Pricing and Risk Management of Basket FX Derivatives and Unit-Linked Life Insurance Contracts

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Preface

The dissertation deals with pure financial derivatives and financial derivatives which are components of life insurance contracts. Pure financial derivatives are financial instruments whose payoff depend on the price development of one or more underlying assets. In the simplest case, the underlying could be a single asset, such as stocks, commodities and foreign currencies, or more complexly, baskets of different assets. Similar payoff structures can be embedded in life insurance contracts. Life insurance contracts provide either survival benefits or death benefits or both. When these benefits are linked to the performance of one or more underlying assets, the policyholders will also have the opportunity to participate in the financial market. The benefits are usually equipped with certain guarantees, so that the policyholders are insured to be protected from the downside development of the financial market. Typical examples of such life insurance contracts are unit-linked life insurance contracts.

Chapter 1 focuses on the complex situation of basket foreign exchange (FX) products. These are financial products whose payoffs depend on the behavior of a basket of foreign currencies at a predetermined time point or within a predetermined time period. The building blocks of these basket FX products are basket options. A well-known feature of basket options is that it is difficult to specify the distribution of the underlying basket starting from the standard assumption that the price processes of the single assets in the basket follow geometric Brownian motions. In the literature, both numerical and approximation methods to price basket options were discussed in great extent. Numerical methods are, for example, the bivariate binomial lattice method by Rubinstein (1994) and the Monte Carlo simulation method proposed by Joy, Boyle and Tang (1996), whose results are considered to be very near to the true prices. However, due to the time consuming of these methods, people have been searching for approximation methods which involve less calculation time without too much sacrifice of the accuracy. These methods are especially important when immediate price calculation is necessary. With regard to the approximation methods of basket option pricing, Beißer (2001) has provided a detailed review which includes the geometric approximation approach[1], the lognormal approximation approach[2], the Edgeworth series expansion approach[3], the Reciprocal Gamma

[1]Confer Gentle (1993).
[2]Confer Levy (1992).
[3]Confer Huynh (1994).

approximation approach[4] and the conditional expectation approach[5]. These methods are transferred from the approximation methods first developed for Asian option pricing. Due to the close structural similarity between basket and Asian options, they can be applied to the pricing of basket options without difficulty. The idea of the first four methods is to approximate the unknown distribution of the underlying basket with a known distribution (the lognormal distribution in the first three cases and the reciprocal gamma distribution in the fourth case), while the conditional expectation approach approximates the price processes of the underlying currencies driven by different Brownian motions with the synthetic processes driven by a common stochastic process by applying the tower property of the conditional expectation and Jensen's inequality.[6] In Chapter 1, within an international financial market model, another approximation method, called the rank one approximation method, is proposed. This method mixes the two approximation categories at two steps. At the first step, it approximates the covariance structure of the uncertain part of the price processes with a rank one matrix and delivers a vector of stochastic processes driven by the same standard normally distributed variable. Then at the second step several adjustment parameters are introduced into the price process of the synthetic underlying basket approximated at the first step for the purpose of correcting the distribution distorted through the first step approximation. The performance of this method concerning the pricing and risk management of basket options will be studied in comparison with one of the popular approximation methods–the lognormal approximation method. By introducing the rank one approximation method, we enlarge the family of approximation methods for the pricing of basket derivatives.

Chapter 2 and 3 are concerned with unit-linked life insurance contracts. Unit-linked life insurance was very popular in household financial planning in the 1990s. The share of unit-linked premiums increased from 20% in 1997 to 36% in 2001 of the total life insurance premiums which accounted for over 10% of the GDP of western Europe, see Re (2003). Although the popularity of unit-linked insurance from the policyholder side declined during financial market crashes, e.g., at the end of 2001 and between 2007 and 2010 , this business is expected to boom again when the capital market recovers from the depression. According to Re (2003), a simple regression analysis shows that a 10% rise in the stock market led to a 15% increase in single-premium unit-linked sales.

Since the payoffs of unit-linked life insurance contracts also depend on the occurrence of the policyholders' death event, the influence of the mortality risk on unit-linked life insurance contracts deserves to be studied properly. In recent years, it has been widely accepted that mortality changes over time in an unpredictable way and stochastic models have been developed to adequately capture the systematic mortality risk. Each mortality model is a possible description of the mortality risk. In Chapter 2, a framework is

[4]Confer Milevsky and Posner (1998).

[5]Confer Curran (1994), Rogers and Shi (1995), Nielsen and Sandmann (2002a).

[6]For more detail, confer Beißer (2001).

proposed for assessing the mortality model risk embedded in unit-linked life insurance contracts arising from different specifications for the mortality intensity. The basic assumption of this framework is that we do not know the exact process of the mortality intensity but are able to figure out its upper and lower bound under the statistical measure. This setup allows us to study the impact of mortality model risk on various contract types more efficiently.

Many unit-linked life insurance contracts also include the provision of surrender options, which allow policyholders to terminate the contracts prematurely. Chapter 3 studies the valuation of unit-linked life insurance contracts with surrender guarantees. The important part in valuing such contract types is to describe the surrender behavior of the policyholders. Surrender decisions are not only triggered by exogenous reasons but also by endogenous reasons. Exogenous reasons are, for instance, the financial stresses of the policyholders, and endogenous reasons are the financial factors which make it monetarily optimal to surrender the contracts at the appropriate moments. In this chapter, the arrival of the surrender event is described by an intensity-based approach and the valuation problem is solved for a representative policyholder. We assume the surrender intensity to be bounded from below and from above. The lower bound represents the surrender base level due to exogenous reasons. And the upper bound represents the maximal surrender intensity that is attributed to exercise of the surrender option when it is financially optimal to do so. The effect of policyholders' monetary rationality on the fair contract design is studied in detail.

Chapter 1

Rank One Approximation Pricing of Basket FX Derivatives

1.1 Introduction

Basket FX derivatives are financial derivatives based on a common base currency and several other risky currencies. The risky currencies build the underlying basket. Depending on the exchange rates of these currencies with the base currency, the payoffs of the derivatives vary.

1.1.1 Main Functions of Basket FX Derivatives

Basket FX derivatives, as other financial derivatives, mainly serve two purposes of the financial participants. Firstly, corporations active on the global market transact between domestic and foreign currencies frequently. Since the exchange rates between the currencies change every moment, these corporations want to hedge against the exposure to the exchange rate risk so as to make more reliable business plans. FX derivatives tailored to their demands enable them to limit the risk exposure at reasonable costs. Secondly, the volatile exchange rate market also provides the opportunities to make profits if the development of the exchange rates is speculated correctly. Institutional as well as private investors are hence attracted to access this market.

Compared to the FX derivatives written on a single exchange rate, basket FX derivatives have two more advantages.

Firstly, corporations having frequent inflows of several foreign currencies are more interested in the overall performance of their currency baskets. Instead of obtaining several plain vanilla options on the single currencies and thus limiting the appreciation of each currency separately to a certain level, the corporations save their costs if a single option on the basket of these currencies is available. The cost saving effect is mainly explained by the convex payoff structure of the options. For example, we consider a basket call option written on the exchange rates $X_1, X_2, \ldots, X_n$ weighted by $w_1, w_2, \ldots, w_n$ respectively. The option with the strike price K pays out $[\sum_{i=1}^{n} w_i X_i - K]^+$ at the maturity date T. Due to the convexity of the payoff structure $[\cdot]^+$, we have

$$\left[\sum_{i=1}^{n} w_i X_i - K\right]^+ = \left[\sum_{i=1}^{n} w_i X_i - \sum_{i=1}^{n} w_i K_i\right]^+ \leq \sum_{i=1}^{n} w_i \left[X_i - K_i\right]^+ .$$

for all the sequences of $K_1, K_2, \ldots, K_n$ with $\sum_{i=1}^{n} w_i K_i = K$. This indicates that a portfolio of plain vanilla options on the single currencies overhedges the risk exposure intended to hedge against, and hence, is also more expensive. This makes it less attractive than the basket option in this situation. Furthermore, transaction costs are saved when only one option is needed instead of several ones.

Secondly, due to the averaging effect, a FX basket is less volatile than a single exchange rate. The averaging effect may be represented by two phenomena. On the one hand, for negatively correlated currencies, the appreciation of one currency is offset by the depreciation of the other one, and vice versa. On the other hand, the effect of a currency with high volatility can be mitigated by other currencies with lower volatilities within the basket.[1] Therefore, basket derivatives are favored by investors who are more confident in predicting the trend of a currency basket than to predict the performance of single currencies. For example, following the optimistic forecast of Goldman-Sachs in 2003 on the development of the BRICs and later in 2005 on the potential of the N-11[2], a number of certificates written on the BRIC or BRIC-plus currency basket[3] have been issued on the German market since 2006. Among them are the BRIC-plus guarantee certificate, the BRIC-plus outperformance certificate and the BRIC-plus certificate with stages issued by Goldman-Sachs, and the BRIC basket partially capital protected notes issued by ABN-AMRO, just to name a few. These certificates bet, on the one side, on the appreciation of the BRIC currencies, and on the other side, are featured with strike levels and participation rates in accordance with the risk appetite and risk tolerance of the investors.

[1]Confer Beißer (2001) p.123.

[2]BRIC refers to the Brazil, Russian, India and China. N-11, the abbreviation for the Next Eleven, refers to the eleven countries as Bangladesh, Egypt, Indonesia, Iran, Korea, Mexico, Nigeria, Pakistan, Philippines, Turkey and Vietnam, which could potentially have a BRIC-like impact in rivalling the G7.

[3]The BRIC-plus currency basket includes the following 10 currencies: Brasilian Real (BRL), Indian Rupie (INR), Korean Won (KRW), Indonesian Rupiah (IDR), Mexican Peso (MXN), Philippine Peso (PHP), Russian Rubel (RUB), Chinese Yuan (CNY), Turkey new Lira (TRY), and South African Rand (ZAR).

1.1.2 Examples of Basket FX Derivatives

In this part, we provide some examples of BRIC-plus certificates on the German market. The underlying of these products is a basket of BRIC-plus currencies, whose exchange rates with one Euro at time t are $X_1(t), X_2(t), \ldots, X_{10}(t)$. Denoting the number of the currencies in the basket respectively as $w_1, w_2, \ldots, w_{10}$, we write the domestic (Euro) price of the currency basket at time t , namely $A(t)$, as $A(t) = \sum_{i=1}^{10} w_i X_i(t)$. We further assume that the issuing date and the maturity date of the products are respectively t_0 and T , and the face value is $A(t_0) = \sum_{i=1}^{10} w_i X_i(t_0)$.[4] Through the examples, we show that the payoffs of FX basket derivatives, which look complicated at the first sight, can be duplicated by a portfolio of zero-coupon bonds and basket FX options.

Example 1.1.1 (Guarantee Certificate)**.** *The Guarantee Certificate WKN: GS5HFX is issued by Goldman-Sachs on April 23, 2007 and matures on April 23, 2010. The composition of the basket is presented in Table 1.1. The certificate matures in three*

i	currency	w_i	$X_i(t_0)$
1	BRL	27.50805	0.36353
2	INR	568.1818	0.01760
3	KRW	12658.23	0.00079
4	IDR	125000	0.00008
5	MXN	148.876	0.06717
6	PHP	647.6684	0.01544
7	RUB	349.406	0.02862
8	CNY	104.7669	0.09545
9	TRY	18.35806	0.54472
10	ZAR	96.11688	0.10404

Table 1.1: The underlying currency basket of the guarantee certificate WKN: GS5HFX

years. If the currency basket value at the maturity date April 23, 2010, i.e., $T = 3$ *, is lower than the initial value, the certificate pays at least the initial value* $A(t_0)$ *back. Otherwise, the investors, besides obtaining the guaranteed amount* $A(t_0)$ *, participate* 6 *times into the return of the currency basket. That is, the payoff in this case would be* $A(t_0) \cdot \left(1 + 6 \cdot \left(\frac{A(T)}{A(t_0)} - 1\right)\right)$ *. To sum up, the payoff of the certificate can be written as* $A(t_0) + 6 \cdot [A(T) - A(t_0)]^+$ *. The graphic illustration of the payoff is shown in Figure 1.1. This payoff can be duplicated by an investment in the domestic zero coupon bonds with the face value* $A(t_0)$ *, and additionally, an investment in* 6 *plain vanilla at-the-money basket call options on the underlying basket.*

[4] In practice, $w_1, w_2, \ldots, w_n$ are determined in such a way that the Euro value of each currency in the basket accounts for $\frac{1}{n}$ of the face value. The face value ist usually equal to 100 . With regard to the BRIC-plus certificates introduced in the examples it indicates that $w_i = \frac{100}{10 X_i(t_0)}$ for $i = 1, 2, \ldots, 10$.

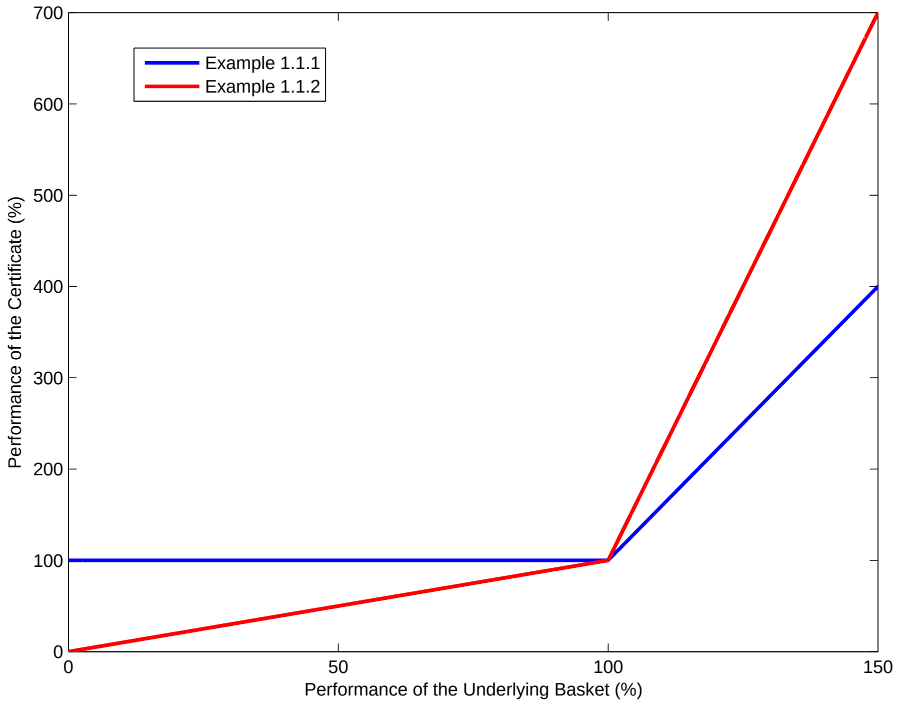

Figure 1.1: The payoff structures of the guarantee certificate (WKN: GS5HFX) and the outperformance certificate (WKN: GS5GFX) on BRIC-plus FX basket

Example 1.1.2 (Outperformance Certificate)**.** *An outperformance certificate is equipped with an outperformance level which is usually equal to* $A(t_0)$ *. When the value of the currency basket at time* T *is no less than the outperformance level, the investor receives, in addition to the initial investment* $A(t_0)$ *, the surplus return of the currency basket over the outperformance level, which is leveraged by a participation rate* α *. This indicates that the payoff in this case would be* $A(t_0)\cdot\left(1+\alpha\cdot\left(\frac{A(T)}{A(t_0)}-1\right)\right)$ *, or equivalently,* $A(T)+(\alpha-1)\cdot(A(T)-A(t_0))$ *. Otherwise, the investor obtains the spot value of the currency basket at time* T *, namely* $A(T)$ *. Overall, the payoff of the certificate depending on* $A(T)$ *can be formulated as* $A(T)+(\alpha-1)\cdot[A(T)-A(t_0)]^+$ *. It is equivalent to the payoff of zero-coupon bonds in the foreign currencies with the same composition as the underlying basket of the certificate in addition to the payoff of* $(\alpha-1)$ *plain vanilla at-the-money basket call options on the underlying basket. The Outperformance Certificate WKN: GS5GFX, which is issued by Goldman-Sachs on April 23, 2007 and matures on April 28, 2010, is written on the BRIC-plus currencies displayed in Table 1.1. It specifies a participation rate of* 1200% *. The payoff structure is also illustrated graphically in Figure 1.1.*

By giving up the guarantee as is provided in Example 1.1.1, the investors of the outperformance certificate in Example 1.1.2 are entitled to participate more overproportionally in the performance of the underlying basket.

Example 1.1.3 (Certificate with Stages). *A certificate with stages displays a more complicated payoff structure which involves more strike prices than the above two products. We suppose that there are* m *stages proportional to* $A(t_0)$ *:* $\beta_1 A(t_0), \beta_2 A(t_0), \ldots, \beta_m A(t_0)$ *where* $\beta_1 < \beta_2 < \cdots < \beta_m$ *. The payoff structure of the certificate is defined as*

$$Z(T) = \begin{cases} \alpha_1 A(t_0) & if \quad A(T) < \beta_1 A(t_0), \\ \alpha_2 A(t_0) & if \quad \beta_1 A(t_0) \leq A(T) < \beta_2 A(t_0), \\ \alpha_3 A(t_0) & if \quad \beta_2 A(t_0) \leq A(T) < \beta_3 A(t_0), \\ \vdots & \\ \alpha_m A(t_0) & if \quad \beta_{m-1} A(t_0) \leq A(T) < \beta_m A(t_0), \\ \alpha_{m+1} A(t_0) & if \quad A(T) \geq \beta_m A(t_0), \end{cases} \tag{1.1}$$

where $\alpha_1 < \alpha_2 < \cdots < \alpha_{m+1}$ *. For the price development of* $A(T)$ *below* $\beta_m A(t_0)$ *, the certificate is specified to outperform the underlying currency basket, that is,* $\alpha_i \geq \beta_i$ *for* $i = 1, \ldots, m$ *; while when the underlying price soars above* $\beta_m A(t_0)$ *, the payoff is limited to a deterministic amount. The payoff structure can be summarized as*

$$\begin{aligned} Z(T) &= \alpha_1 A(t_0) \cdot 1_{\{A(T) < \beta_1 A(t_0)\}} + \alpha_2 A(t_0) \cdot 1_{\{\beta_1 A(t_0) \leq A(T) < \beta_2 A(t_0)\}} + \cdots \\ &\quad \cdots + \alpha_m A(t_0) \cdot 1_{\{\beta_{m-1} A(t_0) \leq A(T) < \beta_m A(t_0)\}} + \alpha_{m+1} A(t_0) \cdot 1_{\{A(T) \geq \beta_m A(t_0)\}} \\ &= \alpha_1 A(t_0) \cdot (1 - 1_{\{A(T) \geq \beta_1 A(t_0)\}}) + \alpha_2 A(t_0) \cdot (1_{\{A(T) \geq \beta_1 A(t_0)\}} - 1_{\{A(T) \geq \beta_2 A(t_0)\}}) + \cdots \\ &\quad \cdots + \alpha_m A(t_0) \cdot (1_{\{A(T) \geq \beta_{m-1} A(t_0)\}} - 1_{\{A(T) \geq \beta_m A(t_0)\}}) + \alpha_{m+1} A(t_0) \cdot 1_{\{A(T) \geq \beta_m A(t_0)\}} \\ &= \alpha_1 A(t_0) + (\alpha_2 - \alpha_1) A(t_0) \cdot 1_{\{A(T) \geq \beta_1\}} + (\alpha_3 - \alpha_2) A(t_0) \cdot 1_{\{A(T) \geq \beta_2 A(t_0)\}} \\ &\quad + \cdots + (\alpha_m - \alpha_{m-1}) A(t_0) \cdot 1_{\{A(T) \geq \beta_{m-1} A(t_0)\}} + (\alpha_{m+1} - \alpha_m) A(t_0) \cdot 1_{\{A(T) \geq \beta_m A(t_0)\}} \\ &= \alpha_1 A(t_0) + \sum_{i=1}^{m} (\alpha_{i+1} - \alpha_i) A(t_0) \cdot 1_{\{A(T) \geq \beta_i A(t_0)\}}. \end{aligned} \tag{1.2}$$

The certificate WKN: GS0QA5, which is issued by Goldman-Sachs on July 18, 2007 and matures on July 16, 2010, is equipped with three stages, i.e., $m = 3$ *. The relevant parameters are respectively* $\beta_1 = 100\%$ *,* $\beta_2 = 105\%$ *,* $\beta_3 = 110\%$ *,* $\alpha_1 = 100\%$ *,* $\alpha_2 = 119\%$ *,* $\alpha_3 = 138\%$ *and* $\alpha_4 = 157\%$ *. Thus, its payoff at the maturity date* $T = 3$ *can be represented as*

$$\begin{aligned} Z(3) &= A(t_0) + 19\% \cdot A(t_0) \cdot 1_{\{A(T) \geq A(t_0)\}} + 19\% \cdot A(t_0) \cdot 1_{\{A(T) \geq 105\% \cdot A(t_0)\}} \\ &+ 19\% \cdot A(t_0) \cdot 1_{\{A(T) \geq 110 \cdot \% A(t_0)\}} \end{aligned} \tag{1.3}$$

The graphic illustration is shown in Figure 1.2. It can be replicated by an investment in the domestic zero-coupon bonds with the face value $A(t_0)$ *and a portfolio of cash-or-nothing basket options, whose strikes are respectively* $A(t_0)$ *(at the money),* $105\% \cdot A(t_0)$ *(out of the money) and* $110\% \cdot A(t_0)$ *(out of the money). Each cash-or-nothing basket option pays* $19\% \cdot A(t_0)$ *back when the currency basket is worth more than the strike price.*

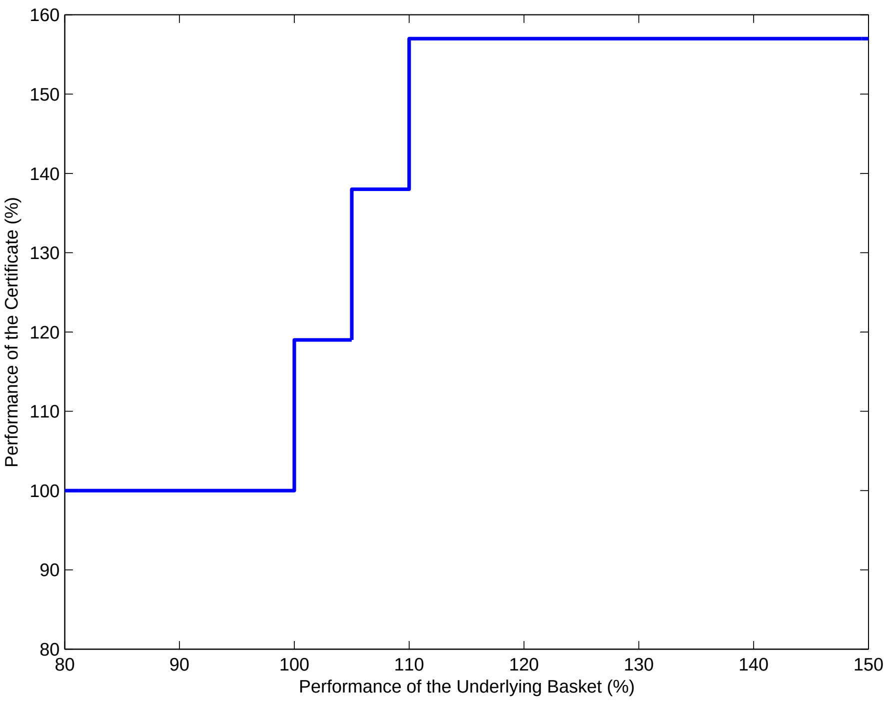

Figure 1.2: The payoff structure of the certificate with stages on BRIC-plus FX basket (WKN: GS0QA5)

The certificate with stages introduced in Example 1.1.3 is suitable for investors who believe in the mild outperformance of the underlying basket. Compared to Example 1.1.1, the payoff of the certificate with stages is higher when the increase in the underlying basket's value is approximately within the $0\% - 3.17\%$ and the $5\% - 6.33\%$ intervals. However, the payoff is limited to 157% of the face value, while in Example 1.1.1 it could theoretically rise to infinity.

1.2 The International Financial Market Model

Prices of basket FX derivatives are not only influenced by the contract parameters like the strike levels and the participation rates. The dynamic changes within the interest rate markets and the exhange rate markets also have a huge impact. Hence, before pricing basket FX derivatives, we need to model the dynamics of the interest rate markets and the exchange rate markets in both the domestic and the foreign countries appropriately.

In the option pricing field, the classical Black-Scholes model enjoys great popularity due to its computational simplicity. Two standard assumptions of the Black-Scholes model are the constant short-term interest rate and the geometric Brownian motion driving the price process of the underlying asset. The assumption about the constant interest rate simplifies the valuation problem, but is inadequate when we deal with FX derivatives with long maturity time. It is very common that a structured FX product has a life time ranging from 1 year to 5 years. The certificates we have introduced in Examples 1.1.1-1.1.3 all have a life time of 3 years. Over such a long time interval, it is unrealistic to assume the interest rates to be constant. More importantly, interest rates in different countries have significant influence on the exchange rates among the currencies of these countries. It is often observed that a raise in the interest rate of one country attracts investors to invest more in that country, and hence, appreciates its currency. The correlation between the interest rates and the exchange rates can hardly be ignored. Hence, we prefer to apply a stochastic model to capture the term structures of the interest rates. On the other hand, with regard to the exchange rates, it is favored in practice to assume that each exchange rate is driven by the geometric Brownian motion. Under this assumption, a closed-form solution can be found for the prices of options written on a single exchange rate. In align with this common practice, we also describe the exchange rates between the domestic currency and the foreign currencies with geometric Brownian motions.

In view of the above considerations, the international financial market model suggested by Amin and Jarrow (1991) provides the appropriate framework, within which we study the valuation problem of basket FX derivatives. In Amin and Jarrow (1991), only two currencies are involved. With regard to basket FX derivatives, however, we need to observe the markets of more than two countries. Hence, in this section, we extend the two-country model of Amin and Jarrow (1991) and generalize it to an $n+1$ country model.

Following the approach of Amin and Jarrow (1991), we assume that trading takes place continuously on the finite time interval $[0, T^*]$. We further assume that the interest rate markets in all the countries display stochastic behaviors and the exchange rates between the domestic country and the foreign countries are stochastic. A formal description of their behaviors is provided in Assumptions 1.2.1 and 1.2.2.

Let $\{\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}\}$ be a filtered probability space where the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T^*]}$ is generated by an m -dimensional Brownian motion $\{W_t\}_{t\in[0,T^*]}$. The m -dimentional Brownian motions can be interpreted as m sources of uncertainty across the $n+1$ economies with $m > n$.

The assumption in Amin and Jarrow (1991) about the interest rate markets can be traced back to Heath, Jarrow and Morton (1992). The Heath-Jarrow-Morton (HJM) model starts with describing the term structure of forward interest rates and provides a very general framework for the interest rate structure. Many famous interest rate models like the continuous time version of the Ho-Lee model (Ho and Lee (1986)) and the Vasicek model (Vasicek (1977)) are special cases of the HJM model.

Assumption 1.2.1 (The forward interest rate dynamics). *The domestic forward interest rate dynamics is*[5]

$$\mathrm{d}f_0(t,T) = \alpha_0(t,T)\mathrm{d}t + \sigma_0(t,T)\cdot \mathrm{d}W_t \quad \forall t\in[0,T] \text{ and } T\in[0,T^*], \tag{1.4}$$

and the forward interest rate dynamics of the i -th foreign country is

$$\mathrm{d}f_i(t,T) = \alpha_i(t,T)\mathrm{d}t + \sigma_i(t,T)\cdot \mathrm{d}W_t \quad \forall t\in[0,T] \text{ and } T\in[0,T^*], \tag{1.5}$$

where $\alpha_0(t,T)$, $\sigma_0(t,T)$, $\alpha_i(t,T)$ and $\sigma_i(t,T)$ are subject to some regularity conditions[6]*. Besides, $\sigma_0(t,T)$ and $\sigma_i(t,T)$ are m -dimensional functions in (t,T) .*

Equations 1.4 and 1.5 indicate that the instantaneous covariance between the domestic forward interest rate and the forward interest rate of the i -th country is

$$cov[\mathrm{d}f_0(t,T),\mathrm{d}f_i(t,T)] = [\sigma_0(t,T)\cdot\sigma_i(t,T)]\mathrm{d}t \quad \forall t\in[0,T] \text{ and } T\in[0,T^*], \tag{1.6}$$

and the instantaneous covariance between the forward interest rates of the i -th and the j -th countries is

$$cov[\mathrm{d}f_i(t,T),\mathrm{d}f_j(t,T)] = [\sigma_i(t,T)\cdot\sigma_j(t,T)]\mathrm{d}t \quad \forall t\in[0,T] \text{ and } T\in[0,T^*]. \tag{1.7}$$

The interest rate market can also be characterized by the price process of the family of zero-coupon bonds.

Lemma 1.2.1. *Under Assumption 1.2.1, the dynamics of the domestic zero-coupon bond price $D_0(t,T)$ is determined by the expression*

$$\mathrm{d}D_0(t,T) = D_0(t,T)(a_0(t,T)\mathrm{d}t + b_0(t,T)\cdot\mathrm{d}W_t) \quad \forall t\in[0,T] \text{ and } T\in[0,T^*], \tag{1.8}$$

where a_0 and b_0 are given by the following formulas

$$a_0(t,T) = f_0(t,t) - \alpha_0^*(t,T) + \frac{1}{2}|\sigma_0^*(t,T)|^2,\ \ b_0(t,T) = -\sigma_0^*(t,T), \tag{1.9}$$

and for any $t\in[0,T]$ we have

$$\alpha_0^*(t,T) = \int_t^T \alpha_0(t,u)du,\ \ \sigma_0^*(t,T) = \int_t^T \sigma_0(t,u)du. \tag{1.10}$$

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[5]The centered dot " · " in the following equations refers to the scalar product of two m -dimensional vectors.

[6]Confer Amin and Jarrow (1991).

Similarly, the dynamics of the i-th foreign zero-coupon bond price is determined by

$$\mathrm{d}D_i(t,T) = D_i(t,T)(a_i(t,T)\mathrm{d}t + b_i(t,T)\cdot \mathrm{d}W_t) \quad \forall t\in[0,T] \text{ and } T\in[0,T^*], \tag{1.11}$$

where a_i and b_i are given by the following formulas

$$a_i(t,T) = f_i(t,t) - \alpha_i^*(t,T) + \frac{1}{2}|\sigma_i^*(t,T)|^2,\ \ b_i(t,T) = -\sigma_i^*(t,T), \tag{1.12}$$

with

$$\alpha_i^*(t,T) = \int_t^T \alpha_i(t,u)\mathrm{d}u,\ \ \sigma_i^*(t,T) = \int_t^T \sigma_i(t,u)\mathrm{d}u \tag{1.13}$$

for any $t\in[0,T]$.

Proof. Confer Heath et al. (1992). □

Remark 1.2.1. *In the following, we use $r_i(t) = f_i(t,t)$, $i\in\{0,1,\dots,n\}$ to denote the instantaneous interest rate at every time t on the domestic and the foreign interest rate markets. The time t value of the money account (with the starting value 1) in the i-th market is denoted as $B_i(t) = \exp\{\int_0^t r_i(u)\,\mathrm{d}u\}$.*

Assumption 1.2.2 (Spot exchange rate dynamics)**.** *The spot exchange rate process between the domestic currency and the i-th foreign currency, denoted as $\{X_i(t)\}_{t\in[0,T^*]}$ for $i\in\{1,\dots,n\}$, follows the geometric Brownian motion*

$$\mathrm{d}X_i(t) = \mu_i(t)X_i(t)\mathrm{d}t + X_i(t)\delta_i(t)\cdot\mathrm{d}W_t, \tag{1.14}$$

where $\mu_i(t)$ and $\delta_i(t)$ satisfy some regularity conditions.

The spot exchange rate processes are influenced by the same m-dimensional Brownian motion $\{W_t\}_{t\in[0,T^*]}$ as the forward interest rate processes presented in (1.4) and (1.5). Hence, correlations between the exchange rate markets and the interest rate markets are captured by this model.

Assumption 1.2.3. *The international financial market is both arbitrage free and complete.*

Under Assumption 1.2.3, the following three conditions should be satisfied.

Condition 1.2.1. *There exists a domestic martingale measure $\mathbb{P}^*$ on the domestic financial market, such that, for $t\in[0,T]$ and $T\in[0,T^*]$, the relative bond price $Z^*(t,T) = \frac{D_0(t,T)}{B_0(t)}$ is a martingale under $\mathbb{P}^*$.*

This condition is satisfied if there exists an adapted R^m-valued process λ such that

$$\mathbb{E}_{\mathbb{P}^*}\left\{\epsilon_{T^*}\left(\int_0^{\cdot} \lambda_u\cdot\mathrm{d}W_u\right)\right\} = 1, \tag{1.15}$$

and for any maturity $T \leq T^*$ there is

$$a_0(t,T) = r_0(t) - b_0(t,T) \cdot \lambda_t. \tag{1.16}$$

The martingale measure $\mathbb{P}^*$ satisfies

$$\frac{\mathrm{d}\mathbb{P}^*}{\mathrm{d}\mathbb{P}} = \epsilon_{T^*}\left(\int_0^\cdot \lambda_u \cdot \mathrm{d}W_u\right), \quad \mathbb{P} - a.s. \tag{1.17}$$

and the $\mathbb{P}^*$ -Brownian motion W^* satisfies

$$W_t^* = W_t - \int_0^t \lambda_u \mathrm{d}u, \quad \forall t \in [0,T]. \tag{1.18}$$

Under $\mathbb{P}^*$ we obtain that for any $T \in [0,T^*]$

$$\mathrm{d}D_0(t,T) = D_0(t,T)(r_0(t)\mathrm{d}t + b_0(t,T) \cdot \mathrm{d}W_t^*). \tag{1.19}$$

Condition 1.2.1 guarantees that the domestic financial market is arbitrage free. The λ that we have defined above indicates the market price of risk on the domestic market.

Condition 1.2.2. *The domestic discounted value of each foreign zero-coupon bond should also be a martingale under* $\mathbb{P}^*$ *, so that there exists no arbitrage between the domestic market and the foreign markets.*

The mathematical meaning of Condition 1.2.2 can be derived through the following steps. First, we convert the foreign bond price to the domestic currency by defining

$$D_i^*(t,T) = D_i(t,T)X_i(t), \tag{1.20}$$

and similarly, the money market account in the unit of the domestic currency is defined by

$$B_i^*(t) = B_i(t)X_i(t). \tag{1.21}$$

Using Itô Lemma we obtain the dynamics of these artificial domestic assets

$$\begin{aligned} \mathrm{d}D_i^*(t,T) &= D_i^*(t,T)[(\mu_i(t) + a_i(t,T) + \delta_i(t)b_i(t,T))\mathrm{d}t \\ &+ (\delta_i(t) + b_i(t,T)) \cdot \mathrm{d}W_t], \qquad (1.22) \\ \mathrm{d}B_i^*(t) &= B_i^*(t)[(\mu_i(t) + r_i(t))\mathrm{d}t + \delta_i(t) \cdot \mathrm{d}W_t]. \qquad (1.23) \end{aligned}$$

Applying the domestic money account as the numeraire, and denoting $Z_i(t,T) = \frac{D_i^*(t,T)}{B_0(t)}$

and $Z_{r_i}(t) = \frac{B_i^*(t)}{B_0(t)}$, we have

$$\begin{aligned}
\mathrm{d}Z_i(t,T) &= Z_i(t,T)[(\mu_i(t) + a_i(t,T) + \delta_i(t)b_i(t,T) - r_0(t))\mathrm{d}t \\
&+ (\delta_i(t) + b_i(t,T)) \cdot \mathrm{d}W_t], \qquad (1.24) \\
\mathrm{d}Z_{r_i}(t) &= Z_{r_i}(t)[(\mu_i(t) + r_i(t) - r_0(t))\mathrm{d}t + \delta_i(t) \cdot \mathrm{d}W_t]. \qquad (1.25)
\end{aligned}$$

Under the assumption that there are no arbitrage opportunities across the countries, the same adapted process λ should satisfy the following two conditions under $\mathbb{P}^*$:

$$\mu_i(t) + a_i(t,T) + \delta_i(t)b_i(t,T) - r_0(t) + (\delta_i(t) + b_i(t,T)) \cdot \lambda_t = 0, \qquad (1.26)$$

$$\mu_i(t) + r_i(t) - r_0(t) + \delta_i(t) \cdot \lambda_t = 0, \qquad (1.27)$$

from which we can obtain

$$a_i(t,T) = r_i(t) - b_i(t,T) \cdot (\delta_i(t) + \lambda_t). \qquad (1.28)$$

We can also say that (1.16), (1.27) and (1.28) specify the no arbitrage conditions for the international market. In addition, we can derive from (1.28) that the market price of risk on the i -th foreign market is $\delta_i(t) + \lambda_t$ at $t \in [0,T]$ with $T \in [0,T^*]$.

Condition 1.2.3. *The λ satisfying (1.16), (1.27) and (1.28) is unique and independent of the particular assets chosen to construct the risk-neutral economy.*

Remark 1.2.2. *We refer to Amin and Jarrow (1991) for the calibration of λ . If there are more than $n+1$ Brownian motions, then bonds with different maturities than T should be chosen to complete the dimension and back out λ . The value of λ that has been found out should be unique and should also fit all the other asset price processes. This guarantees the no-arbitrage and completeness of the international market. However, λ is not required explicitly for the pricing issues.*

Conditions 1.2.1-1.2.3 together with Assumptions 1.2.1 and 1.2.2 guarantee that there exists a unique domestic martingale measure $\mathbb{P}^*$ with

$$\frac{\mathrm{d}\mathbb{P}^*}{\mathrm{d}\mathbb{P}} = \epsilon_{T^*}\left(\int_0^{\cdot} \lambda_u \cdot \mathrm{d}W_u\right), \quad P-a.s.$$

such that equations (1.16) (1.27) and (1.28) are valid for all $t \in [0,T]$ and $T \in [0,T^*]$. Under the domestic martingale measure $\mathbb{P}^*$ there is

$$\mathrm{d}D_i(t,T) = D_i(t,T)\left[(r_i(t) + \sigma_i^*(t,T) \cdot \delta_i(t))\mathrm{d}t - \sigma_i^*(t,T) \cdot \mathrm{d}W_t^*\right], \qquad (1.29)$$

and the dynamics of the exchange rates are described by

$$\mathrm{d}X_i(t) = X_i(t)\left[(r_0(t) - r_i(t))\mathrm{d}t + \delta_i(t)\cdot \mathrm{d}W_t^*\right]. \tag{1.30}$$

The instantaneous covariance of the i -th and the j -th exchange rate is $cov\left(\frac{\mathrm{d}X_i(t)}{X_i(t)}, \frac{\mathrm{d}X_j(t)}{X_j(t)}\right) = \delta_i(t)\cdot\delta_j(t)\mathrm{d}t$, and the instantaneous covariance of the i th and the j th T-maturity zero-coupon bonds is $cov\left(\frac{\mathrm{d}D_i(t,T)}{D_i(t,T)}, \frac{\mathrm{d}D_j(t,T)}{D_j(t,T)}\right) = \sigma_i^*(t,T)\cdot\sigma_j^*(t,T)\mathrm{d}t$.

1.3 The Pricing Problem

Within the international financial market model, we study the pricing of basket derivatives written on the currency basket $A(T) = \sum_{i=1}^n w_i X(T)$. Since the currency basket is composed of the n foreign currencies, we denote the payoff of a basket derivative at the maturity date T as $g(X_1(T),\ldots,X_n(T))$ where $g:\mathbb{R}^n \to \mathbb{R}$ is a bounded Borel-measurable function.[7] Under the martingale measure $\mathbb{P}^*$, the time t price $V(t)$ of this basket derivative is

$$V(t) = B_0(t)\,\mathbb{E}^{\mathbb{P}^*}\left[B_0(T)^{-1} g(X_1(T),\ldots,X_n(T))\middle|\mathcal{F}_t\right]. \tag{1.31}$$

This pricing formula is inconvenient when the domestic interest rate is not a determinant, because the joint probability law of the $\mathcal{F}_T$ -measurable random variables $B_0(T)$ and $X_1(T),\ldots,X_n(T)$ must be known. To circumvent this difficulty, we apply the change-of-numeraire technique introduced in Geman and Rochet (1995). The numeraire is changed from the money account (as is indicated by (1.31)) to the domestic T -maturity zero-coupon-bond. Correspondingly, a T -forward measure can be found so that the forward price of the basket derivative is a martingale under this measure. The definition of the T -forward measure is provided below.

Definition 1.3.1. *The domestic T-forward risk adjusted measure $\mathbb{P}^T$ is an equivalent measure with respect to $\mathbb{P}^*$ on $(\Omega,\mathcal{F})$ with the Radon-Nikodym derivative*

$$\frac{\mathrm{d}\mathbb{P}^T}{\mathrm{d}\mathbb{P}^*} = \frac{1}{B_0(T)D_0(0,T)}. \tag{1.32}$$

$\mathrm{d}W_t^T = \mathrm{d}W_t^* - b_0(t,T)\,\mathrm{d}t$ *defines a vector of Brownian motion under* $\mathbb{P}^T$ [8].

The T -forward measure defined above helps to reformulate the pricing formula (1.31) in a more tractable way.

[7] Confer Musiela and Rutkowski (2005).
[8] This results from the dynamics of the domestic zero-coupon bond and the Girsanov's theorem.

Lemma 1.3.1. *The time t no arbitrage price of a basket FX derivative with the payoff $g(X_1(T), \dots, X_n(T))$ at the maturity date T is*

$$V(t) = D_0(t,T)\mathbb{E}^{\mathbb{P}^T}\left[g(F_{X_1}(T,T), \dots, F_{X_n}(T,T))\big|\mathcal{F}_t\right] \quad (1.33)$$

for any $t \in [0,T]$, where $F_{X_i}(T,T)$ refers to the T-forward exchange rate of the i-th currency with the domestic currency at time T.

Proof. According to Bayes's rule, there is

$$\begin{aligned} V(t) &= \frac{B_0(t)\mathbb{E}^{\mathbb{P}^T}\left[\frac{\mathrm{d}\mathbb{P}^*}{\mathrm{d}\mathbb{P}^T}B_0(T)^{-1}g(X_1(T), \dots, X_n(T))\big|\mathcal{F}_t\right]}{\mathbb{E}^{\mathbb{P}^T}\left[\frac{\mathrm{d}\mathbb{P}^*}{\mathrm{d}\mathbb{P}^T}\big|\mathcal{F}_t\right]} \\ &= \frac{B_0(t)\mathbb{E}^{\mathbb{P}^T}\left[B_0(T)D_0(0,T)B_0(T)^{-1}g(X_1(T), \dots, X_n(T))\big|\mathcal{F}_t\right]}{\mathbb{E}^{\mathbb{P}^T}\left[B_0(T)D_0(0,T)\big|\mathcal{F}_t\right]} \\ &= \frac{B_0(t)D_0(0,T)\mathbb{E}^{\mathbb{P}^T}\left[g(X_1(T), \dots, X_n(T))\big|\mathcal{F}_t\right]}{D_0(0,T)B_0(t)/D(t,T)} \\ &= D_0(t,T)\mathbb{E}^{\mathbb{P}^T}\left[g(X_1(T), \dots, X_n(T))\big|\mathcal{F}_t\right] \end{aligned} \quad (1.34)$$

for any $t \in [0,T]$.

Under the T-forward measure $\mathbb{P}^T$, the dynamics of the exchange rate follows

$$\begin{aligned} \mathrm{d}X_i(t) &= X_i(t)\left[(r_0(t) - r_i(t))\mathrm{d}t + \xi(t) \cdot (\mathrm{d}W_t^T + b_0(t,T)\mathrm{d}t)\right] \\ &= X_i(t)\left[(r_0(t) - r_i(t) + \xi_i(t) \cdot b_0(t,T))\mathrm{d}t + \xi_i(t) \cdot \mathrm{d}W_t^T\right]. \end{aligned} \quad (1.35)$$

As is indicated by the drift term in equation (1.35), $X_i(t)$ is not a martingale under the T-forward measure $\mathbb{P}^T$. The dynamics of the interest rate market needs to be dealt with explicitly, which complicates the pricing problem due to its stochastic feature. To avoid this complexity, we make use of the fact that the T-forward exchange rates of the foreign currencies with the domestic currency are martingale under the T-forward measure $\mathbb{P}^T$.

The no-arbitrage argument implies that the T-forward exchange rate at time $t \in [0,T]$ with $T \in [0,T^*]$, satisfies

$$F_{X_i}(t,T) = \frac{X_i(t)D_i(t,T)}{D_0(t,T)}, \qquad i = 1, \dots, n. \quad (1.36)$$

Its process is obtained by applying the Itô lemma which leads to[9]

$$\mathrm{d}F_{X_i}(t,T) = F_{X_i}(t,T)\eta_i(t,T) \cdot \mathrm{d}W_0^T(t),$$

where $\eta_i(t,T) := \delta_i(t) + b_i(t,T) - b_0(t,T)$. Expressed as a stochastic integral equation,

[9] Confer Nielsen and Sandmann (2002b).

the value of $F_{X_i}(T,T)$ satisfies

$$F_{X_i}(T,T) = F_{X_i}(t,T)\exp\left\{-\frac{1}{2}\int_t^T ||\eta_i(u,T)||^2\mathrm{d}u + \int_t^T \eta_i(u,T)\cdot \mathrm{d}W^T(u)\right\} \tag{1.37}$$

for $i = 1,\cdots,n$.

Since $F_{X_i}(T,T) = \frac{X_i(T)D_i(T,T)}{D_0(T,T)} = X_i(T)$, we can rewrite equation (1.34) into

$$V(t) = D_0(t,T)\mathbb{E}^{\mathbb{P}^T}\left[g(F_{X_1}(T,T),\ldots,F_{X_n}(T,T))\middle|\mathcal{F}_t\right].$$

□

Remark 1.3.1. *The pricing formula (1.33) does not require the knowledge about the interest rate structure explicitly but only the distribution of the forward exchange rate vector* $(F_{X_1}(T,T),\ldots,F_{X_n}(T,T))$ *under the forward measure* $\mathbb{P}^T$ *, which is related to the interest rate dynamics through the volatility parameters* $b_0,b_1,\ldots,b_n$ *as well as the common* m *-dimensional Brownian motion* $\{W_t^T\}_{t\in[0,T]}$ *.*

1.3.1 The Rank One Approximation Method

The pricing formula (1.33) shows that the value of the basket derivative depends on the vector of n forward exchange rates $(F_{X_1}(T,T),F_{X_2}(T,T),\ldots,F_{X_n}(T,T))$. Observing the dynamics of each forward exchange rate as is presented in (1.35) more closely, we notice that the only term that causes the stochastic behavior of the i -th forward exchange rate is $\int_t^T \eta_i(u,T)\cdot \mathrm{d}W^T(u)$, which can also be written as

$$\sqrt{\int_t^T ||\eta_i(u,T)||^2\mathrm{d}u}\left(\int_t^T \frac{\eta_i(u,T)\cdot \mathrm{d}W^T(u)}{\sqrt{\int_t^T ||\eta_i(u,T)||^2\mathrm{d}u}}\right). \tag{1.38}$$

By extracting the term $\sqrt{\int_t^T ||\eta_i(u,T)||^2\mathrm{d}u}$ out of $\int_t^T \eta_i(u,T)\cdot \mathrm{d}W_0^T(u)$, we standardize the stochastic effect and make the performance of the rank one approximation to be implemented later less relevant to the "terminal volatility" $\sqrt{\frac{\int_t^T ||\eta_i(u,T)||^2\mathrm{d}u}{T-t}}$,[10] which is itself a deterministic term. The stochastic features of the contingent claim can be captured by the vector of standard normally distributed and correlated random variables

$$(\varsigma_1(t,T),\ldots,\varsigma_n(t,T)) = \left(\int_t^T \frac{\eta_1(u,T)\cdot \mathrm{d}W_0^T(u)}{\sqrt{\int_t^T ||\eta_1(u,T)||^2\mathrm{d}u}},\ldots,\int_t^T \frac{\eta_n(u,T)\cdot \mathrm{d}W_0^T(u)}{\sqrt{\int_t^T ||\eta_n(u,T)||^2\mathrm{d}u}}\right).$$

[10] Rebonato has defined the term "terminal correlation" in Rebonato (2004). We denote "terminal volatility" in a similar sense.

