Gromov-Witten correspondences, derived categories, and Frobenius manifolds

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Maxim Smirnov

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1. Gutachter: **Prof. Dr. Yuri I. Manin**
2. Gutachter: **Prof. Dr. Daniel Huybrechts**

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Introduction

The leitmotif of this dissertation is the theory of quantum cohomology and its interactions with algebraic geometry and mathematical physics. In this chapter we will give a short overview of this beautiful subject and thereby put the subsequent parts of this thesis on the common footing.

0.1 Quantum cohomology. The appearance of quantum cohomology in the physical literature dates back to the end of 1980s and in the mathematical to the beginning of 1990s.

In the language of physicists, quantum cohomology is a mathematical theory of the topological σ-model with target space X, where X is a smooth projective variety over $\mathbb{C}$. We will not use this language here and refer to [Ko] and references therein for details.

Rigorous mathematical approach to the theory of quantum cohomology has been worked out by Maxim Kontsevich and Yuri I. Manin in their foundational paper [KoMa].

For a smooth projective variety X, the structure of $QH(X)$ is defined via enumerative invariants attached to it by the Gromov-Witten theory. There are two equivalent ways to package this information. Namely, one is to say that $QH(X)$ gives an example of a formal Frobenius manifold, the other is to define it as an algebra over an operad $H_*\overline{M}_0$ naturally attached to homology groups of moduli spaces $\overline{M}_{0,n}$.

Below we will briefly outline both constructions. More details on these constructions and a proof of their equivalence can be found in [Ma].

0.1.1 Formal Frobenius manifolds. A Frobenius manifold is a manifold endowed with a commutative associative multiplication on the tangent sheaf and a flat metric satisfying some compatibility conditions (see Ch. 3, Sec. 1). Quantum cohomology gives a prominent example of such structure in the realm of formal geometry.

Let $\Delta_0, \dots \Delta_n$ be a graded basis in $H := H^*(X, \mathbb{C})$ and $x_0, \dots, x_n$ the respective dual coordinates. Gromov-Witten theory of X gives a formal power series $\Phi \in K := \mathbb{C}[[x_0, \dots, x_n]]$, which encodes enumerative information about X.

Quantum product is a multiplication structure on $H \otimes_{\mathbb{C}} K$ defined in the above basis by

$$\Delta_i \circ \Delta_j = \sum_{k,l} \Phi_{ijk} g^{kl} \Delta_l, \tag{0.1}$$

and then K-linearly extended to $H \otimes_{\mathbb{C}} K$. Here $\Phi_{ijk} = \frac{\partial^3 \Phi}{\partial x_i \partial x_j \partial x_k}$, and g is the Poincaré pairing on H. This endows (the formal completion of) H with a structure of a (formal) Frobenius manifold.[1]

Multiplication (0.1) is clearly commutative, but it is associative iff the power series Φ satisfies the system of non-linear partial differential equations

$$\forall a, b, c, d \qquad \sum_{e,f} \Phi_{abe} g^{ef} \Phi_{fcd} = \sum_{e,f} \Phi_{bce} g^{ef} \Phi_{fda}.$$

which are known as *WDVV-equations*. Properties of Gromov-Witten invariants ensure that these equations hold.

One needs to work in the formal category because Φ is not known to be convergent in general. If it is convergent in some domain in H, then on this domain we get a structure of complex analytic Frobenius manifold.

0.1.2 Operadic viewpoint. In Gromov-Witten theory for each $n \geq 2$ one can naturally consider the *evaluation/stabilization* diagram

$$\begin{tikzcd} \overline{M}_{0,n+1}(X) \arrow[r, "ev_{n+1}"] \arrow[d, "(st, ev_{\{1,\dots,n\}})"] & X \\ \overline{M}_{0,n+1} \times X^n & \end{tikzcd} \tag{0.2}$$

where $\overline{M}_{0,n+1}$ is the moduli space of stable curve of genus zero, $\overline{M}_{0,n+1}(X)$ is the moduli space of stable maps to X, and the arrows $ev_{n+1}, (st, ev_{\{1,\dots,n\}})$ are defined canonically (see Chapter 1 for more details).

Using deformation theory techniques one obtains a class in the Chow group $A_*(\overline{M}_{0,n+1}(X))$ called *virtual fundamental class*. Diagram (0.2) gives a natural morphism

$$\overline{M}_{0,n+1}(X) \to \overline{M}_{0,n+1} \times X^{n+1}.$$

Taking pushforward of the virtual fundamental class with respect to this morphism we obtain class in the Chow group $A_*(\overline{M}_{0,n+1} \times X^{n+1})$. This is a correspondence between $\overline{M}_{0,n+1} \times X^n$ and X (cf. Chapter 1).

On the level of cohomology groups this correspondence gives maps

$$H^*(\overline{M}_{0,n+1} \times X^n, \mathbb{C}) \to H^*(X, \mathbb{C}),$$

which by the Künneth formula can be rewritten as

$$H^*(\overline{M}_{0,n+1}, \mathbb{C}) \otimes H^{\otimes n} \to H,$$

where $H := H^*(X, \mathbb{C})$. By Poicaré duality we can rewrite it as

$$H_*(\overline{M}_{0,n+1}, \mathbb{C}) \otimes H^{\otimes n} \to H. \tag{0.3}$$

There exists an operad $H_*\overline{M}_0$ such that $H_*\overline{M}_0(n) = H_*(\overline{M}_{0,n+1}, \mathbb{C})$. Moreover, maps (0.3) define on H a structure of an algebra over $H_*\overline{M}_0$.

The vector space H with this $H_*\overline{M}_0$-algebra structure is another avatar of $QH(X)$. We refer to [Ma] for more details.

[1] If $H^*(X, \mathbb{C})$ has classes of odd degree, then we need to work in the realm of super geometry. In presence of odd classes quantum product is super commutative. To avoid that we will be considering the pure even case most of the time.

Remark. To simplify the exposition we have swept under the rug some important technical details. Namely, in general the moduli space $\overline{M}_{0,n+1}(X)$ is a disjoint union of infinitely many connected components $\overline{M}_{0,n+1}(X,\beta)$, where β is an element in Mori's cone of effective curves on X. Therefore, to make the above constructions make sense one considers the (co)homology with coefficients in the Novikov ring.

0.2 Quantum cohomology and geometry of X. Having introduced such a rich structure one may wonder which properties of X can be seen from its quantum cohomology. Below we will discuss implications of (generic) semi-simplicity of $QH(X)$.

0.2.1 Semi-simplicity. Let A be an algebra over $\mathbb{C}$ of dimension n. Recall that it is called semi-simple iff it is isomorphic to $\mathbb{C}^n$ with componentwise multiplication.

A Frobenius manifold M is called semi-simple at a point $P \in M$ iff the tangent space $T_P M$ with the induced multiplication $\circ_P$ is a semi-simple algebra. Semi-simple points form an open subset of M. A Frobenius manifold is called generically semi-simple iff the set of semi-simple points is dense in M (see Ch. 3, Sec. 1 for more details).

It turns out that generic semi-simplicity of $QH(X)$ gives very strong restrictions on the geometry of X. Namely, it implies that $H^{p,q}(X) = 0$ unless $p = q$ (cf. Th. 1.3 of [HeMaTe]). It is not known whether generic semi-simplicity of $QH(X)$ implies that the Chow motive of X is a Tate motive.

On the other hand there exist surfaces of general type such that their Chow motive with rational coefficients is Tate (see [GoOr], Prop. 2.2 and 2.3) but its quantum cohomology coincides with the classical cohomology[2], and hence it is not generically semi-simple.

0.2.2 Dubrovin's conjecture. The original formulation (cf. [Du2]) of the conjecture is as follows.

1. For a Fano variety X the following are equivalent:

(i) $\mathcal{D}^b(X)$ has a full exceptional collection

(ii) $QH(X)$ is generically semi-simple.

2. If the first part holds, then the Stokes matrix of $QH(X)$ coincides with the matrix of *Ext*-groups of the exceptional collection (after some suitable choices on both sides).

There are many experimental results supporting this conjecture. For example, its first part has been verified for projective spaces, Grassmanians, rational surfaces and some families of Fano threefolds (cf. [Te] p. 203). Moreover, for some of these varieties the second part of the conjecture is also known to hold (cf. [Ue]).

Despite all this evidence there are almost no conceptual approaches to this conjecture. The only known result is a combination of works of A. Bayer and D. Orlov. It says that if the first part of the conjecture holds for X, then it also holds for the blow-up of X at a point (see [Bay]).

Remark 1. Dubrovin's conjecture would follow from the homological mirror symmetry conjecture (see [Te] p. 203).

[2] All genus zero Gromov-Witten invariants vanish for dimension reasons.

Remark 2. It is known that if $\mathcal{D}^b(X)$ has a full exceptional collection, then the motive of X with rational coefficients is a Tate motive (see [GMSKP]). Therefore, if Dubrovin's conjecture were true, then the generic semi-simplicity of $QH(X)$ would imply that the motive of X with rational coefficients is a Tate motive.

Remark 3. In Section 0.1.2 we have used cohomology to linearise geometry and obtain the operadic description of quantum cohomology.

Instead one could try to mimic those constructions on the level of derived categories. It even appears to be more natural, since the virtual fundamental class can be first considered as an object in the derived category and then pushed down to the Chow group (cf. [BeFa], [Lee]).

Moreover, if we work with canonical DG-enhancements of these categories, then there exists an analogue of the Künneth formula and one could try to define an operadic action on the categorical level. An analogue of Mori's cone is yet to be developed and appears to be one of the central ingredients of this story.

This approach would put both sides of Dubrovin's conjecture on the common ground and maybe give some new insights.

0.3 Moduli spaces of genus zero curves. As we have seen, varieties $\overline{M}_{0,n}$ play a central role in the theory of quantum cohomology. It is especially clear if one uses the operadic viewpoint from Section 0.1.2. Therefore, it appears to be natural to study quantum cohomology of these spaces themselves. Surprisingly enough there is almost no progress in this direction for $n \geq 6$. For $n \leq 5$ quantum cohomology of these spaces is well understood, since in these cases the geometry of $\overline{M}_{0,n}$ is very simple.

In Chapter 1 we study Gromov-Witten correspondences for boundary curves on $\overline{M}_{0,n}$ which can be considered as a small step towards understanding the structure of $QH(\overline{M}_{0,n})$. It is proved that in this case respective Gromov-Witten invariants can be computed geometrically. This chapter is based on the joint work with Yuri I. Manin [MaS1].

In Chapter 2 we study the derived category of $\mathcal{D}^b(\overline{M}_{0,n})$ and, in particular, show the existence of full exceptional collections. This chapter is based on the joint work with Yuri I. Manin [MaS2].

Based on Dubrovin's conjecture and results of Chapter 2 it is expected that $QH(\overline{M}_{0,n})$ is generically semi-simple. This is well known for $n \leq 5$ and unknown for $n \geq 6$.

0.4 Frobenius manifolds and mirrors. Consider a pair (Y, f), where Y is smooth algebraic variety and f a regular function on Y. Under certain assumptions, from these data one can construct a Frobenius manifold M called *a Saito's framework* attached to (Y, f).[3]

Let X be a smooth projective Fano variety. A pair (Y, f) is called a *Landau-Ginzburg model* for X iff there exists a Saito's framework attached to it which is isomorphic to $QH(X)$.

In Chapter 3 we consider Landau-Ginzburg models for odd-dimensional quadrics. The ultimate goal is to construct a Landau-Ginzburg model in the

[3] This procedure is not necessarily unique. We refer to Chapter 3 and references therein for more details.

sense described above. Some partial results in this direction are obtained. This chapter is based on a joint work in progress with Vassily Gorbounov.

Remark. In the literature the term a Landau-Ginzburg model could refer to different notions. Conjecturally they all should be related by homological mirror symmetry.

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Chapter 1

Towards Gromov-Witten theory for $\overline{M}_{0,n}$

In this chapter, we study Gromov-Witten correspondences *of genus zero* for $\overline{M}_{0,n}$ and β being the class of a boundary curve in $\overline{M}_{0,n}$. These results were obtained by Yuri I. Manin and the author in [MaS1]. This is the first step of a much more ambitious program in which all components of the stable family diagrams are allowed to be stacks, and in which we take for targets the stacks $\overline{M}_{g,n}$ and arbitrary β.

We start by briefly recalling relevant notions about stacks, moduli spaces and Gromov-Witten theory. After that we turn to the study of the problem mentioned above, and our presentation follows mostly that of *loc.cit.*.

1 Moduli problems and stacks

This section gives a short introduction to the theory of stacks and its applications to moduli problems. Even though notions of algebraic stacks and Deligne-Mumford stacks are not discussed here, we will use them freely in the rest of the chapter. For a detailed treatment of stacks and orbifolds we refer to [Vi], [LaMo-Ba], [Ma], [Hi], [HiVa].

Throughout this section $\mathcal{C}$ denotes an arbitrary category.

1.1 Yoneda lemma. Let X be an object in $\mathcal{C}$. One can consider a contravariant functor $h_X\colon \mathcal{C}^{\mathrm{op}} \to \underline{Set}$ defined by $h_X(T) := \mathrm{Hom}_{\mathcal{C}}(T, X)$. It is called *functor of points* of X. If $X \to Y$ is a morphism in $\mathcal{C}$, then by composition we get a morphism $h_X \to h_Y$. This defines a functor

$$h\colon \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \underline{Set})$$

which is fully faithful. Functors lying in the (essential) image of h are called *representable.*

This picture allows us to identify objects of $\mathcal{C}$ with representable functors. This construction can be applied to any category. In particular, we can take $\mathcal{C}$ to be $\underline{Sch}$ or $\underline{Sch}_S$ for any base scheme S.

1.2 Topologies and sheaves. A contravariant functor $F\colon \mathcal{C}^{\mathrm{op}} \to \underline{Set}$ should be thought of as a *presheaf* on $\mathcal{C}$ with values in $\underline{Set}$ and we put it as a definition. If we endow $\mathcal{C}$ with a *Grothendieck topology* $\mathcal{T}$(see [Vi]), then one can ask whether F is a *sheaf* with respect to $\mathcal{T}$. Of course, it will depend on $\mathcal{T}$. A topology $\mathcal{T}$ on a category $\mathcal{C}$ is called *subcanonical* if every representable functor on $\mathcal{C}$ is a sheaf with respect to $\mathcal{T}$.

The most prominent topologies on the category $\underline{Sch}_S$ are *the Zariski topology*, *the étale topology*, *the fppf-topology*, and *the fpqc-topology*. We have listed them here according to how fine they are. The Zariski topology is the most coarse one and the fpqc-topology is the finest.

It can be easily checked that a representable functor on $\underline{Sch}_S$ is a sheaf in the Zariski topology. Moreover, a theorem due to Grothendieck (see [Vi], Th. 2.55) asserts that it is a sheaf in fpqc-topology. Hence, it is also a sheaf in the étale and fppf topologies. Thus, all of the above topologies are subcanonical.

1.3 Stacks. Let $\mathcal{C}_{\mathcal{T}}$ be a *site*, i.e. a category $\mathcal{C}$ with a Grothendieck topology $\mathcal{T}$. In the previous paragraph we considered sheaves of sets on it as some generalizations of objects of $\mathcal{C}$. One can go further and consider "sheaves of categories" instead of sheaves of sets. Here a set is being viewed as a category where all morphisms are identities.

Informal Definition: a *stack* on a site $\mathcal{C}_{\mathcal{T}}$ is a sheaf of categories on it.

Though being philosophically correct, this definition is sloppy and one needs to be more precise. There are two equivalent ways to do that. One is using *fibered categories* and the other using *weak* 2*-functors* (or *pseudofunctors*). For a comparison of these notions we refer to [Vi].

1.3.1 Fibered categories. A category $\mathcal{F}$ with a functor $p\colon \mathcal{F} \to \mathcal{C}$ will be called a *category over* $\mathcal{C}$. Let $\varphi\colon \xi \to \eta$ be a morphism in $\mathcal{F}$, $U = p(\xi)$ and $V = p(\eta)$. It is convenient to picture these data as

$$
\begin{tikzcd}
\xi \arrow[r, "\varphi"] \arrow[d, mapsto] & \eta \arrow[d, mapsto] \\
U \arrow[r] & V
\end{tikzcd}
$$

The arrow $\varphi\colon \xi \to \eta$ is called *cartesian* iff for any arrow $\psi\colon \zeta \to \eta$ in $\mathcal{F}$ and any arrow $h\colon p(\zeta) \to p(\xi)$ in $\mathcal{C}$ with $p(\varphi) \circ h = p(\psi)$, there exists a unique arrow $\theta\colon \zeta \to \xi$ with $p(\theta) = h$ and $\varphi \circ \theta = \psi$, as in the commutative diagram

$$
\begin{tikzcd}
\zeta \arrow[dr, dashed, "\theta"'] \arrow[drr, bend left, "\psi"] \arrow[dd, mapsto] & & \\
& \xi \arrow[r, "\phi"'] \arrow[dd, mapsto] & \eta \arrow[dd, mapsto] \\
p(\zeta) \arrow[dr, "h"'] \arrow[drr] & & \\
& p(\xi) \arrow[r] & p(\eta)
\end{tikzcd}
$$

If $\xi \to \eta$ is a cartesian arrow of $\mathcal{F}$ mapping to an arrow $U \to V$ of $\mathcal{C}$, we also say that ξ is *a pullback of* η *to* U.

A *category fibered over* $\mathcal{C}$ is a category $\mathcal{F}$ over $\mathcal{C}$, such that given an arrow $f: U \to V$ in $\mathcal{C}$ and an object η of $\mathcal{F}$ mapping to V, there exists a cartesian arrow $\varphi: \xi \to \eta$ with $p(\varphi) = f$. In other words, in a fibered category $\mathcal{F} \to \mathcal{C}$ we can pull back objects of $\mathcal{F}$ along any arrow of $\mathcal{C}$.

Let $p_1: \mathcal{F}_1 \to \mathcal{C}$ and $p_2: \mathcal{F}_2 \to \mathcal{C}$ be categories fibered over $\mathcal{C}$. A functor $q: \mathcal{F}_1 \to \mathcal{F}_2$ is *a morphism of fibered categories* iff $p_1 = p_2 \circ q$ and it sends cartesian arrows to cartesian.

For more details on fibered categories we refer to [Vi].

1.3.2 Pseudofunctors. Let $\underline{Cat}$ be a category of categories of certain type. For example, the category of groupoids $\underline{Grpd}$.[1]

The category $\underline{Cat}$ is an example of a so called *strict* 2-*category*; every 1-category gives such an example as well. Note that in both examples the composition of 1-morphisms is strictly associative. Therefore, one can define *strict 2-functors* in this situation.

There is a more general notion of *weak 2-categories* (or *bicategories*), the main difference being that composition of 1-morphisms is no longer strictly associative, but associative up to a 2-isomorphism. Of course, a strict 2-category is naturally a weak 2-category.

In the world of weak 2-categories strict 2-functors do not make sense any more and the appropriate notion is *weak 2-functor* (or *pseudofunctor*).

Let $\mathcal{C}$ be a 1-category. A presheaf with values in $\underline{Cat}$ is a weak 2-functor

$$F: \mathcal{C}^{op} \to \underline{Cat}.$$

1.3.3 Descent data and stacks. Let $\mathcal{T}$ be a Grothendieck topology on $\mathcal{C}$ and $F: \mathcal{C}^{op} \to \underline{Cat}$ a weak 2-functor. Then to a covering $\{U_i \to U\}$ in $\mathcal{C}_\mathcal{T}$ one can attach a category $F(\{U_i \to U\})$ – the category of *descent data* – defined as follows: its objects are collections (ξ_i) of objects in $F(U_i)$ together with isomorphisms $\varphi_{ij}: pr_j^*\xi_i \simeq pr_i^*\xi_j$ satisfying a natural cocycle condition; morphism are defined in a natural way (see [Vi], 4.1.2 for details). It is clear that there exist a natural morphism $F(U) \to F(\{U_i \to U\})$.

A weak 2-functor $F: \mathcal{C}_\mathcal{T}^{op} \to \underline{Cat}$ is a *stack* iff for any covering $\{U_i \to U\}$ the induced morphism

$$F(U) \to F(\{U_i \to U\})$$

is an equivalence of categories.

1.4 Moduli problems. Since the time of Grothendieck moduli problems in algebraic geometry (and beyond it) are formulated in the language of functors of points.

Let S be a scheme and put $\mathcal{C} = \underline{Sch}_S$. Then, a *moduli problem* usually gives a weak 2-functor

$$M: \mathcal{C}^{op} \to \underline{Grpd}, \tag{1.1}$$

[1] One should be more careful and define $\underline{Cat}$ to be the category of small categories of certain type and $\underline{Grpd}$ the category of small groupoids. We admit to being a little careless with these set theoretical issues here.

which happens to be a stack with respect to one the standard topologies on $\mathcal{C}$.[2]

An archetypal situation can be described as follows. We consider a classification problem of geometric objects of certain type. Typical examples are algebraic curves, vector bundles, stable morphisms to a fixed scheme X, closed subschemes of a given scheme and so forth. Define a weak 2-functor (1.1) which assigns to an S-scheme U a groupoid $M(U)$ whose objects are families over U of geometric objects of this type and morphisms in $M(U)$ are isomorphisms of such families. For a morphism $\varphi\colon V \to U$ in $\mathcal{C}$ the functor $M(\varphi)\colon M(V) \to M(U)$ is given by the pullback of families. The resulting weak 2-functor (1.1) may or may not be a stack with respect to one of the standard topologies.

In some cases (1.1) factors via the embedding $\underline{Set} \to \underline{Grpd}$ and we get a contravariant functor

$$M'\colon \mathcal{C}^{op} \to \underline{Set}. \tag{1.2}$$

If M' is representable, then the representing scheme is called the *fine moduli space* for moduli problem (1.1). This property is related to the absence of automorphisms of the geometric objects in question.

In the next sections we will encounter examples of such constructions. Most of the time we will be using fibered categories instead of weak 2-functors.

2 Moduli of stable curves.

From now on we assume that the ground field K is of characteristic zero.[3]

Let C be a complete curve over K (i.e. one-dimensional reduced scheme proper over K). It is called *prestable* if it has at most nodal singularities.

Let Σ be a finite set. By a *Σ-labelled prestable curve* we mean a prestable curve endowed with sections $x_j\colon \mathrm{Spec}\,(K) \to C$ where $j \in \Sigma$, and the image of x_j is required to be a smooth point on C.

A Σ-labelled prestable curve C is called *stable* if it is connected and its automorphism group is finite (here we mean automorphisms respecting marked points).

Let T be a scheme over K. By a family over T of objects of one of the above types (i.e. prestable, labelled prestable or stable curves) we mean a flat proper morphism $C \to T$ of schemes over K such that geometric fibers are objects of this type. Since the family is flat, the arithmetic genus is a locally constant function on T.

2.1 Definition. Consider the category $\overline{M}_\Sigma$ over $\underline{Sch}_K$ defined as follows.

(a) Objects: families

$$\begin{tikzcd} C_T \arrow[d] \\ T \arrow[u, bend left, "x_{j,T}"] \end{tikzcd} \tag{2.1}$$

of stable curves labelled by Σ.

[2]Of course, on can give an equivalent definition in terms of categories fibered in groupoids.
[3]Some of the results hold also in positive characteristic.

(b) Morphisms: commutative diagrams

$$\begin{tikzcd} C_S \arrow[r] \arrow[d] & C_T \arrow[d] \\ S \arrow[r, "\varphi"] \arrow[u, bend left, "x_{j,S}"] & T \arrow[u, bend right, "x_{j,T}"'] \end{tikzcd} \tag{2.2}$$

such that the induced diagram

$$\begin{tikzcd} C_S \arrow[r] \arrow[d] & \varphi^*(C_T) \arrow[d] \\ S \arrow[r, "id"] \arrow[u, bend left, "x_{j,S}"] & S \arrow[u, bend right, "\varphi^*(x_{j,T})"'] \end{tikzcd}$$

gives an isomorphism of families of stable curves.

(c) The structure morphism

$$\overline{M}_\Sigma \to \underline{Sch}_K \tag{2.3}$$

sends a diagram of type (2.1) to T and a morphism of type (2.2) to $\varphi\colon S \to T$.

One can check that $\overline{M}_\Sigma$ is a category fibered in groupoids over $\underline{Sch}_K$ with pull-backs given by base change.

2.2 Properties. Since the arithmetic genus g is constant in flat families, we get the decomposition

$$\overline{M}_\Sigma = \coprod_{|\Sigma|+2g\geq 3} \overline{M}_{g,\Sigma}$$

where $\overline{M}_{g,\Sigma}$ is the category fibered in groupoids consisting of families of Σ-labelled stable curves of fixed arithmetic genus. The condition $|\Sigma| + 2g \geq 3$ ensures that $\overline{M}_{g,\Sigma}$ is non-empty.

2.2.1 Theorem ([DeMu]). *If we consider $\underline{Sch}_K$ with étale or fppf topology, then $\overline{M}_{g,\Sigma}$ is a connected smooth proper Deligne-Mumford stack of dimension* $3g - 3 + |\Sigma|$.

2.2.2 Remark. If $\Sigma = \{1, \ldots, n\}$, then we will write $\overline{M}_{g,n}$ instead of $\overline{M}_{g,\Sigma}$. Of course, $\overline{M}_{g,\Sigma}$ is isomorphic to $\overline{M}_{g,|\Sigma|}$ but not canonically.

2.3 Genus 0 case. It turns out that moduli spaces $\overline{M}_{0,n}$ with $|\Sigma| \geq 3$ are actually smooth projective varieties of dimension $|\Sigma| - 3$ (see [Ma], Chapter 3). For first values of n we have: $\overline{M}_{0,3}$ is a point, $\overline{M}_{0,4}$ is a projective line, $\overline{M}_{0,5}$ is a Del-Pezzo surface of degree 5, $\overline{M}_{0,6}$ is a three dimensional variety which is neither Fano nor toric, and so on.

2.3.1 Kapranov's presentation. There is a presentation of the space $\overline{M}_{0,n}$ due to Kapranov (see [Ka]) via consecutive blow-ups of $\mathbb{P}^{n-3}$. First blow up $n-1$ points in general position on $\mathbb{P}^{n-3}$. After that blow up strict transforms of lines connecting pairs of these points. Then blow up strict transforms of planes passing through triples of them and so on.

In Chapter 2 we will see a different presentation of these spaces due to Keel.

3 Moduli of stable maps.

Let X be a smooth projective variety over K and Σ a finite set. Let C be a connected prestable Σ-labelled curve over K. A morphism $f\colon C \to X$ is called *stable* if it has finite automorphism group. It is equivalent to saying that f does not contract unstable irreducible components of C.

By a *family of stable morphisms to* X over a K-scheme T we mean a diagram

$$\begin{tikzcd} C_T \arrow[r, "f_T"] \arrow[d] & X \\ T \arrow[u, bend left, "x_{j,T}"] & \end{tikzcd}$$

such that all geometric fibers are stable morphisms. In particular, $C_T \to T$ is a family of connected Σ-labelled prestable curves. Sometimes, to avoid drawing complicated diagrams, we will denote it simply by $(C_T, (x_{j,T}), f_T)$, and when the base T is understood we will drop it from the notation as well.

3.1 Definition. Consider the category $\overline{M}_\Sigma(X)$ over $\underline{Sch}_K$ defined as follows.

(a) Objects: families

$$\begin{tikzcd} C_T \arrow[r, "f_T"] \arrow[d] & X \\ T \arrow[u, bend left, "x_{j,T}"] & \end{tikzcd} \tag{3.1}$$

of stable morphisms to X.

(b) Morphisms: commutative diagrams

$$\begin{tikzcd} & & X \\ C_S \arrow[r] \arrow[rru, "f_S"] \arrow[d] & C_T \arrow[ru, "f_T"'] \arrow[d] & \\ S \arrow[r, "\varphi"] \arrow[u, bend left, "x_{j,S}"] & T \arrow[u, bend right, "x_{j,T}"'] & \end{tikzcd} \tag{3.2}$$

such that the induced diagram

$$\begin{tikzcd} & & V \\ C_S \arrow[r] \arrow[rru, "f_S"] \arrow[d] & \varphi^*(C_T) \arrow[ru, "\varphi^*(f_T)"'] \arrow[d] & \\ S \arrow[r, "id"] \arrow[u, bend left, "x_{j,S}"] & S \arrow[u, bend right, "\varphi^*(x_{j,T})"'] & \end{tikzcd}$$

gives an isomorphism of stable morphisms over S.

