Fermat Reals

Nilpotent Infinitesimals and Infinite Dimensional Spaces

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If we do not believe in the existence of God, then, from Gödel's ontological theorem it follows that an absolute, necessary and not only possible moral system cannot exists. But I believe that the human kind can achieve this type of moral system, so I have to believe in God.

If you further assume, suitably formalized by a rigorous mathematical language, that any good thing has in God its first cause and that mathematics is a good thing, then you cannot believe in genius anymore. The simple consequences for the everyday work of a mathematician and, more in general, for scientific collaboration are left to the reader.

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Abstract and structure

The main aim of the present work is to start a new theory of actual infinitesimals, called *theory of Fermat reals*. After the work of A. Robinson on nonstandard analysis (NSA), several theories of infinitesimals have been developed: synthetic differential geometry, surreal numbers, Levi-Civita field, Weil functors, to cite only some of the most studied. We will discuss in details of these theories and their characteristics, first of all comparing them with our Fermat reals. One of the most important differences is the philosophical thread that guided us during all the development of the present work: we tried to construct a theory with a strong intuitive interpretation and with non trivial applications to the infinite-dimensional differential geometry of spaces of mappings. This driving thread tried to develop a good dialectic between formal properties, proved in the theory, and their informal interpretations. The dialectic has to be, as far as possible, in both directions: theorems proved in the theory should have a clear and useful intuitive interpretation and, on the other hand, the intuition corresponding to the theory has to be able to suggest true sentences, i.e. conjectures or sketch of proofs that can then be converted into rigorous proofs. Almost all the present theories of actual infinitesimals are either based on formal approaches, or are not useful in differential geometry. As a meaningful example, we can say that the Fermat reals can be represented geometrically (i.e. they can be drawn) respecting the total order relation.

The theory of Fermat reals takes a strong inspiration from synthetic differential geometry (SDG), a theory of infinitesimals grounded in Topos theory and incompatible with classical logic. SDG, also called smooth infinitesimal analysis, originates from the ideas of Lawvere [1979] and has been greatly developed by several categorists. The result is a powerful theory able to develop both finite and infinite dimensional differential geometry with a formalism that takes great advantage of the use of infinitesimals. This theory is however incompatible with classical logic and one is forced to work in intuitionistic logic and to construct models of SDG using very elaborated topoi. The theory of Fermat reals is sometimes formally very similar to SDG and indeed, several proofs are simply a reformulation in our theory of the corresponding proofs in SDG. However, our theory of Fermat reals is fully compatible with classical logic. We can thus describe our work as a way to bypass an impossibility theorem of SDG, i.e. a way considered as impossible by several researchers. The differences between the two theories are due to our constraint to have always a good intuitive interpretation, whereas SDG develops a more formal approach to infinitesimals.

Generally speaking, we have constructed a theory of infinitesimals which does not need a background of logic to be understood. On the contrary, nonstandard analysis and SDG need this non trivial background, and this is a great barrier for potential users like physicists or engineers or even several

mathematicians. This is a goal strongly searched in NSA, so as to facilitate the diffusion of the theory.

Many parts of our construction are completely constructive and this result, also considered by several researchers in NSA, opens good possibilities for a computer implementation of our Fermat reals, with interesting potential applications in automatic proof theory or in automatic differentiation theory.

Our infinitesimals h, like in SDG, are *nilpotent* so that we have $h \neq 0$, but h is "so small" that for some power $n \in \mathbb{N}_{>1}$ we have $h^n = 0$. This permits to obtain an equality between a function and its tangent straight line in a first order infinitesimal neighborhood, i.e.

$$f(x+h) = f(x) + h \cdot f'(x), \tag{0.0.1}$$

where $h^2 = 0$. More generally, we will prove infinitesimal Taylor's formulas without any rest, so that every smooth functions, in our framework, is equal to a k-th order polynomial in every k-th order infinitesimal neighborhood.

The second part of the work is devoted to the development of a theory of smooth infinite dimensional spaces, first of all thinking applications in differential geometry. Our approach is based on a generalization of the notion of diffeology (see e.g. Iglesias-Zemmour [2008]). This permits to obtain a cartesian closed complete and cocomplete category in which the category of smooth manifolds is embedded. Using the above mentioned generalization we can obtain a category containing the extension of all smooth manifolds using our new infinitesimal points. We have hence the category $\boldsymbol{\mathcal{C}}^\infty$ of diffeological spaces, which contains all the smooth manifolds, and a functor ${}^\bullet(-) : \boldsymbol{\mathcal{C}}^\infty \longrightarrow {}^\bullet\boldsymbol{\mathcal{C}}^\infty$, called Fermat functor, which extends every space $X \in \boldsymbol{\mathcal{C}}^\infty$ adding infinitesimal points. E.g. the ring of Fermat reals is ${}^\bullet\mathbb{R} := {}^\bullet(\mathbb{R})$. The above mentioned categorical properties of these categories permits to say that we can construct infinite products, spaces of mapping, infinite sums, quotient spaces and we also have that mappings like composition, insertion, evaluation, and practically all the interesting set-theoretical operations are always smooth. We have hence a flexible framework where infinitesimal methods are also available.

Moreover, the Fermat functor possesses very good properties: it preserves products of manifolds and intersections, unions, inclusions, counter-images of open sets, to cite some of them. We will study in general this preservation properties, discovering some relationships between the Fermat functor and intuitionistic logic.

In the third part of the work we will present the basis for the whole development of the differential and integral calculus both for smooth functions defined on open sets of Fermat reals and on infinitesimal domains. We also give some first results of differential geometry using infinitesimal methods, always considering the case of the space of all the smooth mappings between two manifolds. A very general proof of the Euler-Lagrange equations is also given, with Lagrangians defined on spaces of mappings of the general form

$$^\bullet\mathcal{C}^\infty(^\bullet M_s, {}^\bullet\mathcal{C}^\infty(^\bullet M_{s-1}, \cdots, {}^\bullet\mathcal{C}^\infty(^\bullet M_2, {}^\bullet\mathbb{R})\cdots),$$

where M_i are manifolds. The space $^\bullet M_i \in {}^\bullet\mathcal{C}^\infty$ is the application of the Fermat functor to the manifold M_i, so that it can be thought as the manifold with the adding of our new infinitesimal points. In this section we also give a sketch of some ideas for a further development of the present work.

