Quantum Statistical Mechanics of Shimura Varieties

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Abstract

We investigate from the point of view of quantum statistical mechanics certain groupoids and C*-dynamical systems arising from Shimura varieties. Shimura varieties are higher dimensional analogues of elliptic modular curves, and play an important role in modern number theory.

The starting point of our investigation is the Bost-Connes C*-algebra which is the convolution algebra of functions on the groupoid defined by the (partial) action of $\mathrm{GL}_1^+(\mathbb{Q})$ on $\hat{\mathbb{Z}}$ (the profinite completion of the integers). This C*-algebra is the complexification of a distinguished rational subalgebra, and comes with a canonical time evolution, which moreover commutes with an action by the Galois group $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$. The resulting C*-dynamical system has remarkable arithmetic properties: when regarded as the algebra of observables of a quantum statistical mechanical system, its equilibrium states at zero temperature (more specifically, its KMS_∞ states) take values in $\mathbb{Q}^{\mathrm{ab}}$ when evaluated on the rational subalgebra, and the Galois action on $Q^{\mathrm{ab}}/\mathbb{Q}$ matches the Galois action on equilibrium states. An analogue of the Bost-Connes system having a similarly rich arithmetical structure was recently constructed by Connes-Marcolli for the group GL_2.

The Bost-Connes and Connes-Marcolli systems are seen to be associated to the Shimura varieties for GL_1 and GL_2, respectively, and in this thesis we carry out the construction of Bost-Connes-Marcolli systems (consisting of a groupoid and an associated C*-dynamical system) for general Shimura varieties. We study the detailed structure of the underlying groupoid, attach to it various zeta functions (that coincide with statistical-mechanical partition functions and, in certain cases, classical zeta functions), and analyze its low-temperature KMS states. We also study various special cases. Our Shimura-variety approach provides a unified treatment of such C*-dynamical systems, and for the first time allows for the construction of a Bost-Connes system for a general number field F that admits symmetry by the group of connected components of the idèle class group of F (which is isomorphic to $\mathrm{Gal}(F^{\mathrm{ab}}/F)$ by class field theory), and recovers the Dedekind zeta function as a partition function. One noteworthy (and rather crucial) ingredient in our constructions is a reductive monoid for the reductive group associated to the Shimura variety. Such monoids, which have been studied by Lenner, Putcha, Vinberg, and Drinfeld, are closely related to reductive groups, but (to the best of our knowledge) have hitherto played little role in the theory of Shimura varieties. Our work reveals their relation to noncommutative spaces.

Acknowledgments

I left the writing of the acknowledgments to the very end so that I could finish off with the most pleasant task ...

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Some Notational Conventions

Units and Superunits One potential notational pitfall is the distinction that we make between the superscripts $^\times$ and *. Given an inclusion of rings $A \subset B$, we write $A^\times$ for the multiplicative group of *units* of A, while we write A^* for the multiplicative monoid of elements in A that are invertible in B (which we call the *superunits* of A with respect to B), that is, $A^* = A \cap B^\times$. To take an example of this notational distinction, for the inclusion $\mathbb{Z} \subset \mathbb{Q}$ we have $\mathbb{Z}^\times = \{\pm 1\}$, while $\mathbb{Z}^* = \mathbb{Z} - \{0\}$. In examples such as this one where the given ring A comes with a canonical inclusion, the notation A^* will be employed without explicit mention of the ambient ring B.

Algebraic Groups Regarding algebraic groups, we employ standard notation. Thus, if G is an algebraic group over $\mathbb{Q}$, then for any $\mathbb{Q}$-algebra A, the group of A-valued points of G is denoted by $G(A)$; the connected component of the identity of $G(\mathbb{R})$ is $G(\mathbb{R})^+$, and $G(\mathbb{Q})^+ = G(\mathbb{Q}) \cap G(\mathbb{R})^+$.

Adèles Our notation concerning adèles and such is standard. We denote the profinite completion of the integers $\mathbb{Z}$ by $\hat{\mathbb{Z}}$: this is the (compact) ring

$$\hat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n\mathbb{Z} = \prod_p \mathbb{Z}_p$$

where $\mathbb{Z}_p$ are the p-adic integers. The locally compact ring of *finite adèles* is $\mathbb{A}_\mathrm{f} = \mathbb{Q} \otimes_\mathbb{Z} \hat{\mathbb{Z}}$, while the full ring of *adèles* is $\mathbb{A} = \mathbb{R} \times \mathbb{A}_\mathrm{f}$. Similar conventions apply for any number field F. For example, the ring of F-adèles is $\mathbb{A}_F = \mathbb{A} \otimes_\mathbb{Q} F$, the profinite completion of the F-integers $\mathcal{O}_F$ is $\hat{\mathcal{O}}_F = \mathcal{O}_F \otimes_\mathbb{Z} \hat{\mathbb{Z}}$, and so on.

Chapter 1

Introduction

Arithmetic geometry, as it is usually regarded, is the algebraic geometry of varieties (or schemes) defined over arithmetic fields, often investigated with a view towards Diophantine equations. However, it has recently been discovered that a *noncommutative* geometry of arithmetic varieties sometimes lurks beyond the reach of the usual tools of algebraic geometry. Such is the case, for example, at the boundary of modular curves, where the addition of noncommutative elliptic curves enriches the classical theory of modular symbols (see [33, Chapter 2]).

One passage from the notion of space to the notion of noncommutative space is opened by the (anti) equivalence between the category of locally compact Hausdorff spaces and the category of commutative C*-algebras (Theorem of Gelfand-Naimark, [9]); namely, a noncommutative space is the "space" corresponding to a noncommutative algebra, which is then regarded as a noncommuting algebra of "coordinates". That this is a real mathematical concept, and not merely a synonym for noncommutative algebra, owes in large part to the extensively developed theory of the geometry and topology of such spaces that has arisen around the work of Alain Connes. His approach, based on abstract functional analysis, has thus far yielded the most spectacular results, with applications to foliations, index theory (vastly generalized), the Novikov conjecture, and the Standard Model of particle physics; see [9] for a panorama of this marvelous mathematical world. In short, noncommutative spaces have been found to be abundant in (mathematical) Nature, and their geometry plays a role even in elucidating the finer structure of classical spaces.

This thesis lies in the intersection of number theory, noncommutative geometry, and physics. Much of this work is an outgrowth of a fruitful collaboration with Frédéric Paugam (Institut de Mathématiques de Jussieu, Paris). We show that for any Shimura variety there is a naturally associated quantum statistical mechanical system (QSM). For particular Shimura varieties such systems furnish physical settings for explicit class field theory, as shown in the seminal work of J.-B. Bost and Connes [3], and Connes, M. Marcolli, and N. Ramachandran [14]; it is hoped that this will lead to a new approach to explicit class field theory (Hilbert's 12th problem). Moreover, our generalization of certain aspects of these works uncovers fascinating new facets in the study of Shimura varieties, which play a central role in the Langlands program.

What Bost and Connes discovered is that the following C*-dynamical system (A_1, σ_t) recovers the class field theory of $\mathbb{Q}$ when interpreted as a QSM. Here A_1 is the C*-algebra of the groupoid

$$\mathcal{G}_1 = \{ (r, \rho) \mid r\rho \in \hat{\mathbb{Z}} \} \subset \mathbb{Q}_+^\times \times \hat{\mathbb{Z}}$$

of the partially defined action of $\mathbb{Q}_+^\times$ on $\hat{\mathbb{Z}}$, and σ_t is a certain 1-parameter group of automorphisms of A_1. To regard (A_1, σ_t) as a QSM means that one distinguishes, for each inverse temperature $0 < \beta \leq \infty$, the β-equilibrium states of (A_1, σ_t), which are characterized mathematically by the *KMS_β condition* [21]. Important statistical mechanical functions take on an arithmetic significance in this setting; for example, the partition function coincides with the Riemann zeta function. To make the connection to the class field theory of $\mathbb{Q}$, Bost and Connes showed that:

1. A_1 admits a symmetry by the action of the profinite group $\hat{\mathbb{Z}}^\times$ which commutes with the time evolution σ_t, and which therefore acts on the KMS_β states; and

2. A_1 is generated (over $\mathbb{C}$) by a natural rational subalgebra $A_1^{\mathbb{Q}}$. Importantly, $A_1^{\mathbb{Q}}$ has an explicit presentation and is preserved by the $\hat{\mathbb{Z}}^\times$ action.

In general, for any given β, the space of KMS_β states forms an infinite-dimensional compact convex simplex. The evaluation of the extremal KMS_∞ states on $A_1^{\mathbb{Q}}$ yields the generators of the maximal abelian extension $\mathbb{Q}^{\mathrm{ab}}$ of $\mathbb{Q}$, and the Galois action on these numbers coincides with the $\hat{\mathbb{Z}}^\times$ action on the KMS_∞ states via the class field theory isomorphism $\hat{\mathbb{Z}}^\times \cong \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$. In hindsight, one recognizes the space of extremal KMS_∞ as the Shimura variety $\mathrm{Sh}(\mathrm{GL}_1, \{\pm 1\})(\mathbb{C})$.

The phenomena that appears in the work of Bost-Connes has been formulated as the problem of "Fabulous States" (see [11] and [14]), which makes its relation to Hilbert's 12th problem clearer. Roughly speaking, given a number field F, one seeks a QSM (A, σ_t) such that:

1. As an algebra, $A = A_F \otimes_F \mathbb{C}$ for some F-algebra A_F;
2. The KMS_∞ states yield the maximal abelian extension F^{ab} of F upon evaluation on A_F; and
3. There is a symmetry of (A, σ_t) by the group $G_F = \mathrm{Gal}(F^{\mathrm{ab}}/F)$ such that the evaluation of KMS_∞ states intertwines the G_F action on A_F with the usual Galois action of G_F on F^{ab}/F.

Additionally, one expects to recover the Dedekind zeta function as partition function.

The noncommutative-geometric underpinning of the Bost-Connes system was discovered by Connes and Marcolli. They showed that A_1 is the noncommutative space of $\mathbb{Q}$-lattices in $\mathbb{R}$ (lattices in $\mathbb{R}$ together with a labeling of its torsion points), up to commensurability and scaling by $\mathbb{R}_+^\times$. This realization of the Bost-Connes system as a noncommutative space leads naturally to the higher dimensional analogue of spaces of $\mathbb{Q}$-lattices in $\mathbb{R}^n$. For $n = 2$, this leads to a QSM (A_2, σ_t) having many of the features of the Bost-Connes system:

1. It is the C*-algebra of the quotient of the groupoid of commensurability classes of $\mathbb{Q}$-lattices in $\mathbb{C}$ modulo scaling by $\mathbb{C}^\times$;
2. There is an explicit classification of the KMS_∞ states of (A_2, σ_t);

3. There exists a natural action of the group $\mathbb{Q}^\times \backslash \mathrm{GL}_2(\mathbb{A}_\mathrm{f})$ commuting with the time evolution;

4. There is a rational subalgebra $A_2^\mathbb{Q}$ of the algebra of unbounded multipliers of A_2 to which the $\mathbb{Q}^\times \backslash \mathrm{GL}_2(\mathbb{A}_\mathrm{f})$ action, and also the evaluation of KMS_∞ states, extends; and most importantly for applications

5. The action of the symmetry $\mathbb{Q}^\times \backslash \mathrm{GL}_2(\mathbb{A}_\mathrm{f})$ on the KMS_∞ states recovers the *Shimura Reciprocity Law* upon evaluation on the rational algebra $A_2^\mathbb{Q}$.

Again, one finds that the set of extremal KMS_∞ states is a manifestation of a Shimura variety, in this case

$$\mathrm{Sh}(\mathrm{GL}_2, \mathbb{H}^\pm)(\mathbb{C}) \cong \mathrm{GL}_2(\mathbb{Q}) \backslash \mathrm{GL}_2(\mathbb{A}_\mathrm{f}) \times \mathbb{H}^\pm.$$

One can regard the noncommutative space

$$\mathrm{GL}_2(\mathbb{Q}) \backslash M_2(\mathbb{A}_\mathrm{f}) \times \mathbb{H}^\pm$$

of (not necessarily invertible) $\mathbb{Q}$-lattices up-to-scaling as a deformation of this Shimura variety.

Key to identifying the geometry of the Bost-Connes and Connes-Marcolli systems is the interpretation of the underlying groupoid as a (badly-behaved) generalized equivalence relation on the space of $\mathbb{Q}$-lattices. In fact, given the possible connection between Hilbert's 12th problem and the systems of Bost-Connes-Marcolli, it is almost expected that Shimura varieties should play a more active role in any generalization aimed at reinforcing such a link. Thus, the starting point of our work is to bring the Shimura variety to the fore and incorporate it into the construction of the groupoid. In this way we manage to construct a QSM for general Shimura varieties which, in addition to recovering the constructions of Bost-Connes and Connes-Marcolli, preserves the main features of these systems in general, and has, as we shall briefly discuss below, the primary desirable features in the case of multiplicative Shimura varieties (these features are needed to accommodate so-called "fabulous states").

Let us give a quick overview of our construction, which starts with a Shimura datum (G, X), as defined by Deligne [16]. A Shimura datum consists of a reductive algebraic group G over $\mathbb{Q}$ and a certain Hermitian symmetric domain X associated to G, subject to axioms that ensure that

$$\mathrm{Sh}(G,X)(\mathbb{C}) = \varprojlim_K G(\mathbb{Q})\backslash G(\mathbb{A}_{\mathrm{f}}) \times X/K,$$

where K runs over the compact open subgroups of the adèlic group $G(\mathbb{A}_{\mathrm{f}})$, are the complex points of a scheme $\mathrm{Sh}(G,X)$ defined over a number field.

To promote (G,X) to a QSM, we consider two extra pieces of data (implicit in the constructions of Connes-Marcolli). The first is an *augmentation* of (G,X) by the addition of a so-called *enveloping algebraic monoid* M satisfying $M \subset \mathrm{End}(V)$ and $M^\times = G$, for some representation V of G. The second is the specification of *level structure data* consisting of a compact open subgroup $K \subset G(\mathbb{A}_{\mathrm{f}})$, and a compact open semigroup $K_M \subset M(\mathbb{A}_{\mathrm{f}})$ with $K \subset K_M$ (and, additionally, a lattice in V, for technical reasons).

With these two pieces in hand, we define the following generalization of the Bost-Connes-Marcolli systems. One has the groupoid

$$\mathcal{G} \subset G(\mathbb{A}_{\mathrm{f}}) \times \big(K_M \times \mathrm{Sh}(G,X)(\mathbb{C})\big)$$

of the partially defined action of $G(\mathbb{A}_{\mathrm{f}})$ on $K_M \times \mathrm{Sh}(G,X)(\mathbb{C})$, generalizing the commensurability relation for $\mathbb{Q}$-lattices. The quotient of $\mathcal{G}$ by the $(K\times K)$-action $(k_1,k_2)\cdot(g,(\rho,[x,h])) = (k_1 g k_2^{-1},(k_2\rho,[x,hk_2^{-1}]))$ is a (stack) groupoid

$$\mathcal{Z} := (K\times K)\backslash\mathcal{G} \rightrightarrows K\backslash\big(K_M \times \mathrm{Sh}(G,X)(\mathbb{C})\big),$$

and an appropriate C*-completion of the convolution algebra of $\mathcal{Z}$ is the Shimura variety generalization of the Bost-Connes-Marcolli (BCM) system [19]. We call this the *Shimura BCM system* (A,σ_t).

The systems of Bost-Connes and Connes-Marcolli are obtained from the Shimura BCM system by specializing to the cases $(G,X) = (\mathrm{GL}_1,\{\pm 1\})$ and $(G,X) = (\mathrm{GL}_2,\mathbb{H}^\pm)$, respectively. And as in these cases, the general Shimura BCM system enjoys many of the same statistical mechanical features: there is a naturally defined time evolution, a symmetry by endomorphisms, and a partition function. Likewise, the "invertible" points of $K_M \times \mathrm{Sh}(G,X)(\mathbb{C})$ yield extremal low temperature KMS states (under some convergence assumptions on the partition function): such states take the form

$$\phi_y(f) = \frac{\mathrm{Tr}(\pi_y(f)e^{-\beta H_y})}{Z(\beta)},$$

where y is an "invertible" point of $K_M \times \mathrm{Sh}(G,X)(\mathbb{C})$, π_y is a certain Hilbert space representation of A, H_y is the Hamiltonian in this representation, and Z is the partition function.

We found this construction in the process of formulating the Bost-Connes-Marcolli system in adèlic terms, the motivation for which was the problem of constructing a Bost-Connes-like system for an arbitrary number field F. In view of its possible relation to Hilbert's 12th problem, such a system ought to realize the Dedekind zeta function of F as partition function, and admit symmetry by the group $\pi_0(F^\times\backslash\mathbb{A}_F^\times)$. Open for ten years since the paper of Bost-Connes, this problem, which had only recently been solved in the case of imaginary quadratic F [14], is now solved by our generalization, by specializing the Shimura BCM to the case $(G,X) = (\mathrm{Res}_{F/\mathbb{Q}}(\mathbb{G}_{\mathrm{m},F}), \{\pm 1\}^{\mathrm{Hom}(F,\mathbb{R})})$ (together with a certain choice of monoidal augmentation and level structure data). Earlier partial solutions of this problem (by Paula Cohen [7], Harari and Leichtnam [22], and Laca and van Frankenhuijsen [28]) were hindered either by the absence of the full symmetry group or of the desired partition function, or by the restriction to fields of class number 1. Our solution to this problem is general, and free of such constraints, as far as having the correct partition function and the correct symmetries. There remains, however, the important but difficult problem of classifying KMS states and understanding their arithmetic properties.

An Outline of this Thesis

This thesis consists of four parts, aside from this introduction. It is an outgrowth of the joint article [19] with F. Paugam, and many of the points discussed there are explained more fully here.

In the first part, we supply the background necessary for the rest of the thesis. This background material includes various technical facts about operator algebras, Shimura varieties, and stacks. We have also included an miniature introduction to quantum statistical mechanics by way of (a very simple) example, and we've taken groupoids (which abound in this thesis) as an excuse to make a fun little excursion into noncommutative space.

The second and third parts are the main parts and form the bulk of this thesis. There we lay out our construction of the Shimura-variety generalization of the Bost-Connes and Connes-Marcolli systems, and study some examples. In particular, in the third part it is shown that our constructions unify all the Bost-Connes-like systems studied by Connes and his collaborators. Of particular note is the specialization to toroidal Shimura varieties which yields a Bost-Connes analogue for number fields having the sought-after symmetry and partition function. Part two begins with a rather long motivational section, with the intention that once the basic cases are well-understood, the passage to the general case will seem natural to the reader.

In the fourth and final part, we consider possible directions for future work, though no precise conjectures are made. This part is a bit of a wild romp. While it makes plain the gaps in this thesis, we hope that it also indicates the breadth of interaction between noncommutative geometry and arithmetic that is left to be explored.

Chapter 2

Background

In this chapter we review the core background in operator algebras, groupoids, and Shimura varieties that is necessary to understand the main constructions given in Chapter 3. Aside from some material on stacks (and in particular, on stack-groupoids), all of the material that we review here is well-known. But because there has traditionally been very little intersection between operator algebras and Shimura varieties, the number of people well-versed in both fields is likely rather small. It is our contention that these two fields can profitably interact. Therefore, we have chosen to cover some rather basic material to facilitate reading of the later parts, even when such material is already well-documented elsewhere.

We will also discuss the mathematical framework of quantum statistical mechanics, and the notion of a noncommutative space, as these will play an important role in motivating the construction and analysis of certain algebras that are the main objects of this thesis (see Chapter 3).

2.1 Operator Algebras and Quantum Statistical Mechanics

In this section we give a rapid and concise review of some basic definitions from the theory of C*-algebras, emphasizing those parts relevant to quantum statistical mechanics. Obviously, we have left out a lot. The reader desiring a thorough treatment will have to consult the literature; in particular, the two-volume set [4] and [5] of Bratteli and Robinson is ideal for our purposes. For an overview of the grand physical picture, see [20]. For a more elementary introduction to both the mathematical and physical aspects of quantum statistical mechanics, see [47].

C*-Algebras

Definition 2.1. A *C*-algebra* is a (not necessarily unital) complex algebra A endowed with a conjugate-linear involutive anti-automorphism ${}^*\colon A \to A$, and a norm $\|\cdot\|$, satisfying the following conditions.

1. A is complete with respect to the norm, and $\|ab\| \leq \|a\| \, \|b\|$ for all $a, b \in A$ (i.e., A is a *Banach algebra*); and

2. Every $a \in A$ satisfies the *C*-condition*: $\|aa^*\| = \|a\|^2$.

Actually a C*-algebra is not as abstract as it may seem from this definition, because every C*-algebra can be realized as a norm-closed sub-*-algebra of the algebra of bounded operators on a Hilbert space (Theorem of Gelfand-Naimark [4, Theorem 2.1.10]), and every such subalgebra is a C*-algebra.

The Mathematical Framework of Quantum Statistical Mechanics

The operator algebraic formulation of quantum statistical mechanics (see the introduction to [4]) consists of a C*-algebra A together with a 1-parameter group of automorphism $\sigma_t\colon A \to A$, which is continuous in the sense that $t \mapsto \sigma_t(a)$ is continuous for every $a \in A$. The algebra A is then the algebra of quantum observables, while σ_t is the time evolution. The pair (A, σ_t) is an example of a *C*-dynamical system.* The *states* of the C*-algebra A are the continuous complex-linear functionals Φ of norm 1 which are positive, i.e., $\Phi(a^*a) \geq 0$ for every $a \in A$. The number $\Phi(a)$ is then the *expectation value* of the observable a in the physical state Φ.

To regard the pair (A, σ_t) as a statistical mechanical system we need an appropriate notion of an "equilibrium state" at temperature $T = 1/\beta$. This is provided by the KMS condition.

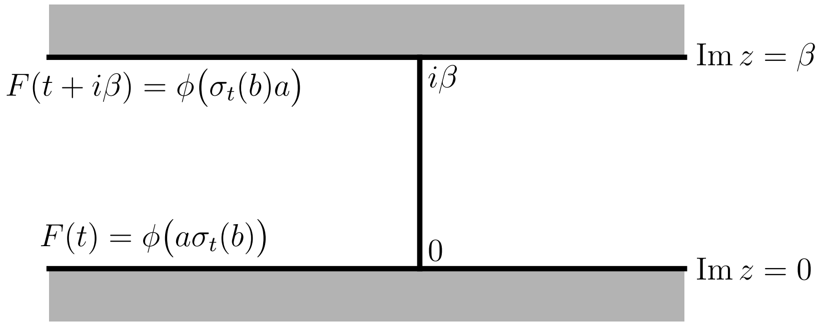

Figure 2.1: The KMS condition

Definition 2.2. Let $0 < \beta < \infty$ be a real number. A state ϕ of a C*-dynamical system (A, σ_t) is said to obey the *KMS$_\beta$-condition* if the following holds: For every pair of elements $a, b \in A$, there is a holomorphic function F on the open horizontal strip

$$\Omega = \{\, z \in \mathbb{C} \mid 0 < \operatorname{Im} z < \beta \,\}$$

that extends to a bounded continuous function on the closure of Ω, where it takes on the boundary values

$$F(t) = \phi\big(a\sigma_t(b)\big) \quad \text{and} \quad F(t+i\beta) = \phi\big(\sigma_t(b)a\big) \quad (t \in \mathbb{R}).$$

(See Figure 2.1.)

This definition of the KMS$_\beta$ condition also makes sense for the value $\beta = \infty$, where it becomes the condition that the function

$$F(t) = \phi\big(a\sigma_t(b)\big) \quad (t \in \mathbb{R})$$

extends to a holomorphic function on the upper half plane. But as noted in [11, §1.2], such a definition of a KMS$_\infty$ state has some undesirable properties. In particular, for the trivial time evolution, any state would then be a KMS$_\infty$ state, whereas weak limits of KMS$_\beta$ states are tracial. A better definition of KMS at $\beta = \infty$ is the following.

Definition 2.3. A *KMS$_\infty$ state* is a weak limit of KMS$_\beta$ states as $\beta \to \infty$.

Definition 2.2 is the usual formulation of the KMS condition that one often sees in the literature, although in practice it is easier to use the following equivalent characterization.

Proposition 2.1. *Let (A, σ_t) be a C*-dynamical system, and let Φ be a state of A.*

1. *([4], Corollary 2.5.23) There is a norm-dense *-subalgebra A^{an} of A such that for every $a \in A^{\mathrm{an}}$, the function $t \mapsto \sigma_t(a)$ can be analytically continued to an entire function.*

2. *([5], Definition 5.3.1 and Corollary 5.3.7) The state Φ is a KMS-β state if and only if*

$$\Phi\big(a\sigma_{i\beta}(b)\big) = \Phi(ba)$$

*for all a, b in a norm-dense σ_t-invariant *-subalgebra of A^{an}.*

Structure of the Set of KMS States

We now proceed to a description of the structure of the set of KMS$_\beta$ states. But before doing so, we need to explain the *GNS construction*, which is a method of getting representations of a C*-algebra from its states; it is fundamental in the theory of operator algebras. We also need to define the notion of a factor state.

Notation 2.1. Given a Hilbert space $\mathcal{H}$, we denote the C*-algebra of all bounded operators on $\mathcal{H}$ by $B(\mathcal{H})$, and the inner product on $\mathcal{H}$ by $\langle \cdot, \cdot \rangle$.

Proposition 2.2. *Let Φ be a state of a C*-algebra A. Then there is a triple*

$$(\mathcal{H}_\Phi, \pi_\Phi, \xi_\Phi)$$

consisting of a representation π_Φ of A on a Hilbert space $\mathcal{H}_\Phi$ and a unit vector $\xi_\Phi \in \mathcal{H}_\Phi$ such that:

1. *$\Phi(a) = \langle \pi_\Phi(a)\xi_\Phi, \xi_\Phi \rangle$ for all $a \in A$; and*

2. *The orbit $\pi_\Phi(A)\xi_\Phi$ is norm-dense in $B(\mathcal{H}_\Phi)$.*

The triple $(\mathcal{H}_\Phi, \pi_\Phi, \xi_\Phi)$ is unique up to unitary equivalence.

Briefly, the GNS representation is the representation of A on itself by right multiplication; a pre-Hilbert space structure is given by the inner product $\langle a, b\rangle = \phi(b^*a)$, and to get a Hilbert space representation one passes to the completion of the corresponding norm (on the quotient by the $\langle\cdot,\cdot\rangle$-kernel).

The states of particular relevance to the KMS theory are the *factor states.* These are the states Φ for which the corresponding GNS representation π_Φ generates a *factor*, which is to say that the weak closure of $\pi_\Phi(A)$ in $B(\mathcal{H}_\Phi)$ has centre consisting of the scalar operators. This weak closure is an example of a *von Neumann algebra*, i.e., a strongly closed unital *-subalgebra of some $B(\mathcal{H})$.

We can now state the main structure theorem for the set of KMS$_\beta$ states.

Proposition 2.3 (Structure of KMS states; [5], Theorem 5.3.30)**.** *The set $\mathcal{E}_\beta$ of KMS$_\beta$ states is a convex, weak*-compact simplex. The extreme points of $\mathcal{E}_\beta$ are precisely those KMS$_\beta$ states that are factor states.*

The KMS Condition and the Notion of Equilibrium

The following example of a (highly simplified) finite quantum system lends credibility to the claim that the KMS condition characterizes the notion of equilibrium.

We take as our algebra A of observables the algebra of n-by-n complex matrices, i.e., the (bounded) operators on the Hilbert space $\mathfrak{H} = \mathbb{C}^n$. Then every state of A is of the form

$$\phi(a) = \mathrm{Tr}(\rho a), \quad a \in A,$$

for some positive operator ρ with trace $\mathrm{Tr}(\rho) = 1$; the operator ρ is called the *density operator* of ϕ. Physically, for a (self-adjoint) observable operator $a \in A$, the real number $\phi(a)$ is interpreted as the expectation value of the observable in the state ϕ. The time evolution of A is determined by a self-adjoint matrix H (the *Hamiltonian operator*), in the following manner:

$$a \mapsto \sigma_t(a) = e^{itH} a e^{-itH}, \quad a \in A, \ t \in \mathbb{R}.$$

(In fact, every 1-parameter family of *-automorphisms of A is of this form.) Notice that σ_t can be defined for complex values of t.

Now let ϕ be a state with density operator ρ. On physical grounds, if ϕ is an equilibrium state, then it is expected to be the most random state of average energy $\phi(H)$. More precisely, the randomness of ϕ is measured by its *entropy*

$$S(\phi) = -\operatorname{Tr}(\rho \log \rho),$$

and an equilibrium state is expected to maximize the scalar $S(\phi) - \beta\phi(H)$. One has the inequality

$$\log Z(\beta) \geq S(\phi) - \beta\phi(H),$$

where

$$Z(\beta) = \operatorname{Tr}(e^{-\beta H})$$

is the *partition function*, and the solution of the variational problem is the familiar *Gibbs state*:

$$\phi_{\text{Gibbs}}(a) = \frac{\operatorname{Tr}(e^{-\beta H} a)}{Z(\beta)}.$$

Thus, having established the Gibbs state as the equilibrium state of our system (A, σ_t), we now show that it is also the *unique* KMS_β state. Indeed, from the invariance of the trace under cyclic permutation it follows easily that the Gibbs state satisfies the KMS_β condition. Conversely, suppose ϕ is a KMS_β state with density operator ρ. The KMS_β condition says that

$$\operatorname{Tr}(\rho ba) = \phi(ba) = \phi\big(a\sigma_{i\beta}(b)\big) = \operatorname{Tr}(\rho a e^{-\beta H} b e^{\beta H}), \quad \text{for all } a, b \in A.$$

So using the invariance of the trace, we get

$$\operatorname{Tr}\big((a\rho - e^{\beta H}\rho a e^{-\beta H})b\big) = 0, \quad \text{for all } a, b \in A,$$

which clearly implies that

$$a\rho e^{\beta H} = e^{\beta H}\rho a, \quad \text{for all } a \in A. \tag{2.1}$$

But by a similar argument, the KMS condition also implies that any operator that commutes with H (and hence is σ_z-invariant) also commutes with the density operator ρ. Hence from Eq. (2.1) we see that $\rho e^{\beta H}$ commutes with all of A, which of course means that $\rho e^{\beta H}$ is a scalar operator. This scalar must be $1/\operatorname{Tr}(e^{-\beta H})$ in order to have $\operatorname{Tr}(\rho) = 1$. This proves that a KMS_β state of our finite system is necessarily the Gibbs state.