This vector has the standard Gaussian distribution $N \sim (0, \Gamma)$ where Γ refers to the covariance matrix of the vector with the form

$$\Gamma = \begin{pmatrix} 1 & \cdots & \cdots & \frac{\int_t^T \eta_1(u,T)\cdot\eta_n(u,T)\mathrm{d}u}{\sqrt{\int_t^T ||\eta_1(u,T)||^2\mathrm{d}u}\sqrt{\int_t^T ||\eta_n(u,T)||^2\mathrm{d}u}} \\ \frac{\int_t^T \eta_1(u,T)\cdot\eta_2(u,T)\mathrm{d}u}{\sqrt{\int_t^T ||\eta_1(u,T)||^2\mathrm{d}u}\sqrt{\int_t^T ||\eta_2(u,T)||^2\mathrm{d}u}} & 1 & \cdots & \frac{\int_t^T \eta_2(u,T)\cdot\eta_n(u,T)\mathrm{d}u}{\sqrt{\int_t^T ||\eta_2(u,T)||^2\mathrm{d}u}\sqrt{\int_t^T ||\eta_n(u,T)||^2\mathrm{d}u}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\int_t^T \eta_1(u,T)\cdot\eta_n(u,T)\mathrm{d}u}{\sqrt{\int_t^T ||\eta_1(u,T)||^2\mathrm{d}u}\sqrt{\int_t^T ||\eta_n(u,T)||^2\mathrm{d}u}} & \cdots & \cdots & 1 \end{pmatrix}.$$

For the ease of illustration in the following, we define

$$\bar{\sigma}_i^2 := \int_t^T ||\eta_i(u,T)||^2\mathrm{d}u, \quad \bar{\sigma}_{ij} := \int_t^T \eta_i(u,T)\cdot\eta_j(u,T)\mathrm{d}u. \tag{1.39}$$

Thus,

$$\Gamma = \begin{pmatrix} 1 & \cdots & \cdots & \frac{\bar{\sigma}_{1n}}{\bar{\sigma}_1\bar{\sigma}_n} \\ \frac{\bar{\sigma}_{12}}{\bar{\sigma}_1\bar{\sigma}_2} & 1 & \cdots & \frac{\bar{\sigma}_{2n}}{\bar{\sigma}_2\bar{\sigma}_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\bar{\sigma}_{1n}}{\bar{\sigma}_1\bar{\sigma}_n} & \cdots & \cdots & 1 \end{pmatrix}. \tag{1.40}$$

The covariance matrix Γ is a square-symmetric matrix. Suppose the rank of Γ is k, then there exists a $k \times n$ matrix $\Theta = [\theta_1, \ldots, \theta_n]$ such that $\Gamma = \Theta^T \cdot \Theta$. Since the elements along the main diagonal of Γ are 1, we should keep in mind that the norm of the vector θ_i satisfies $||\theta_i|| = 1$ for $i = 1, \ldots, n$. We can then set

$$\varsigma_i(t,T) \stackrel{d}{=} \theta_{1i}z_1 + \cdots \theta_{ki}z_k = \theta_i \cdot z := \tilde{\varsigma}_i(t,T), \quad z \in \mathbb{R}^k \ \mathit{and}\ z \sim N(0,I), \ i = 1, \ldots, n. \tag{1.41}$$

The vector $(\tilde{\varsigma_1}(t,T), \ldots, \tilde{\varsigma_n}(t,T))$ displays the same distribution as $(\varsigma_1(t,T), \ldots, \varsigma_n(t,T))$. If $k < n$, then we obtain the nice feature that the number of random variables is reduced.

Much attention has been paid to the rank of the correlation matrix. Most discussions on this topic took place in the context of swaption pricing in fixed income markets where the underlying is a basket of forward LIBOR or swap rates. The most exact and straightforward pricing method for this kind of derivatives is the Monte Carlo simulation method, which means k independent standard normally distributed random variables should be simulated in each simulation round. To increase the simulation speed, efforts have been made to reduce the rank of the correlation matrix, but to keep the approximated rank-reduced matrix as close to the original matrix as possible under certain criteria. Different approaches have been proposed which can be summarized as the solution to the following problem[11]

$$\begin{aligned} Find \ \ \bar{\Theta} \ &\in \ \mathbb{R}^{k\times n} \\ to\, minimize \ \ f(\bar{\Theta}) \ &:= \ \frac{1}{c}\sum_{i<j} w_{ij}(\Gamma_{ij} - \langle \bar{\theta}_i, \bar{\theta}_j\rangle)^2, \\ subject\, to\, ||\bar{\theta}_i||_2 \ &= \ 1, \ \ i = 1,\ldots,n, \end{aligned} \tag{1.42}$$

where w_{ij} are nonnegative weights and $c := 4\sum_{i<j} w_{ij}$. Pietersz and Groenen (2004) reviewed five popular algorithms to solve the above problem and have proposed a new algorithm based on majorization. After this step, the number of random variables is reduced and the Monte Carlo simulation can be carried out more efficiently. In the ideal case, i.e., if we can reduce the correlation matrix to a rank one matrix, we can even obtain closed-form solutions. For example, Brace and Musiela (1994) worked with the covariance matrix of a random vector similar to (1.41). In the setting that they studied, the first eigenvalue of the covariance matrix is approximately 50 times larger than the second, and hence, they assumed that the matrix is of rank one and obtained an analytical solution.

The rank one property is usually not observed in the underlying of basket derivatives. However, we are still attracted by the nice feature of the rank one matrix and try to use such kind of matrix to approximate the original matrix. The idea is that if the price obtained in this way deviates too much from the true value, we proceed additionally with some remedy techniques. In this thesis, we apply the three moment matching technique.

At the first step, we try to find the best approximation of Γ in terms of a rank one matrix. The first candidate may be to follow the scheme of (1.42) while keeping $k = 1$. Because this constraint is very strict, being $||\bar{\theta}_i|| = 1$ for $i = 1,\ldots,n$, the solution is a trivial one with

$$\bar{\Gamma} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \cdots & \cdots & 1 \end{pmatrix}.$$

It can be considered as the correlation matrix of a vector of perfectly correlated random variables. Another way is to consider Γ simply as a covariance matrix.[12] The job to do is to find a matrix that best approximates the overall structure of the covariance matrix. Thinking in this way, we need to solve Problem (1.42) but with no constraint. It is clear that the solution of an unconstrained minimization problem should generate a smaller $f(\bar{\Theta})$ indicating that the approximated matrix is closer to the original one.

[11] Confer Pietersz and Groenen (2004).

[12] However, the covariance matrix is scale-dependent, whereas the correlation matrix is not.

In this thesis, we consider the simple case with equal weights. Then Problem (1.42) is equivalent to the minimization of the Frobenius norm given $k = 1$. This problem can be solved by applying the singular value decomposition to the square-symmetric matrix Γ .

Theorem 1.3.1 (Singular Value Decomposition Theorem[13]). *For any $m \times n$ matrix Γ , there exist the $m \times m$ orthogonal matrices U and V as well as an $n \times n$ diagonal matrix Λ , such that $\Gamma = U\Lambda V^T$. U is composed of the eigenvectors of $\Gamma\Gamma^T$ and is called the left singular value of Γ . While V is the right singular value of Γ which is composed of the eigenvectors of $\Gamma^T\Gamma$. Furthermore, the diagonal matrix Λ has the form*

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & \cdots & \cdots & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & 0 \\ 0 & \cdots & \cdots & \lambda_k & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix} \tag{1.43}$$

and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$. The numbers $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the distinct singular values of Γ which are obtained by calculating the nonzero square roots of eigenvalues of $\Gamma\Gamma^T$ or $\Gamma^T\Gamma$.

Since in our setting the matrix to be discussed is square-symmetric, there is $\Gamma\Gamma^T = \Gamma^T\Gamma$. This implies that for a square-symmetric matrix, there is $U = V$, and therefore, $\Gamma = V\Lambda V^T$. In addition, from the orthogonality of V we infer that $\Gamma V = V\Lambda$, and hence, V is the matrix of the eigenvectors of Γ , namely, $(e_1, \ldots, e_n)$, and the values along the main diagonal of Λ are the eigenvalues of Γ , namely, $(\lambda_1, \ldots, \lambda_k, 0, \ldots, 0)$. This special case of the singular value decomposition of square-symmetric matrix is called the spectral decomposition in matrix analysis.

Based on the singular value decomposition theorem, we define the rank one approximation of Γ in Definition 1.3.2.

Definition 1.3.2 (Rank One Approximation). *For an $n \times n$ matrix Γ with $\Gamma = V\Lambda V^T$, its rank one approximation is $\bar{\Gamma} = V\bar{\Lambda}V^T$, where*

$$\bar{\Lambda} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix} . \tag{1.44}$$

[13] Confer Stewart (1973).

$\bar{\Gamma}$ is the best rank-one approximation of Γ in the sense of the minimization of the Frobenius norm of the difference between Γ and $\bar{\Gamma}$ under the constraint that the rank of $\bar{\Gamma}$ is equal to 1.[14] From $\bar{\Gamma} = V\bar{\Lambda}V^T = V\bar{\Lambda}^{\frac{1}{2}}(V\bar{\Lambda}^{\frac{1}{2}})^T$ and the structure of $\bar{\Lambda}$ we obtain

$$\bar{\Gamma} = (\sqrt{\lambda_1}e_1) \cdot (\sqrt{\lambda_1}e_1)^T. \tag{1.45}$$

Here, $\sqrt{\lambda_1}e_1$ is an $n \times 1$ vector which we denote as $\bar{\Theta}$. The vector of random variables $\bar{\Theta}\bar{z}$, where $\bar{z} \sim N(0,1)$, is $N(0,\bar{\Gamma})$ distributed . It is the best rank one approximation of the original random vector $\varsigma = (\varsigma_1, \ldots, \varsigma_n)$.

Now we come back to the payoff function of the basket derivative $g(F_{X_1}(T,T), \ldots, F_{X_n}(T,T))$. From (1.37), (1.38) and (1.41) we infer that it can be expressed as

$$g(F_{X_1}(t,T)e^{\bar{\sigma}_1\theta_1 \cdot z - \frac{1}{2}\bar{\sigma}_1^2}, \ldots, F_{X_n}(t,T)e^{\bar{\sigma}_n\theta_n \cdot z - \frac{1}{2}\bar{\sigma}_n^2}) := h(z). \tag{1.46}$$

By changing the random variables into the k-dimensional z, we obtain that the time t no-arbitrage price of the basket derivative is

$$V(t) = D_0(t,T) \int_{\mathbb{R}^k} h(z) n_k(z) dz, \tag{1.47}$$

where $n_k(z)$ refers to the density function of the k-dimensional z with $n_k(z) = (2\pi)^{-k/2}e^{-|z|^2/2}$.

Through the rank one approximation, we approximate $g(\cdot)$ in (1.46) with

$$g(F_{X_1}(t,T)e^{\bar{\sigma}_1\bar{\theta}_1 \cdot \bar{z} - \frac{1}{2}\bar{\sigma}_1^2}, \ldots, F_{X_n}(t,T)e^{\bar{\sigma}_n\bar{\theta}_n \cdot \bar{z} - \frac{1}{2}\bar{\sigma}_n^2}) := \bar{h}(\bar{z}). \tag{1.48}$$

Thus, the rank one approximation of $V(t)$ is

$$\bar{V}(t) = D_0(t,T) \int_{\mathbb{R}^1} \bar{h}(\bar{z}) n_1(\bar{z}) \mathrm{d}\bar{z}, \tag{1.49}$$

which usually has a close-formed solution.

Besides, the currency basket $A(T)$ is approximated by

$$\bar{A}_T(\bar{z}) = \sum_{i=1}^{n} w_i F_{X_i}(t,T) e^{-\frac{1}{2}\bar{\sigma}_i^2 + \bar{z}\bar{\theta}_i\bar{\sigma}_i}. \tag{1.50}$$

We call the rank one approximation pricing which finds $\bar{\Theta}$ through the singular value decomposition as the crude rank one approximation method.

[14] Confer Stewart (1973).

1.3.2 Three Moment Matching Technique Based on the Rank One Approximation

Through the rank one approximation presented in Section 1.3.1, the distribution of the currency basket is distorted to some extent. We present the first three moments of the true basket $A(T)$ and of the approximated basket $A_T(\bar{z})$ in Lemma 1.3.2.

Lemma 1.3.2. *For* $l = 1, 2, \ldots$, *we denote*

$$m_l(t) := \mathbb{E}^{\mathbb{P}^T}\left[A(T)^l|\mathcal{F}_t\right], \qquad \bar{m}_l(t) := \mathbb{E}^{\mathbb{P}^T}\left[\bar{A}_T(\bar{z})^l|\mathcal{F}_t\right].$$

Under the forward measure $\mathbb{P}^T$ *, the first three moments of the true basket* $A(T)$ *at time* t *are*

$$m_1(t) = \sum_{i=1}^{n} w_i F_{X_i}(t,T); \tag{1.51}$$

$$m_2(t) = \sum_{i,j=1}^{n} w_i w_j F_{X_i}(t,T) F_{X_j}(t,T) e^{\bar{\sigma}_{ij}}; \tag{1.52}$$

$$m_3(t) = \sum_{i,j,k=1}^{n} w_i w_j w_k F_{X_i}(t,T) F_{X_j}(t,T) F_{X_k}(t,T) e^{\bar{\sigma}_{ij}\bar{\sigma}_{ik}\bar{\sigma}_{jk}}, \tag{1.53}$$

and the first three moments of the approximated basket $A_T(\bar{z})$ *at time* t *are*

$$\bar{m}_1(t) = \sum_{i=1}^{n} w_i F_{X_i}(t,T) e^{-\frac{1}{2}(1-\theta_i^2)\bar{\sigma}_i^2}; \tag{1.54}$$

$$\bar{m}_2(t) = \sum_{i,j=1}^{n} w_i w_j F_{X_i}(t,T) F_{X_j}(t,T) e^{-\frac{1}{2}(\bar{\sigma}_i^2+\bar{\sigma}_j^2)+\frac{1}{2}(\theta_i\bar{\sigma}_i+\theta_j\bar{\sigma}_j)^2}; \tag{1.55}$$

$$\bar{m}_3(t) = \sum_{i,j,k=1}^{n} w_i w_j w_k F_{X_i}(t,T) F_{X_j}(t,T) F_{X_k}(t,T) e^{-\frac{1}{2}(\bar{\sigma}_i^2+\bar{\sigma}_j^2+\bar{\sigma}_k^2)+\frac{1}{2}(\theta_i\bar{\sigma}_i+\theta_j\bar{\sigma}_j+\theta_k\bar{\sigma}_k)^2}. \tag{1.56}$$

Proof. We only proof (1.51) here. The other equations can be verified in the similar way.

$$\begin{aligned} m_1(t) &= \mathbb{E}^{\mathbb{P}^T}\left[A(T)|\mathcal{F}_t\right] \\ &= \mathbb{E}^{\mathbb{P}^T}\left[\sum_{i=1}^{n} w_i F_{X_i}(T,T)\Big|\mathcal{F}_t\right] = \sum_{i=1}^{n} w_i \mathbb{E}^{\mathbb{P}^T}\left[F_{X_i}(T,T)\big|\mathcal{F}_t\right] \end{aligned}$$

$$= \sum_{i=1}^{n} w_i \mathbb{E}^{\mathbb{P}^T}\left[F_{X_i}(t,T)\exp\left\{-\frac{1}{2}\int_t^T ||\eta_i(u,T)||^2 \mathrm{d}u + \int_t^T \eta_i(u,T)\cdot \mathrm{d}W^T(u)\right\}\Big|\mathcal{F}_t\right]$$
$$= \sum_{i=1}^{n} w_i F_{X_i}(t,T).$$

The last equation follows from the $N \sim \left(0, \int_t^T ||\eta_i(u,T)||^2 \mathrm{d}u\right)$ distribution of the stochastic term $\int_t^T \eta_i(u,T)\cdot \mathrm{d}W^T(u)$ and the fact that $\mathbb{E}\left[e^X\right] = e^{\mu+\frac{1}{2}\sigma^2}$ for any $N \sim (\mu, \sigma^2)$ distributed random variable X. □

As the second step of our approximation approach, we try to adjust the approximated basket so that the first three moments of the two baskets are matched. If the new basket approximates the original basket very well, the approximated prices of the basket derivatives will display satisfactory accuracy compared to the true prices. In this part we present the approximation procedure. We will justify in Section 1.5 that the rank one approximation pricing adjusted by the three moment matching procedure can be called the improved rank one approximation method.

To adjust the approximated currency basket, we introduce three parameters α, β and γ into the expression $\bar{A}_T(\bar{z})$ to define a new approximation of $A(T)$, namely $\hat{A}_T(\bar{z})$ with

$$\hat{A}_T(\bar{z}) = \sum_{i=1}^{n} \alpha w_i F_{X_i}(t,T) e^{-\frac{1}{2}\bar{\sigma}_i^2 + \beta \bar{z}\theta_i \bar{\sigma}_i} + \gamma, \text{ where } \alpha > 0. \tag{1.57}$$

The one moment matching is achieved by setting $\beta = 1$ and $\gamma = 0$ while choosing α to match the first moment of $\hat{A}_T(\bar{z})$ with m_1. The two moment matching can be achieved by setting $\gamma = 0$ and finding out the α and β to match the first two moments m_1 and m_2. The three moment matching requires the appropriate choice of the three parameters. We can interpret $\hat{A}_T(\bar{z})$ as a portfolio of perfectly correlated synthetic currencies and the domestic currency with the weight of the i th synthetic currency at time t being αw_i.

Lemma 1.3.3. *Under the forward measure $\mathbb{P}^T$, the first three moments of $\hat{A}_T(\bar{z})$ at time t, where $\hat{A}_T(\bar{z})$ satisfies (1.57), are respectively*

$$\hat{m}_1 = \sum_{i=1}^{n} \alpha w_i F_{X_i}(t,T) e^{-\frac{1}{2}\bar{\sigma}_i^2 + \frac{1}{2}\beta^2\theta_i^2\bar{\sigma}_i^2} + \gamma; \tag{1.58}$$

$$\hat{m}_2 = \sum_{i,j=1}^{n} \alpha^2 w_i w_j F_{X_i}(t,T) F_{X_j}(t,T) e^{-\frac{1}{2}(\bar{\sigma}_i^2+\bar{\sigma}_j^2) + \frac{1}{2}\beta^2(\theta_i\bar{\sigma}_i + \theta_j\bar{\sigma}_j)^2}$$
$$+ 2\sum_{i=1}^{n} \alpha\gamma w_i F_{X_i}(t,T) e^{-\frac{1}{2}\bar{\sigma}_i^2 + \frac{1}{2}\beta^2\theta_i^2\bar{\sigma}_i^2} + \gamma^2; \tag{1.59}$$

and

$$\begin{aligned}\hat{m}_3 &= \sum_{i,j,k=1}^{n} \alpha^3 w_i w_j w_k F_{X_i}(t,T) F_{X_j}(t,T) F_k(t,T) \cdot e^{-\frac{1}{2}(\bar{\sigma}_i^2+\bar{\sigma}_j^2+\bar{\sigma}_k^2)+\frac{1}{2}\beta^2(\theta_i\bar{\sigma}_i+\theta_j\bar{\sigma}_j+\theta_k\bar{\sigma}_k)^2} \\ &+ 3\gamma \sum_{i,j=1}^{n} \alpha^2 w_i w_j F_{X_i}(t,T) F_{X_j}(t,T) e^{-\frac{1}{2}(\bar{\sigma}_i^2+\bar{\sigma}_j^2)+\frac{1}{2}\beta^2(\theta_i\bar{\sigma}_i+\theta_j\bar{\sigma}_j)^2} \\ &+ 3\gamma^2 \sum_{i=1}^{n} \alpha w_i F_{X_i} e^{-\frac{1}{2}\bar{\sigma}_i^2+\frac{1}{2}\beta^2\theta_i^2\bar{\sigma}_i^2} + \gamma^3. \end{aligned} \tag{1.60}$$

Proof. Similar to the proof of Lemma 1.3.2. □

Now we require the first three moments of $\hat{A}_T(\bar{z})$ to meet the first three moments of $A(T)$ and look for the values of α, β and γ. The results are presented in Proposition 1.3.1.

Proposition 1.3.1. *In order that the first three moments of $\hat{A}_T(\bar{z})$ are equal to the first three moments of $A(T)$, the parameters α and γ should satisfy*

$$\alpha = \sqrt{\frac{m_2 - m_1^2}{S_2 - S_1^2}}, \tag{1.61}$$

$$\gamma = m_1 - \alpha S_1 \tag{1.62}$$

with

$$S_1 := \sum_{i=1}^{n} w_i F_{X_i}(t,T) e^{-\frac{1}{2}\bar{\sigma}_i^2+\frac{1}{2}\beta^2\theta_i^2\bar{\sigma}_i^2}, \tag{1.63}$$

$$S_2 := \sum_{i,j=1}^{n} w_i w_j F_{X_i}(t,T) F_{X_j}(t,T) e^{-\frac{1}{2}(\bar{\sigma}_i^2+\bar{\sigma}_j^2)+\frac{1}{2}\beta^2(\theta_i\bar{\sigma}_i+\theta_j\bar{\sigma}_j)^2}. \tag{1.64}$$

The parameter β solves the equation $\hat{m}_3 = m_3$, where α and γ are functions of β following from (1.61) and (1.62).

Proof. To be verified are (1.61) and (1.62). (1.61) can be obtained by solving the equation $\hat{m}_2 - \hat{m}_1^2 = m_2 - m_1^2$. Since $m_2 - m_1^2$ and $S_2 - S_1^2$, as the variances of the variables $A(T)$ and $\bar{A}_T(\bar{z})$ respectively, are both positive, we make sure that α presented in (1.61) does exist. (1.62) follows from $\hat{m}_1 = m_1$. □

By applying the three moment matching technique, we approximate the time t price of the basket derivative with

$$\hat{V}(t) = D_0(t,T) \int_{\mathbb{R}^1} \hat{h}(\bar{z}) n_1(\bar{z}) \mathrm{d}\bar{z}, \tag{1.65}$$

where

$$\hat{h}(\bar{z}) := \hat{g}\left(F_{X_1}(t,T)e^{\beta\bar{\sigma}_1\theta_1\bar{z}-\frac{1}{2}\bar{\sigma}_1^2}, \ldots, F_{X_n}(t,T)e^{\beta\bar{\sigma}_n\theta_n\bar{z}-\frac{1}{2}\bar{\sigma}_n^2}\right). \tag{1.66}$$

$\hat{g}:\mathbb{R}^n \to \mathbb{R}$ is a bounded Borel-measurable function.

Remark 1.3.2. *The three moment matching procedure changes the composition of the artificial currency basket, and hence, the payoff function from* $g(\cdot)$ *to* $\hat{g}(\cdot)$ *.*

1.3.3 Applications to Two Basket Option Types

The examples in Section 1.1.2 show that many basket derivatives can be replicated by a portfolio of plain vanilla basket options or cash-or-nothing basket options. Hence, in the following, we focus on the application of the rank one approximation to the pricing of cash-or-nothing basket options and plain vanilla basket options.

Cash-or-Nothing Basket Options

The payoff of a cash-or-nothing basket option at the maturity date T is characterized by $1_{\{A(T)\geq K\}}$ where K refers to the strike price of the option. This indicates that the payoff function $g(\cdot)$ in (1.33) satisfies

$$g(F_{X_1}(T,T),\ldots,F_{X_n}(T,T)) = 1_{\{\sum_{i=1}^n w_i F_{X_i}(T,T)\geq K\}}. \tag{1.67}$$

According to the approximation procedure introduced in Sections 1.3.1 and 1.3.2, we approximate this payoff function with

$$\hat{g}\left(F_{X_1}(t,T)e^{\beta\bar{\sigma}_1\theta_1\bar{z}-\frac{1}{2}\bar{\sigma}_1^2}, \ldots, F_{X_n}(t,T)e^{\beta\bar{\sigma}_n\theta_n\bar{z}-\frac{1}{2}\bar{\sigma}_n^2}\right) = 1_{\{\alpha\sum_{i=1}^n w_i F_{X_i}(t,T)e^{\beta\bar{\sigma}_i\theta_i\bar{z}-\frac{1}{2}\bar{\sigma}_i^2}+\gamma\geq K\}}. \tag{1.68}$$

Hence, the time t price of the cash-or-nothing basket option is

$$V(t) = D_0(t,T)\cdot\mathbb{P}^T\left(\sum_{i=1}^n w_i F_{X_i}(T,T)\geq K|\mathcal{F}_t\right), \tag{1.69}$$

and its approximated price by following the rank one approximation with the three moment matching is

$$\hat{V}(t) = D_0(t,T)\cdot\mathbb{P}^T\left(\hat{A}_T(\bar{z})\geq K|\mathcal{F}_t\right), \tag{1.70}$$

where $\mathbb{P}^T(\cdot)$ refers to the probability distribution under the forward measure $\mathbb{P}^T$, and $\hat{A}_T(\bar{z})$ satisfies (1.57).

Since there is no closed form solution to (1.69), we look for its approximation with (1.70), where a closed form solution is available. To solve (1.70), we apply the method proposed by Nielsen and Sandmann (2002b). In their work, Nielsen and Sandmann (2002b) have used the conditional expectation approach for the pricing of Asian options. Through this approximation method, they have derived synthetic assets, the exponential parts of whose price processes are similar to what has been approximated above. Similar to Nielsen and Sandmann's approach, We define

$$f_i(\bar{z}) := exp\left\{-\frac{1}{2}\beta\bar{\sigma}_i^2 + \bar{z}\bar{\theta}_i\bar{\sigma}_i\right\}.$$

$f_i(\bar{z})$ is increasingly convex for $\bar{\theta}_i > 0$ and decreasingly convex for $\bar{\theta}_i < 0$. From (1.61) we also know that $\alpha > 0$. Hence, the weighted average $\hat{A}_T(\bar{z})$ is a convex function in $\bar{z}$. Denote by $\mathcal{P}$, $\mathcal{N}$ and $\mathcal{M}$ the sets:

$$\begin{aligned}\mathcal{P} &= \{i|\bar{\theta}_i > 0, i = 1, \ldots, n\},\\ \mathcal{N} &= \{i|\bar{\theta}_i < 0, i = 1, \ldots, n\},\\ \mathcal{M} &= \{i|\bar{\theta}_i = 0, i = 1, \ldots, n\}.\end{aligned}$$

Then

$$\hat{A}_T(\bar{z}) - K = \alpha\sum_{i=1}^{n} w_i F_{X_i}(t,T) f_i(\bar{z}) + \gamma - K = \alpha \sum_{i\in\mathcal{P}\cup\mathcal{N}} w_i F_{X_i}(t,T) f_i(\bar{z}) - \hat{K},$$

with $\hat{K} = K - \gamma - \alpha\sum_{i\in\mathcal{M}} w_i F_{X_i}(t,T) e^{-\frac{1}{2}\beta\bar{\sigma}_i^2}$. Since the sum of convex functions is again convex, the equation

$$0 = \alpha \sum_{i\in\mathcal{P}\cup\mathcal{N}} w_i F_{X_i}(t,T) f_i(\bar{z}) - \hat{K} \tag{1.71}$$

has either zero, one or two solutions. As Nielsen and Sandmann did, we consider four situations:

- $\mathcal{P} \neq \varnothing$, $\mathcal{N} \neq \varnothing$ and $\alpha\sum_{i\in\mathcal{P}\cup\mathcal{N}} w_i F_{X_i}(t,T) f_i(\bar{z}) - \hat{K} < 0$ for some value of $\bar{z}$. Denote the two unique solutions of (1.71) by z^* and z^{**} respectively.
- $\mathcal{P} \neq \varnothing$, $\mathcal{N} \neq \varnothing$ and $\alpha\sum_{i\in\mathcal{P}\cup\mathcal{N}} w_i F_{X_i}(t,T) f_i(\bar{z}) - \hat{K} \geq 0 \;\; \forall \;\; \bar{z}$. Define $z^* = z^{**} := \infty$.
- $\mathcal{P} \neq \varnothing$ but $\mathcal{N} = \varnothing$. Denote the unique solution of (1.71) by z^{**} and define $z^* := -\infty$.
- $\mathcal{P} = \varnothing$ but $\mathcal{N} \neq \varnothing$. Denote the unique solution of (1.71) by z^* and define $z^{**} := \infty$.

It follows that

$$\mathbb{P}^T(\hat{A}_T(\bar{z}) \geq K|\mathcal{F}_t) = N(z^*) + N(-z^{**}). \tag{1.72}$$

We summarize the approximated price of the cash-or-nothing basket option in Proposition 1.3.2.

Proposition 1.3.2. *Following the approximation procedure introduced in Sections 1.3.1 and 1.3.2, the approximated price of the cash-or-nothing basket option, whose payoff at the maturity date T is $1_{\{A(T)\geq K\}}$ with $A(T)$ being the time T value of the underlying basket and K being the strike price, is*

$$\hat{V}(t) = D_0(t,T)[N(z^*) + N(-z^{**})], \tag{1.73}$$

where z^* and z^{**} are the solutions to the equation (1.71).

Plain Vanilla Basket Options

We apply the same procedure to plain vanilla basket options. The payoff function $g(\cdot)$ in (1.33) satisfies

$$g(F_{X_1}(T,T),\ldots,F_{X_n}(T,T)) = \left[\sum_{i=1}^{n} w_i F_{X_i}(T,T) - K\right]^+. \tag{1.74}$$

We approximate it with the payoff function

$$\hat{g}\left(F_{X_1}(t,T)e^{\beta\bar{\sigma}_1\theta_1\bar{z}-\frac{1}{2}\bar{\sigma}_1^2},\ldots,F_{X_n}(t,T)e^{\beta\bar{\sigma}_n\theta_n\bar{z}-\frac{1}{2}\bar{\sigma}_n^2}\right) = \left[\alpha\sum_{i=1}^{n} w_i F_{X_i}(t,T)e^{\beta\bar{\sigma}_i\theta_i\bar{z}-\frac{1}{2}\bar{\sigma}_i^2} + \gamma - K\right]^+. \tag{1.75}$$

Proposition 1.3.3 provides the closed form pricing formula to approximate the true price of the plain vanilla basket options.

Proposition 1.3.3. *Following the approximation procedure introduced in Sections 1.3.1 and 1.3.2, the approximated price of the plain vanilla basket call option, whose payoff at the maturity date T is $[A(T)-K]^+$ with $A(T)$ being the time T value of the underlying basket and K being the strike price, is*

$$\hat{V}(t) = D_0(t,T)\left[\sum_{i=1}^{n}\hat{w}_i(t)F_{X_i}(t,T)(N(z^*-d_i)+N(d_i-z^{**}))-(K-\gamma)(N(z^*)+N(-z^{**}))\right]. \tag{1.76}$$

We define $\hat{w}_i(t) = \alpha w_i e^{\frac{1}{2}(\beta^2\theta_i^2-1)\bar{\sigma}_i^2}$ and $d_i = \beta\theta_i\bar{\sigma}_i$. Besides, z^* and z^{**} are the two solutions to the equation (1.71).

Proof. According to our approximation procedure, there is

$$
\begin{aligned}
\hat{V}(t) &= D_0(t,T)\int_{\mathbb{R}^1}\left(\alpha\sum_{i=1}^{n} w_i F_{X_i}(t,T) f_i(\bar{z}) + \gamma - K\right)^{+} n_1(\bar{z})\mathrm{d}\bar{z} \\
&= D_0(t,T)\left[\int_{-\infty}^{z^*}\left(\alpha\sum_{i=1}^{n} w_i F_{X_i}(t,T) f_i(\bar{z}) + \gamma - K\right) n_1(\bar{z})\mathrm{d}\bar{z}\right. \\
&\qquad \left. + \int_{z^{**}}^{+\infty}\left(\alpha\sum_{i=1}^{n} w_i F_{X_i}(t,T) f_i(\bar{z}) + \gamma - K\right) n_1(\bar{z})\mathrm{d}\bar{z}\right] \\
&= D_0(t,T)\alpha\sum_{i=1}^{n} w_i F_{X_i}(t,T)\left[\int_{-\infty}^{z^*}\frac{1}{\sqrt{2\pi}}e^{-\frac{(\bar{z}-\beta\bar{\sigma}_i\theta_i)^2}{2}+\frac{\bar{\sigma}^2}{2}(\beta^2\bar{\sigma}_i^2-1)}\mathrm{d}\bar{z}\right. \\
&\qquad \left. + \int_{z^{**}}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{(\bar{z}-\beta\bar{\sigma}_i\theta_i)^2}{2}+\frac{\bar{\sigma}^2}{2}(\beta^2\bar{\sigma}_i^2-1)}\mathrm{d}\bar{z}\right] \\
&\quad -D_0(t,T)(K-\gamma)\left[\int_{-\infty}^{z^*}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}\mathrm{d}\bar{z} + \int_{z^{**}}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}\mathrm{d}\bar{z}\right] \\
&= D_0(t,T)\alpha\sum_{i=1}^{n} w_i e^{\frac{\bar{\sigma}_i^2}{2}(\beta_i^2\theta_i^2-1)}\left[\int_{-\infty}^{z^*}\frac{1}{\sqrt{2\pi}}e^{-\frac{(z-\beta\bar{\sigma}_i\theta_i)^2}{2}}\mathrm{d}\bar{z}\right. \\
&\qquad \left. + \int_{z^{**}}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{(\bar{z}-\beta\bar{\sigma}_i\theta_i)^2}{2}}\mathrm{d}\bar{z}\right] \\
&\quad -D_0(t,T)(K-\gamma)\left[N(z^*)+N(-z^{**})\right] \\
&= D_0(t,T)\left[\sum_{i=1}^{n}\hat{w}_i(t)F_{X_i}(t,T)(N(z^*-d_i)+N(d_i-z^{**}))\right. \\
&\qquad \left. -(K-\gamma)(N(z^*)+N(-z^{**}))\right]
\end{aligned}
$$

□

1.4 The Hedging Problem

In this section, we study the hedging problem associated with basket FX derivatives. Similar to Section 1.3.3, we focus on the two basic types of basket options, namely, the cash-or-nothing basket option and the plain vanilla basket option.

The hedging problem is even more important than the pricing problem itself. Without a proper hedging strategy, the risk exposure facing the issuers of the financial derivatives cannot be offset adequately even though the prices are evaluated accurately at the beginning; while an appropriate hedging strategy can help to justify the price settled at the issuing day and quoted during the life time of the product.

In Section 1.4.1 we introduce the delta-hedging and the safe crossing hedging strategies, both of which belong to the category of dynamic hedging strategies. Then, in Section 1.4.2, we are going to study the static hedging strategies. The static hedging does not require the knowledge of the derivative price at the beginning. By building a static hedging position whose price can be observed on the market, it is possible to obtain the price of the original financial derivative by applying the no arbitrage argument. The static hedging saves the costs incurred during the adjustment of the hedging positions and is invariant to volatility and interest rate risks. However, a perfect static hedging is not always available. In most cases, we are only able to build static superhedging positions with non-trivial costs. To decide between dynamic and static hedging strategies, we should weight between the dynamic hedging and the static superhedging costs.

1.4.1 The Dynamic Hedging

According to our assumption about the perfect arbitrage free and complete international financial market, we can replicate the value of a basket FX derivative by a self-financing strategy which takes place either on the forward asset/bond market, the forward/spot asset/bond market, the spot asset/bond market or on the spot asset/cash market.[15] The choice of a certain strategy depends on the availability of the hedging tools as well as the convenience in its implementation. In this chapter, we only discuss the replication strategy on the forward/spot asset/bond market. The similar procedure can be generalized to the other strategies without any difficulty.

Given that the approximated closed form pricing formula obtained through our approximation procedure is very close to the true value (which is to be demonstrated in Section 1.5), we derive the dynamic hedging positions from the approximated prices. However, even if the prices are well approximated, the hedging strategies based on these approximated prices do not necessarily perform so well. This is because the deviations of the

[15] Confer Musiela and Rutkowski (2005) for the brief introduction of these strategies.

approximated prices from the true prices are not fixed under different market conditions. The approximated hedge ratios derived during the life time of the product can hence overestimate or underestimate the true hedge ratios, so that the aggregate effect of the hedging strategies is unclear. We will investigate the performance of the dynamic hedging strategies which are induced from the approximation prices in Section 1.5.2.

From now on, we assume that $\hat{V}(t)$ is the true price of the option. Disregarding the exact structures of the two kinds of options discussed above, we denote their time t price in a general form as

$$\hat{V}(t) = D_0(t,T)\tilde{V}(t, F_X(t,T))$$

with $F_X(t,T) = (F_{X_1}(t,T), \ldots, F_{X_n}(t,T))$. Comparing to the pricing formulas (1.73) and (1.76),

$$\tilde{V}(t, F_X(t,T)) := N(z^*) + N(-z^{**})$$

for the cash-or-nothing basket options and

$$\tilde{V}(t, F_X(t,T)) := \sum_{i=1}^{n} \hat{w}_i(t) F_{X_i}(t,T)(N(\hat{z}^* - \hat{d}_i) + N(\hat{d}_i - \hat{z}^{**})) - (K - \gamma)(N(\hat{z}^*) + N(-\hat{z}^{**}))$$

for the plain vanilla basket options. Both the values of z^* and z^{**} depend on $F_X(t,T)$. Moreover, $\tilde{V}(t, F_X(t,T))$ can be interpreted as the forward price of the option. According to the Itô Lemma, there is

$$\begin{aligned} \mathrm{d}\tilde{V}(t, F_X(t,T)) &= \frac{\partial \tilde{V}}{\partial t}\mathrm{d}t + \sum_{i=1}^{n} \frac{\partial \tilde{V}}{\partial F_{X_i}}\mathrm{d}F_{X_i} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 \tilde{V}}{\partial F_{X_i}\partial F_{X_j}}\mathrm{d}F_{X_i}\mathrm{d}F_{X_j} \\ &= \left(\frac{\partial \tilde{V}}{\partial t} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 \tilde{V}}{\partial F_{X_i}\partial F_{X_j}} F_{X_i}F_{X_j}\eta_i \cdot \eta_j\right)\mathrm{d}t + \sum_{i=1}^{n} \frac{\partial \tilde{V}}{\partial F_{X_i}}\mathrm{d}F_{X_i}. \end{aligned} \tag{1.77}$$

Since the drift of $\tilde{V}(t, F_X(t,T))$ is 0 under the forward measure, we infer that $\tilde{V}(t, F_X(t,T))$ is the solution to the differential equation

$$\frac{\partial \tilde{V}}{\partial t} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 \tilde{V}}{\partial F_{X_i}\partial F_{X_j}} F_{X_i}F_{X_j}\eta_i \cdot \eta_j = 0 \tag{1.78}$$

with the boundary condition that its terminal value equals the payoff of the option at the maturity date T.

Proposition 1.4.1. *The basket option, which is issued at time 0 and the T-forward price of which at time $t \in [0,T]$ is $\tilde{V}(t, F_X(t,T))$, can be replicated by purchasing $\tilde{V}(0, F_X(0,T))$ number of domestic zero coupon bonds at time 0, holding them until the maturity date T, and at the same time, taking continuously $\Delta_i(t) = \frac{\partial \tilde{V}(t, F_X(t,T))}{\partial F_{X_i}(t,T)}$*

positions, where $t \in [0, T)$ *and* $i \in \{1, \ldots, n\}$ *, in the* T *-maturity forward contracts of the* i *-th currency.*

Proof. We infer from (1.77) that the forward value of the replication portfolio suggested above is

$$\begin{aligned}\tilde{V}(0, F_X(0,T)) + \int_0^t \sum_{i=1}^n \frac{\partial \tilde{V}}{\partial F_{X_i}} \mathrm{d}F_{X_i} &= \tilde{V}(0, F_X(0,T)) + \int_0^t \mathrm{d}\tilde{V}(s, F_X(s,T)) \\ &= \tilde{V}(t, F_X(t,T))\end{aligned}$$

for all $t \in [0, T]$. □

Proposition 1.4.1 indicates that $\frac{\partial \tilde{V}(t, F_X(t,T))}{\partial F_{X_i}(t,T)}$ with $i = 1, \ldots, n$ for $t \in [0, T)$ should be the hedging positions we are searching for in the continuous time model. For the cash-or-nothing basket option, the i -th delta Δ_c^i satisfies

$$\Delta_c^i(t) = \frac{\partial \tilde{V}(t, F_X(t,T))}{\partial F_{X_i}(t,T)} = n(z^*) \frac{\partial z^*}{\partial F_{X_i}(t,T)} - n(z^{**}) \frac{\partial z^{**}}{\partial F_{X_i}(t,T)},$$

and for the basket option, the i th delta Δ_b^i is

$$\begin{aligned}\Delta_b^i(t) &= \frac{\partial \tilde{V}(t, F_X(t,T))}{\partial F_{X_i}(t,T)} = \hat{w}_i[N(z^* - d_i) + N(d_i - z^{**})] \\ &+ \sum_{j=1}^n \frac{\partial \hat{w}_j}{\partial F_{X_i}} F_{X_j}[N(z^* - d_j) + N(d_j - z^{**})] \\ &+ \sum_{j=1}^n \hat{w}_j F_{X_i} \left[n(z^* - d_j)(\frac{\partial z^*}{\partial F_{X_i}} - \theta_j \bar{\sigma}_j \frac{\partial \beta}{\partial F_{X_i}}) + n(d_j - z^{**})(\theta_j \bar{\sigma}_j \frac{\partial \beta}{\partial F_{X_i}} - \frac{\partial z^{**}}{\partial F_{X_i}}) \right] \\ &- (K - \gamma) \left[n(z^*) \frac{\partial z^*}{\partial F_{X_i}} - n(z^{**}) \frac{\partial z^{**}}{\partial F_{X_i}} \right] + \frac{\partial \gamma}{\partial F_{X_i}} [N(z^*) + N(-z^{**})].\end{aligned}$$

Although it is possible to find analytical solutions to the above equations, it is a very tedious job. Hence we prefer to compute the hedge ratios numerically by full revaluation, i.e., with the following equation:

$$\Delta_{c,b}^i(t) = \frac{\tilde{V}_{c,b}^{(\epsilon_i)}(t, F_X(t,T)) - \tilde{V}_{c,b}(t, F_X(t,T))}{\epsilon_i} \tag{1.79}$$

with $\epsilon_i > 0$ being a small number. Here $\tilde{V}^{(\epsilon_i)}$ refers to the forward option price when the forward exchange rate of the i th currency is perturbed by a small ϵ_i while the forward exchange rates of the other currencies are kept unchanged. The fast speed of the approximation pricing makes the calculation of the hedge ratios very efficient.

In reality, however, the hedging can only take place at discrete time points with time interval in between. Denoting the discrete time approximation of $\mathrm{d}\tilde{V}(t, F_X(t,T))$ as $\delta\tilde{V}(t, F_X(t,T))$, the discrete time approximation of (1.77) is correspondingly

$$\begin{aligned}
\delta\tilde{V} &= \frac{\partial\tilde{V}}{\partial t} + \sum_{i=1}^{n}\frac{\partial\tilde{V}}{\partial F_{X_i}}\delta F_{X_i} + \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{2}\frac{\partial^2\tilde{V}}{\partial F_{X_i}\partial F_{X_j}}F_{X_i}F_{X_j}(\eta_i\cdot\delta W^T)(\eta_j\cdot\delta W^T) \\
&= \left(\frac{\partial\tilde{V}}{\partial t} + \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{2}\frac{\partial^2\tilde{V}}{\partial F_{X_i}\partial F_{X_j}}F_{X_i}F_{X_j}\eta_i\cdot\eta_j\right)\delta t + \sum_{i=1}^{n}\frac{\partial\tilde{V}}{\partial F_{X_i}}\delta F_{X_i} \\
&+ \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{2}\frac{\partial^2\tilde{V}}{\partial F_{X_i}\partial F_{X_j}}F_{X_i}F_{X_j}\left(\sum_{k=1}^{m}\sum_{l=1}^{m}\eta_{ik}\eta_{jl}(\delta W_k^T\delta W_l^T-\delta t)\right). \qquad (1.80)
\end{aligned}$$

According to (1.78) the first term equals zero. However, the third term is redundant which influences the accuracy of the replication strategy. The key factors in this term are the gamma and cross gamma values which are important indicators of the sensitivity of the hedging positions to the change of the forward exchange rates. To alleviate their effects, it would be ideal to implement the delta-gamma hedging so that the gamma and the cross gamma values reduce to zero and at the same time the delta of the whole portfolio remains zero. However, as was pointed out by Ashraff, Tarczon and Wu (1995), it is hardly possible to completely hedge against the correlation since we need hedging instruments which are also functions of the same correlations but the high transaction costs of buying or selling such options usually prohibit this. As a compromise, Ashraff et al. (1995) have introduced the safe crossing strategy, which aims at building delta neutral and minimum variational portfolio over a short period of time by including the underlying assets and the vanilla options on each of these assets. We apply this strategy to our case. The hedging positions are composed of the domestic zero coupon bonds, the forward contracts on the foreign currencies and the European options written on the respective foreign currencies.

We denote ϕ_0 as the number of domestic zero coupon bonds, ϕ_i as the number of T-forward contracts on the i th currency, ξ_i as the number of forward contracts on the call option with the i th currency as the underlying, and C_i as the forward price of the i -th call option.