The fourth part of the work is composed of an appendix that fixes common notations for the concepts of category theory that we have used, and of a detailed study about the comparison of our theory and other theories of infinitesimals.

The detailed structure of our work is as follows. After a motivational Chapter 1 where we will also give an explanation for the name *Fermat reals*, in Chapter 2 we will define the ring $^\bullet\mathbb{R}$ of Fermat reals and the ideals D_k of k-th order infinitesimals. Having a ring which contains nilpotent elements, one of the most difficult algebraic problem is the dealing with products of powers of these nilpotent numbers. In this chapter we will also prove several effective results that permits to solve these powers (i.e. to decide whether they are zero or not) in an algorithmic way.

The derivative $f'(x)$ in a Taylor's formula like (0.0.1) is determined only up to second order infinitesimals. In Chapter 3 we will deeply study these equality up to k-th order infinitesimals, the corresponding cancellation law and its application to Taylor's formulas.

In Chapter 4 we will define the total order relation. We will show that, generally speaking, the order relation can be total only if the derivative in (0.0.1) is *not* uniquely determined. In this chapter we will also prove that the Fermat reals are in bijective correspondence with suitable curves of the plane $\mathbb{R}^2$, i.e. the geometrical representation of $^\bullet\mathbb{R}$.

Chapter 5 starts the second part of the work, devoted to our approach to infinite dimensional spaces. In this chapter we review the most studied approaches to infinite dimensional spaces used in differential geometry: Banach manifolds and locally convex vector spaces, the convenient vector spaces settings, diffeological spaces and SDG, presenting some of their positive features and some possible deficiencies.

In Chapter 6 we present our generalization of the notion of diffeological space, which permits to define in the same framework both the categories $\mathcal{C}^\infty$ and $^\bullet\mathcal{C}^\infty$, respectively domain and codomain of the Fermat functor. We called this generalization *the cartesian closure of a given category of figures*.

In Chapter 7 the cartesian closure is applied to the category of open sets in spaces of the form $\mathbb{R}^n$ and smooth mappings, obtaining the category $\mathcal{C}^\infty$ of diffeological spaces. We review the embedding of smooth finite dimensional manifolds and give several examples: infinite dimensional manifolds modeled on convenient vector spaces (which include manifolds modeled on Banach spaces), integro-differential operators, set-theoretical operations like compositions and evaluations, and we prove that the space of all the diffeomorphisms between two manifolds is a Lie group.

In Chapter 8 we generalize the construction of the Fermat ring ${}^\bullet\mathbb{R}$ to any smooth diffeological space $X \in \boldsymbol{\mathcal{C}}^\infty$ and we define the category ${}^\bullet\boldsymbol{\mathcal{C}}^\infty$ of smooth Fermat spaces, which includes all the spaces of the form ${}^\bullet X$.

Chapter 9 starts the study of the Fermat functor ${}^\bullet(-) : \boldsymbol{\mathcal{C}}^\infty \longrightarrow {}^\bullet\boldsymbol{\mathcal{C}}^\infty$ that extends every smooth space $X \in \boldsymbol{\mathcal{C}}^\infty$ by adding infinitesimal points. We prove that this functor preserves products of manifolds and we prove that manifolds are also embedded in the category ${}^\bullet\boldsymbol{\mathcal{C}}^\infty$. In this chapter we also prove that a standard part functor, right adjoint of the Fermat functor, does not exists. This correspond to analogous results dealing with the standard part map in constructive NSA.

We then study, in Chapter 10, the logical properties of the Fermat functor, i.e. all the logical operations which are preserved by it. We will see that, even if the theory of Fermat reals is fully compatible with classical logic, the best properties of this functor are present in the case of an intuitionistic interpretation of these logical operations, confirming the good dialectic between smooth differential geometry and intuitionistic logic.

The third part of the work starts with the study, in Chapter 11, of the development of the basis for the differential and integral calculus of smooth functions $f : {}^\bullet U \longrightarrow {}^\bullet\mathbb{R}^d$ defined on an open set ${}^\bullet U \subseteq {}^\bullet\mathbb{R}^n$. These functions generalize the standard smooth functions and can be expressed, locally, as the extension of standard smooth functions ${}^\bullet\alpha(p, -)$ with a fixed parameter $p \in {}^\bullet\mathbb{R}^{\mathsf{p}}$. The differential calculus is based on the analogous, in SDG, of the Fermat-Reyes property, and formalizes perfectly the informal methods used originally by P. de Fermat. In this chapter we also prove the inverse function theorem in ${}^\bullet\mathbb{R}$ and the existence of primitives, which represent a non trivial problem in non-Archimedean fields.

Due to the connections between total order and nilpotent infinitesimals, the differential calculus for function defined on infinitesimal sets, like $D_k = \left\{h \in {}^\bullet\mathbb{R} \,|\, h^{k+1} = 0\right\}$, must be developed using the properties of the equality up to k-th order infinitesimals. This is done in Chapter 12.

The purpose of Chapter 13 is to show the possibilities of the theory of Fermat reals for differential geometry, in particular for spaces of mappings. We essentially develop only tangency theory and the existence of integral curves using infinitesimal methods. We devoted a particular attention to always include in our results spaces of the form ${}^\bullet M^{{}^\bullet N} \in {}^\bullet\boldsymbol{\mathcal{C}}^\infty$ for M and N manifolds. In this chapter we also prove the above mentioned general version of the Euler-Lagrange equations.

In the final Chapter 14 we sketch the ideas of some possible further developments of our work.

Part I

Algebraic and order properties of Fermat reals

Chapter 1

Introduction and general problem

Frequently in work by physicists it is possible to find informal calculations like

$$\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = 1 + \frac{v^2}{2c^2} \qquad \sqrt{1-h_{44}(x)} = 1 - \frac{1}{2}h_{44}(x) \tag{1.0.1}$$

with explicit use of infinitesimals $v/c \ll 1$ or $h_{44}(x) \ll 1$ such that e.g. $h_{44}(x)^2 = 0$. For example Einstein [1926] (pag. 14) wrote the formula (using the equality sign and not the approximate equality sign $\simeq$)

$$f(x, t+\tau) = f(x,t) + \tau \cdot \frac{\partial f}{\partial t}(x,t) \tag{1.0.2}$$

justifying it with the words "*since τ is very small*"; the formulas (1.0.1) are a particular case of the general (1.0.2). Dirac [1975] wrote an analogous equality studying the Newtonian approximation in general relativity.