(c) The structure morphism

$$\overline{M}_\Sigma(X) \to \underline{Sch}_K \tag{3.3}$$

sends a diagram of type (3.1) to T and a morphism of type (3.2) to $\varphi\colon S \to T$.

Again one can check that $\overline{M}_\Sigma(X)$ is a category fibered in groupoids over $\underline{Sch}_K$ with pull-backs given by base change.

3.2 Properties. As in Section 2.2 we get the decomposition with respect to the genus

$$\overline{M}_\Sigma(X) = \coprod_g \overline{M}_{g,\Sigma}(X).$$

Moreover, $\overline{M}_{g,\Sigma}(X)$ further decompose as

$$\overline{M}_{g,\Sigma}(X) = \coprod_\beta \overline{M}_{g,\Sigma}(X,\beta),$$

where β runs over the Mori cone of effective curves.

3.2.1 Proposition. *If we consider $\underline{Sch}_K$ with étale or fppf topology, then $\overline{M}_{g,\Sigma}(X,\beta)$ is a proper Deligne-Mumford stack.*

Proof. See [Ma], V.5. ■

3.2.2 Stabilization morphism. Let $(C_T, (x_{j,T}), f_T)$ be a family of Σ-labelled prestable curves over T with connected fibers and $f_T\colon C_T \to X$ any morphism. There exists in some sense maximal stable morphism $(C'_T, (x'_{j,T}), f'_T)$ to X such that $(C_T, (x_{j,T}), f_T)$ factors through it. It is unique up to canonical isomorphism (cf. [Ma], V.1.7) and called *stabilisation* of $(C_T, (x_{j,T}), f_T)$.

3.2.3 Induced morphism. Let X and Y be smooth projective varieties over K and $f\colon X \to Y$ a morphism. Composing stable maps to X with f and stabilizing (cf. [Ma], V.4.4) we get a functor

$$\overline{M}_\Sigma(X) \to \overline{M}_\Sigma(Y),$$

which is in fact a morphism of fibered categories. It is clear that it preserves the decomposition with respect to the genus and we get a canonical morphism of stacks

$$\overline{M}_{g,\Sigma}(X,\beta) \to \overline{M}_{g,\Sigma}(Y, f_*\beta).$$

3.3 Virtual fundamental classes. Consider the *evaluation/stabilzation diagram*

$$\begin{tikzcd} \overline{M}_{g,\Sigma}(X,\beta) \arrow[r, "ev"] \arrow[d, "st"] & X^\Sigma \\ \overline{M}_{g,\Sigma} & \end{tikzcd}$$

Here

$$ev = (ev_j = f \circ x_j \mid j \in \Sigma): \quad \overline{M}_{g,\Sigma}(X,\beta) \to X^\Sigma$$

and, in case $|\Sigma| + 2g \geq 3$, the absolute stabilization morphism st discards the map f and stabilizes the remaining prestable family of curves (cf. [Ma], V.4.6)

$$st\colon \overline{M}_{g,\Sigma}(X,\beta) \to \overline{M}_{g,\Sigma}.$$

The *virtual fundamental class*, or the J-class $[\overline{M}_{g,\Sigma}(X,\beta)]^{virt}$, is a canonical element in the Chow group $A_*(\overline{M}_{g,\Sigma}(X,\beta))$:

$$J_{g,\Sigma}(X,\beta) \in A_D(\overline{M}_{g,\Sigma}(X,\beta))\,,$$

where D is the virtual dimension (Chow grading degree)

$$(-K_X,\beta) + |\Sigma| + (\dim X - 3)(1-g). \tag{3.4}$$

The respective *Gromov-Witten correspondence*, defined for $|\Sigma| + 2g \geq 3$, is the proper pushforward

$$I_{g,\Sigma}(X,\beta) := (ev \times st)_*(J_{g,\Sigma}(X,\beta)) \in A_D(X^\Sigma \times \overline{M}_{g,\Sigma}).$$

Understanding these correspondences is the content of *motivic* quantum cohomology.

3.3.1 Unobstructed deformations. A stable map $f\colon (C,(x_j)) \to X$ is called *trivially unobstructed* (cf. [Beh3]) iff $H^1(C, f^*\mathcal{T}_X) = 0$. If for every stable morphism of class β this condition holds, then the stack $\overline{M}_{g,\Sigma}(X,\beta)$ is smooth of expected dimension (3.4), and $J_{g,\Sigma}(X,\beta)$ coincides with the usual fundamental class.

3.4 Example: $g = 0$, $\beta = 0$. In this case the natural morphism $ev \times st\colon \overline{M}_{0,\Sigma}(X,0) \to X^\Sigma \times \overline{M}_{0,\Sigma}$ factors through the natural embedding $\Delta_\Sigma \times id\colon X \times \overline{M}_{0,\Sigma} \to X^\Sigma \times \overline{M}_{0,\Sigma}$, where $\Delta_\Sigma\colon X \to X^\Sigma$ is the diagonal. It induces the isomorphism

$$\overline{M}_{0,\Sigma}(X,0) \simeq X \times \overline{M}_{0,\Sigma}.$$

Under this identification the stabilization morphism is simply the projection

$$st = pr_2 :\ X \times \overline{M}_{0,\Sigma} \to \overline{M}_{0,\Sigma},$$

and the evaluation morphism is the projection followed by the diagonal embedding Δ_Σ:

$$ev :\ X \times \overline{M}_{0,\Sigma} \to X \to X^\Sigma.$$

We have (cf. [Beh1], p. 606)

$$J_{0,\Sigma}(X,0) = [\overline{M}_{0,\Sigma}(X,0)] = [X] \otimes [\overline{M}_{0,\Sigma}].$$

Thus, the Gromov–Witten correspondence is the class

$$I_{0,\Sigma}(X,0) = [\Delta_\Sigma(X)] \otimes [\overline{M}_{0,\Sigma}] \in A_*(X^\Sigma \times \overline{M}_{0,\Sigma}). \tag{3.5}$$

(Notice that for $x \in A_*(X), y \in A_*(Y)$ we often denote simply by $x \otimes y \in A_*(X \times Y)$ the image of $x \otimes y \in A_*(X) \otimes A_*(Y)$ with respect to the canonical map $A_*(X) \otimes A_*(Y) \to A_*(X \times Y)$).

3.5 Gromov-Witten invariants. Let γ_i be cohomology classes on X labelled by Σ and define

$$\langle I^X_{0,\Sigma,\beta}\rangle(\otimes_{j\in\Sigma}\gamma_j) = \deg(pr_X^*(\otimes_{j\in\Sigma}\gamma_j)\cap I_{0,\Sigma}(X,\beta)),$$

where $pr_X\colon X^\Sigma\times\overline{M}_{0,\Sigma}\to X^\Sigma$ is the projection. Varying cohomology classes γ_i and classes β we get (possibly infinitely) many numbers called *Gromov-Witten invariants* of X.

4 Boundary curve classes on $\overline{M}_{0,S}$

4.1 Boundary strata of $\overline{M}_{0,S}$. The main combinatorial invariant of an S-pointed stable curve C is its *dual graph* $\tau = \tau_C$. Its set of vertices V_τ is (bijective to) the set of irreducible components of C. Each vertex v is a boundary point of the set of *flags* $f\in F_\tau(v)$ which is (bijective to) the set consisting of singular points and S-labelled points on this component. We put $F_\tau = \cup_{v\in V_\tau}F_\tau(v)$. If two components of C intersect, the respective two vertices carry two flags that are grafted to form an *edge* e connecting the respective vertices; the set of edges is denoted E_τ. The flags that are not pairwise grafted are called *tails.* They form a set T_τ which is naturally bijective to the set of S-labelled points and therefore itself is labelled by S. Stable curves of genus zero correspond to trees τ whose each vertex carries at least three flags.

The space $\overline{M}_{0,S}$ is a disjoint union of locally closed strata M_τ indexed by stable S-labelled trees. Each such stratum M_τ represents the functor of families consisting of curves of combinatorial type τ. In particular, the open stratum $M_{0,S}$ classifies irreducible smooth curves with pairwise distinct S-labelled points. Its graph is a star: tree with one vertex, to which all tails are attached, and having no edges.

Generally, a stratum M_τ lies in the closure $\overline{M}_\sigma$ of M_σ iff σ can be obtained from τ by contracting a subset of edges. Closed strata $\overline{M}_\sigma$ corresponding to trees with nonempty set of edges are called *boundary* ones. The number of edges is the codimension of the stratum.

4.1.1 Lemma. The classes of boundary divisors generate the Chow ring $A^*(\overline{M}_{0,S})$.

Proof. See [Ma], Ch. 3. ■

4.2 Boundary curves of $\overline{M}_{0,S}$. Consider a boundary curve on $\overline{M}_{0,S}$, i.e. 1-dimensional boundary stratum, and denote it C_τ. It is not difficult to see that τ has all vertices of valency 3 except for one vertex of valency 4. Let us call it a *distinguished vertex.* Contracting all edges which are not adjacent to the distinguished vertex one obtains a new stable labelled tree π with up to 4 vertices besides the distinguished one. Such graphs are in one-to-one correspondence with partitions Π of the set S into 4 components. Such partitions will be called *distinguished* as well.

The partition Π (or, equivalently, the associated stable tree π) defines a boundary stratum $\overline{M}_\Pi$, which comes with a closed embedding

$$b_\Pi\colon \overline{M}_\Pi\to\overline{M}_{0,S}. \tag{4.1}$$

Since π is obtained from τ by contraction of some edges, we also have a closed embedding $C_\tau \hookrightarrow \overline{M}_\Pi$. The stratum $\overline{M}_\Pi$ is the product of moduli spaces associated with stars of vertices of π

$$\overline{M}_\Pi \simeq \overline{M}_{0,F(v_0)} \times \prod_{v\neq v_0} \overline{M}_{0,F(v)}.$$

Let $B_\Pi := \prod_{v\neq v_0} \overline{M}_{0,F(v)}$ and consider the natural projection

$$p_\Pi \colon \overline{M}_\Pi \to B_\Pi. \tag{4.2}$$

The boundary curve C_τ is a fiber of this projection.

4.2.1 Lemma ([KeMcK]). *(i) For $n := |S| \geq 4$, each boundary curve C_τ is a fiber of one of the projections (4.2).*

(ii) $[C_{\tau_1}] = [C_{\tau_2}] \in A_1(\overline{M}_{0,S}) \iff$ these curves are fibers of one and the same projection.

Proof. (i) This is clear from the above considerations.

(ii) Since all fibers of p_Π are rationally equivalent, the implication "$\Leftarrow$" is clear. For the implication "$\Rightarrow$" we will give a proof below after collecting some facts. ■

Since all fibers of p_Π are rationally equivalent, a distinguished partition Π defines a class in $A_1(\overline{M}_{0,S})$, which we will denote $\beta(\Pi)$. It also defines a class in $A_1(\overline{M}_\Pi)$ which we denote β_Π.

4.3 Useful facts. Given a distinguished partition Π, denote by $P(\Pi)$ the set of those stable 2-partitions of S, each component of which is a union of two different components of Π. For $|S| \geq 4$ we have $|P(\Pi)| = 3$. Further, denote by $N(\Pi)$ the set of those stable 2-partitions of S whose one component coincides with one component of Π.

It is easy to see that one can reconstruct Π from $P(\Pi)$ (cf. Lemma 3.2.2 in [MaS1]). Hence, one can also reconstruct $N(\Pi)$.

If Π comes from a boundary stratum C_τ as at the beginning of this section, then we will also use notations $P(\tau)$ and $N(\tau)$.

4.3.1 Lemma. We have

$$\overline{M}_\Pi = \bigcap_{\sigma\in N(\Pi)} D_\sigma, \tag{4.3}$$

$$(D_\sigma, C_\tau) = 1, \text{ if } \sigma \in P(\tau), \tag{4.4}$$

$$(D_\sigma, C_\tau) = -1, \text{ if } \sigma \in N(\tau), \tag{4.5}$$

$$(D_\sigma, C_\tau) = 0, \text{ otherwise.} \tag{4.6}$$

Proof. Formula (4.3) follows directly from the definition of $\overline{M}_\Pi$ by looking at combinatorial types of curves parametrized by $\overline{M}_\Pi$ and D_σ with $\sigma \in N(\tau)$.

For formulas (4.4) – (4.6) we reproduce a proof using the notion of *good monomials* from [MaS1], proof of Lemma 3.3.1.(ii) (cf. [KeMcK], Lemma 4.3).

Good monomials are elements of the commutative polynomial ring freely generated by symbols $m(\sigma)$ where σ runs over stable 2-partitions of S. These monomials form a family indexed by stable S-labelled trees τ: $m(\tau) := \prod_{e\in E_\tau} m(\sigma_e)$ where σ_e is the 2-partition of S obtained by cutting e.

Assume first that $m(\sigma)m(\tau)$ is a good monomial so that $(D_\sigma, C_\tau) = 1$. Then it is of the form $m(\rho)$, where ρ is a stable S-labelled tree with all vertices of multiplicity 3 and an edge e such that $m(\sigma) = m(\rho_e)$. This edge is unambiguously characterized by the fact that after collapsing e in ρ to one vertex, we get the labelled tree (canonically isomorphic to) τ. But the vertex to which e collapses must then have multiplicity larger than 3. It follows that e must collapse precisely to the exceptional vertex v_0 of τ. Conversely, the set of ways of putting e back is clearly in a bijection with $P(\tau)$: the 4 flags adjacent to v_0 must be distributed in two groups, 2 flags in each, that will be adjacent to two ends of e.

Assume now that $m(\sigma)$ divides $m(\tau)$. Using Proposition 1.7.1 of [KoMaKa], one sees that $m(\sigma)m(\tau)$ represents zero in the Chow ring (and so $(D_\sigma, C_\tau) = 0$) unless $\sigma = \tau_e$ where e is an edge adjacent to v_0. In this latter case Kaufmann's formula (1.9) from [KoMaKa] implies $(D_\sigma, C_\tau) = -1$. The set of such σ's is in a bijection with $N(\tau)$.

Finally, for any other stable 2-partition σ there exists an $e \in E_\tau$ such that we have $a(\sigma, \tau_e) = 4$ in the sense of [Ma], III.3.4.1. In this case, $(D_\sigma, C_\tau) = 0$ in view of [Ma], III.3.4.2. ■

4.3.2 Proof of Lemma 4.2.1. We have $[C_{\tau_1}] = [C_{\tau_2}]$ iff $(D_\sigma, C_{\tau_1}) = (D_\sigma, C_{\tau_2})$ for all stable 2-partitions σ, because boundary divisors generate A^1. In view of (4.4) – (4.6), this condition implies that

$$P(\tau_1) = P(\tau_2), \quad N(\tau_1) = N(\tau_2).$$

Since Π can be recovered from $P(\Pi)$, we get $\Pi(\tau_1) = \Pi(\tau_2)$. ■

4.3.3 Proposition. (i) Let K_S be the canonical class of $\overline{M}_{0,S}$. Then

$$(-K_S, \beta(\Pi)) = 2 - |N(\Pi)|. \tag{4.7}$$

(ii) Classes of boundary curves are indecomposable in the Mori cone.

Proof. (i) For $2 \le j \le [n/2]$, denote by B_j the sum of all divisors D_σ such that one part of the partition σ is of cardinality j, and by B the sum of all boundary divisors. By Lemma 3.5 of [KeMcK] we have

$$-K_S = 2B - \sum_{j=2}^{[n/2]} \frac{j(n-j)}{n-1} B_j. \tag{4.8}$$

For a stable 2-partition $\sigma = (S_1, S_2)$ of S, put $c(\sigma) := |S_1||S_2|$. Then, combining (4.4) – (4.6) and (4.8), we get

$$(-K_S, \beta(\Pi)) = 2(|P(\tau)| - |N(\tau)|) - \sum_{\sigma \in P(\tau)} \frac{c(\sigma)}{n-1} + \sum_{\sigma \in N(\tau)} \frac{c(\sigma)}{n-1}. \tag{4.9}$$

The most straightforward way to pass from (4.9) to (4.7) is to consider the four cases $|N(\Pi)| = 1, 2, 3, 4$ separately. Here is the calculation for $|N(\Pi)| = 3$; it demonstrates the typical cancellation pattern. We leave the remaining cases to the reader.

We have $2(|P(\Pi)|-|N(\Pi)|)=0$. Let $(1,a,b,c)$ be the cardinalities of the components of Π, where $a,b,c\geq 2$, $a+b+c=n-1$. Then $P(\Pi)$ consists of three partitions, of the following cardinalities respectively

$$(a+1,b+c),\ (b+1,a+c),\ (c+1,a+b).$$

Hence

$$\sum_{\sigma\in P(\Pi)} c(\sigma)=2(ab+ac+bc)+2(a+b+c).$$

Similarly, partitions in $N(\Pi)$ produce the list

$$(a,1+b+c),\ (b,1+a+c),\ (c,1+a+b)$$

so that

$$\sum_{\sigma\in N(\Pi)} c(\sigma)=2(ab+ac+bc)+(a+b+c).$$

Combining it all together, we get $(-K_S,\beta(\Pi))=-1=2-|N(\Pi)|$.

(ii) From formulas (4.4) – (4.7) we get that $(K_S+B,\beta(\Pi))=1$. Since the divisor K_S+B is ample (see Lemma 3.6 of [KeMcK]), we get that $\beta(\Pi)$ is indecomposable. ■

5 Moduli of stable maps for boundary curves

The central result of this chapter is Proposition 5.1 and this section is devoted to its proof.

5.1 Proposition ([MaS1]). *(i) Morphism (4.1) induces an isomorphism of moduli spaces*

$$\widetilde{b}_\Pi\colon \overline{M}_{0,\Sigma}(\overline{M}_\Pi,\beta_\Pi)\to \overline{M}_{0,\Sigma}(\overline{M}_{0,S},\beta(\Pi)). \tag{5.1}$$

(ii) moduli spaces from (5.1) are smooth and of expected dimension (hence, the virtual fundamental class coincides with the usual fundamental class).

5.2 Proof of Proposition 5.1, Part (ii). Let C be a geometric fiber of $p:\overline{M}_\Pi\to B_\Pi$. We already know that it is isomorphic to $\mathbb{P}^1$. Let $j\colon C\to\overline{M}_{0,S}$ be the natural closed embedding. We assert that

$$j^*(\mathcal{T}_{\overline{M}_{0,S}})\cong\mathcal{O}(2)\oplus\mathcal{O}^{n-4-|N(\Pi)|}\oplus\mathcal{O}(-1)^{|N(\Pi)|}, \tag{5.2}$$

where $\mathcal{T}_{\overline{M}_{0,S}}$ is the tangent sheaf and $\mathcal{O}:=\mathcal{O}_C$.

In fact, consider the embedding $i\colon C\to\overline{M}_\Pi$ and the natural filtration

$$\{0\}\subset\mathcal{T}_C\subset i^*(\mathcal{T}_{\overline{M}_\Pi})\subset j^*(\mathcal{T}_{\overline{M}_{0,S}}). \tag{5.3}$$

The consecutive summands in (5.2) correspond to the consecutive quotients of (5.3). Namely, $\mathcal{T}_C\simeq\mathcal{O}(2)$; $i^*(\mathcal{T}_{\overline{M}_\Pi})/\mathcal{T}_C$ is trivial of rank

$$\dim B_\Pi=|S|-4-|N(\Pi)|, \tag{5.4}$$

finally, the last isomorphism follows from (4.5) and (4.3).

From (5.2) we see that $H^1(C,j^*(\mathcal{T}_{\overline{M}_{0,S}}))=0$. Moreover, it is easy to see that $H^1(C,i^*(\mathcal{T}_{\overline{M}_\Pi}))=0$. Therefore, both moduli spaces are smooth of expected dimensions, and the virtual fundamental classes are simply the usual fundamental classes.

5.3 Preparation. Consider the following setup. Let

$$i\colon W \to V \tag{5.5}$$

be a closed embedding of smooth projective varieties. Let $\beta_W \in A_1(W)$ be an effective class in W and put

$$\beta_V = i_*(\beta_W).$$

We get an induced morphism of moduli stacks of stable maps

$$\widetilde{i}\colon \overline{M}_{0,\Sigma}(W,\beta_W) \to \overline{M}_{0,\Sigma}(V,\beta_V). \tag{5.6}$$

5.3.1 Assumption: every stable map $(C_T, (x_{j,T}), f_{V,T})$ to V of class β_V factors through the embedding (5.5), i.e. there exists $(C_T, (x_{j,T}), f_{W,T})$ – a stable map to W of class β_W, such that $f_{V,T} = i \circ f_{W,T}$.

Since closed embeddings are monomorphisms in the category of schemes, such a factorization is unique, if it exists.

5.3.2 Lemma. Under the assumptions of Section 5.3.1, morphism (5.6) is an isomorphism.

Proof. Since we are viewing stacks as fibered categories, to exhibit the desired equivalence we need to construct an inverse functor to $\widetilde{i}$.

(a) Construction of the inverse functor. Using assumptions of Section 5.3.1 one can construct a functor

$$F\colon \overline{M}_{0,\Sigma}(V,\beta_V) \to \overline{M}_{0,\Sigma}(W,\beta_W)$$

in a natural way as follows.

On objects: an object of $\overline{M}_{0,\Sigma}(V,\beta_V)$ given by $(C_T, (x_{j,T}), f_{V,T})$ is mapped to $(C_T, (x_{j,T}), f_{W,T})$ as in Section 5.3.1.

On morphisms: a morphism in $\overline{M}_{0,\Sigma}(V,\beta_V)$ given by the diagram

$$\begin{tikzcd}
 & & V \\
C_S \arrow[r] \arrow[urr, "f_{V,S}"] \arrow[d] & C_T \arrow[ur, "f_{V,T}"'] \arrow[d] & \\
S \arrow[r, "\varphi"] \arrow[u, bend left, "x_{j,S}"] & T \arrow[u, bend right, "x_{j,T}"'] &
\end{tikzcd}$$

gets mapped to

$$\begin{tikzcd}
 & & W \\
C_S \arrow[r] \arrow[urr, "f_{W,S}"] \arrow[d] & C_T \arrow[ur, "f_{W,T}"'] \arrow[d] & \\
S \arrow[r, "\varphi"] \arrow[u, bend left, "x_{j,S}"] & T \arrow[u, bend right, "x_{j,T}"'] &
\end{tikzcd}$$

where, as above, the latter is obtained from the former using assumptions of Section 5.3.1 and is a morphism in $\overline{M}_{0,\Sigma}(W,\beta_W)$.

One sees immediately that $p_W \circ F = p_V$ and hence it is morphism of fibered categories.

(b) Proving the inverseness of F. Consider $(C_T,(x_{j,T}),f_T)$ – a stable map to W of class β_W. Composing it with i, and then factoring it through W again we obtain *precisely* the same stable map. The other way around: we start from $(C_T,(x_{j,T}),f_T)$ – a stable map to V of class β_V, factor it through W, and then compose with i. We again arrive at *precisely* the same stable map.

Analogously one sees that the same holds for morphisms.

These observations mean that

$$F \circ \widetilde{i} = \mathrm{Id}_{\overline{M}_{0,\Sigma}(W,\beta_W)}$$

and

$$\widetilde{i} \circ F = \mathrm{Id}_{\overline{M}_{0,\Sigma}(V,\beta_V)}.$$

This finishes the proof. ■

5.4 Proof of Proposition 5.1.(i). To prove the statement we will apply Lemma 5.3.2. Here $W = \overline{M}_\Pi$, $V = \overline{M}_{0,S}$, $i = b_\Pi$, $\beta_W = \beta_\Pi$ and $\beta_V = \beta(\Pi)$. The only thing that we need to verify is that in this situation the assumption of Section 5.3.1 holds.

Let $(p_T \colon C_T \to T,(x_{j,T}),f_T)$ be a stable map to $\overline{M}_{0,S}$ of class $\beta(\Pi)$ with T being locally Noetherian scheme. By Proposition 4.3.3.(ii) $\beta(\Pi)$ is indecomposable in the Mori cone, and hence, on geometric fibers the morphism f_T contracts all irreducible components except one and on that component it is a closed embedding.

5.4.1 Irreducible geometric fibers. Consider the diagram

$$\begin{tikzcd} C_T \arrow[rr, "f_T \times p_T"] \arrow[dr, "p_T"'] & & \overline{M}_{0,S} \times T \arrow[dl, "pr_T"] \\ & T & \end{tikzcd}$$

provided by the stable map.

Assume that all geometric fibers of p_T are irreducible and hence $f_T \times p_T$ induces closed embeddings on all geometric fibers. By faithfully flat descent it is then a closed embedding on all fibers. Therefore, the fiber of $f_T \times p_T$ at a point $s \in \overline{M}_{0,S} \times T$ is either empty or $\kappa(s)$-isomorphic to Spec$\,(\kappa(s))$, where $\kappa(s)$ is the residue field at s.

Since p_T and pr_T are proper, the morphism $f_T \times p_T$ is also proper. According to [EGA], Proposition 8.11.5 it implies that $f_T \times p_T$ is a closed embedding.

Thus, we see that if we forget the sections $(x_{j,T})$ the stable morphism $(C_T,(x_{j,T}),f_T)$ gives us a T-point of the Hilbert scheme of $\overline{M}_{0,S}$. Moreover, B_Π is a connected component of the Hilbert scheme and the morphism $p_\Pi \colon \overline{M}_\Pi \to$

B_Π is the universal family over it. Therefore, the diagram

$$\begin{tikzcd} C_T \arrow[r, "f_T"] \arrow[d] & \overline{M}_{0,S} \\ T & \end{tikzcd}$$

is obtained from

$$\begin{tikzcd} \overline{M}_\Pi \arrow[r, "b_\Pi"] \arrow[d, "p_\Pi"'] & \overline{M}_{0,S} \\ B_\Pi & \end{tikzcd}$$

by a unique pullback. Therefore, the stable map $(C_T, (x_{j,T}), f_T)$ factors through $\overline{M}_\Pi$.

5.4.2 General case. Let $(C_T, (x_{j,T})_{j\in\Sigma}, f_T)$ be an arbitrary Σ-labelled stable map to $\overline{M}_{0,S}$ of class $\beta(\Pi)$. Let $\Sigma' \subset \Sigma$ be the subset that labels sections that land on the non-contracted component of geometric fibers. Consider the induced *prestable* map $(C_T, (x_{j,T})_{j\in\Sigma'}, f_T)$. Stabilizing it (cf. Section 3.2.2) we get a stable map $(\widetilde{C}_T, (y_{j,T})_{j\in\Sigma'}, g_T)$ to $\overline{M}_{0,S}$ of class $\beta(\Pi)$, such that $f_T = g_T \circ st$. In other words, we get a diagram

$$\begin{tikzcd} C_T \arrow[r, "st"] \arrow[d] & \widetilde{C}_T \arrow[r, "g_T"] \arrow[dl] & \overline{M}_{0,S} \\ T & & \end{tikzcd}$$

where $\widetilde{C}_T \to T$ has irreducible geometric fibers. According to Section 5.4.1 g_T factors through the embedding $i\colon \overline{M}_\Pi \to \overline{M}_{0,S}$, and hence so does f_T.

This finishes our proof of Proposition 5.1.

6 Gromov-Witten correspondences for boundary curves

Rather than addressing Gromov-Witten correspondences for boundary β's directly, we will introduce and work out a more general setup, and then apply it to our problem.