Summarizing what we have just shown, we have:

Proposition 2.4. *Let σ_t be any one-parameter group of automorphisms of the finite-dimensional C^*-algebra $A = M_n(\mathbb{C})$, and let H be the infinitesimal generator of σ_t. Then for every $\beta > 0$, the Gibbs state*

$$\phi_{\beta,H}(a) = \frac{\mathrm{Tr}(e^{-\beta H}a)}{\mathrm{Tr}(e^{-\beta H})}, \quad a \in A \tag{2.2}$$

is the unique KMS$_\beta$ state of the C^*-dynamical system (A, σ_t).

An analysis of more realistic, and accordingly, more complicated, statistical mechanical models (such as quantum lattice gases) lends further weight to the interpretation of the KMS condition as a characterization of equilibrium: See [23] or [46].

2.2 Groupoids and Noncommutative Spaces

Our introduction to groupoids will be through noncommutative geometry; or to noncommutative geometry through groupoids — the distinction is purposely somewhat unclear, as the two are rather tightly tied to one another (at least in Connes' approach). The notion of a noncommutative space will play an important role in motivating the constructions in Chapter 3.

Lots of technical details about groupoids and stack-groupoids will come in the next section, but here we will be a bit fuzzy in order to grasp the basic ideas more quickly.

2.2.1 The Very Simplest Example: The Fuzzy Point

Indeed, fuzzy is one of the key idea. Even without explaining what a noncommutative space is exactly,[1] we can still explain the basic idea by starting with a typical *feature* (or perhaps more aptly, non-feature) that noncommutative spaces share: They have no meaningful points, or at least they are not defined in terms of their points; or talking about points doesn't get one very far (e.g.,

[1] In fact, there's no consensus on what it should be *precisely* — being extremely varied, it has thus far defied axiomatization. In any case, the more important goal is rather to know when to *recognize* a noncommutative space, and when faced with one, to have effective tools at one's disposal.

orbit spaces with dense orbits), etc. Let's see what this means in a setting that might not even seem reasonable according to this creed.

Take the simplest possible space, "the" space with one point. Call it X_1. But now imagine, if a space has no well-defined notion of point, then we ought to be able to "resolve" X_1, to two points, say. Then we get a space X_2 consisting of two points, but the points are now identified. Of course we can continue "resolving" our one point space to X_3, X_4, and so on. X_n will consist of n points, but again, they all have to be identified with each other. And all the X_n's should be regarded as "equivalent".

2.2.2 Groupoids

The gadget that handles this fuzzy space problem in a precise manner is what is known as a *groupoid*. Basically, the idea behind groupoids is that they encode *relations* in a space, and not just the points of a space.

A groupoid is just a small category in which every morphism is invertible. Spelling out what this means puts the definition into the following form (as given in [9]).

Definition 2.4. A *groupoid* is a set G, together with a distinguished subset $G^{(0)} \subset G$, and two maps $s, t \colon G \rightrightarrows G^{(0)}$, and an associative *composition*

$$G^{(2)} := \{\, (g_1, g_2) \in G \times G \mid s(g_1) = t(g_2) \,\} \longrightarrow G, \quad (g_1, g_2) \longmapsto g_1 g_2$$

subject to the conditions:

1. $t(g_1 g_2) = t(g_1)$ and $s(g_1 g_2) = s(g_2)$;
2. $s(g) = t(g) = g$ for all $g \in G^{(0)}$;
3. $gs(g) = g = t(g)g$ for all $g \in G$;
4. Every g has a two-sided inverse g^{-1} satisfying $g^{-1}g = s(g)$ and $gg^{-1} = t(g)$.

Example 2.1. Groups are groupoids in which the unit space consists of one point (the identity element).

Example 2.2. Any equivalence relation on a set X can be regarded as a groupoid: regard the equivalence relation as a subset of $X \times X$, and take the law of transitivity as the composition.

Thus groupoids can be thought of as generalized equivalence relations. It is often useful to think of a groupoid as defining orbits on its space of units.

Example 2.3. If a group G acts on a set X, then one can define a groupoid

$$G \ltimes X,$$

the so-called *cross-product groupoid*, which as a set is just $G \times X$. Its unit space is X itself (as the subset $\{1\} \times X$). The source and target maps are $s(g,x) = x$, $t(g,x) = g \cdot x$. Inversion is given by $(g,x) = (g^{-1}, gx)$. Composition is given by $(g,x)(h,y) = (gh,y)$, which is only defined when $x = hy$.

If G on acts *partially* on X, i.e., if for any given $g \in G$, the action "$g \cdot x$" is only defined for certain $x \in X$, then one can still define the cross-product. It has exactly the same structure as above, but of course, now only the pairs (g,x) for which $g \cdot x$ is defined are allowed.

Partially defined cross-products arise when, for example, G acts on a space X' containing X, but X itself is not stable under the G-action. Then

$$G \ltimes X = \{\, (g,x) \mid g \cdot x \in X \,\}\,.$$

The main groupoids in this thesis are all cross-product groupoids. Also important are quotients of cross-products, which may, however, fail to be groupoids themselves.

The groupoid G_n of our fuzzy space X_n can be regarded as the complete graph on n vertices in which every edge is replaced by arrows in both directions. The description in set-theoretic terms is fairly obvious: G_n corresponds to the set of pairs (i,j) with $i,j \in \{\,1,2,\ldots,n\,\}$, with source and target maps $s(i,j) = j$ and $t(i,j) = i$. The units correspond to the "diagonal" pairs (i,i). The inverse of (i,j) is (j,i). The composition $(i,k)(k,j)$ is (i,j).

2.2.3 Groupoid C*-Algebras

Now that we have the groupoid G_n encoding the equivalences between the points of X_n, we can give Connes' description of the noncommutative "point" as a description of X_n (together with the relations between its points!) in terms of its corresponding function algebra. The motivation for taking this point-of-view comes from the following conceptually important theorem of Gelfand-Naimark, which characterizes locally compact spaces in terms of their function algebras.

Theorem 2.1 (Gelfand-Naimark, [9], §II.1)**.** *Let $\mathcal{S}$ be the category of locally compact Hausdorff spaces, and let $\mathcal{A}$ be the category of (not necessarily unital) commutative C*-algebras. Then the functor*

$$\mathcal{S} \longrightarrow \mathcal{A}, \quad X \mapsto C_0(X)$$

is an (anti-)equivalence of categories. Here $C_0(X)$ is algebra of complex-valued continuous functions of X vanishing at infinity. The inverse functor sends the commutative C-algebra A to its spectrum (the space of characters).*

According to Connes' conception of noncommutative geometry, one can get a finer description of the orbit space by an equivalence relation by considering instead the algebra of functions on the *graph* of the equivalence relation. Likewise, the algebra of functions on a groupoid is, in the spirit of Gelfand-Naimark, the algebraic description of a space in which the relations are described by the groupoid. Of course, the resulting function algebra should remain true to the structure of the groupoid. Thus, we are lead to consider *groupoid C*-algebras.* We describe what this is for our fuzzy space groupoid G_n.

For G_n, the groupoid C*-algebra, call it A_n, consists of the complex-valued functions on G_n, and the multiplication of functions is determined by the groupoid structure, i.e., the product of two functions is their convolution. Precisely, if $f_1, f_2 \in A_n$, then

$$f_1 * f_2(i,j) = \sum_k f_1(i,k) f_2(k,j).$$

But this is simply matrix multiplication. Therefore, we find that

$$A_n = M_n(\mathbb{C}),$$

the algebra of n-by-n matrices with entries in $\mathbb{C}$. The groupoid inversion together with complex-conjugation gives the involution $f^*(i,j) = \overline{f(j,i)}$ in A_n, thereby giving A_n a C*-structure. It obviously coincides with the adjoint operation for matrices.

Rather than going into the general theory of groupoid C*-algebras (for which the canonical reference is [42]), we will define C*-algebras of groupoids later, as the need arises, and only for the particular groupoids of interest (see Section 3.3.2, for example). The point of the above discussion was to explain the geometric ideas with as little clutter as possible.

2.2.4 Morita Equivalence

Finally, we see how to regard all our fuzzy spaces X_1, X_2, etc., as equivalent. The matrix algebras $\mathbb{C} = M_1(\mathbb{C})$, $M_2(\mathbb{C})$, etc., though clearly not isomorphic, are *Morita equivalent*, i.e., they all have equivalent categories of (left) modules (checking this is a pleasant exercise). In noncommutative geometry Morita equivalence is taken as the right notion of equivalence of noncommutative spaces described by noncommutative algebras; in the C*-context, the more appropriate notion is that of Rieffel's *strong Morita equivalence*; we refer the reader to [9, Appendix A of Chapter 2].

Remark 2.1. We worked with the fuzzy one-pointed space as our basic example for reasons of simplicity, but of course it is too simple to illustrate the truly useful aspect of groupoids, namely: Groupoids keep track of points with internal structure, i.e., points with automorphisms. This is a very familiar situation in moduli problems in algebraic geometry where moduli spaces often cannot be represented by schemes because of the presence of non-trivial automorphisms which get wiped out in taking coarse quotients; the solution is the notion of a *stack*, which in the first approximation comes about by replacing the usual functor of points with a functor from (Schemes) (or better, (S-Schemes)) taking values in the category of groupoids (thus, a certain category of categories). To pass from this first approximation to the full notion of a stack, one needs also to consider "descent conditions" (for example, see [18]). In the next Section we will deal with some of these matters in more detail.

Now as for our fuzzy point, a more interesting version would be had by replacing the trivial groupoid G_1 with a non-trivial group H, which corresponds to a point with automorphism group H. Then the Morita equivalence class of our point-with-automorphisms would be the Morita equivalence class of the group C*-algebra $C^*(H)$, which in general is not Morita equivalent to a commutative C*-algebra.

2.3 Some Technical Aspects of Stack-Groupoids

Having now gone through a leisurely introduction to groupoids and groupoid-algebras, we come now to the discussion of the more technical aspects of groupoids and topological stack-groupoids; in fact, our main goal in this section is to set the framework for formulating the proper definitions. In this thesis, stack-groupoids will arise as quotients of groupoids by non-free non-transitive group actions (see Section 3.3.2). The purpose of this section is not to take a long excursion into stack theory, but rather to give some indication of the true nature of the spaces we will encounter in Chapter 3, and to give the technical detail required to make their description rigorous. The exposition that follows is streamlined for reasons of space-time constraints, so the technical prerequisites are greater. If this background section seems more pedantic than the others, that's purposely so, because a rigorous definition of a stack-groupoid (as opposed to just a stack) is difficult to find the literature. To some extent, the main difficulty in understanding stack-groupoids is wrapping one's head around the notion of a higher category.

General references for this section are: [29] (the bible of algebraic stacks), [18], and [40] (topological stacks). The forthcoming book by Behrend et al. is recommendable (though, unfortunately, only small parts of it are currently available). Relevant information on 2-categories can be found in [25], [51], and [49].

2.3.1 Topological Stacks and Stack Groupoids

Roughly speaking, a topological stack is a stack on the site Top of topological spaces with open coverings, i.e., a category fibered in groupoids fulfilling some descent condition (see [29, Définition 3.1]):

- Isomorphisms between two objects form a sheaf;
- Every descent condition with respect to an open covering is effective.

A *stack-groupoid* is a groupoid in the category of topological stacks, i.e., the datum of a tuple

$$(\mathfrak{X}_1, \mathfrak{X}_0, s, t, \epsilon, m) \tag{2.3}$$

composed of two stacks $\mathfrak{X}_1$ and $\mathfrak{X}_0$, equipped with source and target 1-morphisms

$$s\colon \mathfrak{X}_1 \to \mathfrak{X}_0, \quad t\colon \mathfrak{X}_1 \to \mathfrak{X}_0$$

a unit $\epsilon\colon \mathfrak{X}_0 \to \mathfrak{X}_1$, and a composition $m\colon \mathfrak{X}_1 \underset{s,\mathfrak{X}_0,t}{\times} \mathfrak{X}_1 \to \mathfrak{X}_1$:

$$\begin{tikzcd} \mathfrak{X}_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & \mathfrak{X}_0 \arrow[l, bend right, "\epsilon"'] \end{tikzcd}, \qquad \mathfrak{X}_1 \underset{s,\mathfrak{X}_0,t}{\times} \mathfrak{X}_1 \xrightarrow{m} \mathfrak{X}_1.$$

The 1-morphism

$$(\mathrm{Id}_{\mathfrak{X}_1} \times m)\colon \mathfrak{X}_1 \underset{\mathfrak{X}_0}{\times} \mathfrak{X}_1 \to \mathfrak{X}_1 \underset{\mathfrak{X}_0}{\times} \mathfrak{X}_1,$$

that should be thought of as sending a pair (a, b) of composable morphisms to the pair (a, ab), is supposed to be an equivalence (which implies the existence of an inverse for the composition law). This tuple should be equipped with the additional data of an associator

$$\Phi\colon m \circ (m \times \mathrm{Id}_{\mathfrak{X}_1}) \overset{\sim}{\Longrightarrow} m \circ (\mathrm{Id}_{\mathfrak{X}_1} \times m),$$

and two unity constraints

$$U\colon m \circ (\mathrm{Id}_{\mathfrak{X}_1} \times \epsilon) \overset{\sim}{\Longrightarrow} \mathrm{Id}_{\mathfrak{X}_1} \quad \text{and} \quad V\colon m \circ (\epsilon \times \mathrm{Id}_{\mathfrak{X}_1}) \overset{\sim}{\Longrightarrow} \mathrm{Id}_{\mathfrak{X}_1},$$

fulfilling some higher coherence (or cocycle) conditions: pentagon, etc. We do not write these down, due to the typographical challenges involved, but also because we prefer to use Toen's viewpoint of Segal groupoid stacks, which allows one to forget these conditions by including them in the choice of inverses for some equivalences in a simplicial diagram.

In view of the moduli-theoretic origins of the notion of a stack-groupoid, there is, of course, a corresponding notion of a coarse quotient. For convenience of future reference, we record the definition here.

Definition 2.5. Let $(\mathfrak{X}_1, \mathfrak{X}_0, s, t, \epsilon, m)$ be a tuple as before. Its *coarse quotient* is by definition the quotient of the coarse moduli space $|\mathfrak{X}_0|$ (space of isomorphism classes of objects in $\mathfrak{X}_0$) by the equivalence relation generated by

$$x_0 \sim x_0' \Leftrightarrow \exists x_1 \in |\mathfrak{X}_1| \text{ such that } s(x_1) = x_0 \text{ and } t(x_1) = x_0'.$$

2.3.2 Toen's Approach via Segal Stack-Groupoids

Toen has proposed a very concise and elegant definition of the 1-category of stack-groupoids. It is based on the simplicial point of view of 2-categories as explained in Tamsamani's thesis [51] and Simpson [49]. Analogous constructions can also be found in [53], 1.3.4, and [30].

Segal Stack-Groupoids

Notation 2.2. Let Δ be the category whose objects are totally ordered sets $[n] = \{0, \ldots, n\}$ and whose morphisms are increasing maps.

The category of topological stacks can be viewed as a 1-category (Stacks) together with a notion of equivalences.

Definition 2.6. A *Segal stack category* is a simplicial stack $\mathfrak{X}_*\colon \Delta^{op} \to (\mathsf{Stacks})$ such that the *Segal morphisms*

$$\mathfrak{X}_n \to \mathfrak{X}_1 \underset{\mathfrak{X}_0}{\times} \cdots \underset{\mathfrak{X}_0}{\times} \mathfrak{X}_1$$

(given by the n morphisms in Δ, $[1] \to [n]$ that send 0 to i, and 1 to $i+1$) are stack equivalences. The *right multiplication morphism*

$$\mathfrak{X}_2 \to \mathfrak{X}_1 \underset{\mathfrak{X}_0}{\times} \mathfrak{X}_1 \tag{2.4}$$

is given by the two morphisms in Δ

1. $[1] \to [2]$ such that $0 \mapsto 0$, $1 \mapsto 1$, and;

2. $[1] \to [2]$ such that $0 \mapsto 0$, $1 \mapsto 2$.

A Segal stack category is a *Segal stack groupoid* if the right multiplication morphism is a stack equivalence.

Remark 2.2. To help understand the content of this definition, notice that if we were to replace the category (Stacks) by the category of sets, then we would recover the usual notions of category and groupoid.

Recovering the Stack-Groupoid

Let us now show how to get the stack-groupoid 6-tuple (2.3) starting from a Segal stack-groupoid.

First, notice that we can think of the stacks $\mathfrak{X}_n$ as families of n composable morphisms, so that the groupoid condition (the equivalence requirement for the right multiplication map (2.4)) is the condition that the map $(a, b) \mapsto (a, a \circ b)$ is an equivalence, which implies that each a is an isomorphism.

The stack-groupoid data is extracted as follows.

1. The *source* and *target* maps $s, t\colon \mathfrak{X}_1 \to \mathfrak{X}_0$ are induced by the morphisms $s = [0] \to [1]\colon 0 \mapsto 0$ and $t = [0] \to [1]\colon 0 \mapsto 1$.
2. The choice of an inverse ϕ for the Segal morphism $\mathfrak{X}_2 \to \mathfrak{X}_1 \underset{\mathfrak{X}_0}{\times} \mathfrak{X}_1$ allows one to define a *composition* $\mu\colon \mathfrak{X}_1 \underset{s,\mathfrak{X}_0,t}{\times} \mathfrak{X}_1 \to \mathfrak{X}_2 \to \mathfrak{X}_1$ given by composing ϕ with the morphism induced by $[1] \to [2]\colon 0 \mapsto 0, 1 \mapsto 2$.
3. The increasing map $[1] \to [0]\colon 0 \mapsto 0, 1 \mapsto 0$ induces a map $\epsilon\colon \mathfrak{X}_0 \to \mathfrak{X}_1$ called the *unit map*.
4. Choose an inverse ψ to the right multiplication morphism
$$\mathfrak{X}_2 \to \mathfrak{X}_1 \underset{\mathfrak{X}_0}{\times} \mathfrak{X}_1,$$
and compose it with
$$\mathrm{Id}_{\mathfrak{X}_1} \times_{\mathfrak{X}_0} \epsilon\colon \mathfrak{X}_1 \to \mathfrak{X}_1 \times_{\mathfrak{X}_0} \mathfrak{X}_1$$
and
$$d_2\colon \mathfrak{X}_2 \to \mathfrak{X}_1,$$

where d_2 is induced by the map $[1] \to [2] \colon 0 \mapsto 1, 1 \mapsto 2$. Morally, these successive maps send an arrow $a \in \mathfrak{X}_1$ to the pair $(a, 1)$, then to (a, a^{-1}), and finally to a^{-1}. Let $i \colon \mathfrak{X}_1 \to \mathfrak{X}_1$ be this composition.

Thus, up to the two additional choices of ϕ and ψ, we have obtained a tuple $(\mathfrak{X}_1, \mathfrak{X}_0, s, t, \epsilon, i, m)$ giving a diagram

$$\begin{tikzcd} \mathfrak{X}_1 \arrow[loop left, "i"] \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & \mathfrak{X}_0 \arrow[l, bend right, "\epsilon"'] \end{tikzcd}$$

and a multiplication

$$m \colon \mathfrak{X}_1 \underset{s, \mathfrak{X}_0, t}{\times} \mathfrak{X}_1 \longrightarrow \mathfrak{X}_1,$$

which is the basic datum (2.3) necessary to define the notion of stack-groupoid. The problem is now to define an associativity 2-isomorphism and some other 2-conditions, and to find the right notion of coherence conditions for them. The beauty of Toen's construction is that these coherence conditions are already encoded in the simplicial structure. Let us be more explicit.

First, since we work in a 2-category, the inverse of a 1-isomorphism is supposed to be defined up to a *unique* 2-isomorphism. This implies that the multiplication

$$m \colon \mathfrak{X}_1 \underset{\mathfrak{X}_0}{\times} \mathfrak{X}_1 \to \mathfrak{X}_1$$

is well-defined up to a unique 2-isomorphism.

To define the *associator*, we use the following strictly commutative diagrams (products are taken over $\mathfrak{X}_0$)

$$\begin{tikzcd} \mathfrak{X}_3 \arrow[r] \arrow[d] & \mathfrak{X}_2 \arrow[r] \arrow[d] & \mathfrak{X}_1 \\ \mathfrak{X}_2 \times \mathfrak{X}_1 \arrow[r] \arrow[d] & \mathfrak{X}_1 \times \mathfrak{X}_1 & \\ (\mathfrak{X}_1 \times \mathfrak{X}_1) \times \mathfrak{X}_1 & & \end{tikzcd}$$

and

$$\begin{tikzcd}
\mathfrak{X}_3 \arrow[r] \arrow[d] & \mathfrak{X}_2 \arrow[r] \arrow[d] & \mathfrak{X}_1 \\
\mathfrak{X}_1 \times \mathfrak{X}_2 \arrow[r] \arrow[d] & \mathfrak{X}_1 \times \mathfrak{X}_1 & \\
\mathfrak{X}_1 \times (\mathfrak{X}_1 \times \mathfrak{X}_1) & &
\end{tikzcd}$$

whose vertical arrows are equivalences.

The uniqueness of inverses of equivalences up to unique 2-isomorphisms gives natural 2-isomorphisms between the multiplication maps

$$(\mathfrak{X}_1 \times \mathfrak{X}_1) \times \mathfrak{X}_1 \to \mathfrak{X}_1$$

and

$$\mathfrak{X}_1 \times (\mathfrak{X}_1 \times \mathfrak{X}_1) \to \mathfrak{X}_1$$

and the morphism obtained by choosing an inverse of the equivalence

$$\mathfrak{X}_3 \to \mathfrak{X}_1 \times \mathfrak{X}_1 \times \mathfrak{X}_1.$$

This gives the associator 2-isomorphism. To check that the associator fulfills the desired 2-cocycle condition (the so-called *pentagon*), it is necessary to use the simplicial diagram up to $\mathfrak{X}_4$. An explanation of the argument is given in [30].

2.4 Shimura Varieties

Every mathematician already has an idea of what a Shimura variety is: as complex homogeneous varieties, they are just higher dimensional versions of the quotient of the upper half plane by the usual action of $\mathrm{SL}_2(\mathbb{Z})$. Such varieties were studied extensively by Shimura in the 60's and 70's, but it was Deligne who finally codified the definition in terms of Hodge structures and algebraic groups [15]. Briefly, Deligne's strategy was to interpret the varieties of Shimura as parameter spaces for Hodge structures, and then to characterize various features of geometric variation of Hodge structures (e.g., Griffiths transversality) in algebraic-group terms. The resulting set of properties is then taken as the axioms defining Shimura's varieties and generalizations thereof.

In this section we give the relevant definitions — note that our definition of a Shimura variety is slightly different from the standard one — and make a few remarks on the stack quotients that arise for finite level Shimura varieties. But otherwise, our purpose in this section is quite modest. In particular, we don't discuss here any of the deeper aspects, such as the theory of canonical models, the menagerie of compactifications, cohomology, and zeta functions! The reader is referred to the literature (or local expert) to get the real story.

Algebraic Group Preliminaries

For background on algebraic groups we refer the reader to [2] or [50]. We will mostly be concerned here with establishing notation.

Notation 2.3. Let G be a reductive group over $\mathbb{Q}$. If A is a $\mathbb{Q}$-algebra, then the group of A-points of G is denoted $G(A)$. We let $G(\mathbb{R})^+$ denote the connected component of the identity in the real Lie group $G(\mathbb{R})$, and we write $G(\mathbb{Q})^+$ for $G(\mathbb{Q}) \cap G(\mathbb{R})^+$.

If F is an extension of $\mathbb{Q}$, then we write G_F for the base change $G \times_{\mathbb{Q}} F$.

The multiplicative group is denoted $\mathbb{G}_{\mathrm{m}}$.

If we view an algebraic group G over a field k as a functor

$$(k\text{--}\mathsf{Algebras}) \longrightarrow (\mathsf{Groups}), \quad A \longmapsto G(A),$$

then we have an easy description of the *Weil restriction*, which yields an algebraic group over k from an algebraic group over an extension K/k via restriction of scalars. If G is an algebraic group over K, we write $\mathrm{Res}_{K/k}(G)$ for the Weil restriction. Thus, if we write $G' = \mathrm{Res}_{K/k}(G)$, then for an k-algebra A we have $G'(A) = G(A \otimes_k K)$.

An important example of this sort of construction is the *Deligne torus*, the $\mathbb{R}$-algebraic group

$$\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\, \mathbb{G}_{\mathrm{m},\mathbb{C}}.$$

We note that $\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$ and $\mathbb{S}(\mathbb{C}) \cong \mathbb{C}^\times \times \mathbb{C}^\times$.

Shimura Data and Shimura Varieties

Now we recall the definition of Shimura data. The definition that we use in this thesis sits between Deligne's definition [16, 2.1] and Pink's generalization of it [41, 2.1]. In particular, like Deligne, we only consider reductive groups, but following Pink we do allow for finite covers of the Hermitian symmetric domains that form a basic ingredient of the Shimura datum. This extra flexibility will be particularly important when we study the Bost-Connes system and its analogues, in Section 4.2.

Definition 2.7. A *Shimura datum* is a triple (G, X, h), where G is a connected reductive group over $\mathbb{Q}$, X is a left homogeneous space under $G(\mathbb{R})$, and $h\colon X \to \mathrm{Hom}(\mathbb{S}, G_{\mathbb{R}})$ is a $G(\mathbb{R})$-equivariant map[2] with finite fibres, and these are required to satisfy:

1. For $h_x \in h(X)$, (the Hodge structure) $\mathrm{Lie}(G_{\mathbb{R}})$ is of type $\{(-1, 1), (0, 0), (1, -1)\}$;
2. The involution $\mathrm{int}\, h_x(i)$ is a Cartan involution of the adjoint group $G^{\mathrm{ad}}_{\mathbb{R}}$;
3. The adjoint group has no factor G' defined over $\mathbb{Q}$ on which the projection of h_x is trivial.

A Shimura datum is said to be *classical* if it moreover fulfills the following additional axiom:

4. Let $Z_0(G)$ be the maximal split subtorus of the center of G; then $\mathrm{int}\, h_x(i)$ is a Cartan involution of $G/Z_0(G)$.

Note that in Deligne's definition [16, 2.1], X is a conjugacy class of a homomorphism $\mathbb{S} \to G_{\mathbb{R}}$. The above definition allows for equivariant finite covers of such X.

Remark 2.3. We will often denote a Shimura data just by a couple (G, X) when the morphism h is clear from the situation.

Definition 2.8. Let (G, X) be a Shimura datum, and let $K \subset G(\mathbb{A}_\mathrm{f})$ be a compact open subgroup. The *level K Shimura variety* is

$$\mathrm{Sh}_K(G, X) := G(\mathbb{Q})\backslash\big(X \times G(\mathbb{A}_\mathrm{f})/K\big).$$

[2]for the natural conjugation action of $G(\mathbb{R})$ on $\mathrm{Hom}(\mathbb{S}, G_{\mathbb{R}})$

A few remarks regarding the nature of these varieties are in order (cf. [41, §3.2]). First of all, there are only finitely many connected components (this is a nontrivial fact). If K is sufficiently small (for example, if K is a *neat* subgroup), then $\mathrm{Sh}_K(G,X)$ is a bona fide complex-analytic variety [41, §3.3(b)]. More precisely, there is decomposition

$$\mathrm{Sh}_K(G,X) = \coprod_g \Gamma_g \backslash X^+,$$

where g runs over a finite set in $G(\mathbb{A}_\mathrm{f})$, X^+ is a connected component of X (a Hermitian symmetric space), and Γ_g is the congruence subgroup $G(\mathbb{Q}) \cap gKg^{-1}$.

The family of varieties $\left\{\mathrm{Sh}_K(G,X)\right\}_K$ forms a projective system, for given $K_1 \subset K_2$, compact open subgroups of $G(\mathbb{A})$, there is a natural morphism $\mathrm{Sh}_{K_1}(G,X) \to \mathrm{Sh}_{K_2}(G,X)$.

Definition 2.9. The *Shimura variety* for a Shimura datum (G,X) is the projective limit

$$\mathrm{Sh}(G,X) := \varprojlim_K \mathrm{Sh}_K(G,X).$$

The level K Shimura variety can be recovered from the projective limit by quotienting: we have

$$\mathrm{Sh}_K(G,X) = \mathrm{Sh}(G,X)/K \tag{2.5}$$

(see [16, 2.7.1]).

The projective limit is a well-defined scheme over $\mathbb{C}$, and though it is noetherian and regular (cf. [35]), it is not of finite type. We shall however be primarily concerned with the Shimura variety as a topological space; this aspect (in relation to the cohomology of arithmetic groups) has been investigated by J. Rohlfs [44].

Remark 2.4. The Shimura varieties as we've just defined them are varieties over $\mathbb{C}$, but in fact they have models over number fields, which accounts for their arithmetical significance. This is the deep theory of *canonical models*, which we do not touch upon, however see [15], [16], [34], [38].

Hecke Correspondences on the Projective Limit

One important feature of the limit Shimura variety is that it admits an action of $G(\mathbb{A}_{\mathrm{f}})$ (a very big group!): if $g \in G(\mathbb{A}_{\mathrm{f}})$, and K is a compact open subgroup of $G(\mathbb{A}_{\mathrm{f}})$, then we have a map

$$\rho_K(g)\colon \operatorname{Sh}_K(G,X) \longrightarrow \operatorname{Sh}_{g^{-1}Kg}(G,X), \quad [x,h] \mapsto [x,hg]$$

(where we have written $[x,h]$ for the class of $(x,h) \in X \times G(\mathbb{A}_{\mathrm{f}})$ in $\operatorname{Sh}_K(G,X)$). Since the maps ρ_K are compatible with the projective system $\{\operatorname{Sh}_K(G,X)\}_K$, they induce an action on the limit $\operatorname{Sh}_K(G,X)$. These are the so-called *Hecke correspondences* (indeed, they are Hecke correspondences in the modular curve case, i.e., the Shimura variety for $(\mathrm{GL}_{2,\mathbb{Q}}, \mathbb{H})$).

One implication of the fact that $G(\mathbb{A}_{\mathrm{f}})$ acts on the projective limit, but not on individual finite level pieces, is that the limit Shimura variety is better suited for the Shimura-variety generalizations of the Bost-Connes and Connes-Marcolli systems, cf. Section 3.3.2.

Some Examples

For the reader's convenience, we list a few of the Shimura varieties that appear later. (Ours is a very incomplete list!)

Example 2.4. Let T be a torus over $\mathbb{Q}$. Because T is abelian, any $T(\mathbb{R})$-conjugacy class X of homomorphisms $\mathbb{S} \to T_{\mathbb{R}}$ is just a point. Then (T,X) is a Shimura datum in Deligne's sense (X being a point means there are no conditions on T), and therefore in our sense, too. The associated Shimura variety

$$\operatorname{Sh}_K(T,X) = T(\mathbb{Q})\backslash T(\mathbb{A}_{\mathrm{f}})/K$$

is a finite set of points (the proof of this is essentially equivalent to the proof of the finiteness of the class group of number fields).

