h \geq i \geq 1$, the pair (x_{i+1}, x_i) is unstable and
   - $\ell(x_{h+1}) > 1$.

[1] Visy [Vis11] calls these cells *monotone*, because for his master example, the symmetric groups, these cells are tupls of transpositions $(a_i\ b_i)$ with the property that the sequence $(\max\{a_i, b_i\})_{i=1,\dots,n}$ is monotone.

Remark 3.1.7 (a) The empty tuple $[\,] \in \Omega_0$ is essential.

(b) Every element of Ω_1 is essential or redundant. Indeed, for every $x \in S_+$ the 1-cell $[x]$ is essential, and if $x \in X \smallsetminus S$ then $[x]$ is redundant.

(c) For $n > 0$ every n-cell of height 0 is redundant.

(d) Every redundant cell has at least one entry of word length ≥ 2, and thus every cell of elementary partition type is either essential or collapsible.

The map $\mu : \Omega_* \to \Omega_*$ is defined as follows. Let $\underline{x} \in \Omega_n$ be an n-cell of height h. We then set

$$\mu(\underline{x}) = \begin{cases} \underline{x} & \text{if } \underline{x} \text{ essential,} \\ d_h(\underline{x}) & \text{if } \underline{x} \text{ collapsible,} \\ \eta_{h+1}(\underline{x}) & \text{if } \underline{x} \text{ redundant.} \end{cases} \tag{3.1}$$

The aim of this section is to prove the following:

Theorem 3.1.8 *Let (X, S, η) be a factorable monoid. Then the map μ defined in (3.1) defines a noetherian, $\mathbb{Z}X$-compatible matching on the (based) normalized bar resolution $(\overline{\mathbb{E}}_*X, \Omega_*, \overline{\partial}_*)$.*

This statement will be proven in several steps. In Lemma 3.1.9 we show that μ is an involution. In Lemma 3.1.10 we conclude that μ is a $\mathbb{Z}X$-compatible matching. Finally, in Lemma 3.1.20 we show that μ is noetherian.

Lemma 3.1.9 *The map $\mu : \Omega_* \to \Omega_*$ is an involution.*

PROOF. We first show that μ maps redundant cells of height h to collapsible cells of height $h + 1$ and vice versa.

Claim 1. μ maps redundant cells of height h to collapsible cells of height $h + 1$.

Let $\underline{x} = [x_n|\dots|x_1]$ be redundant of height h. We are going to show that $\eta_{h+1}[x_n|\dots|x_1]$ is collapsible of height $h+1$. For convenience we examine this case in some detail. Saying that $\underline{x}$ is redundant of height h means that

- for all i, $h \geq i \geq 1$, we have $\ell(x_i) = 1$ and
- for all i, $h \geq i \geq 1$, the pair (x_{i+1}, x_i) is unstable and
- $\ell(x_{h+1}) > 1$.

We are in the following situation,

$$\begin{array}{c} [\; x_n \mid \dots \mid x_{h+2} \mid \quad x_{h+1} \quad \mid x_h \mid \dots \mid x_1 \;] \\ \downarrow \eta_{h+1} \\ [\; x_n \mid \dots \mid x_{h+2} \mid \overline{x_{h+1}} \mid (x_{h+1})' \mid x_h \mid \dots \mid x_1 \;] \end{array}$$

and we want to show that the cell $\eta_{h+1}(\underline{x}) = [x_n|\ldots|x_{h+2}|\overline{x_{h+1}}|(x_{h+1})'|x_h|\ldots|x_1]$ is collapsible of height $h+1$. First of all, observe that none of the entries of $\eta_{h+1}(\underline{x})$ is trivial. We want to show that $[(x_{h+1})'|x_h|\ldots|x_1]$ is essential and that $(\overline{x_{h+1}}, (x_{h+1})')$ is stable. The latter is trivial. For $h = 0$ the cell $[(x_{h+1})'|x_h|\ldots|x_1] = [(x_{h+1})']$ is obviously essential. Assume now that $h > 0$. It suffices to show that $\ell((x_{h+1})') = 1$ (which is trivial by axiom (F3) for factorization maps, cf. page 56) and that the pair $((x_{h+1})', x_h)$ is unstable. The recognition principle tells us that the latter amounts to saying that (x_{h+1}, x_h) is unstable, which is true because $\underline{x}$ is redundant of height h. Therefore $\eta_{h+1}(\underline{x})$ is collapsible of height $h+1$. Claim 1 is proven.

Claim 2. μ maps collapsible cells of height h to redundant cells of height $h-1$.

Assume now that $[x_n|\ldots|x_1]$ is collapsible of height h. Note that $h > 0$ (see e.g. Remark 3.1.7). We are now going to show that $d_h[x_n|\ldots|x_1]$ is redundant of height $h-1$. We are in the following situation:

$$\begin{array}{c}
[\ x_n \mid \ldots \mid x_{h+2} \mid x_{h+1} \mid x_h \mid x_{h-1} \mid \ldots \mid x_1\ ] \\
\downarrow d_h \\
[\ x_n \mid \ldots \mid x_{h+2} \mid \quad x_{h+1}x_h \quad \mid x_{h-1} \mid \ldots \mid x_1\ ]
\end{array}$$

We have to show that $[x_{h-1}|\ldots|x_1]$ is essential, the pair $(x_{h+1}x_h, x_{h-1})$ is unstable, and $\ell(x_{h+1}x_h) > 1$. Essentiality of $[x_{h-1}|\ldots|x_1]$ is clear. Furthermore, since (x_{h+1}, x_h) is stable, axiom (F2) for factorization maps yields $\ell(x_{h+1}x_h) = \ell(x_{h+1}) + \ell(x_h) \geq 2$. To see that $(x_{h+1}x_h, x_{h-1})$ is unstable we use the recognition principle: Since $\underline{x}$ is collapsible of height h, the pair (x_{h+1}, x_h) is stable, implying that $(x_{h+1}x_h)' = x_h$. Now, the pair $(x_{h+1}x_h, x_{h-1})$ is unstable if and only if $((x_{h+1}x_h)', x_{h-1}) = (x_h, x_{h-1})$ is unstable, which is true because $\underline{x}$ has height h. Claim 2 is proven.

Claim 3. μ is an involution.

For essential cells this is clear. Let $\underline{x}$ be redundant of height h. Then $\mu(\underline{x})$ is collapsible of height $h+1$ and thus

$$\mu^2(\underline{x}) = d_{h+1}\eta_{h+1}(\underline{x}) = \underline{x}.$$

Assume now that $\underline{x}$ is collapsible of height h. In particular, $\underline{x}$ is stable as position h, i.e. $f_h(\underline{x}) = \underline{x}$. Its partner $\mu(\underline{x})$ is redundant of height $h-1$ and we obtain

$$\mu^2(\underline{x}) = \eta_h d_h(\underline{x}) = f_h(\underline{x}) = \underline{x}.$$

The Lemma is proven. □

Lemma 3.1.10 *For a redundant cell $\underline{x} \in \Omega_*$ of height h we have*

$$[\partial\mu(\underline{x}) : \underline{x}] = (-1)^{h+1}.$$

In particular, the matching function μ is $\mathbb{Z}X$-compatible.

The proof of this statement will use μ-invariance of the norm function. More precisely, for every $\underline{x} \in \Omega_*$ we have $\operatorname{norm}(\underline{x}) = \operatorname{norm}(\mu(\underline{x}))$. This is true by axiom (F2) for factorization maps.

Proof. Let $[x_n|\dots|x_1]$ be a redundant n-cell of height h. Its collapsible partner is then given by $\mu(\underline{x}) = \eta_{h+1}(\underline{x})$. Note that $d_{h+1}\eta_{h+1}(\underline{x}) = \underline{x}$ and thus

$$
\begin{aligned}
[\partial\mu(\underline{x}) : \underline{x}] &= \sum_{i=0}^{n}(-1)^i[d_i\eta_{h+1}(\underline{x}) : \underline{x}] \\
&= (-1)^{h+1} + \sum_{\substack{i=0 \\ i\neq h+1}}^{n}(-1)^i[d_i\eta_{h+1}(\underline{x}) : \underline{x}]. \qquad (3.2)
\end{aligned}
$$

We are now going to prove that $[d_i\eta_{h+1}(\underline{x}) : \underline{x}] = 0$ for $i \neq h+1$. Set $\mu(\underline{x}) =: \underline{y} =: [y_{n+1}|\dots|y_1]$.

- For $i = 0$ we have $\operatorname{norm}(d_0\mu(\underline{x})) = \operatorname{norm}(d_0\underline{y}) = \operatorname{norm}([y_{n+1}|\dots|y_2]y_1) = \ell(y_{n+1}) + \dots + \ell(y_2) < \ell(y_{n+1}) + \dots + \ell(y_1) = \operatorname{norm}(\underline{y}) = \operatorname{norm}(\mu(\underline{x})) = \operatorname{norm}(\underline{x})$. This shows that $\operatorname{norm}(d_0\mu(\underline{x})) < \operatorname{norm}(\underline{x})$ and thus $[d_0\mu(\underline{x}) : \underline{x}] = 0$.
- For $i > 0$, the face $d_i\mu(\underline{x})$ is either 0 (if $y_{i+1}y_i = \epsilon$) or an element in Ω_*. We therefore have $[d_i\mu(\underline{x}) : \underline{x}] \neq 0$ if and only if $d_i\mu(\underline{x}) = \underline{x}$.

  We now show that $d_i\mu(\underline{x}) = \underline{x}$ implies $i = h+1$. Assume that $d_i\underline{y} = \underline{x} = d_{h+1}\underline{y}$ and that $i \neq h+1$. Set $j = \min\{i, h+1\}$. Comparing the j-th entries of $d_i(\underline{y})$ and $d_{h+1}(\underline{y})$ we see that $y_{j+1}y_j = y_j$, implying that $\operatorname{norm}(d_j\underline{y}) < \operatorname{norm}(\underline{y})$ and thus $\operatorname{norm}(d_{h+1}\underline{y}) < \operatorname{norm}(\underline{y})$. On the other hand, μ-invariance of the norm yields $\operatorname{norm}(d_{h+1}\underline{y}) = \operatorname{norm}(d_{h+1}\mu(\underline{x})) = \operatorname{norm}(\underline{x}) = \operatorname{norm}(\mu(\underline{x})) = \operatorname{norm}(\underline{y})$. This is a contradiction. Therefore $d_i\underline{y} = d_{h+1}\underline{y}$ implies $i = h+1$.

Altogether we have shown that (3.2) simplifies to $(-1)^{h+1}$. The Lemma is proven. □

Remark 3.1.11 The proof of Lemma 3.1.10 even shows that if $\underline{x}$ is redundant, then there is a unique index i such that $\underline{x} = d_i\mu(\underline{x})$. This property is crucial in the topological setting, cf. Remark 3.1.23.

To conclude Theorem 3.1.8, the only thing that remains to be proven is that there are no infinite descending chains with respect to the relation $\succ\!\!\!-$. This needs some further preparation. Let us briefly outline the plan for the remainder of this section. In Subsection 3.1.2 we introduce the notion of coherence for finite sequences and show that every coherent sequence is right-most, reduced and small. Recall that in Corollary 2.3.29 we proved a boundedness result for such sequences. In Subsection 3.1.3 we argue that an infinite descending chain of redundant cells (with respect to $\succ\!\!\!-$) would give rise to a coherent finite sequence of arbitrary length, contradicting the aforementioned boundedness result.

3.1.2 Coherent sequences

Definition 3.1.12 Let $\underline{x} = [x_n|\dots|x_1] \in \Omega_n$. A reduced finite sequence $(i_s, \dots, i_1) \in F_{n-1}$ is called *$\underline{x}$-coherent* if the following conditions are fulfilled:

(a) $f_{i_s} \circ \dots \circ f_{i_1}$ is *norm-preserving* for $\underline{x}$, that is $\mathrm{norm}(f_{i_s} \circ \dots \circ f_{i_1}(\underline{x})) = \mathrm{norm}(\underline{x})$, and

(b) for all t, $s \geq t \geq 1$, the cell $d_{i_t} \circ f_{i_{t-1}} \circ \dots \circ f_{i_1}(\underline{x})$ is redundant.

Denote by $F_{\underline{x}}$ the set of $\underline{x}$-coherent finite sequences.

To avoid future confusion, we explicitly bring to notice that, by definition, every coherent sequence is reduced.

Remark 3.1.13 Note that if $\underline{x}$ has elementary partition type then (b) implies (a) in Definition 3.1.12. To see this, let $\underline{x} = [x_n|\dots|x_1]$ have elementary partition type and set $\underline{y} := d_i(\underline{x})$. If $\underline{y}$ is redundant then some entry of $\underline{y}$ must have word length ≥ 2, cf. Remark 3.1.7.(d). The only possible candidate is $y_i = x_{i+1}x_i$, implying that d_i, and hence $f_i = \eta_i \circ d_i$ is norm-preserving. Iterating this argument yields the claim.

The following characterization should be regarded as a *Patching Lemma* for coherent sequences. The proof is obvious.

Lemma 3.1.14 *Let $\underline{x} \in \Omega_n$. Consider $(i_s, \dots, i_1) \in F_{n-1}$. Fix $s > t \geq 1$ and set $\underline{y} := f_{i_t} \circ \dots \circ f_{i_1}(\underline{x})$. The following are equivalent.*

(a) $(i_s, \dots, i_1)$ is $\underline{x}$-coherent.

(b) $(i_t, \dots, i_1)$ is $\underline{x}$-coherent, $(i_s, \dots, i_{t+1})$ is $\underline{y}$-coherent, and $i_{t+1} \neq i_t$.

Our aim is to show that for every essential n-cell $\underline{x}$ we have $F_{\underline{x}} \subseteq \overrightarrow{\Lambda}_{n-1}$, i.e. every $\underline{x}$-coherent finite sequence is right-most, reduced and small. In particular, every $F_{\underline{x}}$ is finite, compare Example 2.3.35 (and recall that $\overrightarrow{\Lambda}_{n-1} \cong \overleftarrow{\Lambda}_{n-1}$). We need some preparation.

Lemma 3.1.15 *Let $\underline{x} := [x_n|\dots|x_1] \in \Omega_n$. Denote by h the height of $\underline{x}$. For every $n > i > 0$ we have*

$$\mathrm{ht}(d_i\underline{x}) \geq \min\{i-1,\, h\}.$$

Proof. Clearly, $[x_h|\dots|x_1]$ is essential. We distinguish two cases:

- $i - 1 \geq h$: Applying d_i to $\underline{x}$ does not affect $[x_h|\dots|x_1]$ and therefore $\mathrm{ht}(d_i\underline{x}) \geq h = \min\{h,\, i-1\}$.
- $i - 1 < h$: Clearly, $[x_{i-1}|\dots|x_1]$ is essential, and by the same argument as above, $\mathrm{ht}(d_i\underline{x}) \geq i - 1 = \min\{h,\, i-1\}$.

The Lemma is proven. $\square$

Lemma 3.1.16 *Let $\underline{x} = [x_n|\dots|x_1] \in \Omega_n$ be essential or collapsible cell of height h. For all i, $n > i > 0$, the following holds: If $d_i\underline{x}$ is redundant then $i \leq h+1$.*

Proof. If $\underline{x}$ is essential then $h = n$ and the statement is trivially true. Assume now that $\underline{x}$ is collapsible and that $i > h+1$. Applying d_i to $\underline{x}$ fixes the entries $x_{h+1}, \dots, x_1$, hence the resulting cell $d_i\underline{x}$ is collapsible, and, in particular, not redundant. □

Lemma 3.1.17 *Let $\underline{x} \in \Omega_n$ be essential or collapsible. For all i, $n > i > 0$, the following holds: If $(i) \in F_{n-1}$ is $\underline{x}$-coherent then $\mathrm{ht}(d_i\underline{x}) = i-1$.*

Proof. Denote by h the height of $\underline{x}$. Recall that the sequence $(i) \in F_{n-1}$ is $\underline{x}$-coherent if f_i is norm-preserving for $\underline{x}$ and $d_i(\underline{x})$ is redundant. The former implies that the i-th entry of $d_i(\underline{x})$ has norm ≥ 2, and thus $\mathrm{ht}(d_i(\underline{x})) \leq i-1$. On the other hand, since $d_i(\underline{x})$ is redundant, Lemma 3.1.16 gives $i-1 \leq h$. Lemma 3.1.15 yields $\mathrm{ht}(d_i(\underline{x})) \geq \min\{i-1, h\} = i-1$. The Lemma is proven. □

Lemma 3.1.18 *Let $\underline{x} \in \Omega_n$ be essential or collapsible. Let $(i_s, \dots, i_1)$ be $\underline{x}$-coherent. Then for every t, $s \geq t \geq 1$, the cell $f_{i_t} \circ \dots \circ f_{i_1}(\underline{x})$ is collapsible of height i_t.*

Proof. We first show that for every t, $s \geq t \geq 0$, the cell $f_{i_t} \circ \dots \circ f_{i_1}(\underline{x})$ is essential or collapsible. The first statement then follows from the fact that for $t \geq 1$ the cell $f_{i_t} \circ \dots \circ f_{i_1}(\underline{x})$ is stable at position i_t, hence not essential and thus collapsible. For $t = 0$ we have $f_{i_t} \circ \dots \circ f_{i_1}(\underline{x}) = \underline{x}$, which by assumption is essential or collapsible. Now let $s > t \geq 0$ and assume that the claim is true for t, i.e. the cell $\underline{y} = f_{i_t} \circ \dots \circ f_{i_1}(\underline{x})$ is essential or collapsible. Clearly, the finite sequence (i_{t+1}) is $\underline{y}$-coherent. Lemma 3.1.17 now yields $\mathrm{ht}(d_{i_{t+1}}(\underline{y})) = i_{t+1} - 1$. Thus

$$f_{i_{t+1}}(\underline{y}) = \eta_{i_{t+1}} d_{i_{t+1}}(\underline{y}) = \eta_{\mathrm{ht}(d_{i_{t+1}}(\underline{y}))+1} d_{i_{t+1}}(\underline{y}) = \mu(d_{i_{t+1}}(\underline{y})), \tag{3.3}$$

and hence $f_{i_{t+1}}(\underline{y})$ is collapsible (because its partner $d_{i_{t+1}}(\underline{y})$ is redundant).

It is now easy to compute the height of each $f_{i_t} \circ \dots \circ f_{i_1}(\underline{x})$: Recall that μ matches redundant cells of height h with collapsible cells of height $h+1$. Using (3.3) we see that

$$\mathrm{ht}(f_{i_{t+1}} \circ \dots \circ f_{i_1}(\underline{x})) = \mathrm{ht}(f_{i_{t+1}}(\underline{y})) = \mathrm{ht}(\mu(d_{i_{t+1}}(\underline{y}))) = \mathrm{ht}(d_{i_{t+1}}(\underline{y})) + 1 = i_{t+1},$$

where the last equality follows from Lemma 3.1.17. The Lemma is proven. □

Proposition 3.1.19 *We have $F_{\underline{x}} \subseteq \overrightarrow{\Lambda}_{n-1}$ for every essential or collapsible n-cell $\underline{x}$.*

Proof. Let $\underline{x} \in \Omega_n$ be essential or collapsible and let $(i_s, \dots, i_1)$ be $\underline{x}$-coherent.

Claim 1. $(i_s, \dots, i_1)$ is right-most and reduced.

Clearly, $(i_s, \dots, i_1)$ is reduced, cf. Definition 3.1.12. Lemma 3.1.18 tells us that for every t, $s \geq t \geq 1$, we have $\mathrm{ht}(f_{i_t} \circ \dots \circ f_{i_1}(\underline{x})) = i_t$. Recall that by assumption $d_{i_{t+1}}(\underline{y})$ is redundant. Lemma 3.1.16 now yields

$$i_{t+1} \leq \mathrm{ht}(\underline{y}) + 1 = i_t + 1.$$

By Lemma 2.3.3 this is equivalent to $(i_s, \dots, i_1)$ being right-most.

Claim 2. $(i_s, \dots, i_1)$ is small.

Claim 2 will be proven by contradiction. Assume that there exists a finite sequence $(i_s, \dots, i_1)$ that is $\underline{x}$-coherent but not small. By Claim 1 this sequence is right-most and reduced. W.l.o.g. we may assume minimality of $(i_s, \dots, i_1)$, i.e. we may assume that the subsequence $(i_{s-1}, \dots, i_1)$ is small (and clearly right-most and reduced).

Claim 2.A. $(i_s, \dots, i_1) \sim_P (i_s, i_s - 1, i_s).R$ for some suitable finite sequence R.

From Remark 2.3.31.(b) we conclude that $i_s > i_{s-1}$, hence $i_s = i_{s-1} + 1$ (because our sequence is right-most). Furthermore, there is some t, $s > t \geq 1$, with the property that $i_t = i_s$ and $\max\{i_{s-1}, \dots, i_{t+1}\} < i_t$. In particular, all entries of the connected subsequence $(i_{s-1}, \dots, i_{t+1})$ are $\leq i_{s-1}$. Corollary 2.3.25 and Corollary 2.3.28 imply that $\max\{i_{s-2}, \dots, i_{t+1}\} < i_{s-1}$, hence $\leq i_t - 2$. Therefore, using the relation $\sim_{\mathrm{dist}}$, we can push the entry i_t to the left, cf. Figure 2.16 on page 87. More precisely, set $R = (i_{s-2}, \dots, i_{t+1}).(i_{t-1}, \dots, i_1)$. Then $(i_s, \dots, i_1) \sim_P (i_s, i_s - 1, i_t).R$. Since $i_t = i_s$, this proves Claim 2.A.

Claim 2.B. $d_{i_s} f_{i_{s-1}} \circ \dots \circ f_{i_1}(\underline{x})$ is not redundant.

Claim 2.B gives the desired contradiction, because $\underline{x}$-coherence of our sequence in particular requires $d_{i_s} f_{i_{s-1}} \circ \dots \circ f_{i_1}(\underline{x})$ to be redundant.

We will now prove Claim 2.B. Applying the Evaluation Lemma to Claim 2.A we see that

$$\eta_{i_s} d_{i_s} f_{i_{s-1}} \dots f_{i_1}(\underline{x}) = \eta_{i_s} d_{i_s} f_{i_s - 1} f_{i_s} f_R(\underline{x}).$$

Note that the i_s-th entry of $d_{i_s} f_{i_{s-1}} \dots f_{i_1}(\underline{x})$ has word length ≥ 2, and therefore $d_{i_s} \eta_{i_s}$ stabilizes this tuple. Hence, post-composing both sides by d_{i_s} we obtain

$$d_{i_s} f_{i_{s-1}} \dots f_{i_1}(\underline{x}) = d_{i_s} f_{i_s - 1} f_{i_s} f_R(\underline{x}). \tag{3.4}$$

Recall that every factorable monoid is weakly factorable and therefore we have the following equality in the graded sense: $d_{i_s} f_{i_s - 1} f_{i_s} = d_{i_s} f_{i_s - 1} \eta_{i_s} d_{i_s} \equiv f_{i_s - 1} d_{i_s}$. The right-hand side of (3.4) now simplifies to

$$d_{i_s} f_{i_s - 1} f_{i_s} f_R(\underline{x}) \equiv \eta_{i_s - 1} d_{i_s - 1} d_{i_s} f_R(\underline{x}). \tag{3.5}$$

By $\underline{x}$-coherence of $(i_s, \dots, i_1)$, the left-hand side of (3.5) is norm-preserving, and therefore (3.5) even holds with $\equiv$ replaced by $=$. Tacking together (3.4) and (3.5) we obtain

$$d_{i_s} f_{i_{s-1}} \dots f_{i_1}(\underline{x}) = \eta_{i_s - 1} d_{i_s - 1} d_{i_s} f_R(\underline{x}).$$

This shows that $d_{i_s} f_{i_{s-1}} \dots f_{i_1}(\underline{x})$ is stable at position $i_s - 1$.

On the other hand, Lemma 3.1.15 and Lemma 3.1.18 yield

$$\mathrm{ht}(d_{i_s} f_{i_{s-1}} \circ \dots \circ f_{i_1}(\underline{x})) \geq \min\{i_s - 1, i_{s-1}\} = i_s - 1, \tag{3.6}$$

where the equality follows from the above observation that $i_s = i_{s-1}+1$. Hence, the cell $d_{i_s}f_{i_{s-1}} \circ \ldots \circ f_{i_1}(\underline{x})$ has height $i_s - 1$ and it is stable at position $i_s - 1$. From Remark 3.1.6 we conclude that $d_{i_s}f_{i_{s-1}} \circ \ldots \circ f_{i_1}(\underline{x})$ is collapsible, contradicting the assumption of $(i_s, \ldots, i_1)$ being $\underline{x}$-coherent. $\square$

3.1.3 Noetherianity of μ

Lemma 3.1.20 *The matching $\mu : \Omega_* \to \Omega_*$ is noetherian.*

Proof. We need to show that there is no infinite (strictly) descending chain $\underline{x_1} \rightarrowtail \underline{x_2} \rightarrowtail \underline{x_3} \rightarrowtail \ldots$ of redundant cells. (We use the underbar to emphasize that $\underline{x_i}$ does not denote an entry of some tuple but an element in Ω_*.) Clearly, if such an infinite descending sequence exists, then all the $\underline{x_k}$ live in the same dimension Ω_n. Assume that the assertion is wrong, i.e. there is an infinite (strictly) descending chain of redundant cells in dimension n, say $\underline{x_1} \rightarrowtail \underline{x_2} \rightarrowtail \underline{x_3} \rightarrowtail \ldots$. By definition of $\rightarrowtail$, $\underline{x_{t+1}}$ is a face of $\mu(\underline{x_t})$. We therefore obtain the following diagram:

$$\begin{tikzcd}
\Omega_{n+1}: & & \mu(\underline{x_1}) \arrow[dr, mapsto, "d_{i_1}"] & & \mu(\underline{x_2}) \arrow[dr, mapsto, "d_{i_2}"] & & \mu(\underline{x_3}) \arrow[dr, mapsto, "d_{i_3}"] & & \cdots \\
\Omega_n: & \underline{x_1} \arrow[ur, mapsto, "\mu"] & & \underline{x_2} \arrow[ur, mapsto, "\mu"] & & \underline{x_3} \arrow[ur, mapsto, "\mu"] & & \underline{x_4} \arrow[ur, mapsto, "\mu"] &
\end{tikzcd}$$

Note that for all t we have $\mathrm{norm}(\underline{x_{t+1}}) \leq \mathrm{norm}(\underline{x_t})$: This follows from that fact that μ is norm-preserving and from the triangle inequality for word lengths. Thus there is an index T such that $\mathrm{norm}(\underline{x_{t+1}}) = \mathrm{norm}(\underline{x_t})$ for all $t \geq T$. We may assume that $T = 1$. (Otherwise we replace $\underline{x_1} \rightarrowtail \underline{x_2} \rightarrowtail \ldots$ by $\underline{x_T} \rightarrowtail \underline{x_{T+1}} \rightarrowtail \ldots$.) In particular, every d_{i_t}, $t \geq 1$, preserves the norm. As a consequence, for every $t \geq 1$ the finite sequence $(i_t) \in F_{n-1}$ is coherent with respect to the collapsible cell $\mu(\underline{x_t})$.

Lemma 3.1.17 yields $\mathrm{ht}(\underline{x_{t+1}}) = i_t - 1$, and we obtain

$$\mu(\underline{x_{t+1}}) = \eta_{i_t}(\underline{x_{t+1}}) = \eta_{i_t} d_{i_t}(\mu(\underline{x_t})) = f_{i_t}(\mu(\underline{x_t})).$$

Spelled out, we obtain $\mu(\underline{x_{t+1}})$ from $\mu(\underline{x_t})$ by applying f_{i_t}. We can therefore extend the above diagram as follows:

$$\begin{tikzcd}
\Omega_{n+1}: & & \mu(\underline{x_1}) \arrow[rr, mapsto, "f_{i_1}"] \arrow[dr, mapsto, "d_{i_1}"] & & \mu(\underline{x_2}) \arrow[rr, mapsto, "f_{i_2}"] \arrow[dr, mapsto, "d_{i_2}"] & & \mu(\underline{x_3}) \arrow[rr, mapsto, "f_{i_3}"] \arrow[dr, mapsto, "d_{i_3}"] & & \cdots \\
\Omega_n: & \underline{x_1} \arrow[ur, mapsto, "\mu"] & & \underline{x_2} \arrow[ur, mapsto, "\mu"] & & \underline{x_3} \arrow[ur, mapsto, "\mu"] & & \underline{x_4} \arrow[ur, mapsto, "\mu"] &
\end{tikzcd}$$

Recall that every $(i_t) \in F_{n-1}$ is coherent with respect to $\mu(\underline{x_t})$. Using the Patching Lemma 3.1.14 we see that for every $s \geq 0$ the sequence $(i_s, \ldots, i_1)$ is coherent with respect to the collapsible cell $\mu(\underline{x_1})$. (The fact that $(i_s, \ldots, i_1)$ is reduced follows from the fact that $\underline{x_1} \rightarrowtail \underline{x_2} \rightarrowtail \underline{x_3} \rightarrowtail \ldots$ descends *strictly*.) Proposition 3.1.19 now tells us that for each $s \geq 0$ the finite sequence $(i_s, \dots, i_1) \in F_{n-1}$ is right-most, reduced and small. This contradicts our boundedness result of Corollary 2.3.29. Therefore our assumption of the existence of an infinite (strictly) descending chain of redundant cells was wrong. The Lemma is proven. $\square$

This also finishes the proof of Theorem 3.1.8. $\square$

We can now apply Theorem 1.1.29 to our noetherian matching μ on the (based) normalized bar resolution $(\overline{\mathbb{E}}_* X, \Omega_*, \overline{\partial}_*)$. The Morse complex associated to this matching is our main object of study.

Definition 3.1.21 Let (X, S, η) be a factorable monoid. Let μ be as in (3.1). The *Visy resolution* of X (with respect to the chosen factorability structure) is defined to be the Morse complex associated to the matching μ:

$$(\widetilde{\mathbb{V}}_*, \partial_*^{\mathbb{V}}) := ((\overline{\mathbb{E}}_* X)^\theta, \overline{\partial}_*^\theta) \tag{3.7}$$

For simplicity of notation we suppress X, S, and η in the notation of the Visy resolution.

Recall that, as a $\mathbb{Z}X$-module, $\widetilde{\mathbb{V}}_n$ is freely generated by all n-tuples $[x_n|\dots|x_1]$ satisfying $x_i \in S_+$ for all i and (x_{i+1}, x_i) unstable for all $n > i \geq 1$. In particular, if S is finite, then every $\widetilde{\mathbb{V}}_n$ is finitely generated, and hence X is of type right-FP_∞.

Note that our matching μ does not take into account the "outer" simplicial face maps d_0 and d_n, in the sense that if $\underline{x}$ is a collapsible n-cell then $\mu(\underline{x}) = d_h(\underline{x})$ and $0 < h < n$, cf. Remark 3.1.7. It follows that the construction of μ also applies to the left bar resolution of X. (Compare our discussion before Theorem 1.2.9.)

Theorem 3.1.8 therefore implies the following:

Corollary 3.1.22 *Let X be a monoid. If X admits a factorability structure (X, S, η) with finite generating set S then X satisfies the homological finiteness property* FP_∞.