6.1 Setup, part I. Consider a morphism of smooth irreducible projective varieties $b : E \to W$. Let β_E be an effective curve class on E, and $\beta_W := b_*(\beta_E)$ its pushforward to W. The induced morphism of moduli stacks (cf. Section 3.2.3)

$$\widetilde{b} : \overline{M}_{0,\Sigma}(E, \beta_E) \to \overline{M}_{0,\Sigma}(W, \beta_W)$$

fits into the commutative diagram

$$\begin{tikzcd} \overline{M}_{0,\Sigma}(E,\beta_E) \arrow[r, "\widetilde{b}"] \arrow[d, "(ev_E, st_E)"] & \overline{M}_{0,\Sigma}(W,\beta_W) \arrow[d, "(ev_W, st_W)"] \\ E^\Sigma \times \overline{M}_{0,\Sigma} \arrow[r, "b^\Sigma \times id"] & W^\Sigma \times \overline{M}_{0,\Sigma} \end{tikzcd} \tag{6.1}$$

We assume here and everywhere later on that $|\Sigma| \geq 3$.

6.2 Proposition. *(i) Assume that*

$$J_{0,\Sigma}(W,\beta_W) = \widetilde{b}_*(J_{0,\Sigma}(E,\beta_E)). \tag{6.2}$$

Then

$$I_{0,\Sigma}(W,\beta_W) = (b^\Sigma \times id)_*(I_{0,\Sigma}(E,\beta_E)). \tag{6.3}$$

(ii) Let γ_j be a family of cohomology classes on W marked by Σ. Then from (6.2) it follows that

$$pr_W^*(\otimes_{j\in\Sigma}\gamma_j) \cap I_{0,\Sigma}(W,\beta_W) = (b^\Sigma \times id)_*[pr_E^*(\otimes_{j\in\Sigma} b^*(\gamma_j)) \cap I_{0,\Sigma}(E,\beta_E)]. \tag{6.4}$$

Here we denote by $pr_W : W^\Sigma \times \overline{M}_{0,\Sigma} \to W^\Sigma$ and $pr_E : E^\Sigma \times \overline{M}_{0,\Sigma} \to E^\Sigma$ the respective projection morphisms.

Proof. (i) This follows directly from (6.2) and commutativity of (6.1).

(ii) Using the projection formula we have

$$\begin{aligned} &(b^\Sigma \times id)_*[pr_E^*(\otimes_{j\in\Sigma} b^*(\gamma_j)) \cap I_{0,\Sigma}(E,\beta_E)] = \\ &= (b^\Sigma \times id)_*[(b^\Sigma \times id)^* \circ pr_W^*(\otimes_{j\in\Sigma}\gamma_j) \cap I_{0,\Sigma}(E,\beta_E)] = \\ &= pr_W^*(\otimes_{j\in\Sigma}\gamma_j) \cap (b^\Sigma \times id)_*(I_{0,\Sigma}(E,\beta_E)). \end{aligned}$$

The last expression coincides with l.h.s. of (6.4) in view of (6.3). This completes the proof. ■

6.3 Setup, part II. Keeping notation of Section 6.1, we make the following additional assumptions:

(a) E is explicitly represented as $E = B \times C$ where C is isomorphic to $\mathbb{P}^1$. This identification, including the projections $p = pr_B : E \to B$ and $pr_C : E \to C$, constitutes a part of structure.

(b) β_E is the (numerical) class of any fiber of p.

(c) The deformation problem for any fiber C_0 of p embedded via b_0 in W is trivially unobstructed, i.e.

$$H^1(C_0, b_0^*(\mathcal{T}_W)) = 0\,.$$

(d) The map $\widetilde{b}$ in (6.1) is an isomorphism.

These assumptions are quite strong. In particular, from (a) – (d) it follows that (6.2) holds since the relevant virtual fundamental classes coincide with the ordinary ones. Thus, we can complete the explicit computation of $I_{0,\Sigma}(W,\beta_W)$ starting with the right hand side of (6.3). We will do it in the remaining part of the section.

First of all, we have

$$pr_{B*}(\beta_E) = 0, \quad pr_{C*}(\beta_E) = \mathbf{1},$$

where $\mathbf{1}$ is the fundamental class $[C]$ in the Chow ring of C.

Thus, the two projections induce the map

$$(\widetilde{pr}_B, \widetilde{pr}_C)\colon \overline{M}_{0,\Sigma}(E,\beta_E) \to \overline{M}_{0,\Sigma}(B,0) \times \overline{M}_{0,\Sigma}(C,\mathbf{1})\,.$$

Stabilization maps embed this morphism into the commutative diagram

$$\begin{tikzcd}
\overline{M}_{0,\Sigma}(E,\beta_E) \arrow[r] \arrow[d, "st_E"'] & \overline{M}_{0,\Sigma}(B,0) \times \overline{M}_{0,\Sigma}(C,\mathbf{1}) \arrow[d, "st_B \times st_C"'] \\
\overline{M}_{0,\Sigma} \arrow[r, "\Delta_{\overline{M}_{0,\Sigma}}"] & \overline{M}_{0,\Sigma} \times \overline{M}_{0,\Sigma}
\end{tikzcd}$$

where the lower line is the diagonal embedding (cf. [Beh2], Proposition 5).

Similarly, evaluation maps embed this morphism into the commutative diagram

$$\begin{tikzcd}
\overline{M}_{0,\Sigma}(E,\beta_E) \arrow[r] \arrow[d, "ev_E"'] & \overline{M}_{0,\Sigma}(B,0) \times \overline{M}_{0,\Sigma}(C,\mathbf{1}) \arrow[d, "ev_B \times ev_C"'] \\
E^\Sigma \arrow[r, "s"] & B^\Sigma \times C^\Sigma
\end{tikzcd}$$

where the lower line is now the evident permutation isomorphism induced by $E = B \times C$.

Combining these two diagrams, we get

$$\begin{tikzcd}
\overline{M}_{0,\Sigma}(E,\beta_E) \arrow[r] \arrow[d, "(ev_E, st_E)"'] & \overline{M}_{0,\Sigma}(B,0) \times \overline{M}_{0,\Sigma}(C,\mathbf{1}) \arrow[d, "(ev_B, st_B)\times(ev_C, st_C)"] \\
E^\Sigma \times \overline{M}_{0,\Sigma} \arrow[r, "\widetilde{\Delta}"] & B^\Sigma \times \overline{M}_{0,\Sigma} \times C^\Sigma \times \overline{M}_{0,\Sigma}
\end{tikzcd} \tag{6.5}$$

Here the lower line is an obvious composition of permutations and the diagonal embedding of $\overline{M}_{0,\Sigma}$.

From (6.5) and [Beh2] it follows that

$$I_{0,\Sigma}(E,\beta_E) = \widetilde{\Delta}^!(I_{0,\Sigma}(B,0) \otimes I_{0,\Sigma}(C,\mathbf{1}))\,. \tag{6.6}$$

Furthermore, according to (3.5),

$$I_{0,\Sigma}(B,0) = [\Delta_\Sigma(B) \times \overline{M}_{0,\Sigma}] \in A_*(B^\Sigma \times \overline{M}_{0,\Sigma}). \tag{6.7}$$

Finally, the space $\overline{M}_{0,\Sigma}(C,\mathbf{1})$ and the class $I_{0,\Sigma}(C,\mathbf{1})$ can be described as follows. Recall a construction from [FuMPh]. Let V be a smooth complete algebraic variety. For a finite set Σ, let V^{Σ} be the direct product of a family of V's labeled by elements of Σ. Denote by $\widetilde{V}^{\Sigma}$ the blow-up of the (small) diagonal in V^{Σ}. Finally, define $V^{\Sigma,0}$ as the complement to all partial diagonals in V^{Σ}.

The Fulton-MacPherson *configuration space* $V\langle\Sigma\rangle$ (for curves it was earlier introduced by Beilinson and Ginzburg) is the closure of $V^{\Sigma,0}$ naturally embedded in the product

$$V^{\Sigma} \times \prod_{\Sigma' \subset \Sigma, |\Sigma'| \geq 2} \widetilde{V}^{\Sigma'}.$$

In [FuPa] it was shown that $\overline{M}_{0,\Sigma}(C,\mathbf{1})$ can be identified with $C\langle\Sigma\rangle$ in such a way that the birational morphism ev_C becomes the tautological open embedding when restricted to $C^{\Sigma,0}$.

Therefore, denoting by $D_{\Sigma} \subset C^{\Sigma} \times \overline{M}_{0,\Sigma}$ the closure of the graph of the canonical surjective map $C^{\Sigma,0} \to M_{0,\Sigma}$, we get

$$I_{0,\Sigma}(C,\mathbf{1}) = [D_{\Sigma}]\,. \tag{6.8}$$

Now we can state the main result of this section:

6.4 Proposition. *Assuming 6.3 (a) – (d) we have*

(i) $I_{0,\Sigma}(E,\beta_E) = \widetilde{\Delta}^!([\Delta_{\Sigma}(B) \times \overline{M}_{0,\Sigma} \times D_{\Sigma}])$

(ii) $I_{0,\Sigma}(W,\beta_W) = (b^{\Sigma} \times id)_* \circ \widetilde{\Delta}^!([\Delta_{\Sigma}(B) \times \overline{M}_{0,\Sigma} \times D_{\Sigma}])$

(iii) $\langle I^{E}_{0,\Sigma,\beta_E}\rangle(\otimes_{j\in\Sigma}\gamma_j) = \deg(\cap_{j\in\Sigma}\, pr_{B*}(\gamma_i))$,
where γ_j *is a family of cohomology classes on* E *labelled by* Σ.

Proof. (i) and (ii) are just straightforward implications of (6.6) – (6.8) and (6.3).

(iii) Consider the commutative diagram

$$\begin{tikzcd}
 & & B^{\Sigma} \times C^{\Sigma} \times \overline{M}_{0,\Sigma} \arrow[d, "g"] \arrow[dr, "q"] \arrow[ddl, "p"'] & \\
 & & B^{\Sigma} \times C^{\Sigma} \arrow[dl, "h"] & C^{\Sigma} \times \overline{M}_{0,\Sigma} \\
B \arrow[r, "\Delta_{\Sigma}"] & B^{\Sigma} & &
\end{tikzcd}$$

where all arrows are projections. Identifying $E^{\Sigma} \times \overline{M}_{0,\Sigma}$ and $B^{\Sigma} \times C^{\Sigma} \times \overline{M}_{0,\Sigma}$ we can rewrite $I_{0,\Sigma}(E,\beta_E) = \widetilde{\Delta}^!([\Delta_{\Sigma}(B) \times \overline{M}_{0,\Sigma} \times D_{\Sigma}])$ as

$$p^*[\Delta_{\Sigma}(B)] \cdot q^*[D_{\Sigma}].$$

Let γ_j be cohomology classes on $E = B \times C$, and $\otimes_{j\in\Sigma}\gamma_j$ the corresponding class on E^{Σ}. Identifying E^{Σ} and $B^{\Sigma} \times C^{\Sigma}$ we will view $\otimes_{j\in\Sigma}\gamma_j$ as a class on

$B^\Sigma \times C^\Sigma$. Consecutively applying the projection formula we get

$$\begin{aligned}
&p_*\left(g^*(\otimes_{j\in\Sigma}\gamma_j)\cdot p^*[\Delta_\Sigma(B)]\cdot q^*[D_\Sigma]\right) = \\
&= h_* g_*\left(g^*(\otimes_{j\in\Sigma}\gamma_j)\cdot p^*[\Delta_\Sigma(B)]\cdot q^*[D_\Sigma]\right) = \\
&= h_*\left((\otimes_{j\in\Sigma}\gamma_j)\cdot g_*(p^*[\Delta_\Sigma(B)]\cdot q^*[D_\Sigma])\right) = \\
&= h_*\left((\otimes_{j\in\Sigma}\gamma_j)\cdot h^*[\Delta_\Sigma(B)]\cdot g_*q^*[D_\Sigma]\right) = \\
&= h_*\left((\otimes_{j\in\Sigma}\gamma_j)\cdot h^*[\Delta_\Sigma(B)]\cdot [B^\Sigma\times C^\Sigma]\right) = \\
&= h_*\left((\otimes_{j\in\Sigma}\gamma_j)\cdot h^*[\Delta_\Sigma(B)]\right) = \\
&= h_*(\otimes_{j\in\Sigma}\gamma_j)\cdot [\Delta_\Sigma(B)].
\end{aligned}$$

Taking degree on both sides of the equality finishes the proof. ■

6.5 Applications to $\overline{M}_{0,S}$. Here we return to the settings of Sections 4 and 5. Comparison of notation is as follows: $W = \overline{M}_{0,S}$, $E = \overline{M}_\Pi$, $\beta_E = \beta_\Pi$ and $\beta_W = \beta(\Pi)$. Results of Sections 4 and 5 show that assumptions 6.3 (a) – (d) hold.

6.5.1 Proposition. *(i) Isomorphism* (5.1) *induces the identity*

$$I_{0,\Sigma}(\overline{M}_{0,S},\beta(\Pi)) = (b_\Pi^\Sigma \times id)_*\left(I_{0,\Sigma}(\overline{M}_\Pi,\beta_\Pi)\right),$$

where

$$b_\Pi^\Sigma \times id\colon \overline{M}_\Pi^\Sigma \times \overline{M}_{0,\Sigma} \to (\overline{M}_{0,S})^\Sigma \times \overline{M}_{0,\Sigma}.$$

(ii) Let γ_j be a family of cohomology classes on $\overline{M}_{0,S}$ marked by Σ. Then

$$\langle I_{0,\Sigma,\beta(\Pi)}^{\overline{M}_{0,S}}\rangle(\otimes_{j\in\Sigma}\gamma_j) = deg(\cap_{j\in\Sigma}\, pr_{B_\Pi *}\circ b_\Pi^*(\gamma_i)). \qquad (6.9)$$

Proof. (i) It is a direct consequence of Proposition 6.2.(i).

(ii) It is a direct combination of Propositions 6.2.(ii) and 6.4.(ii). ■

6.5.2 Remark. Here we describe the geometric content of (6.9).

First of all, (6.4) reduces the count to the case of an incidence condition represented by some cycles in $E = \overline{M}_\Pi$: in fact, $b_\Pi^*(\gamma_j)$ are represented by $\Gamma_j \cap \overline{M}_\Pi$ in the case of transversal intersections.

Now, in $\overline{M}_\Pi$ the incidence cycles can be replaced by ones of the form $\Delta_j \times c_j + \Delta_j' \times C$ where c_j are points on a projective line C corresponding to the decomposition $\overline{M}_\Pi = B_\Pi \times C$.

Assume first that $\Delta_j' \neq 0$ for some $j = j_0$. If for such an incidence condition there is a fiber C_0 of $\overline{M}_\Pi \to B_\Pi$ satisfying it at all, then the number of relevant pointed stable maps must be infinite, because x_{j_0} can be chosen arbitrarily along this fiber. Hence decomposable cycles containing at least one factor of the form $\Delta_j' \times C$ give zero contributions to (6.9).

Now consider the case of incidence conditions of the form $\Delta_j \times c_j$ for all $j \in \Sigma$. Let $\Delta_j = pr_{B_\Pi}(\Delta_j \times c_j)$ be in a general position in B_Π so that the intersection cycle $\cap_{j\in\Sigma}\Delta_j$ is a sum of points $y_a \in B_\Pi$, of multiplicity one each. We can also lift Δ_j arbitrarily to $\overline{M}_\Pi$, that is choose all $c_j \in C$ pairwise distinct, and consider $\Delta_j \times c_j$ as a geometric incidence condition representing the initial cohomological incidence condition γ_j.

After that the geometric count becomes straightforward: each point y_a produces one fiber of the class $\beta(\Pi)$ intersecting each $\Delta_j \times c_j$ at one point corresponding to c_j.

The number of y_a's is the right hand side of (6.9), and the curve count interprets the left hand side of (6.9).

Chapter 2

Derived category of $\overline{M}_{0,n}$

In this chapter we study the derived category of moduli spaces $\overline{M}_{0,n}$. These results were obtained by Yuri I. Manin and the author in [MaS2]. In Section 1 we briefly recall relevant notions and then proceed to stating the results.

1 Background notions

Here we recall some facts about semi-orthogonal decompositions. Everything mentioned here can be found in various sources. For example, see [Bo], [Hu], [Ku] and references therein.

Throughout this section $\mathcal{D}$ denotes a K-linear triangulated category with finite dimensional Hom-spaces, where K is a filed. The main example of such category is $\mathcal{D}^b(Coh\, X)$ – the bounded derived category of coherent sheaves on a smooth projective variety X over the field K. We will denote it $\mathcal{D}^b(X)$.

A subcategory of a category is called *strict* iff with an object it also contains all objects isomorphic to it.

1.1 Semi-orthogonal decompositions. An ordered pair of full triangulated subcategories $\mathcal{A}, \mathcal{B}$ is called *semi-orthogonal* iff $\mathrm{Hom}_{\mathcal{D}}(\mathcal{B}, \mathcal{A}) = 0$. It is called a *semi-orthogonal decomposition* iff, moreover, they generate $\mathcal{D}$ as a triangulated category, i.e. the smallest strictly full triangulated subcategory of $\mathcal{D}$ containing $\mathcal{A}$ and $\mathcal{B}$ is $\mathcal{D}$ itself. We will write $\mathcal{D} = \langle \mathcal{A}, \mathcal{B} \rangle$ for such semi-orthogonal decomposition.

Let $\mathcal{A}$ be a full triangulated subcategory of $\mathcal{D}$. Define $\mathcal{A}^\perp$ – the right orthogonal to $\mathcal{A}$ – to be the full subcategory of $\mathcal{D}$ consisting of objects $Y \in \mathcal{D}$ such that $\mathrm{Hom}_{\mathcal{D}}(X, Y) = 0$ for any $X \in \mathcal{A}$. It is a strictly full triangulated subcategory of $\mathcal{D}$. The left orthogonal ${}^\perp\mathcal{A}$ is defined analogously.

This definition gives us two semi-orthogonal pairs

$$\mathcal{A}^\perp, \mathcal{A} \quad \text{and} \quad \mathcal{A}, {}^\perp\mathcal{A}.$$

They form semi-orthogonal decompositions iff $\mathcal{A}$ is *admissible.* More precisely, we call $\mathcal{A}$ *left (resp. right) admissible* iff the inclusion functor $i\colon \mathcal{A} \to \mathcal{D}$ has a left (resp. right) adjoint. We call $\mathcal{A}$ *admissible* is it is both left and right admissible.

The pair $\mathcal{A}^\perp$, $\mathcal{A}$ forms a semi-orthogonal decomposition of $\mathcal{D}$ iff $\mathcal{A}$ is right admissible. Similar statement holds for $\mathcal{A}$, ${}^\perp\mathcal{A}$ and left admissibility.

More generally, a sequence of full triangulated subcategories $\mathcal{A}_1, \dots, \mathcal{A}_n$ is called *semi-orthogonal* iff $\mathrm{Hom}_{\mathcal{D}}(\mathcal{A}_j, \mathcal{A}_i) = 0$ for $j > i$. It is called a *semi-orthogonal decomposition* iff, moreover, they generate $\mathcal{D}$ as a triangulated category. We will write $\mathcal{D} = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$ for such semi-orthogonal decomposition.

1.2 Exceptional collections. An object $E \in \mathcal{D}$ is called *exceptional* iff

$$\mathrm{Hom}_{\mathcal{D}}(E, E) \simeq K \quad \text{and} \quad \mathrm{Hom}_{\mathcal{D}}(E, E[i]) = 0 \quad \text{for } i \neq 0.$$

One can phrase it shorter by saying that the complex $\mathrm{Hom}^\bullet_{\mathcal{D}}(E, E)$ is concentrated in degree zero. Here $\mathrm{Hom}^i_{\mathcal{D}}(E, F) = \mathrm{Hom}_{\mathcal{D}}(E, F[i])$ and the differentials are zero.

Another way of characterising an exceptional object is to say that the strictly full triangulated subcategory $\langle E \rangle$ generated by this object is equivalent to the bounded derived category of finite dimensional vector spaces over the field K.

An ordered sequence of exceptional objects $E_1, \dots E_n$ is called an *exceptional collection* iff $\mathrm{Hom}^\bullet_{\mathcal{D}}(E_j, E_i) = 0$ for $j > i$.

An exceptional collection $E_1, \dots E_n$ is called *full* iff it generates $\mathcal{D}$. It is called *strong* iff $\mathrm{Hom}^\bullet_{\mathcal{D}}(E_j, E_i) = 0$ for $j < i$ is concentrated in degree zero.

2 Derived category of a blow-up

Let X be a smooth projective variety, Y its smooth closed subvariety, $j\colon Y \to X$ the respective closed embedding. Consider the diagram

$$\begin{tikzcd} \widetilde{Y} \arrow[r, "i"] \arrow[d, "\pi"] & \widetilde{X} \arrow[d, "q"] \\ Y \arrow[r, "j"] & X \end{tikzcd} \tag{2.1}$$

describing the blow-up of X along Y.

Let $\mathcal{N}$ be the normal sheaf to Y in X. The exceptional divisor $\widetilde{Y}$ is canonically isomorphic to the relative projective spectrum of the symmetric algebra $S_{\mathcal{O}_Y}(\mathcal{N}^t)$, where $\mathcal{N}^t$ denotes the dual sheaf. The rank $c \geq 2$ of $\mathcal{N}$ equals to the codimension of Y in X; fibers of π are $\mathbb{P}^{c-1}$. This projective bundle carries the standard relative invertible sheaves $\mathcal{O}_\pi(l)$.

From the general definitions, we obtain the canonical exact sequences

$$0 \to \mathcal{N}^t \to j^*(\Omega^1_X) \to \Omega^1_Y \to 0 \tag{2.2}$$

and

$$0 \to \pi^*(\Omega^1_Y) \to \Omega^1_{\widetilde{Y}} \to \Omega^1_{\widetilde{Y}/Y} \to 0. \tag{2.3}$$

The dual exact sequence to that in [Hu], p. 252, reads

$$0 \to \Omega^1_{\widetilde{Y}/Y}(1) \to \pi^*(\mathcal{N}^t) \to \mathcal{O}_\pi(1) \to 0. \tag{2.4}$$

2.1 Orlov's result. The functor

$$\Phi_k = Ri_* \circ (\mathcal{O}_\pi(k) \otimes \,\cdot\,) \circ L\pi^* \colon \mathcal{D}^b(Y) \to \mathcal{D}^b(\widetilde{X})$$

is fully faithful for any k (cf. [Or]). Define

$$\mathcal{D}_k = \begin{cases} \langle \mathrm{Im}(\Phi_k) \rangle, & k = -c+1, \dots, -1 \\ \langle \mathrm{Im}(Lq^*) \rangle, & k = 0 \end{cases}$$

where Im denotes image. These are full triangulated subcategories of $\mathcal{D}^b(\widetilde{X})$. Moreover, they are admissible.

2.1.1 Theorem ([Or]). *The sequence of subcategories*

$$\mathcal{D}_{-c+1}, \dots, \mathcal{D}_{-1}, \mathcal{D}_0 \subset \mathcal{D}^b(\widetilde{X}) \tag{2.5}$$

defines a semi-orthogonal decomposition of $\mathcal{D}^b(\widetilde{X})$.

2.1.2 Exceptional collections. Let $F_1, \dots, F_r$ (resp. $E_1, \dots, E_n$) be full exceptional collections in $\mathcal{D}^b(Y)$ (resp. in $\mathcal{D}^b(X)$), then the objects

$$\begin{aligned} &Ri_*(L\pi^* F_1 \otimes \mathcal{O}_\pi(-c+1)), \dots, Ri_*(L\pi^* F_r \otimes \mathcal{O}_\pi(-c+1)), \\ &\dots \\ &Ri_*(L\pi^* F_1 \otimes \mathcal{O}_\pi(-1)), \dots, Ri_*(L\pi^* F_r \otimes \mathcal{O}_\pi(-1)), \\ &Lq^* E_1, \dots, Lq^* E_n \end{aligned} \tag{2.6}$$

form a full exceptional collection in $\mathcal{D}^b(\widetilde{X})$. This is a direct consequence of Theorem 2.1.1.

2.2 Auxiliary facts. *(a) Adjunction/duality formula.* For a morphism $f \colon U \to V$ of smooth varieties, define $\dim f := \dim U - \dim V$ and put

$$\omega_f := \omega_U \otimes f^*(\omega_V^{-1}).$$

Then for $F \in \mathcal{D}^b(U), E \in \mathcal{D}^b(V)$ we have functorial isomorphisms (cf. [Hu], p. 87)

$$\mathrm{Hom}_{\mathcal{D}^b(V)}(Rf_*(F), E) \cong \mathrm{Hom}_{\mathcal{D}^b(U)}(F, Lf^*(E) \otimes \omega_f[\dim f]) \tag{2.7}$$

(tensor products by an invertible or more general locally free sheaf here and below need not be derived).

(b) Closed embedding. If f is a closed embedding, then the standard short exact sequence

$$0 \to \mathcal{N}^t_{U/V} \to f^*\Omega^1_V \to \Omega^1_U \to 0$$

implies that $\omega_f = \det(\mathcal{N}_{U/V})$. Therefore, in the setting of diagram (2.1) we immediately get

$$\omega_j \simeq \det(\mathcal{N}) \tag{2.8}$$

$$\omega_i \simeq \mathcal{O}_\pi(-1). \tag{2.9}$$

Also $\dim j = -c$ and $\dim i = -1$.

(c) Smooth morphism. If f is smooth, then the standard short exact sequence

$$0 \to f^*\Omega^1_V \to \Omega^1_U \to \Omega^1_{U/V} \to 0$$

implies that

$$\omega_f = \det(\Omega^1_{U/V}). \tag{2.10}$$

(d) The sheaf ω_π*.* Using (2.4) and (2.10) we get

$$\omega_\pi \simeq \pi^* \det(\mathcal{N})^{-1} \otimes \mathcal{O}_\pi(-c).$$

Therefore

$$\omega_\pi[\dim \pi] \cong \pi^*(\det \mathcal{N})^{-1} \otimes \mathcal{O}_\pi(-c)[c-1]. \tag{2.11}$$

(e) The sheaf $\omega_{\widetilde{Y}}$*.* From (2.3) and (2.2) we get

$$\omega_{\widetilde{Y}} \cong \pi^*(j^*(\omega_X) \otimes \det \mathcal{N}) \otimes \omega_{\widetilde{Y}/Y},$$

and from (2.4) we obtain

$$\pi^*(\det \mathcal{N}) \cong \omega^{-1}_{\widetilde{Y}/Y} \otimes \mathcal{O}_\pi(-c).$$

Combining them we get

$$\omega_{\widetilde{Y}} \cong \pi^* \circ j^*(\omega_X) \otimes \mathcal{O}_\pi(-c). \tag{2.12}$$

(f) The sheaf ω_q*.* Using (2.10) and the fact that

$$\omega_{\widetilde{X}} \simeq q^*(\omega_X) \otimes \mathcal{O}_{\widetilde{X}}((c-1)\widetilde{Y})$$

(cf. [Hu], p. 252) we get

$$\omega_q \simeq \mathcal{O}_{\widetilde{X}}((c-1)\widetilde{Y}) \simeq \omega_q[\dim q].$$

2.3 Calculations of Hom's. Below we will need some formulas for the morphism spaces between objects of the collection (2.6) that do not follow directly from the exceptionality of this sequence. In our applications to Keel's tower, all centers of consecutive blow-ups will have codimension $c = 2$. In this case the only relevant value of l is $l = -1$.

Here we take two arbitrary objects $E \in \mathcal{D}^b(X)$ and $F \in \mathcal{D}^b(Y)$.

2.3.1 Proposition. *There is functorial in E and F isomorphism*

$$\mathrm{Hom}_{\mathcal{D}^b(\widetilde{X})}(Ri_*(L\pi^*(F) \otimes \mathcal{O}_\pi(l)), Lq^*(E)) \cong \tag{2.13}$$

$$\mathrm{Hom}_{\mathcal{D}^b(Y)}(F \otimes S^{-l-1}_{\mathcal{O}_Y}(\mathcal{N}))[1], Lj^*(E)) \cong \tag{2.14}$$

$$\mathrm{Hom}_{\mathcal{D}^b(X)}(Rj_*(F \otimes S^{-l-1}_{\mathcal{O}_Y}(\mathcal{N}) \otimes \det \mathcal{N})[1-c], E), \tag{2.15}$$

for $l < 0$.