The discrete time differential of the net portfolio's forward value, denoted by Π is

$$\begin{aligned}
\delta\Pi &= \delta\tilde{V} + \sum_{i=1}^{n}\phi_i\delta F_{X_i} + \sum_{i=1}^{n}\xi_i\delta C_i \\
&= \frac{\partial\tilde{V}}{\partial t}\delta t + \sum_{i=1}^{n}\frac{\partial\tilde{V}}{\partial F_{X_i}}F_{X_i}\eta_i\cdot\delta W^T + \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{2}\frac{\partial^2\tilde{V}}{\partial F_{X_i}\partial F_{X_j}}F_{X_i}F_{X_j}\eta_i\cdot\eta_j\delta t
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{2}\frac{\partial^2\tilde{V}}{\partial F_{X_i}\partial F_{X_j}}F_{X_i}F_{X_j}\sum_{k=1}^{m}\sum_{l=1}^{m}\eta_{ik}\eta_{jl}(\delta W_k^T\delta W_l^T-\delta t)\\
&+ \sum_{i=1}^{n}\phi_i F_{X_i}\eta_i\cdot\delta W^T+\sum_{i=1}^{n}\xi_i\left(\frac{\partial C_i}{\partial t}\delta t+\frac{\partial C_i}{\partial F_{X_i}}F_{X_i}\eta_i\cdot\delta W^T+\frac{1}{2}\frac{\partial^2 C_i}{\partial F_{X_i}}F_{X_i}^2\eta_i^2\delta t\right)\\
&+ \sum_{i=1}^{n}\xi_i\frac{1}{2}\frac{\partial^2 C_i}{\partial F_{X_i}^2}F_{X_i}^2\sum_{j=1}^{m}\sum_{k=1}^{m}\eta_{ij}\eta_{ik}(\delta W_j^T\delta W_k^T-\delta t)\\
&= \left[\frac{\partial\tilde{V}}{\partial t}+\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{2}\frac{\partial^2\tilde{V}}{\partial F_{X_i}\partial F_{X_j}}F_{X_i}F_{X_j}\eta_i\cdot\eta_j+\sum_{i=1}^{n}\xi_i\left(\frac{\partial C_i}{\partial t}+\frac{1}{2}\frac{\partial^2 C_i}{\partial F_{X_i}^2}F_{X_i}^2\eta_i^2\right)\right]\delta t\\
&+ \sum_{i=1}^{n}\left(\frac{\partial\tilde{V}}{\partial F_{X_i}}+\phi_i+\xi_i\frac{\partial C_i}{\partial F_{X_i}}\right)F_{X_i}\eta_i\cdot\delta W^T\\
&+ \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{2}\frac{\partial^2\tilde{V}}{\partial F_{X_i}\partial F_{X_j}}F_{X_i}F_{X_j}\sum_{k=1}^{m}\sum_{l=1}^{m}\eta_{ik}\eta_{jl}(\delta W_k^T\delta W_l^T-\delta t)\\
&+ \sum_{i=1}^{n}\xi_i\frac{1}{2}\frac{\partial^2 C_i}{\partial F_{X_i}^2}F_{X_i}^2\sum_{j=1}^{m}\sum_{k=1}^{m}\eta_{ij}\eta_{ik}(\delta W_j^T\delta W_k^T-\delta t).
\end{aligned}
$$

The portfolio is delta neutral indicating that

$$
\frac{\partial\tilde{V}}{\partial F_{X_i}}+\phi_i+\xi_i\frac{\partial C_i}{\partial F_{X_i}} = 0 \quad for \quad i=1,\ldots,n, \tag{1.81}
$$

which eliminates the terms proportional to δW^T . In addition, because the drift of $\tilde{V}(t,F_X(t,T))$ and C_i , $i=1,\ldots,n$ are 0 under the forward measure, the first term in (1.81) is equal to zero.

Since it is hardly possible to eliminate the variance of the portfolio over the short period δt , we try to minimize it as the second best solution.

Proposition 1.4.2. *The replication portfolio* (ϕ_0,ϕ,ξ) *which minimizes the variance of* $\delta\Pi$ *satisfies*

$$
\begin{aligned}
\xi &= -\Lambda^{-1}\Psi, && (1.82)\\
\phi_i &= -\frac{\partial\tilde{V}}{\partial F_{X_i}}-\xi_i\frac{\partial C_i}{\partial F_{X_i}} \quad for \quad i=1,\ldots,n, && (1.83)\\
\phi_0 &= \tilde{V}(0,F_X(0,T))-\sum_{i=1}^{n}\xi_i\, C_i(0). && (1.84)
\end{aligned}
$$

Here, Λ *is an* $n \times n$ *-matrix whose* (i, r) *-th entry is*

$$\Lambda_{ir} = \frac{1}{2}\frac{\partial^2 C_i}{\partial F_{X_i}^2} F_{X_i}^2 \left(2\sum_{k=1}^{m} \eta_{ik}^2 \eta_{rk}^2 + \sum_{\substack{k,l=1\\k\neq l}}^{m} \eta_{ik}\eta_{il}\eta_{rk}\eta_{rl} \right) \tag{1.85}$$

for $i = 1, \ldots, n$ *and* $r = 1, \ldots, n$ *. Moreover,* Ψ *is an* $n \times 1$ *vector whose* r *-th element is*

$$\Psi_r = \sum_{i,j=1}^{n} \frac{1}{2}\frac{\partial^2 V}{\partial F_{X_i}\partial F_{X_j}} F_{X_i} F_{X_j} \left(2\sum_{k=1}^{m} \eta_{ik}\eta_{jk}\eta_{rk}^2 + \sum_{\substack{k,l=1\\k\neq l}}^{m} \eta_{ik}\eta_{jl}\eta_{rk}\eta_{rl} \right). \tag{1.86}$$

for $r = 1, \ldots, n$ *.*

Proof. The variance of $\delta\Pi$ satisfies

$$\begin{aligned} Var[\delta\Pi] &= 2\sum_{i,j,v,w=1}^{n} \frac{1}{2}\frac{\partial^2 V}{\partial F_{X_i}\partial F_{X_j}} \frac{1}{2}\frac{\partial^2 V}{\partial F_{X_v}\partial F_{X_w}} F_{X_i}F_{X_j}F_{X_v}F_{X_w} \sum_{k=1}^{m} \eta_{ik}\eta_{jk}\eta_{vk}\eta_{wk}\delta t^2 \\ &+ \sum_{i,j,v,w=1}^{n} \frac{1}{2}\frac{\partial^2 V}{\partial F_{X_i}\partial F_{X_j}} \frac{1}{2}\frac{\partial^2 V}{\partial F_{X_v}\partial F_{X_w}} F_{X_i}F_{X_j}F_{X_v}F_{X_w} \sum_{\substack{k,l=1\\k\neq l}}^{m} \eta_{ik}\eta_{jl}\eta_{vk}\eta_{wl}\delta t^2 \\ &+ 2\sum_{i,j=1}^{n} \xi_i\xi_j \frac{1}{2}\frac{\partial^2 C_i}{\partial F_{X_i}^2} \frac{1}{2}\frac{\partial^2 C_j}{\partial F_{X_j}^2} F_{X_i}^2 F_{X_j}^2 \sum_{k=1}^{m} \eta_{ik}^2\eta_{jk}^2\delta t^2 \\ &+ \sum_{i=1}^{n}\sum_{j=1}^{n} \xi_i\xi_j \frac{1}{2}\frac{\partial^2 C_i}{\partial F_{X_i}^2} \frac{1}{2}\frac{\partial^2 C_j}{\partial F_{X_j}^2} F_{X_i}^2 F_{X_j}^2 \sum_{\substack{k,l=1\\k\neq l}}^{m} \eta_{ik}\eta_{il}\eta_{jk}\eta_{jl}\delta t^2 \\ &+ 4\sum_{i,j,v=1}^{n} \frac{1}{2}\frac{\partial^2 V}{\partial F_{X_i}\partial F_{X_j}} F_{X_i}F_{X_j}\xi_v \frac{1}{2}\frac{\partial^2 C_v}{\partial F_{X_v}^2} F_{X_v}^2 \sum_{l=1}^{m} \eta_{il}\eta_{jl}\eta_{vl}^2\delta t^2 \\ &+ 2\sum_{i,j,v=1}^{n} \frac{1}{2}\frac{\partial^2 V}{\partial F_{X_i}\partial F_{X_j}} \xi_v \frac{1}{2}\frac{\partial^2 C_v}{\partial F_{X_v}^2} F_{X_v}^2 \sum_{\substack{k,l=1\\k\neq l}}^{m} \eta_{ik}\eta_{jl}\eta_{vk}\eta_{vl}\delta t^2. \end{aligned} \tag{1.87}$$

To minimize $Var[\delta\Pi]$, we simultaneously calculate

$$\frac{\partial Var[\delta\Pi]}{\partial \xi_r} = 0, \quad for \quad r = 1, \ldots, n$$

to find out the appropriate ξ .

We obtain that

$$\begin{aligned}
\frac{\partial Var[\delta\Pi]}{\partial\xi_r} &= 4\sum_{i=1}^{n}\xi_i\frac{1}{2}\frac{\partial^2 C_i}{\partial F_{X_i}^2}\frac{1}{2}\frac{\partial^2 C_r}{\partial F_{X_r}^2}F_{X_i}^2F_{X_r}^2\sum_{k=1}^{m}\eta_{ik}^2\eta_{rk}^2\delta t^2 \\
&+ 2\sum_{i=1}^{n}\xi_i\frac{1}{2}\frac{\partial^2 C_i}{\partial F_{X_i}^2}\frac{1}{2}\frac{\partial^2 C_r}{\partial F_{X_r}^2}F_{X_i}^2F_{X_r}^2\sum_{\substack{k,l=1\\k\neq l}}^{m}\eta_{ik}\eta_{il}\eta_{rk}\eta_{rl}\delta t^2 \\
&+ 4\sum_{i,j=1}^{n}\frac{1}{2}\frac{\partial^2 V}{\partial F_{X_i}\partial F_{X_j}}F_{X_i}F_{X_j}\frac{1}{2}\frac{\partial^2 C_r}{\partial F_{X_r}^2}F_{X_r}^2\sum_{l=1}^{m}\eta_{il}\eta_{jl}\eta_{rl}^2\delta t^2 \\
&+ 2\sum_{i,j=1}^{n}\frac{1}{2}\frac{\partial^2 V}{\partial F_{X_i}\partial F_{X_j}}F_{X_i}F_{X_j}\frac{1}{2}\frac{\partial^2 C_r}{\partial F_{X_r}^2}F_{X_r}^2\sum_{\substack{k,l=1\\k\neq l}}^{m}\eta_{ik}\eta_{jl}\eta_{rk}\eta_{rl}\delta t^2 \\
&= 0, \qquad (1.88)
\end{aligned}$$

or equivalently,

$$\begin{aligned}
&\sum_{i=1}^{n}\xi_i\frac{1}{2}\frac{\partial^2 C_i}{\partial F_{X_i}^2}F_{X_i}^2(2\sum_{k=1}^{m}\eta_{ik}^2\eta_{rk}^2+\sum_{\substack{k,l=1\\k\neq l}}^{m}\eta_{ik}\eta_{il}\eta_{rk}\eta_{rl}) \\
= &-\sum_{i,j=1}^{n}\frac{1}{2}\frac{\partial^2 V}{\partial F_{X_i}\partial F_{X_j}}F_{X_i}F_{X_j}(2\sum_{k=1}^{m}\eta_{ik}\eta_{jk}\eta_{rk}^2+\sum_{\substack{k,l=1\\k\neq l}}^{m}\eta_{ik}\eta_{jl}\eta_{rk}\eta_{rl}) \qquad (1.89)
\end{aligned}$$

for $r = 1, \ldots, n$.

Denoting Λ_{ir} and Ψ_r as is presented in (1.85) and (1.86), we can write the equation system as

$$\Lambda\xi = -\Psi. \qquad (1.90)$$

Assuming the non-singularity of Λ which is usually the case, we obtain that

$$\xi = -\Lambda^{-1}\Psi.$$

Combining this result with (1.81) we further obtain the value of ϕ_i , i.e.,

$$\phi_i = -\frac{\partial\tilde{V}}{\partial F_{X_i}} - \xi_i\frac{\partial C_i}{\partial F_{X_i}} \quad for \quad i = 1, \ldots, n.$$

At last, ϕ_0 is obtained by investing the rest of the initial money into the domestic zero coupon bonds. □

We compare the performance of the delta hedging and the safe crossing hedging strategies in Section 1.5.

1.4.2 The Static Hedging

The Static Hedging of Cash-or-Nothing Basket Options

We first discuss the static hedging strategy for cash-or-nothing basket options. It is well known that a cash-or-nothing call option can be synthesized using an infinite number of vertical spreads of call options. Denoting $C(t, A(t), K)$ as the time t value of a basket option with strike price K and the same underlying as the cash-or-nothing basket option. The payoff of the cash-or-nothing basket option at T can be replicated by a call spread as follows

$$1_{\{A(T)\geq K\}} = \lim_{h\to 0}\frac{1}{h}(\tilde{V}_b(T, F_X(T,T), K-h) - \tilde{V}_b(T, F_X(T,T), K)). \tag{1.91}$$

We denote $\tilde{V}_b(t, F_X(t,T), K)$ as the T-forward value of a basket option at time t, whose strike price is K and its underlying is the same as the cash-or-nothing basket option.

The call spread displayed in equation (1.91) generally establishes a superhedging position for the cash-or-nothing option. It perfectly hedges the cash-or-nothing option only when an infinite number $\frac{1}{h}$ can be set, which is but impossible. Alternatively, we apply the Richardson extrapolation technique to approximate the right side of equation (1.91) as exactly as possible. The Richardson extrapolation technique was introduced by Geske and Johnson (1984) for the pricing of American put options and applied later by Carr, Ellis and Gupta (1998) for replicating simple cash-or-nothing call options. When the step size h is used, we denote the time t approximation of the T-forward value of the cash-or-nothing option as

$$\tilde{V}(t,h) = \frac{1}{h}(\tilde{V}_b(t, F_X(t,T), K-h) - \tilde{V}_b(t, F_X(t,T), K)).$$

We wish to find $\tilde{V}(t,0)$. Following Geske and Johnson's approach, we assume that $\tilde{V}(t,h)$ takes the form

$$\tilde{V}(t,h) = \tilde{V}(t,0) + a_1h^p + a_2h^q + o(h^s), \tag{1.92}$$

where $s > q > p$. Similarly, we can also write

$$\tilde{V}(t,jh) = \tilde{V}(t,0) + a_1(jh)^p + a_2(jh)^q + o(h^s), \tag{1.93}$$

$$\tilde{V}(t,kh) = \tilde{V}(t,0) + a_1(kh)^p + a_2(kh)^q + o(h^s), \tag{1.94}$$

where $k > j > 1$. By substituting a_1 and a_2 we obtain

$$\tilde{V}(t,0) = \tilde{V}(t,h) + \frac{A}{C}[\tilde{V}(t,h) - \tilde{V}(t,jh)] - \frac{B}{C}[\tilde{V}(t,jh) - \tilde{V}(t,kh)], \tag{1.95}$$

with

$$\begin{aligned} A &= k^q - k^p + j^p - j^q, & (1.96)\\ B &= j^q - j^p, & (1.97)\\ C &= k^q(j^p - 1) - k^p(j^q - 1) + j^q - j^p. & (1.98) \end{aligned}$$

We use $k = 3$, $j = \frac{3}{2}$ and $h = \frac{1}{100}$. If we expand $\tilde{V}(t,h)$ in a Taylor series around $\tilde{V}(t,0)$ and drop the terms of third or higher order, we have $p = 1$ and $q = 2$. It follows that

$$\begin{aligned} \tilde{V}(t) &= \tilde{V}(t,0) \approx 450\,\tilde{V}_b(t, F_X(t,T), K - \frac{1}{100}) - 200\,\tilde{V}_b(t, F_X(t,T), K) \\ &- \frac{800}{3}\,\tilde{V}(t, F_X(t,T), K - \frac{3}{200}) + \frac{50}{3}\,\tilde{V}_b(t, F_X(t,T), K - \frac{3}{100}). \qquad (1.99) \end{aligned}$$

The accuracy of this approximation will be studied in Section 1.5.

In the next subsection we introduce the static hedging of basket options with simple European options. To put it more exactly, we will find a static *superhedging* position by applying the payoff approximation strategy first introduced by Nielsen and Sandmann (2002a) in the pricing of Asian options. Making use of the convex feature of the payoff structure $[\cdot]^+$ and the Jensen's inequality, they find the upper bound of the Asian option price which can be interpreted as a portfolio of plain vanilla options. Su (2006) applies this property to the static superhedging of basket options and introduced different criteria for the choice of the proper strike prices of the vanilla options depending on the risk attitude of the issuer of the financial product. We apply this method to the hedging of basket FX derivatives on the international financial market with more financial risks. We only introduce the basic steps of the upper bound building. To build the hedging positions with the other criteria, readers are referred to Su (2006). The methods introduced there can be implemented in our model without difficulty.

The Static Hedging of Plain Vanilla Basket Options

Now we study the static hedging problem for the plain vanilla basket option at time $t = 0$, with the final payoff being

$$\left[\sum_{i=1}^{n} w_i F_{X_i}(T,T) - K\right]^+$$

under the T -forward measure. Due the convexity of the payoff structure and Jensen's inequality for convex functions, there is

$$\left[\sum_{i=1}^{n} w_i F_{X_i}(T,T) - K\right]^+ = \left[\sum_{i=1}^{n} w_i (F_{X_i}(T,T) - K_i)\right]^+ \leq \sum_{i=1}^{n} w_i [F_{X_i}(T,T) - K_i]^+$$

for all the sequences of $K_1, K_2, \cdots, K_n$ with $\sum_{i=1}^{n} w_i K_i = K$. It tells us that the payoff of a plain vanilla basket call option is dominated by the payoff of a portfolio of n European call options written on the respective underlying assets with the weight of the i -th option being w_i . In addition, the strike price of the i -th option is K_i and the maturity date is T . To build the hedging position with costs as few as possible, we look for the optimal strike price vector $\tilde{K} = [K_1, \ldots, K_n]$ which minimizes the value of the option portfolio. Thus the problem is

$$\min_{\tilde{K}} \sum_{i=1}^{n} w_i \int_{\max(K_i,0)}^{\infty} (F_{X_i}(T,T) - K_i) f_i(F_{X_i}(T,T)) \mathrm{d}F_{X_i}(T,T) \quad s.t. \quad \sum_{i=1}^{n} w_i K_i = K.$$

It can be solved by minimizing the Lagrange function

$$h(\tilde{K}, \lambda) = \sum_{i=1}^{n} w_i \int_{\max(K_i,0)}^{\infty} (F_{X_i}(T,T) - K_i) f_i(F_{X_i}(T,T)) \mathrm{d}F_{X_i}(T,T) + \lambda(\sum_{i=1}^{n} w_i K_i - K).$$

The first order derivative with regard to K_i is[16]

$$\begin{aligned} \frac{\partial h(\tilde{K}, \lambda)}{\partial K_i} &= -w_i \frac{\partial \max(K_i, 0)}{\partial K_i} [max(K_i, 0) - K_i] \, f_i(F_{X_i}(T,T) = \max(K_i, 0)) \\ &- \, w_i \int_{\max(K_i,0)}^{\infty} f_i(F_{X_i}(T,T)) \mathrm{d}F_{X_i}(T,T) + \lambda w_i = 0 \quad \forall \quad i = 1, \ldots, n, \end{aligned} \tag{1.100}$$

and the first order derivative with regard to λ is

$$\frac{\partial h(\tilde{K}, \lambda)}{\partial \lambda} = \sum_{i=1}^{n} w_i K_i - K = 0. \tag{1.101}$$

(1.100) can be simplified into

$$-w_i \int_{\max(K_i,0)}^{\infty} f_i(F_{X_i}(T,T)) \mathrm{d}F_{X_i}(T,T) + \lambda w_i = 0, \quad i = 1, \ldots, n.$$

[16]The general formula for the differentiation of integrals is $\frac{\mathrm{d}f(x)}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x} \int_{a(x)}^{b(x)} f(y,x) \mathrm{d}y = f(b(x),x)b'(x) - f(a(x),x)a'(x) + \int_{a(x)}^{b(x)} \frac{\partial f}{\partial x}(y,x) \mathrm{d}x$. Here, $x = K_i$ and $y = F_{X_i}(T,T)$.

This confirms that $\tilde{K} > 0$, because otherwise there would be at least one K_i^* with $K_i^* \leq 0$ so that

$$\begin{aligned}\int_{\max(K_j,0)}^{\infty} f_j(F_{X_j}(T,T))\mathrm{d}F_{X_j}(T,T) &= \int_{\max(K_i^*,0)}^{\infty} f_i(F_{X_i}(T,T))\mathrm{d}F_{X_i}(T,T)\\ &= \int_0^{\infty} f_i(F_{X_i}(T,T))\mathrm{d}F_{X_i}(T,T) = 1 \quad \forall\ j \neq i^*,\end{aligned}$$

and therefore, $K_j \leq 0 \quad \forall j \neq i^*$, which but violates the condition that $\sum_{i=1}^n w_i K_i = K$.

Since $\tilde{K} > 0$, we have

$$\int_{K_i}^{+\infty} f_i(F_{X_i}(T,T))\mathrm{d}F_{X_i}(T,T) \;=\; \int_{K_j}^{+\infty} f_j(F_{X_j}(T,T))\mathrm{d}F_{X_j}(T,T). \qquad (1.102)$$

Let

$$y_i \;=\; \frac{lnF_{X_i}(T,T) - \mathbb{E}^{\mathbb{P}^T}[lnF_{X_i}(T,T)]}{\sqrt{Var^{\mathbb{P}^T}[lnF_{X_i}(T,T)]}} = \frac{lnF_{X_i}(T,T) - lnF_{X_i}(0,T) - \frac{1}{2}\bar{\sigma}_i^2}{\bar{\sigma}_i},$$

which is standard normally distributed.[17] Then there is

$$\Phi\left(\frac{lnF_{X_i}(0,T) + \frac{1}{2}\bar{\sigma}_i^2 - lnK_i}{\bar{\sigma}_i}\right) \;=\; \Phi\left(\frac{lnF_{X_j}(0,T) + \frac{1}{2}\bar{\sigma}_j^2 - lnK_j}{\bar{\sigma}_j}\right)$$
$$\forall i,j \in \{1,2,\ldots,n\}, \quad \sum_{i=1}^n w_i K_i = K.$$

Since $\Phi(\cdot)$ is a bijective function, we infer that

$$\frac{lnF_{X_i}(0,T) + \frac{1}{2}\bar{\sigma}_i^2 - lnK_i}{\bar{\sigma}_i} \;=\; \frac{lnF_{X_j}(0,T) + \frac{1}{2}\bar{\sigma}_j^2 - lnK_j}{\bar{\sigma}_j}$$
$$\forall i,j \in \{1,2,\ldots,n\}, \quad \sum_{i=1}^n w_i K_i = K.$$

We can express K_i as a strictly increasing function of K_1:

$$K_i = \frac{F_{X_i}(0,T)}{F_{X_1}(0,T)^{\bar{\sigma}_i/\bar{\sigma}_1}} e^{\frac{1}{2}\bar{\sigma}_i(\bar{\sigma}_i - \bar{\sigma}_1)} K_1^{\bar{\sigma}_i/\bar{\sigma}_1}$$

[17] Since we are dealing with the static hedging problem at time $t = 0$, $\bar{\sigma}_i$ is equal to $\int_0^T ||\eta_i(0,T)||^2 \mathrm{d}u$ here.

and hence $\sum_{i=1}^{n} w_i K_i$ as a function of K_1 :

$$k(K_1) = \sum_{i=1}^{n} w_i \frac{F_{X_i}(0,T)}{F_{X_1}(0,T)^{\bar{\sigma}_i/\bar{\sigma}_1}} e^{\frac{1}{2}\bar{\sigma}_i(\bar{\sigma}_i-\bar{\sigma}_1)} K_1^{\bar{\sigma}_i/\bar{\sigma}_1}, \tag{1.103}$$

which is continuously increasing in K_1 .

It is obvious that $k(0) = 0$. Furthermore, we set the index 1 to be the foreign currency with the lowest terminal volatility. Let

$$K_1^* = \frac{F_{X_1}(0,T)D_0(0,T)K}{\min(D_1(0,T),\ldots,D_n(0,T))}.$$

It is easy to prove that

$$\begin{aligned}
k(K_1^*) &= \sum_{i=1}^{n} w_i \frac{F_{X_i}(0,T)}{F_{X_1}(0,T)^{\bar{\sigma}_i/\bar{\sigma}_1}} e^{\frac{1}{2}\bar{\sigma}_i(\bar{\sigma}_i-\bar{\sigma}_1)} \left(\frac{F_{X_1}(0,T)D_0(0,T)K}{\min(D_1(0,T),\ldots,D_n(0,T))}\right)^{\bar{\sigma}_i/\bar{\sigma}_1} \\
&\geq \sum_{i=1}^{n} w_i \frac{X_i(0)D_i(0,T)}{D_0(0,T)F_{X_1}(0,T)^{\bar{\sigma}_i/\bar{\sigma}_1}} e^{\frac{1}{2}\bar{\sigma}_i(\bar{\sigma}_i-\bar{\sigma}_1)} \left(\frac{F_{X_1}(0,T)D_0(0,T)^{\bar{\sigma}_1/\bar{\sigma}_i}K^{\bar{\sigma}_1/\bar{\sigma}_i}}{\min(D_1(0,T),\ldots,D_n(0,T))^{\bar{\sigma}_i/\bar{\sigma}_1}}\right)^{\bar{\sigma}_i/\bar{\sigma}_1} \\
&\geq \sum_{i=1}^{n} w_i \frac{X_i(0)D_i(0,T)}{D_0(0,T)F_{X_1}(0,T)^{\bar{\sigma}_i/\bar{\sigma}_1}} e^{\frac{1}{2}\bar{\sigma}_i(\bar{\sigma}_i-\bar{\sigma}_1)} \left(\frac{F_{X_1}(0,T)^{\bar{\sigma}_i/\bar{\sigma}_1}D_0(0,T)K}{D_i(0,T)}\right) \\
&= \sum_{i=1}^{n} w_i X_i(0) e^{\frac{1}{2}\bar{\sigma}_i(\bar{\sigma}_i-\bar{\sigma}_1)} K \\
&\geq \sum_{i=1}^{n} w_i X_i(0) K = K.
\end{aligned}$$

The last equality is valid due to the common practice we have mentioned in Section 1.1.2. Hence, by applying the Intermediate Value Theorem, there must exist a $K_1 \in (0, K_1^*]$ such that the equation (1.103) is satisfied. Consequently, we can obtain a portfolio of weighted European call options written on the respective foreign currencies with the strike price of the i -th option being K_i as was specified above and its weight in the portfolio being w_i . In Section 1.5 we will explore how well the hedging portfolio works.

1.5 Numerical Results

In Section 1.3 we have presented the rank one approximation method in combination with the three moment matching technique. We have applied this approximation method to the pricing of cash-or-nothing basket options as well as plain vanilla basket options. In this part, we study their performances by comparing the numerical results with the Monte Carlo simulation results. Each simulation is carried out for $200,000$ times, and the antithetic technique is applied to improve its reliability. At the same time, we compare the rank one approximation with one of the popular approximation methods, the lognormal approximation method with three moment matching,[18] whose approximation quality is well approved.

The parameters used in this part are as follows:

- We assume that 5 currencies are included in the basket.

- The volatilities of the forward interest rates are assumed to be constant $(\sigma_0, \ldots, \sigma_5)'$:

$$\begin{pmatrix} \sigma_0 \\ \sigma_1 \\ \vdots \\ \sigma_5 \end{pmatrix} = \begin{pmatrix} 0.01 & 0 & 0 & 0 & 0 & 0 \\ 0.02 & 0.05 & 0.01 & 0.03 & 0.04 & 0.02 \\ 0.05 & 0.02 & 0.03 & 0.04 & 0.06 & 0.07 \\ 0.01 & 0.04 & 0.02 & 0.06 & 0.02 & 0.06 \\ 0.1 & 0.02 & 0.01 & 0.03 & 0.05 & 0.08 \\ 0.01 & 0.03 & 0.04 & 0.05 & 0.02 & 0.05 \end{pmatrix}.$$

- The initial values of the zero coupon bonds are:

$$(D_0(0,T), D_1(0,T), \ldots, D_5(0,T)) = (e^{-y_0 \cdot T}, e^{-y_1 \cdot T}, \ldots, e^{-y_5 \cdot T}),$$

  where $(y_0, y_1, \ldots, y_5) = (0.06, 0.09, 0.10, 0.12, 0.08, 0.07)$, indicating a flat initial yield curve.

- Concerning the volatilities of the exchange rates $(\delta_1, \ldots, \delta_5)$, three groups are dis-

[18] Confer Brigo, Mercurio, Rapisarda and Scotti (2004) for the lognormal approximation method with three moment matching.

cussed. They are respectively

$$\Delta_1 = \begin{pmatrix} 0.05 & 0.08 & 0.02 & 0.01 & 0.03 & 0.02 \\ 0.04 & 0.09 & 0.03 & 0.01 & 0.02 & 0.03 \\ 0.15 & 0.02 & 0.1 & 0.04 & 0.05 & 0.04 \\ 0.02 & 0.01 & 0.02 & 0.03 & 0.05 & 0.03 \\ 0.03 & 0.01 & 0.05 & 0.06 & 0.07 & 0.15 \end{pmatrix};$$

$$\Delta_2 = \begin{pmatrix} 0.05 & 0.08 & 0.02 & -0.01 & -0.03 & 0.02 \\ 0.04 & -0.09 & -0.03 & -0.01 & 0.02 & -0.03 \\ -0.15 & 0.02 & -0.1 & -0.1 & 0.08 & -0.04 \\ -0.02 & -0.01 & 0.02 & -0.01 & -0.05 & -0.03 \\ -0.03 & -0.04 & -0.05 & 0.03 & -0.01 & 0.15 \end{pmatrix};$$

$$\Delta_3 = \begin{pmatrix} 0.05 & 0.08 & 0.02 & -0.01 & -0.03 & 0.02 \\ 0.04 & 0.09 & 0.03 & 0.01 & 0.02 & -0.03 \\ -0.15 & 0.02 & -0.1 & 0.04 & 0.05 & 0.04 \\ 0.02 & 0.01 & 0.02 & -0.03 & 0.05 & 0.03 \\ 0.03 & -0.01 & 0.05 & 0.06 & -0.07 & 0.15 \end{pmatrix}.$$

In the first group, the exchange rates are positively related with each other; the exchange rates of the second group are negatively related; while in the third group, the signs of the correlation coefficients are mixed. The correlation matrices are respectively

$$\rho_1 = \begin{pmatrix} 1 & 0.9796 & 0.6790 & 0.6167 & 0.4684 \\ 0.9796 & 1 & 0.6226 & 0.5697 & 0.4964 \\ 0.6790 & 0.6226 & 1 & 0.7129 & 0.5919 \\ 0.6167 & 0.5697 & 0.7129 & 1 & 0.8586 \\ 0.4684 & 0.4964 & 0.5919 & 0.8586 & 1 \end{pmatrix};$$

$$\rho_2 = \begin{pmatrix} 1 & -0.6089 & -0.4328 & -0.0583 & -0.1546 \\ -0.6089 & 1 & -0.0405 & -0.0688 & -0.0595 \\ -0.4328 & -0.0405 & 1 & -0.0688 & -0.0289 \\ -0.0583 & -0.0688 & -0.0688 & 1 & -0.3840 \\ -0.1546 & -0.0595 & -0.0289 & -0.3840 & 1 \end{pmatrix};$$

$$\rho_3 = \begin{pmatrix} 1 & 0.7501 & -0.4429 & 0.2145 & 0.3227 \\ 0.7501 & 1 & -0.3252 & 0.2658 & -0.1720 \\ -0.4429 & -0.3252 & 1 & -0.1623 & -0.1315 \\ 0.2145 & 0.2658 & -0.1623 & 1 & 0.0523 \\ 0.3227 & -0.1720 & -0.1315 & 0.0523 & 1 \end{pmatrix}.$$

- The option maturities to be observed are $T = 1, 3, 5$ years respectively.

1.5.1 The Performance of the Approximation Pricing

In Table 1.2 we present the approximation prices of cash-or-nothing basket options through the crude rank one approximation, the improved rank one approximation, and the lognormal approximation. We present the relative deviation of these approximation prices from the true prices to demonstrate the quality of the approximation methods.

We see that the crude rank one approximation delivers mostly the low biased prices and the biasness increases with the strike price. The movements of the biasness with the life time of the option are different for different correlation structures and different strike prices. For the correlation structure ρ_1 , the biasness increases for in-the-money options when the option has a longer life time. For at-the-money and nearly out-of-the-money options, the biasness first increases and then decreases with the option's life time. When the option is far out of the money, the biasness decreases with the option's life time. For the correlation structure ρ_2 , the behavior of the biasness is similar for the in-the-money case. However, for at-the-money and out-of-the-money options, we do not observe monotonic performance. The biasness decreases first and then increases with the increase of the option's life time. With regard to the correlation structure ρ_3 , the biasness (with the increase of the option's life time) increases for in-the-money options, decreases and then increases for at-the-money options, decreases for out-of-the-money options. The relative deviation of the crude rank one approximated prices from the true prices (obtained from the Monte Carlo simulation) ranges from 0 to 100% . In this respect, the performance of the crude rank one approximation is not stable.

Table 1.2 shows that the crude rank one approximation method can be improved by the three moment matching technique. The improvement of the approximation performance is dramatic. In most of the cases, the relative deviations of approximated prices from the true prices have not exceeded 1% . High biasness only happens for far-out-of-the-money options close to maturity, when the correlation matrix of the underlying assets displays the correlation structure ρ_3 . The relative deviation amounts to 30% maximally. Moreover, we find that the improved approximation method performs better than the lognormal approximation method in almost all the cases. The absolute relative deviations of the lognormal approximated prices from the true prices range approximated from 0 to 43% . This indicates that to approximate the underlying portfolio with a synthesized lognormal distributed portfolio is not necessarily better than to approximate it with a portfolio of perfectly correlated synthesized assets.

			Monte Carlo	SE (10^{-3})	CrudeRankOne	Rel.Dev.(%)	ImprovedRankOne	Rel.Dev.(%)	Lognormal	Rel.Dev.(%)
		0.70	0.941762	0.002354	0.941764	0.000172	0.941762	-0.000010	0.941753	-0.001026
		0.80	0.936997	0.105680	0.938508	0.161240	0.937062	0.006901	0.935410	-0.169354
		0.90	0.776395	0.566551	0.780246	0.495988	0.776192	-0.026199	0.775494	-0.116144
	Δ_1	1.00	0.300074	0.693821	0.274275	-8.597527	0.300328	0.084407	0.305361	1.761682
		1.05	0.127651	0.509712	0.104137	-18.420601	0.127683	0.024500	0.128072	0.329500
		1.10	0.043020	0.310901	0.029954	-30.370963	0.042863	-0.363644	0.040910	-4.904552
		1.15	0.011685	0.164833	0.006750	-42.232286	0.011690	0.042211	0.010122	-13.371753
		0.70	0.941762	0.002354	0.941765	0.000250	0.941764	0.000184	0.941739	-0.002471
		0.80	0.936455	0.111488	0.941477	0.536253	0.937018	0.0601253	0.933317	-0.335135
		0.90	0.767032	0.578847	0.792537	3.325126	0.765866	-0.152032	0.765334	-0.221295
	Δ_2	1.00	0.301303	0.694574	0.233849	-22.387541	0.301284	-0.006485	0.310624	3.093293
T=1		1.05	0.134317	0.520706	0.080736	-39.891520	0.135699	1.028831	0.136824	1.866668
		1.10	0.049914	0.333599	0.022258	-55.407145	0.050370	0.915270	0.047026	-5.785475
		1.15	0.015923	0.191977	0.005179	-67.476700	0.015858	-0.405121	0.012807	-19.567647
		0.70	0.941765	0.000000	0.941765	0.000000	0.941765	0.000000	0.941765	0.000000
		0.80	0.941753	0.005265	0.941765	0.001250	0.941759	0.000641	0.941753	0.000013
		0.90	0.891778	0.333830	0.941765	5.605263	0.892337	0.062658	0.891237	-0.060682
	Δ_3	1.00	0.219064	0.629122	0.005563	-97.460505	0.218395	-0.305144	0.219076	0.005333
		1.05	0.033602	0.276207	0.000015	-99.956637	0.033415	-0.556305	0.032612	-2.947147
		1.10	0.002300	0.073502	0.000000	-99.998748	0.002232	-2.980107	0.002032	-11.656270
		1.15	0.000099	0.015258	0.000000	-99.999951	0.000069	-29.935138	0.000056	-42.975492
		0.70	0.782268	0.321954	0.775455	-0.870973	0.783308	0.132916	0.778541	-0.476498
		0.80	0.632456	0.566284	0.601751	-4.854933	0.632517	0.009603	0.632715	0.040949
		0.90	0.408345	0.660176	0.360359	-11.751322	0.408139	-0.050513	0.414163	1.424894
	Δ_1	1.00	0.210922	0.573779	0.168391	-20.164260	0.210658	-0.125538	0.215131	1.995451
		1.05	0.141161	0.494927	0.106392	-24.630711	0.140689	-0.333908	0.143003	1.304900
		1.10	0.090662	0.410816	0.064326	-29.049104	0.090129	-0.587845	0.090537	-0.138757
		1.15	0.056631	0.332022	0.037440	-33.887979	0.055670	-1.696792	0.054826	-3.187329
		0.70	0.632659	0.566092	0.604729	-4.414719	0.631522	-0.179739	0.625803	-1.083707
		0.80	0.498556	0.647825	0.464790	-6.772750	0.496891	-0.334073	0.502977	0.886755
		0.90	0.368530	0.655759	0.335624	-8.928833	0.368068	-0.125283	0.380997	3.383028
	Δ_2	1.00	0.260473	0.611799	0.231399	-11.161913	0.260671	0.076203	0.274449	5.365909
		1.05	0.216492	0.578706	0.189652	-12.397563	0.216609	0.054015	0.229244	5.890272
T=3		1.10	0.178211	0.541053	0.154386	-13.369176	0.178811	0.336488	0.189728	6.462650
		1.15	0.145859	0.501390	0.124977	-14.316449	0.146809	0.651167	0.155738	6.773168
		0.70	0.720078	0.455377	0.700597	-2.705441	0.719456	-0.086419	0.714184	-0.818554
		0.80	0.568232	0.615913	0.533456	-6.120131	0.566851	-0.243138	0.568998	0.134811
		0.90	0.392514	0.659143	0.353709	-9.886437	0.392361	-0.039084	0.399806	1.857639
	Δ_3	1.00	0.241961	0.599078	0.208816	-13.698455	0.242668	0.292182	0.249633	3.170698
		1.05	0.183427	0.546731	0.154892	-15.556997	0.184278	0.463443	0.189630	3.381698
		1.10	0.136116	0.487765	0.112694	-17.206971	0.137227	0.816571	0.140727	3.387960
		1.15	0.099445	0.427710	0.080647	-18.903053	0.100471	1.031182	0.102249	2.819682
		0.70	0.444743	0.573754	0.401195	-9.791637	0.440453	-0.964633	0.444609	-0.029996
		0.80	0.348722	0.584664	0.303732	-12.901338	0.344653	-1.166744	0.358644	2.845291
		0.90	0.267367	0.562550	0.223932	-16.245512	0.263902	-1.295935	0.281918	5.442400
	Δ_1	1.00	0.201527	0.521252	0.162222	-19.503659	0.199384	-1.063366	0.217163	7.758912
		1.05	0.174335	0.496885	0.137454	-21.155209	0.172763	-0.901863	0.189436	8.662385
		1.10	0.150423	0.471193	0.116222	-22.736872	0.149496	-0.616469	0.164690	9.484331
		1.15	0.129712	0.445162	0.098113	-24.360519	0.129247	-0.358083	0.142754	10.054497
		0.70	0.363881	0.585577	0.328959	-9.597032	0.361137	-0.753975	0.366423	0.698563
		0.80	0.299487	0.574832	0.266186	-11.119341	0.297025	-0.822127	0.312258	4.264275
		0.90	0.246033	0.551665	0.215318	-12.484005	0.244228	-0.733766	0.264986	7.703521
	Δ_2	1.00	0.202482	0.522024	0.174513	-13.813119	0.201211	-0.627625	0.224303	10.776838
		1.05	0.183641	0.505769	0.157295	-14.346676	0.182847	-0.432696	0.206254	12.313522
T=5		1.10	0.167016	0.489474	0.141909	-15.032272	0.166312	-0.421504	0.189619	13.533850
		1.15	0.151851	0.472851	0.128160	-15.601700	0.151422	-0.282248	0.174309	14.789553
		0.70	0.405070	0.583098	0.371063	-8.395453	0.402742	-0.574714	0.407445	0.586266
		0.80	0.327516	0.581728	0.293429	-10.407563	0.324560	-0.902336	0.337466	3.037998
		0.90	0.261820	0.559936	0.229671	-12.278977	0.259023	-1.068255	0.275899	5.377405
	Δ_3	1.00	0.208157	0.526491	0.178792	-14.107042	0.205681	-1.189294	0.223420	7.332510
		1.05	0.185066	0.507077	0.157597	-14.842820	0.183126	-1.048258	0.200506	8.343051
		1.10	0.164817	0.487173	0.138882	-15.735493	0.163016	-1.093130	0.179684	9.019993
		1.15	0.146830	0.466946	0.122393	-16.643185	0.145123	-1.162775	0.160832	9.536228

Table 1.2: The true prices and approximated prices of cash-or-nothing basket options, the relative deviation of the approximated prices from the true prices.

We present the pricing results for plain vanilla basket options in Table 1.3. Similar to our observations from Table 1.2, the crude rank one approximation delivers mostly low biased prices. The relative deviations from the true prices range from 0 to 100%. Ceteris paribus, the accuracy of the crude rank one approximation decreases with the increase of the strike price. Now we investigate the effects of the options' life time on the performance of the crude rank one approximation. The effects are different when the correlation structures of the underlying currencies are different and when the strike prices are different. With regard to the correlation structure ρ_1, the approximation accuracy decreases with the options' life times for in-the-money options and increases with the options' life times in the other cases. For the correlation structure ρ_2, the accuracy decreases with T for the far-in-the-money options ($K = 0.7, 0.8$) and increases with T for in-the-money ($K = 0.9$), at-the-money ($K = 1.0$) and out-of-the-money ($K = 1.05, 1.10, 1.15$) options. With regard to the correlation structure ρ_3, the accuracy decreases with T for $K = 0.7, 0.8$, decreases and then increases with T for $K = 0.9$, increases with T for $K = 1.0, 1.05, 1.10, 1, 15$.

The improved rank one approximation works better than the crude rank one approximation. Its relative deviations from the true prices are mostly below 2%. Very high biasness (close to 40%) is only observed for out-of-the money options close to maturity ($T = 1$), with the correlation structure of the underlying exchange rates being ρ_3. However, we do not observe monotonic behavior of the performance of the improved rank one approximation with respect to the change of the strike prices and the change of maturity dates. The performance of the improved rank one approximation also dominates the lognormal approximation. The lognormal approximation yields prices whose relatives deviations from the true prices range from 0 to 53%.

			Monte Carlo	SE (10^{-3})	CrudeRankOne	Rel.Dev.(%)	ImprovedRankOne	Rel.Dev.(%)	Lognormal	Rel.Dev.(%)
		0.70	0.253006	0.109699	0.253005	-0.000618	0.253005	-0.000611	0.253005	-0.000565
		0.80	0.158911	0.109379	0.158870	-0.025714	0.158909	-0.001178	0.158952	0.026140
		0.90	0.070294	0.096843	0.069261	-1.469553	0.070284	-0.014089	0.070605	0.442378
	Δ_1	1.00	0.015857	0.052432	0.014160	-10.704397	0.015868	0.069603	0.015773	-0.529316
		1.05	0.005567	0.030778	0.004482	-19.502522	0.005566	-0.015462	0.005335	-4.173911
		1.10	0.001610	0.015994	0.001127	-30.019350	0.001609	-0.0234580	0.001432	-11.029780
		1.15	0.000389	0.007529	0.000230	-40.920530	0.000391	0.458983	0.000308	-20.843160
		0.70	0.252997	0.114248	0.253005	0.002993	0.253005	0.002995	0.253005	0.003119
		0.80	0.158914	0.113893	0.158829	-0.053502	0.158904	-0.006520	0.159005	0.057137
		0.90	0.070706	0.100908	0.067387	-4.693968	0.070689	-0.023683	0.071329	0.880308
	Δ_2	1.00	0.016967	0.056592	0.012319	-27.395195	0.017042	0.444806	0.016868	-0.582717
T=1		1.05	0.006453	0.035002	0.003754	-41.826792	0.006483	0.468777	0.006045	-6.313227
		1.10	0.002126	0.019759	0.000956	-55.013936	0.002114	-0.564901	0.001757	-17.352634
		1.15	0.000624	0.010483	0.000212	-66.040726	0.000602	-3.465430	0.000418	-33.075741
		0.70	0.253007	0.065062	0.253005	-0.000749	0.253005	-0.000749	0.253005	-0.000749
		0.80	0.158830	0.065061	0.158828	-0.001270	0.158828	-0.001245	0.158828	-0.001213
		0.90	0.065477	0.062521	0.064652	-1.259903	0.065458	-0.028226	0.065504	0.042232
	Δ_3	1.00	0.005944	0.023765	0.000076	-98.729123	0.005917	-0.459113	0.005869	-1.269912
		1.05	0.000658	0.007400	0.000000	-99.970379	0.000646	-1.826516	0.000617	-6.226847
		1.10	0.00004	0.001649	0.000000	-99.998966	0.000034	-9.742040	0.000030	-20.15601142
		1.15	0.000001	0.000329	0.000000	-99.999954	0.000000	-39.710980	0.000000	-52.529674
		0.70	0.177615	0.193444	0.176816	-0.449582	0.177447	-0.094658	0.177926	0.175115
		0.80	0.105939	0.171885	0.103877	-1.947183	0.105715	-0.211849	0.106487	0.516561
		0.90	0.053692	0.133594	0.050944	-5.117696	0.053491	-0.374099	0.053902	0.390707
	Δ_1	1.00	0.023304	0.091151	0.020961	-10.055579	0.023111	-0.828515	0.022944	-1.546235
		1.05	0.014577	0.072393	0.012686	-12.975444	0.014413	-1.128197	0.014076	-3.442252
		1.10	0.008864	0.056320	0.007424	-16.237512	0.008717	-1.649073	0.008314	-6.196835
		1.15	0.005240	0.043089	0.004218	-19.496742	0.005131	-2.076657	0.004742	-9.503641
		0.70	0.195825	0.347094	0.195428	-0.202458	0.195907	0.042204	0.200004	2.133938
		0.80	0.139201	0.313702	0.138565	-0.456928	0.139435	0.167775	0.143491	3.081734
		0.90	0.095971	0.274172	0.095140	-0.865475	0.096321	0.364905	0.099373	3.544875
	Δ_2	1.00	0.064734	0.233536	0.063761	-1.502829	0.065094	0.555600	0.066767	3.141296
		1.05	0.052830	0.214012	0.051841	-1.872110	0.053188	0.677924	0.054199	2.590440
$T = 3$		1.10	0.042990	0.195416	0.042001	-2.301552	0.043328	0.786423	0.043748	1.763076
		1.15	0.034907	0.177923	0.033930	-2.798010	0.035211	0.869680	0.035134	0.649393
		0.70	0.183293	0.252785	0.182212	-0.590000	0.183043	-0.136812	0.184273	0.534391
		0.80	0.118582	0.224122	0.116894	-1.422922	0.118318	-0.222735	0.119723	0.962412
		0.90	0.070605	0.184469	0.068758	-2.617145	0.070401	-0.289262	0.071280	0.956175
	Δ_3	1.00	0.039133	0.142373	0.037526	-4.106265	0.038983	-0.381764	0.039100	-0.082518
		1.05	0.028467	0.122740	0.027081	-4.870470	0.028358	-0.384666	0.028165	-1.062013
		1.10	0.020445	0.104788	0.019289	-5.657390	0.020366	-0.387613	0.019952	-2.413764
		1.15	0.014523	0.088766	0.013583	-6.472923	0.014463	-0.409063	0.013918	-4.162549
		0.70	0.163282	0.388559	0.159837	-2.109652	0.162252	-0.630947	0.170571	4.464381
		0.80	0.123705	0.355694	0.119421	-3.462936	0.123104	-0.485132	0.130466	5.465758
		0.90	0.093031	0.321324	0.088186	-5.208522	0.092813	-0.234713	0.098531	5.911152
	Δ_1	1.00	0.069703	0.287757	0.064627	-7.281531	0.069780	0.110692	0.073681	5.706768
		1.05	0.060317	0.271727	0.055225	-8.442681	0.060491	0.288462	0.063528	5.324086
		1.10	0.052216	0.256344	0.047153	-9.695756	0.052448	0.443436	0.054687	4.732254
		1.15	0.045224	0.241677	0.040242	-11.016812	0.045491	0.589600	0.047012	3.953673
		0.70	0.202659	0.626442	0.202671	0.005948	0.203780	0.553477	0.226339	11.684601
		0.80	0.169596	0.595361	0.169218	-0.222925	0.170972	0.811264	0.192462	13.482812
		0.90	0.142420	0.563943	0.141691	-0.511993	0.143997	1.107424	0.163657	14.911058
	Δ_2	1.00	0.120078	0.533115	0.119058	-0.849743	0.121800	1.433770	0.139245	15.961699
		1.05	0.110432	0.518111	0.109291	-1.033071	0.112207	1.606706	0.128487	16.349200
$T = 5$		1.10	0.101675	0.503442	0.100425	-1.229540	0.103485	1.780008	0.118596	16.642137
		1.15	0.093710	0.489145	0.092370	-1.430510	0.095548	1.960883	0.109503	16.852556
		0.70	0.182913	0.486367	0.180424	-1.361177	0.181438	-0.806372	0.192717	5.359402
		0.80	0.146364	0.453758	0.143597	-1.890939	0.145175	-0.812256	0.155534	6.265310
		0.90	0.116983	0.420322	0.114062	-2.496620	0.116102	-0.752844	0.124941	6.802870
	Δ_3	1.00	0.093569	0.387510	0.090598	-3.175989	0.092963	-0.647826	0.100051	6.926681
		1.05	0.083737	0.371646	0.080776	-3.535928	0.083254	-0.577529	0.089461	6.835809
		1.10	0.074983	0.356248	0.072052	-3.908320	0.074610	-0.497707	0.079965	6.644287
		1.15	0.067188	0.341367	0.064306	-4.288957	0.066915	-0.406373	0.071460	6.358230

Table 1.3: The true prices and approximated prices of plain vanilla basket options, the relative deviation of the approximated prices from the true prices.