Using this type of infinitesimals we can write an *equality*, in some infinitesimal neighborhood, between a smooth function and its tangent straight line, or, in other words, a Taylor's formula without remainder. Informal methods based on actual infinitesimals are sometimes used in differential geometry too. Some classical examples are the following: a tangent vector is an infinitesimal arc of curve traced on the manifold and the sum of tangent vectors is made using infinitesimal parallelograms; tangent vectors to the tangent bundle are infinitesimal squares on the manifold; a vector field is sometimes intuitively treated as an "infinitesimal transformation" of the space into itself and the Lie brackets of two vector fields as the commutator of the corresponding infinitesimal transformations.

There are obviously many possibilities to formalize this kind of intuitive reasonings, obtaining a more or less good dialectic between informal and formal thinking, and indeed there are several theories of actual infinitesimals (from now on, for simplicity, we will say "infinitesimals" instead of "actual infinitesimals" as opposed to "potential infinitesimals"). Starting from these theories we can see that we can distinguish between two type of definitions of infinitesimals: in the first one we have at least a ring R containing the real field $\mathbb{R}$ and infinitesimals are elements $\varepsilon \in R$ such that $-r < \varepsilon < r$ for every positive standard real $r \in \mathbb{R}_{>0}$. The second type of infinitesimal is defined using some algebraic property of nilpotency, i.e. $\varepsilon^n = 0$ for some natural number $n \in \mathbb{N}$. For some ring R these definitions can coincide, but anyway they lead, of course, only to the trivial infinitesimal $\varepsilon = 0$ if $R = \mathbb{R}$.

However these definitions of infinitesimals correspond to theories which are completely different in nature and underlying ideas. Indeed these theories can be seen in a more interesting way to belong to two different classes. In the first one we can put theories that need a certain amount of non trivial results of mathematical logic, whereas in the second one we have attempts to define sufficiently strong theories of infinitesimals without the use of non trivial results of mathematical logic. In the first class we have Non-Standard Analysis (NSA) and Synthetic differential geometry (SDG, also called Smooth Infinitesimal Analysis), in the second one we have, e.g., Weil functors, Levi-Civita fields, surreal numbers, geometries over rings containing infinitesimals (see Appendix B for an introduction to several approaches to infinitesimals, together with a first comparison with our approach, and for references). More precisely we can say that to work in NSA and SDG one needs a formal control deeply stronger than the one used in "standard mathematics". In NSA one needs this control to apply the transfer theorem and in SDG one has to be sufficiently formal to be sure that the proofs can be seen as belonging to intuitionistic logic. Indeed to use NSA one has to be able to formally write the sentences one needs to transfer. Whereas SDG does not admit models in classical logic, but in intuitionistic logic only, and hence we have to be sure that in our proofs there is no use of the law of the excluded middle, or e.g. of the classical part of De Morgan's law or of some form of the axiom of choice or of the implication of double negation toward affirmation and any other logical principle which is not valid in intuitionistic logic. Physicists, engineers, but also the greatest part of mathematicians are not used to have this strong formal control in their work, and it is for this reason that there are attempts to present both NSA and SDG reducing as much as possible the necessary formal control, even if at some level this is technically impossible (see e.g. Henson [1997], and Benci and Di Nasso [2003, 2005] for NSA; Bell [1998] and Lavendhomme [1996] for SDG, where using an axiomatic approach the authors try to postpone the very difficult construction of an intuitionistic model of a whole set theory using Topos).

On the other hand NSA is essentially the only theory of infinitesimals with a discrete diffusion and a sufficiently great community of working mathematicians and published results in several areas of mathematics and its applications, see e.g. Albeverio et al. [1988]. SDG is the only theory of infinitesimals with non trivial, new and published results in differential geometry concerning infinite dimensional spaces like the space of all the diffeomorphisms of a generic (e.g. non compact) smooth manifold. In NSA we have only few results concerning differential geometry (we cite Schlesinger [1997] and Hamad [2007], and references therein, where NSA methods are used in problems of differential geometry). Other theories of infinitesimals have not, at least up to now, the same formal strength of NSA or SDG or the same potentiality to be applied in several different areas of mathematics.

One of the aim of the present work is to find a theory of infinitesimals within "standard mathematics" (in the precise sense explained above of a formal control more "standard" and not so strong as the one needed e.g. in NSA or SDG) with results comparable with those of SDG, without forcing the reader to learn a strong formal control of the mathematics he is doing. Because it has to be considered inside "standard mathematics", our theory of infinitesimals must be compatible with classical logic. Let us note that this is not incompatible with the possibility to obtain some results that need a strong formal control (like, e.g., a transfer theorem), because they represent a good potential instrument for the reader that likes such a strong formal control, but they do not force, concretely, all the readers to have such a formal aptitude. For these reasons, we think that it is wrong to frame the present work as in opposition to NSA or SDG.

Concretely, the idea of the present work is to by-pass the impossibility theorem about the incompatibility of SDG with classical logic that forces SDG to find models within intuitionistic logic. This by-pass has to be made, as much as possible, keeping the same properties and final results. We think that the obtained result is meaningful not only for differential geometry, but also for other fields, like the calculus of variations, and we will give a first sketch of results in this direction.

Another point of view about a powerful theory like NSA is that, in spite of the fact that frequently it is presented using opposed motivations, it lacks the intuitive interpretation of what the powerful formalism permits to do. E.g. what is the intuitive meaning and usefulness of $^\circ\sin(I) \in \mathbb{R}$, i.e. the standard part of the sine of an infinite number $I \in {}^*\mathbb{R}$? This and the above-mentioned "strong formal control" needed to work in NSA, together with very strong but scientifically unjustified cultural reasons, may be some motivations for the not so high success of the spreading of NSA in mathematics, and consequently in its didactics.