Remark 3.1.23 One can show that under the assumptions of Corollary 3.1.22, the monoid X satisfies the geometric finiteness property F_∞. Indeed, μ gives a matching on the cells of the classifying space BX, and this matching is a so-called collapsing scheme in the sense of Brown [Bro92, p.140], cf. Remark 3.1.11. Proposition 1 in loc. cit. then tells us that BX collapses onto a CW complex with only finitely many cells in each dimension.

3.2 Computing the differential $\theta^\infty \circ \overline{\partial}$

3.2.1 Θ^∞ and coherent sequences

As a $\mathbb{Z}X$-module, $\widetilde{\mathbb{V}}_*X$ is a direct summand of $\overline{\mathbb{E}}_*X$. Recall from Chapter 1 that the differential $\partial_n^{\mathbb{V}} : \widetilde{\mathbb{V}}_n \to \widetilde{\mathbb{V}}_{n-1}$ is given by the following composition:

$$\begin{tikzcd} \overline{\mathbb{E}}_n \arrow[r, "\overline{\partial}_n"] & \overline{\mathbb{E}}_{n-1} \arrow[d, "\theta^\infty"] \\ \widetilde{\mathbb{V}}_n \arrow[u, "i_n"] \arrow[r, "\partial_n^{\mathbb{V}}"'] & \widetilde{\mathbb{V}}_{n-1} \end{tikzcd}$$

Also recall that $\theta^\infty \circ \overline{\partial}_n = \pi_{n-1} \circ \Theta^\infty \circ \overline{\partial}_n = \pi_{n-1} \circ \overline{\partial}_n \circ \Theta^\infty : \overline{\mathbb{E}}_n X \to \widetilde{\mathbb{V}}_{n-1}$. In this section we are going to give an explicit formula for $\Theta^\infty(\underline{x})$ for every essential cell $\underline{x}$.

Proposition 3.2.1 *Let $\underline{x} \in \Omega_n$ be essential. Denote by $F_{\underline{x}}^{(k)}$ the set of $\underline{x}$-coherent sequences of length at most k. Then for every $k \geq 0$ we have*

$$\Theta^k(\underline{x}) = \sum_{I \in F_{\underline{x}}^{(k)}} (-1)^{\#I} \cdot f_I(\underline{x}). \tag{3.8}$$

Proof. Recall that if $\underline{x}$ has elementary partition type then a reduced finite sequence $(i_s, \ldots, i_1)$ is $\underline{x}$-coherent if and only if for every t the cell $d_{i_t} f_{i_{t-1}} \ldots f_{i_1}(\underline{x})$ is redundant. This is because if $\underline{x}$ has elementary partition type then the latter condition already implies that $f_{i_s} \ldots f_{i_1}$ is norm-preserving for $\underline{x}$. This characterization of $\underline{x}$-coherent sequences will be used at several places in this proof.

We do induction on k. For $k = 0$ the left-hand side of (3.8) simplifies to $\Theta^0(\underline{x}) = \underline{x}$. Furthermore, $F_{\underline{x}}^{(0)} = \{(\,)\}$ and hence the right-hand side of (3.8) also computes to $\underline{x}$.

For $k = 1$ we have $\Theta(\underline{x}) = \underline{x} + \partial V(\underline{x}) + V(\partial \underline{x}) = \underline{x} + V(\partial \underline{x})$. The last equality holds because $\underline{x}$ is essential. Note that $\underline{x} = f_{(\,)}(\underline{x})$. It therefore suffices to show that

$$V(\partial \underline{x}) = - \sum_{\substack{i=1,\ldots,n-1 \\ (i) \in F_{\underline{x}}}} f_i(\underline{x}).$$

We are going to compute $V(\partial \underline{x})$. If $d_i(\underline{x})$ is not redundant, then $V(d_i(\underline{x})) = 0$. Assume now that $d_i(\underline{x})$ is redundant. Then the finite sequence (i) is $\underline{x}$-coherent and Lemma 3.1.17 yields $\mathrm{ht}(d_i(\underline{x})) = i - 1$. Using (1.1) on page 30 and (3.1) on page 95 we compute

$$\begin{aligned} V(d_i(\underline{x})) &= -\mu(d_i(\underline{x})) \cdot [\partial \mu(d_i(\underline{x})) : d_i(\underline{x})]^{-1} \\ &= -f_i(\underline{x}) \cdot [\partial f_i(\underline{x}) : d_i(\underline{x})]^{-1} = (-1)^{i+1} \cdot f_i(\underline{x}). \end{aligned}$$

It follows that $V(\partial \underline{x}) = \sum (-1)^i \cdot (-1)^{i+1} \cdot f_i(\underline{x}) = -\sum f_i(\underline{x})$, where the sum runs over all $\underline{x}$-coherent finite sequences of length 1. Thus, the Proposition holds for $k = 0$ and $k = 1$.

Fix $k \geq 2$ and assume that (3.8) holds true for all indices $\leq k$. By Remark 1.1.19 the chain $\Theta^k(\underline{x})$ is essential-collapsible and thus $V(\Theta^k(\underline{x})) = 0$. This gives

$$\Theta^{k+1}(\underline{x}) = \Theta^k(\underline{x}) + V\overline{\partial}(\Theta^k(\underline{x})).$$

We use the induction hypothesis for k to obtain

$$\Theta^{k+1}(\underline{x}) = \Theta^k(\underline{x}) + \sum_{I \in F_{\underline{x}}^{(k)}} \sum_{j=0}^{n} (-1)^{\#I+j} \cdot V(d_j f_I(\underline{x})).$$

If $d_j f_I(\underline{x})$ is not redundant then it lies in the kernel of V. Hence,

$$\Theta^{k+1}(\underline{x}) = \Theta^k(\underline{x}) + \sum_{I \in F_{\underline{x}}^{(k)}} \sum_{\substack{j=0 \\ d_j f_I(\underline{x}) \text{ red.}}}^{n} (-1)^{\#I+j} \cdot V(d_j f_I(\underline{x})).$$

Since I is $\underline{x}$-coherent, the cell $f_I(\underline{x})$ has elementary partition type. In particular, $f_I(\underline{x})$ is essential or collapsible. Lemma 3.1.17 tells us that if its j-th face $\underline{y} = d_j f_I(\underline{x})$ is redundant then $\mathrm{ht}(\underline{y}) = j - 1$. In particular, $\mu(\underline{y}) = \eta_j(\underline{y})$. We use Lemma 3.1.10 to compute $V(\underline{y}) = -\mu(\underline{y}) \cdot [\partial\mu(\underline{y}) : \underline{y}]^{-1} = (-1)^{j+1} \cdot \eta_j(\underline{y})$. This yields

$$\Theta^{k+1}(\underline{x}) = \Theta^k(\underline{x}) + \underbrace{\sum_{I \in F_{\underline{x}}^{(k)}} \sum_{\substack{j=0 \\ d_j f_I(\underline{x}) \text{ red.}}}^{n} (-1)^{\#I+1} \cdot f_j \circ f_I(\underline{x})}_{=:S}. \tag{3.9}$$

To further simplify the expression S, we distinguish whether or not I is the empty sequence. Set

$$S_0 := -\sum_{\substack{j=0 \\ d_j(\underline{x}) \text{ red.}}}^{n} f_j(\underline{x}).$$

Note that $d_j(\underline{x})$ is redundant if and only if the finite sequence (j) is $\underline{x}$-coherent. Using the induction hypothesis for $k = 1$, we see that $S_0 + \underline{x} = \Theta(\underline{x})$, i.e.

$$S_0 = \Theta(\underline{x}) - \underline{x}. \tag{3.10}$$

We define $S_1 := S - S_0$. Intuitively speaking, S_1 arises from S by summing over all sequences $I \in F_{\underline{x}}^{(k)}$ of positive length,

$$S_1 = \sum_{s=1}^{k} \sum_{(i_s,\dots,i_1) \in F_{\underline{x}}} \sum_{\substack{j=0 \\ d_j f_{i_s} \dots f_{i_1}(\underline{x}) \text{ red.}}}^{n} (-1)^{\#I+1} \cdot f_j \circ f_{i_s} \dots f_{i_1}(\underline{x}). \tag{3.11}$$

To simplify S_1, we distinguish whether or not in the third sum we have $j = i_s$. Set

$$S_1' := \sum_{s=1}^{k} \sum_{(i_s,\ldots,i_1)\in F_{\underline{x}}} \sum_{\substack{j\neq i_s \\ d_j f_{i_s}\ldots f_{i_1}(\underline{x}) \text{ red.}}} (-1)^{\#I+1} \cdot f_j \circ f_{i_s} \ldots f_{i_1}(\underline{x}), \tag{3.12}$$

and $S_1'' = S_1 - S_1'$. Observe that if in (3.12) the cell $d_j f_{i_s} \ldots f_{i_1}(\underline{x})$ is redundant then $(j, i_s, \ldots, i_1)$ is $\underline{x}$-coherent (because $j \neq i_s$). Indeed, using the Patching Lemma 3.1.14 we see that

$$S_1' = \sum_{I\in F_{\underline{x}}^{(k+1)} \smallsetminus F_{\underline{x}}^{(1)}} (-1)^{\#I} \cdot f_I(\underline{x}),$$

and the induction hypothesis (for $k = 1$) yields that

$$S_1' = \sum_{I\in F_{\underline{x}}^{(k+1)}} (-1)^{\#I} \cdot f_I(\underline{x}) \;-\; \Theta(\underline{x}). \tag{3.13}$$

We now simplify $S_1'' = S_1 - S_1'$. Note that if in (3.11) we have $j = i_s$ then $d_j f_{i_s} \ldots f_{i_1}(\underline{x}) = d_{i_s} f_{i_{s-1}} \ldots f_{i_1}(\underline{x})$ is redundant (because $(i_s, \ldots, i_1)$ is $\underline{x}$-coherent). By idempotency of the f_i's we have $f_j f_{i_s} = f_{i_s}$. We can therefore write S_1'' as follows,

$$S_1'' = \sum_{s=1}^{k} \sum_{(i_s,\ldots,i_1)\in F_{\underline{x}}} (-1)^{\#I+1} \cdot f_{i_s} \ldots f_{i_1}(\underline{x}).$$

Clearly, this double sum runs over all $\underline{x}$-coherent sequences of lengths $1, \ldots, k$. Using the induction hypothesis we see that

$$S_1'' = -\Theta^k(\underline{x}) + \underline{x}. \tag{3.14}$$

Putting together (3.9), (3.10), (3.13), and (3.14) we obtain

$$\begin{aligned}
\Theta^{k+1}(\underline{x}) &= \Theta^k(\underline{x}) + S \\
&= \Theta^k(\underline{x}) + S_0 + S_1' + S_1'' \\
&= \sum_{I\in F_{\underline{x}}^{(k+1)}} (-1)^{\#I} \cdot f_I(\underline{x}).
\end{aligned}$$

The Proposition is proven. □

Corollary 3.2.2 *Let $\underline{x} \in \Omega_n$ be essential. In particular, $\underline{x} \in \widetilde{\mathbb{V}}_n$ and*

$$\partial_n^{\mathbb{V}}(\underline{x}) = \pi_{n-1} \circ \overline{\partial}_n \circ \sum_{I\in F_{\underline{x}}} (-1)^{\#I} \cdot f_I(\underline{x}). \tag{3.15}$$

PROOF. We have $\partial_*^{\mathbb{V}} = \theta^\infty \circ \overline{\partial}_* = \pi_{*-1} \circ \Theta^\infty \circ \overline{\partial}_* = \pi_{*-1} \circ \overline{\partial}_* \circ \Theta^\infty$. By Proposition 3.1.19 and Lemma 2.3.33 the set $F_{\underline{x}}$ is finite. Therefore, for k sufficiently large, $F_{\underline{x}}^{(k)} = F_{\underline{x}}^{(k+1)} = \ldots = F_{\underline{x}}$. Applying Proposition 3.2.1 we obtain

$$\sum_{I \in F_{\underline{x}}} (-1)^{\# I} \cdot f_I(\underline{x}) = \Theta^k(\underline{x}) = \Theta^{k+1}(\underline{x}) = \ldots = \Theta^\infty(\underline{x}),$$

whence the Corollary. □

Remark 3.2.3 The reader should compare (3.15) with (2.15) on page 91. Recall that $F_{\underline{x}} \subseteq \overrightarrow{\Lambda}_{n-1}$, and that the latter is a set of representatives for $Q_{n-1} \setminus \square_{n-1}$. Ignoring sources and targets (the formula (3.15) is defined on certain elements in the bar resolution whereas (2.15) is defined on certain elements of the bar complex), the only difference between these two formulas is the indexing set. Note that the indexing set in (3.15) depends on the cell $\underline{x}$, for being $\underline{x}$-coherent depends on the choice of the cell $\underline{x}$. The following subsection is concerned with getting rid of this dependence. This will be achieved by letting the sum run over a larger indexing set. We then have to show that the additional terms cancel out each other. This will occupy the following subsection.

3.2.2 Θ^∞ and right-most, reduced, small sequences

We are going to show that (3.15) still holds true when replacing the indexing set $F_{\underline{x}}$ by the larger set $\overrightarrow{\Lambda}_{n-1}$. We need some preparation first.

Lemma 3.2.4 *Consider $\underline{x} \in \Omega_n$ of elementary partition type and let $(i_s, \ldots, i_1) \in \overrightarrow{\Lambda}_{n-1} \setminus F_{\underline{x}}$, i.e. $(i_s, \ldots, i_1)$ is right-most, reduced and small, but not $\underline{x}$-coherent. Then $f_{i_t} \circ \ldots \circ f_{i_1}(\underline{x}) = 0$ or there exists t, $s \geq t \geq 1$, such that $f_{i_t} \circ \ldots \circ f_{i_1}(\underline{x})$ is stable at position $i_t - 1$.*

PROOF. Let t be the uniquely determined index such that $(i_{t-1}, \ldots, i_1)$ is $\underline{x}$-coherent and $(i_t, \ldots, i_1)$ is not. Set $\underline{y} := f_{i_{t-1}} \circ \ldots \circ f_{i_1}(\underline{x})$. If d_{i_t} is not norm-preserving for $\underline{y}$ then applying η_{i_t} would produce a trivial entry and thus $f_{i_t}(\underline{y}) = 0$. In this case we are finished. So assume that d_{i_t} is norm-preserving for $\underline{y}$. (It follows that $d_{i_t}(\underline{y})$ is not redundant, because otherwise $(i_t, \ldots, i_1)$ would be $\underline{x}$-coherent.) We will now show that $f_{i_t}(\underline{y})$ is stable at position $i_t - 1$.

Claim. $\mathrm{ht}(d_{i_t}(\underline{y})) \geq i_t - 1$.

By Lemma 3.1.18 we have $\mathrm{ht}(\underline{y}) = i_{t-1}$. Note that $i_t \leq i_{t-1} + 1$, since $(i_s, \ldots, i_1)$ is right-most. Lemma 3.1.15 now yields $\mathrm{ht}(d_{i_t}(\underline{y})) \geq \min\{i_t - 1, \mathrm{ht}(\underline{y})\} = i_t - 1$, whence the Claim.

The situation is as follows: $d_{i_t}(\underline{y})$ has height $\geq i_t - 1$ and its i_t-th entry has norm 2. So $\mathrm{ht}(d_{i_t}(\underline{y})) = i_t - 1$. Furthermore, $d_{i_t}(\underline{y})$ is not redundant. It follows that $d_{i_t}(\underline{y})$ is stable at position $i_t - 1$. This yields the following:

$$\begin{aligned} f_{i_t} f_{i_{t-1}} \dots f_{i_1}(\underline{x}) &= \eta_{i_t} d_{i_t} f_{i_{t-1}} \dots f_{i_1}(\underline{x}) \\ &= \eta_{i_t} f_{i_t - 1} d_{i_t} f_{i_{t-1}} \dots f_{i_1}(\underline{x}) \\ &= \eta_{i_t} \eta_{i_t - 1} d_{i_t - 1} d_{i_t} f_{i_{t-1}} \dots f_{i_1}(\underline{x}) \end{aligned}$$

The recognition principle tells us that the latter is stable at position $i_t - 1$. □

The following Lemmas give partial answers to what extend insertion and deletion of entries respect smallness of finite sequences. Recall from Corollary 2.3.25 that a finite sequence is small if and only if all of its connected subsequences are. In particular, a small sequence stays small when deleting its first or last entry. What happens if we delete an "inner" entry?

Lemma 3.2.5 *Let $(i_s, \dots, i_1)$ be a right-most, reduced, small finite sequence. For every k, $s > k > 1$, the following holds: If $i_k < i_{k-1}$ then $(i_s, \dots, \widehat{i_k}, \dots, i_1)$ is right-most, reduced and small.*

Proof. Assume that $i_k < i_{k-1}$.

Claim 1. $(i_s, \dots, \widehat{i_k}, \dots, i_1)$ is right-most and reduced.

We have $i_{k+1} \leq i_k + 1 < i_{k-1} + 1$, i.e. $i_{k+1} \leq i_{k-1}$. So $(i_s \dots, \widehat{i_k}, \dots, i_1)$ is right-most. For "reduced" we have to show that $i_{k+1} \neq i_{k-1}$. Assume that $i_{k+1} = i_{k-1}$. By assumption we have $i_k < i_{k-1}$ and hence $i_{k+1} > i_k$. This implies $i_{k+1} = i_k + 1$ because $(i_s, \dots, i_1)$ is right-most. This gives $(i_{k+1}, i_k, i_{k-1}) = (i_k + 1, i_k, i_k + 1)$, contradicting smallness of $(i_s, \dots, i_1)$, cf. Proposition 2.3.23. Therefore, $(i_s, \dots, \widehat{i_k}, \dots, i_1)$ is right-most and reduced. Claim 1 is proven.

Claim 2. $(i_s, \dots, \widehat{i_k}, \dots, i_1)$ is small.

We will use Proposition 2.3.27. Since $(i_s, \dots, i_1)$ is small, we only have to check the case $t = k - 1$. The successor of i_{k-1} in $(i_s, \dots, \widehat{i_k}, \dots, i_1)$ is i_{k+1}, and we know that $i_{k+1} < i_{k-1}$. We therefore have to show that $\max\{i_s, \dots, i_{k+1}\} < i_{k-1}$. This immediately follows from smallness of the original sequence $(i_s, \dots, i_1)$: By assumption we have $i_k < i_{k-1}$ and smallness of $(i_s, \dots, i_1)$ yields that for every r, $s \geq r > k - 1$, we have $i_r < i_{k-1}$. Claim 2 is proven. □

Lemma 3.2.6 *Let $(i_s, \dots, i_1)$ be right-most, reduced and small. Let j be such that $(i_s, \dots, i_k, j, i_{k-1}, \dots, i_1)$ is right-most and reduced but not small. Then one of the following holds*

- *$j > i_{k-1}$ or*
- *$i_k < j$ and there exists some $r' > k$ such that $i_{r'} = j$.*

Proof. Assume that $j \leq i_{k-1}$, i.e. $j \leq i_{k-1} - 1$ since the sequence is reduced. We are going to verify the second condition. Set

$$(j_{s+1}, \dots, j_1) := (i_s, \dots, i_k).(j).(i_{k-1}, \dots, i_1).$$

Note that $(j_{s+1}, \ldots, \widehat{j_k}, \ldots, j_1) = (i_s, \ldots, i_1)$ and $(j_{k+1}, j_k, j_{k-1}) = (i_k, j, i_{k-1})$, and the assumption $j < i_{k-1}$ reads as $j_k < j_{k-1}$.

Claim 1. $j_{k+1} < j_{k-1}$.

The assumption $j \leq i_{k-1} - 1$ translates to $j_k + 1 \leq j_{k-1}$. The connected subsequence $(j_{k+1}, j_k, j_{k-1}) = (i_k, j, i_{k-1})$ is right-most, yielding $j_{k+1} \leq j_k + 1 \leq j_{k-1}$. This implies $j_{k+1} < j_{k-1}$, because $(j_{s+1}, \ldots, \widehat{j_k}, \ldots, j_1)$ is reduced. Claim 1 is proven.

Recall that $(j_{s+1}, \ldots, j_1)$ is right-most and reduced but not small. Proposition 2.3.27 together with Remark 2.3.31.(c) tells us that there exist indices r and t, $s+1 \geq r > t \geq 1$, such that $j_{t+1} < j_t$ and $j_r = j_t$. Since by assumption $(j_{s+1}, \ldots, \widehat{j_k}, \ldots, j_1)$ is small, we conclude that $t = k-1$ or $t = k$ or $r = k$.

Claim 2. $t = k$.

The case $t = k-1$ can be excluded for the following reason. We would have $r > t = k-1$ and $j_r = j_t = j_{k-1}$. On the other hand, by Claim 1, we have $j_{k+1} < j_{k-1}$. This contradicts smallness of $(j_{s+1}, \ldots, \widehat{j_k}, \ldots, j_1)$.

We now exclude $r = k$. Recall that $r > t$, $j_r = j_t$, $j_{t+1} < j_t$. Furthermore, we have $j_r = j_k < j_{k-1}$. This situation is depicted in Figure 3.1.

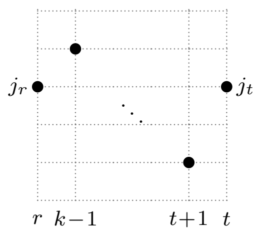


Figure 3.1: Situation for the case $r = k$.

We claim that $t < k-1$. Indeed, we have $t < r = k$ and $t \neq k-1$ follows from $j_t = j_r = j_k < j_{k-1}$. We are in the following situation: $j_{t+1} < j_t$ and $k-1 > t$ and $j_{k-1} > j_t$. Proposition 2.3.27 tells us that $(j_{k-1}, \ldots, j_1) = (i_{k-1}, \ldots, i_1)$ is not small, and hence $(i_s, \ldots, i_1)$ is not small. This contradicts our assumptions, and hence $r \neq k$. Claim 2 is proven.

So we have $t = k$ and from $i_k = j_{k+1} = j_{t+1} < j_t = j_k = j$ we conclude that $i_k < j$. Furthermore, we have $r > k$ and $j_r = j_t = j_k$. Altogether we have $i_{r-1} = j_r = j_k = j$ and we set $r' = r - 1$. (Note that $r' = r - 1 > k$, for $r > k$ and $r \neq k+1$, because $j_r > j_{k+1}$.) The Lemma is proven. □

Proposition 3.2.7 *Let $\underline{x} \in \Omega_n$ be essential. Then*

$$\sum_{I \in \overrightarrow{\Lambda}_{n-1} \setminus F_{\underline{x}}} (-1)^{\# I} \cdot f_I(\underline{x}) = 0.$$

Note that the sum makes sense since $F_{\underline{x}} \subseteq \overrightarrow{\Lambda}_{n-1}$ and $\overrightarrow{\Lambda}_{n-1}$ is finite.

Proof. The proof is quite long and we therefore give a rough outline first. The idea is as follows. We define an action $\mathbb{Z}/2\curvearrowright(\overrightarrow{\Lambda}_{n-1} \setminus F_{\underline{x}})$ such that "the sum over each orbit is zero". More precisely, we define a map $\xi : \overrightarrow{\Lambda}_{n-1} \setminus F_{\underline{x}} \to \overrightarrow{\Lambda}_{n-1} \setminus F_{\underline{x}}$ with the following properties:

(a) $\xi^2 = \mathrm{id}$.

(b) $f_{\xi(I)}(\underline{x}) = f_I(\underline{x})$.

(c) If I is a fixed point of ξ then $f_I(\underline{x}) = 0$.

(d) If I is not a fixed point of ξ then $|\#I - \#\xi(I)| = 1$, i.e. the lengths of I and $\xi(I)$ differ by one. Together with (b) this gives $(-1)^{\#I} f_I(\underline{x}) + (-1)^{\#\xi(I)} f_{\xi(I)}(\underline{x}) = 0$.

We now construct this map ξ. Consider $(i_s, \ldots, i_1) \in \overrightarrow{\Lambda}_{n-1} \setminus F_{\underline{x}}$. If $f_{i_s} \circ \ldots \circ f_{i_1}(\underline{x}) = 0$ then ξ will fix this sequence. (This happens if some d_i multiplies two entries x_{i+1}, x_i that are not geodesic. In this case, post-composition with η_i produces a degenerate cell.) Otherwise, we choose t minimal subject to the condition that $(i_t, \ldots, i_1)$ is not $\underline{x}$-coherent. By Lemma 3.2.4 the cell $f_{i_t} \circ \ldots \circ f_{i_1}(\underline{x})$ is stable at position $i_t - 1$. This observation is central to the construction of ξ. We distinguish three cases.

- Case 1: $[i_t{-}1]_P \in \mathscr{R}[i_s, \ldots, i_{t+1}]_P$.

  The idea is to define $\xi(i_s, \ldots, i_1)$ to be the finite sequence that arises from "deleting" this right-divisor: Let $k \geq t + 1$ be minimal subject to $i_k = i_t - 1$ and set $\xi(i_s, \ldots, i_1) := (i_s, \ldots, \widehat{i_k}, \ldots, i_1)$. We now check that $\xi(i_s, \ldots, i_1)$ lies in $\overrightarrow{\Lambda}_{n-1} \setminus F_{\underline{x}}$.

  Since $[i_k]_P$ is a right-divisor of $[i_s, \ldots, i_{t+1}]_P$, there exists a finite sequence L such that $(i_s, \ldots, i_{t+1}) \sim_P L.(i_k)$. Recall that the monoid P_{n-1} arises from F_{n-1} by quotiening out idempotency relations and distant commutativity. Thus we may choose $L = (i_s, \ldots, i_{k+1}).(i_{k-1}, \ldots, i_{t+1})$:

$$(i_s, \ldots, i_{t+1}) \sim_P (i_s, \ldots, i_{k+1}).(i_{k-1}, \ldots, i_{t+1}).(i_k). \tag{3.16}$$

  We claim that $i_k < i_{k-1}$. For $k = t+1$ this is obvious. For $k > t+1$ we use (3.16) to conclude that $|i_r - i_k| \geq 2$ for every r, $k > r > t$. Since $(i_s, \ldots, i_1)$ is right-most, we must have $i_k < i_{k-1}$.

  Lemma 3.2.5 tells us that $(i_s, \ldots, \widehat{i_k}, \ldots, i_1)$ is indeed right-most, reduced and small. It is obviously not $\underline{x}$-coherent, because $(i_t, \ldots, i_1)$ is not. Condition (c) is void. Condition (d) is fulfilled because $\#\xi(I) = \#I - 1$. Condition (b) follows from the fact that $f_{i_t} \circ \ldots \circ f_{i_1}(\underline{x})$ is stable at position $i_t - 1$, the Evaluation Lemma, and our assumption that $[i_t{-}1]_P \in \mathscr{R}[i_s, \ldots, i_{t+1}]_P$.

- Case 2: $[i_t{-}1]_P \notin \mathscr{R}[i_s, \ldots, i_{t+1}]_P$ and $(i_s, \ldots, i_{t+1}).(i_t{-}1).(i_t, \ldots, i_1)$ is small.

  The idea is to define $\xi(i_s, \ldots, i_1)$ by "inserting" the entry $i_t - 1$ into $(i_s, \ldots, i_1)$. Note that $(i_s, \ldots, i_{t+1}).(i_t{-}1).(i_t, \ldots, i_1)$ need not be right-most. So we define

$$\xi(i_s, \ldots, i_1) := \overrightarrow{(i_s, \ldots, i_{t+1}).(i_t{-}1)}.(i_t, \ldots, i_1), \tag{3.17}$$

where the overlining arrow stands for the right-most, reduced representative. It is easily checked that (3.17) is right-most and reduced. Furthermore, by Remark 2.3.21.(a), $\xi(i_s,\dots,i_1)$ is small. Since $(i_t,\dots,i_1)$ is not $\underline{x}$-coherent, $\xi(i_s,\dots,i_1)$ is neither. Properties (b)-(d) are again obvious.

Before discussing the remaining case let us remark that for all finite sequences I that are covered by Cases 1 and 2 we have $\xi^2(I)=I$.

- Case 3: $[i_t{-}1]_P \notin \mathscr{R}[i_s,\dots,i_{t+1}]_P$ and $(i_s,\dots,i_{t+1}).(i_t{-}1).(i_t,\dots,i_1)$ is not small.

  We are going to find an index $t'>t$ such that $f_{i_{t'}}\cdots f_{i_1}(\underline{x})$ is stable at position $i_{t'}-1$. We then replace t by t' and restart our case distinction. The existence of such an entry t' will follow from Lemma 3.2.6. To have it applicable we need to find the right-most, reduced representative of $(i_s,\dots,i_{t+1}).(i_t-1).(i_t,\dots,i_1)$. Since $(i_s,\dots,i_1)$ is right-most and reduced it follows that there exists some k such that

$$\overrightarrow{(i_s,\dots,i_{t+1}).(i_t{-}1).(i_t,\dots,i_1)} = (i_s,\dots,i_k).(i_t{-}1).(i_{k-1},\dots,i_1). \tag{3.18}$$

  Note that $k\geq t+1$ because $(i_t{-}1,i_t,\dots,i_1)$ is right-most and reduced. Intuitively speaking, we obtain this right-most, reduced representative by successively pushing the entry $(i_t{-}1)$ to the left, and that we do as long as its left neighbour is $\geq i_t+1$.

  Claim 1. $i_t-1<i_{k-1}$.

  If $k-1=t$ then the Claim reads as $i_t-1<i_t$ which is obvious. Otherwise i_t-1 and i_{k-1} commute in the sense of $\sim_{\mathrm{dist}}$, i.e. $|i_t-1-i_{k-1}|\geq 2$. Since our sequence is right-most, this implies $i_t-1<i_{k-1}$. Claim 1 is proven.

  We are in the following situation: The sequence $(i_s,\dots,i_1)$ is right-most, reduced and small. The sequence $(i_s,\dots,i_k).(i_t{-}1).(i_{k-1},\dots,i_1)$ is right-most and reduced but not small. Furthermore, $i_t-1<i_{k-1}$. Lemma 3.2.6 guarantees that $i_k<i_t-1$ and that there exists an entry $t'>k$ such that $i_{t'}=i_t-1$. Amongst all possible choices of t' we choose the smallest one.

  Claim 2. $[i_{t'}{-}1]_Q \in \mathscr{L}[i_{t'},\dots,i_k,i_t{-}1]_Q$.

  The connected subsequence $(i_{t'},\dots,i_k)$ is small and thus by Corollary 2.3.28 there is a unique index where it attains its maximum, namely t': This follows from the facts that $(i_{t'},\dots,i_k)$ is right-most and reduced, $i_{t'}>i_k$ and t' was chosen minimal. In particular, $i_{t'-1}=i_{t'}-1$ and $i_{t'-1}=\max\{i_{t'-1},\dots,i_k\}$. Again by smallness, this maximum is uniquely attained. It follows that $\max\{i_{t'-2},\dots,i_k\}\leq i_t-3$. We can therefore push the entry (i_t-1) to the left:

$$\begin{aligned}(i_{t'},\dots,i_k).(i_t{-}1) &\sim_P (i_{t'},i_{t'-1}).(i_t{-}1).(i_{t'-2},\dots,i_k)\\ &= (i_{t'},i_{t'}{-}1,i_{t'}).(i_{t'-2},\dots,i_k).\end{aligned}$$

  This implies that $[(i_{t'},\dots,i_k,i_t{-}1)]_Q=[(i_{t'},i_{t'}{-}1,i_{t'}).(i_{t'-2},\dots,i_k)]_Q$ from which $[i_{t'}{-}1]_Q$ is a left divisor. Claim 2 is proven.