1.5.2 The Hedging Performance

The Dynamic Hedging Performance

In this section we present the performance of the delta hedging and safe crossing hedging strategies which are induced from our approximation prices. We first study the performance of the dynamic hedging strategies for the plain vanilla basket options. In Figures 1.3-1.5 we have presented the delta as well as safe crossing hedging scenarios of the basket options with the underlying exchange rates being positively correlated, negatively correlated and mixed correlated respectively. The life time of the option is one year and the strike price is set to be 0.8. Moreover, the hedging portfolio is balanced daily.

We see from the figures that both the delta and the safe crossing hedging strategies implemented within our framework work very well for $T = 1$ and $K = 0.8$. The risk exposure is almost eliminated from the market. The advantage of the safe crossing hedging over the delta hedging strategy is not very obvious in this case. This is because, for in-the-money options, the gammas and the cross gammas are small, so that a daily adjustment of the delta hedging portfolio is good enough to replicate the basket option price. However, when the life time of the option is extended or the option is almost at-the-money, we will see the difference in the performances of the two hedging strategies.

The hedging scenarios for $T = 3$ and $K = 0.8$ are shown in Figures 1.6-1.8. Since a longer time to maturity indicates higher gammas and cross gammas when the option is in the money, the delta hedging strategy works badly for not being able to react to the rapid change of the hedge ratios quickly enough. The safe crossing hedging strategy takes the gammas and the cross gammas into account. Although this hedging strategy also incurs high hedging errors for not being able to perfectly neutralize the gammas and cross gammas, the hedging errors are much smaller than those of the delta hedging strategy.

The hedging scenarios for $T = 1$ and $K = 1$ are presented in Figures 1.9-1.11. The hedging errors are high, because the gammas and cross gammas have very high gamma and cross gamma values. For $T = 1$ and $K = 1.1$, the gammas and cross gammas are low again. Hence, the hedging errors are relatively low, which are displayed in Figures 1.12-1.14.[19] In all the cases, the safe crossing hedging dominates the delta hedging with regard to its hedging performance.

[19] The hedging errors look graphically great since the scales used in Figures 1.12-1.14 are different with the other figures.

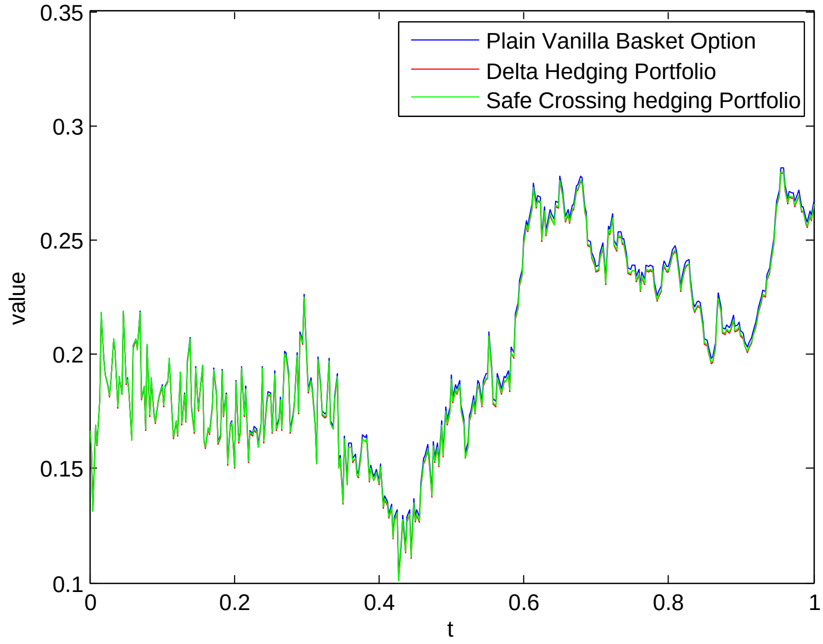


Figure 1.3: Delta and safe crossing hedging scenarios for in-the-money plain vanilla basket option of group 1 with $T = 1$ and $K = 0.8$.

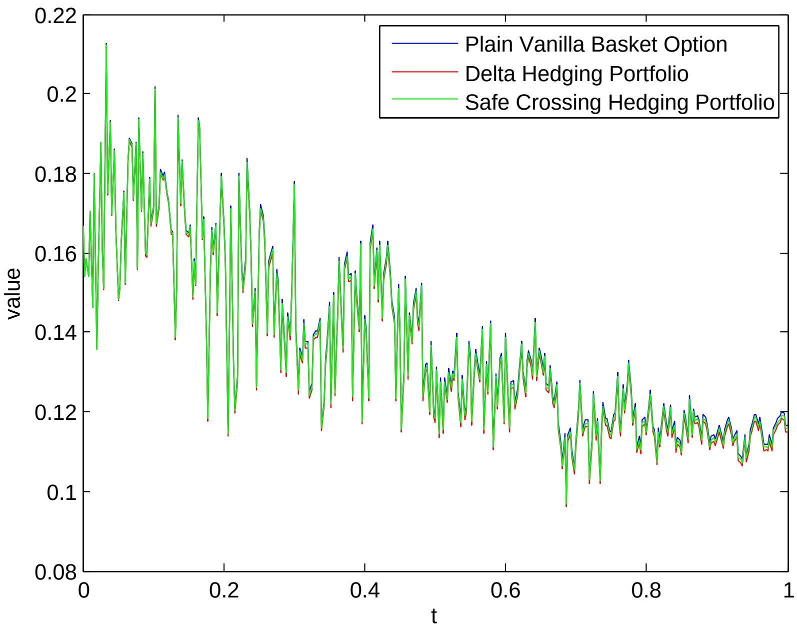


Figure 1.4: Delta and safe crossing hedging scenarios for in-the-money plain vanilla basket option of group 2 with $T = 1$ and $K = 0.8$.

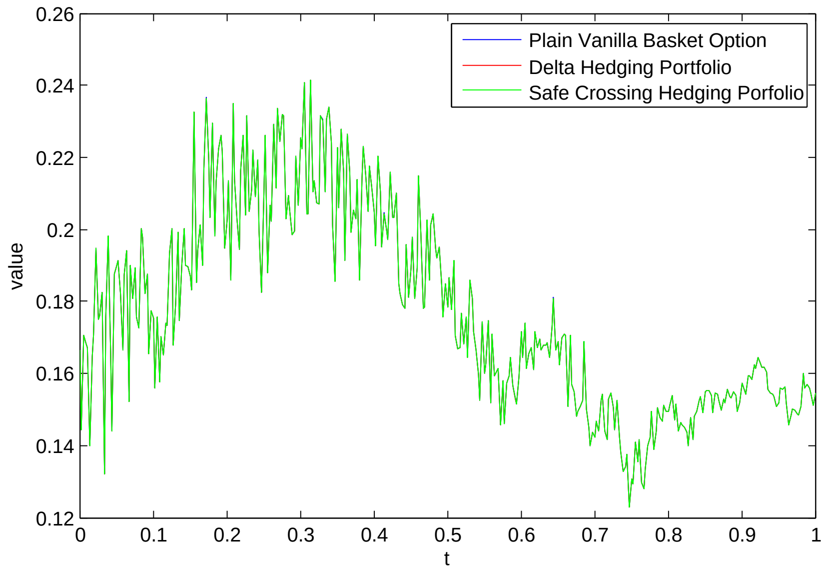


Figure 1.5: Delta and safe crossing hedging scenarios for in-the-money plain vanilla basket option of group 3 with $T = 1$ and $K = 0.8$.

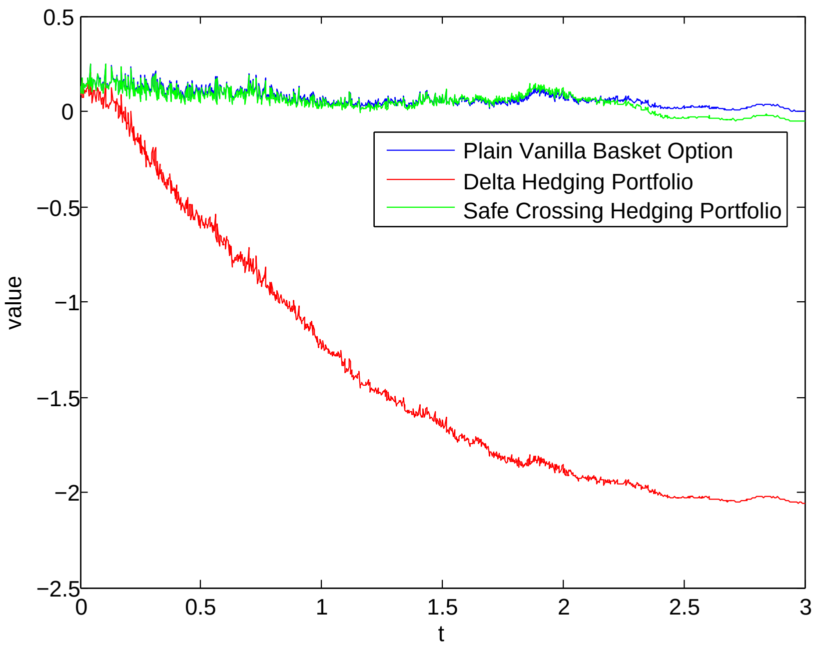


Figure 1.6: Delta and safe crossing hedging scenarios for in-the-money plain vanilla basket option of group 1 with $T = 3$ and $K = 0.8$.

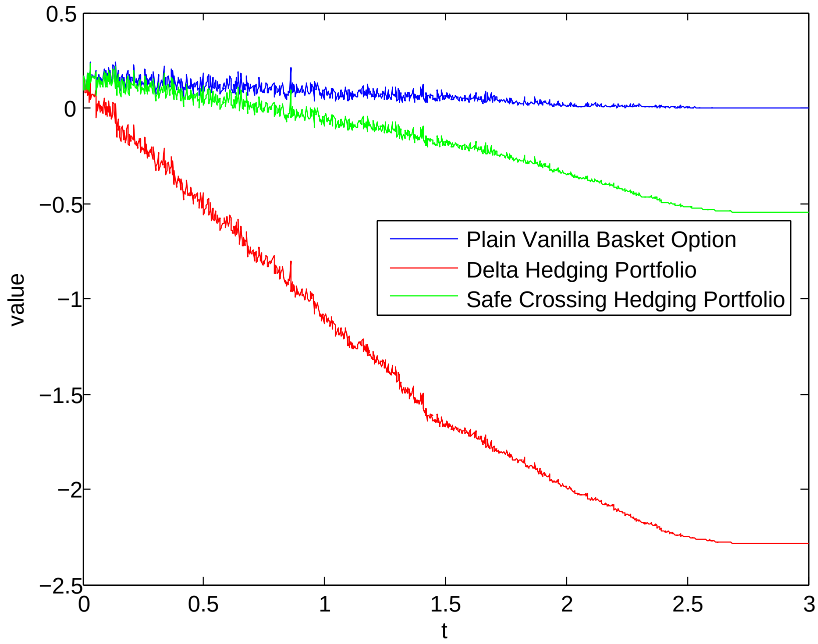


Figure 1.7: Delta and safe crossing hedging scenarios for in-the-money plain vanilla basket option of group 2 with $T = 3$ and $K = 0.8$.

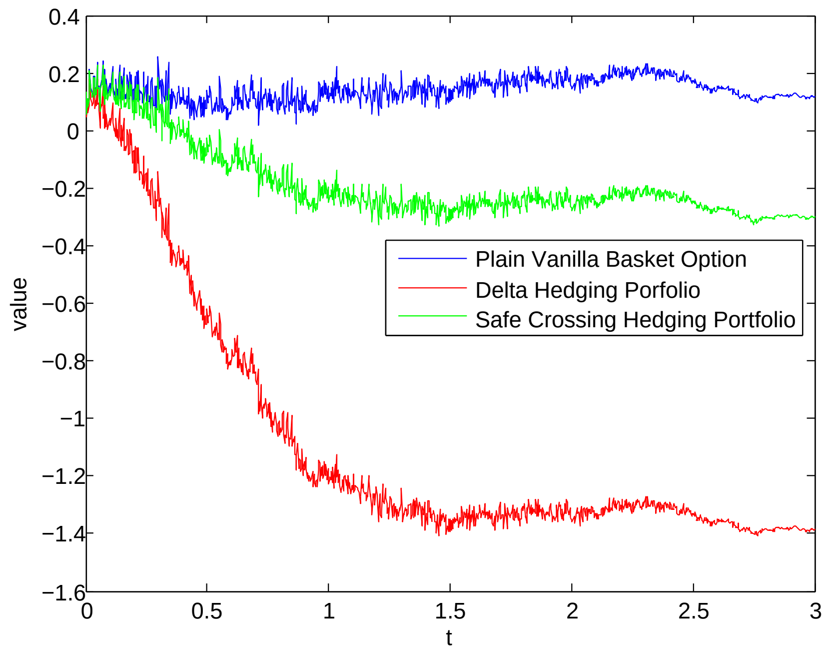


Figure 1.8: Delta and safe crossing hedging scenarios for in-the-money plain vanilla basket option of group 3 with $T = 3$ and $K = 0.8$.

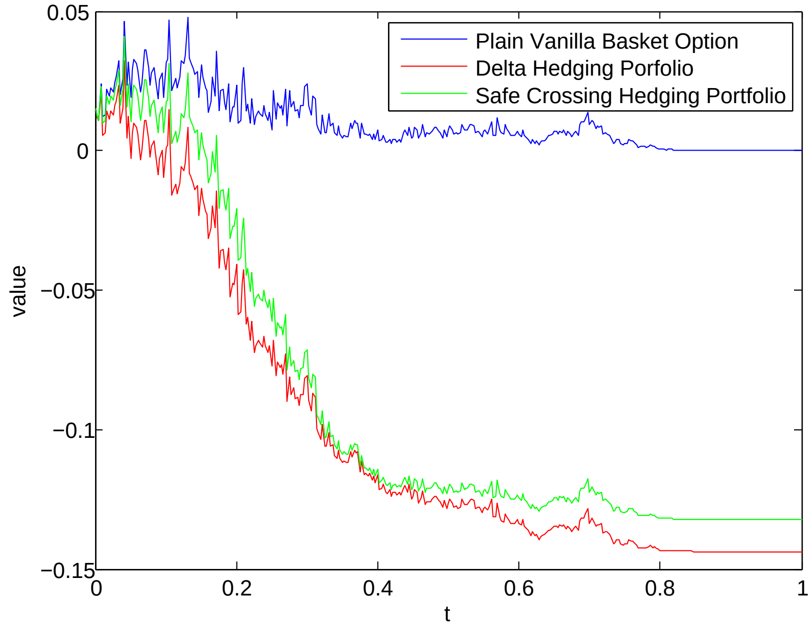


Figure 1.9: Delta and safe crossing hedging scenarios for at-the-money plain vanilla basket option of group 1 with $T = 1$ and $K = 1$.

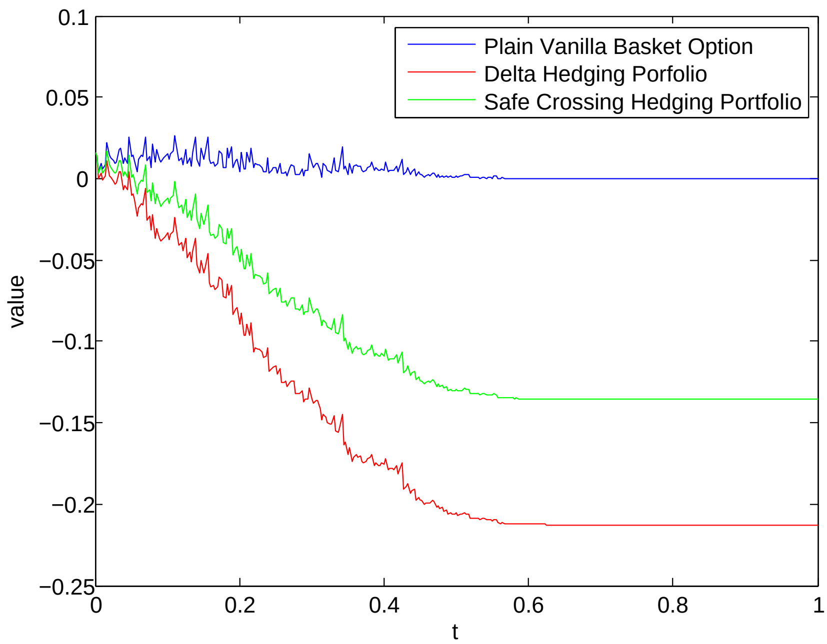


Figure 1.10: Delta and safe crossing hedging scenarios for at-the-money plain vanilla basket option of group 2 with $T = 1$ and $K = 1$.

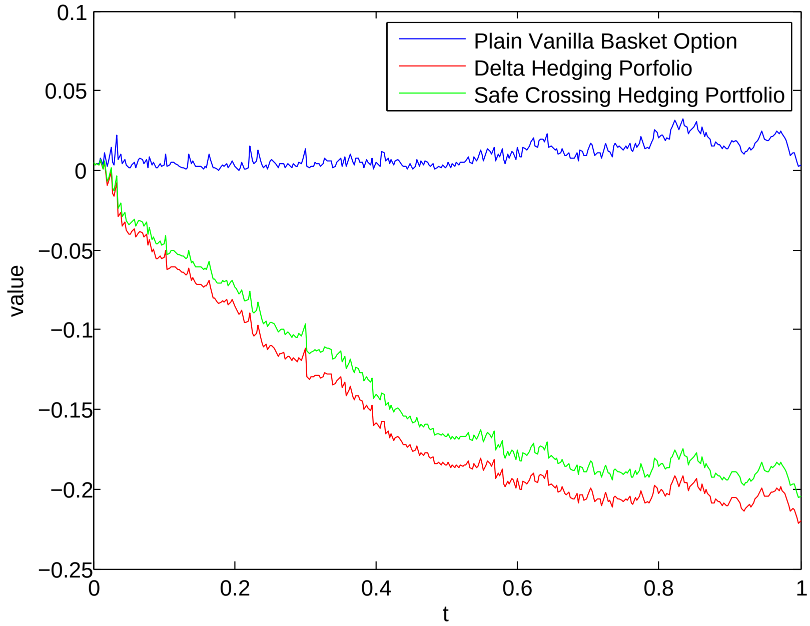


Figure 1.11: Delta and safe crossing hedging scenarios for at-the-money plain vanilla basket option of group 3 with $T = 1$ and $K = 1$.

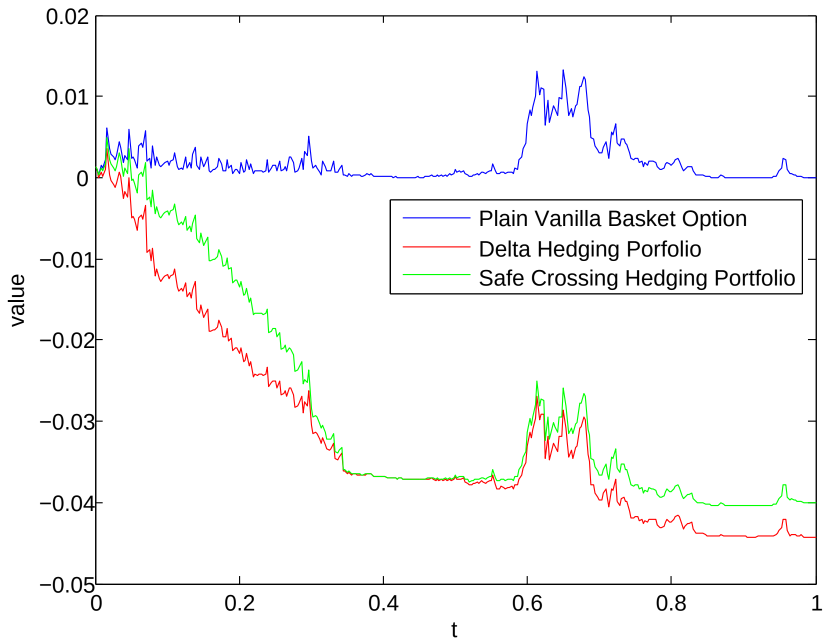


Figure 1.12: Delta and safe crossing hedging scenarios for out-of-the-money plain vanilla basket option of group 1 with $T = 1$ and $K = 1.1$.

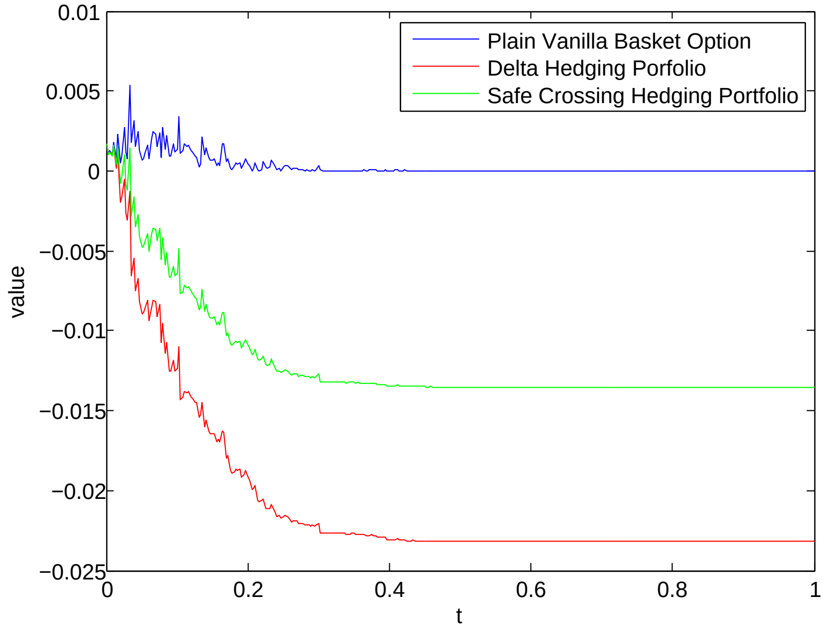


Figure 1.13: Delta and safe crossing hedging scenarios for out-of-the-money plain vanilla basket option of group 2 with $T = 1$ and $K = 1.1$.

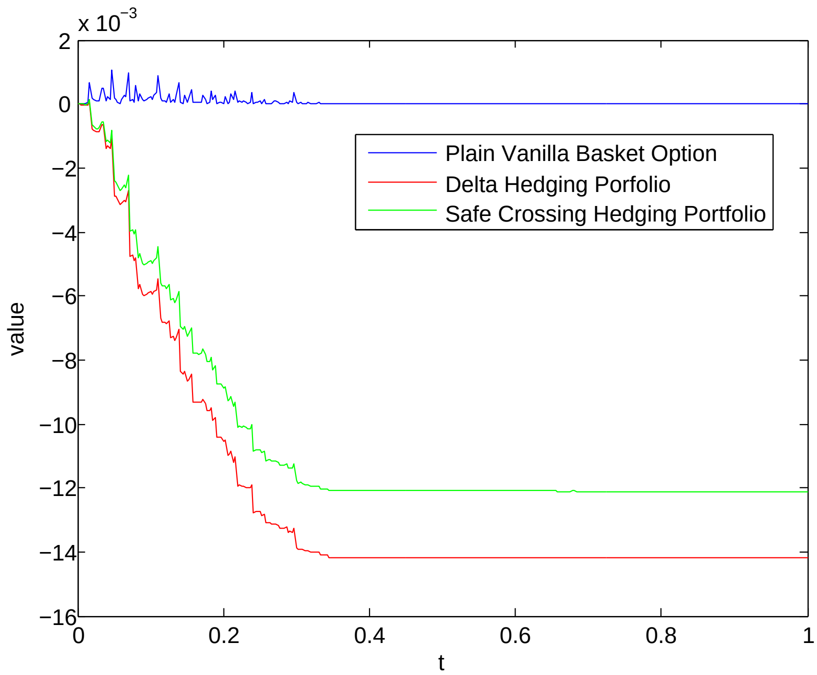


Figure 1.14: Delta and safe crossing hedging scenarios for out-of-the-money plain vanilla basket option of group 3 with $T = 1$ and $K = 1.1$.

For cash-or-nothing basket options, we present the hedging scenarios for $T = 1$ and $K = 0.8$ in Figures 1.15-1.17. From Figures 1.15 and 1.16 we see that the relative deviations of the hedging portfolio's value from the options's payoff (at the maturity date) are about 13% and 11.8% for the delta hedging of the options in group 1 and group 2, 11.5% and 6% for the safe crossing hedging. For group 3, the hedging errors are relatively small, being 0.09% and 0.01% respectively. It is interesting to see that both portfolio types build superhedging positions for $T = 1$ and $K = 0.8$. The reason may lie on the overvaluation of the rank one approximated prices for $T = 1$ and $K = 0.8$ (see Table 1.2). On the one hand, the option prices are undervalued at the beginning, which means that no enough money is injected to build hedging positions. There are two possible reasons for this phenomenon. On the other hand, the option prices are more sensitive to the gammas and cross gammas in these cases. Superhedging positions are however not observed for $T = 1$ and $K = 1$ or for $T = 3$ and $K = 0.8$. The relative performances of the two hedging strategies are similar to those for plain vanilla basket options and hence are omitted here.

The Static Hedging Performance

Since dynamic hedging does not work well in many cases even when transaction costs are neglected, we should consider static hedging as an alternative. In this section we investigate the performance of various static hedging strategies. Since the performance does not vary essentially, we only present the hedging scenarios for $T = 1$ and $K = 0.8$.

In Figures 1.18-1.20 we present the performance of the superhedging portfolios for plain vanilla basket options, which are obtained through the payoff approximation method presented in Section 1.4.2. We simulate the scenarios with the three types of basket underlying. It can be seen that in all the cases the superhedging portfolios build the upper bounds for the basket options. Following our arguments in Section 1.4.2, they are the cheapest among those portfolios of plain vanilla options whose strike prices can be chosen freely to reduce the hedging costs. However, the weight allocation of these plain vanilla options are kept identical with the composition of the original foreign currency basket. Cheaper superhedging portfolio may be obtained with weight adjustment. This is an interesting yet non-trivial question which is beyond the scope of the present work. Hence, we keep it unsolved at the moment. We notice that the superhedging portfolios are very expensive in all these cases, indicating that the hedging costs are very high. To decide whether to hedge statically or dynamically, we should compare the superhedging costs in the static hedging case with the transaction costs to occur during the frequent portfolio adjustment in the dynamic hedging.

With regard to the static hedging of cash-or-nothing basket options, we observe the hedging scenarios with a portfolio of basket options in figures 1.21-1.23. We find that for each group with different underlying baskets the portfolio of basket options obtained in

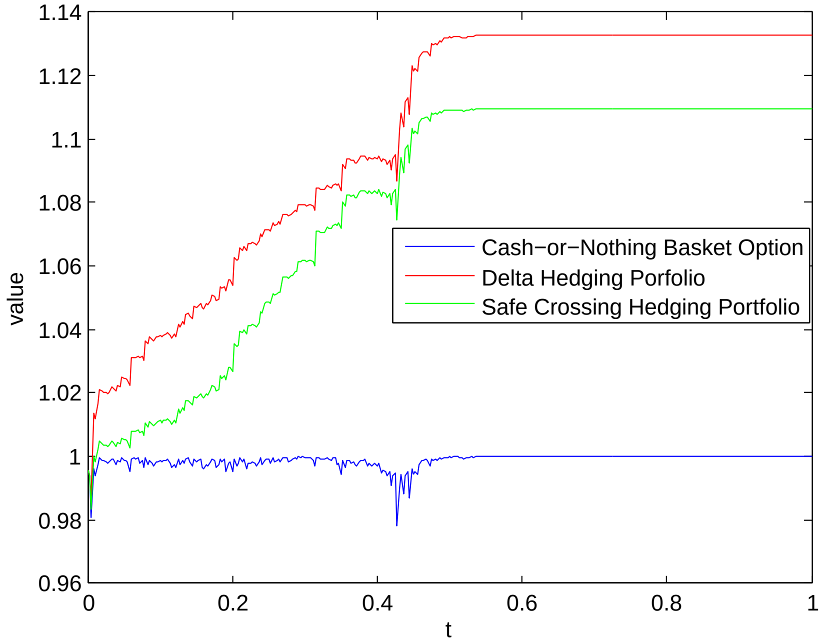


Figure 1.15: Delta and safe crossing hedging scenarios for in-the-money cash-or-nothing basket option of group 1 with $T = 1$ and $K = 0.8$.

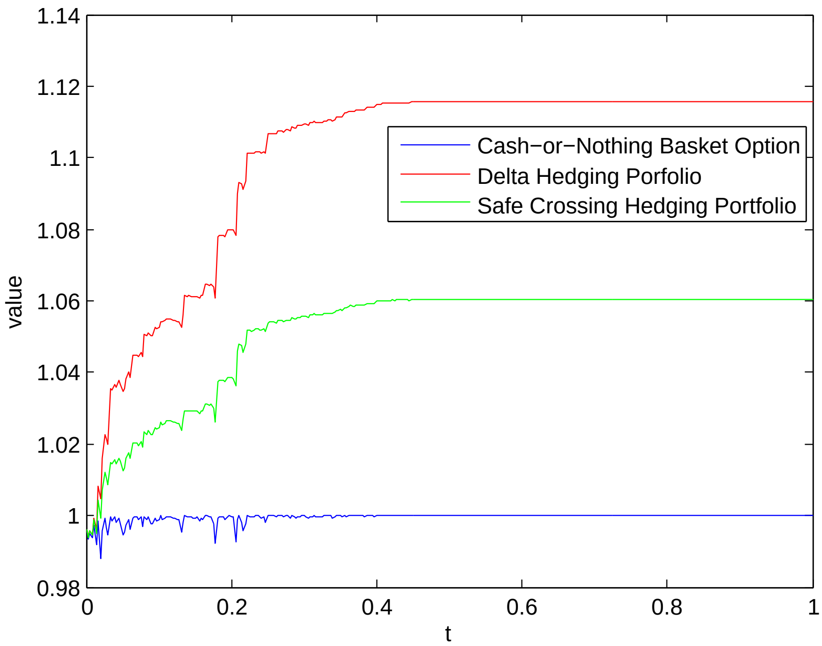


Figure 1.16: Delta and safe crossing hedging scenarios for in-the-money cash-or-nothing basket option of group 2 with $T = 1$ and $K = 0.8$.

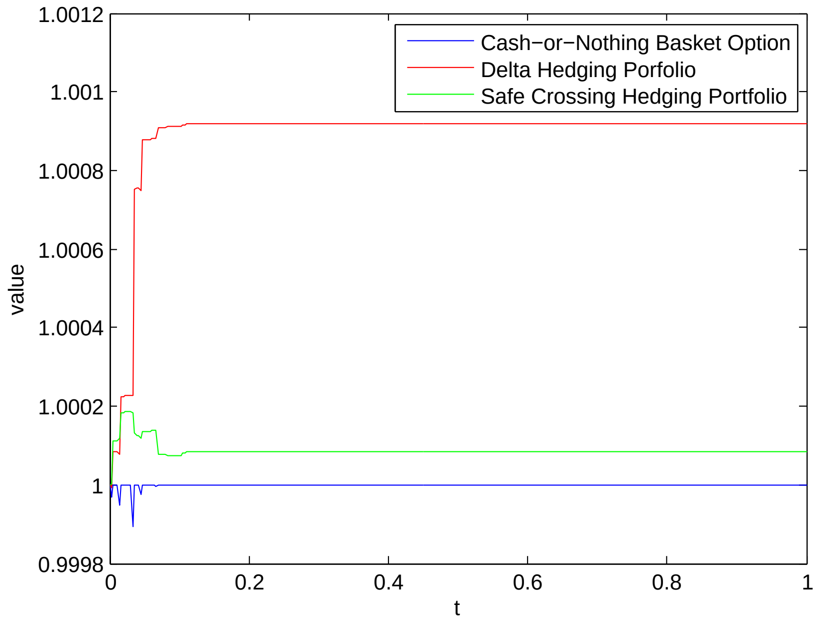


Figure 1.17: Delta and safe crossing hedging scenarios for in-the-money cash-or-nothing basket option of group 3 with $T = 1$ and $K = 0.8$.

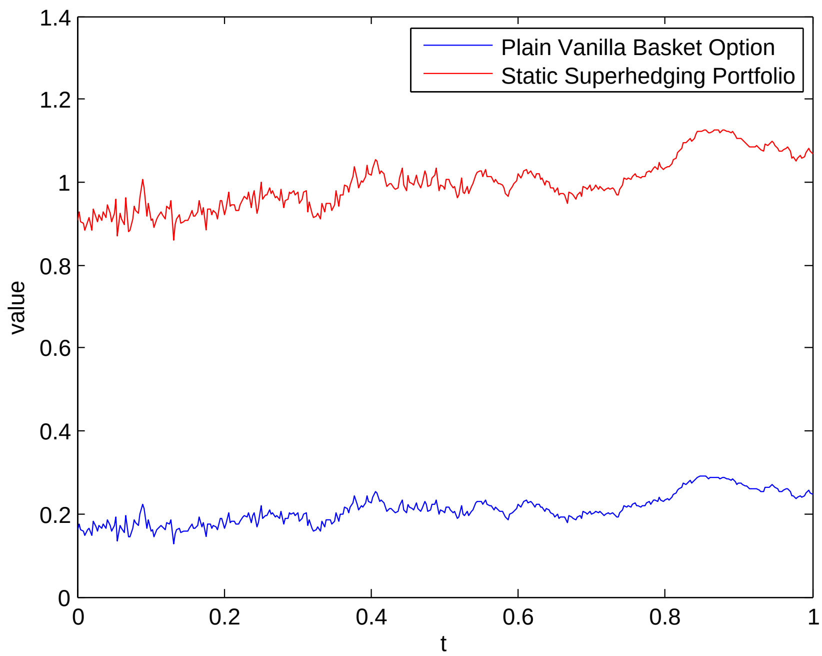


Figure 1.18: Superhedging portfolio v.s. plain vanilla basket option of group 1, $T = 1$, $K = 0.8$.

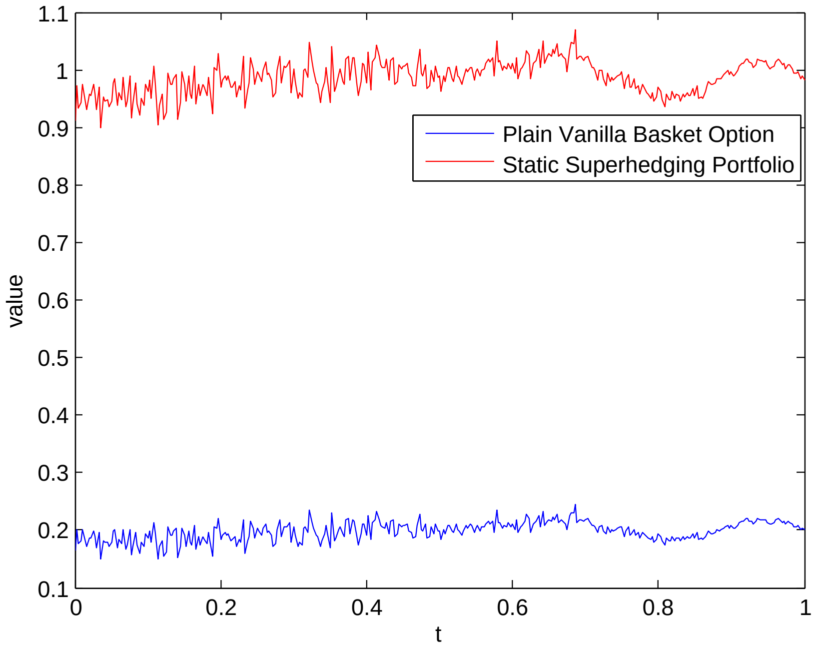


Figure 1.19: Superhedging portfolio v.s. plain vanilla basket option of group 2, $T = 1$, $K = 0.8$.

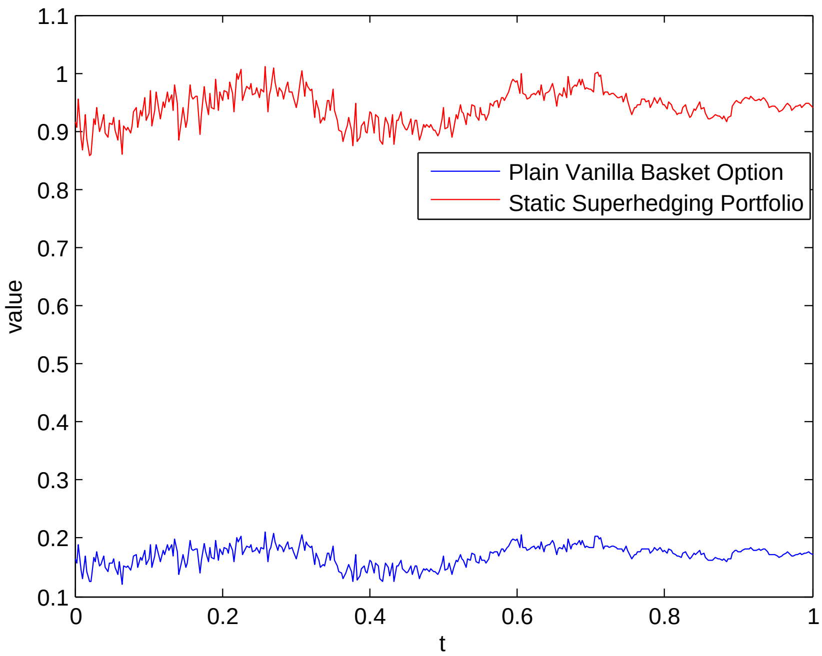


Figure 1.20: Superhedging portfolio v.s. plain vanilla basket option of group 3, $T = 1$, $K = 0.8$.

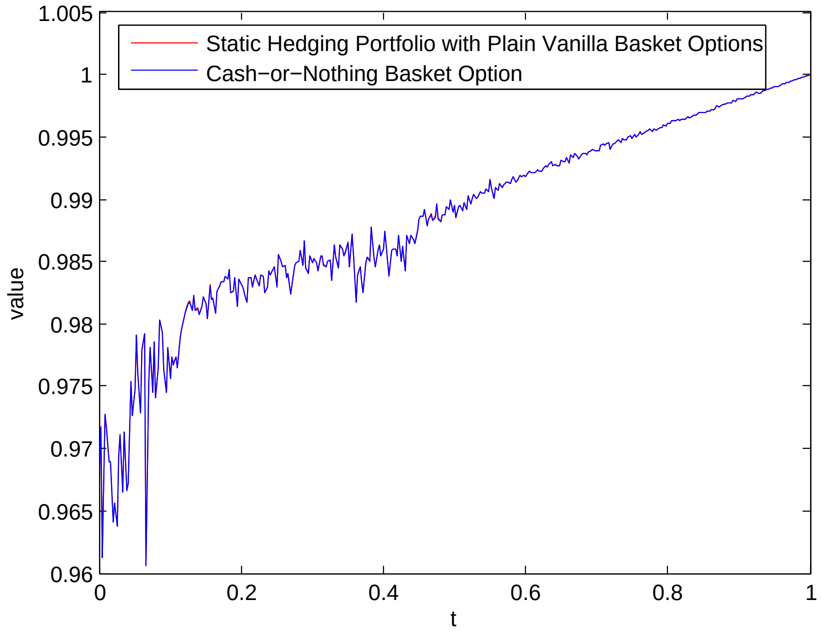


Figure 1.21: Hedging portfolio of basket options v.s. cash-or-nothing basket option of group 1, $T = 1$, $K = 0.8$.

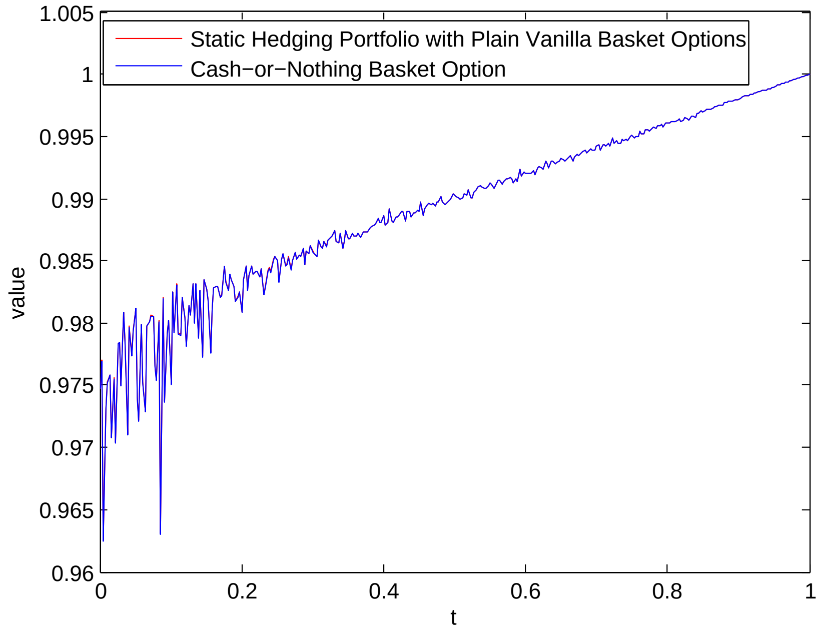


Figure 1.22: Hedging portfolio of basket options v.s. cash-or-nothing basket option of group 2, $T = 1$, $K = 0.8$.

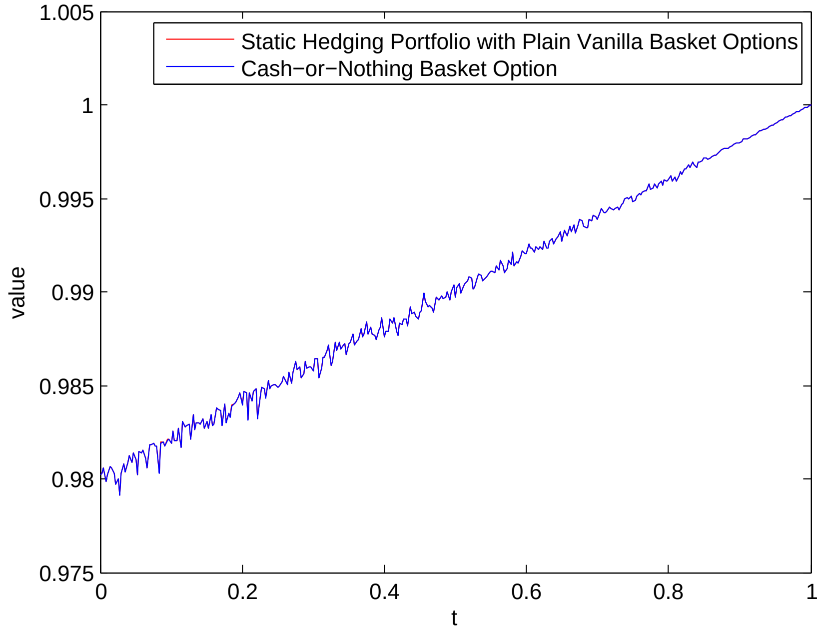


Figure 1.23: Hedging portfolio of basket options v.s. cash-or-nothing basket option of group 3, $T = 1$, $K = 0.8$.

(1.99) is a good replication of the cash-or-nothing basket option. During the life time of the options the values of the hedging portfolio and of the cash-or-nothing basket option almost coincide with each other.

1.6 The Final Terms of Basket FX products

We have introduced three popular types of basket FX certificates in Section 1.1.2. In this part we conduct a short discussion about how the relevant parameters should be chosen so that the issuing value of the certificate is fair at the beginning in the sense that it is identical to the face value. Because the improved rank one approximation pricing delivers the prices efficiently, we can back out the final terms to be settled without difficulty.

The basket FX certificates are usually written at-the-money, which means the strike price is set equal to the initial value of the underlying basket. In this case, the shorter the certificate's life time is, the lower is the probability that the basket value rises above the strike price. On the other hand, under our assumption that the initial yields of the zero-coupon bonds are independent of the maturity time, the discounting effect is lower for shorter life time of the certificates. In Tables 1.4 and 1.5 we present the participation rates α that should be chosen so that the guarantee certificate and the outperformance certificate are issued at par. When the exchange rates are negatively or mixed correlated, the first effect dominates the second. Consequently, the lost in the unfavorable situation should be compensated more when the life time of the certificate is shorter, and hence, a higher participation rate should be chosen. However, the first effect does not dominate the discounting effect when the exchange rates are positively correlated. We see that the participation rate increases when T increases from 1 to 3, indicating a stronger discounting effect. When T further increases, the first effect dominates, and hence, the participation rate should be decreased. With regard to the certificate with stages, the payoff structure is equivalent to a portfolio of at-the-money and out-of-the-money cash-or-nothing call options. When the maturity time T increases, the discounting effect dominates the increase in the probability of the basket to be above the strike prices. The equi-distance[20] between the stages is hence higher when the life time of the certificate increases. The results are displayed in Table 1.6.

	$T = 1$	$T = 3$	$T = 5$
Δ_1	3.67	7.13	3.71
Δ_2	3.42	2.53	2.13
Δ_3	9.84	4.23	2.79

Table 1.4: The participation rate α for guarantee certificate to be issued at fair price at time 0.