Analogously in SDG from the intuitive, classical, point of view, it is a little strange that we cannot exhibit "examples" of infinitesimals (indeed in SDG it is only possible to prove that $\neg\neg\exists\, d \in D$, where $D = \{h \in R \mid h^2 = 0\}$ is the set of first order infinitesimals). Because of this, e.g., we cannot construct a physical theory containing a fixed infinitesimal parameter; another example of a counter intuitive property is that any $d \in D$ is, at the same time, positive $d > 0$ and negative $d < 0$. Similar counter intuitive properties can be found in other theories of infinitesimals that use ideals of rings of polynomials as a formal scheme to construct particular type of infinitesimals. Among these theories we can cite "Weil functors" (see Kolář et al. [1993] and Kriegl and Michor [1996] and Appendix B of the present work for other references) and "differential geometry over general base fields and rings" (see Bertram [2008] and Appendix B). The final conclusion after the establishment of this type of counter intuitive examples (even if, of course, in these theories there are also several intuitively clear examples and concepts), is that if one wants to work in these types of frameworks, sometimes one has to follow a completely formal point of view, loosing the dialectic with the corresponding intuitive meaning.

Another aim of the present work is to construct a new theory of infinitesimals preserving always a very good dialectic between formal properties and intuitive interpretation. A first hint to show this positive feature of our construction is that our is the first theory, as far as we know, where it is possible to represent geometrically its new type of numbers[1], and it is undeniable that to be able to represent standard real numbers by a straight line inspired, and it still inspires, several mathematicians.

More technically we want to show that it is possible to extend the real field adding nilpotent infinitesimals, arriving at an enlarged real line ${}^\bullet\mathbb{R}$, by means of a very simple construction completely inside "standard mathematics". Indeed to define the extension ${}^\bullet\mathbb{R} \supset \mathbb{R}$ we shall use elementary analysis only. To avoid misunderstandings is it important to clarify that the purpose of the present work is not to give an alternative foundation of differential and integral calculus (like NSA), but to obtain a theory of nilpotent infinitesimals and to use it for the foundation of a smooth ($\mathcal{C}^\infty$) differential geometry, in particular in the case of infinite dimensional spaces, like the space of all the smooth functions $\mathbf{Man}(M; N)$ between two generic manifolds (e.g. without compactness hypothesis on the domain M). This focus on the foundation of differential geometry only, without including the whole calculus, is more typical of SDG, Weil functors and geometries over generic rings.

The usefulness of the extension ${}^\bullet\mathbb{R} \supset \mathbb{R}$ can be glimpsed by saying e.g. that using ${}^\bullet\mathbb{R}$ it is possible to write in a completely rigorous way that a smooth function is equal to its tangent straight line in a first order neighborhood; it is possible to use infinitesimal Taylor's formulas without remainder; to define a tangent vector as an infinitesimal curve and sum them using infinitesimal parallelograms; to see a vector field as an infinitesimal transformation, in general to formalize these and many other non-rigorous methods used in physics and geometry. This is important both for didactic reasons

[1] I.e. it is possible to establish a bijective correspondence between suitable lines of the plane and the numbers belonging to a given infinitesimal neighborhood.

and because it was by means of these methods that mathematicians like S. Lie and E. Cartan were originally led to construct important concepts of differential geometry.

We can use the infinitesimals of ${}^\bullet\mathbb{R}$ not only as a good language to reformulate well-known results, but also as a very useful tool to construct, in a simple and meaningful way, a differential geometry in classical infinite-dimensional objects like $\mathbf{Man}(M, N)$ the space of all the $\mathcal{C}^\infty$ mapping between two manifolds M, N. Here with "simple and meaningful" we mean the idea to work directly on the geometric object in an intrinsic way without being forced to use charts, but using infinitesimal points (see Lavendhomme [1996]). Some important examples of spaces of mappings used in applications are the space of configurations of a continuum body, groups of diffeomorphisms used in hydrodynamics as well as in magnetohydrodynamics, electromagnetism, plasma dynamics, and paths spaces for calculus of variations (see Kriegl and Michor [1997], Abraham et al. [1988], Albeverio et al. [1997, 1988], Albeverio [1997] and references therein). Interesting applications in classical field theories can also be found in Abbati and Manià [2000].

1.1 Motivations for the name "Fermat reals"

It is well known that historically two possible reductionist constructions of the real field starting from the rationals have been made. The first one is Dedekind's order completion using sections of rationals, the second one is Cauchy's metric space completion. Of course there are no historical reason to attribute our extension ${}^\bullet\mathbb{R} \supset \mathbb{R}$ of the real field, to be described below, to Fermat, but there are strong motivations to say that, probably, he would have liked the underlying spirit and some properties of our theory. For example:

1. we will see that a formalization of Fermat's infinitesimal method to derive functions is provable in our theory. We recall that Fermat's idea was, roughly speaking and not on the basis of an accurate historical analysis which goes beyond the scope of the present work (see e.g. Bottazzini et al. [1992], Edwards [1979], Eves [1990]), to suppose first $h \neq 0$, to construct the incremental ratio

$$\frac{f(x+h) - f(x)}{h}$$

and, after suitable simplifications (sometimes using infinitesimal properties), to take in the final result $h = 0$.

2. Fermat's method to find the maximum or minimum of a given function $f(x)$ at $x = a$ was to take e to be extremely small so that the value of

$f(x+h)$ was approximately equal to that of $f(x)$. In modern, algebraic language, it can be said that $f(x+h) = f(x)$ only if $h^2 = 0$, that is if e is a first order infinitesimal. Fermat was aware that this is not a "true" equality but some kind of approximation (see e.g. Bottazzini et al. [1992], Edwards [1979], Eves [1990]). We will follow a similar idea to define ${}^\bullet\mathbb{R}$ introducing a suitable equivalence relation to represent this equality.

3. Fermat has been described by Bell [1937] as "the king of amateurs" of mathematics, and hence we can suppose that in its mathematical work the informal/intuitive part was stronger with respect to the formal one. For this reason we can think that he would have liked our idea to obtain a theory of infinitesimals preserving always the intuitive meaning and without forcing the working mathematician to be too much formal.

For these reason we chose the name "Fermat reals" for our ring ${}^\bullet\mathbb{R}$ (note: without the possessive case, to underline that we are not attributing our construction of ${}^\bullet\mathbb{R}$ to Fermat).

We already mentioned that the use of nilpotent infinitesimals in the ring ${}^\bullet\mathbb{R}$ permits to develop many concepts of differential geometry in an intrinsic way without being forced to use coordinates, as we shall see in some examples in the course of the present work. In this way the use of charts becomes specific of stated areas, e.g. where one strictly needs some solution in a finite neighborhood and not in an infinitesimal one only (e.g. this is the case for the inverse function theorem). We can call *infinitesimal differential geometry* this kind of intrinsic geometry based on the ring ${}^\bullet\mathbb{R}$ (and on extensions of manifolds ${}^\bullet M$ and also on more generic object like the exponential objects ${}^\bullet M^{{}^\bullet N}$, see the second part of the present work).