Claim 3. $f_{i_{t'}} \dots f_{i_1}(\underline{x})$ is stable at position $i_{t'} - 1$.

Recall that $f_{i_t} \dots f_{i_1}(\underline{x})$ is stable at position $i_t - 1$, yielding

$$f_{i_{t'}} \dots f_{i_1}(\underline{x}) = f_{i_{t'}} \dots f_{i_{t+1}} \circ f_{i_t - 1} \circ f_{i_t} \dots f_{i_1}(\underline{x}). \tag{3.19}$$

We now apply (3.18) with s replaced by t' to the right-hand side of (3.19). The Evaluation Lemma then gives

$$f_{i_{t'}} \dots f_{i_1}(\underline{x}) = f_{i_{t'}} \dots f_{i_k} \circ f_{i_t - 1} \circ f_{i_{k-1}} \dots f_{i_1}(\underline{x}). \tag{3.20}$$

Claim 2 together with the Evaluation Lemma tells us that the right-hand side of (3.20) is stable at position $i_{t'} - 1$ in the graded sense. Recall that $f_{i_{t'}} \dots f_{i_1}$ is norm-preserving for $\underline{x}$, and thus $f_{i_{t'}} \dots f_{i_1}(\underline{x})$ is (honestly) stable at position $i_{t'} - 1$. Claim 3 is proven.

We can now jump back in the proof and apply the above case distinction with the distinguished index t replaced by t'.

Note that the above iteration finally terminates, since the t' constructed in Case 3 is strictly larger than the t we started with. $\square$

The following example should shed some light on the algorithm described in the proof above.

Example 3.2.8 Consider an essential 4-cell $\underline{x} = [x_4|x_3|x_2|x_1] \in \overline{\mathbb{E}}_4 X$ with the property that the finite sequence $(2,1)$ is $\underline{x}$-coherent and

$$d_3 f_2 f_1(\underline{x}) = [x_4 \overline{x_3 \overline{x_2 x_1}} | (x_3 \overline{x_2 x_1})' | (x_2 x_1)']$$

is not redundant. We write Ξ for the set of all finite sequences in F_3 that are right-most, reduced, small and that are of the form $L.(3,2,1)$ for some $L \in F_3$. Explicitly:

$$\Xi = \{(3,2,1), (1,3,2,1), (2,1,3,2,1), (1,2,1,3,2,1), (2,3,2,1), (1,2,3,2,1)\}.$$

Note that $\Xi \subseteq \overrightarrow{\Lambda}_3 \setminus F_{\underline{x}}$.

Let us assume that every element in Ξ is norm-preserving for $\underline{x}$, because otherwise it would be a fixed point of ξ and nothing interesting happens. We will now explicitly describe the map $\xi : \Xi \to \Xi$. First of all, $f_3 f_2 f_1(\underline{x})$ is clearly stable at position 3, and the proof of Proposition 3.2.7 tells us that it is also stable at position 2.

- $(i_s, \dots, i_1) = (3,2,1) = (\,).(3,2,1)$.

  We will discuss this case in detail. The data is the following: $t = 3$, $i_t - 1 = 2$ and $(i_s, \dots, i_{t+1}) = (\,)$. Clearly, $[i_t - 1]_P = [2]_P \notin \mathscr{R}[\,]_P = \mathscr{R}[i_s, \dots, i_{t+1}]_P$ and $(i_s, \dots, i_{t+1}).(i_t - 1).(i_t, \dots, i_1) = (2).(3,2,1)$ is small. Therefore we are in Case 2, yielding $\xi(3,2,1) = \overrightarrow{(2)}.(3,2,1) = (2,3,2,1)$.

- Let us, for convenience, briefly discuss $(i_s, \ldots, i_1) = (2,3,2,1) = (2).(3,2,1)$.

  Clearly, $[2]_P \in \mathscr{R}[2]_P$. So we are in Case 1, yielding $\xi(2,3,2,1) = (\widehat{2},3,2,1) = (3,2,1)$. This shows that we indeed have $\xi^2(3,2,1) = (3,2,1)$.

- $(i_s, \ldots, i_1) = (1,3,2,1) = (1).(3,2,1)$.

  Clearly, $[2]_P \notin \mathscr{R}[1]_P$ and $(1).(2).(3,2,1)$ is small. Therefore we are in Case 2, yielding $\xi(1,3,2,1) = \overrightarrow{(1).(2)}.(3,2,1) = (1,2,3,2,1)$.

- $(i_s, \ldots, i_1) = (2,1,3,2,1) = (2,1).(3,2,1)$.

  We have $[2]_P \notin \mathscr{R}[2,1]_P$ but $(2,1).(2).(3,2,1)$ is not small. Therefore we are in Case 3. Note that $(\mathbf{2},1,2,3,2,1)$ is already right-most and reduced, the fattened $\mathbf{2}$ being the entry $i_{t'}$. We therefore restart the case distinction with the sequence $(2,1,3,2,1)$ and $t = 5$, $i_t - 1 = 1$. We then have $[1]_P \notin \mathscr{R}[\,]_P$ and $(1,2,1,3,2,1)$ is small. Hence, $\xi(2,1,3,2,1) = (1,2,1,3,2,1)$.

In Figure 3.2 we depict the map $\xi : \Xi \to \Xi$ in terms of the (left) Cayley graph of $\overrightarrow{\Lambda}_3$. Shaded terms that are connected by a dotted curve cancel out each other.

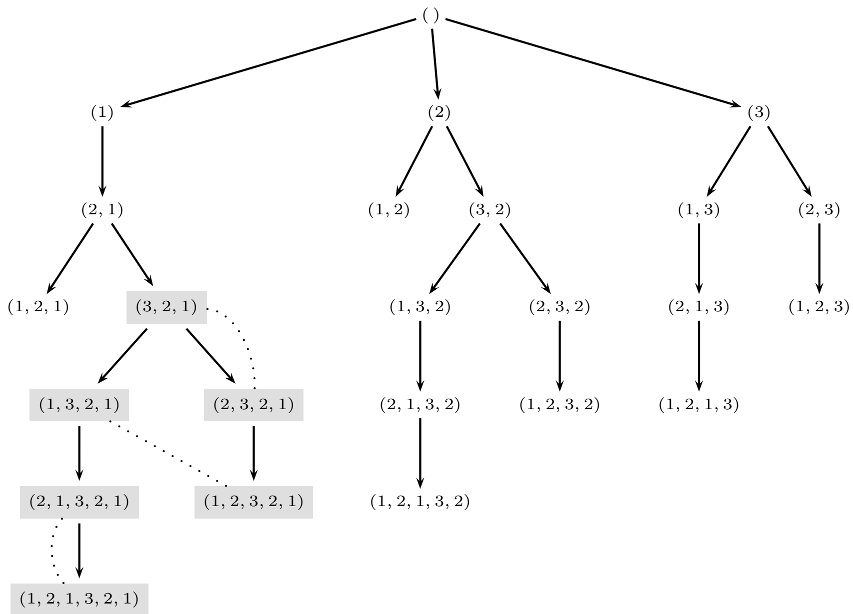


Figure 3.2: Cayley graph of $\overrightarrow{\Lambda}_3$ and the matching induced by ξ.

Recall that $\overrightarrow{\Lambda}_{n-1}$ is a set of representatives for $Q_{n-1} \smallsetminus \square_{n-1}$. Combining Corollary 3.2.2 and Proposition 3.2.7 we obtain the following.

Theorem 3.2.9 *Let (X, S, η) be a factorable monoid and denote by $(\widetilde{\mathbb{V}}_*, \partial^{\mathbb{V}}_*)$ its Visy* *resolution. The differential $\partial_*^{\mathbb{V}} : \widetilde{\mathbb{V}}_* X \to \widetilde{\mathbb{V}}_{*-1} X$ can be written as follows,*

$$\partial_n^{\mathbb{V}} = \pi_{n-1} \circ \overline{\partial}_n \circ \sum_{\alpha \in Q_{n-1} \smallsetminus \square_{n-1}} (-1)^{\ell(\alpha)} \cdot f_\alpha \circ i_n. \tag{3.21}$$

One aim of the following section is to make precise the similarity between formulas (3.21) and (2.15) on page 91.

3.3 $\mathbb{Z}$-coefficients

So far we were working with $\mathbb{Z}X$-coefficients. The Visy complex, however, is over $\mathbb{Z}$. To relate our results to those of Visy and Wang we will now investigate the complex that arises from tensoring $\widetilde{\mathbb{V}}_*$ with $\mathbb{Z}$.

Convention. As long as not stated otherwise, the action of $\mathbb{Z}X$ on $\mathbb{Z}$ is always by augmentation, i.e. X acts trivially on $\mathbb{Z}$.

3.3.1 The E^1-page revisited

Throughout this section we fix a factorable monoid (X, S, η). Recall that the choice of a generating set S gives rise to a filtration of the normalized bar complex, $\mathcal{F}_\bullet \overline{\mathbb{B}}_* X$. Associated to this filtration there is a fourth quadrant spectral sequence, the E^0-page of which having as entries the filtration quotients

$$\mathsf{E}^0_{p,q} = \mathcal{G}_p \overline{\mathbb{B}}_{p+q} X,$$

where $\mathcal{G}_p \overline{\mathbb{B}}_{p+q} X = \mathcal{F}_p \overline{\mathbb{B}}_{p+q} X / \mathcal{F}_{p-1} \overline{\mathbb{B}}_{p+q} X$. Since we are working in the normalized bar complex, every n-cell has norm at least n and thus $\mathcal{F}_h \overline{\mathbb{B}}_n X = 0$ if $h < n$. It follows that $\mathsf{E}^0_{p,q} = 0$ for $q > 0$ and

$$\mathsf{E}^0_{n,0} = \mathcal{F}_n \overline{\mathbb{B}}_n X.$$

In Figure 2.1 we showed how the E^0-page of this spectral sequence looks abstractly. Figure 3.3 gives a more detailed picture.

In order to keep this chapter self-contained, we will now sketch the proof of Theorem 2.1.21 for factorable monoids.

Theorem 2.1.21' (Visy, Wang) *The homology of each vertical complex $\mathcal{G}_h \overline{\mathbb{B}}_* X$ is concentrated in degree h. Equivalently speaking, the E^1-page consists of a single chain complex*

$$0 \longleftarrow \mathsf{E}^1_{1,0} \xleftarrow{\mathsf{d}^1_{2,0}} \mathsf{E}^1_{2,0} \xleftarrow{\mathsf{d}^1_{3,0}} \mathsf{E}^1_{3,0} \xleftarrow{\mathsf{d}^1_{4,0}} \cdots$$

$$
\begin{tikzcd}
 & \mathbf{p=0} & \mathbf{p=1} & \mathbf{p=2} & \mathbf{p=3} & \mathbf{p=4} \\
\mathbf{q=0} & \mathcal{F}_0\overline{\mathbb{B}}_0X & \mathcal{F}_1\overline{\mathbb{B}}_1X & \mathcal{F}_2\overline{\mathbb{B}}_2X \arrow[d, "\mathsf{d}^0_{2,0}"] & \mathcal{F}_3\overline{\mathbb{B}}_3X \arrow[d, "\mathsf{d}^0_{3,0}"] & \mathcal{F}_4\overline{\mathbb{B}}_4X \arrow[d, "\mathsf{d}^0_{4,0}"] \\
\mathbf{q=-1} & 0 & 0 & \mathcal{G}_2\overline{\mathbb{B}}_1X & \mathcal{G}_3\overline{\mathbb{B}}_2X \arrow[d, "\mathsf{d}^0_{3,1}"] & \mathcal{G}_4\overline{\mathbb{B}}_3X \arrow[d, "\mathsf{d}^0_{4,1}"] \\
\mathbf{q=-2} & 0 & 0 & 0 & \mathcal{G}_3\overline{\mathbb{B}}_1X & \mathcal{G}_4\overline{\mathbb{B}}_2X \arrow[d, "\mathsf{d}^0_{4,2}"] \\
\mathbf{q=-3} & 0 & 0 & 0 & 0 & \mathcal{G}_4\overline{\mathbb{B}}_1X
\end{tikzcd}
$$

Figure 3.3: Fourth quadrant spectral sequence $\mathsf{E}^0_{p,q}$.

This Theorem is originally due to Visy, who proved it for factorable groups. Wang generalized his result to weakly factorable monoids. Below we outline an alternative idea of proof for factorable monoids.

Proof. Denote by Ω_n the set of all n-cells $[x_n|\dots|x_1]$ with $x_i \neq \epsilon$ for all i. Then Ω_* is a $\mathbb{Z}$-basis for $\overline{\mathbb{B}}_*X$. We define a matching $\mu : \Omega_* \to \Omega_*$ as in (3.1) on page 95. Clearly, μ is an involution for any ground ring. We can almost literally copy the proof of Lemma 3.1.10 to show that μ is $\mathbb{Z}$-compatible. The proof of noetherianity also goes through without any significant changes. Altogether, μ defines a noetherian matching on the (based) normalized bar complex $(\overline{\mathbb{B}}_*X, \Omega_*, \overline{\partial}_*)$.

We now show that μ descends to a matching on each complex of filtrations quotients $(\mathcal{G}_h\overline{\mathbb{B}}_*X, \overline{\partial}'_*)$. Denote by $\Omega_{n,h}$ the set of n-cells of norm h. Note that $\Omega_{n,h}$ is a $\mathbb{Z}$-basis for $\mathcal{G}_h\overline{\mathbb{B}}_nX$. Furthermore recall that if two cells are matched then their norms coincide. Therefore μ restricts and corestricts to a matching (not necessarily a $\mathbb{Z}$-compatible one) on the based chain complex $(\mathcal{G}_h\overline{\mathbb{B}}_*X, \Omega_{*,h}, \overline{\partial}'_*)$. This matching is clearly noetherian, and for the following reason it is $\mathbb{Z}$-compatible: For every redundant cell $\underline{x} \in \Omega_{n,h}$ we can write $\overline{\partial}\mu(\underline{x})$ uniquely as $g + g' \in \mathcal{F}_h\overline{\mathbb{B}}_nX$, where $g \in \mathcal{G}_h\overline{\mathbb{B}}_nX$ and $g' \in \mathcal{F}_{h-1}\overline{\mathbb{B}}_nX$. (This is possible because the modules $\mathcal{F}_h\overline{\mathbb{B}}_nX$ and $\mathcal{G}_h\overline{\mathbb{B}}_nX$ are freely generated by elements of Ω_*.) Note that $g = \overline{\partial}'_*\mu(\underline{x})$ and that $[g' : \underline{x}] = 0$ for norm reasons. Therefore $[\partial\mu(\underline{x}) : \underline{x}] = [g : \underline{x}] = [\overline{\partial}'\mu(\underline{x}) : \underline{x}]$, proving $\mathbb{Z}$-compatibility.

The essential cells of $\mu : \Omega_{*,h} \to \Omega_{*,h}$ are exactly those essential cells of the original matching $\mu : \Omega_* \to \Omega_*$ that have norm h. In particular, every essential cell has elementary partition type and thus lives in $\mathcal{G}_h\overline{\mathbb{B}}_hX = \mathsf{E}^0_{h,0}$. Theorem 1.1.29 implies that the homology of each complex $\mathcal{G}_h\overline{\mathbb{B}}_*X$ is concentrated in degree h. □

Remark 3.3.1 The above constructed matching $\mu : \Omega_{*,h} \to \Omega_{*,h}$ on $\mathcal{G}_h\overline{\mathbb{B}}_*X$ is *perfect*, meaning that the associated Morse complex $((\mathcal{G}_h\overline{\mathbb{B}}_*X)^\theta, (\overline{\partial}'_*)^\theta)$ is isomorphic to the homology of the original complex $H_*(\mathcal{G}_h\overline{\mathbb{B}}_\bullet X, \overline{\partial}'_\bullet)$.

We now investigate the E^1-page. Recall that $\mathsf{E}^0_{n,1} = 0$ and hence $\operatorname{im}(\mathsf{d}^0_{n,1} : \mathsf{E}^0_{n,1} \to \mathsf{E}^0_{n,0}) = 0$, yielding

$$\mathsf{E}^1_{n,0} = \ker(\mathsf{d}^0_{n,0} : \mathsf{E}^0_{n,0} \to \mathsf{E}^0_{n,-1}).$$

From this we see that $\mathsf{E}^1_{n,0} \subseteq \mathsf{E}^0_{n,0} = \mathcal{F}_n\overline{\mathbb{B}}_nX \subseteq \overline{\mathbb{B}}_n$. In other words, $\mathsf{E}^1_{*,0}$ sits as a graded $\mathbb{Z}$-module in $\overline{\mathbb{B}}_*X$.

Lemma 3.3.2 *The* E^1*-page* $(\mathsf{E}^1_{*,0}, \mathsf{d}^1_{*,0})$ *is a sub-chain complex of the normalized bar complex* $(\overline{\mathbb{B}}_*X, \overline{\partial}_*)$.

PROOF. It only remains to show compatibility of the differentials. To see this, recall that $\mathsf{d}^1_{n,0} : \mathsf{E}^1_{n,0} \to \mathsf{E}^1_{n-1,0}$ is the connecting homomorphism $H_n(\mathcal{G}_n\overline{\mathbb{B}}_*X) \to H_{n-1}(\mathcal{G}_{n-1}\overline{\mathbb{B}}_*X)$ in the long exact homology sequence associated to the following short exact sequence of filtration quotients:

$$0 \longrightarrow \mathcal{G}_{n-1}\overline{\mathbb{B}}_*X \longrightarrow \mathcal{F}_n\overline{\mathbb{B}}_*X/\mathcal{F}_{n-2}\overline{\mathbb{B}}_*X \longrightarrow \mathcal{G}_n\overline{\mathbb{B}}_*X \longrightarrow 0$$

An easy diagram chase shows that $\mathsf{d}^1_{*,0}(c) = \overline{\partial}_*(c)$ for every $c \in \mathsf{E}^1_{*,0}$. □

We proceed by characterizing the module $\mathsf{E}^1_{n,0}$ inside $\overline{\mathbb{B}}_nX$.

Lemma 3.3.3 *For a chain* $c \in \overline{\mathbb{B}}_nX$ *the following are equivalent:*

(a) $c \in \mathsf{E}^1_{n,0}$.

(b) c *has elementary partition type and* $\operatorname{norm}(\overline{\partial}c) < n$.

PROOF. First of all, observe that a chain $c \in \overline{\mathbb{B}}_nX$ has elementary partition type if and only if $c \in \mathcal{F}_n\overline{\mathbb{B}}_nX = \mathsf{E}^0_{n,0}$. It therefore remains to show that for $c \in \mathsf{E}^0_{n,0}$ we have $c \in \ker(\mathsf{d}^0_{n,0} : \mathsf{E}^0_{n,0} \to \mathsf{E}^0_{n,-1})$ if and only if $\operatorname{norm}(\overline{\partial}c) < n$.

The differential $\mathsf{d}^0_{n,0} : \mathsf{E}^0_{n,0} \to \mathsf{E}^0_{n,-1}$ is given as follows,

$$\begin{tikzcd}
\mathsf{E}^0_{n,0} \arrow[r, equal] \arrow[d, "\mathsf{d}^0_{n,0}"'] & \mathcal{F}_n\overline{\mathbb{B}}_nX \arrow[r, "\overline{\partial}"] \arrow[d, "\mathsf{d}^0_{n,0}"'] & \mathcal{F}_n\overline{\mathbb{B}}_{n-1}X \arrow[dl, "\mathrm{pr}"] \\
\mathsf{E}^0_{n,-1} \arrow[r, equal] & \mathcal{F}_n\overline{\mathbb{B}}_{n-1}X/\mathcal{F}_{n-1}\overline{\mathbb{B}}_{n-1}X &
\end{tikzcd}$$

where $\mathrm{pr} : \mathcal{F}_n\overline{\mathbb{B}}_{n-1}X \to \mathcal{F}_n\overline{\mathbb{B}}_{n-1}X/\mathcal{F}_{n-1}\overline{\mathbb{B}}_{n-1}X$ denotes the canonical projection onto the quotient.

Consider $c \in \mathsf{E}^0_{n,0}$. The chain c lies in the kernel of $\mathsf{d}^0_{n,0}$ if and only if $\overline{\partial}c$ lies in the kernel of the projection map, i.e. if and only if $\overline{\partial}c \in \mathcal{F}_{n-1}\overline{\mathbb{B}}_{n-1}X$. The latter is the case if and only if $\operatorname{norm}(\overline{\partial}c) < n$. The Lemma is proven. $\square$

Lemma 3.3.4 *For a chain $c \in \overline{\mathbb{E}}_n X$ the following are equivalent:*

(a) $c \in (\overline{\mathbb{E}}_n X)^\Theta$.

(b) c has elementary partition type and $\operatorname{norm}(\overline{\partial}c) < n$.

Proof. **Claim.** Θ-invariant chains have elementary partition type.

Recall from Remark 3.1.6 that if $\underline{x} \in \Omega_*$ has elementary partition type then $\underline{x}$ is essential or collapsible and hence $V(\underline{x}) = 0$. It follows that if c has elementary partition type then $V(c) = 0$ and thus $\Theta(c) = c + V(\overline{\partial}c)$. The latter has elementary partition type, for if c has elementary partition type then so has $V(\overline{\partial}c)$.

Let $c \in \overline{\mathbb{E}}_n X$ be Θ-invariant. Then by Lemma 1.1.28.(c) we have $\Theta^\infty(\pi(c)) = c$, and $\pi(c)$ is an essential chain. In particular, $\pi(c)$ has elementary partition type. By our above observation $c = \Theta^\infty(\pi(c))$ has elementary partition type. The Claim proven.

Due to the Claim it suffices to prove the Lemma for chains of elementary partition type. So let $c \in \overline{\mathbb{E}}_n X$ have elementary partition type.

"$\Rightarrow$": Assume that c is Θ-invariant. Commutativity of Θ and $\overline{\partial}$ yields $\Theta(\overline{\partial}c) = \overline{\partial}(\Theta(c)) = \overline{\partial}c$, i.e. $\overline{\partial}c$ is Θ-invariant and hence has elementary partition type. On the other hand, $\overline{\partial}c \in \overline{\mathbb{E}}_{n-1}X$. From this we see that $\operatorname{norm}(\overline{\partial}c) \leq n-1$. (Note that $\operatorname{norm}(\overline{\partial}c) < n-1$ if and only if $\overline{\partial}c = 0$.)

"$\Leftarrow$": Let $c \in \overline{\mathbb{E}}_n X$ have elementary partition type. We have $\overline{\partial}c \in \overline{\mathbb{E}}_{n-1}X$. Now, if $\operatorname{norm}(\overline{\partial}c) < n$ then $\overline{\partial}c$ must have elementary partition type. In particular, $\overline{\partial}c$ is an essential–collapsible chain and thus $V(\overline{\partial}c) = 0$. Therefore $\Theta(c) = c + V(\overline{\partial}c) = c$ and c is indeed Θ-invariant. $\square$

Recall from Lemma 3.3.2 that $(\mathsf{E}^1_{*,0}, \mathsf{d}^1_{*,0}) \subseteq (\overline{\mathbb{B}}_* X, \overline{\partial}_*)$. Furthermore, by Theorem 1.1.23, the complex of Θ-invariant chains $((\overline{\mathbb{E}}_* X)^\Theta, \overline{\partial}_*)$ is a retract of the normalized bar resolution $(\overline{\mathbb{E}}_* X, \overline{\partial}_*)$, and thus the complex $((\overline{\mathbb{E}}_* X)^\Theta, \overline{\partial}_*) \otimes_{\mathbb{Z}X} \mathbb{Z}$ also embeds into $(\overline{\mathbb{B}}_* X, \overline{\partial}_*)$. The subsequent Proposition compares those two subcomplexes.

Proposition 3.3.5 *The following map is an isomorphism of chain complexes,*

$$\begin{aligned} (\mathsf{E}^1_{*,0}, \mathsf{d}^1_{*,0}) &\longrightarrow ((\overline{\mathbb{E}}_* X)^\Theta, \overline{\partial}_*) \otimes_{\mathbb{Z}X} \mathbb{Z}, \\ c &\longmapsto c \otimes 1. \end{aligned}$$

Proof. We first show that $\mathsf{E}^1_{*,0}$ and $(\overline{\mathbb{E}}_* X)^\Theta \otimes_{\mathbb{Z}X} \mathbb{Z}$ are isomorphic as graded modules. Recall from (1.3) on page 30 that the map Θ preserves gradings, i.e. $\Theta(\overline{\mathbb{E}}_n X) \subseteq (\overline{\mathbb{E}}_n X)$.

This yields the following:

$$
\begin{aligned}
(\overline{\mathbb{E}}_* X)^\Theta \otimes_{\mathbb{Z}X} \mathbb{Z} &= \left[\bigoplus_{n \geq 0} [(\overline{\mathbb{E}}_n X)^\Theta] \right] \otimes_{\mathbb{Z}X} \mathbb{Z} \\
&= \bigoplus_{n \geq 0} [(\overline{\mathbb{E}}_n X)^\Theta \otimes_{\mathbb{Z}X} \mathbb{Z}]
\end{aligned}
\tag{3.22}
$$

From Lemmas 3.3.3 and 3.3.4 we conclude that for every $n \geq 0$ the map $c \mapsto c \otimes 1$ provides an isomorphism between the modules $\mathsf{E}^1_{n,0}$ and $(\overline{\mathbb{E}}_n X)^\Theta \otimes_{\mathbb{Z}X} \mathbb{Z}$. Plugging this in into (3.22), we obtain $(\overline{\mathbb{E}}_* X)^\Theta \otimes_{\mathbb{Z}X} \mathbb{Z} \cong \mathsf{E}^1_{*,0}$.

Compatibility of the differentials is clear, because on both complexes, $(\mathsf{E}^1_{*,0}, \mathsf{d}^1_{*,0})$ and $((\overline{\mathbb{E}}_* X)^\Theta, \overline{\partial}_*) \otimes_{\mathbb{Z}X} \mathbb{Z}$, the differential is the restriction of the bar differential $\overline{\partial}_*$ (with coefficients in $\mathbb{Z}$). □

3.3.2 κ is an isomorphism

Proposition 3.3.6 *Let (X, S, η) be a factorable monoid. Denote by $(\mathbb{V}_*, \partial^{\mathbb{V}}_*)$ its Visy complex. The following map is an ismorphism of chain complexes,*

$$
\begin{aligned}
(\mathbb{V}_*, \partial^{\mathbb{V}}_*) &\longrightarrow ((\overline{\mathbb{E}}_* X)^\theta, \overline{\partial}^\theta_*) \otimes_{\mathbb{Z}X} \mathbb{Z} \\
c &\longmapsto c \otimes 1.
\end{aligned}
$$

Proof. Denote by Ω^θ_n the set of essential n-cells, i.e. Ω^θ_n consists of all cells $[x_n|\dots|x_1]$ with $x_i \in S_+$ for all i and (x_{i+1}, x_i) unstable for all $1 \leq i < n$. Recall from discrete Morse theory that Ω^θ_* is a $\mathbb{Z}X$-basis for $(\overline{\mathbb{E}}_* X)^\theta$. Furthermore, Ω^θ_* is a $\mathbb{Z}$-basis for the Visy complex $(\mathbb{V}_*, \partial^{\mathbb{V}}_*)$, cf. page 2.1.4. This proves that $c \mapsto c \otimes 1$ is an isomorphism on the underlying graded $\mathbb{Z}$-modules. Compatibility of the differentials follows from comparing Theorem 3.2.9 and Corollary 2.3.37. □

Remark 3.3.7 (a) The above Proposition gives an affirmative answer to the question whether the Visy complex “comes from a resolution” and justifies calling $(\widetilde{\mathbb{V}}_*, \partial^{\mathbb{V}}_*) = ((\overline{\mathbb{E}}_* X)^\theta, \overline{\partial}^\theta_*)$ the Visy resolution of the factorable monoid (X, S, η).

(b) Note that $((\overline{\mathbb{E}}_* X)^\theta, \overline{\partial}^\theta_*) \otimes_{\mathbb{Z}X} \mathbb{Z} = C_*(Y)$, where $C_*(Y)$ is the cellular chain complex of Y, and Y is the quotient of the classifying space BX discussed in Remark 3.1.23. In particular, $C_*(Y) \cong (\mathbb{V}_*, \partial^{\mathbb{V}}_*)$. This affirmatively answers the question whether there exists a CW complex whose cellular chain complex is (isomorphic to) the Visy complex.

We are now ready to state and prove the main result of this thesis.

Theorem 3.3.8 *For every factorable monoid (X, S, η) the following diagram of chain complexes commutes:*

$$\begin{tikzcd}
((\overline{\mathbb{E}}_* X)^\theta, \overline{\partial}^\theta_*) \otimes_{\mathbb{Z}X} \mathbb{Z} \arrow[r, "\Theta^\infty \otimes_{\mathbb{Z}X} \mathbb{Z}"] & ((\overline{\mathbb{E}}_* X)^\Theta, \overline{\partial}_*) \otimes_{\mathbb{Z}X} \mathbb{Z} \\
(\mathbb{V}_*, \partial^{\mathbb{V}}) \arrow[u, "\cong"', "_\otimes 1"] \arrow[r, "\kappa"'] & (\mathsf{E}^1_{*,0}, \mathsf{d}^1_{*,0}) \arrow[u, "\cong"', "_\otimes 1"]
\end{tikzcd}$$

Proof. The vertical arrows are the maps from Propositions 3.3.5 and 3.3.6. Comparing Theorem 3.2.9 with Proposition 2.3.36 and Corollary 2.3.37, we see that $\Theta^\infty \otimes_{\mathbb{Z}X} \mathbb{Z} = \kappa$, whence the Theorem. □

Very vaguely one could summarize the above Theorem by saying that Visy's map κ is nothing but Forman's discrete gradient flow Θ^∞ "in disguise".

Recall that $\Theta^\infty : ((\overline{\mathbb{E}}_* X)^\theta, \overline{\partial}^\theta_*) \to ((\overline{\mathbb{E}}_* X)^\Theta, \overline{\partial}_*)$ is an isomorphism of chain complexes. Thus, by functoriality:

Corollary 3.3.9 *For every factorable monoid (X, S, η), Visy's map*

$$\kappa : (\mathbb{V}_*, \partial^{\mathbb{V}}) \longrightarrow (\mathsf{E}^1_{*,0}, \mathsf{d}^1_{*,0})$$

is an isomorphism of chain complexes.

Remark 3.3.10 Wang [Wan11, Theorem 1.3.3] proved a similar result for certain factorable categories. In case of a monoid (i.e. a category with one object), her result translates to Theorem 2.1.23, namely that κ is an isomorphism if X is right-cancellative and S is finite. Our approach to factorability via monoid actions allows to establish this result in full generality.