For example, take $T = \operatorname{Res}_{F/\mathbb{Q}} \mathbb{G}_{\mathrm{m},F}$, where F is a number field. Then T has dimension $[F:\mathbb{Q}]$, and if we take K to be the maximal compact group $\hat{\mathcal{O}}_F^\times$ (where $\hat{\mathcal{O}}_F$ is the profinite completion of the ring of integers of F), we have

$$\operatorname{Sh}_K(T,X) = \mathbb{A}_F^\times / F^\times F_\infty^\times \hat{\mathcal{O}}_F^\times,$$

which is the ideal class group of F.

Example 2.5. A better version of a toroidal Shimura variety is afforded by the extra flexibility of being allowed to take covers of conjugacy classes in $\mathrm{Hom}(\mathbb{S}, G_{\mathbb{R}})$, cf. Definition 2.7. In the above example, this means replacing the the one-point space by a finite set. Here's an indication of the usefulness of this.

Again, Let F be a number field, and let $T = \mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_{\mathrm{m},F}$. But now let

$$X_F := T(\mathbb{R})/T(\mathbb{R})^+ = \pi_0\big(T(\mathbb{R})\big),$$

i.e., the group of connected components of the multiplicative group of $F_\infty = F \otimes_{\mathbb{Q}} \mathbb{R}$ (product over all archimedean places of F). We then have

$$\mathrm{Sh}(T, X_F) = \varprojlim_K F^\times \backslash \pi_0(F_\infty^\times) \times \mathbb{A}_{F,\mathrm{f}}^\times / K \cong \pi_0(C_F),$$

where C_F is the *idèle class group* $F^\times \backslash \mathbb{A}_F^\times$. By Artin reciprocity,

$$\pi_0(C_F) \cong \mathrm{Gal}(F^{\mathrm{ab}}/F),$$

and so by allowing for finite covers of the "X", we get Shimura varieties that are better suited for applications to (abelian) class field theory.

We can also, of course, give a Hodge theory interpretation for the above datum. We have $F_\infty \cong \mathbb{C}^s \times \mathbb{R}^r$. We put on F_∞ the Hodge structure that is trivial on $\mathbb{R}^r$ and equal to some choice of a complex structure on the remaining factor $\mathbb{C}^s$ (there being 2^s possibilities). This gives a morphism $h_1 : \mathbb{S} \to T_{\mathbb{R}}$. We call the triple $(\mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_{\mathrm{m},F}, X_F, h_1)$ the *multiplicative Shimura datum of the field F*. This Shimura datum is classical (in the sense of Definition 2.7) if and only if $F = \mathbb{Q}$ or F is imaginary quadratic, cf. Section 4.2.1.

Example 2.6. Let $h : \mathbb{S} \to \mathrm{GL}_{2,\mathbb{R}}$ be the morphism given by $h(a+ib) = \left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$. Let $\mathbb{H}^\pm$ be the $\mathrm{GL}_2(\mathbb{R})$-conjugacy class of h. It identifies with the Poincaré double half plane with action of $\mathrm{GL}_2(\mathbb{R})$ by homographies. Then $(\mathrm{GL}_2, \mathbb{H}^\pm)$ is called the *modular Shimura datum*. The limit Shimura variety is the familiar tower of modular curves.

Remarks on Shimura Stacks

From the moduli-space point-of-view, it is more natural to study the *level K Shimura stack*, given by the topological stacky quotient

$$\mathfrak{Sh}_K(G,X) := [G(\mathbb{Q})\backslash X \times G(\mathbb{A}_{\mathrm{f}})/K],$$

and the *Shimura stack*, given by the 2-projective limit

$$\mathfrak{Sh}(G,X) := \varprojlim_K \mathfrak{Sh}_K(G,X).$$

In the case of the multiplicative Shimura datum of a number field, i.e., $G = \mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_{\mathrm{m},F}$, the level K Shimura stack can have infinite isotropy given by the group $G(\mathbb{Q})^+ \cap K$. These isotropy groups are given by generalized congruence relations on the group of units $\mathcal{O}_F^\times$. We will have to keep track of (some of) these isotropy groups in the case of non-classical Shimura data.

Later in Section 3.3.5 we will be concerned with defining *natural* algebras of continuous "functions" on the finite level Shimura varieties. But in order to do that, we have to resolve their stack singularities. The question is now: What sort of obstructions are there to resolving the stack singularities?

Definition 2.10. Let (G,X) be a Shimura datum. A compact open subgroup $K \subset G(\mathbb{A}_{\mathrm{f}})$ is called *neat* if it acts freely on $G(\mathbb{Q})\backslash X \times G(\mathbb{A}_{\mathrm{f}})$.

If K is neat, then observe that the quotient analytic stack

$$\mathfrak{Sh}_K(G,X) = [G(\mathbb{Q})\backslash X \times G(\mathbb{A}_{\mathrm{f}})/K]$$

is an ordinary analytic space, but otherwise it is worthwhile from the moduli viewpoint to keep track of the nontrivial stack structure. For classical Shimura data (Definition 2.7), one can resolve the stack singularities by choosing a smaller compact open subgroup $K' \subset G(\mathbb{A}_{\mathrm{f}})$ that acts freely on $G(\mathbb{Q})\backslash X \times G(\mathbb{A}_{\mathrm{f}})$. This is what we will usually do in order to be able to define continuous "functions" on the stack $\mathfrak{Sh}_K(G,X)$.

If (G,X) is classical, there is, moreover, a relatively simple expression for the limit Shimura variety (see [16, Corollaire 2.1.11]):

$$\mathrm{Sh}(G,X) \cong G(\mathbb{Q})\backslash X \times G(\mathbb{A}_{\mathrm{f}})$$

and the Shimura stacks $\mathfrak{Sh}_K(G,X)$ are in fact algebraic stacks over $\mathbb{C}$.

Unfortunately, things are not generally so simple: non-classical Shimura varieties exist. In fact, for general Shimura data, the quotient $G(\mathbb{Q})\backslash X \times G(\mathbb{A}_{\mathrm{f}})$ is not Hausdorff. This is the case for the zero-dimensional Shimura datum $(\mathrm{Res}_{F/\mathbb{Q}}\,\mathbb{G}_{\mathrm{m},F}, \pi_0(F_\infty^\times))$ whenever F is a number field other than $\mathbb{Q}$ or an imaginary quadratic field. Although we will show this in Section 4.2, let's take a look for the moment at $F = \mathbb{Q}(\sqrt{2})$ for concreteness.

In this case

$$\mathrm{Sh}\big(\mathrm{Res}_{F/\mathbb{Q}}\,\mathbb{G}_{\mathrm{m},F}, \pi_0(F_\infty^\times)\big) \cong F^\times \backslash\, \pi_0(F_\infty^\times) \times \mathbb{A}_{F,\mathrm{f}}^\times,$$

the RHS not even being separated (essentially because $\mathcal{O}_F^\times$ is infinite, by Dirichlet's Unit Theorem). Moreover, the finite resolution of the stack singularities for classical Shimura datum that one gets by passing to sufficiently small neat subgroups is not possible here.

Let $K = \hat{\mathcal{O}}_F^\times$ and consider the stack $\mathfrak{Sh}_K(\mathrm{Res}_{F/\mathbb{Q}}\,\mathbb{G}_{\mathrm{m},F}, \pi_0(F_\infty^\times))$. Its coarse quotient is the ideal class group of F, i.e., the trivial group $\{1\}$. Since F has class number one, this coarse quotient can also be described as $\mathcal{O}_F^\times \backslash\, \pi_0(F_\infty^\times)$. In this case, $\mathcal{O}_F^\times$ is infinite, so that we can not choose a smaller $K' \subset K$ that acts freely on $F^\times \backslash\, \pi_0(F_\infty^\times) \times \mathbb{A}_{F,\mathrm{f}}^\times$. If we want to resolve the stack singularities, we can use the quotient map

$$F^\times \backslash \mathbb{A}_F^\times / K \longrightarrow \mathfrak{Sh}_K\big(\mathrm{Res}_{F/\mathbb{Q}}\,\mathbb{G}_{\mathrm{m},F}, \pi_0(F_\infty^\times)\big)$$

for the scaling action of the connected component of identity D_F in the idèle class group $C_F := F^\times \backslash \mathbb{A}_F^\times$.

Remark 2.5. From the viewpoint of moduli spaces, it is important that the coarse space $\mathrm{Sh}_K(\mathrm{Res}_{F/\mathbb{Q}}\,\mathbb{G}_{\mathrm{m},F}, \pi_0(F_\infty^\times))$, i.e., the big ideal class group, be replaced by the corresponding group stack with infinite stabilizers (given by groups of units with congruence conditions):

$$\mathfrak{Sh}_K(\mathrm{Res}_{F/\mathbb{Q}}\,\mathbb{G}_{\mathrm{m},F}, \pi_0(F_\infty^\times)).$$

This "equivariant viewpoint" of the finite level Shimura variety could also be important to understand geometrically the definition of Stark's zeta functions, and also for the understanding of Manin's real multiplication program [32].

Chapter 3

Shimura Varieties and Bost-Connes-Marcolli Dynamical Systems

In this chapter we construct Shimura-variety generalizations of the Bost-Connes [3] and Connes-Marcolli systems [11]. Aspects of these motivating special cases are reviewed in the first two sections. The remaining sections then carry out a generalization that incorporates a Shimura variety as starting datum. The end result is a family of stack-groupoids and associated C*-algebras, whose general properties are studied (zeta functions, symmetries, KMS states). In the next chapter, various examples are considered. In this chapter we concentrate mostly on formal aspects.

3.1 Arithmetical Quantum Statistical Mechanics in Dimensions 1 and 2

To prepare for the general constructions to be given in Section 3.3, we start with an overview of two examples of arithmetical quantum statistical mechanical systems (arithmetical QSM's, for short). We do not define the terminology "arithmetical QSM" precisely, so one aim in presenting these examples is to explain what we mean by this by showing the type and form of arithmetical phenomena that can occur within the mathematical framework of quantum statistical mechanical systems (in the sense discussed in Section 2.1).

Both examples, the first due to Bost-Connes [3] and second to Connes-Marcolli [11], concern noncommutative moduli of degenerate rational lattices. Both examples are natural, yet highly non-obvious! Once they are understood in the appropriate language — namely, that of adèles — the reader should find the transition to Shimura varieties natural.

3.1.1 Dimension One: The Bost-Connes System

The Bost-Connes system was the first, and still most important, example of a C*-dynamical system (A, σ_t) admitting the action of a Galois group of a number field on which the usual Galois action on fields is recovered upon evaluation of certain canonical states, namely, the KMS states at zero temperature. Actually, to achieve this one must also restrict to an "arithmetic" subalgebra, on which the KMS states evaluate to algebraic numbers.

Let us now describe the system of Bost-Connes. As a C*-algebra, it may be expressed as the C*-algebra of the locally compact groupoid

$$\mathcal{G}_{\mathrm{BC}} = \mathbb{Q}_+^\times \ltimes \hat{\mathbb{Z}} \tag{3.1}$$

where the cross-product symbol signifies that this is the groupoid of the partial action of $\mathbb{Q}_+^\times$ on $\hat{\mathbb{Z}}$ (cf. Section 2.2.2).

The groupoid $\mathcal{G}_{\mathrm{BC}}$ is "amenable" (cf. [42]), which for us means that its C*-algebra has a particularly simple description, which goes as follows. Let $\mathcal{A}_{\mathrm{BC}}$ be the convolution algebra $C_{\mathrm{c}}(\mathcal{G}_{\mathrm{BC}})$ of compactly-supported complex-valued continuous functions on $\mathcal{G}_{\mathrm{BC}}$. Thus for $f_1, f_2 \in \mathcal{A}_{\mathrm{BC}}$, their product is the convolution

$$f_1 * f_2(q, \alpha) = \sum_{q' \,:\, q'\alpha \in \hat{\mathbb{Z}}} f_1(qq'^{-1}, q'\alpha) f_2(q', \alpha).$$

For every point $\chi \in \hat{\mathbb{Z}}$, there is a representation π_χ of $\mathcal{A}_{\mathrm{BC}}$ on the Hilbert space

$$\mathfrak{H}^\chi := \ell^2(\mathcal{G}_{\mathrm{BC}}^\chi)$$

where $\mathcal{G}_{\mathrm{BC}}^\chi$ is the (discrete) "s-fibre"

$$\mathcal{G}_{\mathrm{BC}}^\chi = \left\{ (q, \chi) \in \mathcal{G}_{\mathrm{BC}} \mid q\chi \in \hat{\mathbb{Z}} \right\},$$

and the action of $\mathcal{A}_{\mathrm{BC}}$ on the Hilbert space is by convolution:

$$\pi_\chi \colon \mathcal{A}_{\mathrm{BC}} \longrightarrow B(\mathfrak{H}^\chi)$$
$$\big(\pi_\chi(f)\xi\big)(\gamma) = \sum_{s(\gamma)=\chi} f(\gamma\gamma'^{-1})\xi(\gamma').$$

Now the norm on $\mathcal{A}_{\mathrm{BC}}$

$$\|f\| := \sup_{\chi\in\hat{\mathbb{Z}}} \|\pi_\chi(f)\|_{\mathfrak{H}^\chi}\,. \tag{3.2}$$

satisfies the C*-condition under the involution

$$f^*(q,\alpha) = \overline{f(q^{-1}, q\alpha)}.$$

Definition 3.1. The *Bost-Connes C*-algebra* is the C*-completion of the convolution algebra $\mathcal{A}_{\mathrm{BC}}$ under the norm (3.2).

One of the most interesting aspects of the Bost-Connes algebra is that it admits a natural actions of the Galois group $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ and of $\mathbb{R}$ (i.e., a time evolution σ_t), and these actions commute! Thus $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ is a *dynamical* symmetry of the C*-dynamical system $(A_{\mathrm{BC}}, \sigma_t)$.

We now describe these actions.

The Galois Action on A_{BC} We chose the letter χ for elements of $\hat{\mathbb{Z}}$ to remind us that $\hat{\mathbb{Z}}$ is in fact the Pontrjagin dual of $\mathbb{Q}/\mathbb{Z}$, i.e., the group of continuous (unitary) characters of $\mathbb{Q}/\mathbb{Z}$. Thus there is natural action of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ on $\hat{\mathbb{Z}}$, which under the action of $\sigma \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ sends χ to

$$\sigma\cdot\chi\colon a \mapsto \sigma\big(\chi(a)\big). \tag{3.3}$$

The Galois action therefore induces a Galois action on A_{BC}. It satisfies

$$\sigma\cdot f(q,\chi) = f(q, \sigma^{-1}\cdot\chi).$$

The Time Evolution of A_{BC} The time evolution of A_{BC} simply raises the $\mathbb{Q}$-variable to an imaginary power:

$$\sigma_t(f)(q,\chi) = q^{it} f(q,\chi).$$

It clearly commutes with the Galois action (3.3).

If $\chi \in \hat{\mathbb{Z}}$ is *invertible*, then $\mathfrak{H}^{\chi} \cong \ell^2$ (canonically), and time evolution is generated by the *Hamiltonian*

$$H\xi(n) = \log(n)\xi(n), \quad \xi \in \ell^2 \tag{3.4}$$

which is an unbounded operator on ℓ^2. That is to say,

$$\pi_\chi(\sigma_t f) = e^{iHt}\pi_\chi(f)e^{-iHt}.$$

The Main Theorem of Bost-Connes

The main results of Bost-Connes [3] are as follows. First they give a complete classification of the KMS$_\beta$ states of $(A_{\mathrm{BC}}, \sigma_t)$.

Theorem 3.1 (Bost-Connes)**.** *The following is a complete list of the KMS$_\beta$ states of $(A_{\mathrm{BC}}, \sigma_t)$.*

1. *For each $0 < \beta \leq 1$, there is unique KMS$_\beta$ state. It is therefore a factor state, and the corresponding factor is the Araki-Woods hyperfinite factor of Type III$_1$.*

2. *For each $1 < \beta \leq \infty$, there is a $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$-equivariant homeomorphism between $\hat{\mathbb{Z}}^\times$ and the space $\mathcal{E}_\beta$ of extremal KMS$_\beta$ states:*

$$\hat{\mathbb{Z}}^\times \xrightarrow{\sim} \mathcal{E}_\beta, \quad \chi \longmapsto \phi_{\beta,\chi}(f) = \frac{\mathrm{Tr}\big(\pi_\chi(f)e^{-\beta H}\big)}{Z(\beta)},$$

where H is the Hamiltonian as in (3.4), and the partition function $Z(\beta) = \mathrm{Tr}(e^{-\beta H})$ is the Riemann zeta function.

Each $\phi_{\beta,\chi}$ is a (factor) state of Type I$_\infty$.

Remark 3.1. Notice that the extremal KMS states for $\beta > 1$ take the form of a Gibbs state (cf. (2.2)). As we shall see through other examples, this is typical for KMS states at low temperature.

The second main result, the most interesting aspect of the Bost-Connes system, describes the arithmetical implication of the second part of the above theorem.

Theorem 3.2 (Bost-Connes)**.** *The Bost-Connes algebra A_{BC} is generated by a $\mathbb{Q}$-subalgebra $A_{\mathrm{BC}}^{\mathbb{Q}}$ which has the following amazing property:*

If ϕ is a KMS$_\infty$ state, and if $a \in A_{\mathrm{BC}}^{\mathbb{Q}}$, then

$$\phi(a) \in \mathbb{Q}^{\mathrm{ab}} \quad \textit{and} \quad \sigma(\phi a) = \phi(\sigma a),\ \forall \sigma \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}).$$

In other words, the extremal KMS states at zero temperature intertwine the Galois action on $A_{\mathrm{BC}}^{\mathbb{Q}}$ and the usual Galois action on the field $\mathbb{Q}^{\mathrm{ab}}$. This is very surprising since the Galois action (aside from complex conjugation) is very badly discontinuous with respect to the complex topology (thinking of $\bar{\mathbb{Q}}$ as lying in $\mathbb{C}$), while C*-algebra states are defined to be continuous with respect to the complex topology!

Remark 3.2. The path to this improbable result is as remarkable as the result itself. To reiterate: Starting from the groupoid

$$\mathbb{Q}_+^\times \ltimes \hat{\mathbb{Z}}$$

we get a C*-algebra A_{BC}, a canonical C*-completion of its convolution algebra. This C*-algebra carries a natural $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$-action. Then, considerations from statistical mechanics distinguish a certain set of states, the KMS states. And finally these states realize the equivalence of the Galois action on a generating rational subalgebra $A_{\mathrm{BC}}^{\mathbb{Q}}$ with the Galois action on $\mathbb{Q}^{\mathrm{ab}}$.

We also want to emphasize that although the (abelian) class field theory of $\mathbb{Q}$ is certainly well-understood territory, there is no precedence or anticipation of the Bost-Connes perspective in algebraic number theory.

The Bost-Connes Groupoid in Adèlic Terms

We want to reformulate the description of the Bost-Connes groupoid in adèlic terms, partly in view of the connections to class field theory that we saw in the previous paragraph, but also in order to get a description of the groupoid that will easily generalize to other number fields. In fact, as an extra payoff, the description that we will arrive at will even point the way to higher dimensions.

Let us start by writing down explicitly what the expression

$$\mathcal{G}_{\mathrm{BC}} = \mathbb{Q}_+^\times \ltimes \hat{\mathbb{Z}}$$

means. As a locally compact space, the groupoid $\mathcal{G}_{\mathrm{BC}}$ is the space of pairs $(q, \alpha) \in \mathbb{Q}_+^\times \times \hat{\mathbb{Z}}$ such that $q\alpha \in \hat{\mathbb{Z}}$. The unit space of $\mathcal{G}_{\mathrm{BC}}$ is $\hat{\mathbb{Z}}$, and the source and target maps are

$$s, t\colon \mathcal{G}_{\mathrm{BC}} \rightrightarrows \hat{\mathbb{Z}}, \quad s(q, \alpha) = \alpha, \quad t(q, \alpha) = q\alpha.$$

The composition $(q, \alpha)(q', \alpha')$ is defined whenever $\alpha = q'\alpha'$, in which case the composition is (qq', α'), while the inverse of (q, α) is $(q^{-1}, q\alpha)$.

Now let's rewrite everything adèlically. The first ground rule when working with adèles, indeed their very *raison d'être*, is to work with *all* places simultaneously, whenever possible. This means in particular, making the substitution

$$\hat{\mathbb{Z}}^\times \backslash \mathbb{A}_{\mathrm{f}}^\times \quad \text{for} \quad \mathbb{Q}_+^\times \quad \text{(isomorphic)}.$$

Note that the analogue of this for general number fields does not yield an isomorphism in general (non-trivial class groups exist!). We can get a more symmetric groupoid by adding a piece to the unit space to get the groupoid

$$\mathcal{G}_{\mathbb{A}} = \mathbb{A}_{\mathrm{f}}^\times \ltimes \big(\hat{\mathbb{Z}} \times \hat{\mathbb{Z}}^\times\big).$$

The Bost-Connes groupoid will be a double quotient of this.

In fact, with the groupoid $\mathcal{G}_{\mathbb{A}}$ at hand, we can even consider higher-level analogues of the Bost-Connes groupoid. Let K be any compact open subgroup of $\hat{\mathbb{Z}}$. There are left and right actions of K on $\mathcal{G}_{\mathbb{A}}$ given by

$$k.\big(g, (\alpha, h)\big) = \big(kg, (\alpha, h)\big), \quad \big(g, (\alpha, h)\big).k = \big(gk, (k^{-1}\alpha, hk)\big).$$

Taking the double quotient we get a locally compact groupoid

$$K\backslash\mathbb{A}_{\mathrm{f}}^{\times} \ltimes \big(\hat{\mathbb{Z}} \times \hat{\mathbb{Z}}^{\times}\big)/K,$$

and when $K = \hat{\mathbb{Z}}^{\times}$ we get back the Bost-Connes groupoid.

Proposition 3.1. *Let K be the maximal compact subring $\hat{\mathbb{Z}}^{\times} \subset \mathbb{A}_{\mathrm{f}}^{\times}$. The double quotient $K\backslash\mathcal{G}_{\mathbb{A}}/K$ is then a locally compact groupoid, and the natural inclusion*

$$\mathcal{G}_{\mathrm{BC}} \hookrightarrow \mathcal{G}_{\mathbb{A}}, \quad (q,\alpha) \mapsto \big(q,(\alpha,1)\big).$$

induces a groupoid isomorphism

$$\mathcal{G}_{\mathrm{BC}} \xrightarrow{\sim} K\backslash\mathcal{G}_{\mathbb{A}}/K. \tag{3.5}$$

Proof. Just as the quotient of a group by a subgroup need not be a group, so may the quotient of a groupoid by a group action fail to be a groupoid. It therefore behooves us to check that the groupoid structure on $\mathcal{G}_{\mathbb{A}}$ descends to a groupoid structure on the double quotient $K\backslash\mathcal{G}_{\mathbb{A}}/K$. There is no difficulty in doing this, but let us nonetheless check this explicitly.

First, it is easy to check that the source and target maps of $\mathcal{G}_{\mathbb{A}}$ descend to the double quotient: for example, for the target map it suffices to observe that if $k_1, k_2 \in K$ and $\big(g,(\alpha,k)\big) \in \mathcal{G}_{\mathbb{A}}$, then

$$\begin{aligned} t\big(k_1\big(g,(\alpha,k)\big)k_2\big) &= t\big(k_1gk_2,(k_2^{-1}\alpha,kk_2)\big) \\ &= (k_1g\alpha, kg^{-1}k_1^{-1}) \\ &= t\big(g,(\alpha,k)\big)k_1^{-1}. \end{aligned} \tag{3.6}$$

Evidently the source map satisfies

$$s\big(k_1\big(g,(\alpha,k)\big)k_2\big) = s\big(g,(\alpha,k)\big)k_2, \tag{3.7}$$

and so it too is equivariant (for the $K \times K$-action on $\mathcal{G}_{\mathbb{A}}$, and the right K-action on the unit space). We will continue to write s, resp. t, for the source, resp. target, map on $K\backslash\mathcal{G}_{\mathbb{A}}/K$.

To show that the groupoid multiplication is well-defined on $K\backslash\mathcal{G}_{\mathbb{A}}/K$, we introduce the notation $\pi\colon \mathcal{G}_{\mathbb{A}} \to K\backslash\mathcal{G}_{\mathbb{A}}/K$ for the natural projection. If

$u, v \in \mathcal{G}_{\mathbb{A}}$ are such that $s \circ \pi(u) = t \circ \pi(v)$, then by acting by $K \times K$ (if necessary), we may assume that $s(u) = t(v)$, i.e., that u and v are composable. The multiplication is then defined to be $\pi(u)\pi(v) = \pi(uv)$. It is well-defined, i.e., it does not depend on the particular choice of composable representatives. Checking this is again an easy, though somewhat messy-looking "one-liner", as in (3.6). Explicitly, suppose $u = \big(g, (\alpha, k)\big)$ and $v = \big(g', (\alpha', k')\big)$ are composable, as are $k_1 u k_2$ and $k_1' v k_2'$. From (3.7) and (3.6) we get

$$s(u)k_2 = s(k_1 u k_2) = t(k_1' v k_2') = t(v)k_1'^{-1} = s(u)k_1'^{-1},$$

and hence $k_2 k_1' = 1$. Writing out the product $(k_1 u k_2)(k_1' v k_2')$ we get

$$\begin{aligned}(k_1 u k_2)(k_1' v k_2') &= \big(k_1 g k_2 k_1' g' k_2', (k_2'^{-1}\alpha', k', k_2')\big)\\ &= k_1\big(g k_2 k_1' g', (\alpha', k')\big)k_2'\\ &= k_1\big(g g', (\alpha', k')\big)k_2',\end{aligned}$$

and hence $\pi(uv) = \pi\big((k_1 u k_2)(k_1' v k_2')\big)$. We are now done with showing that $K\backslash\mathcal{G}_{\mathbb{A}}/K$ is a groupoid in the groupoid structure induced by $\mathcal{G}_{\mathbb{A}}$.

It remains to show that the composition

$$\begin{aligned}\phi\colon \mathcal{G}_{\mathrm{BC}} \hookrightarrow \mathcal{G}_{\mathbb{A}} &\xrightarrow{\pi} K\backslash\mathcal{G}_{\mathbb{A}}/K,\\ (q, \alpha) \mapsto \big(q, (\alpha, 1)\big) &\mapsto \pi\big(q, (\alpha, 1)\big)\end{aligned}$$

is an isomorphism of groupoids. The key fact required is the decomposition

$$\mathbb{A}_{\mathrm{f}}^{\times} = \mathbb{Q}_{+}^{\times}.\hat{\mathbb{Z}}^{\times}, \quad \text{with} \quad \mathbb{Q}_{+}^{\times} \cap \hat{\mathbb{Z}}^{\times} = \{1\},$$

which we already used in defining the action of $\mathbb{A}_{\mathrm{f}}^{\times}$ on $\hat{\mathbb{Z}} \times \hat{\mathbb{Z}}^{\times}$ (the unit space of $\mathcal{G}_{\mathbb{A}}$).

Proof of injectivity of ϕ: Suppose $\big(q, (\alpha, 1)\big) = \big(k_1 q' k_2, (k_2^{-1}\alpha', k_2)\big)$, with $k_1, k_2 \in K$ and $(q, \alpha), (q', \alpha') \in \mathcal{G}_{\mathrm{BC}}$. Then clearly $k_2 = 1$ and so $q = k_1 q'$. But because $\mathbb{Q}_{+}^{\times} \cap \hat{\mathbb{Z}}^{\times} = \{1\}$, the only possibility for k_1 is 1, and we get injectivity of ϕ.

Proof of surjectivity of ϕ: Let $\big(g, (\alpha, k)\big)$ be any element in $\mathcal{G}_{\mathbb{A}}$. Since $\mathbb{A}_{\mathrm{f}}^{\times} = \mathbb{Q}_{+}^{\times}.\hat{\mathbb{Z}}^{\times}$, there is a factorization $g = qk'$ with $(q, k') \in \mathbb{Q}_{+}^{\times} \times \hat{\mathbb{Z}}^{\times}$. We get surjectivity, because modulo $K \times K$ we have

$$\big(g, (\alpha, k)\big) = \big(qk', (\alpha, k)\big) \equiv \big(q, (\alpha, k)\big) \equiv \big(q, (k\alpha, 1)\big).$$

Finally, it is clear that the bijection ϕ is continuous. Since both factors of ϕ are open, so is ϕ. Hence ϕ is a topological isomorphism. $\square$

Remark 3.3. Another generalization of the Bost-Connes groupoid to arbitrary number fields suggested by the expression $\mathcal{G}_{\mathrm{BC}} = \mathbb{Q}_+^\times \ltimes \hat{\mathbb{Z}}$, is the following: If F is a number field, let $\mathcal{G}_{\mathrm{BC}}^F$ be the groupoid

$$\mathcal{G}_{\mathrm{BC}}^F = F_+^\times \ltimes \hat{\mathcal{O}}_F \tag{3.8}$$

where $F_+^\times$ is the intersection of $F^\times$ with the connected component of $(F \otimes \mathbb{R})^\times$ (the units of the product of all archimedean completions of F), and $\hat{\mathcal{O}}_F$ is the profinite completion of the ring of integers of F.

Now a natural questions arises: Why not regard this groupoid as the correct generalization of the Bost-Connes groupoid? After all, it is certainly simpler than the adèlic one that we've constructed, and it's resemblance to Bost-Connes is certainly plain to see.

The problem is that $\mathcal{G}_{\mathrm{BC}}^F$ is in fact *too* simple, if one seeks to preserve the main features of Theorems 3.1 and 3.2. In particular, previous work of Harari-Leicthnam [22] and Paula Cohen [7] based on this groupoid yield Bost-Connes analogues that only admit symmetry by $\hat{\mathcal{O}}_F^\times$. Problems or limitations imposed by the existence of non-trivial class group inevitably arise in these approaches.

The adèles lead to a cleaner approach that is relatively robust against simplifying "coincidences" that occur over $\mathbb{Q}$.

Shimura Varieties and the Bost-Connes Groupoid

We return to our latest incarnation of the Bost-Connes groupoid in order to recast it once more, this time bringing the idèle class group to light.