Chapter 2

Definition and algebraic properties of Fermat reals

2.1 The basic idea

We start from the idea that a smooth ($\mathcal{C}^\infty$) function $f : {}^\bullet\mathbb{R} \longrightarrow {}^\bullet\mathbb{R}$ is actually *equal* to its tangent straight line in the first order neighborhood e.g. of the point $x = 0$, that is

$$\forall h \in D : \quad f(h) = f(0) + h \cdot f'(0) \tag{2.1.1}$$

where D is the subset of ${}^\bullet\mathbb{R}$ which defines the above-mentioned neighborhood of $x = 0$. The equality (2.1.1) can be seen as a first-order Taylor's formula without remainder because intuitively we think that $h^2 = 0$ for any $h \in D$ (indeed the property $h^2 = 0$ defines the first order neighborhood of $x = 0$ in ${}^\bullet\mathbb{R}$). These almost trivial considerations lead us to understand many things: ${}^\bullet\mathbb{R}$ must necessarily be a ring and not a field because in a field the equation $h^2 = 0$ implies $h = 0$; moreover we will surely have some limitation in the extension of some function from $\mathbb{R}$ to ${}^\bullet\mathbb{R}$, e.g. the square root, because using this function with the usual properties, once again the equation $h^2 = 0$ implies $|h| = 0$. On the other hand, we are also led to ask whether (2.1.1) uniquely determines the derivative $f'(0)$: because, even if it is true that we cannot simplify by h, we know that the polynomial coefficients of a Taylor's formula are unique in classical analysis. In fact we will prove that

$$\exists!\, m \in \mathbb{R}\ \forall h \in D : \quad f(h) = f(0) + h \cdot m \tag{2.1.2}$$

that is the slope of the tangent is uniquely determined in case it is an ordinary real number. We will call formulas like (2.1.2) *derivation formulas*.

If we try to construct a model for (2.1.2) a natural idea is to think our new numbers in ${}^\bullet\mathbb{R}$ as equivalence classes $[h]$ of usual functions $h : \mathbb{R} \longrightarrow \mathbb{R}$. In this way we may hope both to include the real field using classes generated by constant functions, and that the class generated by $h(t) = t$ could be a first order infinitesimal number. To understand how to define this equivalence relation we have to think at (2.1.1) in the following sense:

$$f(h(t)) \sim f(0) + h(t) \cdot f'(0), \tag{2.1.3}$$

where the idea is that we are going to define $\sim$. If we think $h(t)$ "sufficiently similar to t", we can define $\sim$ so that (2.1.3) is equivalent to

$$\lim_{t \to 0} \frac{f(h(t)) - f(0) - h(t) \cdot f'(0)}{t} = 0,$$

that is

$$x \sim y \quad :\Longleftrightarrow \quad \lim_{t \to 0} \frac{x(t) - y(t)}{t} = 0. \tag{2.1.4}$$

In this way (2.1.3) is very near to the definition of differentiability for f at 0.

It is important to note that, because of de L'Hôpital's theorem we have the isomorphism

$$\mathcal{C}^1(\mathbb{R}, \mathbb{R})/\sim \; \simeq \; \mathbb{R}[x]/(x),$$

the left hand side is (isomorphic to) the usual tangent bundle of $\mathbb{R}$ and thus we obtain nothing new. It is not easy to understand what set of functions we have to choose for x, y in (2.1.4) so as to obtain a non trivial structure. The first idea is to take continuous functions at $t = 0$, instead of more regular ones like $\mathcal{C}^1$-functions, so that e.g. $h_k(t) = |t|^{1/k}$ becomes a k-th order nilpotent infinitesimal ($h^{k+1} \sim 0$); indeed for almost all the results presented in this article, continuous functions at $t = 0$ work well. However, only in proving the non-trivial property

$$(\forall x \in {}^\bullet\mathbb{R} : \quad x \cdot f(x) = 0) \quad \Longrightarrow \quad \forall x \in {}^\bullet\mathbb{R} : \quad f(x) = 0 \tag{2.1.5}$$

(here $f : {}^\bullet\mathbb{R} \longrightarrow {}^\bullet\mathbb{R}$ is a smooth function, in a sense we shall make precise afterwards), we will see that it does not suffice to take continuous functions at $t = 0$. Property (2.1.5) is useful to prove the uniqueness of smooth incremental ratios, hence to define the derivative $f' : {}^\bullet\mathbb{R} \longrightarrow {}^\bullet\mathbb{R}$ of a smooth function $f : {}^\bullet\mathbb{R} \longrightarrow {}^\bullet\mathbb{R}$ which, generally speaking, is not the extension to ${}^\bullet\mathbb{R}$ of an ordinary function defined on $\mathbb{R}$ (like, e.g., the function $t \mapsto \sin(h \cdot t)$, where $h \in {}^\bullet\mathbb{R} \setminus \mathbb{R}$, which is used in elementary physics to describe the small oscillations of the pendulum). To prove (2.1.5) the following functions turned out to be very useful:

Definition 2.1.1. *If $x : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}$, then we say that x is* nilpotent *iff $|x(t) - x(0)|^k = o(t)$ as $t \to 0^+$, for some $k \in \mathbb{N}$. $\mathcal{N}$ will denote the set of all the nilpotent functions.*

In the previous definition, and we will do it also in the following, we have used the Landau notation of little-oh functions (see e.g. Prodi [1970], Silov [1978a]). E.g. any Hölder function $|x(t) - x(s)| \le c \cdot |t - s|^\alpha$ (for some constant $\alpha > 0$) is nilpotent. The choice of nilpotent functions instead of more regular ones establish a great difference of our approach with respect to the classical definition of jets (see e.g. Bröcker [1975], Golubitsky and Guillemin [1973]), that (2.1.4) may recall. Indeed in our approach all the $\mathcal{C}^1$-functions x with the same value and derivative at $t = 0$ generate the same $\sim$-equivalence relation. Only a non differentiable function at $t = 0$ like $x(t) = \sqrt{t}$ generates non trivial nilpotent infinitesimals.