3.4 Connection to Rewriting Systems

In the previous chapters we saw that both, complete rewriting systems as well as factorability structures, give rise to normal forms and in this way induce noetherian matchings on the normalized bar resolution. We are now going to show that factorable monoids fit into the framework of complete rewriting systems. Roughly speaking, we show that in the following diagram the dotted arrow exists:

$$\begin{tikzcd}
\boxed{\begin{array}{c}\text{complete rewriting system}\\ X = (S, \mathcal{R})\end{array}} \arrow[dr] & & \boxed{\begin{array}{c}\text{factorable monoid}\\ (X, S, \eta)\end{array}} \arrow[ll, dotted] \arrow[dl] \\
& \boxed{\begin{array}{c}\text{noetherian matching on}\\ (\overline{\mathbb{E}}_* X, \Omega_*, \overline{\partial}_*)\end{array}} &
\end{tikzcd}$$

More precisely, we prove the following:

Theorem 3.4.1 (H, Ozornova) *Let (X, S, η) be a factorable monoid.*

(a) Then X possesses a complete rewriting system over the alphabet S, say $X = (S, \mathcal{R})$. Moreover, if S is finite, then so is $(S, \mathcal{R})$.

(b) Applying Brown's construction (cf. page 45) to the complete rewriting system associated to (X, S, η) yields the same noetherian matching on the normalized bar resolution as our construction described in Theorem 3.1.8.

The proof of Theorem 3.4.1 will occupy the remainder of this section. We first prove part (a).

Definition 3.4.2 Let (X, S, η) be a factorable monoid. Define the *rewriting system associated to* (X, S, η) as $(S, \mathcal{R})$, where the set of rewriting rules $\mathcal{R}$ is given as follows:

$$\mathcal{R} = \{(s_2 s_1,\, \overline{s_2 s_1}(s_2 s_1)') \mid \text{for all } \eta\text{-unstable pairs } (s_2, s_1) \in S_+ \times S_+\}$$

The intuition is that we write down a rewriting rule $s_2 s_1 \to \overline{s_2 s_1}(s_2 s_1)'$ for every unstable pair (s_2, s_1) of non-trivial generators.

Lemma 3.4.3 *Let (X, S, η) be a factorable monoid. Then the associated rewriting system $(S, \mathcal{R})$ is strongly minimal, and there is at most one irreducible representative in every equivalence class.*

Proof. $(S, \mathcal{R})$ is clearly minimal: Firstly, since the pair $(\overline{s_2 s_1}, (s_2 s_1)')$ is η-stable, the right-hand side of every rewriting rule is irreducible with respect to $\mathcal{R}$. Secondly, given an unstable pair of generators (s_2, s_1), the left-hand side $s_2 s_1 \in S^*$ of the associated rewriting rule is irreducible with respect to $\mathcal{R} \smallsetminus \{(s_2 s_1,\, \overline{s_2 s_1}(s_2 s_1)')\}$. This is because every unstable pair contributes only one rewriting rule in $\mathcal{R}$.

To see that $(S, \mathcal{R})$ is even strongly minimal, we only have to show that every generator $s \in S$ is irreducible. But this is clear, for every reducible word must have length at least 2, because otherwise one cannot apply any rewriting rule.

It remains to show that there is at most one irreducible representative in every $\leftrightarrow_{\mathcal{R}}$-equivalence class. Assume that $(s_n, \ldots, s_1)$ and $(t_m, \ldots, t_1)$ are irreducible words over S representing the same element, say x. We claim that $(s_n, \ldots, s_1)$ and $(t_m, \ldots, t_1)$ are totally η-stable: Since the left-hand side of every rewriting rule has length 2, a word $w \in S^*$ is reducible if and only if some subword of w of length 2 is reducible. We know that $(s_n, \ldots, s_1)$ and $(t_m, \ldots, t_1)$ are irreducible, so no subword of the form $s_{i+1} s_i$ (respectively $t_{i+1} t_i$) is reducible, and thus $(s_n, \ldots, s_1)$ and $(t_m, \ldots, t_1)$ are indeed totally η-stable. The recognition principle now tells us that $(s_n \cdot \ldots \cdot s_2, s_1) = \eta(x) = (t_m \cdot \ldots \cdot t_2, t_1)$. Iterating this argument, we find $(s_n, \ldots, s_1) = (t_m, \ldots, t_1)$. $\square$

To conclude part (a) of Theorem 3.4.1, we need to check noetherianity of the rewriting system $(S, \mathcal{R})$. This requires some preparation.

Recall the finite sequence $I_1^n = (1, 2, \dots, n-1, n) \in F_n$ and denote by $(I_1^n)^n$ the n-fold composition $I_1^n. \dots .I_1^n$. The crucial ingredient for noetherianity is the following result, which has originally been conjectured by the author. The proof is due to Ozornova, see [Ozo].

Proposition 3.4.4 (Ozornova) *Let $I \in F_n$ be a finite sequence and assume that $(I_1^n)^n$ is a subsequence of I. We then have the following equality in the graded sense:*

$$f_I \equiv f_{\overleftarrow{D_n}} : X^{n+1} \longrightarrow X^{n+1}$$

Let us make some remarks on the proof of Proposition 3.4.4. If we could show that $I \sim_Q \overleftarrow{D_n}$, then Proposition 3.4.4 would directly follow from the Evaluation Lemma. However, this is not true in general. For example we have $(1,2,3,2,1,2,3,2,1,2,3) \not\sim_Q \overleftarrow{D_3}$, cf. Example 2.3.8.(b). Ozornova introduces new monoids Q_n' which arise from the monoids Q_n by forcing further cancellation rules: For a finite sequence $J \in F_n$ and a letter $k = 1, \dots, n$ she defines $(k).J \sim_{Q_n'} J$ if there exists a finite sequence $I = (i_s, \dots, i_1)$ that does not contain k (meaning that $i_t \neq k$ for all $s \geq t \geq 1$) and $(k).I.J \sim_{Q_n} I.J$. These new cancellation rules are reasonable in the sense that each Q_n' admits an Evaluation Lemma, that is, $I \sim_{Q'} J$ implies $f_I \equiv f_J$. To conclude Proposition 3.4.4, Ozornova then shows that under the above hypothesis one has $I \sim_{Q'} \overleftarrow{D_n}$. The proof of the latter statement is by a tricky inductive argument.

Example 3.4.5 Let us, for convenience, argue why $(1,2,3,2,1,2,3,2,1,2,3) \sim_{Q'} \overleftarrow{D_3}$. Set $k = 1$, $J = (2,3,2,1,2,3)$ and $I = (3)$. Then

$$I.J = (3,2,3,2,1,2,3) \sim_Q (3,2,3,1,2,3) = \overleftarrow{D_3},$$

and hence $(k).I.J \sim_Q I.J$ (because $\overleftarrow{D_3}$ is a representative of the absorbing element Δ_3). Thus, by the definition of Q', we have $(k).J \sim_{Q'} J$, which reads as $(1,2,3,2,1,2,3) \sim_{Q'} (2,3,2,1,2,3)$. Applying this twice gives

$$\begin{aligned} (1,2,3,2,1,2,3,2,1,2,3) &\sim_{Q'} (2,3,2,1,2,3,2,1,2,3) \\ &\sim_{Q'} (2,3,2,2,3,2,1,2,3) \\ &\sim_Q (3,2,3,1,2,3) = \overleftarrow{D_3}. \end{aligned}$$

Definition 3.4.6 Let $\underline{x} \in X^{n+1}$. A finite sequence $(i_s, \dots, i_1) \in F_n$ is called *$\underline{x}$-effective* if for all t with $s > t \geq 1$ the following holds:

$$f_{i_{t+1}} f_{i_t} \dots f_{i_1}(\underline{x}) \neq f_{i_t} \dots f_{i_1}(\underline{x}).$$

We say that a finite sequence is effective if it is $\underline{x}$-effective for some suitable cell $\underline{x}$.

Clearly, every effective sequence is reduced.

Remark 3.4.7 Consider i, j such that $|i-j| \geq 2$. Then the finite sequence (i, j) is $\underline{x}$-effective if and only if (j, i) is $\underline{x}$-effective. In particular, $\underline{x}$-effectiveness in invariant under the distant commutativity relation $\sim_{\text{dist}}$. It follows that if I is $\underline{x}$-effective then so are its left- and right-most representatives $\overleftarrow{I}$ resp. $\overrightarrow{I}$.

Noetherianity of the rewriting system $(S, \mathcal{R})$ will follow from the fact that in each F_n there are only finitely many effective sequences. More precisely, we show that for each n there is an upper bound c_n for the lengths of effective sequences in F_n.

Definition 3.4.8 Set $c_1 := 1$ and for $n \geq 2$ we inductively define

$$c_n := 2n \cdot (1 + c_{n-1}).$$

Lemma 3.4.9 *Let $\underline{x} \in X^{n+1}$ and consider $I = (i_s, \ldots, i_1) \in F_n$ with the following properties:*

(a) I is left-most.

(b) I is $\underline{x}$-effective.

(c) I is norm-preserving for $\underline{x}$.

Then $\#I \leq c_n$.

Proof. The proof is by induction on n. For $n = 1$ the Lemma is clearly true: Recall that every $\underline{x}$-effective sequence is reduced. Furthermore, the only reduced sequences in F_1 are ( ) and (1), which both have length $\leq c_1$.

Assume now that the Lemma has been proven for all indices $< n$ and let $\underline{x} \in X^{n+1}$ and $I \in F_n$ be as above. We partition I into several building blocks as follows:

$$I = \ldots.(1).I_3.(n).I_2.(1).I_1.(n).I_0, \tag{3.23}$$

where $I_k \in F_{n-1}$ if k is even, and $I_k \in \text{shift}_1(F_{n-1})$ if k is odd. In other words, for even k's the blocks I_k do not contain the entry n, and for odd k's the blocks I_k do not contain the entry 1. Intuitively, we scan I from right to left until the first occurence of the value n. We then define I_0 to be the largest prefix of I that does not contain the entry n. We then continue and scan for the first occurence of the value 1, and so on.

Note that the right-hand side of (3.23) is uniquely determined.

Since I is effective and norm-preserving with respect to $\underline{x}$, I_0 is effective and norm-preserving with respect to $\underline{x}$, and I_1 is effective and norm-preserving with respect to $f_n(f_{I_0}(\underline{x}))$, etc. Furthermore, being a connected subsequence of I, every I_k is left-most. Thus, by the induction hypothesis (possibly applied with a shift), each building block I_k satisfies $\#I_k \leq c_{n-1}$.

Assume now that $\#I > c_n = 2n \cdot (1 + c_{n-1})$. In particular, I consists of at least $2n + 1$ building blocks I_k, and thus the following definition makes sense:

$$J := (1).I_{2n-1}.(n).I_{2n-2}.\ldots.(1).I_1.(n).I_0.$$

Clearly, J is a prefix of I, and we have $J = (i_t, \ldots, i_1)$ for some $t < s$. Also note that J is left-most as well as effective and norm-preserving with respect to $\underline{x}$. It is easily seen that $(1,n)^n$ is a (not necessarily connected) subsequence of J. Since J is left-most, the latter implies that $(I_1^n)^n \subset J$, compare Lemma 2.3.3.

From Proposition 3.4.4 we conclude that $f_J(\underline{x})$ is everywhere stable. Hence $f_{i_{t+1}} \circ f_J(\underline{x}) = f_J(\underline{x})$, contradicting $\underline{x}$-effectiveness of I. Therefore, our assumption $\#I > c_n$ is wrong. This finishes the proof of the induction step. The Lemma is proven. □

Lemma 3.4.10 *Let (X, S, η) be a factorable monoid. Then the associated rewriting system $(S, \mathcal{R})$ is noetherian.*

Proof. Assume that the Lemma is wrong, i.e. there exists an infinite chain or rewritings $w_1 \to_{\mathcal{R}} w_2 \to_{\mathcal{R}} \ldots$. By the definition of the rewriting system $(S, \mathcal{R})$, this is equivalent to saying that there exists an $n \geq 0$, such that there is a cell $\underline{x} \in X^{n+1}$ with entries in the generating set S and an infinite sequence $(\ldots, i_2, i_1)$ which is $\underline{x}$-effective, meaning that every prefix $(i_s, \ldots, i_1)$ is $\underline{x}$-effective.

Recall that applying f_i to a cell either preserves its norm or strictly lowers its norm. It follows that there exists $T \geq 1$ such that the sequence $(\ldots, i_{T+1}, i_T)$ is norm-preserving for $\underline{y} := f_{i_{T-1}} \circ \ldots \circ f_{i_1}(\underline{x})$. Define $I := (i_{T+c_n}, \ldots, i_T)$ and note that I is effective and norm-preserving for $\underline{y}$.

We claim that $\overleftarrow{I}$ provides a counterexample to Lemma 3.4.9 (and thus our above assumption must be wrong): Clearly, $\overleftarrow{I}$ is left-most. By definition we have $\overleftarrow{I} \sim_P I$, and the Evaluation Lemma gives $f_{\overleftarrow{I}} = f_I$, so $\overleftarrow{I}$ is norm-preserving for $\underline{y}$. Furthermore, by Remark 3.4.7, $\overleftarrow{I}$ is $\underline{y}$-effective. To get the desired contradiction, we have to argue that $\#\overleftarrow{I} \geq \#I = c_n + 1$. For this, it suffices to show that I is a representative of minimal length of $[I]_P$. Assume this is wrong. Then there exists a chain of relations $I = J_1 \sim_{\text{dist}} \ldots \sim_{\text{dist}} J_k$ with J_k not reduced. On the other hand, by Remark 3.4.7, effectiveness is invariant under $\sim_{\text{dist}}$, and we know that every effective sequence is reduced. So J_k is reduced, contradicting our previous statement. So $\#\overleftarrow{I} \geq \#I$ and we are done. □

Putting together Lemmas 3.4.3 and 3.4.10, we get that the rewriting system associated to a factorable monoid is complete. Clearly, if the generating set S is finite, then so is the set of rewriting rules $\mathcal{R}$ and in this case the rewriting system $(S, \mathcal{R})$ is finite. Part (a) of Theorem 3.4.1 is proven.

We now prove part (b).

Let (X, S, η) be a factorable monoid and denote by $(S, \mathcal{R})$ the associated complete rewriting system. To avoid confusion, we introduce the following vocabulary.

Definition 3.4.11 Let $\underline{x}$ be a cell.

- We say that $\underline{x}$ is *η-essential*, *η-collapsible* or *η-redundant*, respectively, if it is essential, collapsible or redundant with respect to the matching induced by the factorability structure on X, cf. Theorem 3.1.8.

- We say that $\underline{x}$ is *$\mathcal{R}$-essential*, *$\mathcal{R}$-collapsible* or *$\mathcal{R}$-redundant*, respectively, if it is essential, collapsible or redundant with respect to Brown's matching constructed out of the complete rewriting system on X, cf. page 45.

Part (b) will be proven in two steps. First, we are going to show that a cell is η-essential if and only if it is $\mathcal{R}$-essential, and so on. Using this, we then show that the matching functions are indeed equal.

Lemma 3.4.12 *A cell $\underline{x}$ is η-essential, η-collapsible or η-redundant if and only if it is $\mathcal{R}$-essential, $\mathcal{R}$-collapsible or $\mathcal{R}$-redundant, respectively.*

Proof. Observe that it suffices to prove only one implication, because every cell is either essential or collapsible or redundant. We are going to prove the "only if"-part. Let $[x_n|\dots|x_1]$ be an n-cell and denote by $w_i \in S^*$ the η-normal form of x_i. We distinguish three cases.

- $[x_n|\dots|x_1]$ is η-essential.

  In particular, for every i we have $x_i \in S_+$ and $w_i = x_i$. By η-essentiality, each pair (w_{i+1}, w_i) is unstable, hence contributes a rewriting rule. It follows that the word $w_{i+1}w_i \in S^*$ is reducible. Furthermore, since $w_{i+1}w_i$ has length 2, $w_{i+1}w_i$ cannot have any reducible proper prefix. Therefore, $[x_n|\dots|x_1]$ is $\mathcal{R}$-essential.

- $[x_n|\dots|x_1]$ is η-collapsible of height h.

  Recall that $h > 0$ (see e.g. Remark 3.1.7) and that (x_{h+1}, x_h) is η-stable. The truncated cell $[x_h|\dots|x_1]$ is η-essential, hence $\mathcal{R}$-essential. Note that $x_h \in S$. Denote by $(s_k, \dots, s_1)$ the normal form of x_{h+1}, i.e. $s_k \dots s_1 = w_{h+1} \in S^*$. We need to show that the word $w_{h+1}w_h = s_k \dots s_1 x_h$ is irreducible. For this we need to show that the tuple $(s_k, \dots, s_1, x_h)$ is totally η-stable. Since $(s_k, \dots, s_1)$ is an η-normal form, this is clear at every position > 1. So it remains to show that the pair (s_1, x_h) is η-stable: We have $(x_{h+1})' = s_1$, and the recognition principle tells us that $((x_{h+1})', x_h)$ is stable (because (x_{h+1}, x_h) is).

- $[x_n|\dots|x_1]$ is η-redundant of height h.

  The truncated cell $[x_h|\dots|x_1]$ is η-essential, hence $\mathcal{R}$-essential. In particular, $x_h \in S$. Furthermore, the pair (x_{h+1}, x_h) is unstable and $\ell(x_{h+1}) > 1$. Denote by $(s_k, \dots, s_1)$ the normal form of x_{h+1}. We need to show that some proper prefix of $w_{h+1}w_h = s_k \dots s_1 x_1$ is reducible. We claim that the word $s_1 x_h \in S^*$ is reducible. (Note that $s_1 x_h$ is a proper subword of $w_{h+1}w_h$, since $\ell(x_{h+1}) > 1$.) Indeed, we know that the pair (x_{h+1}, x_h) is η-unstable, and the recognition principle states that the pair $((x_{h+1})', x_h) = (s_1, x_h)$ is η-unstable, hence reducible.

The Lemma is proven. $\square$

To conclude part (b), we need to argue that the matching functions coincide. Recall that every matching is an involution. It therefore suffices to check that the respective matching functions coincide for essential and collapsible cells, and this is easily seen by comparing (1.11) on page 46 and (3.1) on page 95.

Theorem 3.4.1 is proven. □

Remark 3.4.13 Theorem 3.4.1 (together with Theorem 1.2.8) provides an alternative way of proving Theorem 3.1.8 within the world of rewriting systems. However, we found it convenient to explicitly describe the construction of the matching function μ out of a factorability structure: The organization we chose makes this chapter self-contained, and it allows to regard factorability independently of the theory of rewriting systems. Furthermore, constructing μ explicitly allowed us to introduce basic methods and tools for factorability, which were later needed to derive formulas for the differential in the associated Morse complex. Last but not least, the organization of this chapter reflects the chronological development of this thesis: We first found the matching for factorable monoids and only later saw how it fits into the framework of rewriting systems.

4 Applications to generalized Thompson groups and monoids

In this chapter we are going to use our previous results to compute the homology of a certain family of monoids. For some of these monoids our computations carry over to their respective groups of fractions.

4.1 Homology of groups via homology of monoids

Let X be a monoid. A *group of fractions* for X is a group $G(X)$ together with a morphism of monoids $i : X \to G(X)$ satisfying the following universal property: For every group H and every morphism of monoids $f : X \to H$ there is a unique morphism of groups $g : G(X) \to H$ such that $f = g \circ i$.

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & G(X) \arrow[dl, dotted, "\exists!\, g"] \\ H & \end{tikzcd}$$

Evidently, $G(X)$ is unique up to unique isomorphism, and if the map $i : X \to G(X)$ is clear, then we will sometimes just say that $G(X)$ is the group of fractions of X.

It is well-known that every monoid X possesses a group of fractions: Let $X = \langle S \mid R \rangle$ be a presentation of X. Set $G = \langle S \mid R \rangle_{\mathrm{Grp}}$, i.e. here we consider $\langle S \mid R \rangle$ as a group presentation. Then $i : X \to G$ is a group of fractions, where i is the canonical map induced by the identity on S^*.

The map $i : X \to G(X)$ will in general not be injective. For example, in P_n (and also in Q_n) every element of the generating set $S = \{[1]_P, \dots, [n]_P\}$ is idempotent, and hence $G(P_n) = 0$.

An obvious necessary condition for injectivity of $i : X \to G(X)$ is that X is cancellative. However, cancellativity of X is not sufficient. Indeed, in [Mal37, §2], Malcev introduces the following monoid:

$$X = \langle a, b, c, d, a', b', c', d' \mid ac = a'c', ad = a'd', bc = b'c' \rangle$$

Loc. cit. shows that X is cancellative and that $bd \neq b'd'$ (in X), but $i(bd) = i(b'd')$, where $i : X \to G(X)$ is a group of fractions.

Definition 4.1.1 A monoid X satisfies the *right Ore condition*, if any two elements have a common right multiple, i.e. if for all $x, x' \in X$ there exist $y, y' \in X$ such that

$$xy = x'y'.$$

The analogous left Ore condition will not play a role in this work, and thus we will often suppress the adjective "right".

Clearly, every abelian monoid satisfies the Ore condition.

Theorem 4.1.2 (Ore's Embedding Theorem) *Let X be a monoid and $i : X \to G(X)$ a group of fractions. If X is cancellative and satisfies the right Ore condition, then $i : X \to G(X)$ is injective.*

Proof. See e.g. Theorem 1.23 in Clifford–Preston [CP61]. □

The following is Proposition 4.1 in Cartan–Eilenberg [CE99, Chapter X].

Theorem 4.1.3 (Cartan–Eilenberg) *Let X be a monoid and $i : X \to G(X)$ a group of fractions. If X is cancellative and satisfies the right Ore condition, then $i : X \to G(X)$ induces isomorphisms on homology and cohomology groups,*

$$H_*(i) : H_*(X) \xrightarrow{\cong} H_*(G(X)),$$
$$H^*(i) : H^*(G(X)) \xrightarrow{\cong} H^*(X).$$

Theorem 4.1.3 allows us, under certain circumstances, to compute the homology of a group G by computing the homology of a certain submonoid. This has the following advantage. For "most" factorable groups, the associated Visy complex $\mathbb{V}_*$ grows super-linearly, meaning that the rank sequence $(\mathrm{rk}(\mathbb{V}_1), \mathrm{rk}(\mathbb{V}_2), \ldots)$ does not lie in $\mathcal{O}(n)$, cf. Rodenhausen [Rod], [Rod11].[1] If, in contrast, X is a monoid without any non-trivial invertible elements, then it can happen that the Visy complex is finite.

We illustrate this idea with a very basic example. The reader should not mind the fact that the homology of the monoid and group occuring in the example can be computed much easier by writing down explicit resolutions. We will see more interesting examples in Section 4.3.

Example 4.1.4 Consider $X = \mathbb{N}$ with group of fractions $G(X) = \mathbb{Z}$. Clearly, $\mathbb{N}$ is cancellative and commutative, and thus $i_* : H_*(\mathbb{N}) \to H_*(\mathbb{Z})$ is an isomorphism. Write $\mathbb{V}_*\mathbb{Z}$ for the Visy complex associated to the factorability structure on $\mathbb{Z}$ with respect to $S = \{\pm 1\}$, cf. Example 2.1.13. In every positive degree, the Visy complex of $\mathbb{Z}$ is freely generated by the "alternating cells" $[\ldots | -1 | +1 | -1 | +1]$ and $[\ldots | +1 | -1 | +1 | -1]$, whence infinite. Equip $\mathbb{N}$ with the factorability structure discussed in Example 2.2.9 for $m = 1$, i.e. $S = \{1\}$. The tuple $(1, 1)$ is stable, and therefore the Visy complex vanishes in positive degrees. Indeed, the Visy complex $\mathbb{V}_*\mathbb{N}$ is isomorphic to the homology of $\mathbb{N}$.

[1] Indeed, it is indicated in [Rod11] that if G has Visy complex of at most linear growth then G is a so-called *horizontal-vertical group*, for short *hv-group*.

One aim of this chapter is to use the idea described above, to recompute the homology of an infinite family of groups that has been introduced by Brown [Bro87]. One of these groups is Thompson's group F. In particular, we provide a purely combinatorial way of computing the homology of F.

4.2 Braid factorability

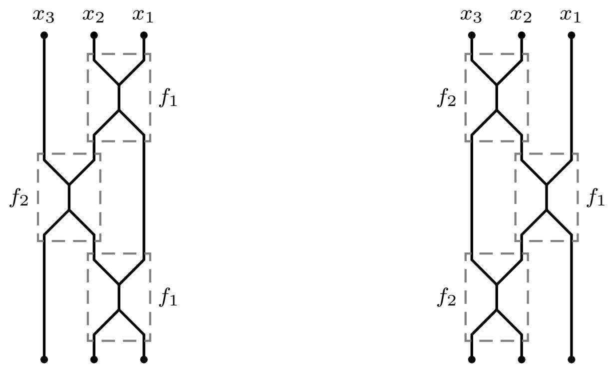
In this section we investigate a very special class of factorable monoids for which the differentials in the associated Visy complex can be described much simpler.

Definition 4.2.1 Let X be a monoid, S a generating set, and $\eta : X \to X \times X$ a factorization map. For $0 < i < n$ set $f_i = \eta_i \circ d_i : X^n \to X^n$, compare (2.1). We say that (X, S, η) is *braid factorable* if

$$f_1 f_2 f_1 \equiv f_2 f_1 f_2 : X^3 \to X^3. \tag{4.1}$$

Clearly, by idempotence of the f_i's, every braid factorable monoid is factorable.

In Figure 4.1 we visualize the compositions occuring in (4.1).



(a) $f_1 f_2 f_1$ (b) $f_2 f_1 f_2$

Figure 4.1: Visualization of the compositions occuring in (4.1).

Example 4.2.2 (a) The trivial factorability structure on a monoid is a braid factorability structure.

(b) Rodenhausen [Rod] proves that the free and direct product of braid factorable monoids is again braid factorable.

In contrast, the semidirect product of two braid factorable monoids will not be braid factorable in general. The reason for this is that the action $X \to \mathrm{End}(Y)$ might not preserve stability, i.e. stable pairs might be sent to unstable ones, cf. [Rod].

(c) It is easily seen that the map $\eta(n) = (n - \min\{n,1\}, \min\{n,1\})$ from Example 2.2.9 (with $m = 1$) endows $(\mathbb{N}, \{1\})$ with a braid factorability structure. Together with part (b) this shows that the canonical factorability structure on free and free abelian monoids is indeed a braid factorability structure.

(d) Ozornova [Ozo] proves that the construction described in Example 2.2.8 actually defines a braid factorability structure on Garside monoids.

(e) Recall from Example 2.1.14 our standard factorability structure on the symmetric groups $\mathcal{S}_n$. For $n \geq 3$ this factorability structure is not a braid factorability structure, since for example

$$f_1 f_2 f_1((23),(13),(12)) = ((12),(12),(13)),$$
$$f_2 f_1 f_2((23),(13),(12)) = (\mathrm{id}, \mathrm{id}, (13)).$$

In particular, $f_1 f_2 f_1$ is norm-preserving for $((23),(13),(12))$, but $f_2 f_1 f_2$ is not. Therefore $f_1 f_2 f_1 \not\equiv f_2 f_1 f_2 : X^3 \to X^3$.

Remark 4.2.3 Rodenhausen [Rod] gives the following characterization of braid factorability. Let (X, S, η) be a factorable monoid. Then (X, S, η) is braid factorable if and only if for every word $(s_n, \dots, s_1) \in S^*$ the following holds: $(s_n, \dots, s_1)$ is an η-normal form if and only if in the opposite monoid the word $(s_1^{-1}, \dots, s_n^{-1})$ is an η-normal form.

4.2.1 The monoids Z_n

As for factorability, we can characterize braid factorability by certain monoid actions. For $n \geq 0$ define the monoid Z_n as the quotient[2]

$$Z_n = P_n \,/\, \langle \sim_{\text{braid}} \rangle,$$

where $[i, i+1, i]_P \sim_{\text{braid}} [i+1, i, i+1]_P$ for all i, $1 \leq i < n$, see Figure 4.2.

Figure 4.2: Visualization of $\sim_{\text{braid}}$.

Explicitly writing down the presentation $P_n \,/\, \langle \sim_{\text{braid}} \rangle$ one finds

$$Z_n \cong \langle z_1, \dots, z_n \mid z_i^2 = z_i,\ z_i z_j = z_j z_i \text{ for } |i-j| \geq 2,\ z_{i+1} z_i z_{i+1} = z_i z_{i+1} z_i \rangle,$$

and hence Z_n is a so-called *Coxeter monoid* of type A_n, see e.g. Tsaranov [Tsa90]. Furthermore, observe that Z_n is a quotient of Q_n. We have the following analogue to Proposition 2.2.17:

[2]We use the letter Z, because Z_n will detect braid factorability (german Zopffaktorabilität).

Proposition 4.2.4 *Let X be a monoid, S a generating set and $\eta : X \to X \times X$ a factorization map. If (X, S, η) is a factorable monoid then for every $n \geq 0$ and all $h \geq 0$ the action $f : P_{n-1} \to \mathrm{End}(\mathcal{G}_h \overline{\mathbb{B}}_n X)$ factors through Z_{n-1}.*

We omit the proof.

Remark 4.2.5 (a) Denote by Γ_n the undirected left Cayley graph of Z_n (with respect to the canonical generating set $\{[1]_Z, \ldots, [n]_Z\}$) without reflexive edges. More precisely, as vertex set we take all elements of Z_n, and we draw an undirected edge between α and β if $\alpha \neq \beta$ and if $\beta = [k]_Z \cdot \alpha$ for some k, $1 \leq k \leq n$. In Figure 4.3 we depict Γ_3.

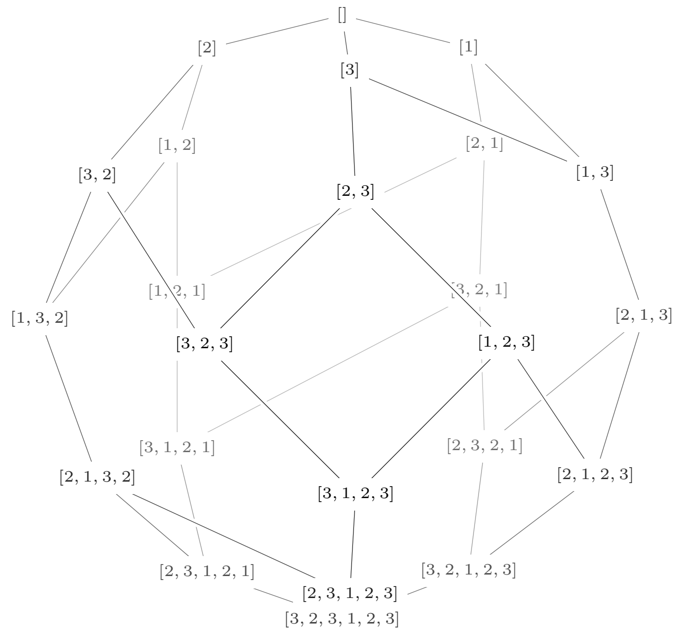


Figure 4.3: Undirected left Cayley graph of Z_3 without reflexive edges.