Proposition 3.2. *The natural injection*

$$\mathcal{G}_{\mathrm{BC}} = \mathbb{Q}_+^\times \ltimes \hat{\mathbb{Z}} \longhookrightarrow \mathbb{A}_{\mathrm{f}}^\times \ltimes \left(\hat{\mathbb{Z}} \times \mathbb{Q}^\times \backslash\, \pi_0(\mathbb{R}^\times) \times \mathbb{A}_{\mathrm{f}}^\times\right)$$

induces an isomorphism of groupoids

$$\mathcal{G}_{\mathrm{BC}} \cong \hat{\mathbb{Z}}^\times \backslash \big(\mathbb{A}_{\mathrm{f}}^\times \ltimes \left(\hat{\mathbb{Z}} \times \mathbb{Q}^\times \backslash\, \pi_0(\mathbb{R}^\times) \times \mathbb{A}_{\mathrm{f}}^\times\right)\big)/\hat{\mathbb{Z}}^\times$$

This is just Proposition 3.1 once again, since

$$\hat{\mathbb{Z}}^\times \cong \mathbb{Q}^\times \backslash \, \pi_0(\mathbb{R}^\times) \times \mathbb{A}_{\mathrm{f}}^\times.$$[1]

But what we have gained is a description in terms of the Shimura variety

$$\mathrm{Sh}\big(\mathrm{GL}_{1,\mathbb{Q}}, \pi_0(\mathbb{R}^\times)\big) = \varprojlim_K \mathbb{Q}^\times \backslash \, \pi_0(\mathbb{R}^\times) \times \mathbb{A}_{\mathrm{f}}^\times / K \cong \mathbb{Q}^\times \backslash \, \pi_0(\mathbb{R}^\times) \times \mathbb{A}_{\mathrm{f}}^\times.$$

In this commutative case, the Shimura variety is just another expression for the group of connected components of the idèle class group

$$\pi_0(\mathbb{Q}^\times \backslash \mathbb{A}^\times)$$

as a profinite group, which under Artin reciprocity is isomorphic to $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$. We see again the advantage of the adèlic approach, for the same holds true for arbitrary number fields F, namely, we have

$$\mathrm{Sh}\big(T, \pi_0(T(\mathbb{R}))\big) \cong \pi_0(F^\times \backslash \mathbb{A}_{F,\mathrm{f}}^\times) \cong \mathrm{Gal}(\bar{F}/F)^{\mathrm{ab}},$$

where T is the torus $\mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_{\mathrm{m},F}$, and the last isomorphism is given by the Artin reciprocity.

Further Directions

So to conclude our review of the Bost-Connes system we see that the Bost-Connes groupoid written in terms of a Shimura variety

$$\mathcal{G}_{\mathrm{BC}} \cong \hat{\mathbb{Z}}^\times \backslash \big(\mathbb{A}_{\mathrm{f}}^\times \ltimes \big(\hat{\mathbb{Z}} \times \mathrm{Sh}(\mathrm{GL}_1, \{\pm 1\})\big)\big) / \hat{\mathbb{Z}}^\times. \tag{3.9}$$

suggests a generalization in two directions: firstly, to analogues for other number fields, and secondly, to Shimura varieties of higher dimension.

In the next section, we will see that this Shimura-variety description of the Bost-Connes groupoid is compatible with the GL_2 analogue of (3.9).

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[1] We will consider this isomorphism again in Section 4.1.2 where we will recognize it as a reflection of certain properties of the algebraic group $\mathrm{GL}_{1,\mathbb{Q}}$.

3.1.2 Dimension Two: The Connes-Marcolli System

The Geometric Meaning of the Bost-Connes Groupoid

One obvious question was left hanging in our consideration of the Bost-Connes groupoid, in view of Section 2.2: What is the geometric meaning of the Bost-Connes groupoid? Does it describe a reasonable noncommutative space? This last question reveals our bias: we seek a groupoid that is not far from an (honest) equivalence relation on an (ordinary) space.

The answer of Connes-Marcolli [11] is the space of $\mathbb{Q}$-lattices in dimension 1. A $\mathbb{Q}$-*lattice* in $\mathbb{R}^n$ is a pair (Λ, ϕ) consisting of a lattice $\Lambda \subset \mathbb{R}^n$ and a $\mathbb{Q}$-structure ϕ for Λ, which is a homomorphism $\phi\colon \mathbb{Q}^n/\mathbb{Z}^n \to \mathbb{Q}\Lambda/\Lambda$ — not necessarily invertible. Two $\mathbb{Q}$-lattices (Λ_1, ϕ_1) and (Λ_2, ϕ_2) are *commensurable* if they are commensurable in the usual sense (the indices $[\Lambda_i : \Lambda_1 \cap \Lambda_2]$ are finite) and $\phi_1 \equiv \phi_2 \pmod{\Lambda_1 + \Lambda_2}$.

Note that a 1-dimensional $\mathbb{Q}$-lattice is simply a pair $(\lambda\mathbb{Z}, \lambda\rho)$ with $\lambda > 0$ and $\rho \in \mathrm{End}(\mathbb{Q}/\mathbb{Z}) \cong \hat{\mathbb{Z}}$. The geometric interpretation of the Bost-Connes groupoid can now be stated.

Proposition 3.3 (Connes-Marcolli)**.** *Let $\mathcal{R}$ be the equivalence relation of commensurability for $\mathbb{Q}$-lattices in $\mathbb{R}$, and let $\mathcal{R}/\mathbb{R}_+^\times$ be its quotient modulo the scaling action. Then the map*

$$\mathcal{G}_{\mathrm{BC}} = \mathbb{Q}_+^\times \ltimes \hat{\mathbb{Z}} \longrightarrow \mathcal{R}/\mathbb{R}_+^\times, \quad (q, \alpha) \mapsto \big((q^{-1}\mathbb{Z}, \rho), (\mathbb{Z}, \rho)\big)$$

is an isomorphism of groupoids.

The Connes-Marcolli Groupoid

The geometric interpretation of the Bost-Connes groupoid directly suggests a generalization to higher dimensions, namely commensurability classes of higher dimensional $\mathbb{Q}$-lattices (possibly up-to scaling, or some other notion of equivalence). The groupoid of the two-dimensional case studied by Connes-Marcolli in [11] is actually a stack-groupoid. We study it as an example of the general construction to come in Section 3.3. We will see in Section 4.1.3 that it coincides with the obvious GL(2) analogue of (3.9).

The GL(2)-system corresponds to the noncommutative space of commensurability classes of 2-dimensional $\mathbb{Q}$-lattice in $\mathbb{C}$, up to scaling by $\mathbb{C}^\times$. As such, it is described by the quotient of a groupoid $\mathcal{G}_2$ by an action of $\mathbb{C}^\times$. In the notation of [11], this groupoid is defined as follows.

Let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, and let $G = \mathrm{GL}(2)$, so that the identity component $G^+(\mathbb{R})$ is the group of invertible matrices of positive determinant, and $G^+(\mathbb{Q}) = G(\mathbb{Q}) \cap G^+(\mathbb{R}) = \mathrm{GL}_2^+(\mathbb{Q})$. The space

$$\begin{aligned} &G^+(\mathbb{Q}) \ltimes \big(M_2(\hat{\mathbb{Z}}) \times G^+(\mathbb{R})\big) \\ &\qquad := \big\{ \, (g, \rho, \alpha) \in G^+(\mathbb{Q}) \times \big(M_2(\hat{\mathbb{Z}}) \times G^+(\mathbb{R})\big) \mid g\rho \in M_2(\hat{\mathbb{Z}}) \, \big\} \end{aligned}$$

is the locally compact groupoid of the partial action of $G^+(\mathbb{Q})$ on $M_2(\hat{\mathbb{Z}}) \times G^+(\mathbb{R})$. The group Γ acts on the left and right of $G^+(\mathbb{Q}) \ltimes \big(M_2(\hat{\mathbb{Z}}) \times G^+(\mathbb{R})\big)$ via the formulas

$$(g, \rho, \alpha) \cdot \gamma = (g\gamma, \gamma^{-1}\rho, \gamma^{-1}\alpha), \quad \gamma \cdot (g, \rho, \alpha) = (\gamma g, \rho, \alpha).$$

Proposition 3.4 (Connes-Marcolli)**.** *The quotient*

$$\mathcal{G}_2' := \Gamma \backslash G^+(\mathbb{Q}) \ltimes \big(M_2(\hat{\mathbb{Z}}) \times G^+(\mathbb{R})\big)/\Gamma$$

is the groupoid of commensurability classes of (Γ-isomorphism classes of) 2-dimensional $\mathbb{Q}$-lattices in $\mathbb{C}$.

However, as in the one dimension case, one should consider $\mathbb{Q}$-lattices up to scaling by $\mathbb{C}^\times$. Unfortunately, the resulting quotient $\mathcal{G}_2'/\mathbb{C}^\times$ is no longer a groupoid due to the existence of lattices with nontrivial isotropy under the action of Γ. In fact, this is the first example of a stack-groupoid, which we discussed in Section 2.3.

Remark 3.4. The quotient $\mathcal{G}_2'/\mathbb{C}^\times$, albeit not a groupoid, is a well-defined stack-groupoid. However, since we are ultimately interested in carrying out an analysis of KMS states, we need to pass to an analytic setting, namely to a function algebra. It is not clear how to extract, in general, a "function algebra" starting from a stack-groupoid. Hence we rely on the trick of Connes-Marcolli, even if it is not so conceptually satisfying.

However, since we still seek to define a convolution C*-algebra, as we did for the Bost-Connes system of the previous section, we examine $\mathcal{G}_2'/\mathbb{C}^\times$ and take one step back in the construction. That is, instead of considering the $\Gamma \times \Gamma$-quotient of

$$\mathcal{V} := G^+(\mathbb{Q}) \ltimes \big(M_2(\hat{\mathbb{Z}}) \times G^+(\mathbb{R})\big),$$

we consider the $\mathbb{C}^\times$-quotient of $\mathcal{V}$,

$$\mathcal{G}_2 := \mathcal{V}/\mathbb{C}^\times = G^+(\mathbb{Q}) \ltimes \big(M_2(\hat{\mathbb{Z}}) \times \mathbb{H}\big),$$

where $\mathbb{H} = G^+(\mathbb{R})/\mathbb{C}^\times$ is the usual upper half plane. Now, the quotient $\mathcal{V}/\mathbb{C}^\times$ *is* a groupoid, so we can consider its C*-algebra $C^*(\mathcal{G}_2)$, and to handle the $\Gamma \times \Gamma$-quotient, we consider only those functions in $C^*(\mathcal{G}_2)$ that are $\Gamma \times \Gamma$-invariant. The resulting C*-algebra is the Connes-Marcolli GL_2 analogue of the Bost-Connes algebra.

The Connes-Marcolli System

More concretely, the Connes-Marcolli algebra $\mathcal{A}_2$ is defined to consist of the functions $f \in C_c(\mathcal{G}_2)$ such that

$$f(\gamma g, y) = f(g, y), \quad f(g\gamma, y) = f(g, \gamma y), \quad \text{for all } \gamma \in \Gamma.$$

The passage to a C*-dynamical system in the spirit of the Bost-Connes system is now straightforward, though as we shall see, there are some surprises.

The *-algebra structure on $\mathcal{A}_2$ is as expected: the involution of $f, f' \in \mathcal{A}_2$ is

$$f^*(g, y) = \overline{f(g^{-1}, gy)},$$

and the convolution product is

$$(f * f')(g, y) = \sum_{h \in \Gamma \backslash G_y} f(gh^{-1}, hy) f'(h, y)$$

where $G_y := \{\, g \in G \mid gy \in M_2(\hat{\mathbb{Z}}) \times \mathbb{H} \,\}$. To simplify notation, set

$$Y := M_2(\hat{\mathbb{Z}}) \times \mathbb{H}.$$

For each $y \in Y$, there is the Hilbert space $\mathfrak{H}_y := \ell^2(\Gamma\backslash G_y)$ and a representation π_y of $\mathcal{A}_2$ on it, namely

$$\pi_y(f)\xi(g) = \sum_{h \in \Gamma\backslash G_y} f(gh^{-1}, hy)\xi(h).$$

Finally, as for the Bost-Connes system, to get a C*-norm on $\mathcal{A}_2$ we consider all the π_y's at once, i.e.,

$$\|f\| := \sup_{y \in Y} \|\pi_y(f)\|_{\mathfrak{H}_y}, \tag{3.10}$$

which is easily shown to be a C*-norm.

Definition 3.2. The *Connes-Marcolli system* is the C*-dynamical system

$$(A_2, \sigma_t)$$

where A_2 is the C*-completion of $\mathcal{A}_2$ with respect to the norm (3.10), and the *time evolution* σ_t is defined on $\mathcal{A}_2$ by

$$(\sigma_t f)(g, y) = (\det g)^{it} f(g, y).$$

(It clearly extends to A_2.)

Having described the GL(2)-system, we now list some of its remarkable properties. As for the Bost-Connes system, there is a nearly complete classification of its KMS$_\beta$ states (due in part to some very recent results of Laca, Larsen, and Neshveyev; only the cases $\beta = 0$ and $\beta = 1$ remain).[2]

Theorem 3.3. *The complete list of KMS$_\beta$ states of (A_2, σ_t) is as follows:*

1. *For $\beta < 1$ there are no KMS$_\beta$ states.*
2. *(Laca-Larsen-Neshveyev) For $1 < \beta \leq 2$ there is a unique KMS$_\beta$ state.*
3. *For $\beta > 2$, the KMS$_\beta$ states are all of the form*

$$\phi_{\beta,y}(f) = Z(\beta)^{-1} \sum_{m \in \Gamma\backslash M_2^+(\mathbb{Z})} (\det m)^{-\beta} f(1, m\rho, mz)$$

[2] I thank Prof. Marcolli for bringing this to my attention.

with $y = (\rho, \tau) \in \Gamma\backslash Y$ an invertible $\mathbb{Q}$-lattice, i.e., $\rho \in G(\hat{\mathbb{Z}})$. Indeed, there is a homeomorphism

$$\mathcal{E}_\beta \cong G(\mathbb{Q})\backslash G(\mathbb{A})/\mathbb{C}^\times$$

between the space $\mathcal{E}_\beta$ of extremal KMS$_\beta$ states, and the space of invertible $\mathbb{Q}$-lattices in $\mathbb{C}$ (up to scaling by $\mathbb{C}^\times$ and up to Γ-isomorphism).

The partition function $Z(\beta)$ is $\zeta(\beta)\zeta(\beta-1)$.

Notice that

$$G(\mathbb{Q})\backslash G(\mathbb{A})/\mathbb{C}^\times$$

is the Shimura variety

$$\begin{aligned}\mathrm{Sh}(\mathrm{GL}_{2,\mathbb{Q}}, \mathbb{H}^\pm) &= \varprojlim_K \mathrm{GL}_2(\mathbb{Q})\backslash\mathbb{H}^\pm \times \mathrm{GL}_2(\mathbb{A}_\mathrm{f})/K \\ &\cong \mathrm{GL}_2(\mathbb{Q})\backslash\mathbb{H}^\pm \times \mathrm{GL}_2(\mathbb{A}_\mathrm{f}).\end{aligned}$$

Explaining why Shimura varieties arise in both the Bost-Connes and Connes-Marcolli systems is part of the goal of the general constructions to come later in this chapter. In [13], the role a Shimura varieties in both of these systems was also discussed.

The Classification of Low-Temperature KMS States

In order to give an idea of the type of technical difficulties that one encounters in classifying KMS states, we want to discuss now in some detail part of the proof of [11, Theorem 1.26]. By examining the steps of the proof in some detail, we hope to extend it to certain analogues of the Connes-Marcolli system associated to other groups, for example to the analogue for $\mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GL}_{2,F})$, F a totally real field. This project has not yet been completely realized.

The starting point of the proof of Theorem 3.3(3) is the following observation. It is a consequence of the Riesz Representation Theorem, after a clever reduction to a simpler C*-dynamical system.

Proposition 3.5 (Connes-Marcolli, [11], Prop. 1.30)**.** *Let ϕ be a KMS$_\beta$ state (for any $\beta > 0$), and let $f \in C_\mathrm{c}(Z)$. Then there is a probability measure $d\lambda_\phi$*

on $\Gamma\backslash Y$ *representing* ϕ*, which is to say that*

$$\phi(f) = \int_{\Gamma\backslash Y} f(1,y)\, d\lambda_\phi(y). \tag{3.11}$$

The proposition is true for any β, but since we are interested in the KMS states for large β (indeed, the limit as $\beta \to \infty$), we fix $\beta > 2$. Then a somewhat technical computation — indeed, the trickiest part of the Connes and Marcolli's classification — shows that the subspace

$$\Gamma\backslash M_2^*(\hat{\mathbb{Z}}) \times \mathbb{H}, \quad M_2^*(\hat{\mathbb{Z}}) := M_2(\hat{\mathbb{Z}}) \cap G(\mathbb{A}_{\mathrm{f}})$$

of invertible $\mathbb{Q}$-lattices has $d\lambda_\phi$-measure 1. Therefore, Eq. (3.11) reduces to the integral

$$\phi(f) = \int_{\Gamma\backslash M_2^*(\hat{\mathbb{Z}})\times\mathbb{H}} f(1,y)\, d\lambda_\phi(y).$$

The key now is to observe that every $\mathbb{Q}$-lattice in $\Gamma\backslash M_2^*(\hat{\mathbb{Z}}) \times \mathbb{H}$ is commensurable to a unique *invertible* $\mathbb{Q}$-lattice, and to show that this implies that $d\lambda_\phi$ is determined by its restriction to the subspace of invertible $\mathbb{Q}$-lattices

$$\Gamma\backslash Y^\times, \quad \text{where } Y^\times = G(\hat{\mathbb{Z}}) \times \mathbb{H}$$

(without, however, being zero on the complement of invertible $\mathbb{Q}$-lattices). To show this explicitly, we express $\Gamma\backslash M_2^*(\hat{\mathbb{Z}}) \times \mathbb{H}$ as a discrete fibration over $Y^\times$, by showing that the action

$$\Gamma\backslash M_2^+(\mathbb{Z}) \times Y^\times \to M_2^*(\hat{\mathbb{Z}}) \times \mathbb{H}, \quad (g,y) \mapsto gy$$

descends to an isomorphism of Γ-quotients:

$$\Gamma\backslash\big(M_2^+(\mathbb{Z}) \times Y^\times\big)/\Gamma \cong \Gamma\backslash M_2^*(\hat{\mathbb{Z}}) \times \mathbb{H}. \tag{3.12}$$

Surjectivity is clear; to get injectivity one exploits the decomposition

$$G(\mathbb{A}_{\mathrm{f}}) = G^+(\mathbb{Q})G(\hat{\mathbb{Z}}).$$

Regarding the measure $d\lambda_\phi$ as a measure $d\lambda_\phi(g,y)$ on the LHS of Eq. (3.12), a judicious application of the KMS condition then shows that

$$\phi(f) = \sum_{m\in\Gamma\backslash M_2^+(\mathbb{Z})} (\det m)^{-\beta} \int_{\Gamma\backslash Y^\times} f(1,my)\, d\lambda_\phi(1,y).$$

To see how the KMS condition applies in this situation, we define a function $f_2 \in C_c(Z)$ by

$$f_2(g,y) = \begin{cases} f(1,gy), & \text{if } y \in Y^\times; \\ 0, & \text{otherwise,} \end{cases}$$

and we choose a function $f_1 \in C_c(Z)$ such that

$$f_1(g^{-1},gy)f(1,gy) = f(1,gy), \quad \text{whenever } (g,y) \in \Gamma\backslash M_2^+(\mathbb{Z}) \times Y^\times,$$

which is essentially a suitable "bump function". We easily check that for $(g,y) \in \Gamma\backslash M_2^+(\mathbb{Z}) \times Y^\times$ we have $f_2 * f_1(1,gy) = f(1,gy)$, and so Eq. (3.11), Eq. (3.12), and the KMS condition combine to yield the equalities

$$\begin{aligned} \phi(f) &= \phi(f_2 * f_1) \\ &= \phi(f_1 * \sigma_{i\beta}(f_2)) \\ &= \int_{\Gamma\backslash(M_2^+(\mathbb{Z})\times Y^\times)/\Gamma} d\lambda_\phi(g,y) \sum_{\substack{h\in\Gamma\backslash G^+(\mathbb{Q}) \\ hgy\in Y}} (\det h)^{-\beta} f_1(h^{-1},hgy) f_2(h,gy) \\ &= \int_{\Gamma\backslash Y^\times} \sum_{m\in\Gamma\backslash M_2^+(\mathbb{Z})} (\det m)^{-\beta} f(1,my)\, d\lambda_\phi(1,y). \end{aligned} \tag{3.13}$$

The definition of f_2 kills the sum in the second last equality, except for the term corresponding to $h = \Gamma$.

At this point the classification is essentially at hand: from Eq. (3.13) we see that KMS$_\beta$ states ($\beta > 2$) correspond 1-to-1 to probability measures on $\Gamma\backslash Y^\times$, and therefore the space of extreme points of the Choquet simplex of KMS$_\beta$ states is precisely the space $\Gamma\backslash Y^\times$ of invertible $\mathbb{Q}$-lattices. But one small step further allows us to express ϕ as an integral over the extremal KMS$_\beta$ states corresponding to the invertible $\mathbb{Q}$-lattices $y \in \Gamma\backslash Y^\times$, in terms of the measure $d\lambda_\phi$ that we started with in Eq. (3.11). Indeed, we simply rewrite the conclusion of Eq. (3.13) as

$$\begin{aligned} \phi(f) &= \sum_{m\in\Gamma\backslash M_2^+(\mathbb{Z})} (\det m)^{-\beta} \int_{\Gamma\backslash Y^\times} f(1,my)\, \frac{d\Omega_\phi(y)}{\zeta(\beta)\zeta(\beta-1)} \\ &= \int_{\Gamma\backslash Y^\times} \phi_{\beta,y}(f)\, d\Omega_\phi(y), \end{aligned}$$

where $d\Omega_\phi = \zeta(\beta)\zeta(\beta-1)\, d\lambda_\phi(1,y)$ is a probability measure on $\Gamma\backslash Y^\times$. Note that the factor

$$\zeta(\beta)\zeta(\beta-1)$$

is the partition function.

Finally, identifying the space $\Gamma\backslash Y^\times$ of (isomorphism classes of) invertible $\mathbb{Q}$-lattices with the Shimura variety

$$\mathrm{Sh}(\mathrm{GL}(2)/\mathbb{Q}, \mathbb{H}^\pm)(\mathbb{C}) = G(\mathbb{Q})\backslash G(\mathbb{A})/\mathbb{C}^\times = G(\mathbb{Q})\backslash G(\mathbb{A}_\mathrm{f}) \times \mathbb{H}^\pm$$

is a straightforward matter of showing that the mapping

$$Y^\times = G(\hat{\mathbb{Z}}) \times \mathbb{H} \to G(\mathbb{Q})\backslash G(\mathbb{A}_\mathrm{f}) \times \mathbb{H}^\pm, \quad (\rho,\tau) \mapsto [\rho,\tau]$$

induces an isomorphism $\Gamma\backslash Y^\times \cong \mathrm{Sh}(G,\mathbb{H}^\pm)(\mathbb{C})$. Again, one uses the decomposition $G(\mathbb{A}_\mathrm{f}) = G^+(\mathbb{Q})G(\hat{\mathbb{Z}})$ to get surjectivity, while injectivity follows from the equality $\Gamma = G(\hat{\mathbb{Z}}) \cap G^+(\mathbb{Q})$.

Adèlic Formulation of the GL(2) System

Remarkably, an almost direct copy of the adèlic formulation of the Bost-Connes groupoid will yield the correct adèlic formulation of the Connes-Marcolli (stack-)groupoid. We will merely state the result here, for the proof will be given later in Section 4.1.

Proposition 3.6. *Let $G = \mathrm{GL}_{2,\mathbb{Q}}$, and let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. The natural inclusion of groupoids*

$$\mathcal{G}_2 = G^+(\mathbb{Q}) \ltimes \big(M_2(\hat{\mathbb{Z}}) \times \mathbb{H}\big) \hookrightarrow G(\mathbb{A}_\mathrm{f}) \ltimes \big(M_2(\hat{\mathbb{Z}}) \times \mathrm{Sh}(G,\mathbb{H}^\pm)\big)$$

induces an equivalence of stack-groupoids

$$\Gamma\backslash\mathcal{G}_2/\Gamma \xrightarrow{\sim} G(\hat{\mathbb{Z}})\backslash G(\mathbb{A}_\mathrm{f}) \ltimes \big(M_2(\hat{\mathbb{Z}}) \times \mathrm{Sh}(G,\mathbb{H}^\pm)\big)/G(\hat{\mathbb{Z}}),$$

where the LHS is the Connes-Marcolli stack-groupoid.

Note the similarity to Proposition 3.2.

3.2 Reductive Monoids

Before moving on to the actual construction of the Shimura-variety analogue of the Bost-Connes-Marcolli systems, we make a small detour into the field of reductive monoids, which are, roughly speaking, the monoidal version of reductive groups. We do not go very far into the thick of this fascinating (somewhat obscure) theory, since a cursory review will suffice for our purposes in the next section. Reductive monoids will be used to build groupoids and noncommutative algebras for Shimura varieties of general reductive groups, following the pattern seen in the last section for the Shimura varieties for GL_1 and GL_2. In the GL_2 case, for example, we saw that the monoid M_2 of 2-by-2 matrices keeps track of *degenerate* rational lattice structures for elliptic curves, and that the space of such degenerations is best described as a noncommutative space (i.e., using a noncommutative algebra of functions). For a general reductive group, we therefore seek an object that plays the role that M_2 does in the GL_2 case.

3.2.1 From Algebraic Groups to Algebraic Monoids

First, an obvious definition: An *algebraic monoid* is an affine algebraic variety which is at the same time a monoid, i.e., a semigroup with unit.

Definition 3.3. Let G be connected reductive group over $\mathbb{Q}$. A *monoid augmentation* of G is an irreducible normal algebraic monoid M over $\mathbb{Q}$ such that $M^\times = G$. We also say that M is a *reductive monoid* (this being the standard terminology).

Remark 3.5. For linear algebraic groups, connectedness coincides with irreducibility, and normality (as an affine variety) automatically holds. However, for linear algebraic *monoids*, this is no longer so, and therefore we need to incorporate irreducibility and normality hypotheses into our definition if we want a monoid that to some extent resembles the group that it "envelopes".

3.2.2 Drinfeld's Classification

In this section we will assume that the field over which our groups are defined is of characteristic 0 and also algebraically closed.

Reductive monoids have been classified by V. Drinfeld. Choose a maximal torus $T \subset G$ and a Borel subgroup $B \supset T$. Let W denote the Weyl group for (G, B, T), and let $X = \mathrm{Hom}(T, \mathbb{G}_{\mathrm{m}})$.

Theorem 3.4 (Drinfeld)**.** *There exists a bijection between*

1. *the set of normal (affine) irreducible monoids M containing G as their group of units, and*
2. *the set of W-invariant rational polyhedral convex cones $K \subset X \otimes_{\mathbb{Z}} \mathbb{R}$ which contain zero and are non-degenerate, i.e., are not contained in a hyperplane.*

This classification implies that a semisimple group G has only one monoid augmentation, namely G itself. This case is for us not very interesting (because, as we will see in Section 3.3.7, a Bost-Connes like system with such a monoid augmentation has a trivial zeta function) and we would like to construct more interesting monoids, in particular, we would like to construct cartesian diagrams

$$\begin{array}{ccc} G & \xrightarrow{\phi} & \mathrm{GL}(V) \\ \downarrow & & \downarrow \\ M & \longrightarrow & \mathrm{End}(V) \end{array}$$

for some fixed representation $\phi \colon G \to \mathrm{GL}(V)$.

For example, for an adjoint Shimura datum (G, X) (i.e., $ZG = \{1\}$), the triviality of the monoid augmentation implies that the BCM systems that we will construct have a trivial partition function. It is then interesting to construct another Shimura datum with adjoint datum (G, X) and such that the monoid augmentation is no longer trivial.

There is a natural method due to Vinberg to "enlarge the center" of a given semisimple simply connected group G in order to have a monoid augmentation that is universal in a certain sense. See [55], and also the recent survey book by Lex Renner [43], which is an excellent all-round reference for algebraic monoids.

3.2.3 Ramachandran's Construction of Chevalley Monoids

There is another way to construct monoid augmentations quite explicitly, which was communicated to us by N. Ramachandran. The construction of N. Ramachandran uses the following theorem of Chevalley (see [50], 5.1).

Theorem 3.5 (Chevalley)**.** *Let G be an algebraic group over $\mathbb{Q}$, and let $\phi : G \to \mathrm{GL}(V)$ be a faithful representation of G. There is a tensor construction $T^{i,j} := V^{\otimes i} \otimes V^{\vee,\otimes j}$ and a line $D \subset T^{i,j}$ such that $\phi(G) \subset \mathrm{GL}(V)$ is the stabilizer of this line.*

Definition 3.4. Let G be an algebraic group over $\mathbb{Q}$, $\phi : G \to \mathrm{GL}(V)$ be a faithful representation of G. Let T and D be as in Chevalley's theorem. Suppose that $T = V^{\otimes i}$ (resp. $T = V^{\vee,\otimes i}$) contains no (resp. only) dual tensor factors. The multiplicative monoid

$$\begin{gathered} M(G,V,T,D) := \{m \in \mathrm{End}(V) \mid m.D \subset D\} \\ (\text{resp. } M(G,V,T,D) := \{m \in \mathrm{End}(V) \mid {}^t m.D \subset D\}) \end{gathered}$$

is called a *Chevalley monoid augmentation* for G in $\mathrm{End}(V)$.

Example 3.1. If $G = \mathrm{GL}_2$ and V is the standard representation, then $D = \bigwedge^2 V \subset V^{\otimes 2}$ is a line as in Chevalley's theorem, and M_2 is the corresponding Chevalley monoid augmentation.

Example 3.2. Let (V, ψ) be a $2g$-dimensional $\mathbb{Q}$-vector space equipped with an alternating form $\psi \in \bigwedge^2(V^\vee)$. Then the line $D = \mathbb{Q}\langle\psi\rangle \subset V^{\vee,\otimes 2}$ is a line as in Chevalley's theorem for the group GSp_{2g} with its standard representation, and the points of the corresponding monoid in a commutative $\mathbb{Q}$-algebra A are given by

$$\begin{aligned} &\mathrm{MSp}_{2g}(A) \\ &:= \{m \in \mathrm{End}(V)(A) \mid \exists\, \mu(m) \in A,\ \psi(m.x, m.y) = \mu(m)\psi(x,y),\ \forall\, x, y \in V_A\}. \end{aligned}$$

3.3 Bost-Connes-Marcolli Systems for Shimura Varieties

3.3.1 The Basic Algebraic Setup

In order to define a generalization of the Connes-Marcolli algebra to general Shimura data, we want to make clear the separation between algebraic and level structure data, which is already implicit in the construction of Connes and Marcolli.