Another problem necessarily connected with the basic idea (2.1.1) is that the use of nilpotent infinitesimals very frequently leads to consider terms like $h_1^{i_1} \cdot \ldots \cdot h_n^{i_n}$. For this type of products the first problem is to know whether $h_1^{i_1} \cdot \ldots \cdot h_n^{i_n} \ne 0$ and what is the order k of this new infinitesimals, that is for what k we have $(h_1^{i_1} \cdot \ldots \cdot h_n^{i_n})^k \ne 0$ but $(h_1^{i_1} \cdot \ldots \cdot h_n^{i_n})^{k+1} = 0$. We will have a good frame if we will be able to solve these problems starting from the order of each infinitesimal h_j and from the values of the powers $i_j \in \mathbb{N}$. On the other hand almost all the examples of nilpotent infinitesimals are of the form $h(t) = t^\alpha$, with $0 < \alpha < 1$, and their sums; these functions have great properties both in the treatment of products of powers and, as we will see, in connection with the order relation. It is for these reasons that we shall focus our attention on the following family of functions $x : \mathbb{R}_{\ge 0} \longrightarrow \mathbb{R}$ in the definition (2.1.4) of $\sim$:

Definition 2.1.2. *We say that x is a* little-oh polynomial*, and we write $x \in \mathbb{R}_o[t]$ iff*

1. $x : \mathbb{R}_{\ge 0} \longrightarrow \mathbb{R}$
2. *We can write*

$$x_t = r + \sum_{i=1}^{k} \alpha_i \cdot t^{a_i} + o(t) \qquad \textit{as} \qquad t \to 0^+$$

for suitable

$$k \in \mathbb{N}$$

$$r, \alpha_1, \ldots, \alpha_k \in \mathbb{R}$$

$$a_1, \ldots, a_k \in \mathbb{R}_{\ge 0}$$

Hence a little-oh polynomial[1] $x \in \mathbb{R}_o[t]$ is a polynomial function with real coefficients, in the real variable $t \ge 0$, with generic positive powers of t, and up to a little-oh function as $t \to 0^+$.

[1] actually in the following notation the variable t is mute

Remark 2.1.3. In the following, writing $x_t = y_t + o(t)$ as $t \to 0+$ we will always mean

$$\lim_{t \to 0^+} \frac{x_t - y_t}{t} = 0 \quad \text{and} \quad x_0 = y_0.$$

In other words, every little-oh function we will consider is continuous as $t \to 0^+$.

Example. Simple examples of little-oh polynomials are the following:

1. $x_t = 1 + t + t^{1/2} + t^{1/3} + o(t)$
2. $x_t = r \quad \forall t$. Note that in this example we can take $k = 0$, and hence α and a are the void sequence of reals, that is the function $\alpha = a : \emptyset \longrightarrow \mathbb{R}$, if we think of an n-tuple x of reals as a function $x : \{1, \dots, n\} \longrightarrow \mathbb{R}$.
3. $x_t = r + o(t)$

2.2 First properties of little-oh polynomials

Little-oh polynomials are nilpotent:

First properties of little-oh polynomials are the following: if $x_t = r + \sum_{i=1}^k \alpha_i \cdot t^{a_i} + o_1(t)$ as $t \to 0^+$ and $y_t = s + \sum_{j=1}^N \beta_j \cdot t^{b_j} + o_2(t)$, then $(x + y) = r + s + \sum_{i=1}^k \alpha_i \cdot t^{a_i} + \sum_{j=1}^N \beta_j \cdot t^{b_j} + o_3(t)$ and $(x \cdot y)_t = rs + \sum_{i=1}^k s\alpha_i \cdot t^{a_i} + \sum_{j=1}^N r\beta_j \cdot t^{b_i} + \sum_{i=1}^k \sum_{j=1}^N \alpha_i \beta_j \cdot t^{a_i} t^{b_j} + o_4(t)$, hence the set of little-oh polynomials is closed with respect to pointwise sum and product. Moreover little-oh polynomials are nilpotent (see Definition 2.1.1) functions; to prove this we firstly prove that the set of nilpotent functions $\mathcal{N}$ is a subalgebra of the algebra $\mathbb{R}^\mathbb{R}$ of real valued functions. Indeed, let x and y be two nilpotent functions such that $|x - x(0)|^k = o_1(t)$ and $|y - y(0)|^N = o_2(t)$, then we can write $x \cdot y - x(0) \cdot y(0) = x \cdot [y - y(0)] + y(0) \cdot [x - x(0)]$, so that we can consider $|x \cdot [y - y(0)]|^k = |x|^k \cdot |y - y(0)|^k = |x|^k \cdot o_1(t)$ and $\frac{|x|^k \cdot o_1(t)}{t} \to 0$ as $t \to 0^+$ because $|x|^k \to |x(0)|^k$, hence $x \cdot [y - y(0)] \in \mathcal{N}$. Analogously $y(0) \cdot [x - x(0)] \in \mathcal{N}$ and hence the closure of $\mathcal{N}$ with respect to the product follows from the closure with respect to the sum. The case of the sum follows from the following equalities (where we use $x_t := x(t)$, $u := x - x_0$, $v := y - y_0$, $|u_t|^k = o_1(t)$ and $|v_t|^N = o_2(t)$ and we have supposed $k \geq N$):

$$u^k = o_1(t), \quad v^k = o_2(t)$$

$$(u + v)^k = \sum_{i=0}^k \binom{k}{i} u^i \cdot v^{k-i}$$

$$\forall i = 0, \dots, k : \quad \frac{u_t^i \cdot v_t^{k-i}}{t} = \frac{\left(u_t^k\right)^{\frac{i}{k}} \cdot \left(v_t^k\right)^{\frac{k-i}{k}}}{t^{\frac{i}{k}} \cdot t^{\frac{k-i}{k}}} = \left(\frac{u_t^k}{t}\right)^{\frac{i}{k}} \cdot \left(\frac{v_t^k}{t}\right)^{\frac{k-i}{k}}.$$

Now we can prove that $\mathbb{R}_o[t]$ is a subalgebra of $\mathcal{N}$. Indeed every constant $r \in \mathbb{R}$ and every power t^{a_i} are elements of $\mathcal{N}$ and hence $r+\sum_{i=1}^{k} \alpha_i \cdot t^{a_i} \in \mathcal{N}$, so it remains to prove that if $y \in \mathcal{N}$ and $w = o(t)$, then $y + w \in \mathcal{N}$, but this is a consequence of the fact that every little-oh function is trivially nilpotent, and hence it follows from the closure of $\mathcal{N}$ with respect to the sum.