Recall that Z_n is a Coxeter monoid of type A_n. The symmetric group $\mathcal{S}_{n+1}$ is a Coxeter group of type A_n, and we claim that Γ_n is isomorphic to the undirected Cayley graph of the $(n{+}1)$-st symmetric group $\mathcal{S}_{n+1}$ with respect to its Coxeter generating set, which consists of the elementary transpositions $\mathcal{E} = \{(12), (23), \ldots, (n\ n{+}1)\}$. Define a map $\mathcal{S}_{n+1} \to Z_n$ as follows. For $\sigma \in \mathcal{S}_{n+1}$ choose a representative $(i_s\ i_s{+}1) \circ \ldots \circ (i_1\ i_1{+}1)$ that is geodesic, i.e. $s = \ell_{\mathcal{E}}(\sigma)$, and set

$$\sigma \longmapsto [i_s, \ldots, i_1]_Z.$$

This is well-defined, and it gives a bijection between the elements of $\mathcal{S}_{n+1}$ and Z_n. It is now not difficult to check that this map realizes an isomorphism between the respective Cayley graphs.

(b) By part (a) we have $\#Z_n = (n+1)!$, and therefore, for a braid factorable monoid (X, S, η), noetherianity of the associated rewriting system $(S, \mathcal{R})$ is immediate, compare Lemma 3.4.10.

Our next aim is to give an easy description of the Visy differential for braid factorable monoids that are 2-balanced. We need some preparation first.

Denote by Ω_n the set of all n-tuples $[x_n|\dots|x_1]$ with non-trivial entries. Recall the projection onto essential cells $\pi_n : \overline{\mathbb{B}}_n \to \overline{\mathbb{B}}_n^\theta$.

Lemma 4.2.6 *Let (X, S, η) be a factorable monoid. Let $\underline{x} \in \Omega_n$ have elementary partition type and consider $I \in F_{n-1}$. If X is 2-balanced with respect to S, then for every i, $n > i > 0$, we have $\pi_{n-1}(d_i f_I(\underline{x})) = 0$.*

Proof. Since X is 2-balanced, we have

$$\mathrm{norm}(\underline{x}) \equiv \mathrm{norm}(f_I(\underline{x})) \mod 2.$$

Note that for $n > i > 0$ we obtain the i-th face by multiplying the $(i+1)$-st and i-th entry of $f_I(\underline{x})$, and thus

$$\mathrm{norm}(f_I(\underline{x})) \equiv \mathrm{norm}(d_i f_I(\underline{x})) \mod 2.$$

We conclude that $\mathrm{norm}(d_i f_I(\underline{x})) \neq \mathrm{norm}(\underline{x}) - 1 = n - 1$ and hence $\pi_{n-1}(d_i(f_I(\underline{x}))) = 0$. The Lemma is proven. $\square$

Lemma 4.2.7 *Let (X, S, η) be a factorable monoid. Let $\underline{x} \in \Omega_n$ have elementary partition type and consider $I \in F_{n-1}$. If one of the following holds,*

- *$f_I(\underline{x})$ is stable at at least two positions or*
- *$f_I(\underline{x})$ is stable at an inner position, i.e. $f_I(\underline{x})$ is stable at some position i with $n-1 > i > 1$,*

then $\pi_{n-1}(d_0 f_I(\underline{x})) = 0$ and $\pi_{n-1}(d_n f_I(\underline{x})) = 0$.

Proof. The proof is immediate, because in any of these cases $d_0 f_I(\underline{x})$ resp. $d_n f_I(\underline{x})$ is stable at at least one position, and hence not essential. $\square$

Recall $\Phi_a^b = f_a \circ \dots \circ f_{b-2} \circ f_{b-1}$ and define its "opposite" $\Psi_b^a = f_{b-1} \circ \dots \circ f_{a+1} \circ f_a$. Note that for $a \geq b$ we have $\Phi_a^b = \mathrm{id}$ and $\Psi_b^a = \mathrm{id}$.

Proposition 4.2.8 *Let (X, S, η) be a braid factorable monoid and assume that X is 2-balanced with respect to S. Then the differential in the Visy complex $(\mathbb{V}_*, \partial_*^{\mathbb{V}})$ can be written as*

$$\partial_n^{\mathbb{V}} = \pi_{n-1} \circ \sum_{i=1}^{n} (-1)^i \cdot \left(d_n \Psi_n^i - d_0 \Phi_1^i\right) \circ i_n. \tag{4.2}$$

We are in a similar situation as we were in Proposition 3.2.7. There we showed that we can actually sum over a larger indexing set, because the additional terms cancelled out orbitwise. Here the situation is a little simpler, because each additional term will already be zero.

Proof. First of all, for $n = 1, 2$ the Proposition is easily seen to hold true: For $n = 1$ the right-hand side of (4.2) computes to $\pi_0 \circ (d_0 - d_1) \circ \mathrm{id} \circ i_1 = \pi_0 \circ \overline{\partial}_1 \circ f_{()} \circ i_1$, which is equal to $\partial_1^{\mathbb{V}}$, because the empty sequence $()$ is the only element in $\overrightarrow{\Lambda}_1$, cf. Theorem 3.2.9. For $n = 2$ the right-hand side of (4.2) computes to $\pi_0 \circ (d_0 + d_2) \circ (\mathrm{id} - f_1) \circ i_2$, which by Lemma 4.2.6 and Theorem 3.2.9 is equal to $\partial_2^{\mathbb{V}}$.

In what follows we assume that $n \geq 3$.

We introduce the following vocabulary. A finite sequence $(i_s, \dots, i_1)$ is called *ascending* (resp. *descending*) if $i_{t+1} = i_t + 1$ (resp. $i_{t+1} = i_t - 1$) for all t, $s > t \geq 1$. Observe that if I is ascending (resp. descending) then $f_I = \Psi_b^a$ (resp. $f_I = \Phi_a^b$) for some $b \geq a$.

Let $\underline{x}$ be an n-cell of elementary partition type.

> **Claim.** If $I \in \overrightarrow{\Lambda}_{n-1}$ is neither ascending nor descending then
> $\pi_{n-1}((d_0 + (-1)^n d_n) \circ f_I(\underline{x})) = 0$.

Obviously, every sequence of length < 2 is ascending (and descending). From now on we only consider sequences $I = (i_s, \dots, i_1)$ with $s \geq 2$. W.l.o.g. we may assume that I is norm-preserving for $\underline{x}$, because otherwise $f_I(\underline{x}) = 0$ in $\overline{\mathbb{E}}_n X$. Furthermore, by Lemma 4.2.7, we may assume that $i_s = 1$ or $i_s = n - 1$ (because otherwise $f_I(\underline{x})$ is stable at an iner position).

Additionally assume that I is neither ascending nor descending. To prove the Claim, it suffices to show that $f_I(\underline{x})$ is stable at at least two positions, cf. Lemma 4.2.7.

Denote by k the smallest index such that the connected subsequence $(i_s, \dots, i_k)$ is ascending or descending, i.e. we consider the largest "tail" of $(i_n, \dots, i_1)$ which is ascending or descending. Clearly, $s > k > 1$. We distinguish three cases. In each of these cases we will use an Evaluation Lemma for $\sim_Z$, compare Remark 2.2.19. It states that, since I is norm-preserving for $\underline{x}$, we have that if $I \sim_Z J$ then $f_I(\underline{x}) = f_J(\underline{x})$.

- Case 1: $i_s = n - 1$.

  Since $(i_s, \dots, i_1)$ is right-most and reduced, it follows that $i_k < i_{k-1}$, see Figure 4.4. We have $(i_s, \dots, i_{k-1}) \sim_Q (i_{k-1}-1).(i_s, \dots, i_{k-1})$, and therefore $f_I(\underline{x})$ is stable at positions $i_s = n-1$ and $i_{k-1}-1 < n-1$. Thus $\pi_{n-1}((d_0 + (-1)^n d_n) \circ f_I(\underline{x})) = 0$.

- Case 2: $i_s = 1$.

  We have $i_k \leq i_{k-1}+1$, because $(i_s, \dots, i_1)$ is right-most. Since $(i_s, \dots, i_1)$ is reduced and $(i_s, \dots, i_{k-1})$ is neither ascending nor descending, it follows that $i_{k-1} > i_k + 1$ or $i_{k-1} = i_k - 1$.

  - Case 2.1: $i_{k-1} > i_k + 1$.

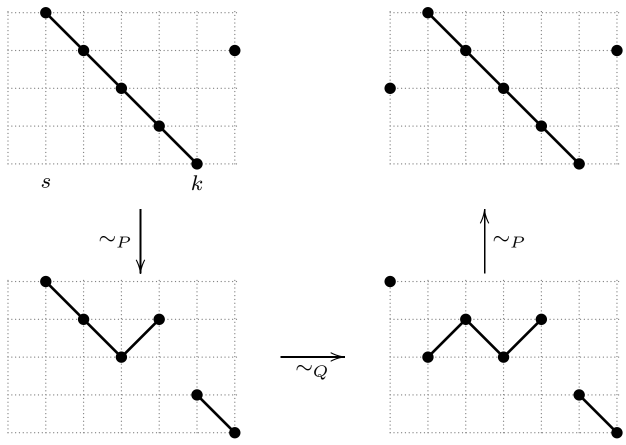


Figure 4.4: Case 1: $i_s = n-1$.

Obviously, $(i_s, \ldots, i_{k-1}) \sim_P (i_{k-1}).(i_s, \ldots, i_k)$, see Figure 4.5. Hence, $f_I(\underline{x})$ is stable at positions $i_s = 1$ and $i_{k-1} > 2$, yielding $\pi_{n-1}((d_0+(-1)^n d_n) \circ f_I(\underline{x})) = 0$.

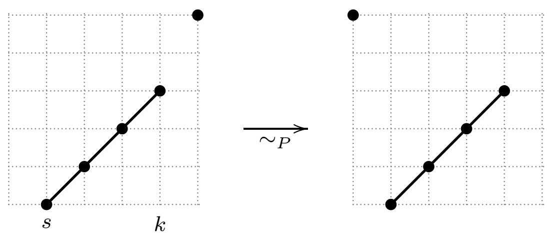


Figure 4.5: Case 2.1: $i_s = 1$ and $i_{k-1} > i_k + 1$.

- Case 2.2: $i_{k-1} = i_k - 1$.

  This is the only case where we actually need that (X, S, η) is *braid* factorable. We have $(i_s, \ldots, i_{k-1}) \sim_Z (i_k).(i_s, \ldots, i_{k-1})$, see Figure 4.6. It follows that $f_I(\underline{x})$ is stable at positions $i_s = 1$ and $i_k > 1$, and therefore we have $\pi_{n-1}((d_0 + (-1)^n d_n) \circ f_I(\underline{x})) = 0$.

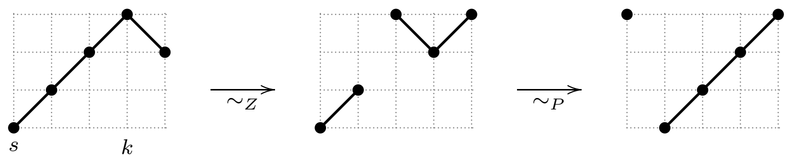


Figure 4.6: Case 2.2: $i_s = 1$ and $i_{k-1} = i_k - 1$.

The Claim is proven.

Denote by $M_{n-1} \subseteq F_{n-1}$ the set of ascending or descending finite sequences $(i_s, \dots, i_1)$ with either $s = 0$ or $i_s = n-1$ or $i_s = 1$. Note that $M_{n-1} \subseteq \overrightarrow{\Lambda}_{n-1}$ and recall that $\overrightarrow{\Lambda}_{n-1}$ is a set of representatives for $Q_{n-1} \smallsetminus \square_{n-1}$. Putting together Theorem 3.2.9, Lemma 4.2.6 and the Claim we obtain

$$\partial_n^{\mathbb{V}} = \pi_{n-1} \circ (d_0 + (-1)^n d_n) \circ \sum_{I \in M_{n-1}} (-1)^{\#I} \cdot f_I \circ i_n, \tag{4.3}$$

Using the above description of ascending resp. descending sequences via Ψ_b^a resp. Φ_a^b we can write

$$\sum_{I \in M_{n-1}} (-1)^{\#I} \cdot f_I = \mathrm{id} + \sum_{i=1}^{n-1} (-1)^{n-i} \Psi_n^i + \sum_{i=2}^{n} (-1)^{i-1} \Phi_1^i. \tag{4.4}$$

Plugging in (4.4) into (4.3) we obtain

$$\partial_n^{\mathbb{V}} = \pi_{n-1} \circ (d_0 + (-1)^n d_n) \circ \left[\mathrm{id} + \sum_{i=1}^{n-1} (-1)^{n-i} \Psi_n^i + \sum_{i=2}^{n} (-1)^{i-1} \Phi_1^i \right] \circ i_n.$$

Now, for $1 \le i \le n-1$ the composition Ψ_n^i ends with f_{n-1} and hence $\Psi_n^i(\underline{x})$ is stable at position $n-1$. In particular, $d_0 \Psi_n^i(\underline{x}) = 0$ is not essential and thus $\pi_{n-1}(d_0 \Psi_n^i(\underline{x})) = 0$. Similarly, $\pi_{n-1}(d_n \Phi_1^i(\underline{x})) = 0$. We therefore get

$$\partial_n^{\mathbb{V}} = \pi_{n-1} \circ \left[(d_0 + (-1)^n d_n) \circ \mathrm{id} + (-1)^n \sum_{i=1}^{n-1} (-1)^{n-i} d_n \Psi_n^i + \sum_{i=2}^{n} (-1)^{i-1} d_0 \Phi_1^i \right] \circ i_n.$$

Recall that $\Phi_1^1 = \mathrm{id}$ and $\Psi_n^n = \mathrm{id}$. This yields

$$\begin{aligned}
\partial_n^{\mathbb{V}} &= \pi_{n-1} \circ \left[d_0 \Phi_1^1 + (-1)^n d_n \Psi_n^n + \sum_{i=1}^{n-1} (-1)^i d_n \Psi_n^i - \sum_{i=2}^{n} (-1)^i d_0 \Phi_1^i \right] \circ i_n \\
&= \pi_{n-1} \circ \left[\sum_{i=1}^{n} (-1)^i d_n \Psi_n^i - \sum_{i=1}^{n} (-1)^i d_0 \Phi_1^i \right] \circ i_n \\
&= \pi_{n-1} \circ \sum_{i=1}^{n} (-1)^i \cdot (d_n \Psi_n^i - d_0 \Phi_1^i) \circ i_n.
\end{aligned}$$

The Proposition is proven. □

4.2.2 A class of braid factorable monoids

In this section we introduce a class of balanced, braid factorable monoids. These monoids arise from an abstract datum that we call “a good set of arrows”, and which should be thought of as a nice set of conjugacy relations. The results of this section will later be used to derive braid factorability of certain monoids.

Definition 4.2.9 Let S be an alphabet, possibly infinite, and let $\mathcal{A} \subseteq S_+ \times S_+ \times S_+$ be a set of triples. The elements of $\mathcal{A}$ are called *arrows*, and for $(a,b,c) \in \mathcal{A}$ we refer to a as its *head* and to (b,c) as its *tail*. (So we think of the arrows as pointing from right to left.) The monoid associated to the pair $(S,\mathcal{A})$ is defined as follows:

$$X(S,\mathcal{A}) := \langle S \mid ab = bc \text{ for every } (a,b,c) \in \mathcal{A}\rangle.$$

Note that $X(S,\mathcal{A})$ is balanced with respect to S.

Define the *source map* $\sigma : S^3 \to S^2$ and *target map* $\tau : S^3 \to S^2$ as follows:

$$\begin{aligned} \sigma(a,b,c) &= (b,c), \\ \tau(a,b,c) &= (a,b). \end{aligned}$$

To avoid confusion, we remark that the source map assigns to every arrow its tail. However, note that the target map assigns to an arrow (a,b,c) the tuple (a,b) and not only its head a.

The idea behind this concept is that we want to define a map $\phi : S^2 \to S^2$ as in Proposition 2.3.17. If (b,c) lies in the image of the source map, that is, there exists $a \in S_+$ such that $(a,b,c) \in \mathcal{A}$, then $\phi(b,c) = (a,b)$. Otherwise ϕ should fix this element. We now give a set of axioms guaranteeing that this map ϕ gives rise to a braid factorability structure on the pair (X,S).

Definition 4.2.10 Let S be an alphabet. We say that $\mathcal{A} \subseteq S_+^3$ is a *good set of arrows* (for S), if the following axioms hold:

(M1) The restriction of the source map $\sigma|_{\mathcal{A}} : \mathcal{A} \to S \times S$ is injective. Spelled out, if $(a,b,c) \in \mathcal{A}$ and $(a',b,c) \in \mathcal{A}$ then $a = a'$. In other words, the head of an arrow is uniquely determined by its tail.

(M2) The images $\sigma(\mathcal{A})$ and $\tau(\mathcal{A})$ are disjoint. Explicitly, if $(a,b,c) \in \mathcal{A}$ then there is no $s \in S_+$ such that $(s,a,b) \in \mathcal{A}$.

(D) "Dropping Axiom": If $(x,a,b) \in \mathcal{A}$ and $(y,b,c) \in \mathcal{A}$ then there exists some z such that $(z,a,c) \in \mathcal{A}$.

(E) "Extension Axiom": Consider $(c,x,c') \in \mathcal{A}$ and $(b,x,b') \in \mathcal{A}$. There exists a such that $(a,b,c) \in \mathcal{A}$ if and only if there exists a' such that $(a',b',c') \in \mathcal{A}$. Furthermore, in this case, $(a,x,a') \in \mathcal{A}$.

We say that a tuple (a,b) is $\mathcal{A}$-*unstable* if it is the tail of an arrow, i.e. if there is z such that $(z,a,b) \in \mathcal{A}$, i.e. if and only if (a,b) lies in $\sigma(\mathcal{A})$. Otherwise we call it $\mathcal{A}$-*stable*. (Note that for $\mathcal{A}$-stability we do not require (a,b) to lie in $\tau(\mathcal{A})$.)

For convenience we discuss these axioms in more detail. Let us introduce the following pictorial language. For every $(a,b,c) \in \mathcal{A}$ we draw a straight arrow of the following form:

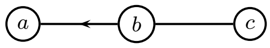
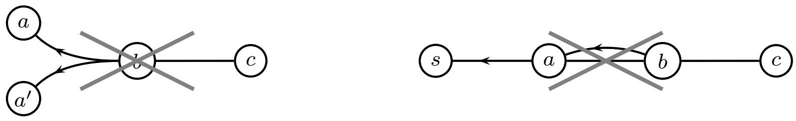


Figure 4.7: The arrow associated to $(a, b, c) \in \mathcal{A}$.

Remark 4.2.11 (a) (M1) and (M2) are minimality axioms: "Minimality" of $(S, \mathcal{A})$ refers to the following. We obtain a rewriting system for $X(S, \mathcal{A})$ by taking S as generating set and writing down a rule $ab \leftarrow bc$ for every arrow $(a, b, c) \in \mathcal{A}$. Denote by $\mathcal{R}$ the rewriting rules obtained this way. In this setting, the axioms (M1) and (M2) are equivalent to minimality of the rewriting system $(S, \mathcal{R})$, meaning that for every tail (b, c) the associated word bc is irreducible with respect to $\mathcal{R} \smallsetminus \{ab \leftarrow bc\}$, and for every target (a, b) the associated word ab is irreducible with respect to $\mathcal{R}$, cf. Figure 4.8.



(a) Axiom (M1)   (b) Axiom (M2)

Figure 4.8: Forbidden constellations.

(b) The dropping axiom can be reformulated as follows. If (a, b, c) is everywhere unstable, i.e. if (a, b) and (b, c) both are unstable, then the tuple (a, c) is also unstable. In other words, whenever we have a totally unstable tuple $(s_n, \ldots, s_1) \in S^n$, then dropping some of the s_i yields again a totally unstable tuple. Figure 4.9 offers a visualization of the dropping axiom. It has to be read as follows: Whenever the two solid arrows exist in $\mathcal{A}$, the dotted arrow also exists.

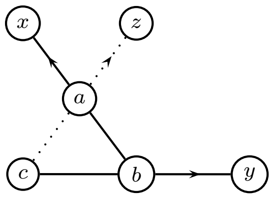
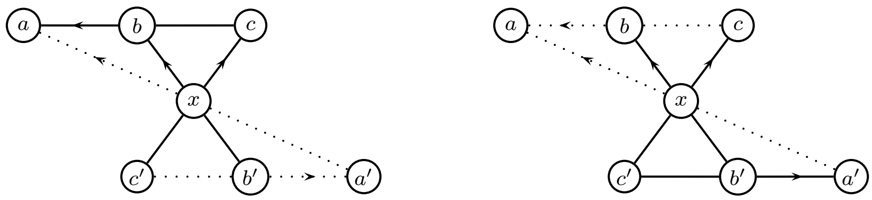


Figure 4.9: The dropping axiom.

(c) The extension axiom assures the existence of certain arrows. We depict the if- and only if-part in two seperate pictures in Figure 4.10.

Assume that (c, x, c'), $(b, x, b') \in \mathcal{A}$. The extension axiom in particular states that if (b', c') is unstable, i.e. if there exists $a' \in S$ such that $(a', b', c') \in \mathcal{A}$, then there exists $a \in S$ such that (a, b, c), and thus (b, c) is unstable.

(d) Note that the dropping axiom is related to the extension axiom. Namely, after applying the dropping axiom, we are in a position to apply extension, cf. Figure 4.10.(b). However, we think that formulating the axioms the way we did it in Definition 4.2.10 is more convenient.



(a) The only if-part. (b) The if-part.

Figure 4.10: The extension axiom.

Example 4.2.12 (a) For every alphabet S, the empty set $\varnothing$ is a good set of arrows. The associated monoid $X(S, \mathcal{A})$ is the free monoid over S.

(b) Let $S = \{s_1, s_2, \ldots\}$ be totally ordered and let $\mathcal{A} = \{(s_j, s_i, s_j) \mid i < j\}$. Then $\mathcal{A}$ is a good set of arrows for S, and the associated monoid $X(S, \mathcal{A})$ is the free abelian monoid over S.

Let S be a formal alphabet and $\mathcal{A}$ a good set of arrows. Define a map $\phi : S^2 \to S^2$ as follows. For $a \in S$ set $\phi(a, \epsilon) = \phi(\epsilon, a) = (\epsilon, a)$. For tuples in S_+ define

$$\phi(b, c) = \begin{cases} (a, b) & \text{if } (a, b, c) \in \mathcal{A} \\ (b, c) & \text{if } (b, c) \notin \sigma(\mathcal{A}). \end{cases}$$

By (M1), a is uniquely determined in the first case. Note that an element of $S_+ \times S_+$ is ϕ-stable if and only if it is $\mathcal{A}$-stable. (For convenience we remark that by axiom (M2), $\mathcal{A}$ cannot contain arrows of the form (a, a, a).)

Proposition 4.2.13 *Let S be a formal alphabet. If $\mathcal{A}$ is a good set of arrows for S then the monoid $X(S, \mathcal{A})$ is braid factorable with respect to S.*

For the proof we will use an analogue of Proposition 2.3.17 for braid factorability. In fact, we are going to show that $\phi_2\phi_1\phi_2 = \phi_1\phi_2\phi_1 : S^3 \to S^3$. It is then easily seen that ϕ gives rise to a factorization map, endowing X with the structure of a braid factorable monoid with respect to the generating set S. In the proof we only show that $\phi_2\phi_1\phi_2$ and $\phi_1\phi_2\phi_1$ are equal when considered as maps $S_+^3 \to S_+^3$. Indeed, if at least one entry of the triple (s_3, s_2, s_1) is trivial, then we automatically have $\phi_2\phi_1\phi_2(s_3, s_2, s_1) = \phi_1\phi_2\phi_1(s_3, s_2, s_1)$.

Proof. We are going to show that $\phi_2\phi_1\phi_2 = \phi_1\phi_2\phi_1 : S_+^3 \to S_+^3$. Let $(a, b, c) \in S_+^3$. The following is easily verified:

$$\phi_2\phi_1\phi_2(a, b, c) = \begin{cases} (a, b, c) & \text{if } (a, b) \text{ stable and } (b, c) \text{ stable} \\ (a, z, b) & \text{if } (a, b) \text{ stable and } (z, b, c) \in \mathcal{A} \text{ and } (a, z) \text{ stable} \\ (y, a, b) & \text{if } (a, b) \text{ stable and } (z, b, c) \in \mathcal{A} \text{ and } (y, a, z) \in \mathcal{A} \\ (x, a, c) & \text{if } (x, a, b) \in \mathcal{A} \text{ and } (a, c) \text{ stable} \\ (x, w, a) & \text{if } (x, a, b) \in \mathcal{A} \text{ and } (w, a, c) \in \mathcal{A} \text{ and } (x, w) \text{ stable} \\ (v, x, a) & \text{if } (x, a, b) \in \mathcal{A} \text{ and } (w, a, c) \in \mathcal{A} \text{ and } (v, x, w) \in \mathcal{A} \end{cases}$$

We now compute $\phi_1\phi_2\phi_1(a,b,c)$. If (a,b) is stable then it is easily seen that

$$\phi_1\phi_2\phi_1(a,b,c) = \begin{cases} (a,b,c) & \text{if } (a,b) \text{ stable and } (b,c) \text{ stable} \\ (a,z,b) & \text{if } (a,b) \text{ stable and } (z,b,c) \in \mathcal{A} \text{ and } (a,z) \text{ stable} \\ (y,a,b) & \text{if } (a,b) \text{ stable and } (z,b,c) \in \mathcal{A} \text{ and } (y,a,z) \in \mathcal{A} \end{cases}$$

We do the remaining cases in more detail.

- Assume that $(x,a,b) \in \mathcal{A}$ and that (a,c) is stable. The dropping axiom tells us that (b,c) is stable, because otherwise we would have that (a,b) and (b,c) are unstable, hence (a,c) must be unstable, contradicting the assumption. We obtain

$$\phi_1\phi_2\phi_1(a,b,c) = \phi_1\phi_2(a,b,c) = \phi_1(x,a,c) = (x,a,c).$$

- Assume that $(x,a,b) \in \mathcal{A}$, $(w,a,c) \in \mathcal{A}$ and that (x,w) is stable, cf. Figure 4.11. The extension axiom tells us that (b,c) is stable (because otherwise (x,w) would not be stable), and we obtain

$$\phi_1\phi_2\phi_1(a,b,c) = \phi_1\phi_2(a,b,c) = \phi_1(x,a,c) = (x,w,a).$$

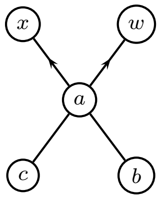


Figure 4.11: $(x,a,b) \in \mathcal{A}$ and $(w,a,c) \in \mathcal{A}$.

- Assume that $(x,a,b) \in \mathcal{A}$, $(w,a,c) \in \mathcal{A}$, $(v,x,w) \in \mathcal{A}$. The extension axiom implies that there exists $v' \in S_+$ such that $(v',b,c) \in \mathcal{A}$ and $(v,a,v') \in \mathcal{A}$. This yields

$$\phi_1\phi_2\phi_1(a,b,c) = \phi_1\phi_2(a,v',b) = \phi_1(v,a,b) = (v,x,a).$$

The Proposition is proven. $\square$

4.3 Thompson monoids

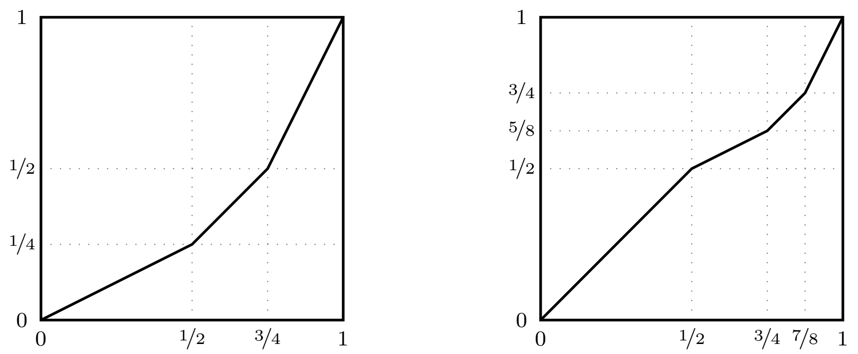
In this section we introduce a large family of monoids generalizing Thompson's famous group F. Thompson's group was the first known example of a torsion-free FP_∞-group that has infinite homological dimension, cf. Brown–Geoghegan [BG84]. The survey article by Cannon, Floyd, Parry [CFP96] serves as a good introduction to F.

We are going to define a 3-parameter family of monoids $t_m(p,q)$ and their groups of fractions $\mathcal{T}_m(p,q)$. For $m = \infty$, $p = 1$, $q = 2$ we will find $\mathcal{T}_\infty(1,2) \cong F$, and for $m = \infty$, $p = n-1$, $q = n$ we find the 1-parameter family of generalized Thompson groups $\mathcal{T}_\infty(n-1,n) \cong F_{n,\infty}$ introduced by Brown [Bro87]. In particular, $F \cong F_{2,\infty}$. (To avoid confusing, we remark that the symbols ∞ occuring in $\mathcal{T}_\infty(n-1,n)$ and $F_{n,\infty}$ have nothing to do with each other.)

The homology of F has first been computed by Brown–Geoghegan [BG84]. Using similar methods, Stein [Ste92] computed the homology of the groups $F_{n,\infty}$.

We are going to compute the homology of the monoids $t_m(p,q)$ by purely combinatorial means for $0 \leq p \leq q$. Together with Theorem 4.1.3, our results in particular provide a recomputation of the homology of the groups $F_{n,\infty}$.

Geometrically, Thompson's group F can be interpreted as the group of all piecewise affine homeomorphisms of the unit interval with break-points at dyadic rationals and slopes which are powers of 2. Using this point of view, F is generated by the functions ξ_1 and ξ_2 depicted in Figure 4.12, see e.g. [CFP96, Corollary 2.6]. This geometric model led to a whole bunch of generalizations of F, most notably by Higman [Hig74] and Stein [Ste92].



(a) ξ_1 (b) ξ_2

Figure 4.12: A generating set for Thompson's group F.

Using the result mentioned in Remark 2.3.19, Rodenhausen [Rod] proved that F is not factorable with respect to the generating set $S = \{\xi_1^{\pm 1}, \xi_2^{\pm 1}\}$.

Introducing further generators $\xi_i = \xi_1^{2-i}\xi_2\xi_1^{i-2}$ for $i \geq 3$, one finds another well-known group presentation for F:

$$F = \langle \xi_1, \xi_2, \xi_3, \ldots \mid \xi_{j-1}\xi_i = \xi_i\xi_j \text{ if } j - i \geq 2 \rangle_{\mathrm{Grp}} \tag{4.5}$$

We use the presentation in (4.5) to define 3-parameter families of monoids and groups generalizing Thompson's group F.

Definition 4.3.1 For $m = 0, \dots, \infty$ and $p, q \geq 0$ define the *(generalized) Thompson monoids* $t_m(p,q)$ as follows:

$$t_m(p,q) = \langle \xi_1, \dots, \xi_m \mid \xi_{j-p}\xi_i = \xi_i\xi_j \text{ if } j - i \geq q \rangle \tag{4.6}$$

We remark that when writing down a relation $\xi_{j-p}\xi_i = \xi_i\xi_j$ we implicitly understand all occuring indices to take values in $1, \dots, m$. From now on, the ξ_i are abstract generators.