	$T = 1$	$T = 3$	$T = 5$
Δ_1	6.53	11.39	6.25
Δ_2	6.13	4.69	4.01
Δ_3	15.83	7.16	4.94

Table 1.5: The participation rate α for outperformance certificate to be issued at fair price at time 0.

	$T = 1$	$T = 3$	$T = 5$
Δ_1	0.12	0.37	0.50
Δ_2	0.11	0.25	0.47
Δ_3	0.23	0.27	0.46

Table 1.6: The equi-distance $\Delta\beta$ between the stages for certificate with stages to be issued at fair price at time 0.

We can also make a comparison between the three types of basket underlying. It is easily seen that a certificate written on a basket of negatively correlated exchange rates (Δ_2) can be equipped with a lower participation rate than a certificate written on a basket of

[20]Compare Example 1.1.3. Usually the distance between two stages, i.e., $\Delta\beta = \beta_{i+1} - \beta_i$ for $i = 0, \ldots, m-1$ is constant.

positively correlated exchange rates (Δ_1). This is because a basket of negatively correlated foreign currencies is better diversified. The less embedded risk explains that the leverage level can be set lower to meet the requirement for fair price at the beginning. With regard to a certificate written on a basket of mixed correlated foreign currencies, we cannot draw a uniform conclusion. When the certificate's life time is long, in our example, $T = 3$ or $T = 5$, the participation rates needed lie between those of the certificates written on positively correlated foreign currencies and the ones written on negatively correlated currencies. This sounds reasonable, since a basket of mixed correlated currencies is better diversified than a basket of positively correlated currencies but less diversified than the negatively correlated ones. However, things are different when the certificate's life time is short, in our example, $T = 1$. The participation rates need to be extraordinarily high to keep the certificate issued at fair price. This indicates that a mixed correlated basket can be more risky in the short run.

1.7 Conclusion

Our main contribution is the introduction of a new approximation pricing method, namely, the rank one approximation method, for basket options. We apply this method to basket FX derivatives. The rank one approximation method starts with the idea of approximating the covariance matrix of the uncertain factors in the underlying basket. We have denoted it as the crude rank one approximation. By applying the singular value decomposition, we can find the best rank one approximation of the observed covariance matrix. However, in comparison with other popular approximation pricing methods, e.g., the lognormal approximation method, the crude rank one approximation is still not satisfactory. Noticing that the true nature of the lognormal approximation is to match the moments of the underlying assets, we have tried to plant the moment-matching technique into our rank one approximation. This improved version is denoted as the improved rank one approximation method. The numerical results have shown that the rank one approximation method with the three moment matching technique has outperformed the lognormal approximation for both the pricing of cash-or-nothing basket options and the basket options.

Based on prices obtained from the rank one approximation method with the three moment matching technique, we have studied both the delta hedging and the safe crossing hedging performances for cash-or-nothing basket options and plain vanilla basket options. We have found that both the hedging strategies work very well for short-termed in-the-money and out-of-the-money plain vanilla basket options, but badly in the other cases. The potential reasons for the high hedging errors have been analyzed in detail. The safe crossing hedging strategy outperforms the delta hedging strategy in general due to the consideration of the gamma and cross gamma values when constructing hedging positions.

The efficiency of the rank one approximation pricing also enables us to study the static hedging strategies with less computation time. Inspired by the three-point Richardson extrapolation method applied by Carr et al. (1998) for the static hedging of cash-or-nothing vanilla options, we have applied this method to the cash-or-nothing basket options and have found accurate static hedging portfolios. We have further considered the static hedging of plain vanilla basket options with plain vanilla European options. We have presented the payoff approximation method which helps to generate the superhedging portfolio for basket options.

With the help of the efficient rank one approximation pricing we can also easily find the parameters to be settled in a structured financial product with basket underlying so that the product is issued at par at the beginning.

The rank one approximation pricing method we have introduced is also suitable for basket derivatives with underlying assets other than foreign currencies. Due to the structural similarity between basket derivatives and Asian options, this method can also be applied to the approximation pricing of Asian options.

Chapter 2

The Uncertain Mortality Intensity Framework: Pricing and Hedging Unit-Linked Life Insurance Contracts[1]

2.1 Introduction

Mortality is a major risk factor for life insurance companies and pension funds that needs to be modeled properly. In recent years, it has been widely accepted that mortality changes over time in an unpredictable way and stochastic models have been developed to adequately capture the systematic mortality risk. For stochastic models valuing of mortality-linked liabilities and determining the required market reserves, see for instance Milevsky and Promislow (2001), Dahl (2004), Biffis (2005), Dahl and Møller (2006), and Young (2008). Stochastic models with an emphasis on securitizing mortality risk by introducing survivor bonds as hedging instruments are discussed by, e.g., Blake, Cairns and Dowd (2006) and Cairns, Blake and Dowd (2006). Each mortality model is a possible description of the mortality risk. Melnikov and Romaniuk (2006) show that different mortality models perform differently in the risk management of a unit-linked pure endowment contract and warns us to be careful when choosing one mortality model against another. In this chapter we provide a framework for assessing the mortality model risk embedded in unit-linked life insurance contracts arising from different specifications for the mortality intensity.

Unit-linked life insurance contracts are popular and widely used on the insurance market. They provide either death benefit or maturity benefit or both. The benefits are linked to an underlying asset with or without certain guarantees so that the policyholders have the opportunity to participate in the financial market and (eventually) be protected from

[1]This chapter is based on Li and Szimayer (2011b)

the downside development of the financial market. Many unit-linked life insurance contracts also embed options in them, e.g., the surrender option allowing the policyholders to terminate the contracts prematurely and the guaranteed annuity option giving the policyholders the right to convert a lump sum payment at the maturity into annuities at a predetermined rate. Depending on the payoff structures of the contracts, the effect of the mortality model risk may also be different. By investigating the effect of the mortality model risk we are able to know whether its importance is under or over-emphasized for different contract types.

In this chapter instead of inputting different mortality models into the same pricing and hedging problem and comparing their performances as Melnikov and Romaniuk (2006), we set up a more flexible framework saying that we do not know the exact process of the mortality intensity but are able to figure out its upper and lower bound under the statistical measure. Further, we restrict the set of equivalent martingale measures such that the same bounds apply to the mortality intensity under these measures. This setup allows us to study various contract types more efficiently and we call it the uncertain mortality intensity framework, see Avellaneda, Levy and Paras (1995) for a related framework for pricing stock options when the volatility process is unknown but bounded.

Within our framework we do not intend to find the fair value of a contract but its price bounds. The price bounds are solutions to the partial differential equations associated to a stochastic control problem. The upper price bound is found by choosing the worst-case mortality intensity at any time during the life time of the contract so that the contract value is maximized. Whereas the lower price bound is found by setting the mortality intensity to the best-case value in the sense that the contract value would be minimized. The effect of our approach is quite similar to that of the practice in traditional life insurance like pure endowment insurance and term insurance. An insurance company usually puts itself on the safe side by adjusting the premium by a loading factor defined as a percentage markup from the actuarially fair value of insurance. This is equivalent to assuming lower mortality intensity for pure endowment insurance and higher mortality intensity for term insurance. However, since our approach chooses the worst (or best) possible mortality intensity dynamically, we are able to deal with more complex contract structures where the safest mortality intensity at any time also depends on the price of the underlying asset. As a result, the higher the difference between the upper and the lower price bounds, the greater impact would the mortality model risk have on the contracts considered. In this way we are able to identify whether model risk is potentially deteriorating the fair evaluation of the contracts.

Further we examine hedging strategies induced by the price bounds. The unsystematic mortality risk is diversified by pooling a large enough number of policyholders together as usually is the case. However, the systematic mortality risk, that is here the random fluctuations of the mortality intensity, can in general not be diversified away by using the pooling rationale. Instead of applying risk-minimizing or mean-variance hedging strategies to minimize either hedging costs or hedging error, see Dahl and Møller (2006) and Young (2008), we suggest using hedging strategies induced by the upper and lower price bounds. By construction, these strategies produce a superhedge and subhedge, respectively, on average for an increasing number of policyholders. We provide numerical examples investigating fixed-term, endowment insurance contracts and their combinations including various guarantee features. The pricing partial differential equation for the upper and lower price bounds is solved by finite difference methods. For our contracts and choice of parameters pricing and hedging is fairly robust with respect to misspecification of the mortality intensity, with at most a mispricing of 4% for single premium contracts and at most 2% for periodic premium payment. We conclude that model risk resulting from the uncertain mortality intensity is of minor importance for these contracts.

The structure of this chapter is as follows. In Section 2.2 we describe both the financial market and the insurance market. In Section 2.3 we formalize the uncertain mortality intensity framework. Based on the model setup, we introduce in Section 2.4 the optimal control rule of the mortality intensity within its upper and lower bounds so that the price bounds are found. This enables us to build in mean superhedging strategies which are discussed in Section 2.5. Section 2.6 illustrates the theoretical results by providing a numerical analysis for different types of unit-linked life insurance contracts. Section 2.7 concludes.

2.2 Setup

The model for the financial market and the insurance market is developed subsequently. Both markets are jointly specified on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$. The probability $\mathbb{P}$ is called the real world measure and is sometimes also referred to as statistical measure. We assume that the probability space is large enough to support an n-dimensional Wiener process $W = [W^1, W^2, \ldots, W^n]$ and a random time τ. The time horizon is denoted by T.

The financial market consists of a risky asset with price process S and a riskless money market account with price process B. The latter is given by $B_t = \exp\{\int_0^t r(u)\,\mathrm{d}u\}$, $0 \leq t \leq T$, where the risk-free interest rate r is a deterministic and continuous function. The risky asset price process S is governed by the stochastic differential equation:

$$\mathrm{d}S_t = a(t, S_t)\, S_t\, \mathrm{d}t + \sigma(t, S_t)\, S_t\, \mathrm{d}W_t^1, \quad 0 \leq t \leq T. \tag{2.1}$$

where a is the local mean rate of return and σ is the volatility. The dividend structure D is given by

$$\mathrm{d}D_t = q(t, S_t) S_t \mathrm{d}t, \quad 0 \leq t \leq T. \tag{2.2}$$

where q is a continuous deterministic function.[2] The financial market modeled in this way is complete and arbitrage free and is called $\mathbb{F}^S$ market. Here, $\mathbb{F}^S = \left(\mathcal{F}_t^S\right)_{0\leq t\leq T}$ is the augmented natural filtration generated by the stock price process S. Since $\sigma > 0$ it follows that the augmented natural filtration generated by the first component W^1 of the Wiener process $\mathbb{F}^1 = (\mathcal{F}_t^{W^1})_{0\leq t\leq T}$ coincides with the market filtration $\mathbb{F}^S$.

The insurance market is modeled by the random time τ denoting the death time of an individual aged x at the starting time 0.[3] For simplicity of notation we will omit the age variable x in the subsequent discussion of mortality related variables. The filtration generated by the right-continuous indicator process $H_t = 1_{\{\tau\leq t\}}$, for $t \in [0,T]$, is denoted $\mathbb{H} = (\mathcal{H}_t)_{0\leq t\leq T}$. The mortality is potentially influenced by an m-dimensional environment process $X = [X_1, \ldots, X_m]$ with dynamics

$$\mathrm{d}X_t = a_X(t, X_t)\,\mathrm{d}t + \Sigma_X(t, X_t)\,\mathrm{d}W_t, \quad 0 \leq t \leq T, \tag{2.3}$$

where a_X is a $\mathbb{R}^m$-valued function and Σ_X is a $\mathbb{R}^{n\times m}$-valued function, both regular enough to ensure the existence of a solution to the SDE. By definition it is clear that X is adapted to the filtration generated by the Wiener process W, say, $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$. Note that $\mathbb{F}^S \subseteq \mathbb{F}$, and further denote the joint filtration by $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$. The financial market model for unit-linked life insurance contracts is then called the $\mathbb{G}$ market.

2.2.1 Dependence of Financial Market and Insurance Market

The probabilistic connection between W and τ is now formalized. In broad terms we assume that we are in a setting frequently used in the credit risk literature, see Bielecki and Rutkowski (2001), part II, for a detailed treatment. In particular, we assume that on $(\Omega, \mathcal{G}, \mathbb{P})$ there exists a unit exponentially distributed random variable E_1 that is independent of W and further that there exists a nonnegative $\mathbb{F}$-adapted process ν such that τ can be represented by

$$\tau := \inf\left\{t \geq 0 : \int_0^t \nu_s\,\mathrm{d}s \geq E_1\right\}, \quad a.s.,$$

with the usual convention that the infimum over the empty set is ∞, and the integrability condition $\int_0^t \nu_s\,\mathrm{d}s < 0$ holds almost surely, for all $t \geq 0$.

[2]We assume that the coefficients a and σ are regular enough to ensure the existence of a solution to the SDE (2.1), see for instance Protter (2004), Ch. V, Sec. 3. Additionally, we assume that a, σ and q are uniformly bounded and σ is bounded away from zero to ensure the integrability of S, related portfolio value processes, and to ensure the existence of the measure change from $\mathbb{P}$ to an equivalent martingale measure $\mathbb{Q}$.

[3]In Section 2.5 we consider the case of a family of random times $(\tau_i)_{i\geq 1}$ and the corresponding contracts.

We then have the following representation

$$M_t \;=\; H_t - \int_0^{t\wedge\tau} \nu_s \mathrm{d}s\,, \quad 0 \le t \le T,$$

where M is a $(\mathbb{P},\mathbb{G})$-martingale, see Bielecki and Rutkowski (2001), p.153, Prop. 5.1.3. In our context the intensity ν is known as mortality intensity.

By specification of τ through E_1 and ν and the assumed independence, the σ-fields $\mathcal{F}_T$ and $\mathcal{H}_t$ are independent given $\mathcal{F}_t$ under the real world probability measure $\mathbb{P}$. Although we may perceive the death probability of the individual, we do not know when the death event really happens. Hence, τ is an inaccessible $\mathbb{G}$ stopping time but not an $\mathbb{F}$ stopping time. On the other hand, the financial market is not influenced by the introduction of τ. Accordingly, the $\mathbb{G}$ market for unit-linked life insurance contracts is free of arbitrage.[4] However, given that there are no products to hedge against the mortality risk (that is the fluctuation of ν and the mortality event indicated by H), the $\mathbb{G}$ market is incomplete, and hence, there should be infinitely many equivalent martingale measures.

2.2.2 Equivalent Martingale Measures

The set of equivalent martingale measures is studied. Given a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ on $(\Omega,\mathcal{G},\mathbb{G})$, the Radon-Nikodym density process η of $\mathbb{Q}$ with respect to $\mathbb{P}$ is

$$\eta_t = \left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{G}_t} = \mathbb{E}_{\mathbb{P}}(Y|\mathcal{G}_t), \quad \mathbb{P}-a.s., \tag{2.4}$$

for some $\mathcal{G}_T$-measurable random variable Y with $\mathbb{P}(Y>0)=1$ and $\mathbb{E}_{\mathbb{P}}(Y)=1$.

Now, we characterize the set of equivalent measure for our setting and also the set of equivalent martingale measures. The set of equivalent measures is given by Prop. 5.3.1 in Bielecki and Rutkowski (2001), p.162. Let $\mathbb{Q}$ be a probability measure equivalent to the real world probability measure $\mathbb{P}$ with the Radon-Nikodym density of $\mathbb{Q}$ with respect to $\mathbb{P}$ defined by (2.4). Then we can write

$$\eta_t = 1 + \int_0^t \eta_{u-}(\varphi_u \mathrm{d}W_u + \phi_u\,\mathrm{d}M_u), \quad 0 \le t \le T\,, \tag{2.5}$$

where φ and ϕ are $\mathbb{G}$-predictable stochastic processes. The change of measure affects

[4] In particular any square integrable $(\mathbb{F},\mathbb{P})$-martingale is also a square integrable $(\mathbb{G},\mathbb{P})$-martingale. This is also known as hypotheses (H), see Jeulin and Yor (1979).

the martingales W and M as follows. Define the processes W^Q and M^Q by

$$W_t^Q = W_t - \int_0^t \varphi_s \, \mathrm{d}s \,, \quad \text{and} \quad M_t^Q = H_t - \int_0^{t\wedge\tau} (1+\phi_u)\, \nu_u \, \mathrm{d}u \,, \quad 0 \le t \le T \,. \tag{2.6}$$

Then W^Q is a $(\mathbb{G},\mathbb{Q})$-Wiener process and M^Q is an $(\mathbb{G},\mathbb{Q})$-martingale, and $\mu = (1+\phi)\,\nu$ is the $\mathbb{Q}$-intensity of τ. Moreover, μ can be chosen to be $\mathbb{F}$-adapted, see Remark following Corollary 5.3.1 in Bielecki and Rutkowski (2001), p.164.

Proposition 2.2.1. *If $\mathbb{Q}$ is an equivalent martingale measure, i.e. $\mathbb{Q} \sim \mathbb{P}$ and S/B is a $(\mathbb{G},\mathbb{Q})$-martingale, then $\{W^{1Q}\}_{t\in[0,T]}$ is uniquely determined by the market price of risk*

$$\varphi_t^1 = -\frac{a(t,S_t) - r(t) + q(t,S_t)}{\sigma(t,S_t)}, \quad 0 \le t \le T \,. \tag{2.7}$$

Proof. This follows from the Second Fundamental Theorem, confer Björk (2009), p.151, Theorem 10.17 and p.204, Sec.14.6. □

Proposition 2.2.1 indicates that when we restrict to the $\mathbb{F}^S$ market, there is a unique martingale measure, which we denote as $\mathbb{Q}^{\mathbb{F}^S}$. Under any equivalent martingale measure $\mathbb{Q}$, the dynamics of the stock price is

$$\mathrm{d}S_t \;=\; (r(t) - q(t,S_t))\, S_t \, \mathrm{d}t + \sigma(t,S_t)\, S_t \, \mathrm{d}W_t^{1Q}, \quad 0 \le t \le T \,. \tag{2.8}$$

However, when we observe the extended market with both the financial and the mortality risks, we cannot find a riskless benchmark security. Hence, ϕ, or equivalently, the risk-neutral mortality intensity is not uniquely defined. Theoretically, among a whole class of equivalent martingale measures, the insurance companies can choose any one depending on their risk attitude. We denote the set of equivalent martingale measures by $\mathcal{Q}$, i.e.

$$\mathcal{Q} \;=\; \left\{ \mathbb{Q} \sim \mathbb{P} : \varphi_t^1 = -\frac{a(t,S_t) - r(t) + q(t,S_t)}{\sigma(t,S_t)} \right\} . \tag{2.9}$$

Remark 2.2.1. *The fair valuation of an insurance liability is carried out under a specific risk-neutral measure. Choosing the valuation measure in an incomplete market is a difficult task. Alternatively, the model can be completed by adding asset that cover the entire risk factors. Biffis and Millossovich (2006) assume that there is a liquid secondary market where the insurers can continuously trade their books of policies making it possible to access both short and long positions. This results into a complete market situation where the valuation measure is unique. Another possibility to uniquely determine the risk-neutral measure is to introduce standardized mortality linked products such as longevity bonds which are liquidly traded on the market, see Blake et al. (2006). However, a fully developed secondary insurance market does not exist yet and the securitization of mortality risk is still at its infancy stage with most of the mortality linked securities only being tai-*

lored to the customers. Hence, in this chapter, we still assume that there is not a unique market price of mortality risk and that there are infinitely many martingale measures $\mathbb{Q}$ *equivalent to* $\mathbb{P}$ *, under which the prices of the insurance contracts do not allow arbitrage.*

2.2.3 Examples

The setup described so far accommodates a large class of models discussed in the literature. We illustrate the use of the environment process X by some prominent examples.

Example 2.2.1. *The mean reverting Brownian Gompertz approach of Milevsky and Promislow (2001) is given by*

$$\begin{aligned} \nu_t &= \nu_0 \, e^{gt+\sigma X_t}, \quad 0 \le t \le T, \\ \mathrm{d}X_t &= -b\, X_t \mathrm{d}t + \mathrm{d}W_t^2, \quad 0 \le t \le T, \quad X_0 = 0, \end{aligned} \tag{2.10}$$

where $g, \sigma, \nu_0 > 0$ *and* $b \ge 0$ *.*

Example 2.2.2. *Dahl (2004) and Dahl and Møller (2006) use the extended CIR model, i.e.* $\nu = X$ *with*

$$\mathrm{d}X_t = (\beta_t - \gamma_t \, X_t)\,\mathrm{d}t + \sigma_t \sqrt{X_t}\mathrm{d}W_t^2, \quad 0 \le t \le T, \tag{2.11}$$

where β_t *,* γ_t *and* σ_t *are positive bounded functions satisfying* $2\,\beta_t \ge \sigma_t^2$ *,* $0 \le t \le T$ *.*

Example 2.2.3. *Biffis (2005) studies affine processes of the form*

$$\begin{aligned} \mathrm{d}X_t^1 &= \gamma_1(X_t^2 - X_t^1)\mathrm{d}t + \sigma_1 \sqrt{X_t^1}\,\mathrm{d}W_t^2, \quad 0 \le t \le T, \\ \mathrm{d}X_t^2 &= \gamma_2(m(t) - X_t^2)\,\mathrm{d}t + \sigma_2 \sqrt{X_t^2 - m^*(t)}\,\mathrm{d}W_t^3, \quad 0 \le t \le T, \end{aligned} \tag{2.12}$$

where $\gamma_i, \sigma_i > 0$ *,* $i = 1, 2$ *, and the mortality intensity is* $\nu = X^1$ *, the stochastic mean reversion level is* $\bar{\nu} = X^2$ *,* $m(t)$ *is a suitable demographic basis, and* $m^*(t)$ *is a time varying lower boundary for the stochastic drift* X^2 *.*

Example 2.2.4. *In Young (2008) we have* $\nu = X$ *with*

$$\mathrm{d}X_t = a(X_t, t)(X_t - \underline{X})\mathrm{d}t + \sigma(t)(X_t - \underline{X})\mathrm{d}W_t^2, \quad 0 \le t \le T, \tag{2.13}$$

where $\underline{X} = \underline{\nu}$ *represents the lowest attainable mortality intensity remaining after the elimination of all causes of death such as accidents and homicide. Moreover,* σ *is a strictly positive and continuous function on* $[0, T]$ *, and the drift* $a(X_t, t)$ *is a suitable Hölder continuous function of* X *and* t *.*

The mortality intensity is typically modeled under the statistical measure $\mathbb{P}$. The life tables are calculated on the basis of real world data. When going to a pricing measure $\mathbb{Q}$, often structure preserving transformations are allowed for, e.g., Dahl and Møller

(2006) relate the $\mathbb{P}$-mortality intensity ν to the $\mathbb{Q}$-mortality intensity μ by assuming $\mu_t = (1+g(t))\,\nu_t$, where g is a deterministic continuously differentiable function. Alternatively, Young (2008) motivates the choice of a specific instantaneous Sharpe ratio α for the mortality risk by constructing a hedging portfolio whose local variance is minimized. The mortality risk charge is defined as α of the local standard deviation of the hedging portfolio. As a result, the drift term in Young's model is modified by $\alpha\sigma$ under the pricing measure $\mathbb{Q}$.

2.3 Uncertain Mortality Intensity

Our setup is a generically incomplete market model and we cannot obtain a unique price for a unit-linked life insurance contract. However, we are able to find its price bounds under certain assumptions. In this chapter, we admit that we cannot perceive the dynamics of the mortality intensity exactly. Instead of applying a specific mortality model, as is done for example in Biffis (2005), Dahl (2004), Dahl and Møller (2006), Milevsky and Promislow (2001), and Young (2008) we assume less stringently that we know the upper and lower bounds of the mortality intensity. As is shown in our numerical section below, this assumption can be motivated by a statistical analysis of survival data and the confidence bounds for the estimated mortality intensity arising there, see, e.g., Lee and Carter (1992). The concept of an uncertain input parameter to a pricing model is related to Avellaneda et al. (1995). They discuss the pricing and hedging of derivative securities in an incomplete market where the incompleteness is attributed to the uncertainty of the future volatility of the underlying asset. As suggested by them, we will use stochastic optimal control techniques to identify the best-case scenario and the worst-case scenario of the mortality intensity dynamics, to derive the upper and lower price bounds of the unit-linked life insurance contracts.

The above assumption on the boundedness of the mortality intensity is now made precise.

Assumption 2.3.1 ($\mathbb{P}$-Bounds for Mortality Intensity)**.** *The mortality intensity ν is an $\mathbb{F}$-adapted stochastic process satisfying*

$$\underline{\mu}(t) \leq \nu_t \leq \bar{\mu}(t)\,,\ \textit{almost surely}\,,\quad 0 \leq t \leq T\,, \tag{2.14}$$

where $0 < \underline{\mu} \leq \bar{\mu} < \infty$ are continuous functions on $[0,T]$.

Remark 2.3.1. *The bounds $\underline{\mu}$ and $\bar{\mu}$ for ν are assumed to hold almost surely whereas the examples in 2.2.3 allow ν to take values in $\mathbb{R}^+$. Accordingly, these examples are not included in our setup once Assumption 2.3.1 is invoked. However, these models satisfy the boundedness condition typically with a high probability when assuming that both, parameters for a model in 2.2.3 and the bounds $\underline{\mu}$ and $\bar{\mu}$, here in terms of confidence bounds, are calculated from the same data set. In fact, by increasing the confidence level* *for the calculation of* $\underline{\mu}$ *and* $\bar{\mu}$ *the probability is increased of a stochastic model for* ν *also fulfilling the boundedness condition. In Section 2.6 numerical results are obtained when working with a 99.9% confidence bound.*

For pricing derivatives the dynamics of the risk factors under an equivalent martingale measure is relevant. Our market model is incomplete and we can choose between an infinite range of equivalent martingale measures, see Proposition 2.2.1 and the discussion thereafter. An incomplete financial market is typically completed by adding assets to the market model such that all risk factors are traded. However, for insurance risk we can use the diversification rationale as an alternative. The diversification applies in our setting to the life insurance risk given by the time of death τ parameterized by the mortality intensity ν. Diversification is driven by the strong law of large numbers and thus tied to the statistical measure $\mathbb{P}$. Based on this rationale, pricing must take all the possible scenarios of death events into account. Consequently, the most suitable equivalent martingale measure should be chosen among all the possible ones so that the diversification works to eliminate the mortality risk. Since the possible scenarios of death events do not change under any pricing measure although their probability distributions are different[5], we impose the boundedness assumption under $\mathbb{P}$ also to any pricing measure $\mathbb{Q} \in \mathcal{Q}$, defined in Equation (2.9).

Definition 2.3.1. *Given that Assumption 2.3.1 holds, denote by* Q^b *the set of equivalent martingale measures* $\mathbb{Q} \in \mathcal{Q}$ *under which the* $\mathbb{F}$ *-adapted mortality intensity* μ *satisfies* $\underline{\mu}(t) \leq \mu_t \leq \bar{\mu}(t)$ *, with* $\underline{\mu}$ *and* $\bar{\mu}$ *being the same functions as in Assumption 2.3.1, i.e.,*

$$\mathcal{Q}^b \quad = \quad \left\{ \mathbb{Q} \in \mathcal{Q} : \underline{\mu} \leq \mu \leq \bar{\mu}, \text{ where } \mu \text{ is the } \mathbb{Q}\text{-intensity of } \tau \right\} . \tag{2.15}$$

In Section 2.4 we establish upper and lower price bounds for specific unit-linked life insurance contracts. Subsequently, in Section 2.5 we show that the upper price bound indeed leads to the cheapest superhedge once diversification is applied such that the biometric risk is eliminated. The respective results for the lower price bound and the most expensive subhedge follows analogously.

2.4 Pricing Unit-Linked Life Insurance Contracts

2.4.1 Payoff Structures

Now we introduce a unit-linked life insurance contract with Markovian payoff structures to the $\mathbb{G}$ market. The contract has the life time of T years. It may be obtained by the

[5]This indicates that given the same the confidence bound, the confidence level under the equivalent martingale measure $\mathbb{Q}$ is not identical with the confidence level under the real world measure $\mathbb{P}$. We assume that the difference in the confidence level is not significantly big.

policyholders upon upfront single payment or a continuous flow of premiums[6]. When the policyholder dies at $\tau < T$, the contract pays $\Psi(\tau, S_\tau)$ immediately. When he survives time T, the payment is $\Phi(S_T)$. The policyholder is entitled to this payoff structure if he pays the premium required. We assume that the cumulated premium payment at time t is $A_t = A_0 + \int_0^t \Gamma(u, S_u)\,\mathrm{d}u$ where Γ refers to the instantaneous premium payment rate on the annual basis. Through a concrete definition of Ψ, Φ and A, we obtain different types of contracts. Some examples are:

- Term insurance: $\Psi(\tau, S_\tau) > 0$, for $\tau \leq T$, and $\Phi(S_T) = 0$, for $\tau > T$.
- Pure endowment insurance: $\Psi(\tau, S_\tau) = 0$, for $\tau \leq T$, and $\Phi(S_T) > 0$, for $\tau > T$.
- Endowment insurance: $\Psi(\tau, S_\tau) > 0$, for $\tau \leq T$, and $\Phi(S_T) > 0$ for $\tau > T$.
- Single premium: $A_t = A_0 = \text{constant}$.
- Periodic premium: A_t is increasing in t.

The contract cash flows specified by the functions Φ, Ψ, and Γ have to satisfy certain integrability conditions. These are summarized below.

Assumption 2.4.1. *The functions Φ, Ψ, and Γ satisfy the following integrability conditions*

$$\mathbb{E}^{\mathbb{Q}}\left[|\Phi(S_T)| + \int_0^T \left(|\Psi(t, S_t)| + |\Gamma(t, S_t)|\right)\,\mathrm{d}t\right] < \infty,$$

where $\mathbb{Q}$ is an equivalent martingale measure.

Note that if the condition holds for a specific $\mathbb{Q} \in \mathcal{Q}$ then it holds for any other equivalent martingale measure. The reason is that all equivalent martingale measures coincide on $\mathcal{F}_T^S$ and the random variable where the expectation is taken is $\mathcal{F}_T^S$-measurable.

Unit-linked life insurance contracts can also have exotic features not covered by our setup, e.g., a surrender guarantee or a guaranteed annuity option. In case of a surrender guarantee the policyholder can surrender the contract and receives the surrender payment replacing all payments afterward originally specified by Φ, Ψ and Γ. The surrender payment may or may not be linked to the underlying asset. If the contract specifies a guaranteed annuity rate a at which the policyholder has the right to convert the terminal payment into annuities at time T, then the terminal value of the contract becomes the original terminal value times a call option on the annuity rate.[7] Unit-linked life insurance contracts with exotic features are important contract types. They can be discussed when extending our framework. However, in this chapter we work with unit-linked life insurance contracts with rather simple payoff structures as was specified at the beginning of this

[6] In reality, periodic premiums are paid monthly or yearly. We assume the continuous flow of premiums just for illustration simplicity.

[7] The payoff is $\Phi(S_T) \cdot \max(1, a\,\mathbb{E}_{\mathbb{Q}}[\int_T^\infty \exp\{-\int_T^u (r + \mu_s) du\}])$.

section. By this we can explain the method we adopt to analyze the risk management of unit-linked life insurance contracts under mortality model risk.

2.4.2 Arbitrage Free Prices and Price Bounds

Fix an equivalent martingale measure $\mathbb{Q} \in \mathcal{Q}$ with mortality intensity μ. An arbitrage free price of the contract (Φ, Ψ, Γ) is given by the conditional expectation of the discounted cashflow under $\mathbb{Q}$, see, e.g., Björk (2009). Decomposing the contract into its components the arbitrage free prices of the death benefit $V^{\mu,\Psi}$, the survival benefit $V^{\mu,\Phi}$, and the premium $V^{\mu,\Gamma}$ are:

$$\begin{aligned} V_t^{\mu,\Psi} &= B_t \mathbb{E}^{\mathbb{Q}}\left[B_\tau^{-1} 1_{\{t<\tau\le T\}} \Psi(\tau, S_\tau) | \mathcal{G}_t\right], \quad V_t^{\mu,\Phi} = B_t \mathbb{E}^{\mathbb{Q}}\left[B_T^{-1} 1_{\{\tau>T\}} \Phi(S_T) | \mathcal{G}_t\right], \\ V_t^{\mu,\Gamma} &= B_t \mathbb{E}^{\mathbb{Q}}\left[\int_t^T B_u^{-1} 1_{\{u<\tau\le T\}} \Gamma(u, S_u)\, \mathrm{d}u \,\middle|\, \mathcal{G}_t\right], \end{aligned}$$

and the arbitrage free price of the aggregate contract V^μ is then

$$V_t^\mu = V_t^{\mu,\Psi} + V_t^{\mu,\Phi} - V_t^{\mu,\Gamma}, \quad 0 \le t \le T.$$

The arbitrage free prices for a life insurance contract and its components under a specific equivalent martingale measure $\mathbb{Q}$ can be given in a more explicit form. Duffie, Schroder and Skiadas (1996) have shown that

$$\begin{aligned} V_t^{\mu,\Psi} &= 1_{\{\tau>t\}} \hat{B}_t \mathbb{E}^{\mathbb{Q}}\left[\int_t^T \hat{B}_u^{-1} \Psi(u, S_u) \mu_u \, \mathrm{d}u \,\middle|\, \mathcal{F}_t\right], V_t^{\mu,\Phi} = 1_{\{\tau>t\}} \hat{B}_t \mathbb{E}^{\mathbb{Q}}\left[\hat{B}_T^{-1} \Phi(S_T) \,\middle|\, \mathcal{F}_t\right], \\ V_t^{\mu,\Gamma} &= 1_{\{\tau>t\}} \hat{B}_t \mathbb{E}^{\mathbb{Q}}\left[\int_t^T \hat{B}_u^{-1} \Gamma(u, S_u)\, \mathrm{d}u \,\middle|\, \mathcal{F}_t\right], \quad 0 \le t \le T, \end{aligned} \tag{2.16}$$

where $\hat{B}_\cdot = \exp\{\int_0^\cdot (r(s) + \mu_s)\, \mathrm{d}s\}$ represents a mortality risk adjusted money market account that also depends on the choice of $\mathbb{Q}$ via μ.

According to the so-called reduced forms above we can consider the contract price as the discounted value of a fictitious security whose dividend payment at t is $\Psi(t, S_t)\, \mu_t - \Gamma(t, S_t)$ and final payment is $\Phi(S_T)$. The fictitious discount factor is $\hat{B}$. In this fictitious world, we can ignore the mortality risk in the form of τ and consider the insurance contract merely as a contingent claim on the fictitious financial market.

From now on, we discuss the pricing problem within the class of equivalent martingale measures $\mathcal{Q}^b$ where the mortality intensity is bounded from below and from above. The worst case that may happen to the insurance company with regard to the death benefit, the survival benefit, the premium and the contract price, respectively, is

$$\bar{V}_t^{\Psi} = \operatorname*{ess\,sup}_{\mathbb{Q}\in\mathcal{Q}^b} V_t^{\mu,\Psi}\,, \quad \bar{V}_t^{\Phi} = \operatorname*{ess\,sup}_{\mathbb{Q}\in\mathcal{Q}^b} V_t^{\mu,\Phi}\,, \quad \underline{V}_t^{\Gamma} = \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{Q}^b} V_t^{\mu,\Gamma}\,,$$

and

$$\bar{V}_t = \operatorname*{ess\,sup}_{\mathbb{Q}\in\mathcal{Q}^b} V_t^{\mu}\,, \quad 0 \le t \le T\,.$$

In view of the results of Duffie et al. (1996) presented above, we can transfer the problem of choosing the best or worst equivalent martingale measure $\mathbb{Q} \in \mathcal{Q}^b$ to the problem of choosing the best or worst mortality intensity $\mu \in \mathcal{U}_t$, where $\mathcal{U}_t$ is the set of $\mathbb{F}$-predictable processes μ on $[t,T]$ such that $\underline{\mu}(s) \le \mu_s \le \bar{\mu}(s)$, for $t \le s \le T$. In particular, we may write

$$\bar{V}_t^{\Psi} = 1_{\{\tau>t\}} \operatorname*{ess\,sup}_{\mu\in\mathcal{U}_t} \hat{B}_t \mathbb{E}^{\mathbb{Q}}\left[\int_t^T \hat{B}_u^{-1}\Psi(u,S_u)\mu_u\,\mathrm{d}u \,\middle|\, \mathcal{F}_t\right], \tag{2.17}$$

$$\bar{V}_t^{\Phi} = 1_{\{\tau>t\}} \operatorname*{ess\,sup}_{\mu\in\mathcal{U}_t} \hat{B}_t \mathbb{E}^{\mathbb{Q}}\left[\hat{B}_T^{-1}\Phi(S_T) \,\middle|\, \mathcal{F}_t\right], \tag{2.18}$$

$$\underline{V}_t^{\Gamma} = 1_{\{\tau>t\}} \operatorname*{ess\,inf}_{\mu\in\mathcal{U}_t} \hat{B}_t \mathbb{E}^{\mathbb{Q}}\left[\int_t^T \hat{B}_u^{-1}\Gamma(u,S_u)\,\mathrm{d}u \,\middle|\, \mathcal{F}_t\right], \tag{2.19}$$

and

$$\bar{V}_t = 1_{\{\tau>t\}} \operatorname*{ess\,sup}_{\mu\in\mathcal{U}_t} \hat{B}_t \mathbb{E}^{\mathbb{Q}}\left[\int_t^T \hat{B}_u^{-1}\Psi(u,S_u)\mu_u\,\mathrm{d}u + \hat{B}_T^{-1}\Phi(S_T) - \int_t^T \hat{B}_u^{-1}\Gamma(u,S_u)\,\mathrm{d}u \,\middle|\, \mathcal{F}_t\right]. \tag{2.20}$$

By specifying the stock price dynamics with (2.8), we have actually fixed the equivalent martingale measure on $\mathbb{F}^S$. Instead of looking for the optimal martingale measure on the $\mathbb{G}$ market, we convert the problem into looking for the $\mathbb{F}$-adapted process μ . The expressions in (2.17-2.20) are stochastic control problems with control process μ . The prices $V^{\mu,\Phi}$ and $V^{\mu,\Gamma}$ depend on μ monotonously. When considering $\mathbb{Q} \in \mathcal{Q}^b$, the highest arbitrage free price for the death benefit $\bar{V}_t^{\Phi}$ is obtained for $\mu = \underline{\mu}$, and the lowest value for the premium income $\underline{V}^{\Gamma}$ is obtained for $\mu = \bar{\mu}$. With regard to $\bar{V}^{\Psi}$ and $\bar{V}$, we apply stochastic control techniques to obtain the respective solutions.

The stock price process S is Markovian and the payoff functions are simple in the sense that they depend on time and the current value of the stock price. This suggest the standard setup of a stochastic control problem with state variable (t,s) , feedback control $\mu \in \mathcal{U}(t,s)$, and maximization problem

$$\bar{v}(t,s) = \sup_{\mu\in\mathcal{U}(t,s)} \mathbb{E}^{t,s}\left[\int_t^T \frac{\hat{B}_t}{\hat{B}_u}\Psi(u,S_u)\mu_u\,\mathrm{d}u + \frac{\hat{B}_t}{\hat{B}_T}\Phi(S_T) - \int_t^T \frac{\hat{B}_t}{\hat{B}_u}\Gamma(u,S_u)\,\mathrm{d}u\right], \tag{2.21}$$

where $\mathcal{U}(t,s) = \{\mu : [t,T] \times \mathbb{R}^+ \mapsto \mathbb{R}^+ : \underline{\mu}(u) \leq \mu(u,x) \leq \bar{\mu}(u)\,,\ \text{all } t \leq u \leq T, x \in \mathbb{R}^+\}$ and $\mathbb{E}^{t,s}$ denotes the expectation conditional on $S(t) = s$ under the measure $\mathbb{Q}^{\mathbb{F}^S}$. Recall, that $\hat{B}$ also depends on μ and in particular $\frac{\hat{B}_t}{\hat{B}_u} = \exp\{-\int_t^u (r(s) + \mu(s, S_s))\,\mathrm{d}s\}$. Observe that the term inside the conditional expectation is $\mathcal{F}_T^S$-measurable.

According to the theorem of the Hamilton-Jacobi-Bellman equation (confer Yong (1997) as well as Yong and Zhou (1999)), $\bar{v}$ is the solution to:

$$0 = \sup_{\mu \in \mathcal{U}(t,s)} \{\mathcal{L}\bar{v}(u,s) + \Psi(u,s)\,\mu(u,s) - \Gamma(u,s) - \bar{v}(u,s)[\mu(u,s) + r(u)]\}\,, \tag{2.22}$$

$$\Phi(s) = \bar{v}(T,s), \tag{2.23}$$

where

$$\mathcal{L}f(u,s) = \frac{\partial f}{\partial u}(u,s) + (r(u) - q(u,s))\,s\,\frac{\partial f}{\partial s}(u,s) + \frac{1}{2}\sigma^2(u,s)\,s^2\,\frac{\partial^2 f}{\partial s^2}(u,s). \tag{2.24}$$

The part of (2.22) that is depending on the control μ is given by $[\Psi(u,s) - \bar{v}(u,s)]\,\mu(u,s)$ and is linear in μ. Hence, we obtain pointwise

$$\sup_{\mu \in \mathcal{U}(t,s)} [\Psi(u,s) - \bar{v}(u,s)]\mu(u,s) = \begin{cases} [\Psi(u,s) - \bar{v}(u,s)]\bar{\mu}(u), & \text{if } \Psi(u,s) \geq \bar{v}(u,s), \\ [\Psi(u,s) - \bar{v}(u,s)]\underline{\mu}(u), & \text{if } \Psi(u,s) < \bar{v}(u,s). \end{cases}$$

The maximizer $\mu^\star$ is thus

$$\mu^\star(t,s) = \begin{cases} \bar{\mu}(t), & \text{if } \Psi(t,s) \geq \bar{v}(t,s), \\ \underline{\mu}(t), & \text{if } \Psi(t,s) < \bar{v}(t,s). \end{cases} \tag{2.25}$$

Plugging the pointwise maximizer in (2.22) gives

$$0 = \mathcal{L}\bar{v}(u,s) + \Psi(u,s)\,\mu^\star(u,s) - \Gamma(u,s) - \bar{v}(u,s)[\mu^\star(u,s) + r(u)]\,, \tag{2.26}$$

$$\Phi(s) = \bar{v}(T,s)\,. \tag{2.27}$$

In fact, the calculation above produces a candidate $\bar{v}$ for solution of the maximization problem (2.21). Moreover, we want this candidate $\bar{v}$ to solve the more general problem (2.20) in the sense that $1_{\{\tau > t\}}\,\bar{v}(t, S_t) = \bar{V}_t = \operatorname{ess\,sup}_{\mathbb{Q} \in \mathcal{Q}^b} V_t^\mu$. We will show this in Theorem 2.4.1 below. To do so we require the following integrability condition.

Assumption 2.4.2. *Denote* $\overline{v} \in C^{1,2}$ *the solution to the partial differential equation in (2.26-2.27). Assume the following integrability condition holds*

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \left(\frac{\partial \overline{v}}{\partial s}(t,S_t)\,\sigma(t,S_t)\,S_t\right)^2 \mathrm{d}t\right] < \infty\,.$$

Theorem 2.4.1. *Given the setup in Section 1.2, suppose Assumptions 2.3.1-2.4.1 hold. Denote* $\overline{v} \in C^{1,2}$ *the solution to the boundary value problem in* (2.26) *with terminal condition* (2.27) *and suppose that Assumption 2.4.2 holds. Then*

$$1_{\{\tau>t\}}\,\overline{v}(t,S_t) \;=\; \operatorname*{ess\,sup}_{\mathbb{Q}\in\mathcal{Q}^b} V_t^{\mu}\,, \quad 0 \le t \le T\,.$$

In particular, the mortality intensity $\mu^\star$ *that maximizes the contract value is given by* (2.25)*.*

Proof. First, we have to establish that indeed $1_{\{\tau>t\}}\,\overline{v}(t,S_t) = V_t^{\mu^\star}$ where $\mu^\star$ is the optimal control given in (2.25). Ito's lemma gives

$$\begin{aligned}
\mathrm{d}\overline{v}(t,S_t) \;=\; & \left(\frac{\partial \overline{v}}{\partial t}(t,S_t) + (r(t)-q(t,S_t))\,S_t\frac{\partial \overline{v}}{\partial s}(t,S_t) + \frac{1}{2}\,\sigma(t,S_t)^2\,S_t^2\,\frac{\partial^2 \overline{v}}{\partial s^2}(t,S_t)\right)\mathrm{d}t \\
& +\sigma(t,S_t)\,S_t\,\frac{\partial \overline{v}}{\partial s}(t,S_t)\,\mathrm{d}W_t^{1Q}\,.
\end{aligned}$$

Now, $\overline{v}$ satisfies (2.26) by assumption and thus can be written as

$$\begin{aligned}
\mathrm{d}\overline{v}(t,S_t) \;=\; & (r(t)\,\overline{v}(t,S_t) - (\Psi(t,S_t) - \overline{v}(t,S_t))\,\mu^\star(t,S_t) + \Gamma(t,S_t))\,\mathrm{d}t \\
& +\sigma(t,S_t)\,S_t\,\frac{\partial \overline{v}}{\partial s}(t,S_t)\,\mathrm{d}W_t^{1Q}\,.
\end{aligned}$$

The differential is a linear stochastic differential equation with formal solution

$$\begin{aligned}
\overline{v}(u,S_u) \;=\; & e^{\int_t^u (r(s)+\mu^\star(s,S_s))\,\mathrm{d}s}\Big(\overline{v}(t,S_t) \\
& - \int_t^u e^{-\int_t^w (r(s)+\mu^\star(s,S_s))\,\mathrm{d}s}\sigma(w,S_w)\,S_w\,\frac{\partial \overline{v}}{\partial s}(w,S_w)\,\mathrm{d}W_w^{1Q} \\
& - \int_t^u e^{-\int_t^w (r(s)+\mu^\star(s,S_s))\,\mathrm{d}s}\,(\Psi(w,S_w)\,\mu^\star(w,S_w) - \Gamma(w,S_w))\,\mathrm{d}w\Big)\,.
\end{aligned}$$

Set $u = T$ and recall that $\overline{v}(T,S_T) = \Phi(S_T)$. Then

$$\begin{aligned}
e^{-\int_t^T (r(s)+\mu^\star(s,S_s))\,\mathrm{d}s}\Phi(S_T) \;=\; & \overline{v}(t,S_t) \\
& - \int_t^T e^{-\int_t^u (r(s)+\mu^\star(s,S_s))\,\mathrm{d}s}\sigma(u,S_u)\,S_u\,\frac{\partial \overline{v}}{\partial s}(u,S_u)\,\mathrm{d}W_u^{1Q} \\
& - \int_t^T e^{-\int_t^u (r(s)+\mu^\star(s,S_s))\,\mathrm{d}s}\,(\Psi(u,S_u)\,\mu^\star(u,S_u) - \Gamma(u,S_u))\,\mathrm{d}u\,.
\end{aligned}$$

Solving for $\overline{v}(t,S_t)$ and taking the expectation with respect to $\mathcal{F}_t$ we obtain

$$\begin{aligned}\overline{v}(t,S_t) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T(r(s)+\mu^\star(s,S_s))\,\mathrm{d}s}\Phi(S_T)\Big|\,\mathcal{F}_t\right]\\ &\quad+\mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^u(r(s)+\mu^\star(s,S_s))\,\mathrm{d}s}\left(\Psi(u,S_u)\,\mu^\star(u,S_u)-\Gamma(u,S_u)\right)\mathrm{d}u\Big|\,\mathcal{F}_t\right],\end{aligned}$$

where the martingale part vanishes because of Assumption 2.4.2. Recalling the reduced form representation of $V^\mu = V^{\mu,\Psi}+V^{\mu,\Phi}-V^{\mu,\Gamma}$ in (2.16) we see that the candidate $\overline{v}$ is indeed a value function, i.e. $\mathbf{1}_{\{\tau>t\}}\,\overline{v}(t,S_t)=V_t^{\mu^\star}$, for $0\le t\le T$.