Closure of little-oh polynomials with respect to smooth functions:

Now we want to prove that little-oh polynomials are preserved by smooth functions, that is if $x \in \mathbb{R}_o[t]$ and $f : \mathbb{R} \longrightarrow \mathbb{R}$ is smooth, then $f \circ x \in \mathbb{R}_o[t]$. Let us fix some notations:

$$x_t = r + \sum_{i=1}^{k} \alpha_i \cdot t^{a_i} + w(t) \quad \text{with} \quad w(t) = o(t)$$

$$h(t) := x(t) - x(0) \quad \forall t \in \mathbb{R}_{\geq 0}$$

hence $x_t = x(0) + h_t = r + h_t$. The function $t \mapsto h(t) = \sum_{i=1}^{k} \alpha_i \cdot t^{a_i} + w(t)$ belongs to $\mathbb{R}_o[t] \subseteq \mathcal{N}$ so we can write $|h|^N = o(t)$ for some $N \in \mathbb{N}$ and as $t \to 0^+$. From Taylor's formula we have

$$f(x_t) = f(r + h_t) = f(r) + \sum_{i=1}^{N} \frac{f^{(i)}(r)}{i!} \cdot h_t^i + o(h_t^N) \tag{2.2.1}$$

But

$$\frac{|o(h_t^N)|}{|t|} = \frac{|o(h_t^N)|}{|h_t^N|} \cdot \frac{|h_t^N|}{|t|} \to 0$$

hence $o(h_t^N) = o(t) \in \mathbb{R}_o[t]$. From this, the formula (2.2.1), the fact that $h \in \mathbb{R}_o[t]$ and using the closure of little-oh polynomials with respect to ring operations, the conclusion $f \circ x \in \mathbb{R}_o[t]$ follows.

2.3 Equality and decomposition of Fermat reals

Definition 2.3.1. *Let x, $y \in \mathbb{R}_o[t]$, then we say that $x \sim y$ or that $x = y$ in ${}^\bullet\mathbb{R}$ iff $x(t) = y(t) + o(t)$ as $t \to 0^+$. Because it is easy to prove that $\sim$ is an equivalence relation, we can define ${}^\bullet\mathbb{R} := \mathbb{R}_o[t]/\sim$, i.e. ${}^\bullet\mathbb{R}$ is the quotient set of $\mathbb{R}_o[t]$ with respect to the equivalence relation $\sim$.*

The equivalence relation $\sim$ is a congruence with respect to pointwise operations, hence ${}^\bullet\mathbb{R}$ is a commutative ring. Where it will be useful to simplify notations we will write "$x = y$ in ${}^\bullet\mathbb{R}$" instead of $x \sim y$, and we will talk directly about the elements of $\mathbb{R}_o[t]$ instead of their equivalence classes; for example we can say that $x = y$ in ${}^\bullet\mathbb{R}$ and $z = w$ in ${}^\bullet\mathbb{R}$ imply $x + z = y + w$ in ${}^\bullet\mathbb{R}$.
The immersion of $\mathbb{R}$ in ${}^\bullet\mathbb{R}$ is $r \longmapsto \hat{r}$ defined by $\hat{r}(t) := r$, and in the sequel

we will always identify $\hat{\mathbb{R}}$ with $\mathbb{R}$, which is hence a subring of ${}^\bullet\mathbb{R}$. Conversely if $x \in {}^\bullet\mathbb{R}$ then the map $^\circ(-) : x \in {}^\bullet\mathbb{R} \mapsto {}^\circ x = x(0) \in \mathbb{R}$, which evaluates each extended real in 0, is well defined. We shall call $^\circ(-)$ the *standard part map*. Let us also note that, as a vector space over the field $\mathbb{R}$ we have $\dim_{\mathbb{R}} {}^\bullet\mathbb{R} = \infty$, and this underlines even more the difference of our approach with respect to the classical definition of jets (see e.g. Bröcker [1975], Golubitsky and Guillemin [1973]). As we will see, more explicitly later on in the course of the present work, our idea is more near to NSA, where standard sets can be extended adding new infinitesimal points, and this is not the point of view of jet theory.

With the following theorem we will introduce the decomposition of a Fermat real $x \in {}^\bullet\mathbb{R}$, that is a unique notation for its standard part and all its infinitesimal parts.

Theorem 2.3.2. *If $x \in {}^\bullet\mathbb{R}$, then there exist one and only one sequence*

$$(k, r, \alpha_1, \dots, \alpha_k, a_1, \dots, a_k)$$

such that

$$k \in \mathbb{N}$$

$$r, \alpha_1, \dots, \alpha_k, a_1, \dots, a_k \in \mathbb{R}$$

and

1. $x = r + \sum_{i=1}^{k} \alpha_i \cdot t^{a_i}$ *in* ${}^\bullet\mathbb{R}$
2. $0 < a_1 < a_2 < \dots < a_k \leq 1$
3. $\alpha_i \neq 0 \quad \forall i = 1, \dots, k$

In this statement we have also to include the void case $k = 0$ and $\alpha = a : \emptyset \longrightarrow \mathbb{R}$. Obviously, as usual, we use the definition $\sum_{i=1}^{0} b_i = 0$ for the sum of an empty set of numbers. As we shall see, this is the case where x is a standard real, i.e. $x \in \mathbb{R}$.

In the following we will use the notations $t^a := \mathrm{d}t_{1/a} := [t \in \mathbb{R}_{\geq 0} \mapsto t^a \in \mathbb{R}]_\sim \in {}^\bullet\mathbb{R}$ so that e.g. $\mathrm{d}t_2 = t^{1/2}$ is a second order infinitesimal[2]. In general, as we will see from the definition of order of a generic infinitesimal, $\mathrm{d}t_a$ is an infinitesimal of order a. In other words these two notations for the same object permit to emphasize the difference between an actual infinitesimal $\mathrm{d}t_a$ and a potential infinitesimal $t^{1/a}$: an actual infinitesimal of order $a \geq 1$ corresponds to a potential infinitesimal of order $\frac{1}{a} \leq 1$ (with respect to the classical notion of order of an infinitesimal function from calculus, see e.g. Prodi [1970], Silov [1978a]).