Denote by $\mathcal{T}_m(p,q)$ the group of fractions of $t_m(p,q)$, i.e. we obtain $\mathcal{T}_m(p,q)$ when considering the right-hand side of (4.6) as group presentation.

Note that $\mathcal{T}_\infty(1,2)$ is Thompson's group F.

Remark 4.3.2 For $q = p+1$ the groups $\mathcal{T}_\infty(p,q)$ have a geometric interpretation in terms of piecewise affine homeomorphisms. More precisely, for every choice of a positive integer r, there is an isomorphism between $\mathcal{T}_\infty(n-1,n)$ and the group of piecewise affine homeomorphisms of the interval $[0,r]$ with break-points in $\mathbb{Z}[\frac{1}{n}]$ and slopes with are powers of n, see Brown [Bro87, Propositions 4.1, 4.4 and 4.8]. These spaces of piecewise affine homeomorphisms are denoted by $F(r, \mathbb{Z}[\frac{1}{n}], \langle n \rangle)$ in Stein [Ste92].

Our next aim is to derive embedding and factorability results for the monoids $t_m(p,q)$, and in order to prove cancellation properties, it will be convenient to have a suitable normal form $t_m(p,q) \to \{\xi_1, \dots, \xi_m\}^*$. Such a normal form will automatically come along with a factorability structure, and we therefore begin our analysis by investigating factorability of the $t_m(p,q)$'s.

4.3.1 Factorability of $t_m(p,q)$

Throughout, let $p, q \geq 0$.

Proposition 4.3.3 *Consider $m = 0, \dots, \infty$. If $q > 0$ and $p \leq q$ then $t_m(p,q)$ is braid factorable with respect to the generating set $\{\xi_1, \dots, \xi_m\}$.*

Proof. We use Proposition 4.2.13. Set $S = \{\xi_1, \dots, \xi_m\}$. In $\mathcal{A}$ we collect all triples (ξ_i, ξ_j, ξ_k) for which $1 \leq i, j, k \leq m$ and $k - j \geq q$ and $i = k - p$. Since we required $p \leq q$, we have $i \geq j$ for every arrow (ξ_i, ξ_j, ξ_k). It follows that a pair (ξ_j, ξ_k) is the tail of an arrow if and only if $k - j \geq q$. In other words, we do not need to care about whether $1 \leq i \leq m$, because this will automatically be fulfilled.

Obviously $X(S, \mathcal{A}) = t_m(p,q)$. To prove the Proposition it suffices to show that $\mathcal{A}$ is a good set of arrows for S.

Axiom (M1) is obvious, because i (and thus ξ_i) is uniquely determined by the index k. We now verify (M2). Assume that $(\xi_i, \xi_j, \xi_k) \in \mathcal{A}$. We need to show that $j - i < q$. Indeed, $j - i = j - (k - p) = p - (k - j) \leq p - q \leq 0 < q$, whence (M2).

We now check the dropping axiom. Assume that $(\xi_y, \xi_b, \xi_c) \in \mathcal{A}$ and $(\xi_x, \xi_a, \xi_b) \in \mathcal{A}$. It suffices to show that $(\xi_y, \xi_a, \xi_c) \in \mathcal{A}$. For this we only need to show that $c - a \geq q$ and $y = c - p$. Indeed, $c - a = c - b + b - a \geq 2q \geq q$ and $y = c - p$ holds because (ξ_y, ξ_b, ξ_c) is an arrow in $\mathcal{A}$. Axiom (D) is proven.

Finally, we check the extension axiom. Assume that $(\xi_b, \xi_x, \xi_{b'}) \in \mathcal{A}$ and $(\xi_c, \xi_x, \xi_{c'}) \in \mathcal{A}$. There exists a, $1 \leq a \leq m$, such that $(\xi_a, \xi_b, \xi_c) \in \mathcal{A}$ if and only if $c - b \geq q$. Observe that $b = b' - p$ and $c = c' - p$. Therefore $c - b \geq q$ if and only if $c' - b' \geq q$, and the latter is equivalent to the existence of some index a', $1 \leq a' \leq m$, with $(\xi_{a'}, \xi_{b'}, \xi_{c'}) \in \mathcal{A}$.

Assume now that there exists a, a' with $(\xi_a, \xi_b, \xi_c) \in \mathcal{A}$ and $(\xi_{a'}, \xi_{b'}, \xi_{c'}) \in \mathcal{A}$. We need to show that $(\xi_a, \xi_x, \xi_{a'}) \in \mathcal{A}$. We have $a' = c' - p$, yielding $a' - x = c' - x - p = c' - b' + b' - x - p \geq 2q - p \geq q$ and thus $(\xi_x, \xi_{a'})$ is $\mathcal{A}$-unstable. To prove that $(\xi_a, \xi_x, \xi_{a'}) \in \mathcal{A}$, it only remains to show that $a = a' - p$. But this is clear, for $a = c - p = c' - p - p = a' - p$. Therefore, (E) is fulfilled. □

Remark 4.3.4 The rewriting system associated to this factorability structure is "opposite" to the rewriting system on $t_\infty(1, 2)$ introduced by Cohen [Coh92], in the sense that we have a rewriting rule $l \to r$ if and only if there is a rule $r \to l$ in Cohen's set of rewriting rules.

What are the essential cells in $\overline{\mathbb{E}}_n t_m(p, q)$ with respect to this braid factorability structure? A pair of generators (ξ_i, ξ_j) is unstable if and only if $j - i \geq q$. Therefore, a cell $[\xi_{i_n}|\dots|\xi_{i_1}]$ is essential if and only if for every t, $n > t \geq 1$, we have $i_t - i_{t+1} \geq q$. This has two important consequences. First, for a cell being essential or not does not depend on the parameter p. Secondly, if $[\xi_{i_n}|\dots|\xi_{i_1}]$ is essential, then $i_1 \geq (n-1) \cdot q + i_n \geq (n-1) \cdot q + 1$.

The latter implies that for $m < \infty$ and $q > 0$ there are only finitely many essential cells. In other words, for $m < \infty$ and $q > 0$ the associated Visy complex $\mathbb{V}_* t_m(p, q)$ is finitely generated. More precisely, from the above estimate $i_1 \geq (n-1) \cdot q + 1$ it follows that an essential cell cannot have more than $1 + (m-1)/q$ entries. As an immediate Corollary we obtain the following:

Corollary 4.3.5 *For $m < \infty$, $q > 0$ and $p \leq q$ we have the following bound on the homological dimension of $t_m(p, q)$:*

$$\operatorname{hodim} t_m(p, q) \leq \left\lfloor \frac{m-1}{q} \right\rfloor + 1$$

We remark that later Proposition 4.5.1 will give an exact computation of the homological dimension.

Example 4.3.6 (Euler characteristic) Let $m < \infty$, $q > 0$ and $p \leq q$. Set

$$r_{n,m}(p, q) := \operatorname{rk} H_n(t_m(p, q)).$$

Clearly, $r_{n,m}(p,q) = 0$ for $n < 0$. Furthermore, by Corollary 4.3.5, $r_{n,m}(p,q) = 0$ for n sufficiently large. We can therefore compute the Euler characteristic $\chi(\boldsymbol{t}_m(p,q))$ as follows,

$$\chi(\boldsymbol{t}_m(p,q)) = \sum_{n=-\infty}^{+\infty} (-1)^n \cdot r_{n,m}(p,q).$$

Note that an n-cell $[\xi_{i_n}|\dots|\xi_{i_1}]$ is essential if and only if $[\xi_{i_n}|\dots|\xi_{i_2}]$ is essential and $i_1 - i_2 \geq q$. Distinguishing whether $i_1 < m$ or $i_1 = m$ yields the following splitting of the Visy modules. For $m \geq q$ and all $n \in \mathbb{Z}$ we have

$$\mathbb{V}_n \boldsymbol{t}_m(p,q) \cong \mathbb{V}_n \boldsymbol{t}_{m-1}(p,q) \oplus \mathbb{V}_{n-1} \boldsymbol{t}_{m-q}(p,q).$$

From this we obtain the following recursion formula for the $r_{n,m}(p,q)$'s:

$$r_{n,m}(p,q) = r_{n,m-1}(p,q) + r_{n-1,m-q}(p,q).$$

Taking on both sides the alternating sum over m running from $-\infty$ to $+\infty$, we conclude that for $m \geq q$ we have

$$\chi(\boldsymbol{t}_m(p,q)) = \chi(\boldsymbol{t}_{m-1}(p,q)) - \chi(\boldsymbol{t}_{m-q}(p,q)). \tag{4.7}$$

For $m \leq q$ the monoid $\boldsymbol{t}_m(p,q)$ is free on m generators (because the set of relations in (4.6) is empty), and hence $\chi(\boldsymbol{t}_m(p,q)) = 1 - m$. For $m > q$ we can use (4.7). We can therefore recursively compute the Euler characteristic for every Thompson monoid $\boldsymbol{t}_m(p,q)$ with parameters $m < \infty$, $q > 0$ and $p \leq q$. Again, note that $\chi(\boldsymbol{t}_m(p,q))$ does not depend on the particular choice of the parameter $p = 0, \dots, q$.

We list some explicit computations in Figure 4.13. An entry is shaded gray if for this choice of parameters there are no relations, i.e. if the corresponding monoid $\boldsymbol{t}_m(p,q)$ is free (non-abelian).

χ	$q=1$	$q=2$	$q=3$	$q=4$	$q=5$	$q=6$	$q=7$
$m=1$	0	0	0	0	0	0	0
$m=2$	0	-1	-1	-1	-1	-1	-1
$m=3$	0	-1	-2	-2	-2	-2	-2
$m=4$	0	0	-2	-3	-3	-3	-3
$m=5$	0	1	-1	-3	-4	-4	-4
$m=6$	0	1	1	-2	-4	-5	-5
$m=7$	0	0	3	0	-3	-5	-6
$m=8$	0	-1	4	3	-1	-4	-6
$m=9$	0	-1	3	6	2	-2	-5
$m=10$	0	0	0	8	6	1	-3

Figure 4.13: Euler characteristic $\chi(\boldsymbol{t}_m(p,q))$ for $p \leq q$.

4.3.2 Embedding $t_m(p,q)$ into $\mathcal{T}_m(p,q)$

We now derive sufficient conditions for $t_m(p,q)$ embedding into its group of fractions. To this end, we first investigate cancellation properties of the $t_m(p,q)$'s.

Clearly, if $m \leq p$ or $m \leq q$ then there are no relations at all. In particular, in this case, the monoids $t_m(p,q)$ are free of rank m, and so are the groups $\mathcal{T}_m(p,q)$. In contrast, if $m > p \geq q$ and $p > 0$, then the monoids $t_m(p,q)$ are not cancellative, because $\xi_1\xi_1 = \xi_1\xi_{p+1}$, but $\xi_1 \neq \xi_{p+1}$, because there is no relation that can be employed to ξ_1.

For the proof of the subsequent Lemma, the following observation is crucial.

Let $(i_s, \dots, i_1) := \overrightarrow{D_n}$. Define the sequence $(i'_s, \dots, i'_1)$ by vertically mirroring $(i_n, \dots, i_1)$, meaning that $i'_k = n+1-i_k$ for all k. We claim that $(i'_n, \dots, i'_1)$ is a representative of the absorbing element in Z_n. This follows from two facts. Firstly, $\overrightarrow{D_n}$ is a representative of the absorbing element in Q_n, and therefore it represents an absorbing element in the quotient Z_n. Secondly, the relations $\sim_{\text{left}}$ and $\sim_{\text{right}}$ have vertically mirrored analogies in Z_n, for example we have $(2,1,2,1) \sim_{\text{braid}} (2,2,1,2) \sim_{\text{idem}} (2,1,2) \sim_{\text{braid}} (1,2,1)$. One can now mimic the proof of Proposition 2.3.14 to show that $[i'_s, \dots, i'_1]_Z$ is indeed an absorbing element for Z_n.

Let $q > 0$ and $p \leq q$. Recall the factorability structure on $t_m(p,q)$ and observe that a tuple $(\xi_{i_n}, \dots, \xi_{i_1})$ is an η-normal form if and only if $i_t - i_{t+1} < q$ for all $n > t \geq 1$.

Lemma 4.3.7 *Assume that $(\xi_{i_n}, \dots, \xi_{i_1})$ is an η-normal form and let $1 \leq j \leq m$. We then have the following.*

(a) There is a unique index k such that $(\xi_{i_n}, \dots, \xi_{i_{k+1}}, \xi_{j-kp}, \xi_{i_k}, \dots, \xi_{i_1})$ is the η-normal form of the word $\xi_{i_n} \dots \xi_{i_1} \cdot \xi_j$.

(b) There is a unique index k such that $(\xi_{i_n - p}, \dots, \xi_{i_{n-k+1}-p}, \xi_j, \xi_{i_{n-k}}, \dots, \xi_{i_1})$ is the η-normal form of the word $\xi_j \cdot \xi_{i_n} \dots \xi_{i_1}$.

PROOF. (a) Every monoid $t_m(p,q)$ is balanced, and thus applying $f_{\overrightarrow{D_n}}$ to the tuple $(\xi_{i_n}, \dots, \xi_{i_1}, \xi_j)$ yields an η-normal form. Furthermore, by the recognition principle, $(\xi_{i_n}, \dots, \xi_{i_1})$ is everywhere stable. Hence

$$\begin{aligned} f_{\overrightarrow{D_n}}(\xi_{i_n}, \dots, \xi_{i_1}, \xi_j) &= f_n \dots f_1 \circ f_{\overrightarrow{D_{n-1}}}(\xi_{i_n}, \dots, \xi_{i_1}, \xi_j) \\ &= f_n \dots f_1(\xi_{i_n}, \dots, \xi_{i_1}, \xi_j). \end{aligned}$$

Now, if $(\xi_{i_n}, \dots, \xi_{i_1}, \xi_j)$ is stable at position 1, then it is everywhere stable and we are done. Otherwise, $f_1(\xi_{i_n}, \dots, \xi_{i_1}, \xi_j) = (\xi_{i_n}, \dots, \xi_{j-p}, \xi_{i_1})$. Iterating this argument, part (a) follows.

(b) works similarly. Indeed, we use the previous discussion on representatives of the absorbing element in Z_n to conclude that $f_1 \dots f_n(\xi_j, \xi_{i_n}, \dots, \xi_{i_1})$ is in η-normal form. We proceed as in (a).

The Lemma is proven. $\square$

Lemma 4.3.8 *Let $q > 0$ and $p \leq q$. We then have the following.*

(a) The monoids $\mathfrak{t}_m(p,q)$ are right-cancellative.

(b) For $p < q$ the monoids $\mathfrak{t}_m(p,q)$ are left-cancellative.

Proof. (a) We show that if for elements $x, y \in \mathfrak{t}_m(p,q)$ we have $x \cdot \xi_j = y \cdot \xi_j$ then $x = y$. Since $\mathfrak{t}_m(p,q)$ is balanced, we know that x, y have the same word length. Let $(\xi_{i_n}, \ldots, \xi_{i_1})$ and $(\xi_{i'_n}, \ldots, \xi_{i'_1})$ be the respective normal forms of x and y.

By our above observation, the normal forms of $x \cdot \xi_j$ and $y \cdot \xi_j$ are given by

$$\begin{aligned} \mathrm{NF}(x \cdot \xi_j) &= (\xi_{i_n}, \ldots, \xi_{i_{k+1}}, \xi_{j-kp}, \xi_{i_k}, \ldots, \xi_{i_1}), \\ \mathrm{NF}(y \cdot \xi_j) &= (\xi_{i'_n}, \ldots, \xi_{i'_{k'+1}}, \xi_{j-k'p}, \xi_{i'_{k'}}, \ldots, \xi_{i'_1}), \end{aligned}$$

for some k, k'. Assume that $x \cdot \xi_j = y \cdot \xi_j$. In particular, the above normal forms coincide. Clearly, if $k = k'$ then $\xi_{i_t} = \xi_{i'_t}$ for all t and we are finished. So assume that $k \neq k'$. W.l.o.g. we may assume that $k < k'$. Consider the pairs $(\xi_{i_{k+1}}, \xi_{j-kp})$ and $(\xi_{i'_{k+1}}, \xi_{j-kp})$. The former is stable, whereas the latter is unstable. Therefore

$$j - kp - i'_{k+1} \geq q. \tag{4.8}$$

Comparing the $(k+1)$-st entries of the above normal forms, we see that $\xi_{j-kp} = \xi_{i'_{k+1}}$, and thus $j - kp = i'_{k+1}$. Together with (4.8) this yields $0 \geq q$, contradicting our assumption.

Part (b) is proven similarly: In this case we find

$$\begin{aligned} \mathrm{NF}(\xi_j \cdot x) &= (\xi_{i_n - p}, \ldots, \xi_{i_{n-k+1} - p}, \xi_j, \xi_{i_{n-k}}, \ldots, \xi_{i_1}), \\ \mathrm{NF}(\xi_j \cdot y) &= (\xi_{i'_n - p}, \ldots, \xi_{i'_{n-k'+1} - p}, \xi_j, \xi_{i'_{n-k'}}, \ldots, \xi_{i'_1}). \end{aligned}$$

Assuming $k < k'$ yields that $(\xi_j, \xi_{i_{n-k}})$ is stable, whereas $(\xi_j, \xi_{i'_{n-k}})$ is unstable. The latter is equivalent to $i'_{n-k} - j \geq q$, and comparing the $(n-k+1)$-st entries yields $j = i'_{n-k} - p$, whence $p \geq q$, contradicting our assumption. ☐

We are now going to show that for $m = \infty$ and $p < q$ the Thompson monoids $\mathfrak{t}_m(p,q)$ satisfy the right Ore condition.

Definition 4.3.9 Let X be a monoid and S a generating set for X. We say that X satisfies the *specific (right) Ore condition* with respect to S if for all $s, s' \in S$ there exist $t, t' \in S$ such that $st = s't'$.

Note that we require t and t' to lie in S. In particular, for an arbitrary generating set S, the specific Ore condition does not in general follow from the Ore condition introduced in Definition 4.1.1. Clearly, taking $S = X$, both notions coincide.

Remark 4.3.10 An inductive argument shows that if X is cancellative and if X satisfies the specific Ore condition for at least one generating set, then X satisfies the Ore condition, see e.g. Clifford-Preston [CP61, §1.10].

Lemma 4.3.11 *Set $S = \{\xi_1, \xi_2, \ldots\}$. Then for $q > 0$ and $q \leq p+1$ the monoids $t_\infty(p,q)$ satisfy the specific Ore condition with respect to S.*

PROOF. Let $s = \xi_i$ and $s' = \xi_j$, where $1 \leq i, j < \infty$. If $i = j$ then we take $t = t' = \epsilon$. Otherwise, we may assume that $j > i$. Set $t' = s$ and $t = \xi_{j+p}$. Note that $j + p - i \geq p + 1 \geq q$, yielding $s't' = \xi_j\xi_i = \xi_i\xi_{j+p} = st$. □

Lemma 4.3.8 tells us that for $q > p$ the monoids $t_m(p,q)$ are cancellative. Thus, by Remark 4.3.10 and Lemma 4.3.11, for $q = p+1$ the monoids $t_\infty(p,q)$ are cancellative and satisfy the Ore condition. Applying Theorem 4.1.3, we obtain the following:

Corollary 4.3.12 *For $q = p+1 > 0$ the map $i_* : H_*(t_\infty(p,q)) \to H_*(\mathcal{T}_\infty(p,q))$ is an isomorphism.*

In particular, for every $n > 1$, $t_\infty(n-1,n)$ and Stein's groups $F(r, \mathbb{Z}[\frac{1}{n}], \langle n \rangle)$ have isomorphic homology, cf. Remark 4.3.2. Taking $n = 2$ and $r = 1$, we see that $t_\infty(1,2)$ has the homology of Thompson's group F

We conclude this section by briefly mentioning several canonical maps between some of the $t_m(p,q)$'s and some of the $\mathcal{T}_m(p,q)$'s.

Remark 4.3.13 (a) Let $p, q > 0$. Recall that for $m \leq p$ or $m \leq q$ the groups $\mathcal{T}_m(p,q)$ are free of rank m. Assume now that $m > \max\{p,q\}$. Then in $\mathcal{T}_m(p,q)$ we have $\xi_{m-p}\xi_{m-q} = \xi_{m-q}\xi_m$, i.e. $\xi_m = \xi_{m-q}^{-1}\xi_{m-p}\xi_{m-q}$, from which the right-hand side can be considered as an element in $\mathcal{T}_{m-1}(p,q)$. Thus, for $m > \max\{p,q\}$, we have a canonical quotient map $\mathcal{T}_{m-1}(p,q) \twoheadrightarrow \mathcal{T}_m(p,q)$. Set $n = \max\{p,q\}$. We have the following infinite sequence of quotients,

$$G(n) \cong \mathcal{T}_n(p,q) \longrightarrow\!\!\!\!\!\rightarrow \mathcal{T}_{n+1}(p,q) \longrightarrow\!\!\!\!\!\rightarrow \mathcal{T}_{n+2}(p,q) \longrightarrow\!\!\!\!\!\rightarrow \cdots,$$

where $G(n)$ denotes the free group of rank n. The colimit of this sequence is $\mathcal{T}_\infty(p,q)$.

(b) Consider the map $\mathrm{id} : S^* \to S^*$. For all values of m, p and q this descends to a surjection $t_m(p,q) \twoheadrightarrow t_m(p,q\text{–}1)$, which should be thought of as "adding relations". We obtain the analogous result for the respective groups of fractions.

(c) Fix $d \geq 1$. Assigning to ξ_i the generator ξ_{di} gives a well-defined map $t_m(p,q) \to t_{dm}(dp,dq)$. Iteratively using the "adding relations"map from part (b), we obtain canonical maps between monoids with parameters $q = p+1$:

$$t_m(p,p+1) \longrightarrow t_{dm}(dp,dp+d) \longrightarrow\!\!\!\!\!\rightarrow t_{dm}(dp,dp+1)$$

4.4 Reducing the Visy complexes $\mathbb{V}_* t_m(p,q)$

Recall that for $q > 0$, $p \leq q$ every $t_m(p,q)$ is braid factorable, and if $m < \infty$ then the associated Visy complex $\mathbb{V}_* t_m(p,q)$ is finite. Yet, if m is large, then the Visy complex becomes too huge to do homology computations straight ahead.

So we further reduce $\mathbb{V}_* t_m(p,q)$ by explicitly defining a noetherian matching on it. We are then going to show that this matching is perfect, which here means that the differentials in the associated Morse complex do all vanish. In other words, every essential cell with respect to our matching constitutes a (free) generator of $H_*(t_m(p,q))$.

The constructions and proofs are not very deep. However, one has to be careful to not be confused by the many indices. For convenience we will accompany our formal constructions by exemplarily discussing them on the Thompson monoid $t_5(1,2)$.

4.4.1 Notational conventions

Throughout, let $q > 0$ and $p \leq q$.

Recall the factorability structure on $t_m(p,q)$. Every essential cell has elementary partition type, and thus is of the form $[\xi_{i_n}|\dots|\xi_{i_1}]$. Cells of elementary partition type are uniquely determined by their sequence of indices $(i_n,\dots,i_1)$. To keep notation simple, we will henceforth write $[i_n|\dots|i_1]$ for the cell $[\xi_{i_n}|\dots|\xi_{i_1}]$.

Our construction involves two different noetherian matchings. The first one lives on the normalized bar complex $\overline{\mathbb{B}}_* t_m(p,q)$ and is induced by the factorability structure on $t_m(p,q)$. By definition, the associated Morse complex is the Visy complex $\mathbb{V}_* t_m(p,q)$. The second matching is defined on the Visy complex $\mathbb{V}_* t_m(p,q)$, and it will be constructed explicitly.

To keep confusion as low as possible we introduce the following notation. Using the above convention, we can write $[i_n|\dots|i_1]$ for a cell of elementary partition type in the bar complex $\overline{\mathbb{B}}_* t_m(p,q)$. Its essential cells generate the Visy complex $\mathbb{V}_* t_m(p,q)$, and its generators will be denoted by double brackets $[\![i_n|\dots|i_1]\!]$. We use this notation to make clear in which complex we consider the respective elements to live, in order to emphasize in which sense the terms “essential”, “collapsible” and “redundant” are to be understood.

Example 4.4.1 We now describe $\mathbb{V}_* t_5(1,2)$ as module, using the above conventions. Recall that a cell of elementary partition type $[i_n|\dots|i_1]$ is essential if and only if $i_t - i_{t+1} \geq 2$ for every $n > t \geq 1$. As a consequence, there are no essential cells in dimensions > 3, cf. Corollary 4.3.5. Figure 4.14 lists all essential cells of $t_5(1,2)$.

$\mathbf{n} =$	**Generators of $\mathbb{V}_{\mathbf{n}} t_{\mathbf{5}}(\mathbf{1},\mathbf{2})$**
0	$[\![\,]\!]$
1	$[\![5]\!]$, $[\![4]\!]$, $[\![3]\!]$, $[\![2]\!]$, $[\![1]\!]$
2	$[\![3\|5]\!]$, $[\![2\|5]\!]$, $[\![1\|5]\!]$, $[\![2\|4]\!]$, $[\![1\|4]\!]$, $[\![1\|3]\!]$
3	$[\![1\|3\|5]\!]$

Figure 4.14: Generators of $\mathbb{V}_* t_5(1,2)$.

4.4.2 Describing the differential $\partial_*^{\mathbb{V}}$

In what follows we fix $m = 0, \ldots, \infty$, $q > 0$ and $p \leq q$.

To define a noetherian matching on the Visy complex, we need a good understanding of the differentials $\partial_n^{\mathbb{V}}$.

Proposition 4.4.2 *Let $[\![i_n|\ldots|i_1]\!]$ be a generator of the Visy complex $\mathbb{V}_*\mathfrak{t}_m(p,q)$, i.e. $i_t - i_{t+1} \geq q$ for all t, $n > t \geq 1$. Then for every k, $1 \leq k \leq n$, the following holds:*

$$\begin{aligned} d_n \Psi_n^k [\![i_n|\ldots|i_1]\!] &= [\![i_n|\ldots|i_{k+1}|i_{k-1}|\ldots|i_1]\!], \\ d_0 \Phi_1^k [\![i_n|\ldots|i_1]\!] &= [\![i_n|\ldots|i_{k+1}|i_{k-1}-p\,|\ldots|i_1-p]\!]. \end{aligned}$$

Proof. Observe that $f_k[\![i_n|\ldots|i_1]\!] = [i_n|\ldots|i_{k+2}|i_k - p|i_{k+1}|i_{k-1}|\ldots|i_1]$ and that $(i_{k+2}, i_k - p)$ and (i_{k+1}, i_{k-1}) are both unstable. The former because of $i_k - p - i_{k+2} = i_k - i_{k+1} + i_{k+1} - i_{k+2} - p \geq 2q - p \geq q$ and the latter due to the dropping axiom. Iteratively applying the f_j's yields

$$\begin{aligned} \Psi_n^k [\![i_n|\ldots|i_1]\!] &= [i_k - (n-k)\cdot p\,|i_n|\ldots|i_{k+1}|i_{k-1}|\ldots|i_1], \\ \Phi_1^k [\![i_n|\ldots|i_1]\!] &= [i_n|\ldots|i_{k+1}|i_{k-1} - p\,|\ldots|i_1 - p\,|i_k]. \end{aligned}$$

Note that the right-hand sides need not to be generators of the Visy complex, for $\Psi_n^k[\![i_n|\ldots|i_1]\!]$ is stable at position $n-1$ (if $k < n$) and $\Phi_1^k[\![i_n|\ldots|i_1]\!]$ is stable at position 1 (if $k > 1$). On all the other positions both these cells are unstable, and hence $d_n\Psi_n^k[\![i_n|\ldots|i_1]\!]$ and $d_0\Phi_1^k[\![i_n|\ldots|i_1]\!]$ are indeed generators of the Visy complex. The Proposition is proven. □

Observe that $d_n\Psi_n^1(\underline{x}) = d_0\Phi_1^1(\underline{x})$. Altogether, (4.2) simplifies as follows:

Corollary 4.4.3 *For every $n \geq 0$ the differential in the Visy complex $\mathbb{V}_*\mathfrak{t}_m(p,q)$ is given by*

$$\partial_n^{\mathbb{V}} = \sum_{k=2}^{n} (-1)^k (d_n \Psi_n^k - d_0 \Phi_1^k). \tag{4.9}$$

Remark 4.4.4 Corollary 4.4.3 states that the Visy complex of $\mathfrak{t}_m(p,q)$ is a cubed complex. More precisely, consider the classifying space of $\mathfrak{t}_m(p,q)$, and denote by Y the associated quotient complex described in Remark 3.1.23. Recall from Remark 3.3.7.(b) that the cellular complex of Y is the Visy complex. Looking at (4.9), we see that Y is built up from cubes (with top face and bottom face glued together). Similar complexes occur in Stein [Ste92, p.489] for the groups $F_{n,\infty} \cong \mathcal{T}_\infty(n-1,n)$.

Example 4.4.5 Corollary 4.4.3 in particular states that $\partial_1^{\mathbb{V}}$ is the zero map. Below we compute $\partial_*^{\mathbb{V}}$ for higher dimensional cells in $\mathbb{V}_* t_5(1,2)$:

$$\begin{aligned}
\partial^{\mathbb{V}}[1|3|5] &= [1|5] - [1|4] - [3|5] + [2|4] \\
\partial^{\mathbb{V}}[3|5] = \partial^{\mathbb{V}}[2|5] = \partial^{\mathbb{V}}[1|5] &= [5] - [4] \\
\partial^{\mathbb{V}}[2|4] = \partial^{\mathbb{V}}[1|4] &= [4] - [3] \\
\partial^{\mathbb{V}}[1|3] &= [3] - [2]
\end{aligned}$$

The Visy complex $(\mathbb{V}_* t_5(1,2), \partial_*^{\mathbb{V}})$ is therefore isomorphic to the following chain complex, where the $\mathbb{Z}$ on the right-hand side sits in degree 0.

$$\mathbb{Z} \xrightarrow[\partial_3^{\mathbb{V}}]{\begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}} \mathbb{Z}^6 \xrightarrow[\partial_2^{\mathbb{V}}]{\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & -1 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}} \mathbb{Z}^5 \xrightarrow[\partial_1^{\mathbb{V}}]{0} \mathbb{Z}$$

Remark 4.4.6 Note from Proposition 4.4.2 that if $p = 0$ then $d_n \Psi_n^k = d_0 \Phi_1^k$ on essential cells. Hence, Corollary 4.4.3 tells us that in this case all differentials $\partial_n^{\mathbb{V}}$ vanish. In other words, the matching induced by the factorability structure on $t_m(0,q)$ is perfect, and thus the Visy complex $\mathbb{V}_* t_m(p,q)$ is isomorphic to the homology $H_*(t_m(p,q))$. Geometrically speaking, for $p = 0$, the complex Y mentioned in Remark 4.4.4 is built up from tori. Indeed, for $p = 0$ the only relations in (4.6) on page 139 are commutativitiy relations, and hence $t_m(0,q)$ is a graph product (say with respect to Γ) of m-many copies of the free monoid on one generator, and every full subgraph of Γ constitutes a torus in Y. In Figure 4.15 we depict the cell structure of Y for $m = 4$, $p = 0$, $q = 2$.