Next, the optimality of $\mu^\star$ and the corresponding value function $\overline{v}$ is established. We fix a measure $\mathbb{Q}\in\mathcal{Q}^b$ and as usual denote by μ the mortality intensity under $\mathbb{Q}$. Define the $\mathbb{F}$-adapted process U^μ by

$$U_t^\mu=\hat{B}_t\mathbb{E}^{\mathbb{Q}}\left[\hat{B}_T^{-1}\Phi(S_T)+\int_t^T\hat{B}_u^{-1}\Psi(u,S_u)\mu_u\,\mathrm{d}u-\int_t^T\hat{B}_u^{-1}\Gamma(u,S_u)\,\mathrm{d}u\Big|\mathcal{F}_t\right],\quad 0\le t\le T,$$

such that $V_t^\mu=\mathbf{1}_{\{\tau>t\}}\,U_t^\mu$, $0\le t\le T$. Define the accompanying martingale M^μ by

$$M_t^\mu=\mathbb{E}^{\mathbb{Q}}\left[\hat{B}_T^{-1}\Phi(S_T)+\int_0^T\hat{B}_u^{-1}\Psi(u,S_u)\mu_u\,\mathrm{d}u-\int_0^T\hat{B}_u^{-1}\Gamma(u,S_u)\,\mathrm{d}u\Big|\mathcal{F}_t\right],\quad 0\le t\le T.$$

Verify that $\mathbb{E}^{\mathbb{Q}}|M_T|<\infty$ by Assumptions 2.3.1 and 2.4.1 and M^μ is indeed an $(\mathbb{Q},\mathbb{F})$-martingale. Now, $\mathbb{Q}$ coincides with $\mathbb{Q}^{\mathbb{F}^S}$ on $\mathcal{F}_T^S$ and for the process $U^{\mu^\star}$ and $M^{\mu^\star}$ defined by

$$\begin{aligned}U_t^{\mu^\star} &= \hat{B}_t^\star\,\mathbb{E}^{\mathbb{Q}}\left[\hat{B}_T^{\star-1}\Phi(S_T)+\int_t^T\hat{B}_u^{\star-1}\Psi(u,S_u)\mu^\star(u,S_u)\,\mathrm{d}u-\int_t^T\hat{B}^{\star-1}\Gamma(u,S_u)\,\mathrm{d}u\Big|\mathcal{F}_t\right],\\ M_t^{\mu^\star} &= \mathbb{E}^{\mathbb{Q}}\left[\hat{B}_T^{\star-1}\Phi(S_T)+\int_0^T\hat{B}_u^{\star-1}\Psi(u,S_u)\mu^\star(u,S_u)\,\mathrm{d}u-\int_0^T\hat{B}_u^{\star-1}\Gamma(u,S_u)\,\mathrm{d}u\Big|\mathcal{F}_t\right],\end{aligned}$$

where $\hat{B}_\cdot^\star=\exp\{\int_0^\cdot(r(s)+\mu^\star(s,S_s))\,\mathrm{d}s\}$, for $0\le t\le T$, it holds that $U_t^{\mu^\star}=\overline{v}(t,S_t)$ and $M^{\mu^\star}$ is an $\mathbb{F}^S$-adapted $(\mathbb{Q},\mathbb{F})$-martingale. For $\eta\in\{\mu,\mu^\star\}$ we can connect U^η and M^η via

$$M_t^\eta=e^{-\int_0^t(r(s)+\eta_s)\,\mathrm{d}s}\,U_t^\eta+\int_0^t e^{-\int_0^u(r(s)+\eta_s)\,\mathrm{d}s}\,\Psi(u,S_u)\,\eta_u\,\mathrm{d}u-\int_0^t e^{-\int_0^u(r(s)+\eta_s)\,\mathrm{d}s}\,\Gamma(u,S_u)\,\mathrm{d}u\,,$$

or, alternatively, in the form of the stochastic differential

$$\begin{aligned}\mathrm{d}M_t^\eta &= e^{-\int_0^t(r(s)+\eta_s)\,\mathrm{d}s}\,\mathrm{d}U_t^\eta-(r(t)+\eta_t)\,e^{-\int_0^t(r(s)+\eta_s)\,\mathrm{d}s}\,U_t^\eta\,\mathrm{d}t\\ &\quad+e^{-\int_0^t(r(s)+\eta_s)\,\mathrm{d}s}\,\Psi(t,S_t)\,\eta_t\,\mathrm{d}t-e^{-\int_0^t(r(s)+\eta_s)\,\mathrm{d}s}\,\Gamma(t,S_t)\,\mathrm{d}t\,,\quad 0\le t\le T.\end{aligned}$$

Solving for $\mathrm{d}U_t^\eta$ yields

$$\mathrm{d}U_t^\eta = (r(t)+\eta_t)\,U_t^\eta\,\mathrm{d}t - \Psi(t,S_t)\,\eta_t\,\mathrm{d}t + \Gamma(t,S_t)\,\mathrm{d}t + \mathrm{d}\hat{M}_t^\eta\,, \quad 0 \le t \le T\,,$$

where $\mathrm{d}\hat{M}_t^\eta = e^{\int_0^t (r(s)+\eta_s)\,\mathrm{d}s}\,\mathrm{d}M_t^\eta$ is a $(\mathbb{Q},\mathbb{F})$-martingale since r and η are uniformly bounded by a deterministic constant. The next step is to define the difference process

$$X_t^\mu = U_t^{\mu^\star} - U_t^\mu\,, \quad 0 \le t \le T\,.$$

The terminal value of X^μ is $X_T^\mu = U_T^{\mu^\star} - U_T^\mu = \Phi(S_T) - \Phi(S_T) = 0$. The stochastic differential of X^μ is given by

$$\begin{aligned}
\mathrm{d}X_t^\mu &= \mathrm{d}U_t^{\mu^\star} - \mathrm{d}U_t^\mu \\
&= \Big[(r(t)+\mu_t)\,X_t^\mu + \Big(U_t^{\mu^\star} - \Psi(t,S_t)\Big)\,(\mu^\star(t,S_t)-\mu_t)\Big]\,\mathrm{d}t + \mathrm{d}\hat{M}_t^{\mu^\star} - \mathrm{d}\hat{M}_t^\mu\,.
\end{aligned}$$

The above differential can be interpreted as a linear stochastic differential with formal solution

$$\begin{aligned}
X_u^\mu &= e^{\int_t^u (r(s)+\mu_s)\,\mathrm{d}s}\Big(X_t^\mu + \int_t^u e^{-\int_t^w (r(s)+\mu_s)\,\mathrm{d}s}(\mathrm{d}\hat{M}_w^{\mu^\star} - \mathrm{d}\hat{M}_w^\mu) \\
&\quad - \int_t^u e^{-\int_t^w (r(s)+\mu_s)\,\mathrm{d}s}\left[U_w^{\mu^\star} - \Psi(w,S_w)\right]\,[\mu^\star(w,S_w)-\mu_w]\,\mathrm{d}w\Big)\,, \quad 0 \le t \le u \le T\,.
\end{aligned}$$

Set $u = T$, solve for X_t^μ and recall $X_T^\mu = 0$. Taking the expectation conditioned on $\mathcal{F}_t$ eliminates the $(\mathbb{Q},\mathbb{F})$-martingale and

$$X_t^\mu = \mathbb{E}^\mathbb{Q}\left[\int_t^T e^{-\int_t^w (r(s)+\mu_s)\,\mathrm{d}s}\left[U_w^{\mu^\star} - \Psi(w,S_w)\right]\,[\mu^\star(w,S_w)-\mu_w]\,\mathrm{d}w\,\middle|\,\mathcal{F}_t\right]$$

Recall that $U_t^{\mu^\star} = \bar{v}(t,S_t)$ and $\mu \in \mathcal{Q}^b$, i.e., $\underline{\mu}(t) \le \mu_t \le \bar{\mu}(t)$, for $0 \le t \le T$. Then by the definition of $\mu^\star$ in (2.25) we have that the integrand inside the conditional expectation is nonnegative. This implies that $X_t^\mu \ge 0$ or, equivalently, $\bar{v}(t,S_t) \ge U_t^\mu$, for $0 \le t \le T$. Multiplying on the indicator process $\mathbf{1}_{\{t<\tau\}}$ we obtain

$$\mathbf{1}_{\{t<\tau\}}\,\bar{v}(t,S_t) \;\ge\; V_t^\mu\,, \quad 0 \le t \le T\,,$$

for all $\mathbb{Q} \in \mathcal{Q}^b$ and corresponding $\mathbb{Q}$-mortality rate μ, establishing the optimality. □

Remark 2.4.1. *The lower bound of the arbitrage free prices* $\underline{V}_t = \operatorname{ess\,inf}_{\mathbb{Q}\in\mathcal{Q}^b} V_t^\mu$ *can be obtained analogously. The minimizer* $\mu_\star$ *is obtained by swapping* $\underline{\mu}$ *and* $\bar{\mu}$ *in* (2.25). *The solution to the partial differential equation* (2.26) *where* $\mu_\star$ *is replaced by* $\mu^\star$ *with terminal condition* (2.27) *is denoted* $\underline{v}$. *Then* $\mathbf{1}_{\{t<\tau\}}\,\underline{v}(t,S_t) = \operatorname{ess\,inf}_{\mathbb{Q}\in\mathcal{Q}^b} V_t^\mu$.

Remark 2.4.2. *The value maximizing mortality* $\mu^\star$ *in*(2.25) *is* $\mathbb{F}^S$*-adapted. However, the set of admissible controls is much larger allowing for* $\mathbb{F}$*-adapted control processes.*

Obviously, the information generated by the environment process X *is not needed for finding the maximal arbitrage free price. This result can be explained by properties of our model. The risk introduced by* X *cannot be hedged since no liquidly traded assets are available for trading and potentially eliminating the associated risk. Further, the bounds* $\underline{\mu}$ *and* $\bar{\mu}$ *in Assumption 2.3.1 and Definition 2.3.1 are almost sure bounds and do not depend on the environment process* X *. Both properties together explain that the optimal control process* $\mu^\star$ *can be determined based on* $\mathbb{F}^S$ *, the information generated by the traded asset with price process* S *.*

Remark 2.4.3. *We can summarize the optimal control rule concerning the death benefit, the survival benefit, the premium and the whole contract as follows so as to obtain their worst-case and best-case values.*

	worst case	*best case*
death benefit	$\bar{\mu}$ *if* $\Psi \geq \bar{v}^\Psi$	$\underline{\mu}$ *if* $\Psi \geq \underline{v}^\Psi$
	$\underline{\mu}$ *if* $\Psi < \bar{v}^\Psi$	$\bar{\mu}$ *if* $\Psi < \underline{v}^\Psi$
survival benefit	$\underline{\mu}$	$\bar{\mu}$
premium	$\bar{\mu}$	$\underline{\mu}$
whole contract	$\bar{\mu}$ *if* $\Psi \geq \bar{v}$	$\underline{\mu}$ *if* $\Psi \geq \underline{v}$
	$\underline{\mu}$ *if* $\Psi < \bar{v}$	$\bar{\mu}$ *if* $\Psi < \underline{v}$

The worst-case value of the contract is its upper price bound and the best-case value is the lower price bound of the contract.

Theorem 2.4.1 indicates that the price bound of an insurance contract usually cannot be obtained by keeping μ to its lower or upper bound. The simple rule of keeping μ to its lower or upper bound is only possible for some special cases. Here are two examples:

Pure endowment insurance with single premium In this case, we have $\Psi = 0$ and $\Gamma = 0$, and hence the value of μ is irrelevant for the death benefit part and the premium part. The maximal value for the survival benefit is obtained by setting $\mu = \underline{\mu}$ and then $\bar{v} = \bar{v}^\Phi$, on $(0, T]$. Similarly, the minimal value for the survival benefit is obtained by setting $\mu = \bar{\mu}$ and then $\underline{v} = \underline{v}^\Phi$, on $[0, T)$.

Term insurance with single premium or periodic premiums The death benefit takes the form $\Psi(t, s) = Ke^{gt}$ with $g \leq r$ or $\Psi(t, s) = S_t$. In the former case, we have

$$\begin{aligned} v^{\mu,\Psi}(t,s) &= K\,\mathbb{E}^{t,s}\left[\int_t^T \exp\left(-\int_t^u (r(s)-g)s\,\mathrm{d}s\right)\exp\left(-\int_t^u \mu_s\,\mathrm{d}s\right)\mu_u\,\mathrm{d}u\right] \\ &\leq K\,\mathbb{E}^{t,s}\left[\int_t^T \exp\left(-\int_t^u \mu_s\,\mathrm{d}s\right)\mu_u\,\mathrm{d}u\right] \\ &\leq K\left(1-\mathbb{E}^{t,s}\left[\exp\left(-\int_t^T \mu_u\,\mathrm{d}u\right)\right]\right) \leq K \leq Ke^{gt} = \Psi(t,s), \end{aligned}$$

and in the latter case, there is

$$\begin{aligned} v^{\mu,\Psi}(t,s) &= \mathbb{E}^{t,s}\left[\exp\left(-\int_t^\tau r(s)ds\right) S_\tau\, 1_{\{\tau\le T\}}\middle|\, \tau>t\right] \\ &\le \mathbb{E}^{t,s}\left[\exp\left(-\int_t^\tau r(s)ds\right) S_\tau \middle|\, \tau>t\right] \;=\; s \;=\; \Psi(t,s). \end{aligned}$$

In both cases we know that $v^{\mu} = v^{\mu,\Psi} - v^{\mu,\Gamma} \le v^{\mu,\Psi} \le \Psi$ and therefore the maximum (minimum) value is obtained when μ is set to $\bar{\mu}$ ($\underline{\mu}$). In the single premium case we have $\bar{v} = \bar{v}^{\Psi}$ and $\underline{v} = \underline{v}^{\Psi}$, on $[0,T)$. In the periodic premium case we have $\bar{v} = \bar{v}^{\Psi} - \underline{v}^{\Gamma}$ and $\underline{v} = \underline{v}^{\Psi} - \bar{v}^{\Gamma}$, on $[0,T)$.

2.4.3 Connection to American-style Financial Contracts

We return to the partial differential equation for a given μ :

$$0 \;=\; \mathcal{L}v(u,s) + \Psi(u,s)\,\mu(u,s) - \Gamma(u,s) - v(u,s)\,[\mu(u,s)+r(u)], \quad t\le u\le T \tag{2.28}$$

with terminal condition $v(T,s) = \Phi(s)$. If we allow μ to move between $[0,\infty)$ which is beyond its original bounds, by setting

$$\mu(u,s) = \begin{cases} 0, & \text{if } \; v(u,s) > \Psi(u,s) \\ \infty, & \text{if } \; v(u,s) \le \Psi(u,s) \end{cases} \tag{2.29}$$

for $t \le u < T$, we force the contract to stop immediately when the contract value reaches the death benefit from above so that $v(u,s) \ge \Psi(u,s)$ is always satisfied. This is equivalent to an optimal stopping problem, whose linear complementarity formulation is

$$\begin{aligned} [\mathcal{L}v(u,s) - \Gamma(u,s) - r(u)\,v(u,s)]\;[v(u,s) - \Psi(u,s)] \;&=\; 0, \\ \mathcal{L}v(u,s) - \Gamma(u,s) - r(u)\,v(u,s) \;\le\; 0, \quad \text{and} \quad v(u,s) - \Psi(u,s) \;&\ge\; 0, \end{aligned}$$

for $t \le u < T$, with adjusted terminal condition $v(T,s) = \max(\Phi(s), \Psi(T,s))$. Similar to Dai, Kwok and You (2007) where the prepayment of mortgage loans is discussed, (2.28) together with (2.29) can be visualized as the penalty approximation to the linear complementarity formulation following the theory of variational inequalities of free boundary problems.

Remark 2.4.4. *Equation (2.29) specifies the optimal control of* μ *within a broader bound which corresponds to the larger set of martingale measure* $\mathcal{Q}$ *defined by (2.9). Setting* $\mu = \mu^\star$ *in (2.28), where* $\mu^\star$ *is given in* (2.25)*, can be viewed as using a suboptimal stopping strategy which does not necessarily yield* $v(u,s) \geq \Psi(u,s)$ *. The value function of the optimal stopping problem will produce a superhedge. In contrast the upper price bound* $\bar{v}$ *based on* $\mathbb{Q}^b$ *cannot produce a superhedge in general. However, a superhedge will arise when diversification of the unsystematic mortality risk is taken into account.*

2.5 Hedging Unit-Linked Life Insurance Contracts

The upper and lower price bounds for unit-linked life insurance contracts in Theorem 2.4.1 and Remark 2.4.1 suggest the implementation of hedging strategies related to these bounds. The financial risk driven by S can be eliminated by these strategies since the risk is represented by a traded asset. In contrast, the mortality risk cannot be eliminated in general. The trading strategies corresponding to the upper (lower) price bound cannot produce a superhedge (subhedge) for a specific single contract. However, mortality risk can be diversified by considering a sufficiently large number of independent policyholders, and then a superhedge (subhedge) can be produced in the limit.

We consider a community of policyholders of size N where the contracts for each individual are identical and given by (Φ, Ψ, Γ) . Further, we assume that the death times of the policyholders $(\tau_i)_{i=1,\dots,N}$ are independent given $\mathcal{F}_T$.[8] The number of individuals that have died until t is denoted by X_t^N and the number of policyholders that are still alive at time t is denoted by $\bar{X}_t^N$, respectively, i.e.

$$X_t^N = \sum_{i=1}^N \mathbf{1}_{\{\tau_i \leq t\}} , \quad \text{and} \quad \bar{X}_t^N = N - X_t^N = \sum_{i=1}^N \mathbf{1}_{\{t < \tau_i\}} , \quad 0 \leq t \leq T . \tag{2.30}$$

Fix as input parameter the potentially misspecified mortality intensity $\tilde{\mu} = (\tilde{\mu}(t, S_t))_{0 \leq t \leq T}$ that is Markovian with state vector (t, S_t) . Compute the price of the contract of a policyholder who is alive at time t as solution $\tilde{v}$ to

$$\mathcal{L}\tilde{v}(t,s) - \Gamma(t,s) + \tilde{\mu}(t,s)\left[\Psi(t,s) - \tilde{v}(t,s)\right] - r(t)\tilde{v}(t,s) \;=\; 0, \qquad \text{and} \qquad \tilde{v}(T,s) = \Phi(s), \tag{2.31}$$

where $\mathcal{L}$ is given in (2.24).

The potentially misspecified value $\tilde{V}^N$ of the outstanding contracts is thus given by the

[8] The underlying model in Section 2.2.1 is extended canonically, i.e., take an i.i.d. family of r.v.s $(E_n)_{n=1,\dots,N}$ that are unit exponentially distributed, pairwise independent, and independent of W . Then define the death times $(\tau_n)_{n=1,\dots,N}$ by $\tau_n = \operatorname{ess\,inf}\{t \geq 0 : \int_0^t \nu_s \, ds \geq E_n\}$, $n = 1, \dots, N$.

number of living policyholders $\bar{X}^N$ times the corresponding value $\tilde{v}$, i.e.

$$\tilde{V}_t^N = \bar{X}_t^N \, \tilde{v}(t, S_t) \,, \quad 0 \le t \le T \,. \tag{2.32}$$

In the formulation of $\tilde{V}^N$ in (2.32) we have two potential sources of error. Firstly, the individual contract may be incorrectly priced by $\tilde{v}$ and secondly, the mortality risk cannot be hedged in our setup and thus the jumps of $\bar{X}^N$ will introduce a further error. In the following we analyze the error when setting up a hedging portfolio based on (2.32).

The hedging strategy and resulting portfolio processes are now specified. Let the left-continuous and adapted process h^N denote the holdings in the risky asset S. (We do not have to specify the holding in the money market account since we are considering self-financing strategies and can use the budget constraint.) The portfolio process V^N with initial value V_0^N is defined by the stochastic differential equation

$$\mathrm{d}V_t^N = (V_t^N - h_t^N \, S_t) \, r(t) \, \mathrm{d}t + h_t^N \, (\mathrm{d}S_t + \mathrm{d}D_t) + \bar{X}_t^N \, \Gamma(t, S_t) \, \mathrm{d}t - \Psi(t, S_t) \, \mathrm{d}X_t^N \,. \tag{2.33}$$

The portfolio process is self-financing given the inflow of premium payment rate Γ of the active contracts $\bar{X}^N$ and the discrete time outflows of the death benefits Ψ at times the individuals pass away given by X^N. The hedge implied by (2.32) is given by

$$h_t^N = \bar{X}_t^N \, \frac{\partial \tilde{v}}{\partial s}(t, S_t) \,, \quad 0 \le t \le T \,. \tag{2.34}$$

It is clear that the price $\tilde{v}$ and the corresponding hedge ratio h are both determined by the assumed mortality intensity $\tilde{\mu}$. For a specific choice of $\tilde{\mu}$ the resulting hedging error is analyzed under the real world measure $\mathbb{P}$. The error has two additive components: a jump-martingale component capturing unsystematic mortality risk, and a predictable finite variation component which is determined by the systematic mortality risk.

Theorem 2.5.1. *Fix $\tilde{\mu} = \tilde{\mu}(t, s)$ and determine $\tilde{v}$ as the solution to* (2.31)*. Fix the size of the community of policyholders N and then define h^N by* (2.34)*. For the corresponding portfolio process V^N in* (2.33) *the hedging error E^N relative to $\tilde{V}^N$ given in* (2.32) *is defined by $E^N = V^N - \tilde{V}^N$. Then the hedging error and has the following $\mathbb{P}$-dynamics:*

$$\mathrm{d}E_t^N = E_t^N \, r(t) \, \mathrm{d}t + \bar{X}_t^N \, [\Psi(t, S_t) - \tilde{v}(t, S_t)] \, [\tilde{\mu}(t, S_t) - \nu_t] \, \mathrm{d}t + [\tilde{v}(t, S_t) - \Psi(t, S_t)] \, \mathrm{d}M_t^N \,,$$

with initial value $E_0^N = V_0^N - N \, \tilde{v}(0, S_0)$. Moreover, $M^N = X^N - \int_0^\cdot \bar{X}_t^N \, \nu_t \, \mathrm{d}t$ is a $\mathbb{P}$-martingale.

Proof. Write the stochastic differential of the portfolio value process using the definitions of the strategy h^N in (2.34) and the risky assets dynamics in (2.1) and (2.2):

$$\begin{aligned}
\mathrm{d}V_t^N &= \left[V_t^N - \bar{X}_t^N \frac{\partial \tilde{v}}{\partial s}(t,S_t)\,S_t\right] r(t)\,\mathrm{d}t + \bar{X}_t^N \frac{\partial \tilde{v}}{\partial s}(t,S_t)\,[a(t,S_t)+q(t,S_t)]\,S_t\,\mathrm{d}t \\
&\quad + \bar{X}_t^N \frac{\partial \tilde{v}}{\partial s}(t,S_t)\,\sigma(t,S_t)\,S_t\,\mathrm{d}W_t + \bar{X}_t^N\,\Gamma(t,S_t)\,\mathrm{d}t - \Psi(t,S_t)\,\mathrm{d}X_t^N \\
&= V_t^N\,r(t)\,\mathrm{d}t + \bar{X}_t^N \frac{\partial \tilde{v}}{\partial s}(t,S_t)\;(a(t,S_t)\,S_t\,\mathrm{d}t + \sigma(t,S_t)\,S_t\,\mathrm{d}W_t) - \Psi(t,S_t)\,\mathrm{d}M_t^N \\
&\quad - \bar{X}_t^N \left[\frac{\partial \tilde{v}}{\partial s}(t,S_t)\,[r(t)-q(t,S_t)]\,S_t - \Gamma(t,S_t) + \Psi(t,S_t)\,\nu_t\right]\mathrm{d}t\,.
\end{aligned}$$

To the last line we apply the partial differential equation (2.31) and then

$$\begin{aligned}
\mathrm{d}V_t^N &= V_t^N\,r(t)\,\mathrm{d}t + \bar{X}_t^N \frac{\partial \tilde{v}}{\partial s}(t,S_t)\;[a(t,S_t)\,S_t\,\mathrm{d}t + \sigma(t,S_t)\,S_t\,\mathrm{d}W_t] - \Psi(t,S_t)\,\mathrm{d}M_t^N \\
&\quad + \bar{X}_t^N \Big[\frac{\partial \tilde{v}}{\partial t}(t,S_t) + \frac{1}{2}\,\sigma^2(t,S_t)\,\frac{\partial^2 \tilde{v}}{\partial s^2}(t,S_t) - r(t)\,\tilde{v}(t,S_t) \\
&\qquad + [\tilde{\mu}(t,S_t)-\nu_t]\,[\Psi(t,S_t)-\tilde{v}(t,S_t)] - \tilde{v}(t,S_t)\,\nu_t\Big]\,\mathrm{d}t\,.
\end{aligned}$$

Using the product rule we obtain the stochastic differential of $\tilde{V}^N = \bar{X}^N\,\tilde{v}(\cdot,S)$:

$$\begin{aligned}
\mathrm{d}\tilde{V}_t^N &= \bar{X}_t^N \left[\frac{\partial \tilde{v}}{\partial t}(t,s) + a(t,S_t)\,S_t\,\frac{\partial \tilde{v}}{\partial s}(t,s) + \frac{1}{2}\sigma(t,S_t)^2\,S_t^2\,\frac{\partial^2 \tilde{v}}{\partial s^2}(t,S_t) - \tilde{v}(t,S_t)\,\nu_t\right]\mathrm{d}t \\
&\quad + \bar{X}_t^N\,\sigma(t,S_t)\,S_t\,\frac{\partial \tilde{v}}{\partial s}(t,s)\,\mathrm{d}W_t - \tilde{v}(t,S_t)\,\mathrm{d}M_t^N\,.
\end{aligned}$$

Now collect the terms from the two equations above to compute stochastic differential of the hedging error $E^N = V^N - \tilde{V}^N$:

$$\begin{aligned}
\mathrm{d}E_t^N &= E^N\,r(t)\,\mathrm{d}t + \bar{X}_t^N\,[\tilde{\mu}(t,S_t)-\nu_t]\,[\Psi(t,S_t)-\tilde{v}(t,S_t)]\,\mathrm{d}t \\
&\quad + [\tilde{v}(t,S_t)-\Psi(t,S_t)]\,\mathrm{d}M_t^N\,.
\end{aligned}$$

Finally, to verify that M^N is a $\mathbb{P}$-martingale see Bielecki and Rutkowski (2001), Proposition 5.1.3., p. 153. □

Remark 2.5.1. *The stochastic differential equation for hedge error* E^N *in Theorem 2.5.1 has the straight-forward solution*

$$E_t^N = e^{\int_0^t r(s)\,\mathrm{d}s}\left(V_0^N - N\,\tilde{v}(0,S_0)\right) + \int_0^t e^{\int_u^t r(s)\,\mathrm{d}s}\,\bar{X}_u^N\,[\Psi(u,S_u)-\tilde{v}(t,S_u)]\,[\tilde{\mu}(t,S_u)-\nu_u]\,\mathrm{d}u$$
$$+\int_0^t e^{\int_u^t r(s)\,\mathrm{d}s}\,[\tilde{v}(u,S_u)-\Psi(u,S_u)]\,\mathrm{d}M_u^N\,.$$

Suppose that the community of policyholders is very large, then the following corollary gives a limit result.

Corollary 2.5.1. *In the setting of Theorem 2.5.1 define the scaled hedging error* $\bar{E}_t^N$ *by*

$$\bar{E}_t^N = \frac{1}{N}\,E_t^N\,,\quad 0\le t\le T\,.$$

Assume that the following limit exists $\lim_{N\to\infty} V_0^N/N =: v$ *, then*

$$\sup_{0\le t\le T}\left|\bar{E}_t^N - \bar{E}_t^\infty\right| \xrightarrow{\mathbb{P}} 0\,,$$

where

$$\bar{E}_t^\infty = e^{\int_0^t r(s)\,\mathrm{d}s}\,(v-\tilde{v}(0,S_0))$$
$$+\int_0^t e^{\int_u^t r(s)\,\mathrm{d}s}\,e^{-\int_0^u \nu_s\,\mathrm{d}s}\,[\Psi(u,S_u)-\tilde{v}(u,S_u)]\,[\tilde{\mu}(u,S_u)-\nu_u]\,\mathrm{d}u\,.$$

Proof. Take the integral representation of the hedge error E^N in Remark 2.5.1 and divide this by N to obtain $\bar{E}^N$. Using the triangular inequality we can study each of the three expressions separately and establish uniform convergence in probability to the corresponding counterpart in $\bar{E}^\infty$. The first expression yields

$$\frac{1}{N}\,e^{\int_0^t r(s)\,\mathrm{d}s}\left(V_0^N - N\,\tilde{v}(0,S_0)\right)\to e^{\int_0^t r(s)\,\mathrm{d}s}\,(v-\tilde{v}(0,S_0))\quad \text{for } N\to\infty\,,$$

by the assumption $v=\lim_{N\to\infty}V_0^N/N$. The convergence is uniform in t since r is deterministic and the integral $\int_0^t r(s)\,\mathrm{d}s$ is a deterministic and continuous function. Accordingly, the expression $e^{\int_0^t r(s)\,\mathrm{d}s}$ is uniformly bounded by a constant on the compact $[0,T]$.

The error in the second expression is then

$$R_t^N = \int_0^t e^{\int_u^t r(s)\,\mathrm{d}s}\left(\frac{\bar{X}_u^N}{N} - e^{-\int_0^t \nu_s\,\mathrm{d}s}\right)[\Psi(u,S_u)-\tilde{v}(u,S_u)]\,[\tilde{\mu}(u,S_u)-\nu_u]\,\mathrm{d}u\,,$$

where we have deducted the integral part of $\bar{E}^\infty$. Then we can establish the following uniform bound

$$\sup_{0\le t\le T} |R_t^N| \;\le\; \sup_{0\le t\le T}\left|\frac{\bar{X}_t^N}{N} - e^{-\int_0^t \nu_s\,\mathrm{d}s}\right| \int_0^T e^{\int_t^T r(s)\,\mathrm{d}s}\,|\Psi(t,S_t)-\tilde{v}(t,S_t)|\,|\tilde{\mu}(t,S_t)-\nu_t|\,\mathrm{d}t\,.$$

The bound depends on N only in the first component. The second component is almost surely finite, and we are left to show that the first component vanishes in probability. To do so, consider

$$e^{-\int_0^t \nu_s\,\mathrm{d}s}\,Y_t^N = \frac{\bar{X}_t^N}{N} - e^{-\int_0^t \nu_s\,\mathrm{d}s}\,, \quad \text{or, equivalently, } Y_t^N = \frac{1}{N}\sum_{i=1}^N \mathbf{1}_{\{t<\tau_i\}}\,e^{\int_0^t \nu_s\,\mathrm{d}s} - 1\,.$$

Then Y^N is a local $\mathbb{P}$-martingale, see Bielecki and Rutkowski (2001), Lemma 5.1.7., p. 152. Moreover, Y^N is square integrable since we assume that ν is uniformly bounded in t, and therefore

$$[Y^N, Y^N]_t = \frac{1}{N^2}\sum_{\{\tau_i\le t\}} e^{2\int_0^{\tau_i}\nu_s\,\mathrm{d}s}\,,$$

is uniformly bounded in t by a deterministic constant on the compact $[0,T]$. Doob's maximal quadratic inequality then gives

$$\mathbb{E}\left(\sup_{0\le t\le T}|Y_t^N|^2\right) \le 4\,\mathbb{E}\left([Y^N,Y^N]_T\right) \le \frac{4}{N}\,e^{2\int_0^T \bar{\mu}(s)\,\mathrm{d}s} = O(1/N)\,,$$

Accordingly, $\sup_{0\le t\le T}|Y_t^N|$ tends to zero in $L^2(\mathbb{P})$ and hence in probability. This establishes the uniform convergence of R^N to zero in probability.

Finally, consider the third expression

$$Z_t^N = \frac{1}{N}\int_0^t e^{\int_u^t r(s)\,\mathrm{d}s}\,[\tilde{v}(u,S_u)-\Psi(u,S_u)]\,\mathrm{d}M_u^N\,.$$

The process Z^N is a $\mathbb{P}$-martingale with quadratic variation

$$[Z^N, Z^N]_t = \frac{1}{N^2}\int_0^t e^{2\int_u^t r(s)\,\mathrm{d}s}\,[\tilde{v}(u,S_u)-\Psi(u,S_u)]^2\,\mathrm{d}X_u^N\,.$$

Note that we can find a localizing sequence of stopping times $(\sigma_n)_{n\ge 1}$ such that the stopped process $(\tilde{v}(\cdot,S)-\Psi(\cdot,S))^{\sigma_n}$ is uniformly bounded by a deterministic constant, say C_n. Thus, without loss of generality we may assume that $(\tilde{v}(\cdot,S)-\Psi(\cdot,S))$ is indeed bounded by real number, say by $C>0$. Then Z^N is square integrable and by Doob's maximal quadratic inequality we obtain

$$\begin{aligned}\mathbb{E}\left(\sup_{0\le t\le T}|Z_t^N|^2\right) &\le 4\,\mathbb{E}\left([Z^N,Z^N]_T\right)\\ &= \frac{4}{N}\,\mathbb{E}\left[\int_0^T e^{2\int_t^T r(s)\,\mathrm{d}s}\,[\tilde{v}(t,S_t)-\Psi(t,S_t)]^2\,e^{-\int_0^t \nu_s\,\mathrm{d}s}\,\nu_t\,\mathrm{d}t\right]\\ &\le \frac{4}{N}\,e^{2\int_0^T |r(s)|\,\mathrm{d}s}\,C^2\int_0^T \bar{\mu}(t)\,\mathrm{d}t = O(1/N)\,.\end{aligned}$$

This implies the uniform convergence of Z^N to zero in $L^2(\mathbb{P})$ and hence in probability. The localizing sequence $(\sigma_n)_{n\ge 1}$ will give in general the result of the uniform convergence of Z^N to zero in probability. □

We can fix $\tilde{\mu}$ in such a way that the scaled hedge error $\bar{E}_t^\infty$ is nonnegative over the life time of the contract.

Remark 2.5.2. *Corollary 2.5.1 can be applied to the upper price bound $\bar{v}$ with mortality intensity $\mu^\star$ defined in (2.25), see Theorem 2.4.1. The normalized hedge error $\bar{E}^N$ converges uniformly in probability to $\bar{E}^\infty$ by Corollary 2.5.1. And the $\mathbb{P}$-dynamics of $\bar{E}^\infty$ are given by*

$$\mathrm{d}\bar{E}_t^\infty = \bar{E}_t^\infty\,r(t)\,\mathrm{d}t + e^{-\int_0^t \nu_u\,\mathrm{d}u}\,[\Psi(t,S_t)-\bar{v}(t,S_t)]\,[\mu^\star(t,S_t)-\nu_t]\,\mathrm{d}t\,,\quad 0\le t\le T\,.$$

Observe that $\bar{E}^\infty$ is of finite variation and, assuming a nonnegative initial value, $\bar{E}_0^\infty \ge 0$, is nondecreasing. To see this, recall that by Assumption 2.3.1 the realized mortality rate ν is bounded, i.e. $\underline{\mu}(t)\le \nu_t \le \bar{\mu}(t)$, $0\le t\le T$. The value function $\bar{v}$ and mortality intensity $\mu^\star$ are specified such that $[\Psi(t,S_t)-\bar{v}(t,S_t)]\,[\mu^\star(t,S_t)-\nu_t]$ is nonnegative. Accordingly, the upper price bound $\bar{v}$ indeed produces a superhedge when we are allowed to diversify the unsystematic mortality risk by the law of large numbers.

2.6 Numerical Results

In this section we analyze several unit-linked life insurance contracts in the uncertain mortality intensity framework. In addition to providing price bounds we also produce the optimally controlled regions of the mortality intensity.

As the underlying asset we take the S&P 500 index in the USA with the starting price $S_0 = 1073$. The dividend rate is assumed to be 0. We adopt the assumption that the volatility is constant over the time with $\sigma = 0.1833$. With regard to the interest rate, we assume that it is constant over the time with $r = 0.03$. The life insurance contracts that we study are described in Table 2.1 where $g_1 = 0.02$ refers to the minimum guarantee rate and $g_2 = 0.06$ refers to capped rate.

	payoff $\Psi(\tau, S_\tau)$ at τ (death)	payoff $\Phi(S_t)$ at T (survival)
I	S_τ	$\max(S_0e^{g_1T}, S_T)$
II	$S_0e^{g_1\tau}$	$\max(S_0e^{g_1T}, S_T)$
III	$\max(S_0e^{g_1\tau}, S_\tau)$	S_T
IV	$\max(S_0e^{g_1\tau}, S_\tau)$	$\max(S_0e^{g_1T}, S_T)$
V	$\min(S_0e^{g_2\tau}, S_\tau)$	$\min(S_0e^{g_2T}, S_T)$
VI	$\min(\max(S_0e^{g_1\tau}, S_\tau), S_0e^{g_2\tau})$	$\min(\max(S_0e^{g_1T}, S_T), S_0e^{g_2T})$

Table 2.1: Payoff structures for different life insurance contracts.

The policyholders are supposed to be 40 years old at the beginning and the contract lasts 30 years. In Figure 2.1, we display the forecast of the mortality intensity $\hat{\nu}$ over the next 30 years based on the model and results of Lee and Carter (1992). Additionally, upper and lower bounds are included corresponding to a pointwise 99.9% confidence interval, see Appendix for details. In the following, these bounds are assumed to be the mortality bounds $\underline{\mu}$ and $\bar{\mu}$ for Assumption 2.3.1 and Definition 2.3.1.

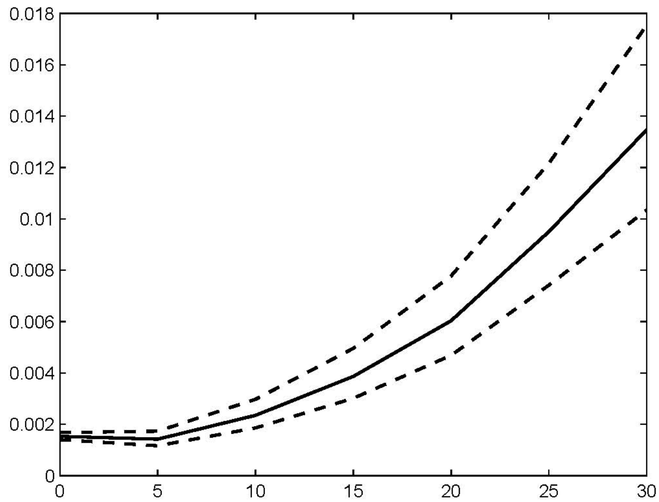


Figure 2.1: The forecast of the mortality intensity including bounds based on pointwise 99.9% confidence level.

We consider both the single premium and the periodic premium cases. In the single premium case, we calculate the lump sum amount a policyholder needs to pay if the mortality intensity moves in the most adverse way from the viewpoint of the insurance company such that the discounted benefit payment is maximized in expectation. In the periodic premium case, we assume that the policyholder pays continuously a prespecified cash flow until he dies. The prespecified cash flow is defined to be a fixed amount which is obtained by meeting the following fair premium principle.

Definition 2.6.1. *A unit-linked life insurance contract is fair if and only if the expected payment to the policyholder equals the expected premium paid by the policyholder at the initial date under the measure* $\mathbb{Q} \in \mathcal{Q}^b$ *with the forecasted mortality intensity* $\hat{\nu}$ *.*

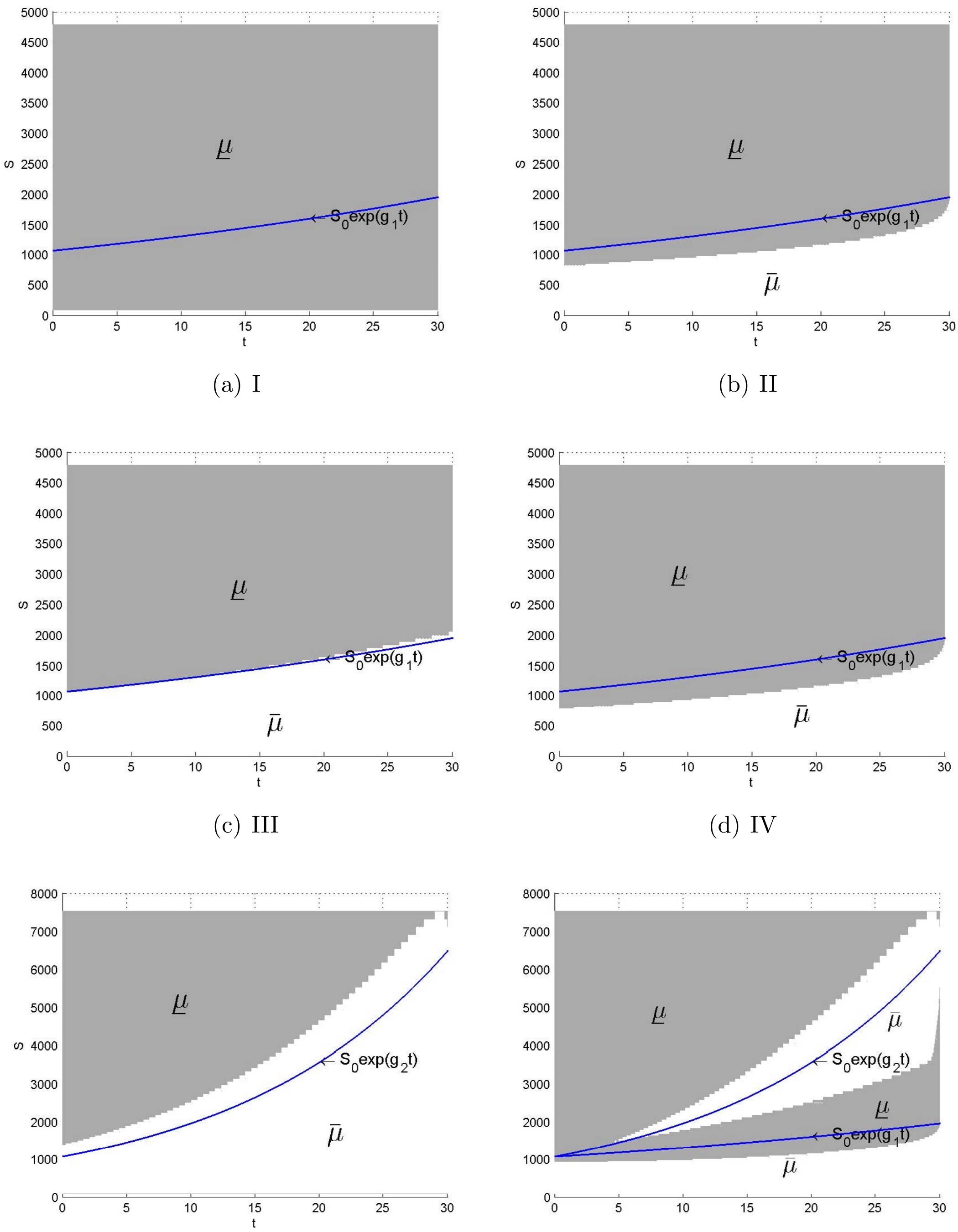
2.6.1 Single Premium Case

We first analyze the single premium case. In Figure 2.2 we present the optimally controlled regions where μ should be set to its lower bound $\underline{\mu}$ or to its upper bound $\bar{\mu}$ so that the upper price bounds of the contracts are obtained.

For contract type I, μ needs merely to be set to its lower bound $\underline{\mu}$ during the whole life time of the contract. By looking more closely at its payoff structure, we see that the reason is self-evident. The present value of the survival payoff exceeds the present value of the death benefit. Therefore, the contract value $v(t, S_t)$ is always greater than the death benefit $\Psi(t, S_t)$. According to Theorem 2.4.1, we will obtain the upper price bound by always setting μ to $\underline{\mu}$. However, in most cases, we cannot follow the simple rule of restricting to one bound of μ . The dynamic control scheme recommended in Theorem 2.4.1 should be implemented to get the price bounds.

For contract types II-IV, the upper bound and the lower bound regions of the mortality intensity are divided slightly below or at the minimum guarantee curve $S_0 e^{g_1 t}$, $0 \le t \le T$. When the asset price remains at a much lower level than the minimum guarantee curve, the chance for the policyholder to participate in the earnings of the risky asset is very low. The policyholder has higher possibility to obtain the minimum guaranteed amount which, however, increases at a lower rate than the investment in the riskless money market amount. Hence, it is optimal if the policyholder is able to quit the contract for the better investment alternative. The mortality intensity μ should be set to $\bar{\mu}$. On the other hand, if the asset price is high enough, the policyholder can benefit more from the risky asset which is even protected by the minimum guarantee, and hence, a lower mortality intensity becomes optimal.

For contract type V, we also see that the upper-left part of the figure is the $\underline{\mu}$ -region and the lower-right part is the $\bar{\mu}$ -region. The two regions are divided near the curve of the capped amount $S_0 \, e^{g_2 t}$, $0 \le t \le T$. When the asset price is well above this curve it is optimal to keep the mortality intensity to its minimum $\underline{\mu}$. The policyholder has a high possibility of obtaining the capped amount growing at rate g_2 which is higher than risk-free rate of r . On the contrary, when the asset price is lower than the capped amount, the immediate death benefit is higher than any later proceeds from the contract. The mortality intensity should be set to its upper bound $\bar{\mu}$. The payoff structure of the contract type VI is the mixture the contract types IV and V, and hence, the optimally controlled regions of μ shown in Figure 2.2 (f) is also the mixture of Figure 2.2 (d) and 2.2 (e).