Monoidal Augmentation of Shimura Data

Definition 3.5. A *BCM datum* is a tuple $\mathcal{M} = (G, X, V, M)$ with (G, X) a Shimura datum, V a faithful representation of G, and M a monoid augmentation for G contained in $\mathrm{End}(V)$.

Notation 3.1. The faithful representation will often be denoted $\phi \colon G \to \mathrm{GL}(V)$.

Level structure data Every Shimura datum (G, X) comes implicitly with a family of level structures given by the family of compact open subgroups $K \subset G(\mathbb{A}_{\mathrm{f}})$. Connes and Marcolli fixed the full level structure $\mathrm{GL}_2(\hat{\mathbb{Z}}) \subset \mathrm{GL}_2(\mathbb{A}_{\mathrm{f}})$ as starting datum for their construction. To avoid the problem they had with stack singularities of their groupoid, we will fix a neat level structure as part of the datum.

The level structure also plays a role in defining the partition function of our system. Consideration of maximal level structures then yields standard zeta functions as partition functions, for example, the Dedekind zeta function of a number field. A technical requirement in the definition of the partition function is the choice of a lattice in the representation of G, which enables us to define a rational determinant for the adèlic matrices in play.

Definition 3.6. Let $\mathcal{M} = (G, X, V, M)$ be a BCM datum. A *level structure on* $\mathcal{M}$ is a triple $\mathcal{L} = (L, K, K_M)$, with $L \subset V$ a lattice, $K \subset G(\mathbb{A}_{\mathrm{f}})$ a compact open subgroup, and $K_M \subset M(\mathbb{A}_{\mathrm{f}})$ a compact open submonoid, such that

- K_M stabilizes $L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$,
- $\phi(K)$ is contained in K_M.

The datum

$$\mathcal{D} = (G, X, V, M; L, K, K_M)$$

will be called a *BCM datum with level structure.*

We can summarize the relation between L, K and K_M by the following diagram:

$$\begin{tikzcd} K \arrow[r, "\phi"] \arrow[d, hook] & K_M \arrow[r, hook] \arrow[d, hook] & \mathrm{End}(L)(\hat{\mathbb{Z}}) \arrow[d, hook] \\ G(\mathbb{A}_{\mathrm{f}}) \arrow[r, "\phi"] & M(\mathbb{A}_{\mathrm{f}}) \arrow[r, hook] & \mathrm{End}(V)(\mathbb{A}_{\mathrm{f}}) \end{tikzcd}$$

Definition 3.7. The *maximal level structure* $\mathcal{L}_0 = (L, K_0, K_{M,0})$ associated with a datum $\mathcal{M} = (G, X, V, M)$ and a lattice $L \subset V$ is defined by setting

$$\begin{array}{rcc} K_{M,0} & := & M(\mathbb{A}_{\mathrm{f}}) \cap \mathrm{End}(L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}), \\ K_0 & := & \phi^{-1}(K_{M,0}^{\times}). \end{array}$$

Definition 3.8. The level structure $\mathcal{L}$ on $\mathcal{M}$ is called *fine* if K acts freely on $G(\mathbb{Q})\backslash X \times G(\mathbb{A}_{\mathrm{f}})$.

The maximal level structure is usually not neat enough to avoid stack singularity problems in the generalization of the Connes-Marcolli algebra. This is why we introduce the additional data of a compact open subgroup $K \subset K_M$. For example, for the Connes-Marcolli case, one takes $K = \mathrm{GL}_2(\hat{\mathbb{Z}})$, $K_M = M_2(\hat{\mathbb{Z}})$, but the fact that this choice of K is not neat implies that the groupoid we introduce in the next section has stack singularities. Thus we instead choose a smaller $K = K(N) \subset \mathrm{GL}_2(\hat{\mathbb{Z}})$ given by the kernel of the mod N reduction of matrices.

3.3.2 The BCM Groupoid

Let $\mathcal{D} = (G, X, V, M; L, K, K_M)$ be a BCM datum with level structure. There are left and right actions of $G(\mathbb{A}_{\mathrm{f}})$ on $M(\mathbb{A}_{\mathrm{f}})$.

Definition

Connes and Marcolli remarked in [11] that, if we want to take a quotient of a groupoid by a group action, it is essential that the action is free on the unit space of the groupoid. If we take the usual quotient set of a groupoid by an action that is not free on the unit space, this will not give a groupoid. We are thus obliged to use unit spaces that are in fact stacks. Some of them have nice singularities (i.e., those with finite stabilizers), though others don't. The language of stacks allows one to work in full generality without bothering about the freeness of actions involved.

Notation 3.2. As in Section 2.5, we denote stacks by German letters, and we denote the corresponding coarse spaces by roman letters.

Let

$$Y_{\mathcal{D}} = K_M \times \mathrm{Sh}(G, X).$$

We denote points of $Y_{\mathcal{D}}$ by triples $y = (\rho, [z, l])$ with $\rho \in K_M$, $[z, l] \in \mathrm{Sh}(G, X)$.

We want to study the equivalence relation on $Y_{\mathcal{D}}$ given by the following partially defined action of $G(\mathbb{A}_{\mathrm{f}})$:

$$g.y = (g\rho, [z, lg^{-1}]), \quad \text{where} \quad y = (\rho, [z, l]).$$

This equivalence relation will be called the *commensurability relation*. This terminology is derived from the notion of commensurability for $\mathbb{Q}$-lattices, cf. [11].

Consider the subspace

$$U_{\mathcal{D}} \subset G(\mathbb{A}_{\mathrm{f}}) \times Y_{\mathcal{D}}$$

of pairs (g, y) such that $gy \in Y_{\mathcal{D}}$, i.e., $g\rho \in K_M$.

The space $U_{\mathcal{D}}$ is a groupoid with unit space $Y_{\mathcal{D}}$. The source and target maps $s\colon U_{\mathcal{D}} \to Y_{\mathcal{D}}$ and $t\colon U_{\mathcal{D}} \to Y_{\mathcal{D}}$ are given by $s(g, y) = y$ and $t(g, y) = gy$. The composition is given, for $y_1 = g_2 y_2$, by $(g_1, y_1) \circ (g_2, y_2) = (g_1 g_2, y_2)$. Notice that the groupoid obtained by restricting this groupoid to the $(g, (\rho, [z, l]))$ such that ρ is invertible is free, i.e., the equality $t(g, y) = s(g, y)$ implies $g = 1$.

There is a natural action of K^2 on the groupoid $U_{\mathcal{D}}$, given by

$$(g, y) \mapsto (\gamma_1 g \gamma_2^{-1}, \gamma_2 y), \tag{3.14}$$

and the induced action on $Y_{\mathcal{D}}$ is given by

$$y \mapsto \gamma_2 y.$$

Remark 3.6. There are two motivations for quotienting $U_{\mathcal{D}}$ by this action. The first one is physical: it is necessary to obtain a reasonable partition function for our system. The second is moduli theoretic: $U_{\mathcal{D}}$ is only a pro-analytic groupoid and the quotient by K^2 is fibered over the Shimura variety $\mathfrak{Sh}_K(G,X)$ which is an algebraic moduli stack of finite type whose definition could be made over $\bar{\mathbb{Q}}$, at least when (G,X) is classical and the Shimura variety has a canonical model.

Let $\mathfrak{Z}_{\mathcal{D}}$ be the quotient stack $[K^2 \backslash U_{\mathcal{D}}]$ and $\mathfrak{S}_{\mathcal{D}}$ be the quotient stack $[K \backslash Y_{\mathcal{D}}]$. The natural maps

$$s, t \colon \mathfrak{Z}_{\mathcal{D}} \to \mathfrak{S}_{\mathcal{D}}$$

define a stack-groupoid structure (see Section 2.3) on $\mathfrak{Z}_{\mathcal{D}}$ with unit stack $\mathfrak{S}_{\mathcal{D}}$.

Definition 3.9. The stack-groupoid $\mathfrak{Z}_{\mathcal{D}}$ is called the *Bost-Connes-Marcolli*[3] groupoid.

Let $Z_{\mathcal{D}} := K^2 \backslash U_{\mathcal{D}}$ be the (classical, i.e., coarse) quotient of $U_{\mathcal{D}}$ by the action of K^2. If K is small enough, i.e., if K acts freely on $G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_{\mathrm{f}})$, then $\mathfrak{Z}_{\mathcal{D}}$ is equal to the classical quotient $Z_{\mathcal{D}}$, which is a groupoid in the usual sense, with units $S = K \backslash Y_{\mathcal{D}}$. Otherwise, suppose that there exists a compact open subgroup $K' \subset K$ that acts freely on $G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_{\mathrm{f}})$ and choose on (G, V, X, M) the level structure $\mathcal{L}' = (L, K', K_M)$. The stack $\mathfrak{Z}_{\mathcal{D}'}$ is a usual topological space that is a finite covering of the coarse space $Z_{\mathcal{D}}$ and such that the stack $\mathfrak{Z}_{\mathcal{D}}$ is the stacky quotient of $Z_{\mathcal{D}'}$ by the projection equivalence relation to $Z_{\mathcal{D}}$.

The reader who prefers to work with usual analytic spaces may thus suppose that K is small enough, but as we remarked before, our basic examples (number fields) do not fulfill this hypothesis! We have also to recall that for nonclassical Shimura data (G,X) in the sense of Section 2.4, there exists no such small enough $K \subset G(\mathbb{A}_{\mathrm{f}})$. This is essentially due to the fact that the "unit group" $Z(\mathbb{Q}) \cap K$ (where Z denotes the center of G) can be infinite.

[3] We will often call it the BCM groupoid, for short.

3.3.3 Stack-Groupoids and the Equivariant Category

We have already explained the general notion of a stack-groupoid in Section 2.5, but here we want to give a more "efficient" alternative description that's possible in the case of stack-groupoids that arise as group-action-quotients of (ordinary) groupoids.

We start with a definition of the relevant category. Let (OSpace) be the category of "spaces with group operation", i.e., pairs (G, X) composed of a topological space X and a group G that acts on X. We also call this the *equivariant category*, which is perhaps more suggestive. A morphism $\phi = (\phi_1, \phi_2) \colon (G_1, X_1) \to (G_2, X_2)$ between two such pairs is a pair composed of a group morphism $\phi_1 \colon G_1 \to G_2$ and a space morphism $\phi_2 \colon X_1 \to X_2$ such that

$$\phi_2(g_1.x_1) = \phi_1(g_1).\phi_2(x_1),\ \forall (g_1, x_1) \in G_1 \times X_1.$$

One can define the notion of groupoid in the category (OSpace). This is the datum of a tuple $((G_1, X_1), (G_0, X_0), s, t, \epsilon, m)$ fulfilling some natural conditions that we will not write explicitly here, because we prefer the geometrical language of stacks. There is a relation between these two languages, which is given by a natural functor called "stacky quotient". Thus one can naturally associate to a groupoid in (OSpace) a stack groupoid.

Now let $\mathcal{D}$ be a BCM datum with level structure, and let

$$(U_{\mathcal{D}}, Y_{\mathcal{D}}, s, t, \epsilon, m)$$

be the groupoid defined in Section 3.3.2. Recall that there is a natural action of K^2 on the groupoid $U_{\mathcal{D}}$, given by

$$(g, y) \mapsto (\gamma_1 g \gamma_2^{-1}, \gamma_2 y).$$

There is also a natural action of K on $Y_{\mathcal{D}}$ given by

$$y \mapsto \gamma y.$$

Let $s_K \colon K^2 \to K, (\gamma_1, \gamma_2) \mapsto \gamma_2$ and $t_K \colon K^2 \to K, (\gamma_1, \gamma_2) \mapsto \gamma_1$ be the two projections. Then the morphisms in (OSpace) given by

$$(s, s_K), (t, t_K) \colon (K^2, U_{\mathcal{D}}) \to (K, Y_{\mathcal{D}})$$

are called the *equivariant source* and *target*, respectively. The fiber product

$$(K^2, U_{\mathcal{D}}) \underset{(s,s_K),(K,Y_{\mathcal{D}}),(t,t_K)}{\times} (K^2, U_{\mathcal{D}})$$

is naturally isomorphic to the (OSpace) object

$$(K^2 \underset{s_K,K,t_K}{\times} K^2, U_{\mathcal{D}} \underset{s,Y_{\mathcal{D}},t}{\times} U_{\mathcal{D}}).$$

This means that m (groupoid multiplication) induces a natural multiplication map

$$m_e = (m_K, m)\colon (K^2, U_{\mathcal{D}}) \underset{(s,s_K),(K,Y_{\mathcal{D}}),(t,t_K)}{\times} (K^2, U_{\mathcal{D}}) \to (K^2, U_{\mathcal{D}})$$

in this equivariant setting, where the map

$$m_K\colon K^2 \underset{s_K,K,t_K}{\times} K^2 \to K^2$$

is given by $m(\gamma_1, \gamma_2, \gamma_2, \gamma_3) = (\gamma_1, \gamma_3)$.

Passing to the stacky quotient, we obtain the multiplication map

$$m\colon \mathfrak{Z}_{\mathcal{D}} \underset{\mathfrak{S}_{\mathcal{D}}}{\times} \mathfrak{Z}_{\mathcal{D}} \to \mathfrak{Z}_{\mathcal{D}}$$

on the stack groupoid of Section 3.3.2. Observe that if the action of K on $Y_{\mathcal{D}}$ is free, then we obtain an ordinary groupoid.

3.3.4 The Commensurability Class Map

Notation 3.3. Recall that ϕ is the representation of G, which we view as the natural inclusion $G \hookrightarrow M$ (for a given monoidal augmentation of G).

For classical Shimura data

We want to give an explicit description of the quotient of $Y_{\mathcal{D}}$ by the commensurability equivalence relation, in the case where (G, X) is classical, i.e., when

$$\mathrm{Sh}(G, X) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_{\mathrm{f}}).$$

Let $K_M A = \phi(G)(\mathbb{A}_\mathrm{f}).K_M \subset M(\mathbb{A}_\mathrm{f})$. There is a natural surjective map of sets

$$\pi\colon Y_\mathcal{D} \to G(\mathbb{Q})\backslash X \times K_M A$$

given by $\pi(\rho, [z, l]) = [z, l\rho]$.

Let $Y_\mathcal{D}^\times = K_M^\times \times (G(\mathbb{Q})\backslash X \times G(\mathbb{A}_\mathrm{f}))$ be the invertible part of $Y_\mathcal{D}$ and let $Z_\mathcal{D}^\times \subset \mathfrak{Z}_\mathcal{D}$ be the corresponding subspace (which is a groupoid in the usual sense because K acts freely on $K_M^\times$); that is, $Z_\mathcal{D}^\times$ is defined just as $\mathfrak{Z}_\mathcal{D}$ is, but with $Y_\mathcal{D}^\times$ in place of $Y_\mathcal{D}$. Let $S_\mathcal{D}^\times := K\backslash Y_\mathcal{D}^\times$ be the unit space of $Z_\mathcal{D}^\times$. Since $K_M^\times \subset M^\times(\mathbb{A}_\mathrm{f}) = \phi(G(\mathbb{A}_\mathrm{f}))$, the map π induces a natural map

$$\begin{array}{cccc} \pi^\times\colon & Y_\mathcal{D}^\times & \to & G(\mathbb{Q})\backslash X \times G(\mathbb{A}_\mathrm{f}) \\ & (\rho, [z, l]) & \mapsto & [z, l\phi^{-1}(\rho)], \end{array}$$

which is complex analytic (for the natural analytic structures induced by the complex structure on X) and surjective. Both π and $\pi^\times$ factor through the quotient of their sources by the left action of K. We will continue to denote this factorization by π and $\pi^\times$.

Definition 3.10. The maps π and $\pi^\times$ are called the *commensurability class maps.*

The last definition is justified by the following lemma. The notion of coarse quotient can be found in Definition 2.5.

Lemma 3.1. *The maps π and $\pi^\times$ are in fact the coarse quotient maps for the groupoids $\mathfrak{Z}_\mathcal{D}$ and $Z_\mathcal{D}^\times$ acting on their unit spaces $\mathfrak{S}_\mathcal{D}$ and $S_\mathcal{D}^\times$.*

Proof. If $(g, \rho, [z, l]) \in U_\mathcal{D}$, then $\pi(g\rho, [z, lg^{-1}]) = [z, l\rho] = \pi(\rho, [z, l])$ which proves that π factors through

$$|\mathfrak{S}_\mathcal{D}/\mathfrak{Z}_\mathcal{D}| \to G(\mathbb{Q})\backslash X \times K_M A.$$

This surjective map is an isomorphism. Indeed, if $(\rho, [z, l]), (\rho', [z', l']) \in Y_\mathcal{D}$ have same image under π, then there exists $g \in G(\mathbb{Q})$ such that $gl\rho = l'\rho'$

and $gz = z'$. We then know that in the quotient space $|\mathfrak{S}_{\mathcal{D}}/\mathfrak{Z}_{\mathcal{D}}|$,

$$\begin{array}{rcl}(\rho,[z,l]) & = & (l^{-1}g^{-1}gl\rho,[z,l]) \\ & = & (l^{-1}g^{-1}l'\rho',[z,l]) \\ & \sim & (l'^{-1}gll^{-1}g^{-1}l'\rho',[z,ll^{-1}g^{-1}l']) \\ & = & (\rho',[z,g^{-1}l']) \\ & = & (\rho',[gz,l']) \\ & = & (\rho',[z',l']).\end{array}$$

This proves injectivity of π and surjectivity was already known. The argument for $\pi^\times$ is completely analogous. $\square$

For commutative Shimura data

Commutative Shimura data form another family of examples for which we can construct the commensurability class map in simple terms. The multiplicative datum of a number field belongs to this family. Thus we now suppose that $\mathcal{M} = (G, X, V, M)$ is a monoidally augmented Shimura datum such that G and M are commutative, and let $\mathcal{L}$ be a level structure on $\mathcal{M}$. For each $K', K \subset G(\mathbb{A}_{\mathrm{f}})$ compact open, there is a natural map

$$\mathfrak{Y}_{K'} := K_M \times \mathfrak{Sh}_{K'}(G,X) \longrightarrow [G(\mathbb{Q})\backslash X \times M(\mathbb{A}_{\mathrm{f}})/K']$$

given by $(\rho,[z,l]) \mapsto [z,l\rho]$. This map is K-equivariant for the trivial action of K on the range because the image of $k.(\rho,[z,l]) = (k\rho,[z,lk^{-1}])$ is equal to the image of $(\rho,[z,l])$. Recall that $\mathfrak{S}_{\mathcal{D}} := [K\backslash Y_{\mathcal{D}}]$ and $S_{\mathcal{D}}^\times = K\backslash Y_{\mathcal{D}}^\times$. If we pass to the limit on $K' \subset G(\mathbb{A}_{\mathrm{f}})$, and then to the quotient by K, we obtain natural maps

$$\pi\colon \mathfrak{S}_{\mathcal{D}} \longrightarrow \varprojlim_{K'} [G(\mathbb{Q})\backslash X \times M(\mathbb{A}_{\mathrm{f}})/K']$$

and

$$\pi^\times\colon S_{\mathcal{D}}^\times \longrightarrow \mathrm{Sh}(G,X).$$

As before, we will call them *commensurability class maps.*

The image of the map π is, as before, the coarse quotient for the action of the groupoid $\mathfrak{Z}$ on its unit space $\mathfrak{S}$.

Let

$$\mathfrak{S}_{K'} := K\backslash(K_M \times \mathfrak{Sh}_{K'}(G,X)).$$

We should remark here that in the commutative case that we have been considering, the space $\mathfrak{S}_{K'}$ is the unit space of a well defined groupoid $\mathfrak{Z}_{K'}$ because the $G(\mathbb{A}_{\mathrm{f}})$ action on $\mathfrak{Sh}_{K'}(G,X)$ is well defined. This shows that

$$\mathfrak{Z} = \varprojlim_{K'} \mathfrak{Z}_{K'}, \tag{3.15}$$

which will be useful for the description of the symmetries of Bost-Connes systems for number fields.

3.3.5 The Bost-Connes-Marcolli Algebra

Functions on BCM Stacks?

Let $\mathcal{M} = (G,X,V,M)$ be a BCM datum, and let $\mathcal{L}_0$ be the associated maximal level structure (Definition. 4.1.4). We would like to define the BCM algebra of $\mathcal{D}_0 = (\mathcal{M};\mathcal{L}_0)$ as a groupoid algebra. Unfortunately, the corresponding groupoid is usually only a stack and there is no canonical notion of continuous functions on such a space. More precisely, if a stack has some nontrivial isotropy group, Connes' philosophy of noncommutative geometry tells us that the "algebra of functions" on it should include this isotropy information in a nontrivial way, and this algebra depends on a presentation of the stack.

If (G,X) is *classical*, there is a very natural way to resolve the stack singularities of $\mathfrak{Z}_{\mathcal{D}_0}$ by choosing a neat level structure $\mathcal{L}$, for which the projection map

$$Z_{\mathcal{D}} \longrightarrow \mathfrak{Z}_{\mathcal{D}_0}$$

is such a resolution, where $\mathcal{D} = (\mathcal{M};\mathcal{L})$. The corresponding convolution algebra of the groupoid $Z_{\mathcal{D}}$ is a completely natural replacement for the groupoid algebra of the stack-groupoid $\mathfrak{Z}_{\mathcal{D}_0}$.

If (G,X) is *nonclassical*, there is no nice resolution of the stack singularities of $\mathfrak{Z}_{\mathcal{D}_0}$. We will thus work with the algebra of functions on the coarse quotient $Z_{\mathcal{D}_0}$. However $Z_{\mathcal{D}_0}$ is *not* a groupoid, and so to define a convolution algebra from the function algebra $C_{\mathrm{c}}(Z_{\mathcal{D}_0})$, we use the trick used by Connes and Marcolli in [11, 1.83] which consists in introducing $G(\mathbb{R})$. Namely, we introduce the groupoid

$$\mathcal{R}_{\mathcal{D}_0} \subset K\backslash G(\mathbb{A}_{\mathrm{f}}) \times_K \left(K_M \times \varprojlim_{K'} G(\mathbb{Q})\backslash G(\mathbb{A})/K'\right)$$

of the partial action of $G(\mathbb{A}_{\mathrm{f}})$ on $K_M \times \varprojlim_{K'} G(\mathbb{Q})\backslash G(\mathbb{A})/K'$ modulo the $K \times K$-action (as in (3.14)); here K' runs over compact open subgroups of $G(\mathbb{A}_{\mathrm{f}})$. Now we identify $C_{\mathrm{c}}(Z_{\mathcal{D}_0})$ with the subalgebra of $C_{\mathrm{c}}(\mathcal{R}_{\mathcal{D}})$ obtained by composing by the projection map $\mathcal{R}_{\mathcal{D}_0} \to Z_{\mathcal{D}_0}$. Since $\mathcal{R}_{\mathcal{D}_0}$ is a groupoid, convolution can be defined on $C_{\mathrm{c}}(Z_{\mathcal{D}})$.

Note that this solution, even if not completely satisfactory from the geometrical viewpoint (because we work on coarse quotients), suffices (and seems to be necessary) for the physical interpretation, i.e., analysis of KMS states.

Now we give the precise definition of the algebra alluded to in the previous paragraph. Let $\mathcal{D} = (G, X, V, M; L, K, K_M)$ be a BCM datum with level structure. Let

$$\mathcal{H}_{\mathcal{D}} := C_{\mathrm{c}}(Z_{\mathcal{D}})$$

be the algebra of compactly supported continuous functions on $Z_{\mathcal{D}}$. As in [11, p. 44] in order to define the convolution of two functions, we consider functions on $Z_{\mathcal{D}}$ as functions on $U_{\mathcal{D}}$ satisfying the following properties:

$$f(\gamma g, y) = f(g, y), \quad f(g\gamma, y) = f(g, \gamma y), \quad \forall \gamma \in K, \quad g \in G(\mathbb{A}_{\mathrm{f}}), \quad y \in Y_{\mathcal{D}}.$$

The convolution product on $\mathcal{H}_{\mathcal{D}}$ is then defined by the expression

$$(f_1 * f_2)(g, y) := \sum_{h \in K\backslash G(\mathbb{A}_{\mathrm{f}}),\, hy \in Y_{\mathcal{D}}} f_1(gh^{-1}, hy) f_2(h, y),$$

and the adjoint by

$$f^*(g, y) := \overline{f(g^{-1}, gy)}.$$

The fact that we consider functions with compact support implies that the sum defining the convolution product is finite.

Definition 3.11. The algebra $\mathcal{H}_{\mathcal{D}}$ (under the convolution product) is called the *Bost-Connes-Marcolli algebra* of the level-structure BCM datum $\mathcal{D}$.

Remark 3.7. We proved in Lemma 3.1 that, if (G, X) is classical, the quotient of $Y_{\mathcal{D}}$ by the commensurability equivalence relation (encoded by the action of the groupoid $Z_{\mathcal{D}}$) does not depend on the choice of K. This implies that in the classical case, the Morita equivalence class of $\mathcal{H}_{\mathcal{D}}$ is independent of the choice of neat level structure K. More precisely, all these algebras are in fact Morita equivalent to the algebra corresponding to the "noncommutative quotient"

$$G(\mathbb{Q})\backslash X \times K_{\mathbb{A}}^{M}, \quad \text{where} \quad K_{\mathbb{A}}^{M} = G(\mathbb{A}_{\mathrm{f}})K_{M}.$$

The BCM C*-Algebra

We now introduce the associated C*-algebra.

Let $\mathcal{D} = (G, X, V, M; L, K, K_M)$ be a BCM datum with neat level structure. On the algebra $\mathcal{H}_{\mathcal{D}}$, we put the following norm: for every $f \in \mathcal{H}_{\mathcal{D}}$, we let

$$\|f\| := \sup_{y \in Y_{\mathcal{D}}} \|\pi_y(f)\|\,. \tag{3.16}$$

Lemma 3.2. *The norm* (3.16) *defines a C*-norm on* $\mathcal{H}_{\mathcal{D}}$*, i.e.,* $\|f^*f\| = \|f\|^2$.

Proof. Indeed, it is easy to check that this is a seminorm satisfying the C*-condition (Definition 2.1): observe that for arbitrarily small $\epsilon > 0$ there is a y such that $\|f\|^2 - \epsilon = \|\pi_y(f)\|^2$. We then have

$$\|f^*f\| \geq \|\pi_y(f^*f)\| = \|\pi_y(f)\|^2 = \|f\|^2 - \epsilon,$$

which of course means that $\|f^*f\| \geq \|f\|^2$. This inequality is easily shown to imply the C*-condition.

That we get a norm (i.e., $\|f\| = 0$ only when $f = 0$), and not just a seminorm, follows from the fact that $f(g, y) \neq 0$ implies that $\pi_y(f) \neq 0$:

$$\langle \pi_y(f)\varepsilon_g, \varepsilon_g\rangle = f(1, gy) = f(g, y) \neq 0.$$

Here $\epsilon_g \in \mathfrak{H}_y$ is the unit vector which takes the value 1 at g, and 0 elsewhere. $\square$

Definition 3.12. The completion of $\mathcal{H}_{\mathcal{D}}$ under the norm $\|\cdot\|$ is denoted $A_{\mathcal{D}}$ and called the *BCM C*-algebra*.

3.3.6 The Time Evolution and Partition Function of the BCM System

Let $\mathcal{D} = (G, X, V, M; L, K, K_M)$ be a BCM datum with neat level.

Definition 3.13. The *time evolution* on $\mathcal{H}_\mathcal{D}$ is defined by

$$\sigma_t(f)(g, y) = \det(\phi(g))^{it} f(g, y). \tag{3.17}$$

Let $y = (\rho, [z, l])$ be in $Y_\mathcal{D}$ and let

$$G_y = \{\, g \in G(\mathbb{A}_\mathrm{f}) \mid g\rho \in K_M \,\} .$$

Let $\mathfrak{H}_y$ be the Hilbert space $\ell^2(K \backslash G_y)$.

Definition 3.14. The representation $\pi_y \colon \mathcal{H}_\mathcal{D} \to B(\mathfrak{H}_y)$ of the Hecke algebra on $\mathfrak{H}_y$ is defined by

$$(\pi_y(f)\xi)(g) := \sum_{h \in K \backslash G_y} f(gh^{-1}, hy)\xi(h), \quad \text{for all } g \in G_y,$$

for $f \in \mathcal{H}_\mathcal{D}$ and $\xi \in \mathfrak{H}_y$.

Lemma 3.3. *The representation π_y is well defined, i.e., $\pi_y(f)$ is bounded for each $f \in \mathcal{H}_\mathcal{D}$.*

Proof. For $f \in \mathcal{H}_\mathcal{D}$, We want to prove that the norm

$$\|\pi_y(f)\| := \sup_{\|\xi\|=1} \|\pi_y(f)\xi\|_2$$

is bounded. This follows from the fact that the functions we consider have compact support. More precisely, denote $Z := Z_\mathcal{D}$. Given $f \in \mathcal{H}_\mathcal{D} = C_\mathrm{c}(Z)$, we need to show that there is a bound $C > 0$ such that for every pair of vectors $\xi, \eta \in \mathcal{H}_y$ we have

$$|\langle \pi_y(f)\xi, \eta \rangle| \le C \,\|\xi\|\, \|\eta\| \,.$$

To this end, we introduce the following notation. We set

$$S_y = \{\, [gh^{-1}, hy] \in Z \mid g, h \in K \backslash G_y \,\},$$

and for each $\gamma \in S_y$ we set

$$R_y(\gamma) = \{\, \gamma' \in Z_y \mid s(\gamma') = t(\gamma) \,\}.$$

These are discrete sets. Here we use the usual notation for groupoids, namely $Z_y = t^{-1}\{y\}$, which we shall identify with $K\backslash G_y$.

Using the Cauchy-Schwarz inequality, we now get a bound on $|\langle \pi_y(f)\xi, \eta\rangle|$ as follows:

$$\begin{aligned}
|\langle \pi_y(f)\xi, \eta\rangle| &\leq \sum_{\gamma_1 \in Z_y} \left|\big(\pi_y(f)\xi\big)(\gamma_1)\overline{\eta(\gamma_1)}\right| \\
&\leq \sum_{\gamma_1,\gamma_2 \in Z_y} \left|f(\gamma_1\gamma_2^{-1})\xi(\gamma_2)\overline{\eta(\gamma_1)}\right| \\
&= \sum_{\gamma \in S_y} |f(\gamma)| \sum_{\gamma' \in R_y(\gamma)} |\xi(\gamma')\eta(\gamma\gamma')| \\
&\leq \sum_{\gamma \in S_y} |f(\gamma)| \Big(\sum_{\gamma' \in R_y(\gamma)} |\xi(\gamma')|^2\Big)^{\frac{1}{2}} \Big(\sum_{\gamma' \in R_y(\gamma)} |\eta(\gamma\gamma')|^2\Big)^{\frac{1}{2}} \\
&\leq \|\xi\|\,\|\eta\| \sum_{\gamma \in S_y} |f(\gamma)|.
\end{aligned}$$

Because f has compact support, the sum $\sum_{\gamma \in S_y} |f(\gamma)|$ is finite, and we thereby get the desired bound. $\square$

Now let's consider the setting of maximal level structure, i.e., let $K_0 = \phi^{-1}(K_M^\times)$, cf. Definition 3.7.