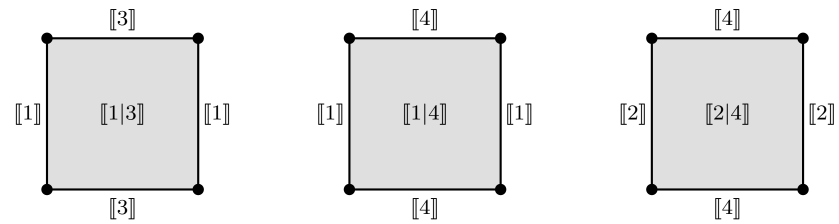


Figure 4.15: Model for the classifying space of $t_4(0,2)$ (after glueing).

If $p > 0$ then our matching will in general not be perfect, and one might ask how to further simplify $\mathbb{V}_* t_m(p,q)$. This is what we are concerned with in the following subsection.

4.4.3 A perfect matching on $\mathbb{V}_* t_m(p,q)$

From now on we additionally assume that $p > 0$, i.e. we consider $m = 0, \ldots, \infty$ and $0 < p \leq q$.

We will define a perfect matching on $V_* t_m(p,q)$. This is done in the usual way. We first say what the essential cells are and then use scanning to match the remaining cells into pairs of one collapsible and one redundant cell.

Consider an arbitrary n-cell $[\![i_n|\dots|i_1]\!]$ in the Visy complex and extend it to an $(n+1)$-tuple $(i_{n+1}, i_n, \dots, i_1)$, where $i_{n+1} = 1-q$. To this we associate the difference sequence

$$(i_n - i_{n+1}, \dots, i_1 - i_2).$$

Where does this n-tuple live? For $n > t \geq 1$ we have $i_t - i_{t+1} \geq q$, because $[\![i_n|\dots|i_1]\!]$ is a generator of the Visy complex (and thus an essential cell with respect to the factorability structure on $t_m(p,q)$.) Furthermore, $i_n - i_{n+1} = i_n - 1 + q \geq q$, and we see that

$$(i_n - i_{n+1}, \dots, i_1 - i_2) \in \{q, q+1, \dots\}^n.$$

Definition 4.4.7 We define the 0-cell $[\![\,]\!]$ to be essential. For $n \geq 1$, an n-cell $[\![i_n|\dots|i_1]\!]$ is defined to be essential if for the associated difference sequence we have

$$(i_n - i_{n+1}, \dots, i_1 - i_2) \in \{q+1, \dots, 2q-1\}^{n-1} \times \{q, \dots, 2q-1\}.$$

Some remarks are in order.

Remark 4.4.8 (a) Essentiality of a cell $[\![i_n|\dots|i_1]\!] \in \mathbb{V}_* t_m(p,q)$ does not depend on the specific choice of the parameter p (provided that $0 < p \leq q$).

(b) Unwinding the conditions posed on the difference sequence, essentiality can be reformulated as follows. The 0-cell $[\![\,]\!]$ is always essential. A 1-cell $[\![i]\!]$ is essential if and only if $i - (1-q) \in \{q, \dots, 2q-1\}$, i.e. if and only if $i \in \{1, \dots, q\}$. For $n \geq 2$, an n-cell $[\![i_n|\dots|i_1]\!]$ is essential if and only if the following conditions are fulfilled:

- $i_n \in \{2, \dots, q\}$,
- $i_t - i_{t+1} \in \{q+1, \dots, 2q-1\}$ for all t, $n > t > 1$, and
- $i_1 - i_2 \in \{q, \dots, 2q-1\}$

(c) Note that if $[\![i_n|\dots|i_1]\!]$ is essential then $i_1 - i_2 \in \{q, \dots, 2q-1\}$ and $[\![i_n|\dots|i_2]\!]$ is again essential. However, the converse need not be true. For example, in $\mathbb{V}_* t_5(1,2)$ the cell $[\![1]\!]$ is essential, but $[\![1|3]\!]$ is not. The first statement is clear, the second can be seen as follows. To $[\![1|3]\!]$ we associate the triple $(1-q, 1, 3) = (-1, 1, 3)$. This has difference sequence $(2,2)$, and

$$(2,2) \notin \{q+1, \dots, 2q-1\} \times \{q, \dots, 2q-1\} = \{3\} \times \{2,3\}.$$

More generally, provided that $m \geq q+1$, the cell $[\![1]\!]$ is essential, but $[\![1|1+q]\!]$ is not.

It is easily read off this definition that for $m \geq 1$ and $p = q = 1$ there are only two essential cells in $\mathbb{V}_* t_m(1,1)$, namely $[\![\,]\!]$ and $[\![1]\!]$: There are no further essential 1-cells, because if $[\![i]\!]$ is essential then $i \in \{1, \dots, q\} = \{1\}$. Similarly, there are no essential cells in dimensions ≥ 2, since the set $\{q+1, \dots, 2q-1\}$ is empty for $q = 1$.

Example 4.4.9 We can also quite easily describe the essential cells in $\mathbb{V}_* t_m(p,q)$ for $q=2$. There is one 0-cell, namely $[\![\,]\!]$. There are two 1-cells, namely $[\![1]\!]$ and $[\![2]\!]$. For $n \geq 2$, an n-cell $[\![i_n|\dots|i_1]\!]$ is essential if and only if $i_n = 2$, $i_t - i_{t+1} = 3$ for all t, $n > t > 1$, and $i_1 - i_2 \in \{2,3\}$. In Figure 4.16 we list the essential cells of $\mathbb{V}_n t_m(p,2)$ for small n.

n =	**Essential cells in the Visy complex $\mathbb{V}_n t_m(p,2)$**	
0	$[\![\,]\!]$	
1	$[\![1]\!]$	$[\![2]\!]$
2	$[\![2\|4]\!]$	$[\![2\|5]\!]$
3	$[\![2\|5\|7]\!]$	$[\![2\|5\|8]\!]$
4	$[\![2\|5\|8\|10]\!]$	$[\![2\|5\|8\|11]\!]$
5	$[\![2\|5\|8\|11\|13]\!]$	$[\![2\|5\|8\|11\|14]\!]$

Figure 4.16: Essential cells in the Visy complex $\mathbb{V}_* t_m(p,2)$ for large m.

The table in Figure 4.16 has to be read as follows: The essential cells of $\mathbb{V}_* t_m(p,2)$ are those for which all occuring indices are $\leq m$. For example, the essential cells of $\mathbb{V}_* t_5(1,2)$ are given by $[\![\,]\!]$, $[\![1]\!]$, $[\![2]\!]$, $[\![2|4]\!]$ and $[\![2|5]\!]$.

Note that none of the indices occuring in Figure 4.16 is divisible by 3. Therefore – provided that our definition of essential cells extends to a noetherian matching on the Visy complex – the Morse complexes $\mathbb{V}^{\theta}_* t_{3m-1}(p,2)$ and $\mathbb{V}^{\theta}_* t_{3m}(p,2)$ are isomorphic as $\mathbb{Z}$-modules. In particular, $t_{3m-1}(p,2)$ and $t_{3m}(p,2)$ have the same Euler characteristic, compare Figure 4.13 on page 141.

Remark 4.4.10 Consider an n-cell $[\![i_n|\dots|i_1]\!]$ in the Visy module $\mathbb{V}_n t_m(p,q)$. Obviously $[\![i_n|\dots|i_1]\!]$ can also be understood as a generator of $\mathbb{V}_n t_{m+1}(p,q)$, and essentiality of $[\![i_n|\dots|i_1]\!]$ does not depend on the specific parameter m of the monoid $t_m(p,q)$ we consider our cell to live in (provided that $m \geq i_1$).

Definition 4.4.11 We define the *height* of an n-cell $[\![i_n|\dots|i_1]\!]$ in $\mathbb{V}_n t_m(p,q)$ as

$$\mathrm{ht}[\![i_n|\dots|i_1]\!] = \max\{h : [\![i_n|\dots|i_{n-h+1}]\!] \text{ is essential}\}.$$

Consider an n-cell $[\![i_n|\dots|i_1]\!]$ and its associated difference sequence $(i_n - i_{n+1}, \dots, i_1 - i_2)$. By the definition of height we have

$$(i_n - i_{n+1}, \dots, i_{n-h+1} - i_{n-h+2}) \in \{q+1, \dots, 2q-1\}^{h-1} \times \{q, \dots, 2q-1\}.$$

Assume that $h < n$ (because otherwise the cell is essential). If $i_{n-h+1} - i_{n-h+2} = q$ then we call $[\![i_n|\dots|i_1]\!]$ collapsible. Otherwise $i_{n-h+1} - i_{n-h+2} \in \{q+1, \dots, 2q-1\}$ and thus, by the definition of height, we must have $i_{n-h} - i_{n-h+1} \geq 2q$, in which case we call $[\![i_n|\dots|i_1]\!]$ redundant.

Very roughly speaking, this classification does the following. We scan our cell $[\![i_n|\dots|i_1]\!]$ from left to right. As long as the entries are "well-distributed" (in the sense that they are $\geq q+1$ and $\leq 2q-1$ apart) we keep scanning. If two successive entries are too close, meaning that they are only distance q apart, then we mark this cell as collapsible. Its partner will arise from removing this entry. Otherwise, if two successive entries are very far apart, i.e. have distance $\geq 2q$, such that we could insert at least one more entry and still obtain a generator in the Visy complex (not necessarily an essential one!), then we mark this entry as redundant. The partner is given by inserting this entry at the earliest possible position, i.e. distance q apart from its left predecessor.

We now make this matching explicit. Define a function μ on the generators of $\mathbb{V}_* t_m(p,q)$ as follows. Let $[\![i_n|\dots|i_1]\!]$ be an n-cell of height h. Set $i_{n+1} = 1-q$ and define

$$\mu([\![i_n|\dots|i_1]\!]) = \begin{cases} [\![i_n|\dots|i_1]\!] & \text{if } [\![i_n|\dots|i_1]\!] \text{ is essential} \\ [\![i_n|\dots|i_{n-h+1}|i_{n-h+1}+q|i_{n-h}|\dots|i_1]\!] & \text{if } [\![i_n|\dots|i_1]\!] \text{ is redundant} \\ [\![i_n|\dots|i_{n-h+2}|i_{n-h}|\dots|i_1]\!] & \text{if } [\![i_n|\dots|i_1]\!] \text{ is collapsible} \end{cases}$$

Note that for collapsible cells we have $\mu([\![i_n|\dots|i_1]\!]) = d_n\Psi_n^{n-h+1}([\![i_n|\dots|i_1]\!])$, cf. Proposition 4.4.2.

Remark 4.4.12 Note that the matching μ does not affect the entries $i_{n-h},\dots,i_1$. Furthermore, if $[\![i_n|\dots|i_1]\!]$ is not essential, then $h < n$, i.e. $n-h > 0$, and therefore the right-most entry of a redundant cell is equal to the right-most entry of its collapsible partner. Since $i_n < \dots < i_1$, this shows that the matching on the generators of $\mathbb{V}_* t_m(p,q)$ restricts and corestricts to a matching on the generators of $\mathbb{V}_* t_{m-1}(p,q)$.

Example 4.4.13 We give a complete classification of the cells in $\mathbb{V}_* t_5(1,2)$. We exemplarily consider the 3-cell $[\![i_3|i_2|i_1]\!] = [\![1|3|5]\!]$. Recall from Remark 4.4.8.(c) that $[\![1]\!]$ is essential, but $[\![1|3]\!]$ is not. Therefore $[\![1|3|5]\!]$ has height $h=1$. We have $i_{n-h+1} - i_{n-h+2} = i_3 - i_4 = 1-(1-q) = q$, hence $[\![1|3|5]\!]$ is collapsible. Its redundant partner is given by deleting the entry $i_{n-h+1} = i_3$. So $\mu([\![1|3|5]\!]) = [\![3|5]\!]$.

Figure 4.17 gives a complete classification of the cells of $\mathbb{V}_* t_5(1,2)$. The matching function μ is indicated by solid lines. Shaded elements correspond to unpaired (i.e. essential) elements, compare Example 4.4.9.

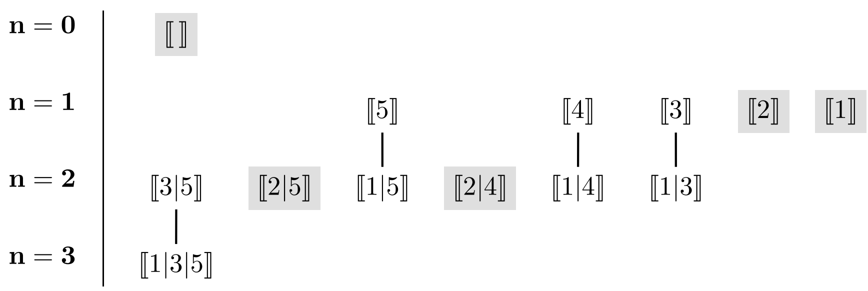


Figure 4.17: Classification of the generators of $\mathbb{V}_* t_5(1,2)$.

Remark 4.4.14 Intuitively, the philosophy behind μ can be described as follows. Let $[\![i_n|\dots|i_1]\!]$ be a collapsible n-cell of height h. In particular, $[i_n|\dots|i_1]$ is a generator of the inhomogeneous bar complex $\overline{\mathbb{B}}_* t_m(p,q)$. We now think of $[i_n|\dots|i_1]$ as a generator of the *homogeneous* bar complex of some abstract, not further specified object. The redundant partner of $[\![i_n|\dots|i_1]\!]$ is $[\![i_n|\dots|\widehat{i_{n-h+1}}|\dots|i_1]\!]$, and it can be interpreted as the $(n-h+1)$-st face of $[\![i_n|\dots|i_1]\!]$ in this abstract homogeneous bar complex.

Very speculatively, this observation might help to further simplify Visy complexes of factorable monoids (X,S,η), if the generating set S is equipped with a reasonable enumeration which is "compatible" with the relations in X. However, we're not able to turn this into a precise statement.

We proceed by showing that μ defines a noetherian matching on $\mathbb{V}_* t_m(p,q)$.

Lemma 4.4.15 *The map μ is an involution.*

Proof. This is done as usual. One first shows that μ maps redundant cells of height h to collapsible cells of height $h+1$ and vice versa. Afterwards, one explicitly computes μ^2 for all redundant and collapsible cells, which is straightforward. □

Lemma 4.4.16 *Let $p>0$. Then for every redundant cell $\underline{i}$ of height h we have*

$$[\partial^{\mathbb{V}}\mu(\underline{i}):\underline{i}] = (-1)^{n-h+1}.$$

In particular, the matching function μ is $\mathbb{Z}$-compatible.

Proof. Let $\underline{i} = [\![i_n|\dots|i_1]\!]$ be redundant of height h. Then its partner

$$\mu(\underline{i}) = [\![i_n|\dots|i_{n-h+1}|i_{n-h+1}+q|i_{n-h}|\dots|i_1]\!]$$

is collapsible of height $h+1$. By Corollary 4.4.3 we have

$$\partial^{\mathbb{V}}_{n+1}(\mu(\underline{i})) = \sum_{k=2}^{n+1}(-1)^k\cdot(d_{n+1}\Psi^k_{n+1} - d_0\Phi^k_1)(\mu(\underline{i})). \tag{4.10}$$

Observe that $d_{n+1}\Psi^{n-h+1}_{n+1}(\mu(\underline{i})) = \underline{i}$. Now, by Proposition 4.4.2, for $k=2,\dots,n+1$, $k\neq n-h+1$ we have $d_{n+1}\Psi^k_{n+1}(\mu(\underline{i})) \neq d_{n+1}\Psi^{n-h+1}_{n+1}$ (compare the total sum of all occuring indices), and furthermore $d_{n+1}\Psi^{n-h+1}_{n+1}(\mu(\underline{i})) \neq d_0\Phi^l_1(\mu(\underline{i}))$ for $l=2,\dots,n+1$ (compare the right-most entries). It follows that $[\partial\mu(\underline{i}):\underline{i}] = \pm 1$, where the sign is the sign of the term $d_{n+1}\Psi^{n-h+1}_{n+1}$ occuring in (4.10), which is $(-1)^{n-h+1}$. The Lemma is proven. □

It remains to prove noetherianity. We need some preparation.

Lemma 4.4.17 *Let $p>0$. Let $\underline{i} = [\![i_n|\dots|i_1]\!]$ be collapsible of height h. Let $1\le k\le n$ and set $[\![j_{n-1}|\dots|j_1]\!] := d_0\Phi^k_1(\underline{i})$. If $k>1$ then $j_1 < i_1$.*

Proof. By Proposition 4.4.2 we have $[\![j_{n-1}|\dots|j_1]\!] = [\![i_n|\dots|i_{k+1}|i_{k-1}-p|\dots|i_1-p]\!]$. Now, if $k>1$, then $j_1 = i_1 - p < i_1$. $\square$

For an n-cell $[\![i_n|\dots|i_1]\!]$ define its *value* as the total sum of (the indices of) its entries,

$$\mathrm{value}[\![i_n|\dots|i_1]\!] := i_n + \dots + i_1.$$

Lemma 4.4.18 *Let $\underline{i} = [\![i_n|\dots|i_1]\!]$ be collapsible of height h. Let $2 \le k \le n$. Then the following holds:*

(a) For $k < n-h+1$, the cell $d_n\Psi_n^k(\underline{i})$ is collapsible.

(b) For $k = n-h+1$, we have $d_n\Psi_n^k(\underline{i}) = \mu(\underline{i})$.

(c) For $k > n-h+1$, the cell $d_n\Psi_n^k(\underline{i})$ is redundant and $\mathrm{value}(\mu(d_n\Psi_n^k(\underline{i}))) < \mathrm{value}(\underline{i})$.

Proof. (a) and (b) are clear. Assume that $k > n-h+1$. Then

$$d_n\Psi_n^k(\underline{i}) = [\![i_n|\dots|i_{k+1}|i_{k-1}|\dots|i_1]\!]$$

has height $n-k$, because $[\![i_n|\dots|i_{k+1}]\!]$ is essential (since $k > n-h+1$) and $i_{k-1} - i_{k+1} = (i_{k-1}-i_k) + (i_k - i_{k+1}) \ge 2q$. Again from $k > n-h+1$ it follows that $i_{k+1} - i_{k+2} > q$, and thus $d_n\Psi_n^k(\underline{i})$ is redundant. Its collapsible partner is given by

$$\mu(d_n\Psi_n^k(\underline{i})) = [\![i_n|\dots|i_{k+1}|i_{k+1}+q|i_{k-1}|\dots|i_1]\!].$$

We see that $\mathrm{value}(\underline{i}) - \mathrm{value}(\mu(d_n\Psi_n^k(\underline{i}))) = i_k - (i_{k+1}+q) = (i_k - i_{k+1}) - q \ge q+1-q = 1$. The Lemma is proven. $\square$

Corollary 4.4.19 *The matching μ is noetherian.*

Proof. Assume that μ is not noetherian, i.e. there is an infinite strictly descending chain of redundant cells, say $\underline{i_1} \to \underline{i_2} \to \underline{i_3} \to \dots$. The situation is depicted in the following diagram:

$$\begin{tikzcd}
 & \mu(\underline{i_1}) \arrow[dr, mapsto] & & \mu(\underline{i_2}) \arrow[dr, mapsto] & & \mu(\underline{i_3}) \arrow[dr, mapsto] & & \dots \\
\underline{i_1} \arrow[ur, mapsto, "\mu"'] & & \underline{i_2} \arrow[ur, mapsto, "\mu"'] & & \underline{i_3} \arrow[ur, mapsto, "\mu"'] & & \underline{i_4} \arrow[ur, mapsto, "\mu"'] &
\end{tikzcd}$$

By the definition of the relation $\nrightarrow$ we have $[\partial^{\mathbb{V}}(\mu(\underline{i_k})) : \underline{i_{k+1}}] \neq 0$. Lemma 4.4.18.(c) and Lemma 4.4.17 now tell us that $\mathrm{value}(\underline{i_{k+1}}) < \mathrm{value}(\underline{i_k})$ or the right-most entry of $\underline{i_{k+1}}$ is strictly smaller than the right-most entry of $\underline{i_k}$. Observe that the right-most entries of $\underline{i_k}$ and $\mu(\underline{i_k})$ coincide (cf. Remark 4.4.12) and that the right-most entry does not increase when passing from $\mu(\underline{i_k})$ to $\underline{i_{k+1}}$ (cf. Proposition 4.4.2). Hence, for some index t, the cell $\underline{i_t}$ has negative right-most entry or negative value. This is a contradiction, because the entries of our tuples are natural numbers. The Corollary is proven. $\square$

Altogether we have shown that for $m = 0, \dots, \infty$ and $0 < p \le q$ the map μ defines a noetherian matching on the Visy complex $(\mathbb{V}_*\boldsymbol{t}_m(p,q), \partial_*^{\mathbb{V}})$. The following subsection is concerned with proving that μ is perfect.

4.4.4 Perfectness of μ

We are going to show that the associated Morse complex $(\mathbb{V}_* \boldsymbol{t}_m(p,q))^\theta$ has no non-trivial differential.

Throughout, let $m = 0, \dots, \infty$ and $0 < p \le q$.

Lemma 4.4.20 *Let $\underline{i} = [\![i_n|\dots|i_1]\!]$ be a redundant cell of height h. Its collapsible partner is then given by $\underline{j} = \mu(\underline{i}) = [\![i_n|\dots|i_{n-h+1}|i_{n-h+1}{+}q|i_{n-h}|\dots|i_1]\!]$. We have the following:*

(a) For $k > n-h+1$, the cell $d_{n+1}\Psi^k_{n+1}(\underline{j})$ is redundant of height $n+1-k$.

(b) $d_{n+1}\Psi^{n-h+1}_{n+1}(\underline{j}) = \underline{i}$.

(c) $\theta(d_{n+1}\Psi^{n-h}_{n+1}(\underline{j})) = \theta(d_0\Phi^{n-h}_1(\underline{j}))$.

(d) For $k < n-h$, the cell $d_{n+1}\Psi^k_{n+1}(\underline{j})$ is collapsible.

(e) For $k < n-h$, the cell $d_0\Phi^k_1(\underline{j})$ is collapsible.

Proof. Using Proposition 4.4.2, parts (a) and (b) are easy. (Indeed, part (b) is equivalent to Lemma 4.4.18.(b).) We now prove (c). Assume first that $h = n-1$. We then have $n-h = 1$, and Proposition 4.4.2 yields $d_{n+1}\Psi^1_{n+1}(\underline{j}) = d_0\Phi^1_1(\underline{j})$. Therefore, (c) holds for $h = n-1$. Let now $h < n-1$. Observe that

$$\begin{aligned} d_{n+1}\Psi^{n-h}_{n+1}(\underline{j}) &= [\![i_n|\dots|i_{n-h+1}|i_{n-h+1}+q|i_{n-h-1}|\dots|i_1]\!], \\ d_0\Phi^{n-h}_1(\underline{j}) &= [\![i_n|\dots|i_{n-h+1}|i_{n-h+1}+q|i_{n-h-1}-p|\dots|i_1-p]\!]. \end{aligned}$$

We see that $d_{n+1}\Psi^{n-h}_{n+1}(\underline{j})$ and $d_0\Phi^{n-h}_{n+1}(\underline{j})$ are collapsible of height $h+1$. In particular, both of them are sent to zero by θ. Part (c) is proven. Parts (d) and (e) are immediate applications of Proposition 4.4.2. □

Proposition 4.4.21 *Let $\underline{i} = [\![i_n|\dots|i_1]\!]$ be redundant of height h. Then*

$$\theta^\infty[\![i_n|\dots|i_1]\!] = \theta^\infty[\![i_n|\dots|i_{n-h+1}|i_{n-h}-p|\dots|i_1-p]\!].$$

Proof. We do induction on the height. First, assume that $h = 0$. We need to show that $\theta^\infty[\![i_n|\dots|i_1]\!] = \theta^\infty[\![i_n-p|\dots|i_1-p]\!]$. The collapsible partner of $\underline{i}$ is $\mu(\underline{i}) = [\![1|i_n|\dots|i_1]\!]$. We use Lemma 4.4.16 to compute $[\partial^{\mathbb{V}}(\mu(\underline{i})) : \underline{i}] = (-1)^{n+1}$. We obtain

$$\begin{aligned} \theta(\underline{i}) &= \underline{i} + \partial^{\mathbb{V}}(V(\underline{i})) \\ &= \underline{i} + \partial^{\mathbb{V}}(-\mu(\underline{i}) \cdot [\partial^{\mathbb{V}}(\mu(\underline{i})) : \underline{i}]^{-1}) \\ &= [\![i_n|\dots|i_1]\!] + (-1)^n \cdot \partial^{\mathbb{V}}[\![1|i_n|\dots|i_1]\!]. \end{aligned}$$

Consider now

$$\begin{aligned} \theta^2(\underline{i}) &= \theta([\![i_n|\dots|i_1]\!] + (-1)^n \cdot \partial^{\mathbb{V}}[\![1|i_n|\dots|i_1]\!]) \\ &= \theta[\![i_n|\dots|i_1]\!] + (-1)^n \cdot \theta\left(\sum_{k=2}^{n+1}(-1)^k(d_{n+1}\Psi^k_{n+1} - d_0\Phi^k_1)[\![1|i_n|\dots|i_1]\!]\right). \end{aligned}$$

Now, for $k \leq n-h=n$ the difference $d_{n+1}\Psi^k_{n+1}[\![1|i_n|\dots|i_1]\!] - d_0\Phi^k_1[\![1|i_n|\dots|i_1]\!]$ lies in the kernel of θ, cf. Lemma 4.4.20.(c),(d),(e). The above therefore simplifies to

$$\theta^2(\underline{i}) = \theta[\![i_n|\dots|i_1]\!] - \theta(d_{n+1}\Psi^{n+1}_{n+1}[\![1|i_n|\dots|i_1]\!]) + \theta(d_0\Phi^{n+1}_1[\![1|i_n|\dots|i_1]\!]).$$

Note that by Lemma 4.4.20.(b) the first and second summand cancel out each other. We use Proposition 4.4.2 to compute $d_0\Phi^{n+1}_1[\![1|i_n|\dots|i_1]\!] = [\![i_n-p|\dots|i_1-p]\!]$. Thus, altogether we have shown that

$$\theta^2(\underline{i}) = \theta([\![i_n-p|\dots|i_1-p]\!]).$$

Applying θ^∞ to both sides, the claim follows for $h=0$.

Assume now that $\underline{i}$ is redundant of height $h>0$ and the Proposition has already been shown for all redundant cells of height $<h$. Set $\underline{j} = \mu(\underline{i}) = [\![i_n|\dots|i_{n-h+1}|i_{n-h+1}+q|i_{n-h}|\dots|i_1]\!]$. Lemma 4.4.16 gives $[\partial^{\mathbb{V}}(\mu(\underline{i})):\underline{i}] = (-1)^{n-h+1}$, and we obtain

$$\begin{aligned}\theta(\underline{i}) &= \underline{i} + (-1)^{n-h}\cdot\partial^{\mathbb{V}}(\underline{j})\\ &= \underline{i} + (-1)^{n-h}\cdot\sum_{k=2}^{n+1}(-1)^k\cdot(d_{n+1}\Psi^k_{n+1}(\underline{j}) - d_0\Phi^k_1(\underline{j})).\end{aligned}\tag{4.11}$$

We now claim that for $k \neq n-h+1$ the difference $d_{n+1}\Psi^k_{n+1}(\underline{j}) - d_0\Phi^k_1(\underline{j})$ lies in the kernel of θ^∞. For $k \leq n-h$ this again follows from Lemma 4.4.20.(c),(d),(e). If $k > n-h+1$ then, by Proposition 4.4.2,

$$\begin{aligned}d_{n+1}\Psi^k_{n+1}(\underline{j}) &= [\![i_n|\dots|i_k|i_{k-2}|\dots|i_{n-h+1}|i_{n-h+1}+q|i_{n-h}|\dots|i_1]\!]\\ d_{n+1}\Phi^k_1(\underline{j}) &= [\![i_n|\dots|i_k|i_{k-2}-p|\dots|i_{n-h+1}-p|i_{n-h+1}+q-p|i_{n-h}-p|\dots|i_1-p]\!].\end{aligned}$$

The former cell is redundant of height $n-k+1$. Since $n-k+1 < n-(n-h+1)+1 = h$, the induction hypothesis applies. The Claim is proven.

Applying θ^∞ to both sides of (4.11) the Claim yields the following:

$$\theta^\infty(\underline{i}) = \theta^\infty(\underline{i}) - \theta^\infty(d_{n+1}\Psi^{n-h+1}_{n+1}(\underline{j})) + \theta^\infty(d_0\Phi^{n-h+1}_1(\underline{j})).$$

Note that by Lemma 4.4.20.(b) we have $d_{n+1}\Psi^{n-h+1}_{n+1}(\underline{j}) = \underline{i}$. The first and second summand therefore cancel out each other, yielding

$$\theta^\infty(\underline{i}) = \theta^\infty(d_0\Phi^{n-h+1}_1(\underline{j})).$$

Observe that $d_0\Phi^{n-h+1}_1(\underline{j}) = [\![i_n|\dots|i_{n-h+1}|i_{n-h}-p|\dots|i_1-p]\!]$, and whence the Proposition. $\square$

Corollary 4.4.22 *For $m=0,\dots,\infty$ and $0<p\leq q$, the noetherian matching μ on $\mathbb{V}_*t_m(p,q)$ is perfect.*

Proof. Let $[\![i_n|\dots|i_1]\!]$ be an essential n-cell in $\mathbb{V}_*t_m(p,q)$. We have to show that $\theta^\infty \circ \partial_n^{\mathbb{V}}[\![i_n|\dots|i_1]\!] = 0$. By Corollary 4.4.3 we have

$$\theta^\infty \circ \partial_n^{\mathbb{V}}[\![i_n|\dots|i_1]\!] = \sum_{k=2}^{n}(-1)^k \cdot \theta^\infty(d_n\Psi_n^k[\![i_n|\dots|i_1]\!] - d_0\Phi_1^k[\![i_n|\dots|i_1]\!]).$$

Recall from Proposition 4.4.2 that

$$\begin{aligned} d_n\Psi_n^k[\![i_n|\dots|i_1]\!] &= [\![i_n|\dots|i_{k+1}|i_{k-1}|\dots|i_1]\!], \\ d_0\Phi_1^k[\![i_n|\dots|i_1]\!] &= [\![i_n|\dots|i_{k+1}|i_{k-1}-p|\dots|i_1-p]\!]. \end{aligned}$$

Observe that $[\![i_n|\dots|i_{k+1}|i_{k-1}|\dots|i_1]\!]$ is redundant of height $h = n-k$. We can therefore apply Proposition 4.4.21, yielding that $\theta^\infty \circ \partial_n^{\mathbb{V}}[\![i_n|\dots|i_1]\!] = 0$, as desired. □

Using Corollary 4.4.22 we can completely describe the homology of $t_m(p,q)$ for $0 < p \le q$. (We remind the reader that for $p = 0$ we could already do this using the Visy complex, which in this case only has zero differentials, cf. Remark 4.4.6.)