(a) I (b) II

(c) III (d) IV

(e) V (f) VI

Figure 2.2: Optimally controlled regions for μ (single premium)

In Table 2.2 we present the pricing results of the single premium case by inserting different scenarios of the mortality intensity to the examples we have presented in Table 2.1. In the fourth to the sixth columns the mortality intensity is not dynamically controlled

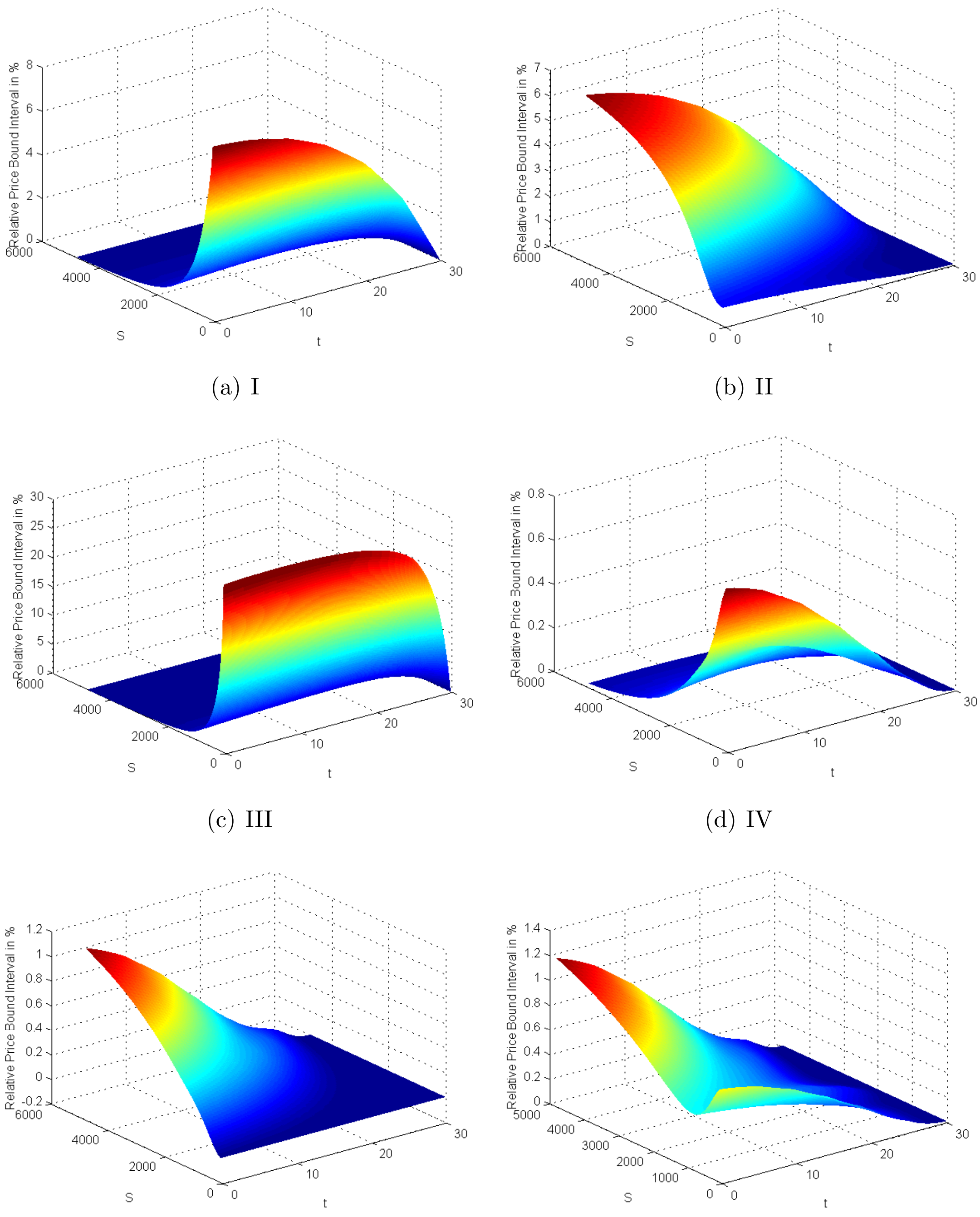
	Ψ	Φ	$\mu=\mu^f$	$\mu=\underline{\mu}$	$\mu=\bar{\mu}$	$\mu\in[\underline{\mu},\bar{\mu}]$		$\mu\in[0,\infty)$	
						lower	upper	lower	upper
I	S_τ	$\max(S_0e^{g_1T},S_T)$	1267.4	1275.2	1257.8	1257.8	1275.2	1073.0	1307.5
II	$S_0e^{g_1\tau}$	$\max(S_0e^{g_1T},S_T)$	1228.4	1242.7	1211.0	1203.8	1248.0	795.1	1357.3
III	$\max(S_0e^{g_1\tau},S_\tau)$	S_T	1109.6	1102.4	1118.4	1102.2	1118.7	1073.0	1357.2
IV	$\max(S_0e^{g_1\tau},S_\tau)$	$\max(S_0e^{g_1T},S_T)$	1303.9	1304.6	1303.2	1301.2	1306.4	1075.3	1357.3
V	$\min(S_0e^{g_2\tau},S_\tau)$	$\min(S_0e^{g_2T},S_T)$	916.4	916.2	916.8	914.3	918.7	855.6	1071.0
VI	$\min(\max(S_0e^{g_1\tau},S_\tau)$ $,S_0e^{g_2\tau})$	$\min(\max(S_0e^{g_1T},S_T)$ $,S_0e^{g_2T})$	1147.3	1147.6	1146.9	1143.8	1150.7	1010.4	1252.9

Table 2.2: Contract prices with different scenarios of the mortality intensity (single premium)

but only set to the forecasted value, its lower bound and its upper bound respectively. The seventh and eighth columns show the results when the mortality intensity μ lies in $[\underline{\mu},\bar{\mu}]$. When μ is controlled least optimally, we obtain the lower price bound in the seventh column; while the upper price bound in the eighth column is obtained when μ is controlled most optimally. Furthermore, we present the case when μ lies in $[0,\infty)$. This is an unrealistic representation of the mortality risk, which corresponds to the choice of the optimal equivalent martingale measure within the whole class of $\mathcal{Q}$ so that the price is minimized (column 9) or maximized (column 10). As we have discussed in Section 2.4.3, the maximized price is equal to the price of a pure American-style financial contract. The upper price is the initial wealth required for setting up a superhedge when diversification of the unsystematic mortality risk is ignored, see Remark 2.4.4.

As we have analyzed previously, for contract type I we follow the simple rule of keeping to $\underline{\mu}$ ($\bar{\mu}$) in order to obtain the upper (lower) price bound. This is can be seen once again in Table 2.2. While for the other contract types, the prices obtained by simply inserting $\underline{\mu}$ and $\bar{\mu}$ as well as μ^f over the time interval $[0,T]$ all lie within the lower and the upper price bounds. However, when we observe the differences between the two price bounds, we find they are not significantly big. To present this issue into more detail we show the differences between the upper and lower price bounds (relative to the upper price bounds) over the life time of the contracts depending on the price of the underlying in Figure 2.3.

The price difference is relatively significant for contract type III, which is close to 20% of the upper price bound when the underlying asset price is close to zero. This is not crucial when we take it into consideration that the possibility for the underlying asset price to decrease to such a low level from the starting value of 1073 is quite small. For contract types I and II, the price differences have not exceeded 6% of the upper price bounds. Once again, the probability for the maximal possible difference is very low. For contract types IV-VI, the price differences are even below 1.2% . Hence, for the pricing purpose,



(a) I (b) II

(c) III (d) IV

(e) V (f) VI

Figure 2.3: Differences between the upper and lower price bounds in relation to the upper price bounds(single premium).

it does not matter too much which scenario of the mortality intensity we implement into the pricing problem as long as it is a reasonable scenario within its confidence interval. This result indicates that mortality model risk does not have a huge effect on the risk management of unit-linked life insurance contracts we focus on, see Table 2.1. This argument is also valid when we increase the confidence level of the mortality intensity. In Table 2.3 we present the lower and upper price bounds at the different confidence levels of 99.9%, 99.99% and 99.999%. We notice that the differences between the lower and the upper price bounds are although wider but have not varied too much over the different confidence levels.

	Ψ	Φ	99.9%		99.99%		99.999%	
			lower	upper	lower	upper	lower	upper
I	S_τ	$\max(S_0e^{g_1T}, S_T)$	1257.8	1275.2	1254.8	1277.2	1252.2	1278.7
II	$S_0e^{g_1\tau}$	$\max(S_0e^{g_1T}, S_T)$	1203.8	1248.0	1198.7	1251.1	1194.0	1253.7
III	$\max(S_0e^{g_1\tau}, S_\tau)$	S_T	1102.2	1118.7	1100.4	1121.5	1099.0	1124.0
IV	$\max(S_0e^{g_1\tau}, S_\tau)$	$\max(S_0e^{g_1T}, S_T)$	1301.2	1306.4	1300.7	1306.9	1300.2	1307.3
V	$\min(S_0e^{g_2\tau}, S_\tau)$	$\min(S_0e^{g_2T}, S_T)$	914.3	918.7	913.9	919.1	913.5	919.5
VI	$\min(\max(S_0e^{g_1\tau}, S_\tau), S_0e^{g_2\tau})$	$\min(\max(S_0e^{g_1T}, S_T), S_0e^{g_2T})$	1143.8	1150.7	1143.1	1151.3	1142.5	1151.9

Table 2.3: Price bounds for different confidence levels (single premium)

Melnikov and Romaniuk (2006) show that different mortality models display different risk management performances for unit-linked pure endowment contracts. These are not contradictory results but give us the hint that mortality model risk can be alleviated by contract design. For contract type I and contract types III-VI, both the death benefit and the terminal payment are strongly associated with the performance of the underlying asset. Hence the contracts are more a financial product than an insurance product. For contract type II as well as the unit-linked pure endowment insurance in Melnikov and Romaniuk (2006), the death benefit is either a deterministic amount independent of the index performance or is zero. The risk profiles of the death benefit before time T and the survival benefit at time T are quite different which makes it crucial to know whether the death event may take place earlier or later.

2.6.2 Periodic Premium Case

Now we come to the periodic premium case. We consider the same payoff structures as before, see Table 2.1. However, the policyholder does not need to pay the premium at the beginning but pays it in arrears during the life time of the contract but maximally till his death time. For simplicity, we assume that the premium is paid continuously at a constant instantaneous rate Γ which is determined according to the fair contract principle given in Definition 2.6.1. If the mortality intensity develops as forecasted, the contract price should be 0 at the beginning. Since it is usually not the case, a different scenario of the mortality intensity will ex post lead to the situation that the premiums are either overpaid or underpaid by the policyholders on average. In the former case, the insurance company earns on average a surplus due to the misspecification of the mortality risk. However, in the latter case, it will find itself losing money. Because ex ante the insurance company has no complete information about the future, it is safe for it to be pessimistic and to assume the worst-case scenario for the pricing purpose.

In Figure 2.4 we see how μ should be optimally controlled so that the contract prices obtained enable the insurance company to stay on the safe side. Due to the introduction of the periodic premium payment, the insurance company bears higher risk that earlier death of the policyholder stops it from collecting the initial investment but does not reduces its obligation of benefit payment. Hence, we see that the optimally controlled regions look totally different in comparison with Figure 2.2.

For contract type I, it is not optimal to keep to the lower bound of μ any more. During the early life time of the contract, the policyholder has not paid too much premium. This indicates that once he dies prematurely, he has the right to get the value of the underlying asset only at trivial costs. Hence, it is optimal if μ takes the upper bounded value. As the policyholder survives most part of the contract's life, he should have already paid a great part of the premium. At this time, the optimal μ depends on the spot price of the underlying asset again. When S is very high, it is still profitable to stop the contract as immediate as possible, that is, to set μ to $\bar{\mu}$, because to go on paying the premium will not bring more benefit in expectation. On the contrary, if S is very low, the policyholder would prefer to pay the premium so that he gets the chance to receive a higher survival benefit. In this case, μ should optimally be set to $\underline{\mu}$.

For contract type II, the higher the underlying asset price is, the higher is the possibility that the policyholder will obtain a higher survival benefit, whose advantage outweighs the premium to be paid, and hence, $\underline{\mu}$ is optimal. In contrast, the lower the underlying asset price is, the higher is the possibility that the policyholder can only receive the guaranteed amount at the increasing rate of g_1. Since $g_1 < r$, further premium payment is not worthwhile for the policyholder and a higher mortality intensity, namely, $\bar{\mu}$ would be better. We also see that the critical asset price that divides the two regions decreases with time. This is due to the fact that the premium that has already been paid are sunk costs and the choice of the optimal scenario only depends on the balance between the future benefit and premium payment. Hence, as time moves on, the advantage of the higher survival benefit over premium payment already reveals at a lower level of S in comparison with the previous stage.

For contract type III, the mortality intensity should always be set to $\bar{\mu}$, meaning that it is always optimal to stop the contract as soon as possible. The present value of the death benefit is always greater than or equal to the present value of the survival benefit. Hence, an optimal mortality intensity should also be $\bar{\mu}$.

Contract type IV has similar payoff structure as contract type III except that the survival benefit also provides a minimum guarantee. The minimum guarantee for the terminal date T has trivial effect during most life time of the contract where the optimally controlled region is identical to the region in contract type III. Only when the contract is close to the maturity date does the minimum guarantee for the survival benefit matter for the optimal choice of μ. When the asset price is very close to the minimum guarantee, the death benefit is close to the spot asset price and is lower than the continuation value even when the periodic premium payment is taken into account. In this case, μ should be set to its lower bound $\underline{\mu}$.

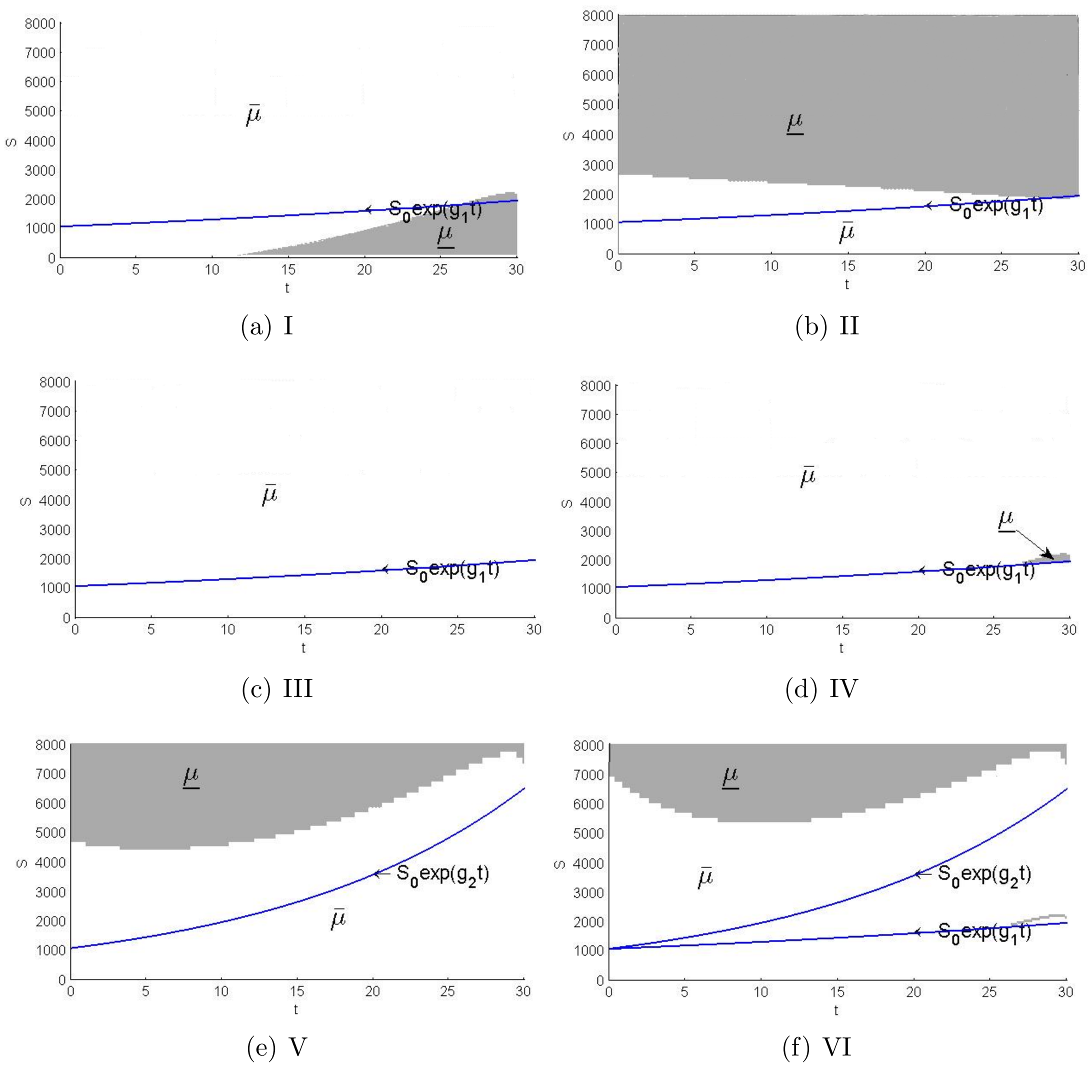


Figure 2.4: Optimally controlled regions of μ (periodic premium)

Contract type V sets a limit to both the death benefit and the survival benefit. When the asset price during the early life time of the contract is high enough, the return rate of the insurance contract is $g_2 > r$, and the policyholder would be willing to keep paying the premium so that the contract keeps alive and that he earns more than a pure investment in the financial market. In this case, μ is optimally set to $\underline{\mu}$ to count for this worst-case scenario from the perspective of the insurance company. Moreover, we see the non-monotonicity in the critical asset price. This is due to two effects. On the one hand, as the cap increases with the time, the critical value of S should also increase with the time. On the other hand, when the contract is farther away from the maturity date, the policyholder has more premium to pay. His incentive of continuing the contract is big only when he knows the possibility that the return of the contract keeps at a higher level. When the asset price is not high enough, due to the premium payment, it is usually optimal if the contract stops as soon as possible and μ should be set to $\bar{\mu}$.

Contract type VI is a mixture of contract type IV and contract type V. Therefore, the optimally controlled regions are a combination of Figure 2.4 (d) and (e).

				Profit and Loss of Insurance Company				
				Realized Mortality Intensity				
	Ψ	Φ	Γ	μ^f	$\mu = \underline{\mu}$	$\mu = \bar{\mu}$	$\mu \in [\underline{\mu}, \bar{\mu}]$	
							lower	upper
I	S_τ	$\max(S_0 e^{g_1 T}, S_T)$	67.02	0.00	3.36	-3.40	10.84	-10.67
II	$S_0 e^{g_1 \tau}$	$\max(S_0 e^{g_1 T}, S_T)$	65.04	0.00	-1.67	-0.33	22.95	-17.85
III	$\max(S_0 e^{g_1 \tau}, S_\tau)$	S_T	58.32	0.00	10.36	-26.80	10.36	-26.80
IV	$\max(S_0 e^{g_1 \tau}, S_\tau)$	$\max(S_0 e^{g_1 T}, S_T)$	68.94	0.00	10.63	-12.91	10.63	-12.91
V	$\min(S_0 e^{g_2 \tau}, S_\tau)$	$\min(S_0 e^{g_2 T}, S_T)$	48.46	0.00	8.32	-9.83	8.81	-10.32
VI	$\min(\max(S_0 e^{g_1 \tau}, S_\tau)$ $, S_0 e^{g_2 \tau})$	$\min(\max(S_0 e^{g_1 T}, S_T)$ $, S_0 e^{g_2 T})$	60.66	0.00	9.56	-11.62	9.95	-12.00

Table 2.4: Average profit and loss for different scenarios of the mortality intensity (periodic premium).

				Profit and Loss of Insurance Company				
				Realized Mortality Intensity				
	Ψ	Φ	Γ	μ^f	$\mu = \underline{\mu}$	$\mu = \bar{\mu}$	$\mu \in [\underline{\mu}, \bar{\mu}]$	
							lower	upper
I	S_τ	$\max(S_0 e^{g_1 T}, S_T)$	67.02	0.00%	0.27%	-0.27%	0.85%	-0.85%
II	$S_0 e^{g_1 \tau}$	$\max(S_0 e^{g_1 T}, S_T)$	65.04	0.00%	-0.14%	-0.03%	1.86%	1.46%
III	$\max(S_0 e^{g_1 \tau}, S_\tau)$	S_T	58.32	0.00%	0.93%	2.46%	0.93%	2.46%
IV	$\max(S_0 e^{g_1 \tau}, S_\tau)$	$\max(S_0 e^{g_1 T}, S_T)$	68.94	0.00%	0.81%	-1.00%	0.81%	-1.00%
V	$\min(S_0 e^{g_2 \tau}, S_\tau)$	$\min(S_0 e^{g_2 T}, S_T)$	48.46	0.00%	0.90%	-1.08%	0.95%	1.14%
VI	$\min(\max(S_0 e^{g_1 \tau}, S_\tau)$ $, S_0 e^{g_2 \tau})$	$\min(\max(S_0 e^{g_1 T}, S_T)$ $, S_0 e^{g_2 T})$	60.66	0.00%	0.83%	1.02%	0.86%	-1.06%

Table 2.5: Average profit and loss in % of the premium collected during the life time of the contracts for different scenarios of the mortality intensity (periodic premium).

As we have shown in Section 2.5, the hedging strategies based on the upper price bound will ensure the insurance company to build a superhedging position if enough policyholders are pooled together. In Table 2.4 we present the average profit and loss for the insurance company on a single contract for different scenarios of the realized mortality intensity. In Table 2.5 the average profit and loss are presented as the percentage of the premium accumulated during the life time of the contract. We first look at Table 2.4. In column 4 the constant instantaneous premium rate Γ is given which enables the contract price to be zero under the assumption that the mortality intensity moves as is forecasted in the future, see also column 5 for zero profit and loss. However, in reality, mortality risk cannot be forecasted with certainty. In column 6 and 7 we show what the contract price would be if the mortality intensity keeps to its lower bound (the extreme case of the longevity risk) and to its upper bound respectively. Column 8 and 9 display the lower price bound when the mortality intensity develops in the most favorable way for the insurance company and the upper price bound when it develops the most unfavorably. The results are qualitatively similar for all contracts. Exemplarily we discuss contract type I. We see that a lower mortality intensity than the forecast is of no risk to the insurance company. The benefit to be paid is less than the premium to be collected on average. The insurance company also does not suffer from the model risk if the mortality intensity develops in the most favorable way as is indicated in column 8. If the mortality intensity is higher than forecasted (column 7) or changes its value in the most unfavorable way (column 9) to the insurance company, the insurance company will find itself not being able to fulfill its obligation totally with the premium collected. All the prices under different scenarios of the mortality intensity lie within the lower price bound and the upper price bound that have been found dynamically according to Theorem 2.4.1. The upper price bound theoretically enables the insurance company to manage the financial risk dynamically under the model risk concerning the mortality intensity. When we look at Table 2.5, we see it counts only for about 0.85% of the whole amount of premium expected to be collected, or equivalently, about 2 month's premium, which for the insurance company may not be a large amount. Similar results can be found in the other contract types which indicates once again that mortality model risk has little price impact for contracts considered here.

2.7 Conclusion

We have investigated the influence of mortality model risk on unit-linked life insurance contracts. This investigation is undertaken within an uncertain mortality intensity framework where we assume reasonable bounds for the unknown mortality intensity. The magnitude of the mortality model risk can be easily identified by carrying out a stochastic control analysis and establishing upper and lower price bounds of unit-linked life insurance contracts, see Theorem 2.4.1. The hedging strategy induced by the upper (lower) price bound produces a superhedge (subhedge) under the statistical measure when pooling together an increasing number of similar contracts, see Corollary 2.5.1 of Theorem 2.5.1 and Remark 2.5.1. The unsystematic mortality risk is diversified away by the pooling rationale. The systematic mortality risk is addressed by dynamically assuming the worst (best) case for the stochastic mortality intensity within the given bounds. If the worst (best) case scenario does not occur then the hedging strategy generates a positive (negative) cash flow. In addition, superhedging strategies are suggested under mortality model risk when assuming that the number of policyholders is large, see Theorem 2.5.1 and Corollary 2.5.1.

We show that when the risk profiles of the death benefit and the survival benefit are not significantly different, the effect of the mortality model risk may not be very large indeed. The contract prices in our examples have little sensitivity with respect to changes in the mortality intensity. For the single premium version the overall contract price differences were well below 4%. In the periodic premium case the deviation from the fair price was in the same range, and was not exceeding a six month premium income. In this case, other risk sources such as interest rate risk and equity risk deserve more attention than mortality model risk.

Our framework can be extended in many useful directions. The setup can be directly extended to include an American feature where the policyholder has the right to quit the contract for a pre-specified payoff, the surrender guarantee. This is studied in Chapter 3. We particularly investigate the effect of policyholders' monetary rationality concerning the exercise of the surrender option on the contract value. Further, other risk factors such as interest rate risk and other facets of equity risk such as volatility risk can be included in the setup. The so extended framework can then be used to analyze the impact of various financial risk factors on mortality model risk. This research question is however beyond the scope of the dissertation and would be studied in the near future.

Chapter 3

The Effect of Policyholders' Rationality on Unit-linked Life Insurance Contracts with Surrender Guarantees[1]

3.1 Introduction

Most unit-linked life insurance contracts entitle the policyholders to terminate the contract before the maturity date and receive a certain cash refund called the surrender value. In the literature, at least four approaches are found to evaluate such contracts. The first approach is to consider the surrender decision as caused by exogenous reasons and a surrender table can be constructed to capture the statistics on surrenders, see Bacinello (2005). The second approach is to work within the contingent-claim framework and consider the surrender option as an American-style contingent claim to be exercised rationally. This approach is favored by most literature in recent years. Examples are Grosen and Jørgensen (1997) Grosen and Jørgensen (2000), Bacinello (2003) Bacinello (2005), and Bacinello, Biffis and Millossovich (2010), to just name a few. The argument is that the policyholder should not complain about the contract depreciation caused by his own non-optimal surrender, even due to exogenous reasons like financial difficulties, when he does have the right to do it optimally. The third approach takes suboptimal surrender into consideration. This is suggested by Bernard and Lemieux (2008). They consider a single policyholder's decision behavior, which is characterized by a decision parameter. The policyholder is assumed to exercise the surrender option only when the ratio between the surrender value and the continuation value exceeds the decision parameter. The fourth approach is carried out on the portfolio level. It is first proposed by Albizzati and Geman (1994) who incorporate both the exogenous and the endogenous surrender reasons into the valuation problem. They assume that the proportion of sur-

[1]This chapter is based on Li and Szimayer (2011a)

render among the active contracts is an increasing function of the ratio of the surrender value and the value when holding the contract until maturity. In case the ratio is below one, the surrender rate is set to its minimum reflecting base level surrender due to exogenous reasons. The surrender rate is then linear increasing with increasing ratio until a fixed upper bound is reached. The upper bound represents the maximal surrender rate. Recently, similar idea is implemented by DeGiovanni (2010) to model the policyholders' rationality in contract surrender.

We consider the approaches of Albizzati and Geman (1994) as well as DeGiovanni (2010) as more realistic than the other three approaches. The first two approaches only address part of the story. Surrender decisions are not only triggered by exogenous reasons but also by endogenous reasons. The empirical study conducted by Kuo, Tsai and Chen (2003) shows that not only the unemployment rate (which corresponds to the exogenous surrender reason) but also the interest rate (which corresponds to the endogenous surrender reason) has impact on surrender behavior. Without treating the endogenous surrender risk properly, the policy issuer will suffer an underestimated loss when disadvantageous financial market movement brings about more surrender cases than that have been summarized by the surrender table. However, it has never been observed that all the policyholders simultaneously take the same surrender action when it is optimal to do so. Treating the surrender action merely as an optimal stopping problem will overestimate the funds needed to manage the contracts. Overall, it is difficult to identify each policyholder's decision rule and to figure out the proportion of policyholders who are characterized by the same decision parameter. Since the policy issuers cannot identify the monetary rationality of the policyholders separately, all the policyholders should be charged the same at the beginning. The premiums charged by considering both the exogenous and the endogenous surrender reasons can be argued to be reasonable on the portfolio level.[2]

Although we tend to follow Albizzati and Geman (1994), we also bear in mind that there are some limitations in their approaches that we try to avoid. In Albizzati and Geman (1994), mortality is considered as one of the surrender events. However, in most cases death benefit and surrender benefit are not equal to each other. Surrender is usually accompanied by a penalty in payment which does not apply to death benefit. Hence, the distinction between the death event and the surrender event should be considered. In addition, Albizzati and Geman (1994) assume that a policyholder surrenders the contract by comparing the surrender value and the value of initiating a new contract which he holds till the maturity. A closed-form solution is obtained by assuming independence between the surrender probabilities at different time points. However, usually a new contract also allows for surrender. In this case, a surrender probability in the future also has influence

[2]For those competent policyholders who are able to exercise their surrender option optimally, less premiums are charged than those are needed to support the contracts. Those policyholders who surrender the contracts monetarily suboptimally have born the extra costs.

on the surrender probability at present. This effect should be taken into consideration when evaluating a contract with surrender guarantees. If the assumption about the independence between the surrender probabilities is suspended, the Monte Carlo simulation method is suggested by them to solve the valuation problem which is very time consuming.

In this chapter we propose the intensity-based valuation of unit-linked life insurance contracts with surrender guarantees. Surrender is not modeled as a binary event but randomized where the surrender intensity reflects the local likelihood of surrender. The intensity based approach was first used in credit risk modeling to describe the arrival of the credit event. Recently, a similar approach has been adopted in other areas. For example, the mortality risk embedded in insurance contracts is characterized by the mortality intensity, (e.g. Milevsky and Promislow (2001), Dahl (2004), Dahl and Møller (2006)) and the prepayment risk embedded in mortgage loans is captured by the prepayment intensity (e.g. Stanton (1995), Dai et al. (2007)). In this chapter, we describe the arrival of the surrender event also by an intensity-based approach and solve the valuation problem for a representative policyholder. We assume that the surrender intensity of the policyholder is bounded from below and from above. As in Albizzati and Geman (1994) and DeGiovanni (2010) the lower bound represents the surrender base level due to exogenous reasons. And the upper bound represents the maximal surrender rate that is attributed to exercise of the surrender option when it is financially optimal to do so. Since the optimal decision will not be made by all the policyholders simultaneously and equivalently not by the representative policyholder, both the lower and the upper bound of the surrender intensity are finite numbers between zero and infinity.[3] They can be easily backed out from the relevant statistics in the past. By capturing the surrender risk with the surrender intensity, and similarly, the mortality risk with the mortality intensity, we are able to establish a partial differential equation whose solution is the contract value we are looking for. The finite difference method is then applied to solve the problem. In this sense, our approach is quite similar to DeGiovanni (2010) but is also different from him in two aspects. We have incorporated the mortality risk in our model which is but ignored by DeGiovanni (2010). In addition, we emphasize the fair contract design.

To formalize the problem, we introduce the model setup in Section 3.2. The valuation of the contracts is carried out in Section 3.3. In Section 3.4 we study the impact of the policyholders' rationality on the contract value through numerical examples. Moreover, the relationship of the parameters in the contract will be analyzed. Section 3.5 concludes.

[3]If the surrender option is exercised optimally, the surrender intensity switches between zero and infinity.

3.2 Setup

The model setup is similar to the description in Section 2.2 with minor difference. Again, unit-linked life insurance contracts link the financial market and the insurance market together.

On the financial market, there is a non-dividend paying risky asset with the price process S and a riskless money market account with the price process B. Under the real world measure $\mathbb{P}$, the two asset price processes are governed respectively by the stochastic differential equations

$$\mathrm{d}S_t = a(t, S_t)\, S_t\, \mathrm{d}t + \sigma(t, S_t)\, S_t\, \mathrm{d}W_t\,, \tag{3.1}$$

and

$$\mathrm{d}B_t = r(t)\, B_t\, \mathrm{d}t\,, \tag{3.2}$$

for $0 \leq t \leq T$, where a is the local mean rate of return of the risky asset and σ is the volatility of the risky asset. Both of them are Markovian. The risk-free interest rate r is assumed to be deterministic. Moreover, W refers to the 1-dimensional Brownian motion under $\mathbb{P}$ and generates the financial market filtration $\mathcal{F} = (\mathcal{F}_t)_{0\leq t\leq T}$. The financial market is complete and arbitrage free, which is equivalent to the existence of a risk-neutral martingale measure $\mathbb{Q}$ so that the price process S is described as

$$\mathrm{d}S_t = r(t)\, S_t\, \mathrm{d}t + \sigma(t, S_t)\, S_t\, \mathrm{d}\hat{W}_t,\ 0 \leq t \leq T, \tag{3.3}$$

where $\hat{W}$ is a Brownian motion under $\mathbb{Q}$ which satisfies $\mathrm{d}\hat{W}_t = \mathrm{d}W_t + \frac{a-r}{\sigma}\mathrm{d}t$.

The insurance market is modeled by two random times τ and λ potentially ending the financial contract. The time τ refers to the death time of an individual aged x at time $t = 0$ when the contract is signed. The time λ refers to the time when the policyholder decides to terminate the contract.

The jump process associated with τ is H with $H_t = 1_{\{\tau\leq t\}}$, for $0 \leq t \leq T$, and generates the filtration $\mathbb{H} = (\mathcal{H}_t)_{0\leq t\leq T}$. The hazard rate of the random time τ (or the mortality intensity) is denoted by μ. In recent literature, the mortality intensity is often assumed to be stochastic based on the observation of the systematic longevity risk in recent decades. However, in Chapter 2 we find that the stochastic feature of the mortality intensity is of minor impact on unit-linked life insurance contracts when the risk profiles at death and at maturity are not dramatically different. We assume here, therefore, that the mortality intensity is described by a deterministic function $\mu(t)$, for $t \in [0, T]$. In fact, the mortality risk is then unsystematic and can be diversified away over a large pool of policyholders.

The jump process associated with λ is J with $J_t = 1_{\{\lambda\leq t\}}$, for $0 \leq t \leq T$. It generates the filtration $\mathbb{J} = (\mathcal{J}_t)_{0\le t\le T}$. The hazard rate of the random time λ is denoted by γ , and is also called the surrender intensity. By introducing the random time λ , and correspondingly, the surrender intensity γ , we can actually represent a large family of insurance contracts. For the degenerate case where $\gamma = 0$, the insurance contracts are European style. When γ is allowed to take positive values, the policyholder can walk away from the contract. In contrast to the mortality intensity μ , the surrender intensity γ is not deterministic but depends on the monetary rationality of the policyholder in making surrender decisions by comparing the the contract value and the surrender value. Since the contract value and eventually also the surrender value are linked to the risky asset S , γ is assumed to be $\mathbb{F}$ -measurable. The exact form of γ will be specified in Section 3.3.

To model the information on the linked market, the filtrations $\mathbb{F}$, $\mathbb{H}$ and $\mathbb{J}$ need to be combined. Bielecki and Rutkowski (2001) give an account on the technicalities to combine these filtrations.[4] We give a brief summary of their key results relevant to our situation.

Starting under the original probability space $(\Omega, \mathcal{G}, \mathbb{P})$ we first specify the enlarged filtration $\mathbb{G} = (\mathcal{G}_t)_{0\le t\le T}$ carrying all the relevant information by $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t \vee \mathcal{J}_t$, for $0 \le t \le T$. Recalling that $\mathbb{F}$ is the filtration generated by the Wiener process W we assume that W remains a Wiener process for the enlarged filtration $\mathbb{G}$. The processes H and J both admit intensities μ and λ that are $\mathbb{F}$ -adapted. Now, we additionally assume that μ and λ are the respective $\mathbb{G}$ -intensities, i.e. the processes $\hat{M}^H = (\hat{M}^H_t)_{0\le t\le T} = (H_t - \int_0^{t\wedge\tau} \mu(u)\,\mathrm{d}u)_{0\le t\le T}$ and $\hat{M}^J = (\hat{M}^J_t)_{0\le t\le T} = (J_t - \int_0^{t\wedge\lambda} \gamma_u\,\mathrm{d}u)_{0\le t\le \mathrm{T}}$ are both $\mathbb{G}$ -martingales, and that joint jumps of H and J occur with zero probability, i.e. $\mathbb{P}(\tau = \lambda) = 0$.

The Radon-Nikodym density process for the measure change from $\mathbb{P}$ to $\mathbb{Q}$ is defined as

$$\eta_t = \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\bigg|_{\mathcal{G}_t} = \mathbb{E}[Y|\mathcal{G}_t] \quad \mathbb{P} - a.s., \tag{3.4}$$

where Y is a $\mathcal{G}_T$ -measurable random variable with $\mathbb{P}(Y > 0) = 1$ and $\mathbb{E}^{\mathbb{P}}[Y] = 1$. According to Bielecki and Rutkowski (2001), Proposition 7.1.3, p. 201, it has the following integral representation

$$\eta_t = 1 + \int_{]0,t]} \eta_{u-}(\varphi_u \mathrm{d}\hat{W}_u + \xi^H_u \mathrm{d}\hat{M}^H_u + \xi^J_u \mathrm{d}\hat{M}^J_u), \tag{3.5}$$

where φ , ξ^H and ξ^J are $\mathbb{G}$ -predictable processes.

Set $\varphi = -\frac{a-r}{\sigma}$ and $\xi^H = \xi^J = 1$, then by Proposition 7.2.1. in Bielecki and Rutkowski

[4] See Bielecki and Rutkowski (2001), Section 7, pp.197.

(2001), $\hat{W}$ in the risk-neutral dynamics of the risky asset in (3.3) is also $\mathbb{Q}$-Brownian motion on the enlarged filtration $\mathbb{G}$. Further, μ and γ are the intensities of τ and λ under the equivalent martingale measure $\mathbb{Q}$ and filtration $\mathbb{G}$. Thus, valuation under the risk-neutral measure $\mathbb{Q}$ and on the extended filtration $\mathbb{G}$ is possible and carried out in Section 3.3.

3.3 Contract Valuation

In this section we introduce the contract and derive the valuation equation. The contract is comprised of a survival benefit, a death benefit and a surrender benefit. Survival benefit and death benefit both offer a guaranteed rate and the possibility to participate in a potentially profitable development of the risky asset. The surrender benefit depends on time only, effectively representing a put option, see Bernard and Lemieux (2008) for a similar approach. The contract value is derived using the balance law of financial economics, see Dai et al. (2007).

We assume that the policyholder pays at the beginning time 0 the single premium P for the contract with the maturity date T. The payoff of the contract is linked to the underlying asset S. When the policyholder survives time T, the payment to him is

$$\Phi(S_T) = P \max\left(\alpha\,(1+g)^T, \left(\frac{S_T}{S_0}\right)^k\right), \tag{3.6}$$

where α refers to the percentage of the initial premium which is provided with the minimum guaranteed rate g and k refers to the policyholder's participation rate in the performance of the underlying asset. When the policyholder dies at time $\tau < T$, the death benefit is

$$\Psi(\tau, S_\tau) = P \max\left(\alpha\,(1+g_d)^\tau, \left(\frac{S_\tau}{S_0}\right)^{k_d}\right), \tag{3.7}$$

where the parameters g_d and k_d refer respectively to the minimum guaranteed rate and the participation rate in the asset performance upon the occurrence of the death event. They need not be identical with g and k. However, in practice, death as a natural event is neither penalized nor rewarded, so that $g = g_d$ as well as $k = k_d$ is very common. Furthermore, the surrender benefit is introduced into the contract. Similar to Bernard and Lemieux (2008) we set the surrender benefit L to be independent of the asset performance.[5] If the policyholder surrenders the contract at time λ, he obtains

$$L(\lambda) = (1-\beta_\lambda)\,P\,(1+h)^\lambda, \tag{3.8}$$

[5] In practice, the surrender benefit is independent of the asset performance. Theoretically, it could also depend on the asset performance. In this case, the numerical results may differ from the results which we present later on. However, the valuation method that we introduce in this section is still applicable.

where β_λ is a penalty charge for the surrender action at time λ and h refers to the minimum guaranteed rate for the surrender benefit. The penalty β_λ is typically constant over one calender year and a decreasing function of time such that early surrender is more penalized.[6] In practice the minimum guaranteed rate h is not allowed to fall below the minimum guaranteed rate g for the survival benefit, see Bernard and Lemieux (2008).

Following our rationale in Section 3.1 we describe the arrival of the surrender action at a random time λ by a generalized Poisson process with stochastic intensity γ. The intensity γ depends on the relationship between the surrender benefit L and the present value of the contract V. When the surrender benefit is smaller than the contract value, the surrender intensity takes a lower value $\underline{\rho}$. On the contrary, a higher value $\bar{\rho}$ is taken. Formally, γ can be expressed as

$$\gamma_t = \begin{cases} \underline{\rho}, & \text{for } L(t) < V_t\,, \\ \bar{\rho}, & \text{for } L(t) \geq V_t\,. \end{cases} \tag{3.9}$$

This formulation is inspired by Dai et al. (2007) and can be traced back to Stanton (1995) who deals with the prepayment terms in mortgage loans. In this way, we are not explicitly solving an optimal stopping problem but a randomized version of it. However, in the limiting case, when $\underline{\rho} \searrow 0$ and $\bar{\rho} \nearrow \infty$, we obtain the solution to the accompanying optimal stopping problem. Accordingly, our approach includes in the limit the aforementioned American-style contingent claim analysis of Grosen and Jørgensen (1997) Grosen and Jørgensen (2000), Bacinello (2003) Bacinello (2005) and Bacinello et al. (2010).

The next step is to establish the contract value V. We derive the contract value by the PDE characterization using the balance law, see Dai et al. (2007).

The balance law is based on the no-arbitrage condition

$$r(t)V_t\mathrm{d}t \;\; = \;\; \mathbb{E}^{\mathbb{Q}}\left[\mathrm{d}V_t|\mathcal{G}_t\right]\,, \tag{3.10}$$

on $\{t < \lambda \wedge \tau \wedge T)\}$ with $x \wedge y := \min(x, y)$, that is, the balance law applies to the case when the contract still exists. Provided that the policyholder is still alive at time t and has not surrendered the contract yet, we consider the following cases under the assumption that the two stopping times τ and λ are conditionally independent of each other:

1) The conditional probability that death occurs over $(t, t+\mathrm{d}t)$ while the surrender does not is $\mu_t\mathrm{d}t(1-\gamma_t\mathrm{d}t) = \mu_t\mathrm{d}t$.

[6] In practice, the penalty charges are imposed to offset the costs associated with the issuance of the contracts. Theses costs may otherwise be compensated during the life time of the contracts if they are held till the maturity date. For examples of penalty functions please refer to Palmer (2006).

2) The conditional probability that surrender occurs over $(t, t+\mathrm{d}t)$ while the death event has not happened is $\gamma_t \mathrm{d}t(1-\mu_t \mathrm{d}t) = \gamma_t \mathrm{d}t$.

3) The conditional probability that both the surrender and the death events occur over $(t, t+\mathrm{d}t)$ is 0 .

Suppose that the contract value at time t is of the form

$$V_t = 1_{\{t<\lambda\wedge\tau\}}\, v(t,S_t) + 1_{\{t=\lambda;\lambda<\tau\}}\, L(\lambda) + 1_{\{t=\tau;\tau\le\lambda\}}\, \Psi(\tau, S_\tau)\,, \tag{3.11}$$

for a suitably differentiable function $v : [0,T]\times\mathbb{R}^+ \to \mathbb{R}_0^+, (t,s) \mapsto v(t,s)$. Thus we can also express γ_t as a function of the state variables, i.e. $\gamma_t = \gamma(t,S_t)$ where $\gamma : [0,T]\times\mathbb{R}^+ \to \mathbb{R}_0^+, (t,s)\mapsto \gamma(t,s)$. Upon the occurrence of death there is a change in the payment liability of the amount $\Psi(t,s)-v(t,s)$ and upon the occurrence of surrender the change in the payment liability is $L(t)-v(t,s)$. Hence, we can rewrite (3.10) on $\{t<\lambda\wedge\tau\wedge T\}$ as

$$r(t)v(t,S_t)\mathrm{d}t \;=\; \mathbb{E}^{\mathbb{Q}}[\mathrm{d}v(t,S_t)|\mathcal{F}_t] + (\Psi(t,S_t)-v(t,S_t))\mu_t\mathrm{d}t + (L(t)-v(t,S_t))\gamma(t,S_t)\mathrm{d}t.$$

Applying Ito's Lemma to $\mathrm{d}v(t,S_t)$ and assuming sufficient integrability we obtain

$$\mathbb{E}^{\mathbb{Q}}[\mathrm{d}v(t,S_t)|\mathcal{F}_t] \;=\; \mathbb{E}^{\mathbb{Q}}\left[\mathcal{L}v(t,S_t)\,\mathrm{d}t + \sigma(t,S_t)S_t\,\frac{\partial v}{\partial s}v(t,S_t)\,\mathrm{d}\hat{W}_t\right|\left.\mathcal{F}_t\right] \;=\; \mathcal{L}v(t,S_t)\,\mathrm{d}t\,,$$

where $\mathcal{L}$ is the differential operator comprised of the partial derivative with respect to time and the generator of the process S defined in (3.3), i.e.

$$\mathcal{L}f(t,s) \;=\; \frac{\partial f}{\partial t}(t,s) + r(t)s\frac{\partial f}{\partial s}(t,s) + \frac{1}{2}\sigma^2(t,s)s^2\frac{\partial^2 f}{\partial s^2}(t,s)\,.$$

Then we obtain

$$\mathcal{L}v(t,s) + \mu(t)\Psi(t,s) + \gamma(t,s)L(t) - (r(t)+\mu(t)+\gamma(t,s))\,v(t,s) \;=\; 0\,.$$

By no-arbitrage, we must also have $v(T,s)=\Phi(s)$, for all $s>0$. We have just derived the pricing PDE summarized in the following proposition.

Proposition 3.3.1. *For the contract value* V *given by* (3.11) *the price function* v *is the solution of the partial differential equation*

$$\mathcal{L}v(t,s) + \mu(t)\Psi(t,s) + \gamma(t,s)L(t) - (r(t)+\mu(t)+\gamma(t,s))\,v(t,s) \;=\; 0\,, \tag{3.12}$$

for $(t,s)\in[0,T)\times\mathbb{R}^+$ *with terminal condition* $v(T,s)=\Phi(s)$ *, for* $s\in\mathbb{R}^+$ *. The solution of (3.12) together with equation* (3.9) *characterizes the surrender intensity* γ *.*

Remark 3.3.1. *The results derived in Proposition 3.3.1 can be generalized. We have assumed that the bounds of the surrender intensity,* $\underline{\rho}$ *and* $\bar{\rho}$ *, respectively, are constant. In fact, we can allow the bounds being driven by the financial market and other non-financial state variables* X *, i.e.* $\underline{\rho}_t = \underline{\rho}(t, S_t, X_t)$ *and* $\bar{\rho}_t = \bar{\rho}(t, S_t, X_t)$ *. Further, we can include stochastic interest rates and stochastic volatility in our model. Under this extended setup the valuation PDE in* (3.12) *carry over.*

The contract value V is influenced by the bounds $\underline{\rho}$ and $\bar{\rho}$. Intuitively it is clear that a lower value for $\underline{\rho}$ leads to less frequent surrender due to exogenous reasons and accordingly increases the contract value. Likewise, a higher value for $\bar{\rho}$ allows a higher surrender activity when it is financially profitable to do so and therefore increases the contract value. The following proposition states this fact precisely.