We view the Hamiltonian as a virtual operator on $\ell^2(K_0\backslash G_y)$. By this we mean that the Hamiltonian does not depend on the choice of K and there is a minimal space on which it is defined: the space

$$\ell^2(K_0\backslash G_y).$$

Consequently, its trace must be computed as a virtual (i.e., equivariant) trace, which is to say that it must be divided by $\#(K\backslash K_0)$.

These considerations are related to the fact that, if (G, X) is classical, we prefer to define BCM algebras using neat level structures in order to resolve the stack singularities of $\mathfrak{Z}_{\mathcal{D}}$.

Proposition 3.7. *The operator on $\mathfrak{H}_y$ given by*

$$(H_y\xi)(g) = \log\det(\phi(g)) \cdot \xi(g) \tag{3.18}$$

is the Hamiltonian, *i.e., the infinitesimal generator of the time evolution, meaning that we have the equality*

$$\pi_y(\sigma_t(f)) = e^{itH_y}\pi_y(f)e^{-itH_y} \tag{3.19}$$

for all $f \in \mathcal{H}_\mathcal{D}$.

Proof. This is just a matter of unwinding the definitions. Let $\xi \in \mathfrak{H}_y$, and let $g \in G_y$. On the one hand we have

$$\begin{aligned}\big(\pi_y(\sigma_t f)\xi\big)(g) &= \sum_{h\in K\backslash G_y} (\sigma_t f)(gh^{-1}, hy)\xi(h) \\ &= \sum_{h\in K\backslash G_y} \det(\phi(g))^{it}\det(\phi(h))^{-it} f(gh^{-1}, hy)\xi(h),\end{aligned}$$

while on the other hand we have

$$\begin{aligned}\big(e^{itH_y}(\pi_y f)e^{-itH_y}\xi\big)(g) &= \det(\phi(g))^{it}\big((\pi_y f)e^{-itH_y}\xi\big)(g) \\ &= \det(\phi(g))^{it} \sum_{h\in K\backslash G_y} f(gh^{-1}, hy)(e^{-itH_y}\xi)(h) \\ &= \det(\phi(g))^{it} \sum_{h\in K\backslash G_y} f(gh^{-1}, hy)\det(\phi(h))^{-it}\xi(h).\end{aligned}$$

We thereby obtain the desired equality. □

Definition 3.15. Let $y \in Y_\mathcal{D}$ and $\beta > 0$. The *partition function* of the system $(\mathcal{H}_\mathcal{D}, \sigma_t, \mathfrak{H}_y, H_y)$, is

$$\zeta_y(\beta) := \frac{1}{\#(K\backslash K_0)} \operatorname{Tr}(e^{-\beta H_y}).$$

3.3.7 Symmetries and Zeta Function of the BCM System

The symmetries of the Connes-Marcolli system play an important role in its relations with arithmetic. The analogous symmetry in our generalization is the following (which will be justified in Subsection 3.3.7).

Definition 3.16. Let $\mathcal{D}$ be a BCM datum with level structure. The semigroup

$$\mathcal{S}_{\mathrm{f}}(\mathcal{D}) := \phi^{-1}(K_M)$$

is called the *finite symmetry semigroup of* $\mathcal{D}$. We will denote by $\mathcal{S}_{\mathrm{f}}{}^{\times}(\mathcal{D})$ the group of invertible elements in $\mathcal{S}_{\mathrm{f}}(\mathcal{D})$.

We included in $\mathcal{L}$ the datum of a lattice in the representation ϕ in order to define a determinant map.

Lemma 3.4. *The determinant* $\det\colon \mathrm{GL}(L) \to \mathbb{G}_{\mathrm{m},\mathbb{Q}}$ *induces a natural map,*

$$(\det\circ\phi)\colon K\backslash G(\mathbb{A}_{\mathrm{f}})/K \longrightarrow \mathbb{Q}_{+}^{\times}.$$

The image of $\mathcal{S}_{\mathrm{f}}(\mathcal{D})$ *under this map is contained in* $\mathbb{N}^{\times}$*.*

Proof. Since $\phi(K) \subset K_M^{\times} \subset \mathrm{GL}(L)(\hat{\mathbb{Z}})$, the representation $\phi : G \to \mathrm{GL}(L \otimes_{\mathbb{Z}} \mathbb{Q})$ induces a map

$$\phi\colon K\backslash G(\mathbb{A}_{\mathrm{f}})/K \to \mathrm{GL}(L)(\hat{\mathbb{Z}})\backslash \mathrm{GL}(L)(\mathbb{A}_{\mathrm{f}})/\mathrm{GL}(L)(\hat{\mathbb{Z}}).$$

The determinant map $\mathrm{GL}(L) \to \mathbb{G}_{\mathrm{m},\mathbb{Q}}$ induces a natural map

$$\det\colon \mathrm{GL}(L)(\hat{\mathbb{Z}})\backslash \mathrm{GL}(L)(\mathbb{A}_{\mathrm{f}})/\mathrm{GL}(L)(\hat{\mathbb{Z}}) \longrightarrow \hat{\mathbb{Z}}^{\times}\backslash\mathbb{A}_{\mathrm{f}}^{\times}/\hat{\mathbb{Z}}^{\times} \cong \hat{\mathbb{Z}}^{\times}\backslash\mathbb{A}_{\mathrm{f}}^{\times} \cong \mathbb{Q}_{+}^{\times}.$$

The composition $\det\circ\phi$ gives us the desired map. The image of $\mathcal{S}_{\mathrm{f}}$ under this map is contained in the image of $\mathrm{GL}(L)(\mathbb{A}_{\mathrm{f}}) \cap \mathrm{End}(L)(\hat{\mathbb{Z}})$ under the determinant map, which is exactly $\hat{\mathbb{Z}}^{*} := \mathbb{A}_{\mathrm{f}}^{\times} \cap \hat{\mathbb{Z}}$. The quotient $\hat{\mathbb{Z}}^{\times}\backslash\hat{\mathbb{Z}}^{*}$ is identified with $\mathbb{Z}^{\times}\backslash\mathbb{Z} \cong \mathbb{N}^{\times} \subset \mathbb{Q}^{\times}$. □

Definition 3.17. Let $\mathcal{D}$ be a BCM datum with level structure. The *zeta function* of $\mathcal{D}$ is the complex valued series

$$\zeta_{\mathcal{D}}(\beta) := \sum_{g\in\mathcal{S}_{\mathrm{f}}{}^{\times}\backslash\,\mathcal{S}_{\mathrm{f}}} \det(\phi(g))^{-\beta}.$$

The level-structure BCM datum $\mathcal{D}$ is called *summable* if there exists $\beta_0 \in \mathbb{R}$ such that $\zeta_{\mathcal{D}}(\beta)$ converges in the right plane $\{\beta \in \mathbb{C} \mid \mathrm{Re}(\beta) > \beta_0\}$ and extends to a meromorphic function on $\mathbb{C}$.

Let $Y_{\mathcal{D}}^{\times} \subset Y_{\mathcal{D}}$ be the set of *invertible* $y = (\rho, [z, l])$, i.e., $\rho \in K_M^{\times}$. From the definition of the Hamiltonian (3.18), the following fact is immediate.

Proposition 3.8. *Suppose that $y \in Y_{\mathcal{D}}^{\times}$. Then $G_y = \mathcal{S}_{\mathrm{f}} := \phi^{-1}(K_M)$. The partition function of the system $(\mathcal{H}_{\mathcal{D}}, \sigma_t, \mathfrak{H}_y, H_y)$, coincides with the zeta function $\zeta_{\mathcal{D}}(\beta)$ of $\mathcal{D}$ (see Definition 3.17).*

Moreover, it follows from 3.4 that the Hamiltonian has *positive energy* in the representation π_y.

Symmetries

Let $\mathcal{D} = (G, X, V, M; L, K, K_M)$ be a BCM datum with neat level. We will denote the center of G by C.

Recall that $\mathcal{S}_{\mathrm{f}}$ is the semigroup $\phi^{-1}(K_M)$. For $m \in \mathcal{S}_{\mathrm{f}}$ and $c \in C(\mathbb{R})$, we define

$$\theta_{(m,c)}(f)(g, \rho, [z, l]) := f(g, \rho\phi(m), [cz, l]). \tag{3.20}$$

Lemma 3.5. *The expression* (3.20) *gives a well-defined right action of*

$$\mathcal{S}(\mathcal{D}) := \mathcal{S}_{\mathrm{f}}(\mathcal{D}) \times C(\mathbb{R})$$

on $\mathcal{H}_{\mathcal{D}}$ which moreover commutes with the time evolution.

Proof. The action is well defined because K acts on $Y_{\mathcal{D}}$ on the left, while $\mathcal{S}$ acts on the right. Recalling that the time evolution is given by the formula $(\sigma_t f)(g, y) = \big(\det \phi(g)\big)^{it} f(g, y)$, it is clear that the action of $\mathcal{S}$ commutes with σ_t. $\square$

Notation 3.4. Let CK_M be the center of K_M.

Definition 3.18. Let $\mathrm{Inn}(\mathcal{D})$ be the subsemigroup of $\mathcal{S}$ defined by

$$\mathrm{Inn}(\mathcal{D}) := C(\mathbb{Q}) \cap \phi^{-1}(CK_M).$$

Remark 3.8. There is a (diagonal) inclusions of semigroups

$$\mathrm{Inn}(\mathcal{D}) \subset \mathcal{S}(\mathcal{D}).$$

This gives a natural action of $\mathrm{Inn}(\mathcal{D})$ on $\mathcal{H}_{\mathcal{D}}$.

Definition 3.19. The semigroup

$$\mathrm{Out}(\mathcal{D}) := \mathrm{Inn}(\mathcal{D})\backslash\, \mathcal{S}(\mathcal{D})$$

is called the *outer symmetry semigroup of the BCM system* $(\mathcal{H}_{\mathcal{D}}, \sigma_t)$.

In practical situations, the following hypotheses will often be fulfilled (see Propositions 4.2 and 4.6).

Definition 3.20. The level structure $\mathcal{L} = (L, K, K_M)$ is called *faithful* if the image $\phi(C(\mathbb{Q}))$ of the center of G commutes with K_M, i.e., $\phi(C(\mathbb{Q})) \subset CK_M$. The level structure $\mathcal{L}$ is called *full* if the natural morphism $\mathrm{Out} \to C(\mathbb{Q})\backslash G(\mathbb{A}_{\mathrm{f}})$ is surjective; if this morphism is an isomorphism, the level structure $\mathcal{L}$ is called *fully faithful*.

These symmetries are symmetries up to inner automorphisms, as we now show.

Proposition 3.9 (cf. [11], Prop. 1.34)**.** *There is a morphism*

$$\mathrm{Out}(\mathcal{D}) \to \mathrm{Out}(\mathcal{H}_{\mathcal{D}}, \sigma_t)$$

to the quotient of the automorphism group of the BCM system by inner automorphisms of the algebra.

Proof. We have to prove that Inn acts by inner automorphisms. For $n \in \mathrm{Inn}$, we let μ_n be

$$\mu_n(g, y) = 1 \text{ if } g \in K.n^{-1},\ \mu_n(g, y) = 0 \text{ if } g \notin K.n^{-1}.$$

We will show that

$$\theta_{(n,n)}(f) = \mu_n f \mu_n^*,$$

i.e., the action of $\theta_{(n,n)}$ is given by the inner automorphism corresponding to μ_n.

We have, for all $y \in Y_{\mathcal{D}}$,

$$\begin{aligned}(\mu_n f \mu_n^*)(g,y) &= \sum_{h \in K \backslash G(\mathbb{A}_{\mathrm{f}}), hy \in Y} \mu_n(gh^{-1}, hy)(f\mu_n^*)(h,y), \\ &= \sum_{h \in K \backslash G(\mathbb{A}_{\mathrm{f}}), hy \in Y} \mu_n(gh^{-1}, hy) \sum_{k \in K \backslash G(\mathbb{A}_{\mathrm{f}}), ky \in Y} f(hk^{-1}, ky)\mu_n^*(k,y), \\ &= \sum_{h,k \in K \backslash G(\mathbb{A}_{\mathrm{f}}), hy, ky \in Y} \mu_n(gh^{-1}, hy) f(hk^{-1}, ky)\mu_n(k^{-1}, ky).\end{aligned}$$

Now, by definition of μ_n, the only nontrivial term of this sum is obtained when $k^{-1} = n^{-1}$ and $gh^{-1} = n^{-1}$, i.e., $k = n$ and $h = ng$. Since n is central, we get

$$\begin{aligned}(\mu_n f \mu_n^*)(g,y) &= f(ngn^{-1}, ny), \\ &= f(g, n\rho, [z, ln^{-1}]), \\ &= f(g, \rho n, [nz, l]), \\ &= \theta_{(n,n)}(f)(g,y).\end{aligned}$$

□

3.3.8 KMS States at Low Temperature

Let $\mathcal{D} = (G, X, V, M; L, K, K_M)$ be a summable BCM datum with level structure. Recall that we have defined

$$Y_{\mathcal{D}}^{\times} = \{\, (g, \rho, [z,l]) \in Y_{\mathcal{D}} \mid \rho \text{ is invertible} \,\}.$$

Lemma 3.6. *Let $y \in Y_{\mathcal{D}}^{\times}$. Let β be such that the zeta function $\zeta_{\mathcal{D}}(\beta)$ converges. The state*

$$\Phi_{\beta,y}(f) := \frac{\operatorname{Tr}\big(\pi_y(f) e^{-\beta H_y}\big)}{\zeta_{\mathcal{D}}(\beta)}, \quad f \in A_{\mathcal{D}}$$

is a KMS$_\beta$ state for the system $(A_{\mathcal{D}}, \sigma_t)$.

Remark 3.9. By Lemma 3.4, we know that $\zeta_{\mathcal{D}}(\beta) \neq 0$.

Proof. By construction, the algebra $\mathcal{H}_{\mathcal{D}}$ is a norm-dense subalgebra of $A_{\mathcal{D}}$, which is also σ_z-invariant. Thus, to verify the KMS$_\beta$ condition, it is enough to show that

$$\Phi_{\beta,y}\big(f_1\sigma_{i\beta}(f_2)\big) = \Phi_{\beta,y}(f_2f_1)$$

for every pair of functions $f_1, f_2 \in \mathcal{H}_{\mathcal{D}}$; see Proposition 2.1. The convergence of the zeta function implies that the operator $e^{-\beta H_y}$ is trace-class. The invariance of the trace under cyclic permutations implies that

$$\begin{aligned}\zeta_{\mathcal{D}}(\beta).\Phi_{\beta,y}\big(f_1\sigma_{i\beta}(f_2)\big) &= \mathrm{Tr}\big(f_1e^{-\beta H_y}f_2e^{\beta H_y}e^{-\beta H_y}\big)\\ &= \mathrm{Tr}(f_1e^{-\beta H_y}f_2)\\ &= \mathrm{Tr}(f_2f_1e^{-\beta H_y})\\ &= \zeta_{\mathcal{D}}(\beta).\Phi_{\beta,y}(f_2f_1).\end{aligned}$$

This finishes the proof of the KMS condition. □

Recall that the *commutant* of a subset $S \subset B(\mathfrak{H}_y)$ is by definition the set

$$S' = \{\, a \in B(\mathfrak{H}_y) \mid as = sa \text{ for all } s \in S \,\}\,.$$

(cf. Section 2.1)

Lemma 3.7. *If $y \in Y_{\mathcal{D}}^{\times}$, then the commutant $\pi_y(A_{\mathcal{D}})'$ consists only of scalar operators.*

Proof. In general, if $y \in Y_{\mathcal{D}}$, then the von Neumann algebra $\pi_y(A_{\mathcal{D}})'$ is generated by the right regular representation of the isotropy group

$$Z_{y,y} := \{\, [g,y] \in Z \mid s[g,y] = [y] = [gy] = t[g,y] \,\}$$

— see [8, Proposition VII.5]. If y is now in $Y_{\mathcal{D}}^{\times}$, then the isotropy group $Z_{y,y}$ is trivial. Therefore, the commutant $\pi_y(A_{\mathcal{D}})'$ consists only of scalar operators. □

Recall that the set of KMS$_\beta$ states is a convex simplex (see Proposition 2.3).

Definition 3.21. The extreme points of the simplex of KMS$_\beta$ states are called the *extremal KMS$_\beta$ states.*

Proposition 3.10. *Let $y \in Y_{\mathcal{D}}^{\times}$ be an invertible element of $Y_{\mathcal{D}}$. Let $\beta > 0$ be such that the zeta function $\zeta_{\mathcal{D}}(\beta)$ converges. Then the KMS$_\beta$ state*

$$\Phi_{\beta,y}(f) := \frac{\mathrm{Tr}\big(\pi_y(f)e^{-\beta H_y}\big)}{\zeta_{\mathcal{D}}(\beta)}$$

is an extremal KMS$_\beta$ state of Type I_∞.

Proof. By Proposition 2.3, the property, for $\Phi_{\beta,y}$, of being extremal is equivalent to the property of being a factor state, i.e., the algebra $A_{\mathcal{D}}$ generates a factor in the GNS representation of $\Phi_{\beta,y}$. Following Harari-Leichtnam, [22, Theorem 5.3.1], the GNS representation is (up to unitary equivalence)

$$\tilde{\pi}_y = \pi_y \otimes \mathrm{Id}_{\mathfrak{H}_y} \colon A_{\mathcal{D}} \to B(\mathfrak{H}_y \otimes \mathfrak{H}_y),$$

and the associated cyclic vector is

$$\Omega_{\beta,y} = \zeta_{\mathcal{D}}(\beta)^{-1/2} \sum_{h \in K \backslash G_y} \det(\phi(h))^{-1/2} \epsilon_h \otimes \epsilon_h,$$

where ϵ_h is the basis vector of $\mathfrak{H}_y$ that takes the value 1 at h, and 0 elsewhere.

The properties that characterize the triple $(\mathfrak{H}_y \otimes \mathfrak{H}_y, \tilde{\pi}_y, \Omega_{\beta,y})$ as the GNS representation of $\Phi_{\beta,y}$ are precisely:

1. $\Phi_{\beta,y}(f) = \langle \tilde{\pi}_y(f)\Omega_{\beta,y}, \Omega_{\beta,y}\rangle$, for every $f \in A_{\mathcal{D}}$; and

2. The orbit $\tilde{\pi}_y(A_{\mathcal{D}})\Omega_{\beta,y}$ is dense in the Hilbert space $\mathfrak{H}_y \otimes \mathfrak{H}_y$.

These two properties are verified by direct calculation. For example, to verify the second condition first observe that

$$\pi_y(f)\epsilon_h = \sum_{g \in K \backslash G_y} f(gh^{-1}, hy)\, \epsilon_g,$$

and so

$$\tilde{\pi}_y(f)\Omega_{\beta,y} = \zeta_{\mathcal{D}}(\beta)^{-1/2} \sum_{g,\, h \in K \backslash G_y} \det(\phi(h))^{-\beta/2} f(gh^{-1}, hy)\, \epsilon_g \otimes \epsilon_h.$$

But since $G_y = \mathcal{S}_{\mathrm{f}}$, every $\det(\phi(h))$ is positive, and we can choose f to have sufficiently small support about (gh^{-1}, hy) to see that the basis vector $\epsilon_g \otimes \epsilon_h$ lies in the closure of $\tilde{\pi}_y(A_{\mathcal{D}})\xi_{\beta,y}$.

By Lemma 3.7, we know that the commutant $\pi_y(A_{\mathcal{D}})'$ consists of scalar operators. It is then clear that

$$\tilde{\pi}_y(A_{\mathcal{D}})' = \pi_y(A_{\mathcal{D}})' \otimes B(\mathfrak{H}_y) = \mathbb{C} \otimes B(\mathfrak{H}_y),$$

and so

$$\tilde{\pi}_y(A_{\mathcal{D}})'' = B(\mathfrak{H}_y) \otimes \mathbb{C}\mathrm{Id}_{\mathfrak{H}_y} \cong B(\mathfrak{H}_y).$$

This proves that $\Phi_{\beta,y}$ is a Type I_∞ factor state. □

Question 3.1. Let $\mathcal{D} = (G, X, V, M; L, K, K_M)$ be a BCM datum with level structure. Is is true that for $\beta \gg 0$, the map $y \mapsto \Phi_{\beta,y}$ induces a bijection from the Shimura variety $\mathrm{Sh}(G, X)$ to the space $\mathcal{E}_\beta$ of extremal KMS_β states on $(\mathcal{H}_{\mathcal{D}}, \sigma_t)$?

This is true for the systems of Bost-Connes [3], Connes-Marcolli [11], and Connes-Marcolli-Ramachandran [14].

Chapter 4

Examples of the General Theory

In this chapter we specialize the general theory of the previous chapter to various cases. We show, in particular, that the Shimura BCM recovers all previous constructions of Connes and his collaborators. We also consider the framework for Hilbert modular surfaces, and a variation of the Bost-Connes system that accommodates Dirichlet characters.

Moreover, in the case of toroidal Shimura varieties, we show that the Shimura BCM yields an analogue of the Bost-Connes system for an arbitrary number field F, which unlike earlier constructions by P. Cohen, Harai-Leichtnam, Laca et al., gives the Dedekind zeta function as partition function *and* admits an action of $\pi_0\big(F^\times\backslash\mathbb{A}_{F,\mathrm{f}}^\times\big)$, which by class field theory is isomorphic to $\mathrm{Gal}(F^{\mathrm{ab}}/F)$. Such a system therefore admits the key features one expects in order to generalize the arithmetical intertwining properties of the extremal KMS_∞ states of the Bost-Connes system over $\mathbb{Q}$ (see Theorem 3.1, Theorem 3.2, and (1) from the Introduction).

4.1 Bost-Connes and Connes-Marcolli Revisited

The Shimura BCM of the previous chapter does indeed generalize the systems of Bost-Connes and Connes-Marcolli. This is actually rather clear from the discussion in Chapter 3 leading up to the abstract constructions of Section 3.3, but here we want to explain this coincidence as a consequence of a more general condition on the Shimura datum, namely, that essentially only the following hypotheses are needed:

1. The natural map
$$G(\mathbb{Q}) \cap K \longrightarrow G(\mathbb{Q})/\big(G(\mathbb{Q}) \cap G(\mathbb{R})^+\big)$$
is surjective; and
2. The class number $\#\big(G(\mathbb{Q})\backslash G(\mathbb{A}_{\mathrm{f}})/K\big)$ is 1.

4.1.1 Principal BCM Systems

We want to understand how our systems are related to the usual Bost-Connes-Marcolli systems in the class number one case. We call these class number one systems *principal BCM systems.* They are directly related to Connes-Marcolli systems defined in [11], as we will shortly see.

Let us setup the relevant apparatus for $\mathcal{D} = (G, X, V, M; L, K, K_M)$ a BCM pair with (G, X) *classical.* Let $\Gamma := G(\mathbb{Q}) \cap K$, and let

$$U^{\mathrm{princ}} := \{ (g, \rho, z) \in G(\mathbb{Q}) \times K_M \times X \mid g\rho \in K_M \}.$$

Let X^+ be a connected component of X, $G(\mathbb{Q})^+$ be $G(\mathbb{Q}) \cap G(\mathbb{R})^+$ (where $G(\mathbb{R})^+$ is the identity component of $G(\mathbb{R})$) and $\Gamma_+ = G(\mathbb{Q})^+ \cap K$. Finally, let

$$U^+ := \big\{ (g, \rho, z) \in G(\mathbb{Q})^+ \times K_M \times X^+ \mid g\rho \in K_M \big\}.$$

We have a natural action of Γ^2 (resp. Γ_+^2) on U^{princ} (resp. U^+) given (as usual) by
$$(g, \rho, z) \longmapsto (\gamma_1 g \gamma_2^{-1}, \gamma_2\rho, \gamma_2 z).$$
Let $\mathfrak{Z}_{\mathcal{D}}^{\mathrm{princ}}$ (resp. $\mathfrak{Z}_{\mathcal{D}}^+$) be the stacky quotient of U^{princ} (resp. U^+) by Γ^2 (resp. Γ_+^2).

Definition 4.1. The stack groupoid $\mathfrak{Z}_{\mathcal{D}}^{\mathrm{princ}}$ is called the *principal BCM groupoid* for the level-structure BCM datum $\mathcal{D}$.

Proposition 4.1. *Suppose that the natural map $\Gamma \to G(\mathbb{Q})/G(\mathbb{Q})^+$ is surjective. Then the natural map*
$$\mathfrak{Z}_{\mathcal{D}}^+ \xrightarrow{\sim} \mathfrak{Z}_{\mathcal{D}}^{\mathrm{princ}}$$
is an isomorphism.

Proof. *Surjectivity*: Let $u = (g, \rho, z) \in U^{\mathrm{princ}}$. We want to show that there exists $\gamma_1, \gamma_2 \in \Gamma$ such that $(\gamma_1, \gamma_2).u = (\gamma_1 g \gamma_2^{-1}, \rho, \gamma_2 z) \in U^+$. There exists $\gamma_2 \in \Gamma$ with $\gamma_2 z \in X^+$ because: 1) the definition of a Shimura datum implies that $\pi_0(X)$ is a $\pi_0(G(\mathbb{R}))$-homogeneous space; and 2) from our hypothesis and the theorem of real approximation, we get a surjection $\Gamma \to G(\mathbb{Q})/G(\mathbb{Q})^+ \cong G(\mathbb{R})/G(\mathbb{R})^+$. Our hypothesis now implies that there exists $\gamma_1 \in \Gamma$ such that $\gamma_1 g \gamma_2^{-1} \in G(\mathbb{Q})^+$. This proves surjectivity.

Injectivity: Now suppose that two points (g_1, ρ_1, z_1) and (g_2, ρ_2, z_2) have the same image in the quotient. Then there exists $\gamma_1, \gamma_2 \in \Gamma$ such that $(g_1, \rho_1, z_1) = (\gamma_1 g_2 \gamma_2^{-1}, \gamma_2 \rho_2, \gamma_2 z_2)$. Since γ_2 stabilizes X^+, it is in $G(\mathbb{R})^+$, and therefore also in Γ_+. This implies that γ_1 is in Γ_+. This proves injectivity. □

Notation 4.1. The cardinality of the finite set $G(\mathbb{Q})\backslash G(\mathbb{A}_{\mathrm{f}})/K$ is denoted by $h(G, K)$.

Proposition 4.2. *Suppose $h(G, K) = 1$. Then the principal and the full BCM groupoids are the same, i.e., the natural map*

$$\mathfrak{Z}_{\mathcal{D}}^{\mathrm{princ}} \xrightarrow{\sim} \mathfrak{Z}_{\mathcal{D}}$$

is an isomorphism.

Proof. There is a natural map

$$\begin{array}{rccc} \psi: & (\Gamma\backslash G(\mathbb{Q})) \times K_M \times X & \to & (K\backslash G(\mathbb{A}_{\mathrm{f}})) \times K_M \times G(\mathbb{Q})\backslash(X \times G(\mathbb{A}_{\mathrm{f}})) \\ & (g, \rho, z) & \mapsto & (g, \rho, [z, 1]) \end{array}.$$

The action of $\gamma_2 \in \Gamma$ on the source is given by $(g, \rho, z) \mapsto (g\gamma_2^{-1}, \gamma_2\rho, \gamma_2 z)$ and on the range by $(g, \rho, [z, l]) \mapsto (g\gamma_2^{-1}, \gamma_2\rho, [z, l\gamma_2^{-1}])$. Since $\Gamma = K \cap G(\mathbb{Q})$, we have

$$\begin{aligned} \psi(\gamma_2 \cdot (g, \rho, z)) &= (g\gamma_2^{-1}, \gamma_2\rho, [\gamma_2 z, 1]), \\ &= (g\gamma_2^{-1}, \gamma_2\rho, [z, \gamma_2^{-1}]), \\ &= \gamma_2 \cdot \psi(g, \rho, z). \end{aligned}$$

This proves that ψ, being Γ-equivariant, induces a well defined map

$$\overline{\psi} : (\Gamma\backslash G(\mathbb{Q})) \underset{\Gamma}{\times} [K_M \times X] \to (K\backslash G(\mathbb{A}_{\mathrm{f}})) \underset{K}{\times} [K_M \times G(\mathbb{Q})\backslash(X \times G(\mathbb{A}_{\mathrm{f}}))].$$

Let us prove that $\overline{\psi}$ is surjective. This will essentially follow from the equalities $G(\mathbb{A}_\mathrm{f}) = K.G(\mathbb{Q}) = G(\mathbb{Q}).K$ (the class number one hypothesis $h(G,K) = 1$).

For $(g, \rho, [z, l]) \in (K\backslash G(\mathbb{A}_\mathrm{f})) \underset{K}{\times} [K_M \times G(\mathbb{Q})\backslash(X \times G(\mathbb{A}_\mathrm{f}))]$, there exists $\gamma_2 \in K$ and $l_2 \in G(\mathbb{Q})$ such that $l = l_2\gamma_2$. Then, we have the equalities in our quotient space

$$\begin{aligned}(g, \rho, [z, l]) &= (g, \rho, [z, l_2\gamma_2]) \\ &= (g\gamma_2^{-1}, \gamma_2\rho, [z, l_2]) \\ &= (g\gamma_2^{-1}, \gamma_2\rho, [l_2^{-1}z, 1]).\end{aligned}$$

There exists $\gamma_1 \in K$ and $g_1 \in G(\mathbb{Q})$ such that $\gamma_1 g_1 = g\gamma_2^{-1}$ and we have the following equalities in our quotient space

$$\begin{aligned}(g, \rho, [z, l]) &= (g\gamma_2^{-1}, \gamma_2\rho, [l_2^{-1}z, 1]) \\ &= (\gamma_1 g_1, \gamma_2\rho, [l_2^{-1}z, 1]) \\ &= \psi(g_1, \gamma_2\rho, l_2^{-1}z).\end{aligned}$$

Thus $\overline{\psi}$ is surjective.