Proof. Using (2.7) for the morphism i we see that (2.13) is isomorphic to

$$\mathrm{Hom}_{\mathcal{D}^b(\widetilde{Y})}(L\pi^*(F) \otimes \mathcal{O}_\pi(l), Li^* \circ Lq^*(E) \otimes \omega_i[-1]).$$

Replacing here ω_i by (2.9) and $Li^* \circ Lq^*$ by $L\pi^* \circ Lj^*$, we rewrite it as

$$\mathrm{Hom}_{\mathcal{D}^b(\widetilde{Y})}(L\pi^*(F) \otimes \mathcal{O}_\pi(l), L\pi^* \circ Lj^*(E) \otimes \mathcal{O}_\pi(-1)[-1]). \tag{2.16}$$

Multiply both arguments of (2.16) by the same invertible sheaf $\pi^*(\det \mathcal{N})^{-1} \otimes \mathcal{O}_\pi(1-c)$ and then apply to them the same shift $[c]$, without changing Hom. Using (2.11) to rewrite the new second argument, we see that the result will be

$$\mathrm{Hom}_{\mathcal{D}^b(\widetilde{Y})}(L\pi^*(F) \otimes \pi^*(\det \mathcal{N})^{-1} \otimes \mathcal{O}_\pi(l+1-c)[c], L\pi^* \circ Lj^*(E) \otimes \omega_\pi[\dim \pi]).$$

Using (2.7) for the morphism π we can rewrite it as

$$\mathrm{Hom}_{\mathcal{D}^b(Y)}(R\pi_* \left[L\pi^*(F \otimes (\det \mathcal{N})^{-1}) \otimes \mathcal{O}_\pi(l+1-c)\right][c], Lj^*(E)). \tag{2.17}$$

By the projection formula for π (cf. [Hu], (3.11) on p. 83) we have

$$R\pi_* \left[L\pi^*(F \otimes (\det \mathcal{N})^{-1}) \otimes \mathcal{O}_\pi(l+1-c)\right][c] \cong$$
$$F \otimes (\det \mathcal{N})^{-1} \overset{L}{\otimes} R\pi_*(\mathcal{O}_\pi(l+1-c))[c].$$

Note that for any $m < 0$ the complex $R\pi_*(\mathcal{O}_\pi(m))$ is quasi-isomorphic to the complex

$$R^{c-1}\pi_*(\mathcal{O}_\pi(m))[1-c].$$

Therefore, (2.17) becomes

$$\mathrm{Hom}_{\mathcal{D}^b(Y)}(F \otimes (\det \mathcal{N})^{-1} \otimes R^{c-1}\pi_*(\mathcal{O}_\pi(l+1-c))[1], Lj^*(E)).$$

Finally, relative Serre's duality implies that for $l < 0$ we have

$$R^{c-1}\pi_*(\mathcal{O}_\pi(l+1-c)) \cong S^{-l-1}_{\mathcal{O}_Y}(\mathcal{N}) \otimes \det \mathcal{N},$$

and we get (2.14).

Multiplying both arguments by $\det \mathcal{N}$, shifting by $-c$ and using (2.7) for the morphsim j we get (2.15). ■

Remarks. (i) For $l = -1$, $c = 2$, (2.14) becomes simply

$$\mathrm{Hom}_{\mathcal{D}^b(Y)}(F[1], Lj^*(E)). \tag{2.18}$$

This is the only case that must be considered, when $c = 2$. We will use this formula below.

(ii) We could have started our proof of Proposition 2.3.1 by applying (2.7) to the morphism q, rather than to i, thus following diagram (2.1) clockwise rather than counter-clockwise.

2.4 Exceptional collections of locally free sheaves. Let Y be of codimension 2 in X. If $F_1, \ldots, F_r$ (resp. $E_1, \ldots, E_n$) is a full exceptional collection of locally free sheaves on Y (resp. on X), then (2.6) takes form

$$Ri_*(\pi^* F_1 \otimes \mathcal{O}_\pi(-1)), \ldots, Ri_*(\pi^* F_r \otimes \mathcal{O}_\pi(-1)), q^* E_1, \ldots, q^* E_n. \tag{2.19}$$

Even though we started from locally free collections, for the blow-up we get a collection of coherent sheaves. The following proposition gives an explicit full exceptional collection of locally free sheaves under certain assumptions on $E_1, \ldots, E_n$.

2.4.1 Proposition. *Let $E_1, \dots E_n$ be a full exceptional collection of locally free sheaves on X. If $j^*E_1, \dots, j^*E_r$ form a full exceptional collection on Y, then the collection*

$$q^*E_1,\, q^*E_1(\widetilde{Y}), \dots, q^*E_r\,, q^*E_r(\widetilde{Y}), q^*E_{r+1}, \dots, q^*E_n \tag{2.20}$$

is a full exceptional collection on $\widetilde{X}$.

Here and below we use notation of the type $q^*L(\widetilde{Y})$ as a shorthand for $q^*L \otimes \mathcal{O}_{\widetilde{X}}(\widetilde{Y})$.

Proof. The strategy of our proof is simple. We consider exceptional collection (2.19), with $F_a := q^*E_a$ for $1 \le a \le r$, and show that it can be transformed into (2.20) by an explicit sequence of mutations (cf. [Bo], [Kuz]).

Let $A_a := Ri_*(\pi^*j^*E_a \otimes \mathcal{O}_\pi(-1))$ and $B_a := q^*E_a$. Then we can rewrite (2.19) as

$$A_1, \dots, A_r, B_1, \dots, B_n. \tag{2.21}$$

First, we will check that

$$\mathrm{Hom}^\bullet(A_b, B_a) = 0 \quad \text{for} \quad b > a, \tag{2.22}$$

so that the right mutation of such an exceptional pair simply reduces to the permutation $(A_b, B_a) \mapsto (B_a, A_b)$. This shows, that we may consecutively move A_r in (2.21) to the right, until it reaches the position directly to the left of B_r; then move A_{r-1} to the right, until it reaches the position directly to the left of B_{r-1}; and so on. The result will be the exceptional collection

$$A_1, B_1, A_2, B_2, \dots, A_r, B_r, B_{r+1}, \dots, B_n.$$

Second, we will check that

$$R_{B_a}(A_a) \simeq B_a(\widetilde{Y}) \tag{2.23}$$

Therefore, additional r right mutations will transform the latter collection into

$$B_1, B_1(\widetilde{Y}), \dots, B_r, B_r(\widetilde{Y}), B_{r+1}, \dots, B_n,$$

which gives us the claim.

Proof of (2.22). Consider the isomorphism (2.13) $\cong$ (2.18) written for $F := j^*E_b$, $E := E_a[i]$ where i is an arbitrary shift and $b > a$. Its left hand side will represent one of the components of $\mathrm{Hom}^\bullet(A_b, B_a)$. Hence it suffices to prove that the right hand side vanishes. But it is simply $\mathrm{Hom}_{\mathcal{D}^b(Y)}(j^*(E_b)[1], j^*E_a[i])$, $1 \le a < b \le r$, and all these groups vanish, because we assumed that $j^*E_1, \dots, j^*E_r$ is an exceptional collection on Y.

Proof of (2.23). First of all, recall that $R_{B_a}(A_a)$ is defined as the cone $C(\alpha_a)$ of the canonical morphism in $\mathcal{D}^b(\widetilde{X})$

$$\alpha_a \colon A_a \to \mathrm{Hom}^\bullet(A_a, B_a)^t \otimes B_a, \tag{2.24}$$

where t means linear dual in the category of of graded linear spaces.

As above, to calculate $\mathrm{Hom}^\bullet(A_a, B_a)$ consider the isomorphism (2.13) $\cong$ (2.18) written for $F := j^*E_a$ and $E := E_a[i]$. The right hand side is simply $\mathrm{Hom}_{\mathcal{D}^b(Y)}(j^*(E_a)[1], j^*E_a[i])$. It is of dimension one for $i = 1$ and vanishes otherwise because j^*E_a is exceptional by assumption.

Note that A_a fits into the exact sequence

$$0 \to q^*E_a \to q^*E_a(\widetilde{Y}) \to A_a \to 0,$$

and therefore is quasi-isomorphic to the complex

$$0 \to q^*E_a \to q^*E_a(\widetilde{Y}) \to 0$$

(with the first non-zero term in degree -1).

Therefore, (2.24) can be represented by the morphism of complexes

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & q^*E_a \arrow[r, "g"] \arrow[d, "id"] & q^*E_a(\widetilde{Y}) \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & q^*E_a \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

Its cone is the complex

$$0 \longrightarrow q^*E_a \xrightarrow{(id,-g)} q^*E_a \oplus q^*E_a(\widetilde{Y}) \longrightarrow 0\ ,$$

where $q^*E_a \oplus q^*E_a(\widetilde{Y})$ is in degree -1. There exists a short exact sequence of sheaves

$$0 \longrightarrow q^*E_a \xrightarrow{(id,-g)} q^*E_a \oplus q^*E_a(\widetilde{Y}) \xrightarrow{\psi} q^*E_a(\widetilde{Y}) \longrightarrow 0\ ,$$

where $\psi(v, w) = g(v) + w$. Therefore, the cone is quasi-isomorphic to the complex with one non-zero term $q^*E_a(\widetilde{Y})$ placed in degree -1, i.e. $q^*E_a(\widetilde{Y})[1]$. Hence $R_{B_a}(A_a)[1] \simeq q^*E_a(\widetilde{Y})[1]$ and finally $R_{B_a}(A_a) \simeq B_a(\widetilde{Y})$. ■

2.4.2 Corollary. *Let $E_1, \dots, E_n$ be a full exceptional collection of locally free sheaves on X. Assume that for some $s < r$, $j^*E_s, \dots, j^*E_r$ form a full exceptional collection on Y, then*

$$q^*E_1(\widetilde{Y}), \dots, q^*E_{s-1}(\widetilde{Y}), q^*E_s,\ q^*E_s(\widetilde{Y}), \dots, q^*E_r\ , q^*E_r(\widetilde{Y}),$$
$$q^*E_{r+1}, \dots, q^*E_n$$

is a full exceptional collection on $\widetilde{X}$.

Proof. First, let us recall a general fact. Let $\mathcal{D}$ be a triangulated category with a Serre functor $S\colon \mathcal{D} \to \mathcal{D}$ and $\mathcal{A}$ an admissible subcategory (cf. [BoKa1], [Ku]). In this situation we have two semi-orthogonal decompositions $\langle \mathcal{A}, {}^\perp\mathcal{A}\rangle$ and $\langle \mathcal{A}^\perp, \mathcal{A}\rangle$. By [BoKa1], Proposition 3.6, we obtain that

$$R_{{}^\perp\mathcal{A}}(\mathcal{A}) = S^{-1}(\mathcal{A}), \tag{2.25}$$
$$L_{\mathcal{A}^\perp}(\mathcal{A}) = S(\mathcal{A}). \tag{2.26}$$

Our proof of Corollary 2.4.2 will consist of applications of formulas (2.25), (2.26) and Proposition 2.4.1.

Let $\mathcal{D} = \mathcal{D}^b(X)$, $\mathcal{A} = \langle E_1, \dots, E_{s-1} \rangle$, ${}^{\perp}\mathcal{A} = \langle E_s, \dots, E_n \rangle$ and the Serre functor is $S_X = \cdot \otimes \omega_X[\dim X]$. Applying (2.25) we get the full exceptional collection

$$E_s, \dots, E_n, S_X^{-1}(E_1), \dots, S_X^{-1}(E_{s-1}).$$

Applying to it Proposition 2.4.1 we get

$$\begin{aligned} &q^*E_s, q^*E_s(\widetilde{Y}), \dots, q^*E_r, q^*E_r(\widetilde{Y}), q^*E_{r+1}, \dots, q^*E_n, \\ &Lq^* \circ S_X^{-1}(E_1), \dots, Lq^* \circ S_X^{-1}(E_{s-1}), \end{aligned}$$

which is a full exceptional collection on $\widetilde{X}$.

Let now $\mathcal{D} = \mathcal{D}^b(\widetilde{X})$, $\mathcal{A} = \langle Lq^* S_X^{-1}(E_1), \dots, Lq^* S_X^{-1}(E_{s-1}) \rangle$ and the Serre functor is $S_{\widetilde{X}} = \cdot \otimes \omega_{\widetilde{X}}[\dim \widetilde{X}]$. Applying (2.26) and using that

$$S_{\widetilde{X}} \circ Lq^* \circ S_X^{-1} \simeq Lq^* \otimes \mathcal{O}_{\widetilde{X}}(\widetilde{Y})$$

we get the desired statement. ■

3 Keel's tower and semi-orthogonal decompositions

3.1 Notation: combinatorics of marks. Let S be a finite set, $|S| = n \geq 3$. We will call *an inductive structure on* S the choice of a three-element subset $P \subset S$. Sometimes, we will denote $\Sigma := S \setminus P$ so that $S = \Sigma \sqcup P$.

Recall that boundary strata of $\overline{M}_{0,S}$ are bijectively numbered by the (isomorphism classes of) stable S-labelled trees. Such a tree describes the dual combinatorial type of the curve parametrized by the generic point of the respective stratum. The number of edges of such a tree equals the codimension of the stratum.

In particular, boundary divisors, that is, trees with one edge, are determined by *unordered 2-partitions* $S = S_1 \sqcup S_2$, stable in the sense that $|S_i| \geq 2$. Furthermore, codimension two strata are determined by *3-partitions* $S = S_1 \sqcup S_2 \sqcup S_3$ in which the middle term S_2 is uniquely defined, whereas S_1 and S_2 can be interchanged. Stability condition here means that $|S_1|, |S_3| \geq 2, |S_2| \geq 1$.

Whenever an inductive structure P is chosen on S, and $|S| \geq 4$, we may and will *order each stable 2-partition* by the condition $|S_1 \cap P| \leq 1$, and for $|S| \geq 4$ we will *order each stable 3-partition* by the condition $|(S_1 \cup S_2) \cap P| \leq 1$.

3.2 Keel's blow-down. Now, assuming $n := |S| \geq 4$ and given an inductive structure on S, consider a one-point set $\{\bullet\}$ disjoint from S and the diagram of two forgetful morphisms, forgetting respectively sections marked by Σ and the one marked by $\bullet$:

$$\begin{tikzcd} \overline{M}_{0,S\sqcup\{\bullet\}} \arrow[r, "f_\Sigma"] \arrow[d, "f_{\{\bullet\}}"] & \overline{M}_{0,P\sqcup\{\bullet\}} \\ \overline{M}_{0,\Sigma\sqcup P} & \end{tikzcd}$$

Notice that $\overline{M}_{0,P\sqcup\{\bullet\}}$ is $\mathbb{P}^1$ endowed with three boundary points. They are canonically marked by unordered partitions of $P\sqcup\{\bullet\}$ into two parts of cardinality 2. Such a partition, in turn, is determined by an element p of P (one part is $\{\bullet, p\}$) or a two-element subset of P (the part not containing $\bullet$).

We summarize below some results of [Ke], showing, in particular, that the morphism

$$(f_{\{\bullet\}}, f_\Sigma)\colon \overline{M}_{0,S\sqcup\{\bullet\}} \to \overline{M}_{0,S} \times \overline{M}_{0,P\sqcup\{\bullet\}} \tag{3.1}$$

is a composition of blow-ups of smooth codimension two subvarieties isomorphic to boundary divisors of $\overline{M}_{0,S}$. These blow-ups naturally form a sequence of $n-3$ steps $b_k : B_{k,S} \to B_{k-1,S}$, $k = 2, \dots, n-2$. At each step, a union of pairwise disjoint connected smooth submanifolds of codimension two is blown up.

In order to bridge our notation with Keel's, the reader should have in mind the following case:

$$S := \{1, \dots, n\}, \quad \bullet := n+1, \quad P := \{1,2,3\}, \quad \Sigma := \{4, \dots, n\}. \tag{3.2}$$

3.3 Exceptional divisors of $(f_{\{\bullet\}}, f_\Sigma)$. In our notation, Lemma 1 of [Ke] establishes that exceptional divisors of the morphism $(f_{\{\bullet\}}, f_\Sigma)$ are exactly all boundary divisors of $\overline{M}_{0,S\sqcup\{\bullet\}}$ corresponding to the stable 2-partitions of $S\sqcup\{\bullet\}$, satisfying the following condition:

(*) The part containing $\bullet$ contains no more than one element of P and has cardinality ≥ 3.

As an independent check, the reader can convince oneself that the number of such partitions coincides with the difference of ranks of the Picard groups

$$rk\, Pic\, \overline{M}_{0,S\sqcup\{\bullet\}} - rk\, Pic\, (\overline{M}_{0,S} \times \overline{M}_{0,P\sqcup\{\bullet\}}) = 2^{n-1} - n - 1.$$

Let $\overline{\sigma}$ be a partition of $S\sqcup\{\bullet\}$ satisfying (*) above, and let σ be be the respective partition of S obtained by deleting $\bullet$. Obviously, we have

$$f_{\{\bullet\}}(D_{\overline{\sigma}}) = D_\sigma.$$

We will call the cardinality of the second part of $\overline{\sigma}$ (and of σ) *the height* of $D_{\overline{\sigma}}$.

3.4 Keel's tower. The main result of [Ke], Sec. 1, can now be stated in the following way.

Morphism (3.1) can be represented as a composition of blow-downs

$$\overline{M}_{0,S\sqcup\{\bullet\}} =: B_{n-2,S} \to B_{n-3,S} \to \dots \to B_{1,S} := \overline{M}_{0,S} \times \overline{M}_{0,P\sqcup\{\bullet\}} \tag{3.3}$$

satisfying the following conditions:

(i) Image of any exceptional divisor $D_{\overline{\sigma}}$ of height h remains a divisor in $B_{h,S}$, but becomes a closed subscheme of codimension 2 in $B_{h-1,S}, \dots, B_{1,S}$. The composition of the subsequent arrows, followed by the projection $B_{1,S} \to \overline{M}_{0,S}$ identifies this subscheme with D_σ.

(ii) Each morphism $B_{h+1,S} \to B_{h,S}$ is the blow-up of the disjoint union of those subschemes in $B_{h,S}$ that are images of exceptional divisors of height $h+1$. Connected components of the center of the respective blow-up are isomorphic to $\overline{M}_{0,p+1} \times \overline{M}_{0,q+1}$, $p, q \leq n-2$.

3.5 Inductive construction of semi-orthogonal decompositions.

3.5.1 The inductive step I: functoriality in S. In order to calculate $\mathcal{D}^b(\overline{M}_{0,S\sqcup\{\bullet\}})$ assuming that the derived categories of the respective moduli spaces of smaller dimension are already known, we will apply Orlov's results summarized in Section 2 to Keel's tower.

More precisely, Keel's tower depends on the choice of an *inductive structure* on S in the sense of 3.1. In order to be able to induce a given inductive structure on the subsets of S, we will adopt here the following convention, essentially returning us to the Keel's choice (3.2).

S is totally ordered, and for $|S| \geq 3$, $P \subset S$ consists of the first three elements of S. The inductive structure induced on subsets of S is defined then by the induced order.

With this conditions, one easily sees that a bijection of two sets of marks compatible with their respective orders lifts to a unique isomorphism of Keel's towers.

3.5.2 The inductive step II: Keel's blow-ups. Our "inductive leap" from S to $S \sqcup \{\bullet\}$ breaks down into the sequence of small inductive steps corresponding to the consecutive floors of Keel's tower (3.3). They will allow us to obtain an inductive description of a class of semi-orthogonal decompositions of $\mathcal{D}^b(\overline{M}_{0,S})$.

Each floor $q_k\colon B_{k+1,S} \to B_{k,S}$ of Keel's tower gives the blow-up diagram

$$\begin{tikzcd} \widehat{Y}_{k,S} \arrow[r, "i_k"] \arrow[d, "\pi_k"] & B_{k+1,S} \arrow[d, "q_k"] \\ Y_{k,S} \arrow[r, "j_k"] & B_{k,S} \end{tikzcd}$$

where $Y_{k,S}$ is a smooth (often non-connected) subvariety of codimension two in $B_{k,S}$. By Orlov's theorem we get the semi-orthogonal decomposition

$$\mathcal{D}^b(B_{k+1,S}) = \langle \mathcal{D}_{-1}^{k+1},\, \mathcal{D}_0^{k+1} \rangle.$$

Moreover, since q_k blows up a disjoint union of smooth subvarieties Y_σ, where $\sigma = (S_1, S_2)$ is a stable 2-partition of S, each of the subcategories $\mathcal{D}_{-1}^{k+1}$ admits the *orthogonal* decomposition

$$\mathcal{D}_{-1}^{k+1} = (\mathcal{D}(Y_\sigma)_{-1} \,|\, \mathrm{card}\, S_2 = k+2),$$

where $\mathcal{D}(Y_\sigma)_{-1}$ is the subcategory of $\mathcal{D}^b(B_{k+1,S})$ given by the connected component Y_σ (cf. Sec. 2.1).

Finally, we have the canonical identification

$$Y_\sigma = \overline{M}_{0,S_1\sqcup\{\bullet_\sigma\}} \times \overline{M}_{0,S_2\sqcup\{\bullet_\sigma\}}$$

where $\bullet_\sigma$ corresponds to the intersection point of two components. Therefore, $\mathcal{D}^b(Y_\sigma)$ is generated by the external product $\boxtimes$ of any two exceptional collections generating respectively $\overline{M}_{0,S_1\sqcup\{\bullet_\sigma\}}$ and $\overline{M}_{0,S_2\sqcup\{\bullet_\sigma\}}$.

The base $B_{1,S}$ has a similar decomposition (cf. (3.3)). Thus, Keel's tower provides a tool to generate semi-orthogonal decompositions (and exceptional collections) for $\mathcal{D}^b(\overline{M}_{0,n+1})$ from similar objects for $\mathcal{D}^b(\overline{M}_{0,m})$, $m \leq n$.

3.6 Exceptional collections for small n. On $\overline{M}_{0,4} = \mathbb{P}^1$ there is a standard full strong exceptional collection $\langle \mathcal{O}(-1), \mathcal{O} \rangle$.

If we represent $\overline{M}_{0,5}$ as a blow-up $p\colon \overline{M}_{0,5} \to \mathbb{P}^2$ at four points, and denote by l_i, $i = 1, \ldots, 4$ the respective exceptional divisors, then for any choice $\langle F_0, F_1, F_2 \rangle$ of a full strong exceptional collection on $\mathbb{P}^2$, e. g. $F_i = \mathcal{O}(i-2)$, Orlov's theorem will provide a full strong exceptional collection on $\overline{M}_{0,5}$ of the form

$$\langle \mathcal{O}_{l_1}(-1), \ldots, \mathcal{O}_{l_4}(-1), p^*F_0, p^*F_1, p^*F_2 \rangle. \tag{3.4}$$

Taking its $\boxtimes$ with the standard collection on $\mathbb{P}^1$ and then using Keel's blow-up, one can get a full exceptional collection for $\overline{M}_{0,6}$.

4 Example: moduli space $\overline{M}_{0,6}$

Here we will give an example of a full exceptional collection on $\overline{M}_{0,6}$ consisting of invertible sheaves.

4.1 Preparation: moduli space $\overline{M}_{0,5}$. Let $S = \{1,2,3,4,5\}$, $S' = \{1,2,3,4\}$ and $P = \{1,2,3\}$. Consider Keel's tower

$$\begin{tikzcd} B_{2,S'} = \overline{M}_{0,S} \arrow[d, "q_{1,S'}"] \\ B_{1,S'} = \overline{M}_{0,S'} \times \overline{M}_{0,P\sqcup\{5\}} \end{tikzcd}$$

The map $q_{1,S'}$ contracts 3 boundary divisors D_{σ_i} corresponding to unordered partitions

$$\sigma_1 = (5,1,4|2,3), \quad \sigma_2 = (5,2,4|1,3), \quad \sigma_3 = (5,3,4|2,1).$$

Let us identify $\overline{M}_{0,S'}$ with $\overline{M}_{0,P\sqcup\{5\}}$ using the bijection of labels identical on $\{1,2,3\}$ and mapping 4 to 5. Then images of D_{σ_i} become three points on the diagonal, corresponding to the partitions with one part $\{i,4\}$, resp. $\{i,5\}$. Let us imagine $\overline{M}_{0,P\sqcup\{5\}}$ as the horizontal axis $\mathbb{P}^1$, and $\overline{M}_{0,S'}$ as the vertical one. Let H_i, resp. V_i, be the horizontal, resp. vertical fiber, passing through the image of D_{σ_i}.

Now denote by $\widetilde{H}_i$ and $\widetilde{V}_i$ the proper transforms of H_i, resp. V_i, in $\overline{M}_{0,S}$, and let $\widetilde{Z}$ be the proper transform of the diagonal Z. Divisors D_{σ_i},$\widetilde{H}_i$, $\widetilde{V}_i$ and $\widetilde{Z}$ are all isomorphic to $\mathbb{P}^1$ and have self-intersection (-1). There are precisely 10 such curves on $\overline{M}_{0,S}$.

Let F_0, F_1 and G_0, G_1 be full exceptional collections of locally free sheaves on $\overline{M}_{0,S'}$ and $\overline{M}_{0,P\sqcup\{5\}}$ respectively. We know that

$$F_0 \boxtimes G_0, F_1 \boxtimes G_0, F_0 \boxtimes G_1, F_1 \boxtimes G_1$$

is a full exceptional collection on $\overline{M}_{0,S'} \times \overline{M}_{0,P\sqcup\{5\}}$. Denote it as

$$L_0, L_1, L_2, L_3. \tag{4.1}$$

Consider the decomposition $q_{1,S'} = f_1 \circ f_2 \circ f_3$ where f_i contracts only D_{σ_i}. Each f_i is a blow-up of a surface at a point. Applying Corollary 2.4.2 to the blow-up f_3 and using restriction of L_0 as a full exceptional collection in $\mathcal{D}^b(pt)$ we obtain a full exceptional collection on the resulting surface

$$f_3^* L_0, f_3^* L_0(D_{\sigma_3}), f_3^* L_1, f_3^* L_2, f_3^* L_3.$$

Continuing in the same way and always using restriction of the first element as a full exceptional collection in $D(pt)$ we obtain a full exceptional collection on $\overline{M}_{0,S}$

$$q_{1,S'}^* L_0, q_{1,S'}^* L_0(D_{\sigma_1}), q_{1,S'}^* L_0(D_{\sigma_2}), q_{1,S'}^* L_0(D_{\sigma_3}), q_{1,S'}^* L_1, q_{1,S'}^* L_2, q_{1,S'}^* L_3. \tag{4.2}$$

Of course, this is just an example of a full exceptional collection on $\overline{M}_{0,S}$. If we used restrictions of other elements of (4.1) we would have obtained a different answer.

4.2 Preparation: moduli space $\overline{M}_{0,6}$. Let $S = \{1,2,3,4,5\}$, $\{\bullet\} = \{6\}$ and $P = \{1,2,3\}$. Consider Keel's tower

$$\begin{tikzcd} B_{3,S} = \overline{M}_{0,S\sqcup\{6\}} \arrow[d, "q_{2,S}"] \\ B_{2,S} \arrow[d, "q_{1,S}"] \\ B_{1,S} = \overline{M}_{0,S} \times \overline{M}_{0,P\sqcup\{6\}} \end{tikzcd} \tag{4.3}$$

The map $q_{1,S} \circ q_{2,S}$ contracts 10 boundary divisors E_i, $1 \le i \le 10$. At the height 3 level it contracts 7 boundary divisors corresponding to the following unordered partitions

$$E_4 \leftrightarrow (6,1,4|5,2,3), \quad E_6 \leftrightarrow (6,2,4|5,1,3), \quad E_8 \leftrightarrow (6,3,4|5,2,1),$$

$$E_5 \leftrightarrow (6,1,5|4,2,3), \quad E_7 \leftrightarrow (6,2,5|4,1,3), \quad E_9 \leftrightarrow (6,3,5|4,2,1),$$

and $E_{10} \leftrightarrow (6,4,5|1,2,3)$. At the height 2 level it contracts images under $q_{2,S}$ of 3 boundary divisors corresponding to the following unordered partitions

$$E_1 \leftrightarrow (6,1,4,5|2,3), \quad E_2 \leftrightarrow (6,2,4,5|1,3), \quad E_3 \leftrightarrow (6,3,4,5|2,1).$$

The divisors E_1, E_2, E_3 are pairwise disjoint, and $E_4, \dots E_{10}$ are pairwise disjoint as well. We list below all non-empty intersections

$$E_1 \cdot E_4 = P_1, \quad E_1 \cdot E_5 = Q_1,$$

$$E_2 \cdot E_6 = P_2, \quad E_2 \cdot E_7 = Q_2,$$

$$E_3 \cdot E_8 = P_3, \quad E_3 \cdot E_9 = Q_3,$$

$$E_1 \cdot E_{10} = R_1; \quad E_2 \cdot E_{10} = R_2; \quad E_3 \cdot E_{10} = R_3,$$

where P_i, Q_i, R_i are isomorphic to $\mathbb{P}^1$ and all intersections are transversal.