Proposition 3.3.2. *Suppose that* v *is the value function of the contract with bounds* $\underline{\rho}$ *and* $\bar{\rho}$ *, and that* w *is the value function of the contract with bounds* $\underline{\zeta}$ *and* $\bar{\zeta}$ *. Assume that* $\underline{\zeta} \leq \underline{\rho}$ *and* $\bar{\rho} \leq \bar{\zeta}$ *. Then we have* $w(t,s) \geq v(t,s)$ *, for* $(t,s) \in [0,T] \times \mathbb{R}^+$ *.*

Proof. The function v is the solution of the PDE (3.12) with terminal condition $v(T,s) = \Phi(s)$ and bounds $\underline{\rho}$ and $\bar{\rho}$. The function w is the solution of the same PDE (3.12) with identical terminal condition $w(T,s) = \Phi(s)$ but different bounds $\underline{\zeta}$ and $\bar{\zeta}$. Assume that $\underline{\zeta} \leq \underline{\rho}$ and $\bar{\rho} \leq \bar{\zeta}$. Now define $z = w - v$. It follows directly that $z(T,s) = w(T,s) - v(t,s) = \Phi(s) - \Phi(s) = 0$. To obtain the dynamics of z take the difference of the PDEs describing w and v , i.e.:

$$\begin{aligned} 0 &= \mathcal{L}w(t,s) + \mu(t)\Psi(t,s) + \gamma^{\zeta}(t,s)L(t) - (r(t) + \mu(t) + \gamma^{\zeta}(t,s))\, w(t,s) \\ &\quad - (\mathcal{L}v(t,s) + \mu(t)\Psi(t,s) + \gamma^{\rho}(t,s)L(t) - (r(t) + \mu(t) + \gamma^{\rho}(t,s))\, v(t,s)) \\ &= \mathcal{L}z(t,s) + (\gamma^{w}(t,s) - \gamma^{v}(t,s))\,(L(t) - w(t,s)) - (r(t) + \mu(t) + \gamma^{v}(t,s))\, z(t,s)\,, \end{aligned}$$

were γ^v and γ^w , respectively, are given by (3.9) using the appropriate bounds. By Feynman-Kac we obtain the stochastic representation of z as follows

$$z(t,s) = \mathbb{E}^{t,s}_{\mathbb{Q}}\left[\int_t^T e^{-\int_t^u (r(x)+\mu(x)+\gamma^v(x,S_x))\,\mathrm{d}x} (\gamma^w(u,S_u) - \gamma^v(u,S_u))\,(L(u) - w(u,S_u))\mathrm{d}u\right],$$

where $\mathbb{E}^{t,s}_{\mathbb{Q}}$ denotes the expectation conditioned on $S_t = s$. From the definition of γ^w in (3.9) and the assumption $\bar{\zeta} \geq \bar{\rho}$ we see that if $(L - w) \geq 0$ we have $\gamma^w = \bar{\zeta} \geq \bar{\rho} \geq \gamma^v$ and thus $(\gamma^w - \gamma^v) \geq 0$. On the other hand, if $(L - w) < 0$ then $\gamma^w = \underline{\zeta}$. By assumption we have $\underline{\zeta} \leq \underline{\rho}$ and thus $\gamma^w \leq \underline{\rho} \leq \gamma^v$, or, $(\gamma^w - \gamma^v) \leq 0$. Thus, we see that the integrand in the above equation is nonnegative and therefore $z \geq 0$. Since $z = w - v$ we obtain $w \geq v$. □

Corollary 3.3.1. *In the setting of Proposition 3.3.2 define the sets where exclusively exogenous surrender occurs by* $C^v = \{(t,s) \in [0,T] \times \mathbb{R}^+ : L(t) < v(t,s)\}$ *and* $C^w = \{(t,s) \in [0,T] \times \mathbb{R}^+ : L(t) < w(t,s)\}$ *, respectively. Then* $C^v \subseteq C^w$ *.*

Proof. This is an immediate consequence of Proposition 3.3.2. □

3.4 Numerical Analysis

In this section we study the life insurance contract we have specified above closely through numerical analysis. We assume that the underlying of the contract is the S&P 500 with volatility $\sigma(t,s) = 0.2$. The market interest rate is constant at $r = 0.04$. At the beginning $P = \$100$ is paid. The contract life time is 10 years. For the moment we assume that the participation rate into the minimum guaranteed amount is $\alpha = 0.85$. The minimum guaranteed rates at survival, at death and at surrender satisfy $g = g_d = h = 0.02$. The participation coefficient at survival and at death satisfy $k = k_d = 0.9$. The penalty rates are $\beta_1 = 0.05$, $\beta_2 = 0.04$, $\beta_3 = 0.02$, $\beta_4 = 0.01$ and $\beta_t = 0$ for $t > 4$. We further assume that the mortality intensity follows the deterministic process $\mu(t) = A + Bc^{x+t}$ for the policyholder aged x at time $t = 0$ with $A = 5.0758 \times 10^{-4}$, $B = 3.9342 \times 10^{-5}$, $c = 1.1029$. The pool of policyholders are assumed to be 40 -aged at the moment they enter into the contract.

3.4.1 Monetary Rationality and Contract Price

We first study the effect of the policyholders' monetary rationality on the contract price. Table 3.1 displays contract values V_0 for various rationalities of the policyholders that are parameterized by the lower and upper bound of the surrender intensity γ . The lower bound $\underline{\rho}$ is the base level surrender intensity representing surrender due to exogenous reasons, and takes the values 0 , 0.03 , 0.3 . The upper bound $\bar{\rho}$ limits the local exercise probability in case exercising the surrender option is financially advantageous, see (3.9). It takes the values 0 , 0.03 , 0.3 , and ∞ . We may say that a policyholders acts financially more rational the lower the lower bound $\underline{\rho}$ and the higher the upper bound $\bar{\rho}$. It is clear that a higher degree of rationality leads to a higher contract price, see Proposition 3.3.2.

	$\bar{\rho}$				
$\underline{\rho}$	0.00	0.03	0.30	3.00	∞
0	102.7630	103.9335	108.2971	110.6107	110.9602
0.03	-	99.4447	103.5910	105.5440	105.8250
0.3	-	-	92.7071	94.4926	94.9999

Table 3.1: Contract value V_0 for various bounds $\underline{\rho}$ and $\bar{\rho}$

For $\underline{\rho} = \bar{\rho} = 0.00$ the surrender option is never exercised. Therefore we obtain a European-style contract with value 102.7630 . Keeping $\underline{\rho} = 0.00$ and increasing the upper bound $\bar{\rho}$ to the limit ∞ results in a contract where the surrender option is exercised optimally. The value of the American-style contract is 110.9602 , and is about 8% higher than the value of the corresponding European-style contract. In general we can observe that the contract values are increasing with increasing $\bar{\rho}$ as stated in Proposition 3.3.2. Purely exogenous surrender can be presented by assuming that the upper and lower bound are identical, i.e. $\underline{\rho} = \bar{\rho}$. The values on the diagonal of Table 3.1 are decreasing with increasing surrender rate. This is not a general effect but due to the fact that for this contract the surrender value L is on average lower than the value V of a contract that is still alive. Fixing the upper bound and varying the lower bound representing the exogenous surrender the contract values are increasing with decreasing lower bound $\underline{\rho}$ what is in line with Proposition 3.3.2.

Let us now focus on the benchmark parameters for the subsequent fair contract analysis in Section 3.4.3, i.e. set $\underline{\rho} = 0.03$ and $\bar{\rho} = 0.30$. The resulting contract value is 103.5910. To obtain the corresponding purely exogenous surrender situation the upper bound is set to $\bar{\rho} = 0.03$ and the value decreases to 99.4447. In contrast, for optimal exercise of the surrender option the upper bound is set to ∞. The corresponding contract value increases to 105.8250. We can interpret the benchmark value of 103.5910 as a weighted average of the purely exogenous surrender value and the value obtained when the surrender option is optimally exercised, with weights 35% and 65%, respectively.

3.4.2 The Separating Boundary

For the insurance company writing the contract it is instructive to identify the actual surrender intensity γ for any given time t and asset value $S_t = s$. According to (3.9) γ is determined by the current contract value and surrender benefit. Once the value function v is obtained by solving the pricing PDE in Propostion 3.3.1 we can identify the region C where purely exogenous surrender occurs, $\gamma(t,s) = \underline{\rho}$, i.e. $C = \{(t,s) \in [0,T] \times \mathbb{R}^+ : v(t,s) > L(t)\}$, and its complement C^c where surrender occurs at the maximal intensity, $\gamma(t,s) = \bar{\rho}$, i.e. $C^c = \{(t,s) \in [0,T] \times \mathbb{R}^+ : v(t,s) \leq L(t)\}$. The separating boundary is then the set $\partial C = \{(t,s) \in [0,T] \times \mathbb{R}^+ : v(t,s) = L(t)\}$. Moreover, Corollary 3.3.1 characterizes the relationship of C when the bounds of the surrender intensity $\underline{\rho}$ and $\bar{\rho}$ are varied. Decreasing $\underline{\rho}$ or, alternatively, increasing $\bar{\rho}$ expands the set C where purely exogenous surrender occurs.

Figure 3.1 displays the separating boundary for the benchmark parameters on the left and for the case when the upper bound of the surrender intensity is set to ∞ on the right. For both figures we observe that a higher underlying price makes the participation in it more attractive, and hence indicates a lower surrender rate in this region. While a lower underlying price suggests that it is not promising to benefit from the growth of the underlying price. In addition, three factors affect the separating boundary. One is the interest rate effect. In our example, the minimum guaranteed rates at death, at survival and at surrender are all smaller than the interest rate on the market. An early surrender enables the policyholders to invest their money into a riskless asset with a higher rate

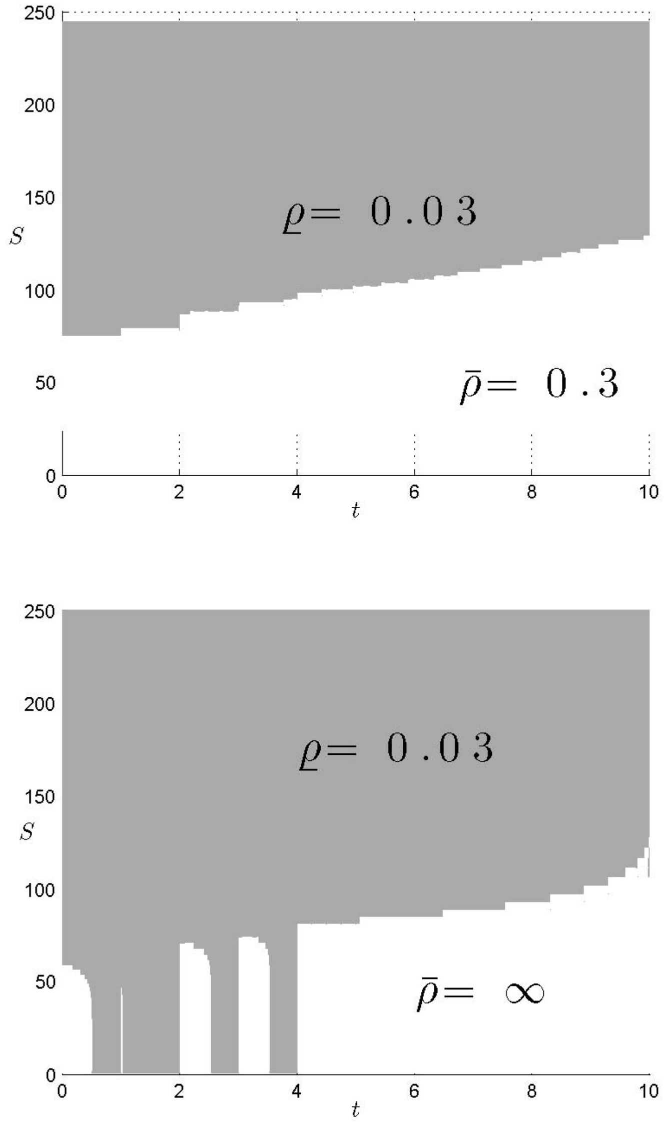


Figure 3.1: The separating boundary ∂C for $\underline{\rho} = 0.03$, $\bar{\rho} \in \{0.3, \infty\}$, $\alpha = 0.85$, $g = g_d = h = 0.02$, $k = k_d = 0.9$, $\beta_1 = 0.05$, $\beta_2 = 0.04$, $\beta_3 = 0.02$, $\beta_4 = 0.01$ and $\beta_t = 0$ for $t \geq 5$.

of return than the minimum guaranteed rate and is hence preferred. The incentive to surrender the contract earlier can be reduced if the asset price is high enough so that the probability of receiving a higher payoff increases which offsets the interest rate effect. The second one is the time effect. For the same asset price level, the earlier it is, the more higher is the possibility that the asset price at a certain time point in the future will rise to a higher level, and hence, the higher is the continuation value of the contract. Thus, a lower asset price at the early stage can be more tolerated and the separating boundary can be lower at this stage due to the time effect. The third one is the penalty effect. In our example, there is $\alpha P(1+g)^t < (1-\beta_t)P(1+h)^t$, for all $t \geq 0$. Besides, $(1-\beta_t)P(1+h)^t \leq (1-\beta_{t'})P(1+h)^{t'}$ for $t \leq t'$. This indicates that for S small enough, the surrender value is always higher than the minimum guarantee. As time increases, the dominance of the surrender value is more obvious, and hence, the asset price must be higher to compensate the disadvantage of the relatively lower guaranteed amount. Figure 3.1 results from the three effects mentioned. Within one year, the interest rate effect dominates, while between the different years, the other two effects dominate. Consequently, the separating boundary is not smooth in the first 4 years and it is smooth and monotonically increasing afterwards. Comparing the benchmark case (top in Figure 3.1) with the case where the upper bound $\bar{\rho}$ is set to ∞ (bottom in Figure 3.1) we observe that the set indicating purely exogenous surrender C expands. This is expected due to Corollary 3.3.1.

Now, the penalty term is eliminated by setting $\beta_t = 0$ for all t . Then we obtain a separating boundary as displayed in Figure 3.2. The penalty and the time effect dominate the interest rate effect. Hence, we observe the monotonic increase of the separating boundary over the life time of the contract. Moreover, the boundary is now smooth, since the penalty parameters for different years are identical. Again, the set C where purely exogenous surrender occurs expands when the upper bound $\bar{\rho}$ is increased from 0.30 (top) to ∞ (bottom).

3.4.3 Fair Contract Analysis

In this section we study how the parameters should be specified to ensure a fair contract, i.e. $V_0 = P = 100$. Since the contract price depends on the assumption about the monetary rationality of the policyholders in our model, our fair contract analysis is conducted in a narrow sense by fixing the monetary rationality of the policyholders. The price obtained is the amount that should be charged on average based on this assumption. We assume in this part that $\underline{\rho} = 0.03$ and $\bar{\rho} = 0.30$. Furthermore, we compare the result with $\bar{\rho} = 0.03$ and with $\bar{\rho} = \infty$. Here, $\bar{\rho} = \infty$ represents the worst case from the viewpoint of the insurance company writing the contract. In presence of exogenous surrender the surrender option is exercised optimally. In contrast, $\bar{\rho} = 0.03$ characterizes the case of purely exogenous surrender. For our original parameters chosen in Section 3.4.1 the contract value is 103.5910 and is therefore over par. To reduce to the contract value,

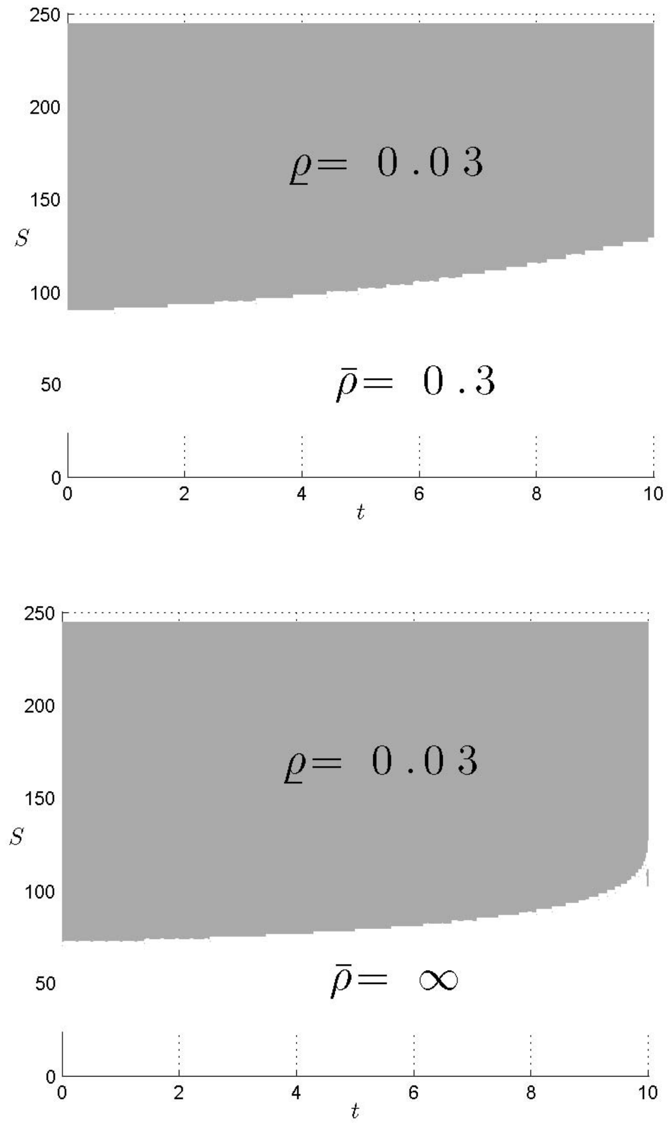


Figure 3.2: The separating boundary of ∂C for $\underline{\rho} = 0.03$, $\bar{\rho} \in \{0.3, \infty\}$, $\alpha = 0.85$, $g = g_d = h = 0.02$, $k = k_d = 0.9$, $\beta_t = 0$ for $t \geq 0$.

there are potentially three ways. The first way is to reduce the minimum guarantee at survival or at death or in both cases. The second way is to enhance the penalty in the early surrender case. The third way is to reduce the participation in the performance of the underlying asset.

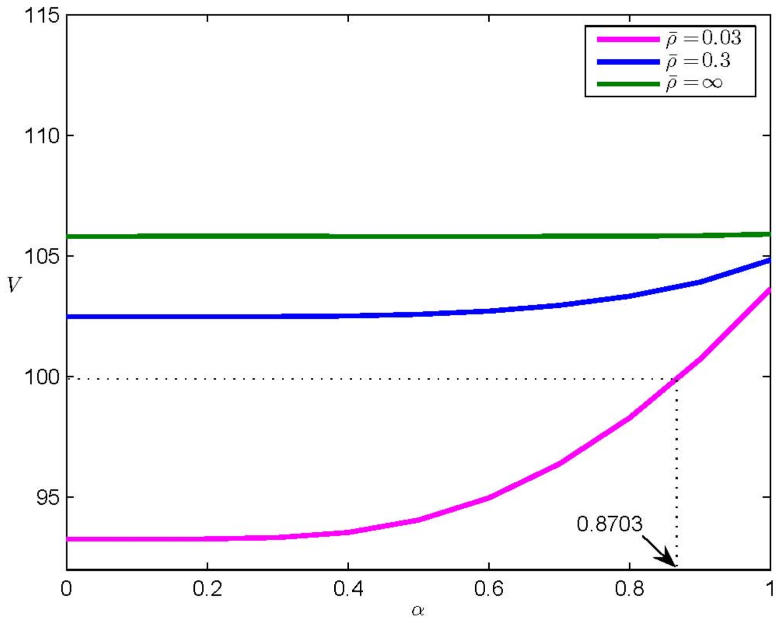


Figure 3.3: The contract value V_0 depending on the participation rate in the minimum guarantee α for $\underline{\rho} = 0.03$, $\bar{\rho} \in \{0.03, 0.3, \infty\}$, $g = g_d = h = 0.02$, $k = k_d = 0.9$, $\beta_1 = 0.05$, $\beta_2 = 0.04$, $\beta_3 = 0.02$, $\beta_4 = 0.01$, and $\beta_t = 0$, for $t \geq 5$.

We investigate the effect of a reduction of the minimum guarantee on the contract value. The reduction of minimum guarantee can be achieved either by reducing the participation rate α , the minimum guarantee rate g_1 , or g_2 . Since their effects are similar, we only focus on the participation rate α . In Figure 3.3 we present the contract values with different choices of α while other parameters are kept the same as we chose at the beginning. We notice from Figure 3.3 that the effect of the minimum guarantee on the contract value depends on the monetary rationality of the policyholders. For the completely rational policyholders (i.e., $\bar{\rho} = \infty$), the minimum guarantee hardly has any effect on the contract value. When the policyholders are on average more rational than those who only surrender for exogneous reasons, the effect of the minimum guarantee is also minor. This is because a reasonable surrender guarantee is supplied in the contract. If it is unprofitable to go on holding the contract, the policyholders can simply terminate the contract and obtain the guaranteed surrender value which may be higher than the minimum guarantee. On the contrary, for irrational policyholders (i.e., when $\bar{\rho} = 0.03$), their surrender decisions do not depend on the surrender guarantee. The effect of the minimum guarantee on the contract value is hence much higher. We can verify this rationale by setting the surrender value to zero.

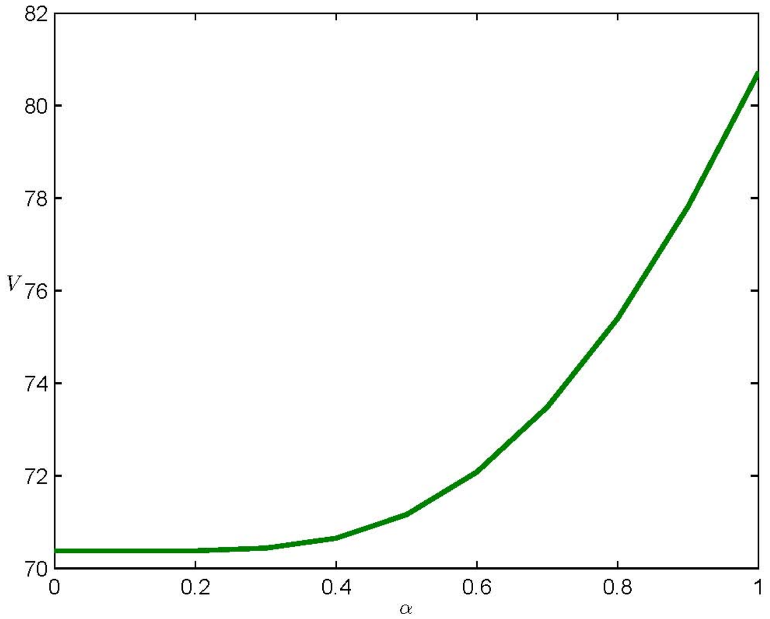


Figure 3.4: The contract value V_0 depending on the participation rate in the minimum guarantee α for $\underline{\rho} = 0.03$, $\bar{\rho} \in \{0.03, 0.3, \infty\}$, $g = g_d = h = 0.02$, $k = k_d = 0.9$, $\beta_t = 1$, for $t \geq 0$.

The contract values for $\beta = 1$, and hence $L = 0$, and various values for α are displayed in Figure 3.4. We see that the contract value in this case is actually independent of the monetary rationality. This is because the surrender value is zero so that always the lowest surrender intensity $\underline{\rho}$ applies which is identical for the different choices for $\bar{\rho}$. We also see that when the surrender guarantee is small the participation rate α plays a more important role in determining the contract value. The contract values for $\alpha = 0$ and $\alpha = 1$ differ by 10.3529 whereas in the previous setting the difference was just 2.3552 , both for $\bar{\rho} = 0.30$. The pattern is similar for $\bar{\rho} = 0.03$ and $\bar{\rho} = \infty$. On the other hand we can interpret from Figure 3.4 that to ensure the contract to be issued at par the policyholders should not be overpenalized. In Bernard and Lemieux (2008), the participation rate α is included both in the minimum guarantee and in the asset performance. Hence, the variation of the parameter works simultaneously on both parts which may display a more significant effect. However, when we observe these two parts separately we are more clear about the specific effect of each parameter and gain insight into the design of effective contracts. According to the contract that we have designed we can simply keep $\alpha = 1$

so that the contract looks more attractive to the policyholders. While other parameters should be adjusted more carefully.

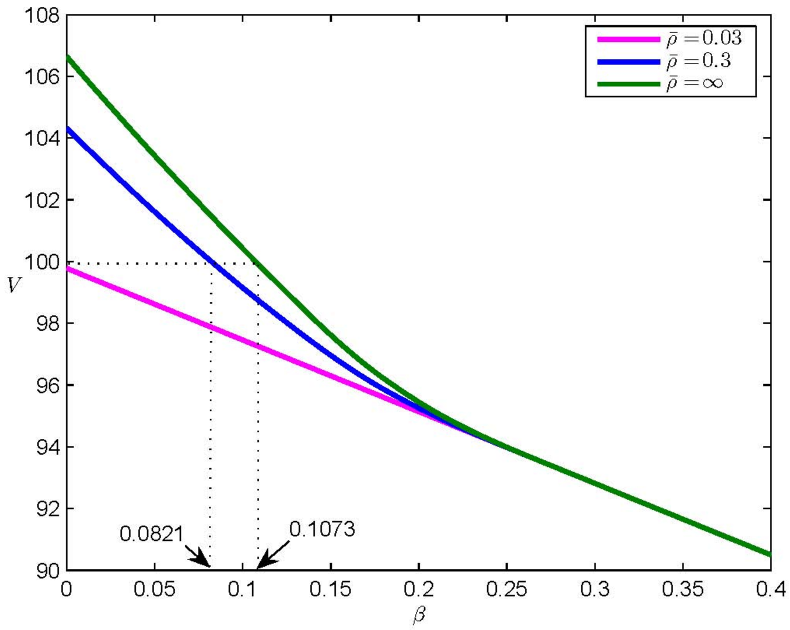


Figure 3.5: The contract value V_0 depending on the penalty parameter β for $\underline{\rho} = 0.03$, $\bar{\rho} \in \{0.03, 0.3, \infty\}$, $\alpha = 0.85$, $g = g_d = h = 0.02$, $k = k_d = 0.9$.

Next, we investigate the relationship between the penalty parameter and the contract value. In Figure 3.5 we display the contract value V_0 as a function of penalty parameter β graphically for different degrees of monetary rationality, $\bar{\rho} = 0.03, 0.30, \infty$. The contract value is monotonically decreasing in the penalty parameter. For the contract to be fairly issued the penalty parameter should be 0.0821 for $\bar{\rho} = 0.30$. In case of rational surrender, i.e. $\bar{\rho} = \infty$, in presence of exogenous surrender with $\underline{\rho} = 0.03$ the penalty parameter has to be increased to 0.1073 for the contract to be fair. While for purely exogenous surrender, i.e. $\bar{\rho} = 0.03$, the contract value is always under par in our example. This means that other parameters must be adjusted so as to take the policyholders' monetary irrationality into account properly.

Finally, we analyze the effect of the participation rate in the asset performance on the contract value. For simplicity we assume the participation rates for both, the survival and the death events, to be the same namely, $k = k_d$. Other parameters are consistent with the values detailed at the beginning of Section 3.4. In Figure 3.6 we display the relationship of the participation rates in the asset performance with the contract value graphically for $\bar{\rho} = 0.03, 0.30, \infty$. We see that the contract value increases monotoni-

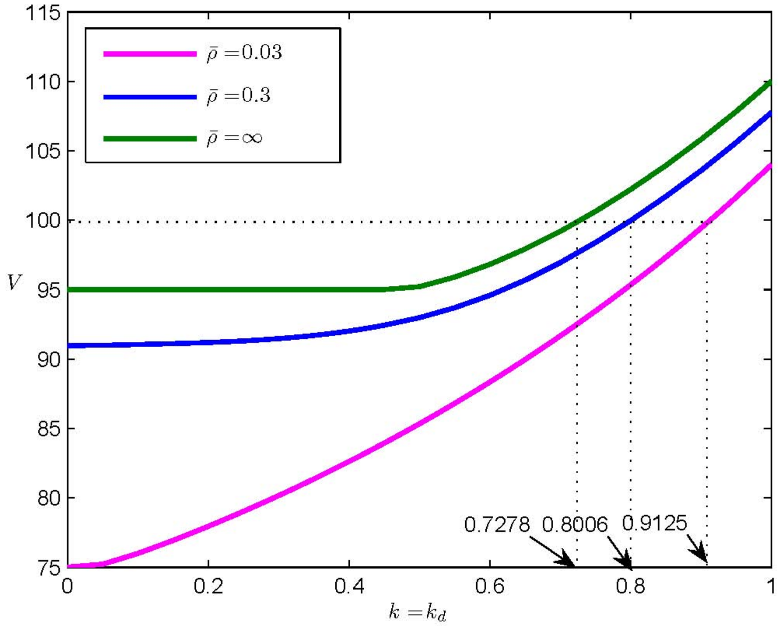


Figure 3.6: The contract value V_0 depending on the participation rates $k = k_d$ for $\underline{\rho} = 0.03$, $\bar{\rho} \in \{0.03, 0.3, \infty\}$, $\alpha = 0.85$, $g = g_d = h = 0.02$, $k = k_d = 0.9$, $\beta_1 = 0.05$, $\beta_2 = 0.04$, $\beta_3 = 0.02$, $\beta_4 = 0.01$, and $\beta_t = 0$, for $t \geq 5$.

cally with the participation rates. For $\bar{\rho} = 0.30$ and ∞ the increase is not that large for small values of the participation rates $k = k_d$. This is because in these cases holding the contract generally brings lower benefit to the policyholders than surrendering the contract prematurely. The surrender benefit thus plays a dominant role in determining the contract value. Since the surrender guarantee is independent of k and k_d in our numerical example, the contract value does not vary too for small values of $k = k_d$. Also notice that that in this case the contract value is under par. On the contrary, when k and k_d are large the survival benefit and the death benefit dominate the contract value, the contract value increases is more sensitive to changes of $k = k_d$. However, when the policyholders surrender due to exogenous reasons (indicated by $\bar{\rho} = 0.03$) the survival and death benefit are driving the contract value. Hence the effect of an increase in $k = k_d$ on the contract value is nearly linear. To obtain a fair contract the participation rates $k = k_d$ should be set to 0.8006 for $\bar{\rho} = 0.30$. Increasing $\bar{\rho}$ to its limit ∞ requires a lower participation of $k = k_d = 0.7278$ for $\bar{\rho} = 0.03$ for the contract to be fair. In contrast, for the cases of purely exogenous surrender, i.e. $\bar{\rho} = 0.03$, the participation rate has to increase to 0.9125 to constitute a fair contract.

In the remainder of this section we focus on the design of a fair contract and investigate the interaction of various parameters. First, we study the relationship between participation rate in the minimum guarantee α and the minimum guaranteed rate at survival and at death $g = g_d$. To produce realistic results we alter the benchmark parameters by setting $k = k_d = 0.7$ to ensure the existence of a fair contract. We present the relationship between α and $g = g_d$ in Figure 3.7. We see that α is decreasing in $g = g_d$. For α below 0.9 the minimum guaranteed rate of return at survival and at death must be higher than the market interest rate for the contract value to be higher. Further note that the higher the monetary rationality of policyholders is, the lower is the $\alpha - g$ level in Figure 3.7. Since the more rational policyholders can judge the situation more correctly and make the better out of it, they need less compensation offered by the minimum guarantee.

Next we study pairs of the participation rate in the minimum guarantee α and the participation parameters in the asset performance k and k_d such that a fair contract is obtained. The other parameters are kept as in the benchmark case. A graphical illustration for this setting is given in Figure 3.8. We observe that for the same level of α , a lower (higher) $k = k_d$ is required to account for the higher (lower) monetary rationality of the pool of policyholders. Moreover, when the policyholders act more rational, the sensitivity of α with respect to $k = k_d$ is higher, or in other words, the sensitivity of $k = k_d$ with respect to α is lower.

We have mentioned in Section 3.3 that the growth rate h for the surrender case is, in practice, not allowed to fall below the minimum guaranteed rate g for the survival benefit. For our numerical analysis, however, we loose this restriction and study the relationship of h with other parameters. As an example, we present in Figure 3.9 the relationship

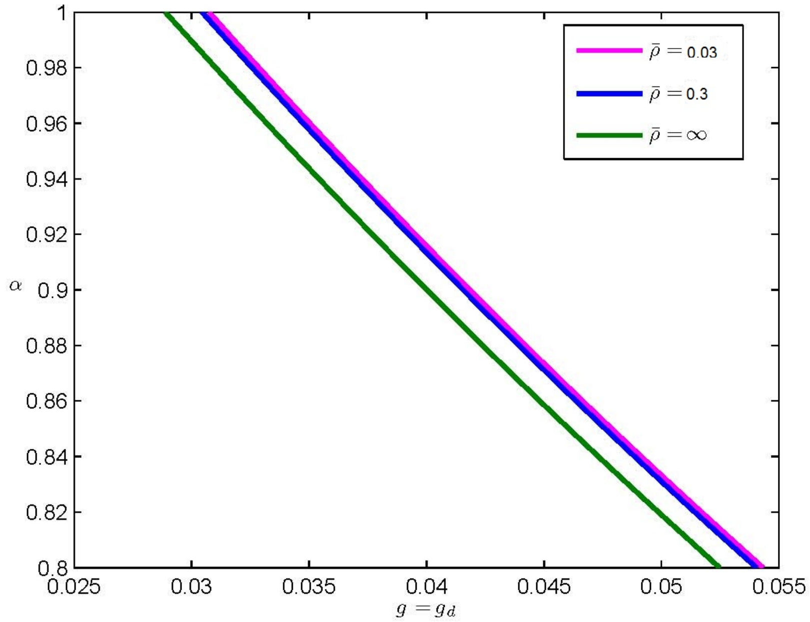


Figure 3.7: Parameter combinations of the participation rate in the minimum guarantee α and the minimum guaranteed rates at survival and at death $g = g_d$ ensuring a fair contract, for $\underline{\rho} = 0.03$, $\bar{\rho} \in \{0.03, 0.3, \infty\}$, $g = g_d = h = 0.02$, $k = k_d = 0.7$, $\beta_1 = 0.05$, $\beta_2 = 0.04$, $\beta_3 = 0.02$, $\beta_4 = 0.01$, and $\beta_t = 0$, for $t \geq 5$.

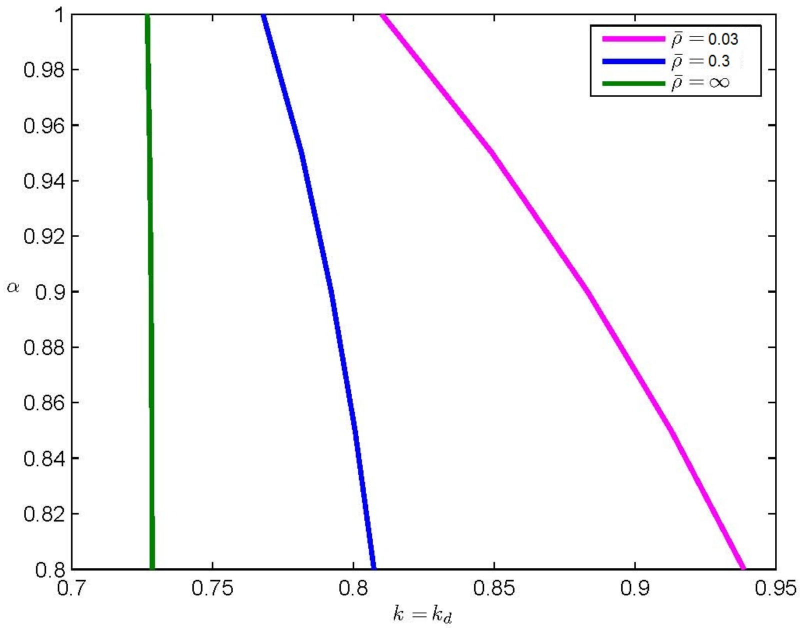


Figure 3.8: Parameter combinations of the participation rate in the minimum guarantee α and participation rates in the asset performance at survival and at death $k = k_d$ ensuring a fair contract, for $\underline{\rho} = 0.03$, $\bar{\rho} \in \{0.03, 0.3, \infty\}$, $g = g_d = h = 0.02$, $k = k_d = 0.9$, $\beta_1 = 0.05$, $\beta_2 = 0.04$, $\beta_3 = 0.02$, $\beta_4 = 0.01$, and $\beta_t = 0$, for $t \geq 5$.

between h and $k = k_d$. It is obvious that for given $k = k_d$, h must be set lower (higher) to account for the higher (lower) monetary rationality of the policyholders. For the policyholders with low monetary rationality, a fair contract may not even exist if we keep h at the same level as g , and at the same time, only allow the policyholders to participate in the asset performance less than proportionally.

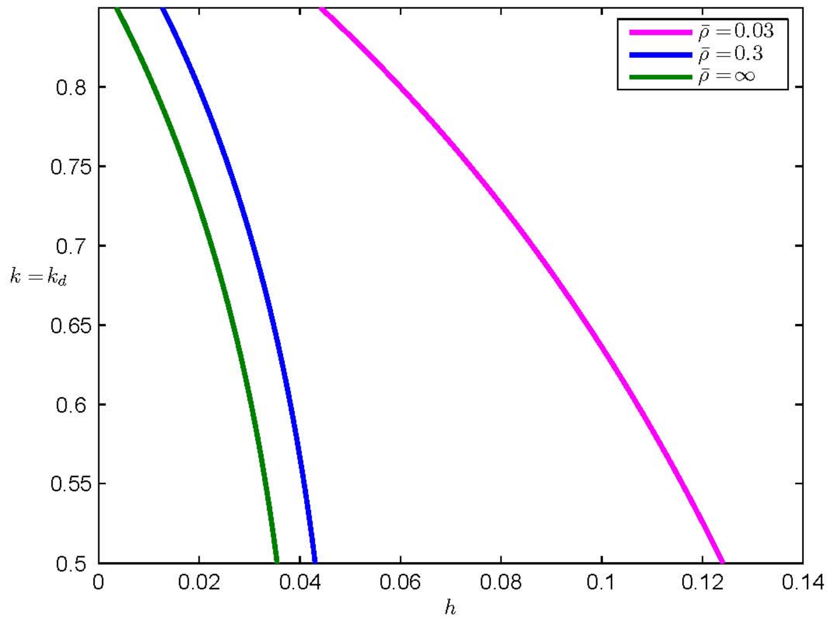


Figure 3.9: Parameter combinations of the minimum guaranteed rate h for the surrender benefit and the participation rates in the asset performance at survival and at death $k = k_d$ ensuring a fair contract, for $\underline{\rho} = 0.03$, $\bar{\rho} \in \{0.03, 0.3, \infty\}$, $\alpha = 0.85$, $g = g_d = 0.02$, $\beta_1 = 0.05$, $\beta_2 = 0.04$, $\beta_3 = 0.02$, $\beta_4 = 0.01$, and $\beta_t = 0$, for $t \geq 5$.

For the insurance company the level of the policyholders' monetary rationality indicated by $\bar{\rho}$ is an important parameter affecting valuation and hedging of the respective contract. Assuming the highest possible rationality ($\bar{\rho} = \infty$) is specifying a worst case scenario and is therefore of particular interest. Compared to the benchmark case ($\bar{\rho} = 0.30$) the contract design has to be modified accordingly to produce a fair contract. Focusing on the surrender component, the increase of the rationality to the highest possible level can be compensated by either increasing the penalty β by 0.025 (see Figure 3.5) or by decreasing the minimum guaranteed rate h by 0.01 (see Figure 3.9). Alternatively, death and survival benefit can be adjusted either by decreasing the participation α by 0.02 (see Figure 3.7), by decreasing the minimum guaranteed rates $g = g_d$ by 0.002 (see Figure 3.7), or by decreasing the participation rate in the asset performance $k = k_d$ by 0.07 (see Figure 3.6 and Figure 3.9). Overall, the design of our contract is fairly robust with respect to variations in the rationality. The parameter that has perhaps the highest sensitivity with respect to variations in the level of rationality is the participation rate in the asset performance $k = k_d$ (see Figure 3.8).

3.5 Conclusion

In this chapter we have studied the valuation of unit-linked life insurance contracts with surrender guarantees. Instead of solving an optimal stopping problem, we have proposed a more realistic approach accounting for policyholders' monetary rationality in exercising their surrender option. The valuation is conducted at the portfolio level by assuming the surrender intensity to be bounded from below and from above. The lower bound corresponds to purely exogenous surrender and the upper bound represents the limited monetary rationality of the policyholders. In practice, the lower and the upper bounds can be obtained from historical data. We have shown that for different degrees of monetary rationality the average contract value can vary significantly. Hence, it is important to judge the monetary rationality of the potential policyholders realistically. Based on the realistic estimation of their monetary rationality, the contract can be designed more reasonably and an average overvaluation can be avoided. We provide the separating boundary between purely exogenous surrender and surrender due to financial reasons. This may help insurance companies to better understand the surrender activity of their policyholders affecting also the companies' hedge programs. In addition, our fair contract analysis has revealed specific contract designs that are fairly robust with respect to the degree of monetary rationality of the policyholders.

This chapter can be extended in several ways. The bounds $\underline{\rho}$ and $\bar{\rho}$ need not be constant but can be driven by market variables and non-financial factors. An extension in this direction has been carried out in Uzelac and Szimayer (2012) where the bounds of the surrender intensity further depend on the occurrence of two possible economic states. Further, as indicated in Remark 3.3.1 we can extend the model to allow for stochastic interest rates and stochastic volatility. The general results in Proposition 3.3.1 and Proposition 3.3.2 and the respective corollaries are likely to carry over. However, in a multi-factor model solving the valuation PDE can easily become a high dimensional problem. In this case, least-squared Monte Carlo simulation following Longstaff and Schwartz (2001) can be adapted. This issue will be addressed in our future research. A further interesting perspective is to incorporate a secondary market where the policyholders are given the additional option to sell their contracts to a third party. The impact of a secondary market on contract valuation and fair contract design could be significant. This problem is addressed in Hilpert, Li and Szimayer (2011).

Chapter 4

Concluding Remarks

In this dissertation we have studied two types of financial derivatives, namely, basket FX derivatives and unit-linked life insurance contracts.

For the pricing and risk management of basket FX derivatives, two types of financial risk are influential, the interest rate risk and the exchange rate risk. We have first presented the international financial market model based on Amin and Jarrow (1991) to describe these two risk factors. The underlying of basket FX derivatives is a basket of foreign currencies, whose exchange rates to the domestic currency follow geometric Brownian motions within our international financial market model. To cope with the difficulty that the distribution of such basket is unknown to us and thus the closed-form solution to the pricing problem is not available, we have suggested the rank one approximation method in combination with three moment matching. We have shown that this method outperforms one of the popular approximation methods–the lognormal approximation method which is often used for the approximation pricing of basket derivatives. Based on the prices obtained through the rank one approximation method, both the dynamic and the static hedging strategies were examined with regard to their hedging performances.

When studying unit-linked life insurance contracts, we have neglected the interest rate risk to simplify the problem and only assumed the financial risk related to the underlying asset. The focus has been set on the two other risk sources which are determinant for the contract valuation. One is the mortality risk, and the other one is the surrender risk.

The mortality risk is represented by the mortality intensity, which is deterministic when there is only unsystematic mortality risk and stochastic when systematic mortality risk also exists. In recent years, there has been the consensus that the mortality intensity is governed by certain stochastic processes. In this dissertation, we have focused on model risk arising from different specifications for the mortality intensity. To do so we have assumed that the mortality intensity is almost surely bounded under the statistical measure. Further, we have restricted the equivalent martingale measures and applied the same bounds to the mortality intensity under these measures. For this setting we have derived upper and lower price bounds for unit-linked life insurance contracts using stochastic control techniques. We have also shown that the induced hedging strategies indeed produce a dynamic superhedge and subhedge under the statistical measure in the limit when the number of contracts increases. This has justified the bounds for the mortality intensity under the pricing measures. We have provided numerical examples investigating fixed-term, endowment insurance contracts and their combinations including various guarantee features. The pricing partial differential equation for the upper and lower price bounds has been solved by finite difference methods. For our contracts and choice of parameters the pricing and hedging has been fairly robust with respect to misspecification of the mortality intensity. The model risk resulting from the uncertain mortality intensity has been of minor importance.

The surrender risk is relevant when unit-linked life insurance contracts allow for premature termination. Instead of solving an optimal stopping problem, we have proposed a more realistic approach accounting for policyholders' monetary rationality in exercising their surrender option. The valuation has been conducted at the portfolio level by assuming the surrender intensity to be bounded from below and from above. The lower bound corresponds to purely exogenous surrender and the upper bound represents the limited monetary rationality of the policyholders. The valuation problem has been formulated by a valuation PDE and solved with the finite difference method. We have shown that the monetary rationality of the policyholders has a significant effect on average contract value and hence on the fair contract design. We have also presented the separating boundary between purely exogenous surrender and endogenous surrender. This has provided implications on the predicted surrender activity of the policyholders.

The real world could be more complicated either by nature or by man in various aspects. In the long run, interest rate is more likely to be stochastic, and so is the volatility of the underlying asset. These financial risk factors may have impact on the mortality model risk and the surrender risk we have investigated above. Moreover, the bounds of the surrender intensity could be driven by market variables and non-financial factors. These issues will be studied in our future research. In a multi-factor model, it is not suitable to apply the PDE approach any more due to the high dimensionality of the problem. We will adapt the least-squared Monte Carlo simulation method following Longstaff and Schwartz (2001) to study the open questions we have proposed above.

Appendix: The Bounds of the Mortality Intensity

The bounds of the mortality intensity are obtained through the Lee-Carter model. The Lee-Carter model in Lee and Carter (1992) and its various extensions, see, e.g., Lee (2000), have been used successfully to forecast the death rates of the population in many developed countries, such as USA, Canada, Japan, Chile, Belgium, Austria and Australia.

Lee and Carter (1992) describe the logs of the age-specific death rates $m(t, x)$ by a linear function of an unobserved period-specific intensity index k_t with age-specific parameters a_x and b_x .[1] After the estimation of a_x and b_x as well as k_t in the past, a time series model is applied to describe the dynamics of k so as to forecast its future development, based on which the central death rate is forecasted and its confidence interval was found. The link between the central death rate and the mortality intensity is $m_t = \int_t^{t+1} \nu_u \, \mathrm{d}u$.[2]

Assuming that there would be no extreme change with the mortality intensity within one year, the same confidence interval that bounds the death rate should also be a suitable bound for the mortality intensity. The age-specific parameters we use in this paper are presented in Table A.

Concerning the mortality index k , we take the ARIMA time series model estimated by Lee and Carter but without the dummy term, namely, $k_t = k_{t-1} - 0.365 + e_t$. The standard error of the estimation (see) is assumed to be 0.651 and k_0 is equal to -18 . Also for the sake of simplicity, we make the assumption that there are no estimation errors with the age-specific parameters. The uncertainty of the mortality forecast is supposed to be only attributed to the random behavior of the mortality index k . Under this assumption,

[1] The constraint serves to normalize the solutions, see Lee and Carter (1992). It should be pointed out that here x denotes the age of an individual at time t instead at time 0 as we have referred to in the previous part. We allow this abuse of notation to keep consistent with the literature on the Lee-Carter model. In the later part, we always refer x to the age of an individual at time 0 .

[2] Focusing only on the policyholders who are aged x at time 0 , we omit x in the index and denote the central death rate at time t as m_t .

Table A: The age-specific parameters a_x and b_x

x	a_x	b_x
40-44	-5.51323	0.05279
45-49	-5.09024	0.04458
50-54	-4.65680	0.03830
55-59	-4.25497	0.03382
60-64	-3.85608	0.02949
65-69	-3.47313	0.02880
70-74	-3.06117	0.02908
75-79	-2.63023	0.03240
80	-2.20498	0.03091

we obtain that the confidence interval of the mortality intensity at confidence level p by

$$\begin{aligned}\mu_{40+t} \;\in\; & \Big[\exp\Big(a_{40+\lfloor t\rfloor} + b_{40+\lfloor t\rfloor}\,(k_0 - \lfloor t\rfloor\, 0.365) - q\, b_{40+\lfloor t\rfloor}\,\text{see}\,\sqrt{\lfloor t\rfloor}\Big)\,, \\ & \exp\Big(a_{40+\lfloor t\rfloor} + b_{40+\lfloor t\rfloor}\,(k_0 - \lfloor t\rfloor\, 0.365) + q\, b_{40+\lfloor tu\rfloor}\,\text{see}\,\sqrt{\lfloor t\rfloor}\Big)\Big]\,,\end{aligned}$$

for $0 \le t \le 30$, and $q = \Phi^{-1}((1+p)/2)$.

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