Now we prove that $\overline{\psi}$ is injective. Suppose that

$$\overline{\psi}(g_1, \rho_1, z_1) = \overline{\psi}(g_2, \rho_2, z_2).$$

Then there exists $\gamma_1 \in K$, $\gamma_2 \in K$, $\gamma_3 \in G(\mathbb{Q})$ such that

$$(\gamma_1 g_1 \gamma_2^{-1}, \gamma_2\rho_1, [\gamma_3 z_1, \gamma_3\gamma_2^{-1}]) = (g_2, \rho_2, [z_2, 1]).$$

This implies that $\gamma_3 = \gamma_2$, and so $\gamma_2 \in K \cap G(\mathbb{Q}) = \Gamma$. But we also have $\gamma_1 = g_2\gamma_2 g_1^{-1} \in G(\mathbb{Q}) \cap K = \Gamma$. This shows that

$$(g_2, \rho_2, z_2) = (\gamma_1 g_1 \gamma_2^{-1}, \gamma_2\rho_1, \gamma_2 z_1)$$

with $\gamma_1, \gamma_2 \in \Gamma$, i.e., (g_2, ρ_2, z_2) and (g_1, ρ_1, z_1) are the same in $(\Gamma\backslash G(\mathbb{Q})) \underset{\Gamma}{\times} [K_M \times X]$. This proves injectivity.

To finish, we prove that the bijection $\overline{\psi} : \mathfrak{Z}_\mathcal{D}^{\mathrm{princ}} \to \mathfrak{Z}_\mathcal{D}$ is compatible with the groupoid structures. Let $Y^{\mathrm{princ}} = K_M \times X$, and $Y = K_M \times \mathrm{Sh}(G, X)$. If $(g, \rho, z) \in \mathfrak{Z}^{\mathrm{princ}}$, the image of $(\rho, z) \in Y^{\mathrm{princ}}$ under $g \in G(\mathbb{Q})$ is given by $(g\rho, gz) \in Y^{\mathrm{princ}}$. The image of $(\rho, [z, 1]) \in Y$ under g is given by $(g\rho, [z, g^{-1}]) \in Y$, which is equal to $(g\rho, [gz, 1])$. This finishes the proof. □

4.1.2 The Bost-Connes System Revisited

Let F be a number field. Consider the following datum.

$$\begin{aligned} G &= \mathrm{Res}_{F/\mathbb{Q}}\,\mathbb{G}_{\mathrm{m},F} & L &= \mathcal{O}_F \\ X_F &= G(\mathbb{R})/G(\mathbb{R})^+ \cong \{\pm 1\}^{\mathrm{Hom}(F,\mathbb{R})} & K &= \hat{\mathcal{O}}_F^\times \\ V &= F & K_M &= \hat{\mathcal{O}}_F = M_1(\hat{\mathcal{O}}_F) \\ M &= \mathrm{Res}_{F/\mathbb{Q}}\,M_{1,F} \end{aligned}$$

Definition 4.2. The level-structure BCM datum

$$\mathcal{P}(\mathrm{Res}_{F/\mathbb{Q}}\,\mathbb{G}_{\mathrm{m},F}, X_F) = (\mathbb{G}_{\mathrm{m},F}, X_F, F, \mathrm{Res}_{F/\mathbb{Q}}\,M_{1,F}; \mathcal{O}_F, \hat{\mathcal{O}}_F^\times, \hat{\mathcal{O}}_F)$$

is called the *Bost-Connes pair* for F. The corresponding algebra

$$\mathcal{H}(\mathrm{Res}_{F/\mathbb{Q}}\,\mathbb{G}_{\mathrm{m},F}, X_F)$$

is called the *Bost-Connes algebra* for F.

Proposition 4.3. *In the case $F = \mathbb{Q}$, $\mathcal{H}(\mathbb{G}_{\mathrm{m},\mathbb{Q}}, \{\pm 1\})$ is the original Bost-Connes algebra.*

Proof. Recall from Section 3.1.1 that the Bost-Connes algebra is the convolution algebra of the groupoid $Z_{BC} \subset \mathbb{Q}_+^\times \times \hat{\mathbb{Z}}$ of pairs (g, ρ) with $g\rho \in \hat{\mathbb{Z}}$; thus we need only to show that Z_{BC} coincides with the BCM groupoid Z of the Bost-Connes pair. Indeed, in the notation of Section 4.1.1, we have

$$U^+ = \{\, (g, \rho, 1) \in \mathbb{Q}_+^\times \times \hat{\mathbb{Z}} \times \{1\} \mid g\rho \in \hat{\mathbb{Z}}\,\}, \quad \Gamma = \{\pm 1\}, \text{ and } \Gamma_+ = 1.$$

Therefore $\mathbb{Z}^+ := \Gamma^+ \backslash U^+ = Z_{BC}$; the map $\Gamma \to G(\mathbb{Q})/G(\mathbb{Q})^+$ is an isomorphism of $\{\pm 1\}$; and $h(\mathbb{G}_{\mathrm{m},\mathbb{Q}}, \hat{\mathbb{Z}}^\times) = 1$, since it is the usual class number of $\mathbb{Q}$. The proposition follows from Propositions 4.1 and 4.2. □

4.1.3 The Connes-Marcolli System Revisited

We now show that in the $\mathrm{GL}_{2,\mathbb{Q}}$ case, we obtain exactly the same groupoid as Connes and Marcolli [11]. This groupoid is only a stack-groupoid, not a usual groupoid. This restriction was circumvented by Connes and Marcolli using functions of weight 0 for the scaling action (see [11], remark shortly preceding 1.83). Such a scaling action is not canonically defined in the general case we consider. As explained before, we deliberately chose to view this groupoid as a stack-groupoid in order to define a natural groupoid algebra for it that depends on the resolution of stack singularities given by the choice of K.

Consider the following datum.

$$\begin{aligned} G &= \mathrm{GL}_{2,\mathbb{Q}} & L &= \mathbb{Z}^2 \\ X &= \mathbb{H}^{\pm} & K &= \mathrm{GL}_2(\hat{\mathbb{Z}}) \\ V &= \mathbb{Q}^2 & K_M &= M_2(\hat{\mathbb{Z}}) \\ M &= M_{2,\mathbb{Q}} \end{aligned}$$

Definition 4.3. The level-structure BCM datum

$$\mathcal{P}(\mathrm{GL}_2, \mathbb{H}^{\pm}) := \big(\mathrm{GL}_2, \mathbb{H}^{\pm}, \mathbb{Q}^2, M_{2,\mathbb{Q}}; \mathbb{Z}^2, \mathrm{GL}_2(\hat{\mathbb{Z}}), M_2(\hat{\mathbb{Z}})\big)$$

is called *the modular BCM pair*. The corresponding BCM stack-groupoid is denoted by $\mathfrak{Z}_{\mathrm{GL}_2,\mathbb{H}^{\pm}}$.

The stack-groupoid $\mathfrak{Z}^{+}_{\mathrm{GL}_2,\mathbb{H}^{\pm}}$ is defined as in Section 4.1.1. This is exactly the groupoid studied by Connes and Marcolli in [11].

Lemma 4.1. *The stack-groupoid for the modular BCM pair is the same as Connes and Marcolli's. In other words, the natural map*

$$\mathfrak{Z}^{+}_{\mathrm{GL}_2,\mathbb{H}} \to \mathfrak{Z}_{\mathrm{GL}_2,\mathbb{H}^{\pm}}$$

is an isomorphism.

Proof. We have in this case $h(G, K) = 1$ so by Proposition 4.2, we have $[Z^{\mathrm{princ}}_{\mathrm{GL}_2,\mathbb{H}^{\pm}}] \cong [Z_{\mathrm{GL}_2,\mathbb{H}^{\pm}}]$. The map $\mathrm{GL}_2(\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{R})/\mathrm{GL}_2(\mathbb{R})^{+}$ is surjective, so that we can apply proposition 4.1, which tells us that

$$\mathfrak{Z}^{+}_{\mathrm{GL}_2,\mathbb{H}^{pm}} \cong \mathfrak{Z}^{\mathrm{princ}}_{\mathrm{GL}_2,\mathbb{H}^{\pm}}.$$

□

4.2 Toroidal Shimura Varieties

In this section we investigate the BCM systems for toroidal Shimura varieties. This is the case relevant for generalizations of the Bost-Connes system to arbitrary number fields F. The main results are in Sections 4.2.2 and 4.2.2, where we show that the generalized Bost-Connes systems that we get when specializing to the BCM system for the Shimura datum $(\mathrm{Res}_{F/\mathbb{Q}}\,\mathbb{G}_{\mathrm{m},F}, \pi_0(F_\infty^\times))$ have particularly nice features: symmetry by the full group $\pi_0(\mathbb{A}_F^\times/F^\times)$, and Dedekind zeta function as partition function. Moreover, unlike previous approaches, our Bost-Connes generalization enjoys these properties without conceding to class number restraints on the field F.

4.2.1 Some Facts about the Idèle Class Group

As observed by Richard Pink in his Bonn thesis [41, Page xiii and §11.4], Deligne's definition of a (pure) Shimura variety is better suited to abelian class field if one makes the following adaptation: when defining a Shimura datum for a reductive group G over $\mathbb{Q}$, allow not only for arithmetic quotients of a $G(\mathbb{R})$-conjugacy class X of homomorphism $h\colon \mathbb{S} \to G_{\mathbb{R}}$, but also finite $G(\mathbb{R})$-equivariant covers $Y \to X$ on which $G(\mathbb{R})$ acts transitively.

That this adaptation is better suited for applications to class field theory can be clearly seen in the case of toroidal Shimura varieties.

Proposition 4.4 (Deligne [16])**.** *Let T be a torus (over $\mathbb{Q}$). The group*

$$\pi_0\big(T(\mathbb{Q})\backslash T(\mathbb{A})\big)$$

of connected components of $T(\mathbb{Q})\backslash T(\mathbb{A})$ is a profinite group, namely the projective limit

$$\varprojlim_K T(\mathbb{Q})\backslash T(\mathbb{A})/T(\mathbb{R})^+K$$

over the compact open subgroups K of $T(\mathbb{A}_{\mathrm{f}})$.

In particular, we have the following description of the group of components of the idèle class group.

Corollary 4.1. *Let F be a number field, and let T be the torus $\mathrm{Res}_{F/\mathbb{Q}}\,\mathbb{G}_{\mathrm{m},F}$ of dimension $[F:\mathbb{Q}]$. Then $T(\mathbb{Q})\backslash T(\mathbb{A})$ is the idèle class group C_F of F, and its group of connected components is a profinite group. More precisely,*

$$\pi_0(C_F) \cong \varprojlim_K C_F/(F_\infty^\times)^+ K =: \mathrm{Sh}\big(T, \pi_0\, T(\mathbb{R})\big).$$

where K runs over the compact open subgroups of $\mathbb{A}_{F,\mathrm{f}}^\times$.

Of course, this Corollary is an immediate consequence of the Proposition. However, since no proof of the corollary is actually written down in [16], we've included a sketch for the reader's convenience.

Sketch of Proof. To show that $\pi_0(C_F)$ is profinite, we need to show that it is compact and totally disconnected. This later property is alone a consequence of the fact that C_F is a topological group. To show compactness, we look at the exact sequence of groups

$$1 \longrightarrow F^\times\backslash\mathbb{A}_F^{1,\times} \longrightarrow C_F \longrightarrow \mathbb{R}_+^\times \longrightarrow 1,$$

where $\mathbb{A}_F^{1,\times}$ is the group of norm-1 idèles. Now since $F^\times\backslash\mathbb{A}_F^{1,\times}$ is compact and $\mathbb{R}_+^\times$ is connected, the compactness of $\pi_0(C_F)$ immediate. Hence $\pi_0(C_F)$ is a profinite group.

For the identification of the projective limit see [24, §III.7]. Essentially, what is required is a somewhat explicit description of the connected components of the identity of C_F (Artin) together with general facts about projective limits of groups. □

Expression of the Projective Limit

Although we now have an expression for the limit Shimura variety

$$\mathrm{Sh}(\mathrm{Res}_{F/\mathbb{Q}}\,\mathbb{G}_{\mathrm{m},F}, \pi_0(F_\infty^\times)) = \varprojlim_K F^\times\backslash\,\pi_0(F_\infty^\times) \times \mathbb{A}_{F,\mathrm{f}}^\times/K$$

as the connected-components groups of $\mathbb{A}_F^\times/F^\times$, this is not necessarily a big simplification, because of the somewhat mysterious nature of the identity component of the idèle class group. The issue we want to investigate is when is it that one can give an alternative description of the limit, as in the case for $\mathbb{Q}$ where the limit is

$$\mathbb{Q}^\times \backslash \{\pm 1\} \times \mathbb{A}_{\mathrm{f}}^\times = \mathbb{Q}_+^\times \backslash \mathbb{A}_{\mathrm{f}}^\times = \hat{\mathbb{Z}}^\times.$$

Consider the Shimura datum $(T, \pi_0(T(\mathbb{R})))$ for a torus T defined over $\mathbb{Q}$. Let $\overline{T(\mathbb{Q})}$ is the closure of $T(\mathbb{Q})$ in $T(\mathbb{A}_{\mathrm{f}})$. It is claimed in [16, §2.2.3],

$$\overline{T(\mathbb{Q})} \backslash \, \pi_0\big(T(\mathbb{R})\big) \times T(\mathbb{A}_{\mathrm{f}}), \tag{4.1}$$

is isomorphic to

$$\mathrm{Sh}\big(T, \pi_0\, T(\mathbb{R})\big) = \varprojlim_K \, T(\mathbb{Q}) \backslash \, \pi_0\big(T(\mathbb{R})\big) \times T(\mathbb{A}_{\mathrm{f}})/K. \tag{4.2}$$

But this is *false* in general, because there are tori for which (4.1) is not even Hausdorff, while (4.2) is a profinite group, and therefore necessarily Hausdorff.

Indeed, this discrepancy is precisely what complicates the analysis of the Bost-Connes system for general number fields. In fact, it is rather rare for a (limit) toroidal Shimura variety to have an expression as a simple quotient. A precise statement in this direction is the following.

Proposition 4.5. *For a number field F, the following are equivalent:*

1. *F is either $\mathbb{Q}$ or an imaginary quadratic field;*
2. *The quotient space $F^\times \backslash \mathbb{A}_{F,\mathrm{f}}^\times$ is Hausdorff;*
3. *The (limit) Shimura variety takes the form*

$$\mathrm{Sh}(\mathrm{Res}_{F/\mathbb{Q}}\, \mathbb{G}_{\mathrm{m},F}, \pi_0\, F_\infty^\times) \cong F^\times \backslash \, \pi_0(F_\infty^\times) \times \mathbb{A}_{F,\mathrm{f}}^\times.$$

Proof. The Proposition is essentially a consequence of Dirichlet's Unit Theorem, which, as we recall, states that the group of units $\mathcal{O}_F^\times$ of the ring of integers of F is a finitely generated (abelian) group of rank $r + s - 1$, where r, resp. $2s$, is the number of real, resp. complex, embeddings of F. Thus, $\mathcal{O}_F^\times$ is an infinite group if and only if F is *neither* $\mathbb{Q}$ *nor* an imaginary quadratic field.

(3) $\Longrightarrow$ (2): This is clear, because $\mathrm{Sh}(\mathrm{Res}_{F/\mathbb{Q}}\,\mathbb{G}_{\mathrm{m},F}, \pi_0\, F_\infty^\times)$ is, by definition, a profinite group, and therefore also Hausdorff.

(2) $\Longrightarrow$ (1): Suppose that $\mathcal{O}_F^\times$ is infinite. Since $\mathcal{O}_F^\times$ is an infinite set which is discretely embedded in the compact space $\hat{\mathcal{O}}_F^\times$, it has an accumulation point $a \in \hat{\mathcal{O}}_F^\times$ lying outside of $\mathcal{O}_F^\times$. The $F^\times$-orbit of a cannot coincide with $F^\times \subset \mathbb{A}_{F,\mathrm{f}}^\times$, for if it did, then a would lie in $F^\times \cap \hat{\mathcal{O}}_F^\times = \mathcal{O}_F^\times$. Hence 1 and a represent distinct non-separable classes in the quotient $F^\times\backslash\mathbb{A}_{F,\mathrm{f}}^\times$.

(1) $\Longrightarrow$ (2): From the fundamental exact sequence of class field theory,

$$1 \longrightarrow \overline{F^\times.(F_\infty^\times)^+} \longrightarrow \mathbb{A}_F^\times \longrightarrow \mathrm{Gal}(F^{\mathrm{ab}}/F) \longrightarrow 1,$$

it follows that the quotient $\mathbb{A}_F^\times/\overline{F^\times.(F_\infty^\times)^+}$ is isomorphic to the Hausdorff (indeed, profinite) group $\mathrm{Gal}(F^{\mathrm{ab}}/F)$. Now if F is either $\mathbb{Q}$ or an imaginary quadratic field, then the finiteness of $\mathcal{O}_F^\times$ implies the closedness of $F^\times.(F_\infty^\times)^+$ (cf. [45, §5.2]) and therefore the Hausdorff property for

$$\begin{aligned} F^\times\backslash\mathbb{A}_F^\times/F_\infty^\times &= F^\times\backslash\mathbb{A}_{F,\mathrm{f}}^\times \times \pi_0(F_\infty^\times) \\ &= \begin{cases} \mathbb{Q}^\times\backslash\mathbb{A}_{\mathrm{f}}^\times \times \{\pm 1\}, & \text{if } F = \mathbb{Q}; \\ F^\times\backslash\mathbb{A}_{F,\mathrm{f}}^\times, & \text{if } F \text{ is imaginary quadratic.} \end{cases} \end{aligned}$$

(1) $\Longrightarrow$ (3): When F is either $\mathbb{Q}$ or an imaginary quadratic field, then one checks that the natural projection

$$F^\times\backslash\, \pi_0(F_\infty^\times) \times \mathbb{A}_{F,\mathrm{f}}^\times \longrightarrow \varprojlim_K F^\times\backslash\, \pi_0(F_\infty^\times) \times \mathbb{A}_{F,\mathrm{f}}^\times/K$$

is an isomorphism; again, the finiteness of $\mathcal{O}_F^\times$ is crucial. $\square$

The upshot is that because in general

$$\mathrm{Sh}(\mathrm{Res}_{F/\mathbb{Q}}\,\mathbb{G}_{\mathrm{m},F}, \pi_0\, F_\infty^\times) \not\cong F^\times\backslash\, \pi_0(F_\infty^\times) \times \mathbb{A}_{F,\mathrm{f}}^\times,$$

the analysis of the symmetries of the Bost-Connes algebra $\mathcal{A}_F$ for F (see Definition 4.4) is complicated. In particular, it will be necessary to use the finer structure of the Bost-Connes groupoid $\mathfrak{Z}_F$ as a *stack-groupoid* in order to recover $\pi_0(\mathbb{A}_F^\times/F^\times)$ as the outer symmetry semigroup of $\mathcal{A}_F$. The coarse quotient of $\mathfrak{Z}_F$ simply won't do.

4.2.2 The Bost-Connes System for Number Fields

Having already discussed the Bost-Connes system for $\mathbb{Q}$ in numerous sections, we now move on to its analogue for arbitrary number fields. First, let us recall the setup. (For the reader's convenience some of the definitions for the general BCM system are repeated in this special case.)

Let F be a number field, and let $F_\infty := F \otimes_{\mathbb{Q}} \mathbb{R}$. The BCM datum that we attach to F is:

$$\begin{aligned}&(G, X, V, M; L, K, K_M)\\ &\qquad = (\mathrm{Res}_{F/\mathbb{Q}}\,\mathbb{G}_{\mathrm{m},F}, \pi_0(F_\infty^\times), F, \mathrm{Res}_{F/\mathbb{Q}}\, M_{1,F}; \mathcal{O}_F, \hat{\mathcal{O}}_F^\times, \hat{\mathcal{O}}_F) =: \mathcal{P}_F.\end{aligned}$$

To avoid typographical monstrosities, let us first set

$$Y_F := \hat{\mathcal{O}}_F \times \mathrm{Sh}(\mathrm{Res}_{F/\mathbb{Q}}\,\mathbb{G}_{\mathrm{m},F}, \pi_0(F_\infty^\times)).$$

To define the BCM groupoid, we first consider the cross product groupoid

$$\mathcal{G}_F := \mathbb{A}_{F,\mathrm{f}}^\times \ltimes Y_F \rightrightarrows Y_F$$

of the natural partial action of $\mathbb{A}_F^\times$ on Y_F. The group $\hat{\mathcal{O}}_F^\times$ acts on the left and right of $\mathcal{G}_F$ like so:

$$k.\big(g, (\rho, [x, l])\big) = \big(kg, (\rho, [x, l])\big), \quad \big(g, (\rho, [x, l])\big).k = \big(gk, (k^{-1}\rho, [k^{-1}x, lk])\big).$$

Definition 4.4. The *BCM groupoid for F* is the quotient

$$\mathfrak{Z}_F := \hat{\mathcal{O}}_F^\times \backslash Y_F / \hat{\mathcal{O}}_F^\times,$$

and the *BCM algebra for F* is the convolution algebra $\mathcal{A}_F = C_{\mathrm{c}}(Z_F)$.

Recall also that we have a *time evolution* for $\mathcal{A}_F$ given by

$$\sigma_t(f)(g, y) = N_{\mathbb{A}_F^\times}(g)^{-it} f(g, y), \quad f \in \mathcal{A}_F,\ (g, y) \in \mathbb{A}_{F,\mathrm{f}}^\times \ltimes Y_F,$$

where $N_{\mathbb{A}_F^\times}$ is the usual norm on idèles. Let $y \in Y_F$. Recall that the Hilbert space over this point is

$$\mathfrak{H}_y := \ell^2(\hat{\mathcal{O}}_F^\times \backslash G_y),$$

where $G_y = \{ g \in \mathbb{A}_{F,\mathrm{f}}^\times \mid gy \in Y_F \}$, and that $\mathcal{A}_F$ has a representation on it via convolution:

$$\pi_y(f)\xi(g) = \sum_{[h] \in \hat{\mathcal{O}}_F^\times \backslash G_y} f(gh^{-1}, hy)\, \xi(h).$$

In this representation, it is straightforward to show that the time evolution is generated by the unbounded *Hamiltonian* operator on $\mathfrak{H}_y$ defined by

$$(H_y\xi)(g) = \log\bigl(-N_{\mathbb{A}_F^\times}(g)\bigr)\, \xi(g).$$

That is,

$$\pi_y\bigl(\sigma_t(f)\bigr) = e^{iH_y t}\pi_y(f)e^{-iH_y t}.$$

For $F = \mathbb{Q}$, we recover all the ingredients of the original Bost-Connes system (Section 4.1.2). For F an imaginary quadratic field, we recover the system of Connes-Marcolli-Ramachandran [14].

In the following two subsections we completely analyze the symmetries and partition function of $\mathcal{P}_F$, and find that our Bost-Connes system for F has two very desirable features:

1. It recaptures the Dedekind zeta function as its partition function; and

2. Its outer symmetry semigroup is the full group $\pi_0(\mathbb{A}_F^\times/F^\times)$, which is isomorphic to $\mathrm{Gal}(F^{\mathrm{ab}}/F)$ by Artin reciprocity.

We emphasize, however, that there remains the difficult problem of classifying its KMS states and determining its F-rational subalgebra. This is what will be needed to extend the full arithmetic of the Bost-Connes system (Theorems 3.1 and 3.2) — intertwining of Galois actions through the evaluation of KMS_∞ states — to the case of general number fields. Aside from $F = \mathbb{Q}$, this has thus far only been completely carried out for imaginary quadratic F (see [14], where the analysis is drastically simpler than in the general case because the associated Shimura datum is classical, in the sense discussed in Section 2.4 and elsewhere).

Finite Symmetries and the Partition Function

Recall that for $(G, X, V, M; L, K, K_M)$, a BCM datum with level structure, the finite symmetry semigroup is $\mathcal{S}_\mathrm{f} = K_M \cap G(\mathbb{A}_\mathrm{f})$.

Theorem 4.1. *Let $T = \operatorname{Res}_{F/\mathbb{Q}} \mathbb{G}_{\mathrm{m},F}$. Then the finite symmetry semigroup*

$$\mathcal{S}_\mathrm{f}\big(T, \pi_0\, T(\mathbb{R})\big)$$

of the Bost-Connes system $\mathcal{P}_F$ is

$$\hat{\mathcal{O}}_F^* := \hat{\mathcal{O}}_F \cap \mathbb{A}_{F,\mathrm{f}}^\times,$$

and for $\beta > 1$, its partition function is the Dedekind zeta function

$$\zeta_F(\beta) = \sum_{\mathfrak{a} \subset \mathcal{O}_F} N(\mathfrak{a})^{-\beta}$$

(sum over integral ideals of F).

We begin by giving a more convenient description of the integral ideals of a number field.

Lemma 4.2. *Let F be a number field, and let $\mathcal{I}_F$ denote the semigroup of integral ideals of F. The semigroup homomorphism*

$$\begin{aligned} \psi\colon \hat{\mathcal{O}}_F \cap \mathbb{A}_{F,\mathrm{f}}^\times &\longrightarrow \mathcal{I}_F \\ a = (a_\nu)_\nu &\longmapsto \prod_{\nu \text{ finite}} \mathfrak{p}_\nu^{\nu(a_\nu)} \quad (\nu\text{: place of } F) \end{aligned}$$

is compatible with the (absolute) norm on ideals and the inverse (absolute) norm on idèles. It fits into the semigroup exact sequence

$$1 \longrightarrow \hat{\mathcal{O}}_F^\times \longrightarrow \hat{\mathcal{O}}_F \cap \mathbb{A}_{F,\mathrm{f}}^\times \xrightarrow{\psi} \mathcal{I}_F \longrightarrow 1.$$

In particular,

$$\mathcal{I}_F \cong \hat{\mathcal{O}}_F^\times \backslash \hat{\mathcal{O}}_F \cap \mathbb{A}_{F,\mathrm{f}}^\times.$$

Proof. Clearly the map ψ is surjective. The kernel of ψ is $\hat{\mathcal{O}}_F^\times$ since

$$\psi(a) = \mathcal{O}_F \iff |a_\nu|_\nu = 1 \text{ for all } \nu \iff a \in \hat{\mathcal{O}}_F^\times.$$

The compatibility of the absolute norms is the assertion

$$[\mathcal{O}_F : \psi(a)] =: N_{F/\mathbb{Q}}(\psi a) = N_{\mathbb{A}_F^\times}(a)^{-1} := \prod_\nu |a_\nu|_\nu^{-1},$$

whose validity follows from the multiplicativity of the norms and the equality $|a_\nu|_\nu = [\mathcal{O}_F : \mathfrak{p}_\nu]^{-\nu(a_\nu)}$. □

Proof of 4.1. The statement about the finite symmetry semigroup is immediate from its definition. As for the statement about the partition function, what we need to show is that

$$\mathrm{Tr}(e^{-\beta H_y}) = \zeta_F(\beta)$$

whenever $y = (\rho, z) \in Y_F = \hat{\mathcal{O}}_F \times \mathrm{Sh}(\mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_{\mathrm{m},F}, \pi_0(F_\infty^\times))$ is invertible, i.e., $\rho \in \hat{\mathcal{O}}_F^\times$. When this holds, then $g \in G_y$ if and only if $g\rho \in \hat{\mathcal{O}}_F$ if and only if $g \in \hat{\mathcal{O}}_F^*$, and so

$$\mathrm{Tr}(e^{-\beta H_y}) = \sum_{g \in \hat{\mathcal{O}}_F^\times \backslash G_y} N_{\mathbb{A}_F^\times}(g)^\beta = \sum_{g \in \hat{\mathcal{O}}_F^\times \backslash \hat{\mathcal{O}}_F^*} N_{\mathbb{A}_F^\times}(g)^\beta.$$

By the Lemma, this last sum is just another expression for $\zeta_F(\beta)$. □

The Full Symmetry Semigroup

We now identify the action of the full symmetry semigroup

$$\mathcal{S}(\mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_{\mathrm{m},F}, X_F) = \hat{\mathcal{O}}_F^* \times \mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_{\mathrm{m},F}(\mathbb{R}),$$

which contains archimedean information. That our Shimura-variety formalism for Bost-Connes-Marcolli systems is able to yield the following theorem is a strong indication of its basic correctness.

Theorem 4.2. *We have* $\mathrm{Inn}(\mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_{\mathrm{m},F}, X_F) = \mathcal{O}_F^* := \mathcal{O}_F - \{0\}$ *and the outer symmetry semigroup* $\mathrm{Out}(\mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_{\mathrm{m},F}, X_F)$ *acts on the BCM algebra* $\mathcal{H}_F$ *through*

$$\pi_0(F^\times \backslash \mathbb{A}_F^\times).$$

Proof. Recall from Eq. (3.15) that $\mathfrak{Z}_F$ can be written as a projective limit of groupoids $\mathfrak{Z}_{K'}$ for $K' \subset G(\mathbb{A}_\mathrm{f})$ compact open. The $\mathcal{S}$-action therefore induces an action of the projective limit semigroup $\varprojlim_{K'} \mathcal{S}/K'$ over all compact open $K' \subset G(\mathbb{A}_\mathrm{f})$. That is to say, $\mathcal{S}/K'$ acts on the corresponding piece $\mathfrak{Z}_{K'}$, and that this action is compatible with the projective systems. Recall that

$$\begin{aligned}\varprojlim_{K'} F^\times \backslash \big(\mathbb{A}_{F,\mathrm{f}}^\times / K' \times \pi_0(F_\infty^\times)\big) &=: \mathrm{Sh}(\mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_{\mathrm{m},F}, \pi_0(F_\infty^\times)) \\ &\cong \pi_0(F^\times \backslash \mathbb{A}_F^\times).\end{aligned}$$

(See Corollary 4.1.)

It thus remains to prove that the natural map

$$\mathcal{O}_F^* \backslash (\hat{\mathcal{O}}_F^* \times \pi_0(F_\infty^\times)) \longrightarrow F^\times \backslash (\mathbb{A}_{F,\mathrm{f}}^\times \times \pi_0(F_\infty^\times))$$

is an isomorphism. The injectivity of this map is clear because $\mathcal{O}_F^* := \mathcal{O}_F - \{0\} = \hat{\mathcal{O}}_F^* \cap F^\times$. Since $F^\times$ acts transitively on $\pi_0(F_\infty^\times)$, to prove surjectivity it suffices to prove surjectivity of the upper map of the following diagram:

$$\begin{tikzcd} \hat{\mathcal{O}}_F^* \arrow[r] \arrow[d] & F^\times \backslash \mathbb{A}_{F,\mathrm{f}}^\times \arrow[d] \\ \mathcal{O}_F^* \backslash \hat{\mathcal{O}}_F^* / \hat{\mathcal{O}}_F^\times \arrow[r, "\sim"] & F^\times \backslash \mathbb{A}_{F,\mathrm{f}}^\times / \hat{\mathcal{O}}_F^\times \end{tikzcd}$$

The lower arrow is an isomorphism because these two groups are equal to the ideal class group of F. Let $g \in \mathbb{A}_{F,\mathrm{f}}^\times$ be a finite idèle. Then its image by the vertical projection gives an ideal class, which is the image of some $m \in \hat{\mathcal{O}}_F^*$. We have $[m] = [g]$ in the right quotient so that there exists $k \in \hat{\mathcal{O}}_F^\times$ such that $g = mk \mod F^\times$. Then $mk \in \hat{\mathcal{O}}_F^*$ is in the preimage of the upper arrow of the diagram, which proves surjectivity. □

Remark 4.1. As mentioned before, analogous results were already obtained for F imaginary quadratic by Connes-Marcolli-Ramachandran (see [14]). In that case, the Shimura datum

$$(\mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_{\mathrm{m},F}, X_F)$$

is classical, so its analysis is simpler.