Let $L_0, \dots, L_6$ be an exceptional collection of invertible sheaves on $\overline{M}_{0,S}$ and G_0, G_1 an exceptional collection of invertible sheaves on $\overline{M}_{0,P\sqcup\{6\}}$. From them we construct a collection on $\overline{M}_{0,S} \times \overline{M}_{0,P\sqcup\{6\}}$

$$L_0 \boxtimes G_0, \dots L_6 \boxtimes G_0, L_0 \boxtimes G_1, \dots L_6 \boxtimes G_1. \tag{4.4}$$

We will need some assumptions on the collection $L_0, \dots, L_6$. We use notations for $\overline{M}_{0,S'}$ introduced earlier in section 4.1.

Assumption 1. L_1, L_2 restricted to D_{σ_1} form a full exceptional collection on D_{σ_1}; L_2, L_3 restricted to D_{σ_2} form a full exceptional collection D_{σ_2}; L_3, L_4 restricted to D_{σ_3} form a full exceptional collection D_{σ_3}.

Assumption 2. L_0, L_1 restrict to a full exceptional collection on $\widetilde{H}_1$ and $\widetilde{V}_1$. The same holds for L_1, L_2 on $\widetilde{H}_2, \widetilde{V}_2$ and L_2, L_3 on $\widetilde{H}_3, \widetilde{V}_3$ and L_5, L_6 on $\widetilde{Z}$.

These assumptions are satisfied, for example, for

$$\mathcal{O}, \mathcal{O}(D_{\sigma_1}), \mathcal{O}(D_{\sigma_2}), \mathcal{O}(D_{\sigma_3}), q_{1,S'}^*\mathcal{O}(0,1), q_{1,S'}^*\mathcal{O}(1,0), q_{1,S'}^*\mathcal{O}(1,1),$$

where we used notations of Section 4.1 and identification of $\overline{M}_{0,S'} \times \overline{M}_{0,P\sqcup\{5\}}$ with $\mathbb{P}^1 \times \mathbb{P}^1$.

4.3 Collection. Similar to Section 4.1, in view of Keel's tower (4.3), consecutively applying Corollary 2.4.2 to full exceptional collection (4.4) one can obtain a full exceptional collection on $\overline{M}_{0,6}$. We start with listing its elements, and afterwards give some indications about how it was obtained.

Let $q = q_{1,S} \circ q_{2,S}$ and $E_{\geq i} = \sum_{k=i}^{k=10} E_k$. The exceptional collection is

$$
\begin{aligned}
&q^*(L_0 \boxtimes G_0)(E_{\geq 1}),\\
&q^*(L_1 \boxtimes G_0)(E_{\geq 2}), q^*(L_1 \boxtimes G_0)(E_{\geq 1})\\
&q^*(L_2 \boxtimes G_0)(E_{\geq 2}), q^*(L_2 \boxtimes G_0)(E_1 + E_{\geq 3}), q^*(L_2 \boxtimes G_0)(E_{\geq 1})\\
&q^*(L_3 \boxtimes G_0)(E_{\geq 3}), q^*(L_3 \boxtimes G_0)(E_2 + E_{\geq 4}), q^*(L_3 \boxtimes G_0)(E_{\geq 2})\\
&q^*(L_4 \boxtimes G_0)(E_{\geq 4}), q^*(L_4 \boxtimes G_0)(E_{\geq 3})\\
&q^*(L_5 \boxtimes G_0)(E_{\geq 4}),\\
&q^*(L_6 \boxtimes G_0)(E_{\geq 4}),
\end{aligned}
$$

$$
\begin{aligned}
&q^*(L_0 \boxtimes G_1)(E_{\geq 5}), q^*(L_0 \boxtimes G_1)(E_4 + E_{\geq 6}), q^*(L_0 \boxtimes G_1)(E_{\geq 4}),\\
&q^*(L_1 \boxtimes G_1)(E_{\geq 6}), q^*(L_1 \boxtimes G_1)(E_{\geq 5}), q^*(L_1 \boxtimes G_1)(E_4 + E_{\geq 7}),\\
&q^*(L_1 \boxtimes G_1)(E_4 + E_6 + E_{\geq 8}), q^*(L_1 \boxtimes G_1)(E_4 + E_{\geq 6}),\\
&q^*(L_2 \boxtimes G_1)(E_{\geq 8}), q^*(L_2 \boxtimes G_1)(E_{\geq 7}), q^*(L_2 \boxtimes G_1)(E_6 + E_{\geq 9}),\\
&q^*(L_2 \boxtimes G_1)(E_6 + E_8 + E_{10}), q^*(L_2 \boxtimes G_1)(E_6 + E_{\geq 8}),\\
&q^*(L_3 \boxtimes G_1)(E_{10}), q^*(L_3 \boxtimes G_1)(E_{\geq 9}), q^*(L_3 \boxtimes G_1)(E_8 + E_{10}),\\
&q^*(L_4 \boxtimes G_1)(E_{10}),\\
&q^*(L_5 \boxtimes G_1), q^*(L_5 \boxtimes G_1)(E_{10}),\\
&q^*(L_6 \boxtimes G_1), q^*(L_6 \boxtimes G_1)(E_{10}).
\end{aligned}
$$

Below we describe the algorithm that was used to obtain this collection.

4.3.1 Algorithm, step I. Due to Assumption 1, the restriction of the pair

$$L_1 \boxtimes G_0, L_2 \boxtimes G_0$$

to $q(E_1)$ is a full exceptional collection. The same holds for pairs $L_2 \boxtimes G_0, L_3 \boxtimes G_0$ on $q(E_2)$ and $L_3 \boxtimes G_0, L_4 \boxtimes G_0$ on $q(E_3)$.

Represent $q_{1,S} = f_1 \circ f_2 \circ f_3$ as a composition of three blow-ups (cf. Section 4.1) in such a way that the (preimage of) $q(E_i)$ is blown up at the i-th step for $1 \leq i \leq 3$. At the first step we use the pair $L_1 \boxtimes G_0, L_2 \boxtimes G_0$ to apply Corollary 2.4.2. At the second step we use the pair

$$f_1^*(L_2 \boxtimes G_0)(f_2 \circ f_3 \circ q_{2,S}(E_1)), f_1^*(L_3 \boxtimes G_0),$$

which restricts to a full exceptional collection on $f_2 \circ f_3 \circ q_{2,S}(E_2)$ because $q(E_1)$ and $q(E_2)$ are disjoint and $L_2 \boxtimes G_0, L_3 \boxtimes G_0$ restricts to an exceptional collection on $q(E_2)$ as was pointed out above.

One proceeds similarly at the third step and obtains the following exceptional collection on $B_{2,S}$

$q_{1,S}^*(L_0 \boxtimes G_0)(E_1' + E_2' + E_3')$,
$q_{1,S}^*(L_1 \boxtimes G_0)(E_2' + E_3'), q_{1,S}^*(L_1 \boxtimes G_0)(E_1' + E_2' + E_3')$,
$q_{1,S}^*(L_2 \boxtimes G_0)(E_2' + E_3'), q_{1,S}^*(L_2 \boxtimes G_0)(E_1' + E_3'), q_{1,S}^*(L_2 \boxtimes G_0)(E_1' + E_2' + E_3')$,
$q_{1,S}^*(L_3 \boxtimes G_0)(E_3'), q_{1,S}^*(L_3 \boxtimes G_0)(E_2'), q_{1,S}^*(L_3 \boxtimes G_0)(E_2' + E_3')$,
$q_{1,S}^*(L_4 \boxtimes G_0), q_{1,S}^*(L_4 \boxtimes G_0)(E_3')$,
$q_{1,S}^*(L_5 \boxtimes G_0), q_{1,S}^*(L_6 \boxtimes G_0)$,
$q_{1,S}^*(L_0 \boxtimes G_1), q_{1,S}^*(L_1 \boxtimes G_1) \ldots$,

where $E_i' = q_{2,S}(E_i)$.

4.3.2 Algorithm, step II. Due to Assumption 2, the restriction of the pair

$$L_0 \boxtimes G_1, L_1 \boxtimes G_1$$

to $q_{2,S}(E_4)$ and $q_{2,S}(E_5)$ gives full exceptional collections on them. The same holds for the pair $L_1 \boxtimes G_1, L_2 \boxtimes G_1$ on $q_{2,S}(E_6)$ and $q_{2,S}(E_7)$; for $L_2 \boxtimes G_1, L_3 \boxtimes G_1$ on $q_{2,S}(E_8)$ and $q_{2,S}(E_9)$; for $L_5 \boxtimes G_1, L_6 \boxtimes G_1$ on $q_{2,S}(E_{10})$.

Represent $q_{2,S} = g_4 \circ \cdots \circ g_{10}$ as the composition of blow-downs g_i of E_i for $4 \leq i \leq 10$. To apply Corollary 2.4.2 to a single blow-up one needs to choose an exceptional pair. We always choose pairs related to those described in the beginning of this section(in fact, they are pull-backs of those followed by a twist with $\mathcal{O}(D)$, where D is a divisor disjoint from the exceptional divisor considered at this step).

Chapter 3

Mirror picture for odd-dimensional quadrics

In mathematical literature the meaning of the phrase "Landau-Ginzburg model of a Fano variety" differs from one source to another. Nonetheless, all these notions can be (partially conjecturally) deduced from its presumably strongest version used in the homological mirror symmetry for Fano varieties (see, for example, [Ba], [Or2], [Pr] and references therein).

Here we are working with one of such weaker structures. Namely, we consider mirror symmetry on the level of Frobenius manifolds. In this framework the mirror statement consists of an isomorphism of two Frobenius manifolds: one given by the quantum cohomology of a Fano variety, the other coming from (generalizations of) singularity theory. A conjectural relation to the homological mirror symmetry goes via Hochschild (co)homology.

In Section 1 we recall notions of Frobenius manifolds, Saito's frameworks and Landau-Ginzburg models in this context.

In Section 2 we motivate the usual LG potential for odd-dimensional quadrics given in [EgHoXi]. Namely, we try to argue that it naturally integrates the table of (small) quantum multiplication by the anti-canonical class. More precisely, it only takes into account points of the spectral cover of $QH(Q_{2n+1})$ lying on a certain torus $\mathbf{T}$. Therefore, it cannot be an honest LG model for Q_{2n+1} in the framework of Frobenius manifolds and requires a partial compactification.

In Section 3 we overview structures appearing in works of A. Douai and C. Sabbah, which we apply in Section 4 to the case of three-dimensional quadrics.

This chapter is based on a joint work in progress with Vassily Gorbounov.

1 Background and notation

1.1 F-manifolds. Let M be a complex manifold and $\mathcal{T}_M$ its holomorphic tangent sheaf. The structure of *F-manifold* on M consists of a commutative associative multiplication

$$\circ\colon \mathcal{T}_M \otimes_{\mathcal{O}_M} \mathcal{T}_M \to \mathcal{T}_M$$

with identity $e \in \mathcal{T}_M$ subject to some constraint (so called *F-identity*).

This notion has been introduced in [HeMa] as a weakening of the notion of Frobenius manifold. For further details we refer to [HeMa], [Ma], [He].

An *Euler field* on an F-manifold $(M, \circ, e)$ is a vector field E such that

$$Lie_E(\circ) = d_0 \circ \quad ,$$

for some $d_0 \in \mathbb{C}$.

1.1.1 Spectral cover. Let M be an F-manifold. Considering $(\mathcal{T}_M, \circ)$ as an $\mathcal{O}_M$-algebra one can consider its relative analytic spectrum Specan $(\mathcal{T}_M, \circ)$, called *spectral cover* of M, which is an analytic space endowed with a canonical morphism to M

$$\mathrm{Specan}\,(\mathcal{T}_M, \circ) \to M.$$

Moreover, since there is a canonical surjective morphism $S^\bullet(\mathcal{T}_M) \to (\mathcal{T}_M, \circ)$ of sheaves of $\mathcal{O}_M$-algebras we get a commutative diagram

$$\begin{tikzcd} \mathrm{Specan}\,(\mathcal{T}_M, \circ) \arrow[r, "i"] \arrow[dr] & T^*M \arrow[d] \\ & M \end{tikzcd}$$

where $T^*M \to M$ is the cotangent bundle, and i is a closed embedding. For more details on the spectral cover we refer to [He], [Ma].

1.2 Frobenius manifolds. A *Frobenius manifold* is a tuple

$$(M, \circ, e, g)$$

where M is a complex manifold, $\circ \colon \mathcal{T}_M \otimes \mathcal{T}_M \to \mathcal{T}_M$ is a commutative associative multiplication with unity $e \in \mathcal{T}_M$ and g is a symmetric non-degenerate $\mathcal{O}_M$-bilinear pairing on $\mathcal{T}_M$ with the following properties:

- the metric g is flat and multiplication invariant, i.e.

$$g(X \circ Y, Z) = g(Y, X \circ Z) \qquad \text{for} \quad X, Y, Z \in \mathcal{T}_M$$

- the multiplication $\circ$ is potential, i.e. locally on M there exists a function Φ such that

$$\partial_i \circ \partial_j = \sum_{k,l} \Phi_{ijk} g^{kl} \partial_l, \tag{1.1}$$

  where $\Phi_{ijk} = \frac{\partial^3 \Phi}{\partial x_i \partial x_j \partial x_k}$ and $\partial_i = \frac{\partial}{\partial x_i}$ for some local flat coordinate system $(x_1, \ldots, x_r)$.

- the identity vector field e is flat.

1.2.1 Remark. On can drop flatness of the identity element e from the definition. In this case one needs to modify the rest of the chapter accordingly. Since in our application to quantum cohomology and mirror symmetry the identity is always flat, we assume this from the beginning.

1.2.2 Euler field. A vector field E on a Frobenius manifold $(M, \circ, e, g)$ is called *Euler field* iff

$$Lie_E(\circ) = d_0 \circ \qquad \text{and} \qquad Lie_E(g) = Dg, \tag{1.2}$$

where d_0 and D are some complex numbers.

1.2.3 First structure connection. Consider $(M, \circ, e, g, E)$ a Frobenius manifold with an Euler field, and let $p\colon M \times \mathbb{P}^1_\lambda \to M$ be the natural projection. One introduces a meromorphic connection on $p^*\mathcal{T}_M$ called the *first structure connection* defined as

$$\widehat{\nabla} = p^*\nabla + \lambda\, p^*C + \left(p^*\mathcal{U} + \frac{1}{\lambda}(p^*\mathcal{V} + \frac{D}{2}\mathrm{Id})\right) d\lambda, \tag{1.3}$$

where $C\colon \mathcal{T}_M \to \Omega^1_M \otimes \mathcal{T}_M$ is an $\mathcal{O}_M$-linear morphism defined by $C_X Y = -X \circ Y$; $\mathcal{U}$ and $\mathcal{V}$ are endomorphisms of $\mathcal{T}_M$ given by

$$\mathcal{U}(X) = E \circ X \tag{1.4}$$

$$\mathcal{V}(X) = \nabla_X(E) - \frac{D}{2}X, \tag{1.5}$$

where D is defined in (1.2).

Connection (1.3) has a pole of order less or equal to 1 along $M \times \{0\}$ and of order less or equal to 2 along $M \times \{\infty\}$. For more details we refer to [Ma].

1.2.4 Semi-simplicity and initial conditions. Let M be a Frobenius manifold. A point $p \in M$ is called semi-simple iff the algebra $(T_pM, \circ_p)$ is semi-simple, i.e. isomorphic to $\mathbb{C}^n$. Semi-simple points form an open subset of M.

Let $p \in M$ be a semi-simple point of a Frobenius manifold with an Euler field $(M, \circ, e, g, E)$. In a neighborhood of this point the tuple $(M, \circ, e, g, E)$ is uniquely determined by the data

$$(T, U, V, g, e), \tag{1.6}$$

where $T = T_pM$, U and V are endomorphisms of T induced by (1.4) and (1.5) respectively, g is a non-degenerate symmetric bilinear pairing on T induced by the metric, and e is an element in T induced by the identity vector field. This follows from [Du, Main Th., p.188] or [Sa1, Th. VII.4.2] .

1.3 Quantum cohomology. Let X be a smooth projective complex algebraic variety and $QH(X)$ its big quantum cohomology (see [Ma]). If $\Delta_0, \dots \Delta_n$ is a graded basis and $x_0, \dots, x_n$ dual coordinates, then the quantum product is defined as

$$\Delta_i \circ \Delta_j = \sum_{k,l} \Phi_{ijk} g^{kl} \Delta_l, \tag{1.7}$$

where Φ is the Gromov-Witten potential, $\Phi_{ijk} = \frac{\partial^3 \Phi}{\partial x_i \partial x_j \partial x_k}$, and g is the Poincaré pairing on H. The full structure of the quantum cohomology of X endows (the formal completion of) H with a structure of (formal) Frobenius manifold. One needs to work in the formal category because Φ is not known to be convergent.

Disregarding convergence issues one can think of $QH(X)$ as a family of multiplications on $H = H^*(X, \mathbb{C})$ parametrized by H itself, i.e. it is a multiplication on $\mathcal{T}_H$, where we consider H as a complex manifold. Poincaré pairing on H defines a constant pairing on $\mathcal{T}_H$ which is multiplication invariant and flat.

Under small quantum cohomology one means the restriction of the above picture to $H^2(X, \mathbb{C}) \subset H$. In terms coordinates it means that we reduce all our formulas modulo an ideal generated by coordinates dual to Δ_i's not lying in $H^2(X, \mathbb{C})$.

1.3.1 Initial conditions. At a semi-simple point in the small quantum cohomology the initial conditions take form (cf. (1.6))

$$T = H, \quad U = -K_V \circ\,, \quad V = \frac{1}{2}(n - deg), \quad g, \quad e = 1,$$

where K_V is the canonical class of V, and g is the Poincaré pairing. Here we used the standard Euler field in the quantum cohomology for which $d_0 = 1$ and $D = 2 - \dim X$ (see [Ma]).

1.3.2 Spectral cover. Assume that $H = H^*(X, \mathbb{C})$ is of dimension one in each even degree and zero otherwise. In this situation one can consider an algebraic torus $\mathbf{T} \subset H^t$, which is a locally closed subvariety of H^t. Here H^t is the dual of H.

Namely, let $\Delta_0, \ldots, \Delta_r$ be a graded basis of H, such that Δ_0 is the identity element, and consider

$$H^t = \operatorname{Spec}(S^\bullet(H)) = \operatorname{Spec}(\mathbb{C}[\Delta_0, \ldots, \Delta_r]).$$

In H^t we have an affine subspace $\{\Delta_0 = 1\} \simeq \operatorname{Spec}(\mathbb{C}[\Delta_1, \ldots, \Delta_r])$, and inside this affine subspace we have the torus $\mathbf{T} = \operatorname{Spec}(\mathbb{C}[\Delta_1^{\pm 1}, \ldots, \Delta_r^{\pm 1}])$. This torus does not depend on the choice of $\Delta_1, \ldots, \Delta_r$ and will play an important role in construction of LG models.

Equations that define the spectral cover $\operatorname{Spec}(QH(X))$ as a subvariety of $H \times H^t$ are given just by the multiplication table. One of the equations is always $\Delta_0 = 1$. Hence, the spectral cover always lives inside the affine space $\{\Delta_0 = 1\} \simeq \operatorname{Spec}(\mathbb{C}[\Delta_1, \ldots, \Delta_r])$.

One can summarize it in the diagram

$$\begin{tikzcd}
\operatorname{Spec}(QH(X)) \arrow[r, "i"] \arrow[dr] & H \times H^t \arrow[d] & H \times \mathbf{T} \arrow[l, "j"'] \arrow[dl] \\
& H &
\end{tikzcd}$$

where i and j are embeddings.

In some cases $\operatorname{Spec}(QH(X))$ lies in $H \times \mathbf{T}$. For example, it is true for projective spaces (at least in the small quantum cohomology). It is not true for the case of odd-dimensional quadrics that we consider.

1.4 Saito's framework. Here we recall the setup introduced by K. Saito (see [SaTa] and references therein). Our presentation follows [Ma, III.8].

Let $N \to M$ be a submersion of complex manifolds of relative dimension n and $F \colon N \to \mathbb{C}$ a holomorphic function. We view it as a family of functions on fibers of p parametrized by M. Let $C \subset N$ be the fiberwise critical locus of F and J its ideal sheaf, i.e. it is the zero locus of $d_{N/M}(F) \in \Gamma(N, \Omega^1_{N/M})$. Let $i_C \colon C \to N$ be the natural closed embedding and $p_C = p \circ i_C$. We can collect all these data into the diagram

$$\begin{tikzcd} C \arrow[r, "i_C"] \arrow[dr, "p_C"'] & N \arrow[r, "F"] \arrow[d, "p"] & \mathbb{C} \\ & M & \end{tikzcd} \tag{1.8}$$

1.4.1 Multiplication. Consider the morphism

$$s \colon \mathcal{T}_M \to p_{C*}(\mathcal{O}_C) \tag{1.9}$$

defined by $X \mapsto (\widehat{X}F) \, mod \, J$, where X is a local vector field on M and $\widehat{X}$ its arbitrary lifting to N (cf. [Ma], III.8.2).

If (1.9) is an isomorphism, then we can endow $\mathcal{T}_M$ with a multiplication $\circ \colon \mathcal{T}_M \otimes_{\mathcal{O}_M} \mathcal{T}_M \to \mathcal{T}_M$ by transferring it from $p_{C*}(\mathcal{O}_C)$; the identity element is $e = s^{-1}(1)$. If p_C is generically étale, then $(M, \circ, e)$ is an F-manifold.

On the F-manifold $(M, \circ, e)$ there is a natural Euler field E with $d_0 = 1$. Namely, it is the vector field that corresponds to F under identification (1.9).

1.4.2 Digression: coherent duality. Following [Bea] consider the diagram

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[dr, "g"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

where X, Y, Z are analytic spaces, f is smooth of relative dimension n, g is finite, and i is a regular closed embedding.

Since g is proper, the functor $g^!$ is right adjoint to Rg_*, and there exists a canonical morphism

$$\mathrm{Res}_{X/Y} \colon Rg_* g^!(\mathcal{O}_Y) \to \mathcal{O}_Y. \tag{1.10}$$

Here we adopted the notation $\mathrm{Res}_{X/Y}$ from *loc.cit.*; another common notation is Tr_g (cf. [Ha1]).

Since $g^! \simeq \mathbf{D}_Z \circ Lg^* \circ \mathbf{D}_Y$ we can rewrite (1.10) as

$$\mathrm{Res}_{X/Y} \colon Rg_*(\omega_Z \otimes g^* \omega_Y^{-1}) \to \mathcal{O}_Y,$$

and by the adjunction formula for the closed embedding i we get

$$\mathrm{Res}_{X/Y} \colon Rg_* \left(i^* \omega_{X/Y} \otimes \Lambda^n (I/I^2)^\vee \right) \to \mathcal{O}_Y. \tag{1.11}$$

Assume that Z is globally defined by the regular sequence $t_1, \ldots, t_n$. This sequence defines an element

$$\tau \in \Gamma(Z, \Lambda^n (I/I^2)^\vee) = \mathrm{Hom}(\Lambda^n (I/I^2), \mathcal{O}_Z),$$

by the formula

$$\tau(t_1 \wedge \cdots \wedge t_n) = 1.$$

Plugging τ into (1.11) we get the morphism[1]

$$\mathrm{Res}^{\tau}_{X/Y} \colon g_*(i^*\omega_{X/Y}) \to \mathcal{O}_Y. \tag{1.12}$$

Consider $\omega \in \Gamma(X, \omega_{X/Y})$ and define

$$\mathrm{Res}_{X/Y}\left[\begin{array}{c} \omega \\ t_1 \dots t_n \end{array}\right] = \mathrm{Res}_{X/Y}(i^*\omega \otimes \tau) = \mathrm{Res}^{\tau}_{X/Y}(i^*\omega).$$

It lies in $\Gamma(Y, \mathcal{O}_Y)$ and is called "the residue of ω with respect to $t_1, \dots, t_n$" (cf. [Bea]).

1.4.3 Metric. Here we return to the setting of diagram (1.8) and apply constructions from the previous section.

Let $\omega = dx_1 \wedge \cdots \wedge dx_n$ be a nowhere vanishing global section of $\Omega^n_{N/M}$. Constructions from the above section allow us to define a symmetric bilinear pairing

$$p_{C_*}(i_C^*\Omega^n_{N/M}) \times p_{C_*}(i_C^*\Omega^n_{N/M}) \to \mathcal{O}_M, \tag{1.13}$$

by the formula

$$\eta(\omega_1, \omega_2) := \mathrm{Res}_{N/M}\left[\begin{array}{c} \varphi_1\varphi_2\omega \\ \frac{\partial F}{\partial x_1} \cdots \frac{\partial F}{\partial x_n} \end{array}\right],$$

where $\omega_i = \varphi_i\omega$. The pairing does not depend on the choice of ω and is non-degenerate (cf. [He]).

Let ω be again a nowhere vanishing global section of $\Omega^n_{N/M}$. It gives us the isomorphism

$$\begin{aligned} \mathcal{O}_C &\to i_C^*\Omega^n_{N/M} \\ 1 &\mapsto i_C^*\omega. \end{aligned}$$

Applying p_{C*} we get the isomorphism

$$p_{C*}(\mathcal{O}_C) \to p_{C*}(i_C^*\,\Omega^n_{N/M}).$$

Combining it with (1.9) we get identifications

$$\mathcal{T}_M \overset{\sim}{\rightrightarrows} p_{C*}(\mathcal{O}_C) \overset{\sim}{\rightrightarrows} p_{C*}(i_C^*\,\Omega^n_{N/M}). \tag{1.14}$$

Using these identifications we can transport pairing (1.13) to $\mathcal{T}_M$. This gives us a symmetric bilinear non-degenerate pairing

$$g_\omega \colon \mathcal{T}_M \times \mathcal{T}_M \to \mathcal{O}_M, \tag{1.15}$$

which is multiplication invariant. As the notation suggests the pairing g_ω depends on the choice of ω. On the other hand, flatness of g_ω does not depend on ω because (1.13) is independent of ω. Therefore, flatness of g_ω is defined by unfolding diagram (1.8).

[1] Since g is finite, all higher direct images vanish. Therefore we simply write g_* instead of Rg_*.

1.4.4 Summary. Given a generically étale unfolding (1.8) satisfying (1.9) we get an F-manifold $(M, \circ, e)$. It has a natural Euler field E with $d_0 = 1$.

Moreover, if we pick a nowhere vanishing relative from ω, then we get a multiplication invariant symmetric non-degenerate $\mathcal{O}_M$-bilinear pairing g_ω.

To assert that this stricture is Frobenius we need to show that g_ω is flat, the multiplication $\circ$ is potential and the identity e is flat. We will discuss this under certain assumptions in Section 3.3.

1.4.5 LG models. Let X be a Fano variety and $QH(X)$ its quantum cohomology.

A Saito's framework is called a Landau-Ginzburg model for X iff it is isomorphic to $QH(X)$ as a Frobenius manifold.

Consider a pair (U, f) consisting of a complex smooth affine variety and a regular function on it. It is called a Landau-Ginzburg model for X iff there exists a deformation of (U, f) and a Saito's framework attached to it which is isomorphic to $QH(X)$ as a Frobenius manifold.