4.5 Homological results on Thompson monoids

In this section we derive recursion formulas for the homology groups of the Thompson monoids with parameters $m = 0,\dots,\infty$ and $0 < p \le q$. For $m = \infty$ we can give explicit results. In particular, this section provides computations of the homology groups $H_*(\mathcal{T}_\infty(p,q))$ of all generalized Thompson groups with parameters $q = p+1 > 1$, cf. Corollary 4.3.12.

Throughout, let $m = 0,\dots,\infty$ and $0 < p \le q$.

4.5.1 Homological dimension

Let $[\![i_n|\dots|i_1]\!]$ be an essential cell in $\mathbb{V}_*t_m(p,q)$, i.e. the difference sequence associated to $(1-q, i_n, \dots, i_1)$ lies in $\{q+1,\dots,2q-1\}^{n-1} \times \{q,\dots,2q-1\}$ and hence

$$i_1 \ge 1 - q + (n-1)\cdot(q+1) + q = n\cdot(q+1) - q.$$

It follows that in $\mathbb{V}_*t_m(p,q)$ there exist essential n-cells only if $m \ge n\cdot(q+1)-q$. Thus, for fixed m, every essential cell in $\mathbb{V}_*t_m(p,q)$ has at most $(m+q)/(q+1)$ entries. On the other hand, if $q > 1$, then in $\mathbb{V}_*t_m(p,q)$ there is at least one essential cell of dimension $n = \lfloor (m+q)/(q+1)\rfloor$, namely $[\![2|q+3|2q+4|\dots|(n-2)\cdot q+n|(n-1)\cdot q+n]\!]$. (Indeed, for $n = \lfloor (m+q)/(q+1)\rfloor$ we have $(n-1)\cdot q+n = \lfloor (m+q)/(q+1)\rfloor\cdot(q+1)-q \le m+q-q = m$, and thus this cell lies in $\mathbb{V}_*t_m(p,q)$. Essentiality is obvious.) We have proven the following.

Proposition 4.5.1 *Let $m < \infty$, $q > 1$ and $0 < p \leq q$. Then $t_m(p,q)$ has homological dimension*

$$\operatorname{hodim} t_m(p,q) = \left\lfloor \frac{m-1}{q+1} \right\rfloor + 1.$$

For $q = 1$ the set $\{q+1, \dots, 2q-1\}$ is empty. It follows that there are no essential cells in dimensions > 1. Furthermore, since $\{q, \dots, 2q-1\} = \{1\}$, the only essential cell in dimension 1 is $[\![1]\!]$. This shows that $\operatorname{hodim} t_m(1,1) = 1$ for $m > 0$.

4.5.2 The homology of Thompson monoids

Explicit computations for small m

We will explicitly compute $H_*(t_m(p,q))$ for $m \leq 2q$. Recall that we have a one-to-one correspondence between essential n-cells in $\mathbb{V}_* t_m(p,q)$ and free generators of $H_n(t_m(p,q))$.

If $m \leq q$ then $t_m(p,q)$ is free on m generators, and we know its homology. It is easy to check that for $m = q+1$ there are no other essential cells in $\mathbb{V}_* t_{q+1}(p,q)$ than $[\![\,]\!], [\![1]\!], \dots, [\![q]\!]$. (By Remark 4.4.8.(b) there are no further 1-cells, and by Proposition 4.5.1 there are no essential cells in dimension ≥ 2.)

For $m > q+1$ and $m \leq 2q$ we can determine all essential cells of $\mathbb{V}_* t_m(p,q)$ by an easy combinatorial argument. First of all, Proposition 4.5.1 tells us that there are no essential cells of dimension ≥ 3. Furthermore, there is exactly one 0-cell, namely $[\![\,]\!]$, and there are exactly q-many 1-cells, namely $[\![1]\!], \dots, [\![q]\!]$. How many essential 2-cells are there?

Recall that $[\![i_2|i_1]\!]$ is an essential cell in $\mathbb{V}_* t_m(p,q)$ if and only if $i_2 \in \{2, \dots, q\}$ and $i_1 - i_2 \in \{q, \dots 2q-1\}$. For convenience we first discuss the question how many essential 2-cells of the form $[\![2|i_1]\!]$ we have. $[\![2|i_1]\!]$ is essential in $\mathbb{V}_* t_m(p,q)$ if and only if $i_1 \leq m$ and $i_1 - 2 \in \{q, \dots, 2q-1\}$. Note that, since $q+1 < m \leq 2q$, the cell $[\![2|m]\!]$ is always essential. We conclude that in $\mathbb{V}_* t_m(p,q)$ all essential 2-cells with left-most entry being 2 are given by

$$[\![2|q+2]\!], [\![2|q+3]\!], \dots, [\![2|m]\!].$$

Thus, there's a total of $m-q-1$ such essential 2-cells. Similarly one shows that there are exactly $(m-q-2)$-many essential 2-cells with left-most entry being 3. Continuing this way, we see that there is exactly one essential 2-cell with left-most entry being $m-q$. (Note that $m - q \leq q$.) Altogether we obtain that the total number of essential 2-cells in $\mathbb{V}_* t_m(p,q)$ is given by

$$\sum_{i=1}^{m-q-1} i = \frac{(m-q)\cdot(m-q-1)}{2}.$$

We have proven the following.

Proposition 4.5.2 *Let $m < \infty$ and $0 < p \leq q$ subject to $m \leq 2q$. Then*

$$H_0(\boldsymbol{t}_m(p,q)) \cong \mathbb{Z},$$
$$H_1(\boldsymbol{t}_m(p,q)) \cong \begin{cases} \mathbb{Z}^m & \text{if } m < q, \\ \mathbb{Z}^q & \text{if } q \leq m \leq 2q, \end{cases}$$
$$H_2(\boldsymbol{t}_m(p,q)) \cong \begin{cases} 0 & \text{if } m < q, \\ \mathbb{Z}^{\frac{(m-q)\cdot(m-q-1)}{2}} & \text{if } q \leq m \leq 2q. \end{cases}$$

and $H_i(\boldsymbol{t}_m(p,q)) = 0$ for $i \geq 3$.

A recursion formula for large m

In the preceding subsection we explicitly computed $H_*(\boldsymbol{t}_m(p,q))$ for $m \leq 2q$. We now derive recursion formulas for $H_*(\boldsymbol{t}_m(p,q))$ for $m > 2q$. In what follows we assume $m \geq 2q+1$ to be fixed.

Recall that, as a $\mathbb{Z}$-module, $H_*(\boldsymbol{t}_m(p,q))$ is freely generated by all tuples of the form $[\![i_n|\dots|i_1]\!]$ with associated difference sequence in $\{q+1,\dots,2q-1\}^{n-1} \times \{q,\dots,2q-1\}$. For $a = q+1,\dots,2q-1$ define

$$\begin{aligned} \zeta_{n,a} : H_n(\boldsymbol{t}_{m-a}(p,q)) &\longrightarrow H_{n+1}(\boldsymbol{t}_m(p,q)) \\ [\![i_n|\dots|i_1]\!] &\longmapsto [\![1-q+a|i_n+a|\dots|i_1+a]\!] \end{aligned} \tag{4.12}$$

and extend linearly. Note that this is indeed well-defined, for if $[\![i_n|\dots|i_1]\!]$ is an essential n-cell in $\mathbb{V}_*\boldsymbol{t}_{m-a}(p,q)$ then $[\![1-q+a|i_n+a|\dots|i_1+a]\!]$ is an essential $(n+1)$-cell in $\mathbb{V}_*\boldsymbol{t}_m(p,q)$.

We will study the simultaneous images of these maps:

$$\begin{array}{c} H_n(\boldsymbol{t}_{m-(q+1)}(p,q)) \oplus \dots \oplus H_n(\boldsymbol{t}_{m-(2q-1)}(p,q)) \\ \Big\downarrow{\scriptstyle \zeta_n := \zeta_{n,q+1} \oplus \dots \oplus \zeta_{n,2q-1}} \\ H_{n+1}(\boldsymbol{t}_m(p,q)) \end{array} \tag{4.13}$$

Proposition 4.5.3 *Let ζ_n be defined as above.*

(a) For all $n \geq 0$, the map ζ_n is injective.

(b) ζ_n is an isomorphism for $n \geq 1$.

(c) $\operatorname{coker}(\zeta_0) \cong \mathbb{Z}$.

Proof. (a) Injectivity is clear by closely looking at (4.12): The parameter a is uniquely determined by the left-most entry of the tuple $[\![1-q+a|i_n+a|\dots|i_1+a]\!]$, and once we know a, we can completely recover the preimage $[\![i_n|\dots|i_1]\!]$.

(b) For $n \geq 1$ the inverse map is given by sending an $(n+1)$-cell $[\![i_{n+1}|\dots|i_1]\!]$ to the n-cell $[\![i_n - a|\dots|i_1 - a]\!] \in H_*(t_{m-a}(p,q))$, where $a = i_{n+1} + q - 1$.

(c) $H_1(t_m(p,q))$ is freely generated by the elements $[\![1]\!], \dots, [\![q]\!]$. For $a = q+1, \dots, 2q-1$, $H_0(t_{m-a}(p,q))$ is cyclic with generator $[\![\ ]\!]$, which under ζ_0 is mapped to $[\![1-q+a]\!]$. Thus, every generator in $H_1(t_m(p,q))$ is hit, except for $[\![1]\!]$. □

Corollary 4.5.4 *Let $m \geq 2q+1$, $0 < p \leq q$. Then*

$$H_n(t_m(p,q)) \cong \begin{cases} \bigoplus_{a=q+1}^{2q-1} H_{n-1}(t_{m-a}(p,q)) & \text{if } n \geq 2, \\ \mathbb{Z}^q & \text{if } n = 1, \\ \mathbb{Z} & \text{if } n = 0. \end{cases}$$

Proof. The cases $n = 0, 1$ are clear, and for $n = 2$ the statement follows from Proposition 4.5.3. □

The homology of some generalized Thompson groups

For $m = \infty$ we may consider ζ_n as a map

$$H_n(t_\infty(p,q)) \oplus \dots \oplus H_n(t_\infty(p,q)) \longrightarrow H_{n+1}(t_\infty(p,q)),$$

yielding isomorphisms

$$H_{n+1}(t_\infty(p,q)) \cong [H_n(t_\infty(p,q))]^{q-1}$$

for $n \geq 1$. Note that $H_1(t_\infty(p,q)) \cong \mathbb{Z}^q$. An easy inductive argument gives the following.

Proposition 4.5.5 *Let $0 < p \leq q$. Then*

$$H_n(t_\infty(p,q)) \cong \begin{cases} \mathbb{Z} & \text{if } n = 0, \\ \mathbb{Z}^{(q-1)^{n-1} \cdot q} & \text{if } n > 0. \end{cases}$$

Recall that for $m = \infty$ and $q = p+1$ the monoids $t_\infty(p,q)$ are cancellative and satisfy the Ore condition. Combining Proposition 4.5.5 and Corollary 4.3.12 we obtain:

Corollary 4.5.6 *Let $q > 1$. Then*

$$H_n(\mathcal{T}_\infty(q-1,q)) \cong \begin{cases} \mathbb{Z} & \text{if } n = 0, \\ \mathbb{Z}^{(q-1)^{n-1} \cdot q} & \text{if } n > 0. \end{cases}$$

Recall from Remark 4.3.2 that $\mathcal{T}_\infty(n-1,n) \cong F(r, \mathbb{Z}[\frac{1}{n}], \langle n \rangle)$. The above Corollary therefore provides a recomputation of the homology of the groups $F(r, \mathbb{Z}[\frac{1}{n}], \langle n \rangle)$. The homology of these groups has first been computed by Stein [Ste92, Theorem 2.6].

For $r = 1$, $p = 1$, $q = 2$, we find that in every positive degree the homology of Thompson's group F is free abelian on two generators. In degree 1, these generators are represented by $[\![1]\!]$ and $[\![2]\!]$, in degrees $n > 1$ they are represented by the elements $[\![2|5|\dots|3n-4|3n-1]\!]$ and $[\![2|5|\dots|3n-4|3n-2]\!]$.

A stabilization result

The inclusion of generating sets $\{\xi_1,\dots,\xi_m\} \subset \{\xi_1,\dots,\xi_{m+1}\}$ induces a well-defined map $\boldsymbol{t}_m(p,q) \to \boldsymbol{t}_{m+1}(p,q)$, compare Remark 4.3.13.(a). From this we obtain an infinite sequence of n-th homology groups,

$$H_n(\boldsymbol{t}_0(p,q)) \to H_n(\boldsymbol{t}_1(p,q)) \to H_n(\boldsymbol{t}_2(p,q)) \to \dots$$

This sequence stabilizes. More precisely we have the following.

Proposition 4.5.7 *Let $0 < p \le q$. Then for $m \ge n\cdot(2q-1)+1-q$ the above maps induce an isomorphism*

$$H_n(\boldsymbol{t}_m(p,q)) \cong H_n(\boldsymbol{t}_\infty(p,q)).$$

Proof. It suffices to show that for $m \ge n\cdot(2q-1)+1-q$ the essential cells in the respective Visy modules $\mathbb{V}_n\boldsymbol{t}_m(p,q)$ and $\mathbb{V}_n\boldsymbol{t}_\infty(p,q)$ coincide. By Remark 4.4.10, every essential n-cell in $\mathbb{V}_n\boldsymbol{t}_m(p,q)$ can be considered as an essential n-cell in $\mathbb{V}_n\boldsymbol{t}_\infty(p,q)$. Vice versa, assume that $[\![i_n|\dots|i_1]\!]$ is essential in $\mathbb{V}_n\boldsymbol{t}_\infty(p,q)$. Then $i_1 \le (n-1)\cdot(2q-1)+q = n\cdot(2q-1)+1-q \le m$, and thus we may consider $[\![i_n|\dots|i_1]\!]$ as an essential cell in $\mathbb{V}_n\boldsymbol{t}_m(p,q)$. □

4.5.3 A remark on the monoids $\boldsymbol{t}_m(1,q),\dots,\boldsymbol{t}_m(q,q)$

Recall that for fixed $m > 0$, $q > 0$ and $p = 1,\dots,q$ the monoids $\boldsymbol{t}_m(p,q)$ have the same homology. We are now going to show that if m is large enough with respect to q, then the monoids $\boldsymbol{t}_m(1,q),\dots,\boldsymbol{t}_m(q,q)$ are pairwise not isomorphic. (Of course, we cannot expect this to hold for all m, since for $m \le q$ the monoids $\boldsymbol{t}_m(p,q)$ are free on m generators.)

We need a Lemma about permutations with prescribed bounds. Denote by $\mathcal{S}_n$ the n-th symmetric group.

Lemma 4.5.8 *Let d be such that $1 \le 2d+1 \le n$. If $\pi \in \mathcal{S}_n$ satisfies the following condition,*

$$j - i \le d \implies \pi(j) - \pi(i) \le d \tag{4.14}$$

then $\pi = \mathrm{id}$.

Before we give the proof, note that the Lemma does not hold for $d \geq n/2$. For example, taking $n = 2d$, the transposition $(d\ d{+}1)$ satisfies condition (4.14), because whenever $\pi(j) - \pi(i) > d$ then we necessarily have $\pi(j) > d + 1$ and $\pi(i) < d$, yielding $\pi(j) = j$, $\pi(i) = i$ and thus $j - i > d$, cf. Figure 4.18.

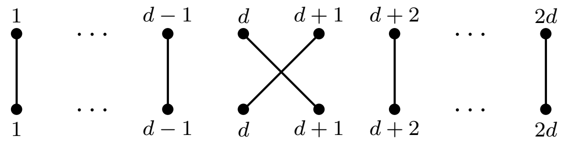


Figure 4.18: Counterexample to Lemma 4.5.8 for $n = 2d$.

We now prove Lemma 4.5.8.

Proof. Let $\pi \in \mathcal{S}_n$ satisfy (4.14).

Claim 1. For every k, π maps $\{\max\{k{-}d, 1\}, \dots, n\}$ into $\{\max\{\pi(k){-}d, 1\}, \dots, n\}$, cf. Figure 4.19.

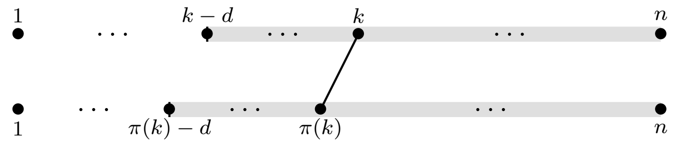


Figure 4.19: π maps entries $\geq k - d$ to entries $\geq \pi(k) - d$.

Let $j \in \{\max\{k{-}d, 1\}, \dots, n\}$. In particular, $j \geq 1$ and $j \geq k{-}d$. The latter is equivalent to $k - j \leq d$. Thus, by (4.14), we have $\pi(k) - \pi(j) \leq d$, yielding $\pi(j) \geq \pi(k) - d$ and hence $\pi(j) \in \{\max\{\pi(k) - d, 1\}, \dots, n\}$. Claim 1 is proven.

Claim 1 has two important consequences.

Claim 2. For $k \geq d + 1$ we have $\pi(k) \leq k$.

Clearly, if $k \geq d + 1$ then $k - d \geq 1$ and thus π maps $\{k - d, \dots, n\}$ into $\{\max\{\pi(k) - d, 1\}, \dots, n\}$. Since π is injective, the latter must contain at least $n-(k-d)+1$ elements, forcing $\pi(k) \leq k$. Claim 2 is proven.

Claim 3. For $k \leq d + 1$ we have $\pi(k) \leq d + 1$.

Clearly, if $k \leq d + 1$ then $k - d \leq 1$ and thus $\{\max\{k - d, 1\}, \dots, n\} = \{1, \dots, n\}$, which under π is mapped bijectively onto $\{1, \dots, n\}$. We therefore must have $\max\{\pi(k) - d, 1\} \leq 1$ and hence $\pi(k) \leq d + 1$. Claim 3 is proven.

There are "opposite" statements to Claims 1-3, reading as follows. First, for every k, π maps $\{1, \dots \min\{n, k + d\}\}$ into $\{1, \dots, \min\{n, \pi(k) + d\}\}$. From this we conclude that, secondly, for $k \leq n-d$ we have $\pi(k) \geq k$, and thirdly, for $k \geq n-d$ we have $\pi(k) \geq n-d$.

Putting everything together, we see that every $k = d + 1, \dots, n - d$ is a fixed point of π. Also note that there is at least one such fixed point, because $d + 1 \leq n - d$ (since $n \geq 2d+1$). Furthermore, we conclude that π fixes $\{1, \dots, d\}$ and $\{n - d + 1, \dots, n\}$ (as sets). This situation is depicted in Figure 4.20.

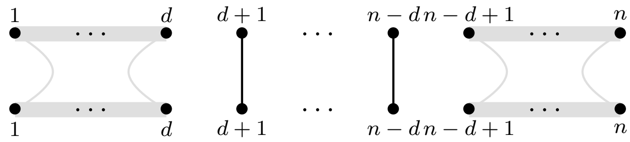


Figure 4.20: Situation in the proof of Lemma 4.5.8.

By a simple counting argument, the restriction $\pi| : \{n-d+1, \ldots, n\} \to \{n-d+1, \ldots, n\}$ is a bijection. On the other hand, by Claim 2, for every $k \in \{n-d+1, \ldots, n\}$ we have $\pi(k) \le k$. It follows that π is the identity on $\{n-d+1, \ldots, n\}$. Using the opposite statements, one shows that the restriction $\pi| : \{1, \ldots, d\} \to \{1, \ldots, d\}$ is again the identity.

Altogether, for every $k = 1, \ldots, n$ we have $\pi(k) = k$. The Lemma is proven. □

Corollary 4.5.9 *Let q be such that $1 \le 2q - 1 \le n$. If $\sigma \in \mathcal{S}_n$ satisfies the following condition,*

$$j - i \ge q \implies \sigma(j) - \sigma(i) \ge q \tag{4.15}$$

then $\sigma = \mathrm{id}$.

Proof. Assume $\sigma(j) - \sigma(i) < q$ and apply Lemma 4.5.8 with $\pi = \sigma^{-1}$, $d = q - 1$. □

Proposition 4.5.10 *Fix $m < \infty$ and let q be such that $1 \le 2q - 1 \le m$, i.e. $q > 0$ and $q \le (m+1)/2$. Then the monoids $t_m(p, q)$ are pairwise not isomorphic for every $p = 1, \ldots, q$.*

Proof. Recall that an element of a monoid is said to be an atom if it cannot be factored into two non-trivial elements. The atoms of $t_m(p, q)$ are $\xi_1, \ldots, \xi_m$.

Let $1 \le p, p' \le q$ and let $\varphi : t_m(p, q) \to t_m(p', q)$ be an isomorphism of monoids. Clearly, φ must map atoms bijectively onto atoms, and therefore φ gives rise to a permutation $\sigma \in \mathcal{S}_m$ as follows. For $k = 1, \ldots, m$ let $\sigma(k)$ be the element uniquely determined by

$$\varphi(\xi_k) = \xi_{\sigma(k)}.$$

Now, for $j - i \ge q$ we have a relation $\xi_{j-p}\xi_i = \xi_i\xi_j$ in $t_m(p, q)$, and thus we must have a relation $\xi_{\sigma(j-p)}\xi_{\sigma(i)} = \xi_{\sigma(i)}\xi_{\sigma(j)}$ in $t_m(p', q)$. This has two consequences. First, $\sigma(j-p) = \sigma(j) - p'$. Secondly, if $j - i \ge q$ then $\sigma(j) - \sigma(i) \ge q$. So we are in the situation of Corollary 4.5.9, yielding $\sigma = \mathrm{id}$, and from $\sigma(j-p) = \sigma(j) - p'$ we conclude $p = p'$. □

Remark 4.5.11 Note that for $m < \infty$ and $m > q$ the monoid $t_m(0, q)$ is not isomorphic to any of the monoids $t_m(1, q)$, …, $t_m(q, q)$. This can be seen by comparing the respective first homology groups: From Remark 4.4.6 it follows that $H_1(\boldsymbol{t}_m(0,q)) \cong \mathbb{Z}^m$, because every 1-cell $[\![1]\!], \dots, [\![m]\!]$ is a generator of the Visy complex $\mathbb{V}_*\boldsymbol{t}_m(0,q)$ and thus (since $p = 0$) a free generator in homology. For $p = 1, \dots, q$ our analysis in Subsection 4.5.2 shows that for $m \geq q$ we have $H_1(\boldsymbol{t}_m(p,q)) \cong \mathbb{Z}^q$. Thus, for $m > q$ and every $p = 1, \dots, q$, the monoids $\boldsymbol{t}_m(0,q)$ and $\boldsymbol{t}_m(p,q)$ have different homology.

Bibliography

[ABE08] J. Abhau, C.-F. Bödigheimer, and R. Ehrenfried. Homology of the mapping class group $\Gamma_{2,1}$ for surfaces of genus 2 with a boundary curve. In *The Zieschang Gedenkschrift*, volume 14 of *Geom. Topol. Monogr.*, pages 1–25. Geom. Topol. Publ., Coventry, 2008.

[Ani86] D. J. Anick. On the homology of associative algebras. *Trans. Amer. Math. Soc.*, 296(2):641–659, 1986.

[BB97] M. Bestvina and N. Brady. Morse theory and finiteness properties of groups. *Invent. Math.*, 129(3):445–470, 1997.

[BG84] K. S. Brown and R. Geoghegan. An infinite-dimensional torsion-free FP_∞ group. *Invent. Math.*, 77(2):367–381, 1984.

[Böd] C.-F. Bödigheimer. Hilbert Uniformization of Riemann Surfaces : I. in preparation.

[Böd90] C.-F. Bödigheimer. The topology of moduli spaces, part I: Hilbert Uniformization. *Math. Gott. (Preprint Series SFB 170, Göttingen)*, 7+8, 1990.

[Bro87] K. S. Brown. Finiteness properties of groups. In *Proceedings of the Northwestern conference on cohomology of groups (Evanston, Ill., 1985)*, volume 44, pages 45–75, 1987.

[Bro92] K. S. Brown. The geometry of rewriting systems: a proof of the Anick-Groves-Squier theorem. In *Algorithms and classification in combinatorial group theory (Berkeley, CA, 1989)*, volume 23 of *Math. Sci. Res. Inst. Publ.*, pages 137–163. Springer, New York, 1992.

[BV] C.-F. Bödigheimer and B. Visy. Factorable groups and their homology. in preparation.

[CE99] H. Cartan and S. Eilenberg. *Homological algebra*. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 1999. With an appendix by David A. Buchsbaum, Reprint of the 1956 original.

[CFP96] J. W. Cannon, W. J. Floyd, and W. R. Parry. Introductory notes on Richard Thompson's groups. *Enseign. Math. (2)*, 42(3-4):215–256, 1996.

[Cha00] M. K. Chari. On discrete Morse functions and combinatorial decompositions. *Discrete Math.*, 217(1-3):101–113, 2000. Formal power series and algebraic combinatorics (Vienna, 1997).

[CMW04] R. Charney, J. Meier, and K. Whittlesey. Bestvina's normal form complex and the homology of Garside groups. *Geom. Dedicata*, 105:171–188, 2004.

[Coh92] D. E. Cohen. A monoid which is right FP_∞ but not left FP_1. *Bull. London Math. Soc.*, 24(4):340–342, 1992.

[Coh97] D. E. Cohen. String rewriting and homology of monoids. *Math. Structures Comput. Sci.*, 7(3):207–240, 1997.

[CP61] A. H. Clifford and G. B. Preston. *The algebraic theory of semigroups. Vol. I.* Mathematical Surveys, No. 7. American Mathematical Society, Providence, R.I., 1961.

[Deh02] P. Dehornoy. Groupes de Garside. *Ann. Sci. École Norm. Sup. (4)*, 35(2):267–306, 2002.

[Ebe] J. Ebert. Hilbert Uniformization of Riemann Surfaces : II. in preparation.

[Fie84] Z. Fiedorowicz. Classifying spaces of topological monoids and categories. *Amer. J. Math.*, 106(2):301–350, 1984.

[For98] R. Forman. Morse theory for cell complexes. *Adv. Math.*, 134(1):90–145, 1998.

[Gro90] J. R. J. Groves. Rewriting systems and homology of groups. In *Groups—Canberra 1989*, volume 1456 of *Lecture Notes in Math.*, pages 114–141. Springer, Berlin, 1990.

[Hig74] G. Higman. *Finitely presented infinite simple groups.* Department of Pure Mathematics, Department of Mathematics, I.A.S. Australian National University, Canberra, 1974. Notes on Pure Mathematics, No. 8 (1974).

[HO] A. Heß and V. Ozornova. A new proof of a Theorem by Charney–Meier–Whittlesey. unpublished.

[JW09] M. Jöllenbeck and V. Welker. Minimal resolutions via algebraic discrete Morse theory. *Mem. Amer. Math. Soc.*, 197(923):vi+74, 2009.

[Koz08] D. Kozlov. *Combinatorial algebraic topology*, volume 21 of *Algorithms and Computation in Mathematics*. Springer, Berlin, 2008.

[Mal37] A. Malcev. On the immersion of an algebraic ring into a field. *Math. Ann.*, 113(1):686–691, 1937.

[Meh11] S. Mehner. Homologieberechnungen von Modulräumen Riemannscher Flächen durch diskrete Morse-Theorie. Diploma thesis, Universität Bonn, 2011.

[NR93] G. A. Niblo and M. A. Roller, editors. *Geometric group theory. Vol. 1*, volume 181 of *London Mathematical Society Lecture Note Series*, Cambridge, 1993. Cambridge University Press.

[Ozo] V. Ozornova. PhD thesis, Universität Bonn, in preparation.

[Rod] M. Rodenhausen. PhD thesis, Universität Bonn, in preparation.

[Rod11] M. Rodenhausen. Cohomology of hv-Groups. Preprint, 2011.

[Skö06] E. Sköldberg. Morse theory from an algebraic viewpoint. *Trans. Amer. Math. Soc.*, 358(1):115–129 (electronic), 2006.

[Squ87] C. C. Squier. Word problems and a homological finiteness condition for monoids. *J. Pure Appl. Algebra*, 49(1-2):201–217, 1987.

[Ste92] M. Stein. Groups of piecewise linear homeomorphisms. *Trans. Amer. Math. Soc.*, 332(2):477–514, 1992.

[Tsa90] S. V. Tsaranov. Representation and classification of Coxeter monoids. *European J. Combin.*, 11(2):189–204, 1990.

[Vis11] B. Visy. Factorable groups and their homology. PhD thesis, Universität Bonn, 2011.

[Wan11] R. Wang. Homology computations for mapping class groups, in particular for $\Gamma^0_{3,1}$. PhD thesis, Universität Bonn, 2011.

[Wei94] C. A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.

Index

$\overline{\partial}$ (reduced bar differential), 40
V (discrete gradient vector field), 30
Θ (discrete gradient flow), 30
C_*^{Θ} (harmonic chains), 32
θ (reduced gradient flow), 33
C_*^{θ} (essential chains), 34
2-balanced, 53
R-graph, 27
FP-property, 18
F-property, 19

absorbing element, 80
atomic, 47

balanced, 53
bar complex, 42
bar resolution, 39
based chain complex, 28
braid factorable, 127

cell
  collapsible, 94
  essential, 94
  height of, 94
  monotone, 94
  redundant, 94
collapsible, 29
collapsing scheme, 37
connected subsequence, 72
Coxeter monoid, 128

discrete gradient flow, 30
discrete gradient vector field, 30
discrete Morse function, 37

essential, 29

Evaluation Lemma, 75

factorable group, 57
factorable monoid, 65
factorization map, 56
finite sequence, 72
  ascending, 131
  coherent, 98
  descending, 131
  effective, 120
  left-most, 76
  length of, 72
  norm-preserving, 98
  reduced, 76
  right-most, 76
  small, 85
  value of, 77

Garside element, 48
Garside monoid, 48
geodesic (tuple), 54
good set of arrows, 134
graded
  equality in the graded sense, 54
  graded map, 54
graph
  directed, 26
  noetherian, 27
  thin, 27
graph product (of monoids), 72
graph with labels, 27
group of fractions, 125

height (of a vertex), 27

incidence graph, 29

incidence number, 28
irreducible word, 43

labelling (of a graph), 27

matching, 27
  R-compatible, 27
  noetherian (on a based chain complex), 29
  noetherian (on a graph), 28
  perfect, 115
monoid, 17
monoid action, 17
Morse complex (discrete), 36

norm (of a cell), 93
norm (of a chain), 93
normal form (of an element), 44
normal form (section), 18
normal form (w.r.t. η), 63

Ore condition
  right, 126
  specific, 143

partition type, 93
Patching Lemma (for coherent sequences), 98
path (in a graph), 27
prefix, 56

recognition principle, 64
reduced gradient flow, 33
reducible word, 43
redundant, 29
remainder, 56
rewriting system, 43
  complete, 44
  confluent, 44
  convergent, 44
  finite, 46
  minimal, 43
  noetherian, 44
  strongly minimal, 44
rewriting system associated to a factorable monoid, 119

scanning (of tuples), 40
stable (pair), 57

Thompson groups (generalized), 139
Thompson monoids (generalized), 139

Visy complex $\mathbb{V}_*$, 62
Visy resolution $\widetilde{\mathbb{V}}_*$, 102

weakly factorable monoid, 59