4.2.3 Dirichlet Characters for Bost-Connes Systems

Review of Zeta Functions of Dirichlet Characters

We here recall from Neukirch's book [39], p. 501, some facts about characters.

Definition 4.5. A *Hecke character* is a character of the idèle class group $C_F := \mathbb{A}_F^\times/F^\times$, i.e., a continuous homomorphism $\chi\colon C_F \to S^1$ to the group S^1 of complex numbers of norm 1. A *Dirichlet character* is a Hecke character that factors through the quotient group $(F_\infty^\times)^+\backslash\mathbb{A}_F^\times/F^\times$ where recall that "+" denotes the connected identity component for the real topology.

Let $\mathfrak{m} = \prod_\mathfrak{p} \mathfrak{p}^n$ be a full ideal of $\mathcal{O}_F$ and let $K(\mathfrak{m})$ be the kernel of the natural map

$$\hat{\mathcal{O}}_F^\times \to (\hat{\mathcal{O}}_F/\mathfrak{m})^\times.$$

We say that $\mathfrak{m}$ is *a module of definition* for the Dirichlet character χ if $\chi\big(K(\mathfrak{m})\big) = 1$. We then call $K(\mathfrak{m})$ *a subgroup of definition* for χ.

Each Dirichlet character has a module of definition and for such an $\mathfrak{m}$, we have a factorization $\chi\colon C(\mathfrak{m}) \to S^1$, where $C(\mathfrak{m}) = \big((F_\infty^\times)^+ \times K(\mathfrak{m})\big)\backslash\mathbb{A}_F^\times/F^\times$ is the big ray class group modulo $\mathfrak{m}$. Such an $\mathfrak{m}$ that is moreover minimal (among the modules of definition) is called *the conductor of the Dirichlet character.*

Recall that $\hat{\mathcal{O}}_F^* = \mathbb{A}_{F,\mathrm{f}}^\times \cap \hat{\mathcal{O}}_F$. If $\chi\colon \mathbb{A}_F^\times \to S^1$ is a Dirichlet character, we factor it through $(F_\infty^\times)^+\backslash\mathbb{A}_F^\times$, and thus restrict it to $\pi_0(F_\infty^\times) \times \hat{\mathcal{O}}_F^*$. Let $K(\mathfrak{m}) \subset \hat{\mathcal{O}}_F^\times$ be a primitive subgroup of definition for χ and let $K^*(\mathfrak{m}) := \{n \in \hat{\mathcal{O}}_F^* \mid \bar{n} = 1 \in \hat{\mathcal{O}}_F/\mathfrak{m}\}$.

There is an injective map $K(\mathfrak{m})\backslash K^*(\mathfrak{m}) \hookrightarrow \hat{\mathcal{O}}_F^\times\backslash\hat{\mathcal{O}}_F^*$ whose image is the semigroup of all ideals of F prime to $\mathfrak{m}$.

At least if χ is trivial at infinity, it induces $\chi\colon K(\mathfrak{m})\backslash K^*(\mathfrak{m}) \to S^1$. Now, we can define the L-function of our Dirichlet character χ as

$$L_F(s,\chi) = \sum_{n\in K(\mathfrak{m})\backslash K^*(\mathfrak{m})} \frac{\chi(n)}{N(n)^s},$$

where N is the norm map. In the particular case of a class character, we have

$$L_F(s,\chi) = \sum_{n\in\hat{\mathcal{O}}_F^\times\backslash\hat{\mathcal{O}}_F^*} \frac{\chi(n)}{N(n)^s}.$$

A Bost-Connes Algebra for Dirichlet Characters

Let $\chi\colon \mathbb{A}_F \to S^1$ be a Dirichlet character that we assume is trivial at infinity. Let $G = \mathrm{Res}_{F/\mathbb{Q}}\,\mathbb{G}_{\mathrm{m},F}$, and let $X := G(\mathbb{R})/G(\mathbb{R})^+ \cong \{\pm 1\}^{\mathrm{Hom}(F,\mathbb{R})}$. Let $\mathfrak{m}$ be the conductor of χ, and let $K_M(\mathfrak{m}) \subset \hat{\mathcal{O}}_F$ be the multiplicative semigroup defined by

$$K_M(\mathfrak{m}) = \ker_{\mathrm{mult}}(\hat{\mathcal{O}}_F \to \hat{\mathcal{O}}_F/\mathfrak{m}) := \{n \in \hat{\mathcal{O}}_F \mid \bar{n} = 1 \in \hat{\mathcal{O}}_F/\mathfrak{m}\}.$$

Recall that we denoted by $K(\mathfrak{m}) \subset \hat{\mathcal{O}}_F^\times$ the subgroup $K(\mathfrak{m}) = \ker\big(\hat{\mathcal{O}}_F^\times \to (\hat{\mathcal{O}}_F/\mathfrak{m})^\times\big)$. Let $L = \mathcal{O}_F$ and $\phi\colon G \to \mathrm{GL}_{\mathbb{Q}}(F)$ be the regular representation.

Definition 4.6. The level-structure BCM datum

$$\mathcal{M}_{F,\mathfrak{m}} := \big(\mathrm{Res}_{F/\mathbb{Q}}\,\mathbb{G}_{\mathrm{m},F}, X, F, \hat{\mathcal{O}}_F; L, K(\mathfrak{m}), K_M(\mathfrak{m})\big)$$

is called the *Bost-Connes datum of conductor* $\mathfrak{m}$.

The time evolution and Hamiltonian are the same as in the Bost-Connes case studied in Section 4.2.2.

Let a_χ be the operator on $\mathcal{H}_y$ defined by

$$(a_\chi \xi)(g) = \chi(g).\xi(g).$$

Definition 4.7. The *χ-twisted trace* Tr_χ on $B(\mathfrak{H}_y)$ is defined by

$$\mathrm{Tr}_\chi(D) = \mathrm{Tr}(a_\chi.D).$$

Definition 4.8. The *χ-twisted partition function* of $\mathcal{M}_{F,\mathfrak{m}}$ is defined as

$$\zeta_{\mathcal{M}_{F,\mathfrak{m}},\chi}(s) = \mathrm{Tr}_\chi(e^{-\beta H_y}).$$

Lemma 4.3. *The χ-twisted partition function of $\mathcal{M}_{F,\mathfrak{m}}$ is equal to the Dirichlet L-function $L_F(s,\chi)$.*

Proof. This follows from the definition and Subsection 4.2.3. □

Notice that in this case, the symmetry semigroup is not full in the sense of Definition 3.20.

4.3 Hilbert Modular Surfaces

We now specialize the general formalism of Section 3.3 to the case of Hilbert modular Shimura data. This is a good training ground in preparation for dealing with general Shimura data. It is clear that there is still much to be done.

4.3.1 Construction

Let F be a totally real number field, and let r be the number of real embeddings of F. Formally, the BCM system for a Hilbert modular surface — the Shimura variety for the datum $(\mathrm{Res}_{F/\mathbb{Q}}\,\mathrm{GL}_{2,F}, (\mathbb{H}^{\pm})^r)$ — is the same as the BCM system for elliptic modular curves (i.e., the Connes-Marcolli GL_2-system). Thus, consider the following datum.

$$\begin{aligned} G &= \mathrm{Res}_{F/\mathbb{Q}}\,\mathrm{GL}_{2,F} & L &= \mathcal{O}_F^2 \subset V \\ X &= (\mathbb{H}^{\pm})^r & K_0 &= \mathrm{GL}_2(\hat{\mathcal{O}}_F) \subset G(\mathbb{A}_{\mathrm{f}}) \\ V &= F^2 & K_M &= M_2(\hat{\mathcal{O}}_F) \subset M_2(\mathbb{A}_{F,\mathrm{f}}) \\ M &= \mathrm{Res}_{F/\mathbb{Q}}\,M_{2,F} \end{aligned}$$

Definition 4.9. Let K be a neat subgroup of K_0. The level-structure BCM datum

$$\mathcal{P}(G, X, K) = (G, X, V, M; L, K, K_M)$$

is called the *Hilbert modular BCM pair* for F. The BCM algebra $\mathcal{H}(\mathcal{P})$ is called a *Hilbert modular BCM algebra*.

Lemma 4.4. *If we suppose that F has class number one, then the natural morphism*

$$\mathcal{H}(\mathcal{P}) \longrightarrow \mathcal{H}_{\mathrm{princ}}(\mathcal{P})$$

from the principal to the full Bost-Connes-Marcolli algebra is an isomorphism.

Proof. The hypothesis implies (in fact is equivalent to) $h(G, K) = 1$. The result then follows from proposition 4.2. $\square$

We now describe more explicitly the time evolution whose construction was made in Section 3.3.6.

Let $C := \operatorname{Res}_{F/\mathbb{Q}} \mathbb{G}_{\mathrm{m},\mathbb{Q}}$, which is the center of $G = \operatorname{Res}_{F/\mathbb{Q}} \mathrm{GL}_2$. The natural determinant map $\det\colon G \to C$ induces $\det\colon K\backslash G(\mathbb{A}_{\mathrm{f}}) \to C(\hat{\mathbb{Z}})\backslash C(\mathbb{A}_{\mathrm{f}})$. The norm map $N\colon C \to \mathbb{G}_{\mathrm{m},\mathbb{Q}}$ induces

$$N\colon C(\hat{\mathbb{Z}})\backslash C(\mathbb{A}_{\mathrm{f}}) \longrightarrow \hat{\mathbb{Z}}^\times\backslash\mathbb{A}_{\mathrm{f}}^\times \cong \mathbb{Z}^\times\backslash\mathbb{Q}^\times \cong \mathbb{Q}_+^\times \subset \mathbb{R}_+^\times.$$

Lemma 4.5. *For the Hilbert modular BCM algebra $\mathcal{H}(G, X, K)$, the time evolution equals*

$$\sigma_t(f)(g, y) = N(\det(g))^{it} f(g, y).$$

4.3.2 Symmetries

We apply the general definitions of Section 3.3.7 to this case. We see that

$$\mathcal{S}_{\mathrm{f}}(\mathcal{P}) = M_2(\hat{\mathcal{O}}_F)^* := \mathrm{GL}_2(\mathbb{A}_{F,\mathrm{f}}) \cap M_2(\hat{\mathcal{O}}_F).$$

The center of $G = \operatorname{Res}_{F/\mathbb{Q}} \mathrm{GL}_{2,F}$ is $C = \operatorname{Res}_{F/\mathbb{Q}} \mathbb{G}_{\mathrm{m},\mathbb{Q}}$ and the center of $M_2(\hat{\mathcal{O}}_F)$ is $\hat{\mathcal{O}}_F$ as a diagonal subsemigroup. We also have

$$\operatorname{Inn}(\mathcal{P}) = \mathcal{O}_F^* := \mathcal{O}_F \cap F^\times$$

and an inclusion of semigroups $\mathcal{O}_F^* \subset M_2(\hat{\mathcal{O}}_F)^*$.

The following lemma explains what the symmetries are in the case of Hilbert modular BCM systems.

Proposition 4.6. *The outer symmetry semigroup* $\operatorname{Out}(\mathcal{P})$ *of the Hilbert modular BCM system is isomorphic to*

$$F^\times\backslash \mathrm{GL}_2(\mathbb{A}_{F,\mathrm{f}}) \times \operatorname{Res}_{F/\mathbb{Q}} \mathbb{G}_{\mathrm{m},F}(\mathbb{R}).$$

More precisely, the natural map

$$\mathcal{O}_F^*\backslash \mathcal{S}_{\mathrm{f}}(\mathcal{P}) = \mathcal{O}_F^*\backslash M_2(\hat{\mathcal{O}}_F)^* \longrightarrow F^\times\backslash \mathrm{GL}_2(\mathbb{A}_{F,\mathrm{f}})$$

is an isomorphism.

Proof. The injectivity of this map is clear because,

$$\mathcal{O}_F^* = F^\times \cap M_2(\hat{\mathcal{O}}_F)^*.$$

Let $(M_2(\hat{\mathcal{O}}_F)^*)^{-1} = \{m \in G(\mathbb{A}_\mathrm{f}) \mid m^{-1} \in M_2(\hat{\mathcal{O}}_F)\}$ be the semigroup of inverses of elements in $M_2(\hat{\mathcal{O}}_F)^*$. We then have

$$M_2(\hat{\mathcal{O}}_F)^*.(M_2(\hat{\mathcal{O}}_F)^*)^{-1} = \mathrm{GL}_2(\mathbb{A}_{F,\mathrm{f}}).$$

Let $m \in \mathcal{O}_F^* \backslash M_2(\hat{\mathcal{O}}_F)^*$. We only need to prove that $m^{-1} \in \mathcal{O}_F^* \backslash M_2(\hat{\mathcal{O}}_F)^*$. Moreover, to invert a matrix it is enough to prove that its determinant is invertible. We have $\det(m) \in \mathcal{O}_F^* \backslash \hat{\mathcal{O}}_F^*$. The nonarchimedean part of Proposition 4.2 gives $\mathcal{O}_F^* \backslash \hat{\mathcal{O}}_F^* \cong F^\times \backslash \mathbb{A}_{F,\mathrm{f}}^\times$, which implies that $\det(m)^{-1} \in \mathcal{O}_F^* \backslash \hat{\mathcal{O}}_F^* \subset \mathcal{O}_F^* \backslash M_2(\hat{\mathcal{O}}_F)^*$. This finishes the proof. $\square$

Chapter 5

Prospects

We close this thesis by considering some possible avenues for future research. The presentation is very informal, and no precise conjectures are made. Nevertheless, we thought it worthwhile to compile these ideas. At the very least, putting them down on paper makes is more likely that others will suggest more promising directions while discarding whatever rotten ideas lurk within.

To start, we seem to be happy about having found a good generalization of the Bost-Connes system for arbitrary number fields, but in truth, the really hard part is still left to be done! In particular, it still remains to classify the KMS states and understand their arithmetic properties. Moreover, the most interesting aspect of the Bost-Connes system (and also of the complex multiplication system of Connes-Marcolli-Ramachandran) is the way in which its low-temperature KMS states intertwine the Galois action on fields with the action of the idèle class group on the C*-dynamical system. In order to realize that for the general Bost-Connes system constructed in this thesis, it will be necessary to construct an appropriate rational subalgebra on which extremal zero-temperature KMS states evaluate to algebraic numbers. This seems to be a very difficult problem, but it seems not entirely unlikely that a moduli interpretation of the underlying Shimura variety — either as a parameter space for Hodge structures, or (conjecturally) for motifs (cf. [16]), if such sophistication is warranted — will be of use in finding this rational structure. For higher dimensional Shimura varieties it may be worthwhile to consider the same problem with the Milne-Shih reciprocity [37] in mind.

In the last chapter we setup the bare framework for the Shimura BCM in the case of Hilbert modular surfaces, and as further testing ground for the general constructions we are working out in more explicit detail the general features of the Shimura BCM in this "classical" setting. Since Hilbert modular surfaces are defined starting from a real quadratic number field, it would also be of interest to investigate the relation between the Hilbert modular BCM system, and Manin's Real Multiplication Program [32], which seeks to develop a geometric analogue for real quadratic fields of the theory of complex multiplication, by replacing elliptic curves by noncommutative 2-tori.

The Shimura BCM system is more than just a repackaging of the underlying Shimura variety, for the mere metamorphosis into a physical system reveals new lines of inquiry coming from statistical mechanics.

For example, the KMS theory enlarges the scope of the *Equidistribution Conjecture* regarding CM (aka special) points on Shimura varieties (see, for example [56] and [54]). This is already to be seen in the GL_2 system of Connes-Marcolli. W. Duke [17] showed that for the curve $M = \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$, suitably generic sequences $\{x_n\}$ of complex multiplication points of M have equidistributed Galois orbits $\mathcal{O}(x_n)$, i.e., for any bounded continuous function f on M, one has the asymptotic distribution

$$\lim_{N\to\infty} \frac{1}{|\mathcal{O}(x_N)|} \sum_{x\in\mathcal{O}(x_N)} f(x) = \int_M f(x)\,dx,$$

for the usual (appropriately normalized) measure dx on M. Interpreted ergodically, the LHS is a "time average" while the RHS is, in fact, the KMS_2 state of the Connes-Marcolli system evaluated on an easy extension of f to an element of the Connes-Marcolli C*-algebra A_2 (see [11, Prop. 1.25]). The Equidistribution Conjecture implies one of the central conjectures concerning the geometry of Shimura varieties, namely that of *André-Oort* which asserts that every closed subvariety that is the Zariski closure of its CM points is a "special" subvariety. Thus, we are lead immediately to the following question: Can arbitrary KMS states (in particular, the extremal and near-critical ones) also be understood ergodically as "time averages" in a manner that extends the equidistribution phenomena to aspects of the geometry of the Shimura variety more general than those of special subvarieties? (Part of the problem itself is to make sense of this vague formulation.)

It also remains to understand systematically the arithmetic significance of the zeta functions that arise in our construction, namely

$$Z(s) = \sum_{g \in S^\times \backslash S} (\det g)^{-s}, \quad \big(S = K_M \cap G(\mathbb{A}_\mathrm{f})\big).$$

Recall that the Shimura BCM system is known to realize as partition function the Riemann zeta function (Bost-Connes), and more generally, Dedekind zeta functions, and also $\zeta(\beta)\zeta(\beta-1)$ (Connes-Marcolli), which is the zeta function of $\mathbb{P}^1_{\mathbb{Q}}$. A natural question is now: How are the Shimura BCM zeta (partition) functions related to the very complicated zeta functions of Shimura varieties? This is at present not even very well-posed since we have made no use of canonical models in our Shimura BCM systems.

We do not yet know how to reap an arithmetical payoff from renormalization group methods in condensed matter physics. However, such a possibility is plausible after the work of Connes-Kreimer and Connes-Marcolli, [10], [12], on the geometry of renormalization in Quantum Field Theory, and its relation to certain categories of mixed Tate motives.

Finally, though we are far from exhausting the supply of problems, we mention just one more research prospect for the Shimura BCM, suggested to us by Yu. I. Manin. Recall that an essential ingredient of our Shimura BCM system was the augmentation of the starting Shimura datum (G, X) by an enveloping monoid (which stands in relation to G as M_n stands in relation to GL_n). Drinfeld, Renner, and Putcha have classified such monoids in terms of certain polyhedral cones in $\mathrm{Hom}(T, \mathbb{G}_\mathrm{m}) \otimes_{\mathbb{Z}} \mathbb{R}$ invariant under the Weyl group action (where $T \subset G$ is a maximal torus). When the Shimura variety $\mathrm{Sh}(G, X)$ itself has a moduli interpretation (e.g., abelian varieties with level structure), then variation through Drinfeld's rigidity condition leads, loosely speaking, to a sort of moduli of moduli that is reminiscent of the moduli of "stability conditions" recently developed by T. Bridgeland [6] in the setting of an arbitrary triangulated category, motivated by the work of string theorist Michael Douglas. It would be interesting to explore such an analogy in view of the many interesting features of the Shimura BCM system, such as zeta functions, that depend on the choice of enveloping monoid.

In Summary

It is not so far-fetched that noncommutative geometry should be relevant to the study of Shimura varieties, for as soon as one tries to define a naïve analogue of a Shimura variety by replacing the reductive group by a reductive monoid, one no longer encounters a variety in the classical sense — a noncommutative-geometric viewpoint, of one kind or another, is necessary to understand the resulting construction. We are reminded that, more than any other field of mathematics, number theory shows the worth, indeed necessity, of a diversity of viewpoints.

We have chosen to follow the philosophy of Alain Connes, but that is not the only perspective available,[1] nor even the most appropriate one necessarily, as we at present only have tools to treat our noncommutative varieties topologically/analytically, whereas Shimura varieties are arithmetic (having models over number fields).

The analytic approach, though, does have some powerful advantages that are absent in the purely algebraic realm. For example, by the Tomita-Connes-Takesaki theory [9, §5.3], every noncommutative von Neumann algebra has a canonical time evolution — up to inner automorphism — so measure-theoretically, every noncommutative space is a *nontrivial* dynamical space! Implicitly, this is what made the link to quantum statistical mechanics in our work.

[1]For an excellent survey of perspectives in noncommutative *algebraic* geometry, the reader should consult the recent Diplomarbeit of Snigdhayan Mahanta [31].

Bibliography

[1] J. Arledge, M. Laca, and I. Raeburn. Semigroup crossed products and Hecke algebras arising from number fields. *Doc. Math.*, 2:115–138 (electronic), 1997.

[2] A. Borel. *Linear Algebraic Groups* GTM. Springer-Verlag, Berlin, Second Edition, 1991.

[3] J.-B. Bost and A. Connes. Hecke algebras, type III factors and phase transitions with spontaneous symmetry breaking in number theory. *Selecta Math. (N.S.)*, 1(3):411–457, 1995.

[4] O. Bratteli and D. W. Robinson. *Operator algebras and quantum statistical mechanics. 1.* Texts and Monographs in Physics. Springer-Verlag, New York, second edition, 1987. C^*- and W^*-algebras, symmetry groups, decomposition of states.

[5] O. Bratteli and D. W. Robinson. *Operator algebras and quantum statistical mechanics. 2.* Texts and Monographs in Physics. Springer-Verlag, Berlin, second edition, 1997. Equilibrium states. Models in quantum statistical mechanics.

[6] T. Bridgeland. Stability conditions on triangulated categories. *arXiv*, (math.AG/0212237), 2004.

[7] P. B. Cohen. A C^*-dynamical system with Dedekind zeta partition function and spontaneous symmetry breaking. *J. Théor. Nombres Bordeaux*, 11(1):15–30, 1999. Les XXèmes Journées Arithmétiques (Limoges, 1997).

[8] A. Connes. Sur la théorie non commutative de l'intégration. In *Algèbres d'opérateurs (Sém., Les Plans-sur-Bex, 1978)*, volume 725 of *Lecture Notes in Math.*, pages 19–143. Springer, Berlin, 1979.

[9] A. Connes. *Noncommutative geometry.* Academic Press Inc., San Diego, CA, 1994.

[10] A. Connes and D. Kreimer. Hopf algebras, renormalization and noncommutative geometry. *Comm. Math. Phys.*, 199(1):203–242, 1998.

[11] A. Connes and M. Marcolli. From Physics to Number Theory via Noncommutative Geometry. Part I: Quantum Statistical Mechanics of $\mathbb{Q}$-lattices. *arXiv*, (math.NT/0404128), 2004.

[12] A. Connes and M. Marcolli. From Physics to Number Theory via Noncommutative Geometry, Part II: Renormalization, the Riemann-Hilbert correspondence, and motivic Galois theory. *arXiv*, (hep-th/0411114), 2004.

[13] A. Connes and M. Marcolli. $\mathbb{Q}$-lattices: Quantum statistical mechanics and Galois theory. *Journal of Geometry and Physics*, 56 (2006) 2–23.

[14] A. Connes, M. Marcolli, and N. Ramachandran. KMS states and complex multiplication. *arXiv*, (math.OA/0501424), 2005.

[15] P. Deligne. Travaux de Shimura. Sém. Bourbaki Février 71, Exposé 389, Lecture Notes in Math., vol. 244, Springer-Verlag, Berlin, 1971.

[16] P. Deligne. Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2*, pages 247–289. Amer. Math. Soc., Providence, R.I., 1979.

[17] W. Duke. Hyperbolic distribution problems and half-integral weight Maass forms. *Invent. Math.*, 92(1):73–90, 1988.

[18] T. Gómez. Algebraic stacks. *Proc. Indian Acad. Sci. (Math. Sci.)*, Vol. 111, No. 1, February 2001, pp. 1–31.

[19] E. Ha and F. Paugam. Bost-Connes-Marcolli systems for Shimura varieties. Part I. Definitions and formal analytic properties. *IMRP*, 5:237–286, 2005.

[20] R. Haag. *Local quantum physics.* Texts and Monographs in Physics. Springer-Verlag, Berlin, second edition, 1996. Fields, particles, algebras.

[21] R. Haag, N. M. Hugenholtz, and M. Winnink. On the equilibrium states in quantum statistical mechanics. *Comm. Math. Phys.*, 5:215–236, 1967.

[22] D. Harari and E. Leichtnam. Extension du phénomène de brisure spontanée de symétrie de Bost-Connes au cas des corps globaux quelconques. *Selecta Math. (N.S.)*, 3(2):205–243, 1997.

[23] N. M. Hugenholtz. C*-Algebras and Statistical Mechanics. *Proc. Symp. Pure Math.*, Vol. 38 (1982), Part 2, 407–465.

[24] S. Iyanaga. *The Theory of Numbers.* North-Holland mathematical library; v. 8, North-Holland, Amsterdam, 1969.

[25] M. M. Kapranov and V. A. Voevodsky. 2-categories and Zamolodchikov tetrahedra equations. In *Algebraic groups and their generalizations: quantum and infinite-dimensional methods (University Park, PA, 1991)*, volume 56 of *Proc. Sympos. Pure Math.*, pages 177–259. Amer. Math. Soc., Providence, RI, 1994.

[26] M. Laca. Semigroups of *-endomorphisms, Dirichlet series, and phase transitions. *J. Funct. Anal.*, 152(2):330–378, 1998.

[27] M. Laca and I. Raeburn A semigroup crossed product arising in number theory. *J. London Math. Soc. (2)*, 59(1):330–344, 1999.

[28] M. Laca and M. Frankenhuijsen. Phase transitions on Hecke C*-algebras and class-field theory over Q. *arXiv*, (math.OA/0410305), 2004.

[29] G. Laumon and L. Moret-Bailly. *Champs algébriques*, volume 39 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related*

Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer-Verlag, Berlin, 2000.

[30] T. Leinster. Up-to-Homotopy Monoids. *arXiv*, (math.QA/9912084), 2000.

[31] S. Mahanta. A Brief Survey of Non-Commutative Algebraic Geometry. *arXiv*, (math.QA/0501166), 2005.

[32] Yu. I. Manin. Real Multiplication and noncommutative geometry. *arXiv*, (math.AG/0202109), 2002.

[33] M. Marcolli. Lectures on Arithmetic Noncommutative Geometry. *arXiv*, (math.QA/0409520), 2004.

[34] J. S. Milne. The action of an automorphism of $\mathbb{C}$ on a Shimura variety and its special points. in *Arithmetic and geometry, Vol. I*, Vol. 35 Progr. Math., Birkhäuser Boston, Boston, p. 239–265.

[35] J. S. Milne. The points on a Shimura variety modula a prime of good reduction. in *The zeta functions of Picard modular surfaces*, Univ. Montreal, Montreal, QC, p. 151–253.

[36] J. S. Milne. Introduction to Shimura Varieties. 2004.

[37] J. S. Milne and K.-y. Shih. Automorphism groups of Shimura varieties and reciprocity laws. *Amer. J. Math.*, 103(5):911–935, 1981.

[38] B. Moonen. Models of Shimura varieties in mixed characteristics. in *Galoisrepresentations in arithmetic algebraic geometry (Durham, 1996)*, 267-350, London Math. Soc. Lecture Note Ser. 254, Cambridge Univ. Press, Cambridge, 1998.

[39] J. Neukirch. *Algebraische Zahlentheorie*. Springer-Verlag, Berlin, 1992.

[40] B. Noohi. Foundations of Topological Stacks I. *arXiv*, (math.AG/0503247), 2005.

[41] R. Pink. *Arithmetical compactification of mixed Shimura varieties*, volume 209 of *Bonner Mathematische Schriften [Bonn Mathematical Publications]*. Universität Bonn Mathematisches Institut, Bonn, 1990. Dissertation, Rheinische Friedrich-Wilhelms-Universität Bonn, Bonn, 1989.

[42] J. Renault. *A groupoid approach to C*-algebras*. Lecture Notes in Math., 793, Springer-Verlag, Berlin, 1980.

[43] L. Renner. *Linear Algebraic Monoids*. Springer-Verlag, Berlin, 2005.

[44] J. Rohlfs. Projective limits of locally symmetric spaces and cohomology. *J. reine. angew. Math.*, 479 (1996) 149–182.

[45] G. Shimura. *Introduction to the Arithmetic Theory of Automorphic Functions*. Princeton Univ. Press, Princeton, 1971.

[46] B. Simon. *The Statistical Mechanics of Lattice Gases*. Princeton Series in Physics, Princeton Univ. Press, Princeton, 1993.

[47] W. Thirring. *Quantum Mathematical Physics*. Springer-Verlag, Berlin, 2002.

[48] *Théorie des topos et cohomologie étale des schémas. Tome 3*. Springer-Verlag, Berlin, 1973. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck et J. L. Verdier. Avec la collaboration de P. Deligne et B. Saint-Donat, Lecture Notes in Mathematics, Vol. 305.

[49] C. Simpson. A closed model structure for n-categories, internal Hom, n-stacks and generalized Seifert-Van Kampen. *arXiv*, (math.AG/9704006), 1997.

[50] T. A. Springer. *Linear algebraic groups*, volume-9 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1981.

[51] Z. Tamsamani. On non-strict notions of n-category and n-groupoid via multisimplicial sets. *arXiv*, (math.AG/9512006), 1995.

[52] J. T. Tate. Fourier analysis in number fields, and Hecke's zeta-functions. In *Algebraic Number Theory (Proc. Instructional Conf., Brighton, 1965)*, pages 305–347. Thompson, Washington, D.C., 1967.

[53] B. Toen and G. Vezzosi. Homotopical Algebraic Geometry II: geometric stacks and applications. *arXiv*, (math.AG/0404373), 2004.

[54] E. Ullmo. Théorie ergodique et géométrie arithmétique. Proceedings of the ICM, Beijing 2002, vol. 2, 197–206. *arXiv*, (math.NT/0304236), 2003.

[55] E. B. Vinberg. On reductive algebraic semigroups. Amer. Math. Soc. Transl., Series 2, 169(1994), 145–182.

[56] S.-W. Zhang. Elliptic curves, L-functions, and CM-points. In *Current developments in mathematics, 2001*, pages 179–219. Int. Press, Somerville, MA, 2002.