These definitions are quite restrictive. We will relax both of them as follows. We require the existence of a point in $QH(X)$ and a point in Saito's framework such that the germs of Frobenius manifolds at these points are isomorphic.

2 Construction of LG potentials

2.1 Quantum cohomology of Q_{2n+1}. Let $V = Q_{2n+1}$ be a smooth Fano hypersurface in $\mathbb{P}^{2n+2}$ which is given by a non-degenerate homogeneous polynomial of degree 2, and let $r = 2n+1$ be its dimension.

The classical cohomology groups $H^i(V, \mathbb{Z})$ are of rank one in each even degree and vanish in odd degrees. Consider a graded basis $\Delta_0, \Delta_1, \Delta_2, \ldots, \Delta_r$ of $H^*(V, \mathbb{Z})$, such that Δ_0 is the identity, Δ_1 is the hyperplane class, $\Delta_i = \Delta_1^{\cup i}$ for $i \leq n$, and $\Delta_i \cup \Delta_{r-i} = \Delta_{2n+1}$, where Δ_{2n+1} is Poincaré dual to the class of a point.

The table of quantum multiplication by Δ_1 in the small quantum cohomology is

$$\begin{aligned}
&\Delta_1^2 = \Delta_2 \\
&\ldots \\
&\Delta_1\Delta_{n-1} = \Delta_n \\
&\Delta_1\Delta_n = 2\Delta_{n+1} \\
&\Delta_1\Delta_{n+1} = \Delta_{n+2} \\
&\ldots \\
&\Delta_1\Delta_{2n} = \Delta_{2n+1} + q\Delta_0 \\
&\Delta_1\Delta_{2n+1} = q\Delta_1.
\end{aligned} \tag{2.1}$$

Therefore, the spectral cover defined by this system consists of $2n+2$ reduced points

$$\begin{aligned}
&P_0 = (0, \ldots, 0, -q) \\
&P_i = (\xi_i, \ldots, \xi_i^n, \frac{1}{2}\xi_i^{n+1}, \ldots, \frac{1}{2}\xi_i^{2n}, q),
\end{aligned}$$

where ξ_i are roots of $\xi^{2n+1} = 4q$, and $1 \le i \le 2n+1$. The point P_0 does not lie on the torus $\mathbf{T}$, as we already mentioned (cf. Section 1.3.2).

2.1.1 Initial conditions. We consider here only the case of Q_3 because this is the only case we will need. The anti-canonical class $-K_X = 3\Delta_1$ and in the basis of Δ_i's we get

$$U = \begin{pmatrix} 0 & 0 & 3q & 0 \\ 3 & 0 & 0 & 3q \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{pmatrix}$$

and

$$V = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} + \frac{1}{2}$$

2.2 Standard Landau-Ginzburg potential. Restricting to the torus $\mathbf{T}$ we can rewrite system (2.1) as

$$\begin{aligned}
\Delta_1 &= \frac{\Delta_2}{\Delta_1} \\
&\dots \\
\Delta_1 &= \frac{\Delta_n}{\Delta_{n-1}} \\
\Delta_1 &= \frac{2\Delta_{n+1}}{\Delta_n} \\
\Delta_1 &= \frac{\Delta_{n+2}}{\Delta_{n+1}} \\
&\dots \\
\Delta_1 &= \frac{\Delta_{2n+1} + q}{\Delta_{2n}} \\
\Delta_1 &= \frac{q\Delta_1}{\Delta_{2n+1}}.
\end{aligned}$$

After some manipulations one can find a potential integrating these equations. First, consecutively substitute the right hand side of the current equation into the place of the left hand side of the next one for the first $r-2$ equations. Replace the last two equations by new ones generating the same ideal

$$\begin{aligned}
\frac{\Delta_{2n}}{\Delta_{2n-1}} &= \frac{(\Delta_{2n+1} + q)^2}{2\Delta_{2n}\Delta_{2n+1}} \\
\frac{\Delta_{2n+1}^2}{2\Delta_{2n}\Delta_{2n+1}} &= \frac{q^2}{2\Delta_{2n}\Delta_{2n+1}}
\end{aligned}$$

After these operations the system takes form

$$
\begin{aligned}
&\Delta_1 = \frac{\Delta_2}{\Delta_1} \\
&\frac{\Delta_2}{\Delta_1} = \frac{\Delta_3}{\Delta_2} \\
&\dots \\
&\frac{\Delta_n}{\Delta_{n-1}} = \frac{2\Delta_{n+1}}{\Delta_n} \\
&\dots \\
&\frac{\Delta_{2n}}{\Delta_{2n-1}} = \frac{(\Delta_{2n+1}+q)^2}{2\Delta_{2n}\Delta_{2n+1}} \\
&\frac{\Delta_{2n+1}^2}{2\Delta_{2n}\Delta_{2n+1}} = \frac{q^2}{2\Delta_{2n}\Delta_{2n+1}}
\end{aligned}
$$

and can be integrated. Indeed, the potential is

$$f = \Delta_1 + \frac{\Delta_2}{\Delta_1} + \dots + \frac{\Delta_n}{\Delta_{n-1}} + \frac{2\Delta_{n+1}}{\Delta_n} + \frac{\Delta_{n+2}}{\Delta_{n+1}} + \dots + \frac{\Delta_{2n}}{\Delta_{2n-1}} + \frac{(\Delta_{2n+1}+q)^2}{2\Delta_{2n}\Delta_{2n+1}}.$$

The system $\Delta_i \frac{\partial f}{\partial \Delta_i} = 0$ coincides with the above system.

Considering another coordinate system on the torus $\mathbf{T} = \{\Delta_1 \dots \Delta_{2n+1} \neq 0\}$ given by

$$Y_1 = \Delta_1, Y_2 = \frac{\Delta_2}{\Delta_1}, \dots, Y_n = \frac{\Delta_n}{\Delta_{n-1}}, Y_{n+1} = \frac{2\Delta_{n+1}}{\Delta_n},$$

$$Y_{n+2} = \frac{\Delta_{n+2}}{\Delta_{n+1}}, \dots, Y_{2n} = \frac{\Delta_{2n}}{\Delta_{2n-1}}, Y_{2n+1} = \Delta_{2n+1}$$

we recover the potential proposed in [EgHoXi]

$$f = Y_1 + \dots + Y_{2n} + \frac{(Y_{2n+1}+q)^2}{Y_1 \dots Y_{2n+1}}.$$

2.3 Compactification. The LG potential f considered above has one critical point less than the spectral cover of $QH(Q_{2n+1})$. Here we will fix this problem by extending f to a new potential $\widetilde{f}$. We do it in a very *ad hoc* manner. This will be justified by results of Section 4 where, in the case of a three-dimensional quadric, we construct a conjectural Saito's Frobenius manifold attached to an unfolding of $\widetilde{f}$, and match it with the quantum cohomology of that quadric.[2]

Let $Y_1, \dots Y_{2n+1}$ be the standard coordinates on $\mathbb{C}^{2n+1}$. The LG potential we start with is

$$f = Y_1 + \dots + Y_{2n} + \frac{(Y_{2n+1}+q)^2}{Y_1 \dots Y_{2n+1}},$$

which is a regular function on the torus $\{Y_1 \dots Y_{2n+1} \neq 0\}$.

[2] One can also compare $\widetilde{f}$ to quantum cohomology by using a part of the special initial conditions u_i, η_i in the sense of [Ma].

Consider functions $x, y_1, \dots, y_n, z_1, \dots z_n$ given by

$$\begin{aligned} x &= \frac{Y_{2n+1}+q}{qY_1 \dots Y_{2n}} \\ y_1 &= Y_1 \\ &\dots \\ y_n &= Y_n \\ z_1 &= \frac{Y_{n+1}}{Y_1} - 1 \\ &\dots \\ z_n &= \frac{Y_{2n}}{Y_n} - 1. \end{aligned} \tag{2.2}$$

They define a coordinate system on $\{Y_1 \dots Y_{2n+1} \neq 0\}$. Rewriting f in terms of these coordinates we get the expression

$$\widetilde{f} = \sum_{i=1}^{n} y_i(2+z_i) + \frac{qx^2}{(xy_1 \dots y_n - 1)(1+z_1)\dots(1+z_n)}. \tag{2.3}$$

This expression defines a regular function on an open subvariety of $\mathbb{C}^{2n+1}$ with standard coordinates $x, y_1, \dots, y_n, z_1, \dots z_n$. The torus $\{Y_1 \dots Y_{2n+1} \neq 0\}$ is embedded into this space by formulas (2.2).

This is the partial compactification we were looking for. Indeed, the critical locus of $\widetilde{f}$ has $2n+2$ points

$$P_0 = (0, 0, \dots, 0, -2, \dots, -2) \quad \text{and} \quad P_i = (\frac{2}{\xi_i^n}, \xi_i, \dots, \xi_i, 0, \dots, 0),$$

where $\xi_i^{2n+1} = 4q$. Therefore, it might give a Saito's framework isomorphic to $QH(Q_{2n+1})$.

3 Overview of the Douai-Sabbah construction

In [DoSa1] authors have found a way to construct a Saito's Frobenius manifold for a certain class of regular functions on smooth affine varieties. Below we briefly review this construction. Our presentation follows mostly [Do1, App. A].

Once and for all we fix $\mathbb{C}$ as the ground field. If not mentioned otherwise, all algebraic varieties are considered with the Zariski topology.

3.1 Structure at one point. Let X be a smooth affine variety of dimension n with a regular function h on it. The main example relevant for mirror symmetry: $X = (\mathbf{G}_m)^n$ and h is a Laurent polynomial. To such a function h one can attach its *Gauss-Manin system*

$$G = \Omega^n(X)[\theta, \theta^{-1}]/(\theta d - dh\wedge)\Omega^{n-1}(X)[\theta, \theta^{-1}],$$

which is a free $\mathbb{C}[\theta, \theta^{-1}]$-module of finite rank with a flat connection ∇ defined as follows. Let $\sum_i \omega_i \theta^i$ be a representative of some class $\gamma \in G$, i.e. $\gamma = [\sum_i \omega_i \theta^i]$. Then

$$\theta^2 \nabla_{\frac{\partial}{\partial \theta}}(\gamma) = \left[\sum_i h\omega_i\theta^i + \sum_i i\omega_i\theta^{i+1} \right],$$

where the brackets [ ] denote taking class in G.

Let $\mathbb{P}^1_\theta = U_0 \cup U_\infty$ be the standard open cover, i.e. $U_0 = \mathbb{A}^1_\theta$, $U_1 = \mathbb{A}^1_{\theta^{-1}}$, and $W = U_0 \cap U_\infty = \mathbb{A}^1_\theta - \{0\}$. Here $\mathbb{A}^1_\theta$ stands for $\mathrm{Spec}(\mathbb{C}[\theta])$ and $\{0\}$ for $\{\theta = 0\}$. We will use this sort of notation repeatedly in what follows.

Using the above notation, $G = \Gamma(W, \mathcal{F}_W)$ for some locally free sheaf $\mathcal{F}_W$ with a flat connection ∇_W; moreover, $\mathcal{F}_W$ is globally free.[3]

3.1.1 Extension to $\mathbb{P}^1_\theta$. The goal is to extend the pair $(\mathcal{F}_W, \nabla_W)$ defined on W to a pair $(\mathcal{F}, \nabla)$ defined on $\mathbb{P}^1_\theta$ such that $\mathcal{F}$ is a free $\mathcal{O}_{\mathbb{P}^1_\theta}$-module and ∇ is a flat *meromorphic* connection on $\mathcal{F}$ with a pole of order less or equal to 2 at zero and of order less or equal to 1 at infinity. This type of question is known as *the Birkhoff problem* (cf. [Sa1, Ch. 4]).

More concretely one needs to do the following. To extend $\mathcal{F}_W$ to a locally free sheaf $\mathcal{F}$ on $\mathbb{P}^1_\theta$ we just need $\mathcal{F}_{U_0}$ and $\mathcal{F}_{U_\infty}$ – free $\mathcal{O}_{U_0}$ and $\mathcal{O}_{U_\infty}$-modules respectively. The resulting sheaf $\mathcal{F}$ is globally free iff

$$G_0 = (G_0 \cap G_\infty) \oplus \theta G_0,$$

where $G_0 = \Gamma(U_0, \mathcal{F}_{U_0})$ and $G_\infty = \Gamma(U_\infty, \mathcal{F}_{U_\infty})$. Moreover, since U_0 and U_∞ are affine, giving $\mathcal{F}_{U_0}$ and $\mathcal{F}_{U_\infty}$ is equivalent to giving G_0 and G_∞.

Since all of the above sheaves are free the restriction maps are injective. Therefore, ∇_W extends to $\mathcal{F}$ uniquely, and it has prescribed poles iff

$$\theta^2 \nabla_{\frac{\partial}{\partial\theta}}(G_0) \subset G_0 \tag{3.1}$$

$$\tau \nabla_{\frac{\partial}{\partial\tau}}(G_\infty) \subset G_\infty, \tag{3.2}$$

where $\tau = \theta^{-1}$.

In our situation there is a natural candidate for G_0 (cf. [Do1, App. A]). Namely, consider the module

$$G_0 = \Omega^n(X)[\theta]/(\theta d - dh\wedge)\Omega^{n-1}(X)[\theta], \tag{3.3}$$

the *Brieskorn lattice* of h, which is free if we assume h to be *tame*.[4] Condition (3.1) holds automatically. Later on, whenever we consider G_0 we tacitly assume that h is tame and use (3.3) as the extension to U_0.

As for G_∞ satisfying (3.2), the situation is more delicate. There always exists a canonical G_∞ which satisfies (3.2) and it is constructed using the Hodge theory (cf. [Do1, App. A]). In practice though, it appears to be hard to work with.

[3]The natural framework to use here is the theory of $\mathcal{D}$-modules. For more details on these constructions see [DoSa1], [Do2] and references therein.

Let V be a finite dimensional vector space and V^* its dual. The Fourier transform is a certain functor from the category of $\mathcal{D}$-modules on $\mathbb{A}(V)$ to the category of $\mathcal{D}$-modules on $\mathbb{A}(V^*)$, where $\mathbb{A}(V) = \mathrm{Spec}(S(V^*))$ and $\mathbb{A}(V^*) = \mathrm{Spec}(S(V))$.

Let V be a 1-dimensional vector space, $\tau \in V$ its basis element, and $t \in V^*$ the dual basis. Then we have natural identifications $\mathbb{A}(V) = \mathbb{A}^1_t$ and $\mathbb{A}(V^*) = \mathbb{A}^1_\tau$. With these identifications in mind we proceed to the definition.

Consider h as the morphsim $h\colon X \to \mathbb{A}^1_t$ given by $t \mapsto h$. One first defines the *Gauss-Manin complex* $h_*\mathcal{O}_X$, where h_* is the direct image of $\mathcal{D}$-modules. Since h_* preserves holonomic $\mathcal{D}$-modules with regular singularities and $\mathcal{O}_X$ is such, cohomology groups of $h_*\mathcal{O}_X$ are holonomic $\mathcal{D}$-modules with regular singularities. Consecutively taking H^0, applying Fourier transform, restricting to $\mathbb{A}^1_\tau - \{0\}$, and denoting $\theta = \tau^{-1}$ we get we get our $\mathcal{F}_W$.

[4]There are different notions of tameness, see Section 4.1.

On the other hand, one can always try to find G_∞ as follows. Pick a $\mathbb{C}[\theta]$-basis of G_0, consider it inside of G and it will give us a basis of G. Now let G_∞ be the submodule of G generated by these elements over $\mathbb{C}[\theta^{-1}]$. This process automatically gives the desired extension $\mathcal{F}$. As for condition (3.2), it is equivalent to the connection matrix in this basis being of the form

$$\left(\frac{A_0}{\theta} + A_\infty\right)\frac{d\theta}{\theta}, \tag{3.4}$$

where A_0 and A_∞ are constant matrices. This has to be verified in each case individually. Of course, bases of G_0 which differ by a $\mathbb{C}$-linear transformation give the same G_∞.

3.1.2 Pairing. If h is a tame function, then there exists a non-degenerate bilinear pairing (cf. [DoSa1])

$$S_W \colon \mathcal{F}_W \otimes j^*\mathcal{F}_W \to \mathcal{O}_W, \tag{3.5}$$

where $j\colon W \to W$ is given by $\theta \mapsto -\theta$.[5]

It satisfies

$$\frac{d}{d\tau}S_W(g_1, g_2) = S_W(\partial_\tau g_1, g_2) + S_W(g_1, \partial_\tau g_2), \tag{3.6}$$

i.e. it is a horizontal section of the sheaf $\mathcal{H}om_{\mathcal{O}_W}(\mathcal{F}_W \otimes j^*\mathcal{F}_W, \mathcal{O}_W)$ equipped with its natural connection, and

$$S_W(g_1, g_2) = (-1)^n\overline{S_W(g_2, g_2)}, \tag{3.7}$$

where we used the notation $\overline{P(\tau, \tau^{-1})} := P(-\tau, -\tau^{-1})$ for a Laurent polynomial $P(\tau, \tau^{-1})$.

Moreover, (3.5) has the property

$$S_W(\mathcal{F}_{U_0}, j^*\mathcal{F}_{U_0}) \subset \theta^n\mathcal{O}_{U_0} \subset \mathcal{O}_W, \tag{3.8}$$

and therefore we get a natural extension

$$S_{U_0} \colon \mathcal{F}_{U_0} \otimes j^*\mathcal{F}_{U_0} \to \mathcal{O}_{U_0}. \tag{3.9}$$

On $\mathcal{F}_{U_0}$ we can write

$$S_{U_0} = \sum_{i \geq n} S_i\theta^i,$$

where $S_i \colon \mathcal{F}_{U_0} \otimes j^*\mathcal{F}_{U_0} \to \mathbb{C}_{U_0}$ are *higher residue pairings* of K. Saito; S_n is the Grothendieck residue pairing.[6] For a modern overview of K. Saito's works on this subject we refer to [SaTa].

[5] The morphism j extends uniquely to a morphism $\mathbb{P}^1_\theta \to \mathbb{P}^1_\theta$. Abusing notation we will denote this morphism again by j, and we will use the same notation for its restrictions if it does not lead to confusion.

[6] We apologize for the seemingly highbrow usage of sheaves in such a simple affine situation but it is this notation that can be easily adopted to the general case. A completely analogous treatment on the level of global sections is done in [Do1, App. A].

So far, assuming h is tame, we have exhibited a canonical pairing (3.5) on $\mathcal{F}_W$ which naturally extends to $\mathcal{F}_{U_0}$. Assume now that we have an arbitrary pair $(\mathcal{F}, \nabla)$ extending $(\mathcal{F}_{U_0}, \nabla_{U_0})$ as in Section 3.1.1. The goal now is to extend (3.9) to a pairing

$$S\colon \mathcal{F} \otimes j^*\mathcal{F} \to \mathcal{O}_{\mathbb{P}^1_\theta}.$$

There exists $d \in \mathbb{Z}$ such that $S_W(\mathcal{F}_{U_\infty}, j^*\mathcal{F}_{U_\infty}) \subset \tau^{-d}\mathcal{O}_{U_\infty}$ and therefore (3.9) always extends to

$$S\colon \mathcal{F} \otimes j^*\mathcal{F} \to \mathcal{O}_{\mathbb{P}^1_\theta}(-n \cdot \{0\} + d \cdot \{\infty\}). \tag{3.10}$$

Here $\mathcal{O}_{\mathbb{P}^1_\theta}(-n \cdot \{0\} + d \cdot \{\infty\})$ is an invertible subsheaf of $K_{\mathbb{P}^1_\theta}$ which consists of rational functions of $\mathbb{P}^1_\theta$, generated by θ^n and τ^{-d}. It is isomorphic to $\mathcal{O}_{\mathbb{P}^1_\theta}$ if and only if $d = n$. The choice of d in (3.10) is not unique but there exists the minimal possible d.

By (3.8) we know that $d \geq n$ and therefore (3.10) produces a pairing with values in $\mathcal{O}_{\mathbb{P}^1_\theta}$ iff $d = n$. The latter condition is equivalent to the existence of a global basis $e_1, \ldots, e_\mu$ of $\mathcal{F}$ such that $S_{U_0}(e_{i|U_0}, e_{j|U_0}) \in \theta^n\mathbb{C}$, i.e. $\mathcal{F}$ is given by a basis with such property.

3.1.3 ***V*-filtration.** Let $\mathcal{M}$ be a $\mathcal{D}$-module on a smooth algebraic variety X. If Y is a smooth subvariety of codimension 1, then one has the notion of *V-filtration* (or *Kashiwara-Malgrange filtration*) along Y. It may or may not exist in general but it is known to exist for some classes of $\mathcal{D}$-modules (cf. [DoSa1], [DoSa2], [PeSt], [Bu]).

Let I be the ideal sheaf of Y in X. First define an increasing filtration $V_\bullet\mathcal{O}_X$ by putting $V_i\mathcal{O}_X = \mathcal{O}_X$ if $i \geq 0$ and $V_i\mathcal{O}_X = I^{-i}$ if $i < 0$. Now let $V_\bullet\mathcal{D}_X$ be an increasing filtration defined as

$$V_i\mathcal{D}_X = \{P \in \mathcal{D}_X \mid P(V_m\mathcal{O}_X) \subset V_{m+i}\mathcal{O}_X, \quad \forall\, m \in \mathbb{Z}\}.$$

One can locally describe it more explicitly as follows (cf. [PeSt]). Let $(y_1, \ldots, y_n, x)$ be a local coordinate system on X such that in this neighbourhood Y is given by the equation $x = 0$. Then $V_0\mathcal{D}_X$ is a subsheaf of rings of $\mathcal{D}_X$ locally generated by $\mathcal{O}_X$, vector fields $\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}$ and $x\frac{\partial}{\partial x}$. If we denote $\partial_x = \frac{\partial}{\partial x}$, then $V_i\mathcal{D}_X$ is a $V_0\mathcal{D}_X$-module generated by $x^i\partial_x^j$ with $i - j \geq -k$.

Let $\mathcal{M}$ be a (left) $\mathcal{D}_X$-module and $V_\bullet\mathcal{M}$ a discrete exhaustive increasing filtration indexed by $\mathbb{Q}$. It is called *V-filtration* iff

1. it is compatible with the $V_\bullet\mathcal{D}_X$, i.e. $(V_i\mathcal{D}_X)(V_\alpha\mathcal{M}) \subset V_{\alpha+i}\mathcal{M}$ for all α and i; furthermore, the inclusion $I\,(V_\alpha M) \subset V_{\alpha-1}M$ should be an equality for $\alpha < 0$.

2. the action of $x\partial_x + \alpha$ on $\mathrm{Gr}^V_\alpha M$ is nilpotent.

If such a filtration exists, then it is unique (cf. [Bu]).

Application. The Gauss-Manin system G considered as a $\mathbb{C}[\tau]\langle\partial_\tau\rangle$-module[7] always has a V-filtration along $\{\tau = 0\}$, and pairing (3.5) satisfies

$$S_W(V_0G, \overline{V_{<1}G}) \subset \mathbb{C}[\tau]. \tag{3.11}$$

For more details we refer to [DoSa1].

[7] This just means that we consider not G itself but its push-forward as a D-module with respect to the open inclusion $U_0 \cap U_\infty \to U_\infty$.

3.1.4 Summary. Assume that h is a tame function. Consider its Gauss-Manin system $(\mathcal{F}_W, \nabla_W)$. It can be always extended to a pair $(\mathcal{F}_{U_0}, \nabla_{U_0})$ using (3.3), and let $\mathcal{F}$ be an extension satisfying (3.2). The canonical pairing (3.5) always extends to $\mathcal{F}_{U_0}$; it extends to $\mathcal{F}$ iff there exists a basis of global sections $e_1, \ldots, e_\mu$ of $\mathcal{F}$ such that $S_{U_0}(e_{i|U_0}, e_{j|U_0}) \in \theta^n\mathbb{C}$.

3.2 Adding parameters. Let F be a deformation of h over some parameter space (M, x_0), i.e. we have a diagram of type (1.8)

$$\begin{tikzcd} C \arrow[r, "i_C"] \arrow[dr, "p_C"'] & N \arrow[r, "F"] \arrow[d, "p"] & \mathbb{C} \\ & M & \end{tikzcd} \tag{3.12}$$

where assumptions on the objects are the same as for (1.8) plus additionally we have a point $x_0 \in M$. The fiber $p^{-1}(x_0)$ together with $F_{|p^{-1}(x_0)}$ are analytizations of X and h from Section 3.1.

The simplest example is when M is an open ball in $\mathbb{C}^m$, $N = X \times M$ and F is given by

$$F = f + \sum_{i=1}^{m} z_i g_i,$$

where z_i are standard coordinates on $\mathbb{C}^m$ and g_i are functions on X.

The product $M \times \mathbb{P}^1_\theta$ (considered with the analytic topology) has an open cover $M \times \mathbb{P}^1_\theta = (M \times U_0) \cup (M \times U_\infty)$. Analogously to what we had before $(M \times U_0) \cap (M \times U_\infty) = M \times W$.

Assume that (3.12) satisfies conditions from [DoSa1, Sec. 2.a], then first, as in Sec. 2.d of *loc.cit.*, one can attach to it

$$(\mathcal{F}_{M\times W}, \nabla_{M\times W})$$

a free sheaf on $M \times W$ of rank μ with connection $\nabla_{M\times W}$. Moreover, one can extend these objects to $M \times U_0$, i.e. we have a pair

$$(\mathcal{F}_{M\times U_0}, \nabla_{M\times U_0}), \tag{3.13}$$

where $\mathcal{F}_{M\times U_0}$ is a free sheaf on $M\times U_0$ of rank μ with a meromorphic connection with the pole of order less or equal to 2 along $M \times \{0\}$, such that its restriction to $M \times W$ is $(\mathcal{F}_{M\times W}, \nabla_{M\times W})$. As before (3.13) is called *the Brieskorn lattice.*[8]

The problem of extending the above objects to a pair $(\mathcal{F}, \nabla)$ on $M \times \mathbb{P}^1_\theta$ with the pole of order less or equal to 1 along $M \times \{\infty\}$ is called the *Birkhoff problem in family* (cf. [Sa1]).

Analogously to Section 3.1.2 we can add a pairing to the setup, then eventually we have the triple

$$(\mathcal{F}, \nabla, S). \tag{3.14}$$

[8]In *loc.cit.* these objects are denoted G and G_0. Besides, here we tacitly did some additional analytization in the θ-direction.

3.2.1 Summary. Starting from an appropriate unfolding F of a tame function $h\colon X \to \mathbb{A}^1$ and finding a solution of the Birkhoff problem with metric we get (3.14). Here $\mathcal{F}$ is a free $\mathcal{O}_{M\times\mathbb{P}^1_\theta}$-module of rank μ, ∇ is a flat meromorphic connection on $\mathcal{F}$ with poles of order less or equal to 2 along $M \times \{0\}$ and logarithmic pole along $M \times \{\infty\}$. This means that ∇ is a $\mathbb{C}$-linear morphism of sheaves

$$\nabla\colon \mathcal{F} \to \Omega^1_{M\times\mathbb{P}^1_\theta}(2D_0 + D_\infty) \otimes \mathcal{F} \tag{3.15}$$

satisfying the Leibniz rule, where $D_0 = M \times \{0\}$ and $D_\infty = M \times \{\infty\}$. The pairing S is a ∇-flat non-degenerate bilinear form

$$S\colon \mathcal{F} \times j^*\mathcal{F} \to \mathcal{O}_{M\times\mathbb{P}^1_\theta}(n \cdot D_\infty - n \cdot D_0), \tag{3.16}$$

where $j\colon M \otimes \mathbb{P}^1_\theta \to M \times \mathbb{P}^1_\theta$ is defined by $j(x,\theta) = (x,-\theta)$ and n is the dimension of the fiber of p (which is also equal to the dimension of X).

3.3 Frobenius manifold construction. Here we continue our discussion from where we have left it in Section 1.4.4. Namely, we will explain how to see that g_ω is flat if we have a solution to the Birkhoff problem.

Assume that we have found a solution to the Birkhoff problem in family (3.14). Let $\pi\colon M \times \mathbb{P}^1_\theta \to M$ be the projection and consider $E := \pi_*\mathcal{F}$, which is a free $\mathcal{O}_M$-module of rank μ. Also let $i_0\colon M \to M \times \mathbb{P}^1_\theta$ and $i_\infty\colon M \to M \times \mathbb{P}^1_\theta$ be the natural closed embeddings. There exist isomorphisms $E \to i_0^*\mathcal{F}$ and $E \to i_\infty^*\mathcal{F}$ given by restriction of sections to D_0 and D_∞ respectively. They give us the isomorphism

$$i_0^*\mathcal{F} \to i_\infty^*\mathcal{F}. \tag{3.17}$$

Applying i_∞^* to (3.16) we obtain the pairing

$$g_\infty\colon i_\infty^*\mathcal{F} \otimes i_\infty^*\mathcal{F} \to \mathcal{O}_M.$$

Moreover, since (3.15) has a regular singularity along D_∞, on $i_\infty^*\mathcal{F}$ we have the residual connection

$$\nabla_\infty\colon i_\infty^*\mathcal{F} \to \Omega^1_M \otimes i_\infty^*\mathcal{F},$$

which is flat because (3.15) is flat. The pairing g_∞ is ∇_∞-horizontal (cf. [Sa1], VI.2.10).

There exists a natural isomorphism $p_{C*}(i_C^*\,\Omega^n_{N/M}) \to i_0^*\mathcal{F}$. Composing it with (3.17) and (1.14) we get the sequence of isomorphisms

$$\mathcal{T}_M \xrightarrow{\simeq} p_{C*}(\mathcal{O}_C) \xrightarrow{\simeq} p_{C*}(i_C^*\,\Omega^n_{N/M}) \xrightarrow{\simeq} i_0^*\mathcal{F} \xrightarrow{\simeq} i_\infty^*\mathcal{F}. \tag{3.18}$$

The pullback of g_∞ to $\mathcal{T}_M$ with respect to this sequence is equal to g_ω, and the pull-back of ∇_∞ is its Levi-Civita connection. Therefore, g_ω is a flat metric.

The identity vector field e is not necessarily flat. Its flatness is equivalent to ω being mapped to a ∇_∞-flat section of $i_\infty^*\mathcal{F}$. We assume this from now on.

Thus, the tuple $(M, \circ, e, g_\omega)$ is an *associative pre-Frobenius manifold* in the sense of [Ma, Ch.I]. The only thing which is left to check is potentiality.

3.3.1 Potentiality. Considering in (3.15) only covariant derivatives along vector fields tangent to M and using identifications (3.18) we get a family of flat connections on $\mathcal{T}_M$ parametrized by $\mathbb{P}^1_\theta - \{0, \infty\}$ as in [Ma], I.1.4.[9] Applying Theorem I.1.5 of *loc. cit.* we get potentiality of the multiplication.

Therefore, $(M, \circ, e, g_\omega, E)$ is a Frobenius manifold with an Euler field.

3.3.2 Objects encoded in (3.14). As we have just seen (3.16) and (3.15) "know" g_ω and its Levi-Civita connection respectively. In fact, from (3.14) one can extract more objects which we have already encountered in Section 1.2. Below we list all of them (see [Ma], [Sa1] for details).

Since the singularity along D_∞ is regular, one can attach to it the residual connection ∇_∞ and the residual endomorphism $R_\infty \in \mathrm{End}_{\mathcal{O}_M}(i_\infty^* \mathcal{F})$. These objects are well defined.

Since the singularity along D_0 is of order less or equal to 2, one can attach to it an endomorphism $R_0 \in \mathrm{End}_{\mathcal{O}_M}(i_0^* \mathcal{F})$ and an element $\Phi \in \mathrm{Hom}_{\mathcal{O}_M}(i_0^*\mathcal{F}, \Omega^1_M \otimes i_0^*\mathcal{F})$. Since we are working with a fixed coordinate θ on $\mathbb{P}^1_\theta$, they are also well defined.

Using these objects, (3.15) can be rewritten as

$$\nabla = \nabla_\infty + \frac{\Phi}{\theta} + \left(\frac{R_0}{\theta} - R_\infty\right)\frac{d\theta}{\theta} \tag{3.19}$$

Under identifications (3.18) the objects R_0, Φ correspond to $\mathcal{U}, C$ from Section 1.2 respectively. As we already mentioned ∇_∞ corresponds to ∇^ω – the Levi-Civita connection of g_ω.

If we assume that $R_\infty(\omega) = -a\,\omega$, for some complex number a, then

$$\nabla^\omega(E) = R_\infty + (1+a)\,\mathrm{Id}\,,$$

where E is the natural Euler field. Therefore, for this Euler field

$$D = 2(a+1) - n.$$

From now on we will assume that $R_\infty(\omega) = -a\,\omega$.

3.3.3 Initial conditions. From the identifications in Section 3.3.2 it follows that the first structure connection for the Frobenius manifold $(M, \circ, e, g_\omega, E)$ can be identified with

$$\widehat{\nabla} = \nabla_\infty + \lambda\Phi + \left(R_0 + \frac{1}{\lambda}(R_\infty + (1+a)\,\mathrm{Id}\,)\right)d\lambda. \tag{3.20}$$

Restricting to $\{x_0\} \times \mathbb{P}^1_\lambda$ and writing its matrix it in the basis of flat sections we get

$$\left(A_0 + \frac{1}{\lambda}(-A_\infty + (1+a)\,\mathrm{Id}\,)\right)d\lambda.$$

Thus, the initial conditions for $(M, \circ, e, g_\omega, E)$ at the point x_0 written in the same basis are

$$T = T_{x_0}M, \quad U = A_0, \quad V = -A_\infty + \frac{n}{2}\mathrm{Id}, \quad g_\omega, \quad e.$$

[9] To match notations put $\lambda = \frac{1}{\theta}$.

4 Gauss-Manin system of $\widetilde{f}$

For further use we repeat here the partial compactification in this particular case in a somewhat backwards order. Consider $\mathbb{C}^3$ with standard coordinates x, y, z. Let $\widetilde{U}$ be the open submanifold defined by $\{(xy-1)(1+z) \neq 0\}$. On $\widetilde{U}$ we have the regular function

$$\widetilde{f} = y(2+z) + \frac{qx^2}{(xy-1)(1+z)}, \tag{4.1}$$

which is our partially compactified potential (2.3).

Consider functions $\Delta_1, \Delta_2, \Delta_3$ on $\mathbb{C}^3$ given by

$$\Delta_1 = y, \quad \Delta_2 = \frac{y^2}{2}(z+1), \quad \Delta_3 = qxy - q,$$

which form a coordinate system on the subset $\{y \neq 0\} \subset \mathbb{C}^3$. The inverse coordinate change is given by

$$x = \frac{\Delta_3 + q}{q\Delta_1}, \quad y = \Delta_1, \quad z = \frac{2\Delta_2}{\Delta_1^2} - 1. \tag{4.2}$$

Let U be the intersection $\widetilde{U} \cap \{y \neq 0\}$. On U function (4.1) can be rewritten in terms of $\Delta_1, \Delta_2, \Delta_3$ as

$$f := \widetilde{f}_{|U} = \Delta_1 + \frac{2\Delta_2}{\Delta_1} + \frac{(\Delta_3+q)^2}{2\Delta_2\Delta_3}. \tag{4.3}$$

Formulas (4.2) give an isomorphism of U with a torus $(\mathbb{C}^*)^3$, such that the standard coordinates on $(\mathbb{C}^*)^3$ correspond to coordinates $\Delta_1, \Delta_2, \Delta_3$ on U.

4.1 Tameness. Let X be a smooth algebraic variety and $h\colon X \to \mathbb{A}^1$ a morphism. By a partial compactification we mean a commutative diagram

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "h"] & \overline{X} \arrow[dl, "\overline{h}"] \\ \mathbb{A}^1 & \end{tikzcd}$$

where $\overline{X}$ is an algebraic variety(not necessarily smooth), j is an open embedding, and $\overline{h}$ is proper.

The morphism h is called *cohomologically tame* iff there exists a partial compactification such that the support of $\Phi_{\overline{h}-a}(Rj_*\mathbb{C}_X)$ is finite and contained in X_a, for all $a \in \mathbb{A}^1$. We refer to [Sa2] for more details.

4.1.1 Lemma. *Function (4.1) is cohomologically tame.*[10]

Proof. See Appendix A. ■

4.1.2 Conjecture. There is a notion of M-tameness, and this is the one which is used in [DoSa1]. We conjecture that $\widetilde{f}$ is M-tame but we do not have a proof of this yet.

[10]It is also true that f has isolated singularities at infinity in the sense of [Do2].

4.1.3 Lemma. *The Gauss-Manin system $G^{\widetilde{f}}$ has the following properties:*

(i) $G^{\widetilde{f}}$ is a free $\mathbb{C}[\theta, \theta^{-1}]$-module of rank 4;

(ii) $G_0^{\widetilde{f}}$ is a free $\mathbb{C}[\theta]$-module of rank 4.

Proof. For both properties it is essential that $\widetilde{f}$ is cohomologically tame.

(i) For a function with isolated critical points the module G is always free of finite rank. If, moreover, the function is cohomologically tame, then the rank is equal to the Milnor number ([Do2], Th. 5.2.3). In our case it is 4.

(ii) For a function with (cohomologically) isolated critical points at infinity Corollary 5.2.6 of [Do2] states, that G_0 is free and of finite type iff the function is cohomologically tame.

Applying this corollary to $\widetilde{f}$ we get that $G_0^{\widetilde{f}}$ is a free $\mathbb{C}[\theta]$-module of finite rank. Hence, its rank equals to the dimension of the fiber at zero. Using Proposition 5.1.1 of [Do2] we see that the rank is equal to the Milnor number. ■

4.1.4 Lemma. *The natural morphism of $\mathcal{D}_W$-modules $G^{\widetilde{f}} \to G^f$ given by the restriction of differential forms from $\widetilde{U}$ to U is an isomorphism.*[11]

Proof. Restriction of differential forms from $\widetilde{U}$ to U defines the morphism

$$\Omega^i(\widetilde{U}) \to \Omega^i(U),$$

which is injective but not surjective; it is the localization morphism given by inverting Δ_1. One can check directly that the induced morphism $G^{\widetilde{f}} \to G^f$ on the Gauss-Manin systems is also injective.

By Theorem 5.2.3 of [Do2] the rank of G^f is 4 (we use here that f has one isolated singularity at infinity).

Consider the short exact sequence of $\mathcal{O}_W$-coherent $\mathcal{D}_W$-modules

$$0 \to G^{\widetilde{f}} \to G^f \to G^f/G^{\widetilde{f}} \to 0.$$

Since rk $G^{\widetilde{f}} =$ rk G^f the quotient is an $\mathcal{O}_W$-module of rank zero. Therefore, by the standard fact that for a smooth algebraic variety X any $\mathcal{O}_X$-coherent $\mathcal{D}_X$-module is a locally free $\mathcal{O}_X$-module (see [Be], Lect. 2, 1.a), we get that $G^f/G^{\widetilde{f}}$ is locally free of rank zero and hence vanishes. ■

4.2 Birkhoff problem. Conisder the following 3-form on $\widetilde{U}$

$$\omega_0 = \frac{dx \wedge dy \wedge dz}{(xy-1)(z+1)},$$

and let $\omega_i = \Delta_i \omega_0$. Note also that

$$\omega_{0|U} = \frac{d\Delta_1}{\Delta_1} \wedge \frac{d\Delta_2}{\Delta_2} \wedge \frac{d\Delta_3}{\Delta_3}.$$

If ω is a 3-form, then let $[\omega]$ denote its class in G_0. In the above formulas by Δ_i we mean $\Delta_{i|\widetilde{U}}$ and $\Delta_{i|U}$ respectively. We will continue to use this notation, if it does not lead to confusion.

[11] Recall from Section 3.1 that $W = \mathbb{A}^1_\theta - \{0\}$

4.2.1 Lemma. *In G^f we have the following identities*

$$\begin{aligned}
&[\Delta_i f'_{\Delta_i}\omega_0] = 0\\
&[\Delta_i\Delta_j f'_{\Delta_i}\omega_0] = 0\\
&[\Delta_i^2 f'_{\Delta_i}\omega_0] = \theta[\omega_i].
\end{aligned}$$

Proof. Let us only prove the third identity for $i=2$. The other cases are analogous.

We have the following equality of differential forms

$$\Delta_2^2 f'_{\Delta_2}\omega_0 = -df\wedge\left(\Delta_2\frac{d\Delta_1}{\Delta_1}\wedge\frac{d\Delta_3}{\Delta_3}\right),$$

hence $[\Delta_2^2 f'_{\Delta_2}\omega_0] = \theta[\omega_2]$ in G^f. ∎

4.2.2 Lemma. *Elements $[\omega_0],\dots,[\omega_3]$ are $\mathbb{C}[\theta]$-linearly independent in $G_0^{\widetilde{f}}$.*

Proof. The vector space $G_0^{\widetilde{f}}/\theta G_0^{\widetilde{f}}$ can be identified with the Milnor ring by mapping 1 to the class of $[\omega_0]$. Under this isomorphism the class of Δ_i goes to the class of $[\omega_i]$. Since $1,\Delta_1,\Delta_2,\Delta_3$ form a basis in the Milnor ring, classes of $[\omega_0],\dots,[\omega_3]$ form a basis in $G_0^{\widetilde{f}}/\theta G_0^{\widetilde{f}}$. This implies the statement. ∎

4.2.3 Lemma. *(i) Elements $[\omega_0],\dots,[\omega_3]$ freely generate in $G^{\widetilde{f}}$ an $\mathcal{O}_W$-submodule $H^{\widetilde{f}}$ of rank 4;*

(ii) The following identities hold

$$\begin{aligned}
&\theta^2\partial_\theta[\omega_0] = 3[\omega_1]\\
&\theta^2\partial_\theta[\omega_1] = 6[\omega_2]+\theta[\omega_1]\\
&\theta^2\partial_\theta[\omega_2] = 3[\omega_3]+3q[\omega_0]+2\theta[\omega_2]\\
&\theta^2\partial_\theta[\omega_3] = 3q[\omega_1]+3\theta[\omega_3],
\end{aligned}$$

and therefore $H^{\widetilde{f}}$ is a $\mathcal{D}_W$-submodule;

(iii) $G^{\widetilde{f}} = H^{\widetilde{f}}$.

(iv) The connection matrix in the basis $[\omega_0],\dots,[\omega_3]$ takes the form

$$\left(\frac{A_0}{\theta}+A_\infty\right)\frac{d\theta}{\theta},$$

where

$$A_0 = \begin{pmatrix} 0 & 0 & 3q & 0\\ 3 & 0 & 0 & 3q\\ 0 & 6 & 0 & 0\\ 0 & 0 & 3 & 0\end{pmatrix}$$

and

$$A_\infty = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 2 & 0\\ 0 & 0 & 0 & 3\end{pmatrix}$$

Proof. (i) By Lemma 4.2.2 $[\omega_0], \ldots, [\omega_3]$ are linearly independent in $G_0^{\widetilde{f}}$, and hence also in $G^{\widetilde{f}}$ and G^f. Therefore, they generate a submodule of rank 4 in $G^{\widetilde{f}}$ (and in G^f).

(ii) Because of the natural isomorphism $G^{\widetilde{f}} \to G^f$ we can check these identities in G^f.

First, note that the following identities hold in the ring of functions on U

$$
\begin{aligned}
&f = 3\Delta_1 - 2\Delta_1 f'_{\Delta_1} - \Delta_2 f'_{\Delta_2}\\
&\Delta_1\Delta_1 = 2\Delta_2 + \Delta_1^2 f'_{\Delta_1}\\
&\Delta_1\Delta_2 = (\Delta_3 + q) + \Delta_2(\Delta_1 f'_{\Delta_1} + \Delta_2 f'_{\Delta_2} - \Delta_3 f'_{\Delta_3})\\
&\Delta_1\Delta_3 = q\Delta_1 + \Delta_3(\Delta_1 f'_{\Delta_1} + \Delta_2 f'_{\Delta_2} + \Delta_3 f'_{\Delta_3}) - q(\Delta_1 f'_{\Delta_1} + \Delta_2 f'_{\Delta_2} - \Delta_3 f'_{\Delta_3}).
\end{aligned}
$$

These identities can be checked by direct computations.

Using the first identity we get

$$
\begin{aligned}
&\theta^2\partial_\theta[\omega_0] = [f\omega_0] = [(3\Delta_1 - 2\Delta_1 f'_{\Delta_1} - \Delta_2 f'_{\Delta_2})\omega_0] =\\
&3[\Delta_1\omega_0] - 2[\Delta_1 f'_{\Delta_1}\omega_0] - [\Delta_2 f'_{\Delta_2}\omega_0].
\end{aligned}
$$

Applying Lemma 4.2.1 we get

$$\theta^2\partial_\theta[\omega_0] = 3[\Delta_1\omega_0] = 3[\omega_1].$$

Using the first two identities and Lemma 4.2.1 we get

$$\theta^2\partial_\theta[\omega_1] = [f\Delta_1\omega_0] = [6\Delta_2\omega_0 + \Delta_1^2 f'_{\Delta_1}\omega_0 - \Delta_1\Delta_2 f'_{\Delta_2}\omega_0] = 6[\omega_2] + \theta[\omega_1].$$

The remaining two formulas are obtained analogously.

(iii) Since $H^{\widetilde{f}}$ and $G^{\widetilde{f}}$ are $\mathcal{O}_W$-coherent $\mathcal{D}_W$-modules of the same rank, they coincide (as in the proof of Lemma 4.1.4).

(iv) It follows from (ii). ■

4.2.4 Lemma. *The classes $[\omega_0], \ldots, [\omega_3]$ form a $\mathbb{C}[\theta]$-basis in $G_0^{\widetilde{f}}$.*

Proof. Let $H_0^{\widetilde{f}}$ be the $\mathcal{O}_{\mathbb{A}^1_\theta}$-submodule of $G_0^{\widetilde{f}}$ generated by $[\omega_0], \ldots, [\omega_3]$. We have the short exact sequence of $\mathcal{O}_{\mathbb{A}^1_\theta}$-modules

$$0 \to H_0^{\widetilde{f}} \to G_0^{\widetilde{f}} \to Q_0^{\widetilde{f}} \to 0, \tag{4.4}$$

and we need to show that $Q_0^{\widetilde{f}} = 0$.

Since $Q^{\widetilde{f}}_{0\,|\mathbb{A}^1_\theta - \{0\}} = 0$ by Lemma 4.2.3, and $Q_0^{\widetilde{f}}$ is finitely generated, it is enough to prove that the fiber at zero vanishes, i.e. $Q_0^{\widetilde{f}} \otimes_{\mathbb{C}[\theta]} \mathbb{C}[\theta]/(\theta) = 0$.

Tensoring (4.4) with $\mathbb{C}[\theta]/(\theta)$ we get a short exact sequence (tensor product is right exact)

$$H_0^{\widetilde{f}} \otimes_{\mathbb{C}[\theta]} \mathbb{C}[\theta]/(\theta) \to G_0^{\widetilde{f}} \otimes_{\mathbb{C}[\theta]} \mathbb{C}[\theta]/(\theta) \to Q_0^{\widetilde{f}} \otimes_{\mathbb{C}[\theta]} \mathbb{C}[\theta]/(\theta) \to 0,$$

which can be rewritten as

$$H_0^{\widetilde{f}} \otimes_{\mathbb{C}[\theta]} \mathbb{C}[\theta]/(\theta) \to \Omega^n(\widetilde{U})/d\widetilde{f} \wedge \Omega^{n-1}(\widetilde{U}) \to Q_0^{\widetilde{f}} \otimes_{\mathbb{C}[\theta]} \mathbb{C}[\theta]/(\theta) \to 0.$$

Since classes of $[\omega_0], \ldots, [\omega_3]$ generate $\Omega^n(\widetilde{U})/d\widetilde{f} \wedge \Omega^{n-1}(\widetilde{U})$, the first map is surjective. Therefore, $Q_0^{\widetilde{f}} \otimes_{\mathbb{C}[\theta]} \mathbb{C}[\theta]/(\theta) = 0$, and, finally, $Q_0^{\widetilde{f}} = 0$. ■

4.3 Pairing. In this section we study pairing (3.5) in our setup. Since it will make no difference here, we are dropping the subscripts in the notation of the Gauss-Manin systems, and just write G.

4.3.1 Lemma. *The V-filtration on G along $\{\tau = 0\}$ is given by*

$$V_0 G = \bigoplus_{i=0}^{3} \mathbb{C}[\tau] e_i$$
$$V_p G = \tau^{-p} V_0 G,$$

where $e_i = \tau^i [\omega_i]$ (cf. Section 3.1.3).

Proof. We just show that this filtration satisfies conditions from the definition of V-filtration.

1. Compatibility of filtrations.

1a. Using that $\partial_\tau = -\theta^2 \partial_\theta$ and applying Lemma 4.2.3 it is easy to see that

$$\partial_\tau \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{pmatrix} = \frac{1}{\tau} \operatorname{diag}(0,1,2,3) \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{pmatrix} - (A_0 + \frac{1}{\tau} A_\infty) \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{pmatrix} \tag{4.5}$$

and therefore $\partial_\tau (V_0 G) \subset V_1 G$. Analogously one shows that $\partial_\tau (V_p G) \subset V_{p+1} G$.

It is clear that $\tau(V_p G) \subset V_{p-1} G$.

These two facts imply that $(V_m \mathcal{D}_{\mathbb{A}_\tau})(V_p G) \subset V_{p+m} G$.

1b. It is clear that the condition $\tau V_p G = V_{p-1} G$ for $p < 0$ holds.

2. Nilpotence.

Formula (4.5) actually gives more than we used so far. Indeed, it implies that $\partial_\tau (V_0 G) \subset V_0 G$ and, thus, $\tau \partial_\tau$ is nilpotent on $\mathrm{Gr}_0^V G$.

Analogous direct computations shows that $\tau \partial_\tau + p$ is nilpotent on $\mathrm{Gr}_p^V G$. ■

4.3.2 Lemma. *The pairing S_W satisfies*[12]

$$S_W([\omega_k], \overline{[\omega_l]}) = \begin{cases} S_W([\omega_0], \overline{[\omega_3]}) & \text{if } \; k + l = 3 \\ 0 & \text{otherwise} \end{cases} \tag{4.6}$$

and $S_W([\omega_0], \overline{[\omega_3]}) \in \tau^{-3} \mathbb{C}$.

Proof. To simplify the notation we will be writing S instead of S_W.

By (3.8) we know that

$$S([\omega_k], \overline{[\omega_l]}) \in \tau^{-3} \mathbb{C}[\tau^{-1}].$$

On the other hand, by (3.11) we get

$$S(e_k, \overline{e_l}) = \tau^k (-\tau)^l S([\omega_k], \overline{[\omega_l]}) \in \mathbb{C}[\tau],$$

therefore $S([\omega_k], \overline{[\omega_l]}) \in \tau^{-(k+l)} \mathbb{C}[\tau]$. Hence

$$\begin{aligned} S([\omega_k], \overline{[\omega_l]}) &= 0 && \text{if} \quad k + l < 3, \\ S([\omega_k], \overline{[\omega_l]}) &\in \tau^{-3} \mathbb{C} && \text{if} \quad k + l = 3. \end{aligned} \tag{4.7}$$

We consider explicitly vanishing in the remaining 4 cases with $k + l > 3$.

[12] The overline over the second argument of S_W stresses that the element is considered as a section of $j^* \mathcal{F}_W$. Therefore, τ and $\nabla_{\frac{\partial}{\partial \tau}}$ act with the opposite sign.

- Applying (3.6) to $S([\omega_0],\overline{[\omega_3]})\in\tau^{-3}\mathbb{C}$ and using (4.7) we get

$$\begin{aligned}-3\tau^{-1}S([\omega_0],\overline{[\omega_3]}) &= \frac{d}{d\tau}S([\omega_0],\overline{[\omega_3]}) = S(\partial_\tau[\omega_0],\overline{[\omega_3]}) - S([\omega_0],\overline{\partial_\tau[\omega_3]}) = \\ &= -S(3[\omega_1],\overline{[\omega_3]}) + S([\omega_0],\overline{3q[\omega_1]+3\theta[\omega_3]}) = \\ &= -3S([\omega_1],\overline{[\omega_3]}) - 3\tau^{-1}S([\omega_0],\overline{[\omega_3]}),\end{aligned}$$

and therefore

$$S([\omega_1],\overline{[\omega_3]}) = 0. \tag{4.8}$$

- Applying (3.6) to $S([\omega_1],\overline{[\omega_3]})=0$, and using (4.7) and (4.8) we get

$$\begin{aligned}0 &= \frac{d}{d\tau}S([\omega_1],\overline{[\omega_3]}) = S(\partial_\tau[\omega_1],\overline{[\omega_3]}) - S([\omega_1],\overline{\partial_\tau[\omega_3]}) = \\ &= -S(6[\omega_2]+\theta[\omega_1],\overline{[\omega_3]}) + S([\omega_1],\overline{3q[\omega_1]+3\theta[\omega_3]}) \\ &= -6S([\omega_2],\overline{[\omega_3]}).\end{aligned}$$

Hence

$$S([\omega_2],\overline{[\omega_3]}) = 0.$$

- Applying (3.6) to $S([\omega_2],\overline{[\omega_1]})$ we get

$$\begin{aligned}&-3\tau^{-1}S([\omega_2],\overline{[\omega_1]}) = \frac{d}{d\tau}S([\omega_2],\overline{[\omega_1]}) = S(\partial_\tau[\omega_2],\overline{[\omega_1]}) - S([\omega_2],\overline{\partial_\tau[\omega_1]}) = \\ &= -S(3[\omega_3]+3q[\omega_0]+2\theta[\omega_2],\overline{[\omega_1]}) + S([\omega_2],\overline{6[\omega_2]+\theta[\omega_1]}) = \\ &= -2\tau^{-1}S([\omega_2],\overline{[\omega_1]}) + 6S([\omega_2],\overline{[\omega_2]}) - \tau^{-1}S([\omega_2],\overline{[\omega_1]}),\end{aligned}$$

and therefore

$$S([\omega_2],\overline{[\omega_2]}) = 0.$$

- Applying (3.6) to $S([\omega_3],\overline{[\omega_3]})$ we get

$$\begin{aligned}&\frac{d}{d\tau}S([\omega_3],\overline{[\omega_3]}) = S(\partial_\tau[\omega_3],\overline{[\omega_3]}) - S([\omega_3],\overline{\partial_\tau[\omega_3]}) = \\ &= -S(3q[\omega_1]+3\theta[\omega_3],\overline{[\omega_3]}) + S([\omega_3],\overline{3q[\omega_1]+3\theta[\omega_3]}) \\ &= -3\theta S([\omega_3],\overline{[\omega_3]}) - 3\theta S([\omega_3],\overline{[\omega_3]}) = -6\theta S([\omega_3],\overline{[\omega_3]}).\end{aligned}$$

Therefore $S([\omega_3],\overline{[\omega_3]})\in\tau^{-6}\mathbb{C}$.

Now, apply (3.6) to $S([\omega_2],\overline{[\omega_3]})$ and get

$$\begin{aligned}0 &= \frac{d}{d\tau}S([\omega_2],\overline{[\omega_3]}) = S(\partial_\tau[\omega_2],\overline{[\omega_3]}) - S([\omega_2],\overline{\partial_\tau[\omega_3]}) = \\ &= -S(3[\omega_3]+3q[\omega_0]+2\theta[\omega_2],\overline{[\omega_3]}) + S([\omega_2],\overline{3q[\omega_1]+3\theta[\omega_3]}) \\ &= -3S([\omega_3],\overline{[\omega_3]}) - 3qS([\omega_0],\overline{[\omega_3]}) + 3qS([\omega_2],\overline{[\omega_1]}).\end{aligned}$$

Therefore $S([\omega_3],\overline{[\omega_3]})=0$ and $S([\omega_0],\overline{[\omega_3]})=S([\omega_2],\overline{[\omega_1]})$.

Moreover, by (3.7) we have

$$S([\omega_0],\overline{[\omega_3]}) = S([\omega_3],\overline{[\omega_0]}) \qquad \text{and} \qquad S([\omega_2],\overline{[\omega_1]}) = S([\omega_1],\overline{[\omega_2]}).$$

This finishes the proof. ■

4.4 Conjectural Frobenius manifold. Results of Sections 4.2 and 4.3 mean that we have exhibited a solution of the Birkhoff problem at x_0.

Assume that Conjecture 4.1.2 holds and consider any unfolding F of $\widetilde{f}$ satisfying assumptions of Section 3.2. Then by Theorem VI.2.1 and Proposition VI.2.7 of [Sa1] the above solution can be extended to a neighbourhood of x_0 in M, and we get the triple $(\mathcal{F}, \nabla, S)$. If needed, we make M small enough so that the solution is defined over the whole M. Hence, we have established flatness of metrics g_ω in this Saito's framework and we get Frobenius manifold structures on M.

Choose ω so that in (3.18) the identity vector field goes to $\widetilde{\omega_0}$, where $\widetilde{\omega_0}$ is the basis element of $\mathcal{F}$ that restricts to ω_0. This will ensure that the identity vector field of the resulting Frobenius manifold is flat. It also implies that the constant a for the Euler field is equal to 0, and thus, $D = -1$.

The initial data of this Frobenius manifold at the point x_0 is given by Lemma 4.2.3 and coincide with those of $QH(Q_3)$ at the point $q = 1$ in the small quantum cohomology. Therefore, germs of these manifolds at respective points are isomorphic.

A Proof of Lemma 4.1.1

The proof of Lemma 4.1.1 given here has been kindly explained to me by Claude Sabbah and András Némethi.

A.1 Vanishing cycles. Here we recall some basic facts about functors of vanishing cycles. For a comlpex algebraic variety X we denote by $D^b_c(X)$ the bounded derived category of $\mathbb{C}_{X^{an}}$-modules with constructible cohomology, where X^{an} the associated analytic space.

A.1.1 Functor of vanishing cycles. Let X be a complex algebraic variety and $g\colon X \to \mathbb{A}^1_t$ a morphism. The functor of vanishing cycles to the fiber over 0 of the morphism g is denoted Φ_g. If one considers the fiber over a, then one shifts g and considers Φ_{g-a}.

If we denote X_0 the fiber of g over 0, then the functor of vanishing cycles to this fiber is a triangulated functor

$$\Phi_g\colon D^b_c(X) \to D^b_c(X_0),$$

i.e. it maps distinguished triangles to distinguished triangles. See [Di] for a precise definition.

Below we collect some basic properties of these functors. They are standard and can be found in [Di].

A.1.2 Proper morphism. Let $\pi\colon X \to Y$ be a morphism of algebraic varieties and consider the following commutative diagram

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d, "\widehat{\pi}"] & X \arrow[d, "\pi"] \\ Y_0 \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\ \{0\} \arrow[r] & \mathbb{A}^1 \end{tikzcd}$$

where X_0 and Y_0 are fibers over 0. Naturally one can attach to it the diagram of derived categories and functors between them

$$\begin{tikzcd} D^b_c(X_0) \arrow[d, "R\widehat{\pi}_*"] & D^b_c(X) \arrow[l, "\Phi_{g\circ\pi}"'] \arrow[d, "R\pi_*"] \\ D^b_c(Y_0) & D^b_c(Y) \arrow[l, "\Phi_g"'] \end{tikzcd}$$

Fact: if π is *proper*, then the above diagram is commutative (e.g. this is true if π is a closed embedding).

A.1.3 Restriction to an open subset. Let $U \subset X$ be an open subset and let j be the natural inclusion. We have a commutative diagram

$$\begin{tikzcd} U_0 \arrow[r] \arrow[d, "\widehat{j}"] & U \arrow[d, "j"] \\ X_0 \arrow[r] \arrow[d] & X \arrow[d, "g"] \\ \{0\} \arrow[r] & \mathbb{A}^1 \end{tikzcd}$$

with U_0 and X_0 being fibers over 0. Consider the associated diagram of derived categories and functors between them

$$\begin{tikzcd} D^b_c(U_0) & D^b_c(U) \arrow[l, "\Phi_{g\circ j}"'] \\ D^b_c(X_0) \arrow[u, "\widehat{j}^{-1}"] & D^b_c(X) \arrow[l, "\Phi_g"'] \arrow[u, "j^{-1}"'] \end{tikzcd}$$

Fact: the above diagram is commutative.

A.1.4 Duality. For any complex algebraic variety Y there exists a functor $\mathbf{D}_Y : D^b(Y) \to D^b(Y)$ defined by

$$\mathbf{D}_Y(\mathcal{F}^\bullet) := R\mathcal{H}om(\mathcal{F}^\bullet, \omega_Y),$$

where ω_Y is the dualizing complex. It has the property $\mathbf{D}_Y \circ \mathbf{D}_Y \simeq \mathrm{Id}_{D^b(Y)}$. Moreover, this functor restricts to $D^b_c(Y)$ and we will use the same notation for this restriction. If we do not want specify the space, we will just denote it $\mathbf{D}$.

One can show that the following properties hold:

1. For an arbitrary morphism $f\colon X \to Y$ of algebraic varieties we have

$$\mathbf{D}_Y \circ Rf_* \circ \mathbf{D}_X \simeq Rf_!$$

and

$$Rf_* \simeq \mathbf{D}_Y \circ Rf_! \circ \mathbf{D}_X. \tag{A.1}$$

One can be obtained from the other using $\mathbf{D} \circ \mathbf{D} \simeq \mathrm{Id}$.

2. If $g\colon Y \to \mathbb{A}^1$ is a morphism, then there exists a non-functorial isomorphism

$$\mathbf{D} \circ \Phi_g(\mathcal{F}^\bullet) \simeq (\Phi_g \circ \mathbf{D}(\mathcal{F}^\bullet))\,[-2]. \tag{A.2}$$

3. Duality commutes with restriction to an open subvariety.

4. If Y is smooth, then

$$\omega_Y \simeq \mathbb{C}_Y[2\dim Y]. \tag{A.3}$$

Hence, $\mathbf{D}_Y(\mathbb{C}_Y) \simeq \mathbb{C}_Y[2\dim Y]$. Here $\dim Y$ is the complex dimension.

5. The functor $\mathbf{D}$ preserves the support.

A.2 Setup I. Let X be smooth complex algebraic variety, U an open smooth subvariety, and Z the complement to U. Let $\overline{f}\colon X \to \mathbb{A}^1$ be a proper morphism. Then we have a commutative diagram

$$\begin{tikzcd} U \arrow[r, "j"] \arrow[dr, "f_U"'] & X \arrow[d, "\overline{f}"] & Z \arrow[l, "i"'] \arrow[dl, "f_Z"] \\ & \mathbb{A}^1 & \end{tikzcd} \tag{A.4}$$

(i) By (A.1) we have the isomorphism

$$Rj_*(\mathbb{C}_U) \simeq \mathbf{D}_X \circ Rj_! \circ \mathbf{D}_U(\mathbb{C}_U).$$

Applying $\Phi_{\overline{f}}$ to both sides and using (A.3) we get

$$\Phi_{\overline{f}} \circ Rj_*(\mathbb{C}_U) \simeq \Phi_{\overline{f}} \circ \mathbf{D}_X \circ Rj_!(\mathbb{C}_U[2\dim U]).$$

Using (A.2) we can rewrite it as

$$\Phi_{\overline{f}} \circ Rj_*(\mathbb{C}_U) \simeq \Big(\mathbf{D}_X \circ \Phi_{\overline{f}} \circ Rj_!(\mathbb{C}_U[2\dim U])\Big)\,[-2].$$

If the support of the complex $\Phi_{\overline{f}} \circ Rj_!(\mathbb{C}_U)$ is contained in U, then the same is true for $\Phi_{\overline{f}} \circ Rj_*(\mathbb{C}_U)$ (cf. Property 5 from Section A.1.4). Therefore, it does not matter which one to consider for cohomological tameness, since we are interested only in the support.

(ii) There exists a standard short exact sequence (see [Ha2], Chapter II, exercise 1.19.c)

$$0 \to Rj_!\mathbb{C}_U \to \mathbb{C}_X \to i_*\mathbb{C}_Z \to 0.$$

Applying $\Phi_{\overline{f}}$ to it we get a distinguished triangle

$$\Phi_{\overline{f}} \circ Rj_!(\mathbb{C}_U) \to \Phi_{\overline{f}}(\mathbb{C}_X) \to \Phi_{\overline{f}} \circ i_*(\mathbb{C}_Z) \to \Phi_{\overline{f}} \circ Rj_!(\mathbb{C}_U)[1].$$

Since i is *proper*, we can rewrite it as

$$\Phi_{\overline{f}} \circ Rj_!(\mathbb{C}_U) \to \Phi_{\overline{f}}(\mathbb{C}_X) \to \hat{i}_* \circ \Phi_{\overline{f}\circ i}(\mathbb{C}_Z) \to \Phi_{\overline{f}} \circ Rj_!(\mathbb{C}_U)[1],$$

where $\hat{i}\colon Z_0 \to X_0$ is the natural closed embedding.

In the future applications we will need to prove that $\Phi_{\overline{f}} \circ Rj_!(\mathbb{C}_U)$ is supported on U_0. This would follow if we prove that $\Phi_{\overline{f}}(\mathbb{C}_X)$ is supported on U_0 and $\Phi_{\overline{f}\circ i}(\mathbb{C}_Z)$ has empty support (since $\hat{i}_*$ does not change support).

Complexes $\Phi_{\overline{f}}(\mathbb{C}_X)$ and $\Phi_{\overline{f}\circ i}(\mathbb{C}_Z)$ compute vanishing cycles of $\overline{f}$ and f_Z with values in $\mathbb{C}_X$ and $\mathbb{C}_Z$ respectively. Therefore, their support can be computed geometrically.

A.3 Setup II. It is clear that proving cohomological tameness of (4.1) is equivalent to proving cohomological tameness of

$$g(x, y, z) = y(z+1) + \frac{qx^2}{(xy-1)z}.$$

Here x, y, z are coordinates on $\mathbb{A}^3 = \operatorname{Spec}(\mathbb{C}[x, y, z])$ and g is defined on $U \subset \mathbb{A}^3$ given by the equation $(xy-1)z \neq 0$. By a simple computation of partial derivatives one can see that all critical points of g lie in the subset $\{xy - z - 1 = 0\}$.

Consider the map $U \to U \times \mathbb{A}^1_t$ given by the graph of g and denote its image $\Gamma_g \subset U \times \mathbb{A}^1_t$. Further, consider the natural embedding of $U \times \mathbb{A}^1_t$ into $\mathbb{A}^3 \times \mathbb{A}^1_t \subset (\mathbb{P}^1)^3 \times \mathbb{A}^1_t$. Let $\overline{\Gamma}_g$ be the closure of Γ_g in $(\mathbb{P}^1)^3 \times \mathbb{A}^1_t$. Thus, we have a commutative diagram completely analogous to (A.4)

$$\begin{tikzcd} \Gamma_g \arrow[r, "j"] \arrow[dr, "g_U"'] & \overline{\Gamma}_g \arrow[d, "\overline{g}"] & \overline{\Gamma}_g \setminus \Gamma_g \arrow[l, "i"'] \arrow[dl, "g_Z"] \\ & \mathbb{A}^1_t & \end{tikzcd} \tag{A.5}$$

where $\overline{g}$ is induced by projection $(\mathbb{P}^1)^3 \times \mathbb{A}^1_t \to \mathbb{A}^1_t$, and we will identify g_U with g.

It is easy to see that $\Gamma_g \subset \mathbb{A}^3 \times \mathbb{A}^1_t$ is defined by the equation

$$y(z+1) + \frac{qx^2}{(xy-1)z} = t,$$

and $\overline{\Gamma}_g \subset (\mathbb{P}^1)^3 \times \mathbb{A}^1_t$ is given by the homogeneous equation

$$x_0 z_1 (x_1 y_1 - x_0 y_0) [y_1 (z_1 + z_0) - t y_0 z_0] + q z_0^2 x_1^2 y_0^2 = 0. \tag{A.6}$$

A.4 Open cover. Consider an open cover of $(\mathbb{P}^1)^3 \times \mathbb{A}^1_t$ by 8 open subsets

$$V_{i,j,k} = \{x_i y_j z_k \neq 0\},$$

each of which is just $\mathbb{A}^3 \times \mathbb{A}^1_t$. As standard local coordinates on these open subsets we will be always using fractions $\frac{x_{i+1}}{x_i}$, $\frac{y_{i+1}}{y_i}$, $\frac{z_{i+1}}{z_i}$ and t (in the subscripts here we mean mod 2 sum: $1+1=0$).

On each $V_{i,j,k}$ one can write down equations of $\Gamma_g \cap V_{i,j,k}$ and $\overline{\Gamma}_g \cap V_{i,j,k}$ in terms of local coordinates and we will be referring to these equations simply as "equation of Γ_g in the chart $V_{i,j,k}$" and "equation of $\overline{\Gamma}_g$ in the chart $V_{i,j,k}$" respectively.

A.5 How to compute $\Phi_{\overline{g}}(\mathbb{C}_{\overline{\Gamma}_g})$? Decompose $\overline{\Gamma}_g$ into the disjoint union

$$\overline{\Gamma}_g = W \sqcup S,$$

where S is the singular locus of $\overline{\Gamma}_g$ and W is the smooth locus; S is a closed subvariety of codimension at least 1, W is open in $\overline{\Gamma}_g$ and smooth.

According to Section A.1.3, computing vanishing cycles commutes with restriction to an open subset, and therefore, to investigate the support at points of W, we can restrict to it from the beginning. Since W is smooth the support of $\Phi_{\overline{g}}(\mathbb{C}_X)$ can be non-zero only in singular points of fibers of $\overline{g}_{|W}$ (by the implicit function theorem). Thus, on W we just need to worry about critical points of $\overline{g}_{|W}$.

There exists a decomposition into disjoint union

$$(\mathbb{P}^1)^3 \times \mathbb{A}^1_t = V_{0,0,0} \sqcup \{x_0 y_0 z_0 = 0\}.$$

A.5.1 Lemma. *Let $Q \in \overline{\Gamma}_g$ be a smooth point, i.e. $Q \in W$. If $Q \in \{x_0 y_0 z_0 = 0\}$, then Q is not a critical point of $\overline{g}_{|W}$.*

Proof. Consider a chart $V_{i,j,k}$ and temporarily denote the local coordinates just by x, y, z. Let $P(x, y, z, t) = 0$ be equation (A.6) written in these coordinates.

In this notation the intersection $V_{i,j,k} \cap S$ is defined by the system

$$\begin{aligned} &x_0 z_1 (x_1 y_1 - x_0 y_0) y_0 z_0 = 0 \\ &P_x(x, y, z, t) = 0 \\ &P_y(x, y, z, t) = 0 \\ &P_z(x, y, z, t) = 0 \\ &P(x, y, z, t) = 0, \end{aligned} \tag{A.7}$$

where we have written the first equation, which comes from the derivative with respect to t, in the original homogeneous coordinates.

On the other hand, the intersection of the fiber of $\overline{g}$ over the point $d \in \mathbb{A}^1_t$ with $V_{i,j,k}$ is given by the equation $P(x, y, z, d) = 0$. Therefore, on $V_{i,j,k}$ the singular locus of this fiber is given by the system

$$\begin{aligned} &P_x(x, y, z, d) = 0 \\ &P_y(x, y, z, d) = 0 \\ &P_z(x, y, z, d) = 0 \\ &P(x, y, z, d) = 0. \end{aligned} \tag{A.8}$$

From (A.6) it is easy to see that if a point $Q \in \{x_0 y_0 z_0 = 0\}$ with coordinates (a, b, c, d) satisfies (A.8), then it satisfies it for arbitrary d, i.e. we get singular points simultaneously in all fibers. Thus, on the locus $\{x_0y_0z_0 = 0\}$ systems (A.8) and (A.7) coincide.

Therefore, if $Q \in \{x_0y_0z_0 = 0\}$ is a smooth point of $\overline{\Gamma}_g$, then it is also a smooth point in the respective fiber of $\overline{g}$ (and $\overline{g}_W$). ■

A.5.2 Strategy. In the rest of this section we will treat points in $V_{0,0,0}$ and in $\{x_0y_0z_0 = 0\}$ separately. In the latter case, by the above lemma, it is enough to look at vanishing cycles at singular points of $\overline{\Gamma}_g$.

A.6 Chart $V_{0,0,0}$. Recall that $\Gamma_g \subset V_{0,0,0}$ and in this chart we need to prove that the support is contained in Γ_g. In this chart $\overline{\Gamma}_g$ is given by

$$z(xy-1)[y(1+z)-t] + qx^2 = 0$$

and Γ_g is defined by the intersection with $z(xy-1) \neq 0$. Consider functions

$$\begin{aligned} x_1 &= x \\ y_1 &= y(1+z) - t \\ z_1 &= z(xy-1) \\ t_1 &= t. \end{aligned} \tag{A.9}$$

Computing the Jacobian we get

$$J = xy - z - 1,$$

and hence on the complement to the closed subset $\{xy - z - 1 = 0\}$ formulas (A.9) define a new coordinate system. Notice that $\left(\overline{\Gamma}_g \setminus \Gamma_g\right) \cap V_{0,0,0}$ lies in this open set.

On this open set the equation for $\overline{\Gamma}_g$ can be rewritten as

$$y_1z_1 + qx^2 = 0, \tag{A.10}$$

and hence the restriction of $\overline{g}$ to this open set is a projection and therefore has no vanishing cycles. Thus, we conclude that in this chart all vanishing cycles of $\overline{g}$ with coefficients in $\mathbb{C}_{\overline{\Gamma}_g}$ live in Γ_g.

Using (A.10) it is easy to see that

$$V_{0,0,0} \cap S = \{x = 0,\, z = 0,\, y = t\}.$$

As we will see in the next section the singular locus S consists of six disjoint lines.

A.7 Set $\{x_0y_0z_0 = 0\}$. To check vanishing of stalks of $\Phi_{\overline{g}}(\mathbb{C}_X)$ at points of S we will be restricting to different charts and work in local coordinates. There are 7 charts left to be checked.

A.7.1 Chart $V_{0,1,0}$. In this chart the equation of $\overline{\Gamma}_g$ takes the form

$$z(x-y')[(z+1) - y't] + qx^2y'^2 = 0. \tag{A.11}$$

Finding singular points of $\overline{\Gamma}_g$ with $y' = 0$ (we need to check only them) we get that there are two singular lines given by

$$\begin{aligned} &x = 0,\, y' = 0,\, z = 0 \\ &x = 0,\, y' = 0,\, z = -1. \end{aligned} \tag{A.12}$$

Consider functions

$$\begin{aligned} x_1 &= x \\ y_1 &= y' \\ z_1 &= (1 + z - ty')z \\ t_1 &= t. \end{aligned} \tag{A.13}$$

Computing the Jacobian we get

$$J = 2z + 1 - ty',$$

and we see that it is not zero in the neighborhood of (A.12). Thus, we get a new coordinate system in this neighborhood. In terms of these coordinates (A.11) takes form

$$z_1(x_1 - y_1) + qx_1^2 y_1^2 = 0,$$

and is independent of t. Hence, no vanishing cycles.

A.7.2 Chart $V_{1,0,0}$. In this chart the equation of $\overline{\Gamma}_g$ takes the form

$$zx'(y - x')[y(z + 1) - t] + q = 0.$$

We see that $\overline{\Gamma}_g \cap V_{1,0,0} \cap \{x' = 0\} = \emptyset$ and there is nothing to check in this chart.

A.7.3 Chart $V_{0,0,1}$. In this chart the equation of $\overline{\Gamma}_g$ takes the form

$$(xy - 1)[y(1 + z') - tz'] + qx^2 z'^2 = 0.$$

Solving the system for singular points of $\overline{\Gamma}_g$ with $z' = 0$ one sees that there are none. Hence, nothing to check here.

A.7.4 Chart $V_{1,1,0}$. In this chart the equation of $\overline{\Gamma}_g$ takes the form

$$x'z(1 - x'y')[(z + 1) - ty'] + qy'^2 = 0. \tag{A.14}$$

Finding singular points with $x' = 0$ and $y' = 0$ we get that there are two singular lines given by

$$\begin{aligned} &x' = 0,\ y' = 0,\ z = 0 \\ &x' = 0,\ y' = 0,\ z = -1. \end{aligned} \tag{A.15}$$

Consider functions

$$\begin{aligned} x_1 &= x' \\ y_1 &= y' \\ z_1 &= (1 + z - ty')z \\ t_1 &= t. \end{aligned} \tag{A.16}$$

Computing the Jacobian we get

$$J = 2z + 1 - ty',$$

and see that it is not zero in the neighborhood of (A.15). Thus, we get a new coordinate system in this neighborhood. In terms of these coordinates (A.14) takes form

$$x_1(1 - x_1 y_1)z_1 + qy_1^2 = 0,$$

and is independent of t. Hence, no vanishing cycles.

A.7.5 Chart $V_{0,1,1}$. In this chart the equation of $\overline{\Gamma}_g$ takes form

$$(x-y')[(1+z')-y'z't]+qx^2z'^2y'^2=0.$$

Looking for singular points with $z'=0$ and $y'=0$ we get that there are none. Hence, there is nothing to check.

A.7.6 Chart $V_{1,0,1}$. In this chart the equation of $\overline{\Gamma}_g$ takes form

$$x'(y-x')[y(1+z')-tz']+qz'^2=0.$$

The singular locus with $x'=0$ and $z'=0$ is the line

$$x'=0,\ y=0,\ z'=0.$$

This is the only case where the argument is a bit more involved.

Namely, we see that for each value of t the fiber over it has an isolated singularity at the origin, and we need to ensure that there are no vanishing cycles to these points. Our family is of the form $\alpha(x',y,z')+t\beta(x',y,z')=0$, where

$$\begin{aligned}\alpha(x',y,z')&=x'(y-x')y(1+z')+qz'^2\\ \beta(x',y,z')&=-x'(y-x')z'.\end{aligned}$$

According to Corollary 2.1 of [Pa], if this family of isolated hypersurface singularities is μ-constant (i.e. Milnor numbers coincide for all fibers), then it is topologically locally trivial over the base. In particular, this ensures the absence of vanishing cycles.

Let us compute the Milnor numbers. The partial derivatives of $P(x',y,z')=x'(y-x')[y(1+z')-tz']+qz'^2$ with respect to x',y,z' are

$$\begin{aligned}&P_{x'}=(y-2x')[y(1+z')-tz']\\ &P_y=x'[y(1+z')-tz']+x'(y-x')(1+z')=x'[(2y-x')(1+z')-tz']\\ &P_{z'}=x'(y-x')[y-t]+2qz'.\end{aligned}\tag{A.17}$$

By definition the Milnor number is

$$\mu=\dim\left(\mathbb{C}\{x',y,z'\}/(P_{x'},P_y,P_{z'})\right).$$

From (A.17) we see that $2qz'=-x'(y-x')[y-t]$ in this quotient, and hence we can rewrite it as

$$\mathbb{C}\{x',y\}/(\widetilde{P}_{x'},\widetilde{P}_y),\tag{A.18}$$

where

$$\begin{aligned}\widetilde{P}_{x'}&=(y-2x')[y(1-\frac{x'(y-x')[y-t]}{2q})+t\frac{x'(y-x')[y-t]}{2q}]=f_1f_2\\ \widetilde{P}_y&=x'[(2y-x')(1-\frac{x'(y-x')[y-t]}{2q})+t\frac{x'(y-x')[y-t]}{2q}]=f_3f_4.\end{aligned}$$

It easy to check that any pair of functions out of $f_1,\dots,f_4$ forms a coordinate system around the origin in the x',y-plane.

Let $u = f_1, v = f_2$ be such a coordinate system. Then (A.18) can be rewritten as

$$\mathbb{C}\{u, v\}/(uv, \widetilde{P}_y), \tag{A.19}$$

where $\widetilde{P}_y$ is written as a power series in u and v; it starts from terms quadratic in u and v, since $\widetilde{P}_y = f_3 f_4$. Moreover, as a vector space (A.19) can be further rewritten as

$$\mathbb{C}\{u\}/(u^2) \oplus \mathbb{C}\{v\}/(v^2).$$

Thus,

$$\mu = 4,$$

and is independent of t.

A.7.7 Chart $V_{1,1,1}$. In this chart the equation of $\overline{\Gamma}_g$ takes form

$$x'(1 - x'y')[(1 + z') - tz'y'] + qz'^2 y'^2 = 0.$$

Looking for singular points with $x' = 0$, $y' = 0$ and $z' = 0$ we get that there are none. Hence nothing to check.

A.8 Computation of $\Phi_{\overline{f} \circ i}(\mathbb{C}_Z)$. Recall that $Z = \overline{\Gamma}_g \setminus \Gamma_g$ and $\overline{\Gamma}_g \subset (\mathbb{P}^1)^3 \times \mathbb{C}$ is given by the homogeneous equation

$$x_0 z_1 (x_1 y_1 - x_0 y_0) [y_1 (z_1 + z_0) - t y_0 z_0] + q z_0^2 x_1^2 y_0^2 = 0.$$

The subvariety Γ_g is defined by additionally putting

$$\begin{aligned} &x_0 y_0 z_0 \neq 0 \\ &z_1 (x_1 y_1 - x_0 y_0) \neq 0. \end{aligned}$$

Thus, $\overline{\Gamma}_g \setminus \Gamma_g$ is defined by the system

$$\begin{aligned} &x_0 z_1 (x_1 y_1 - x_0 y_0) [y_1 (z_1 + z_0) - t y_0 z_0] + q z_0^2 x_1^2 y_0^2 = 0 \\ &x_0 y_0 z_0 z_1 (x_1 y_1 - x_0 y_0) = 0. \end{aligned}$$

This system is equivalent to

$$\begin{aligned} &x_0 z_1 y_1 (x_1 y_1 - x_0 y_0)(z_1 + z_0) + q z_0^2 x_1^2 y_0^2 = 0 \\ &x_0 y_0 z_0 z_1 (x_1 y_1 - x_0 y_0) = 0, \end{aligned}$$

which is independent of t. Therefore $Z = \overline{\Gamma}_g \setminus \Gamma_g$ is a product and $\Phi_{\overline{g} \circ i}(\mathbb{C}_Z)$ has empty support.

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Summary

In this thesis we consider questions arising in Gromov-Witten theory, quantum cohomology and mirror symmetry. The first two chapters deal with Gromov-Witten theory and derived categories for moduli spaces $\overline{M}_{0,n}$. In the third chapter we consider Landau-Ginzburg models for odd-dimensional quadrics.

In the first chapter it is proved that the moduli spaces of stable maps $\overline{M}_{0,\Sigma}(\overline{M}_{0,S},\beta)$ are smooth of expected dimension for β a class of a boundary curve on $\overline{M}_{0,S}$. Moreover, this moduli space is naturally isomorphic to the moduli space of stable maps $\overline{M}_{0,\Sigma}(\overline{M}_{\Pi},\beta_{\Pi})$, where $\overline{M}_{\Pi}$ is a certain boundary stratum in $\overline{M}_{0,S}$ of the form $B \times C$, and β_{Π} is the class of a fiber of the natural projection $B \times C \to B$. These results allow explicit computations of Gromov-Witten correspondences in this case.

In the second chapter we consider inductive constructions of semi-orthogonal decompositions and exceptional collections in the derived category of moduli spaces $\overline{M}_{0,n}$ based on a nice presentation of these spaces as consecutive blow-ups along smooth codimension two subvarieties each of which is isomorphic to the product $\overline{M}_{0,p} \times \overline{M}_{0,q}$ with $p, q \leq n-2$.

In the third chapter we give an ad hoc partial compactification of the standard Landau-Ginzburg potential for an odd-dimensional quadric, and study its Gauss-Manin system in the case of three dimensional quadrics. We show that under some hypothesis this Landau-Ginzburg potential would give a Frobenius manifold isomorphic to the quantum cohomology of a three dimensional quadric.