[2] Let us point out that we make hereby an innocuous abuse of language using the same notation both for the value of the function, $t^a \in \mathbb{R}$, and for the equivalence class, $t^a \in {}^\bullet\mathbb{R}$.

Remark 2.3.3. Let us note that $\mathrm{d}t_a \cdot \mathrm{d}t_b = \mathrm{d}t_{\frac{ab}{a+b}}$, moreover $\mathrm{d}t_a^\alpha := (\mathrm{d}t_a)^\alpha = \mathrm{d}t_{\frac{a}{\alpha}}$ for every $\alpha \geq 1$ and finally $\mathrm{d}t_a = 0$ for every $a < 1$. E.g. $\mathrm{d}t_a^{[a]+1} = 0$ for every $a \in \mathbb{R}_{>0}$, where $[a] \in \mathbb{N}$ is the integer part of a, i.e. $[a] \leq a < [a]+1$.

Existence proof:

Since $x \in \mathbb{R}_o[t]$, we can write $x_t = r + \sum_{i=1}^k \alpha_i \cdot t^{a_i} + o(t)$ as $t \to 0^+$, where r, $\alpha_i \in \mathbb{R}$, $a_i \in \mathbb{R}_{\geq 0}$ and $k \in \mathbb{N}$. Hence $x = r + \sum_{i=1}^k \alpha_i \cdot t^{a_i}$ in ${}^\bullet\mathbb{R}$ and our purpose is to pass from this representation of x to another one that satisfies conditions 1, 2 and 3 of the statement. Since if $a_i > 1$ then $\alpha_i \cdot t^{a_i} = 0$ in ${}^\bullet\mathbb{R}$, we can suppose that $a_i \leq 1$ for every $i = 1, \ldots, k$. Moreover we can also suppose $a_i > 0$ for every i, because otherwise, if $a_i = 0$, we can replace $r \in \mathbb{R}$ by $r + \sum\{\alpha_i \,|\, a_i = 0,\, i = 1, \ldots, k\}$.
Now we sum all the terms t^{a_i} having the same a_i, that is we can consider

$$\bar{\alpha}_i := \sum\{\alpha_j \,|\, a_j = a_i \,,\, j = 1, \ldots, k\}$$

so that in ${}^\bullet\mathbb{R}$ we have

$$x = r + \sum_{i \in I} \bar{\alpha}_i \cdot t^{a_i}$$

where $I \subseteq \{1, \ldots, k\}$, $\{a_i \,|\, i \in I\} = \{a, \ldots, a_k\}$ and $a_i \neq a_j$ for any i, $j \in I$ with $i \neq j$. Neglecting $\bar{\alpha}_i$ if $\bar{\alpha}_i = 0$ and renaming a_i, for $i \in I$, in such a way that $a_i < a_j$ if i, $j \in I$ with $i < j$, we obtain the existence result. Note that if $x = r \in \mathbb{R}$, in the final step of this proof we have $I = \emptyset$.

Uniqueness proof:

Let us suppose that in ${}^\bullet\mathbb{R}$ we have

$$x = r + \sum_{i=1}^k \alpha_i \cdot t^{a_i} = s + \sum_{j=1}^N \beta_j \cdot t^{b_j} \tag{2.3.1}$$

where α_i, β_j, a_i and b_j verify the conditions of the statement. First of all ${}^\circ x = x(0) = r = s$ because a_i, $b_j > 0$. Hence $\alpha_1 t^{a_1} - \beta_1 t^{b_1} + \sum_i \alpha_i \cdot t^{a_i} - \sum_j \beta_j \cdot t^{b_j} = o(t)$. By reduction to the absurd, if we had $a_1 < b_1$, then collecting the term t^{a_1} we would have

$$\alpha_1 - \beta_1 t^{b_1 - a_1} + \sum_i \alpha_i \cdot t^{a_i - a_1} - \sum_j \beta_j \cdot t^{b_j - a_1} = \frac{o(t)}{t} \cdot t^{1-a_1}. \tag{2.3.2}$$

In (2.3.2) we have that $\beta_1 t^{b_1 - a_1} \to 0$ for $t \to 0^+$ because $a_1 < b_1$ by hypothesis; $\sum_i \alpha_i \cdot t^{a_i - a_1} \to 0$ because $a_1 < a_i$ for $i = 2, \ldots, k$; $\sum_j \beta_j \cdot t^{b_j - a_1} \to 0$ because $a_1 < b_1 < b_j$ for $j = 2, \ldots, N$, and finally t^{1-a_1} is limited because $a_1 \leq 1$. Hence for $t \to 0^+$ we obtain $\alpha_1 = 0$, which conflicts with condition 3 of the statement. We can argue in a corresponding way if we had $b_1 < a_1$.

In this way we see that we must have $a_1 = b_1$. From this and from equation (2.3.2) we obtain

$$\alpha_1 - \beta_1 + \sum_i \alpha_i \cdot t^{a_i - a_1} - \sum_j \beta_j \cdot t^{b_j - a_1} = \frac{o(t)}{t} \cdot t^{1-a_1} \tag{2.3.3}$$

and hence for $t \to 0^+$ we obtain $\alpha_1 = \beta_1$. We can now restart from (2.3.3) to prove, in the same way, that $a_2 = b_2$, $\alpha_2 = \beta_2$, etc. At the end we must have $k = N$ because, otherwise, if we had e.g. $k < N$, at the end of the previous recursive process, we would have

$$\sum_{j=k+1}^{N} \beta_j \cdot t^{b_j} = o(t).$$

From this, collecting the terms containing $t^{b_{k+1}}$, we obtain

$$t^{b_{k+1}-1} \cdot [\beta_{k+1} + \beta_{k+2} \cdot t^{b_{k+2}-b_{k+1}} + \cdots + \beta_N \cdot t^{\beta_N - \beta_{k+1}}] \to 0. \tag{2.3.4}$$

In this sum $\beta_{k+j} \cdot t^{b_{k+j}-b_{k+1}} \to 0$ as $t \to 0^+$, because $b_{k+1} < b_{k+j}$ for $j > 1$ and hence $\beta_{k+1} + \beta_{k+2} \cdot t^{b_{k+2}-b_{k+1}} + \cdots + \beta_N \cdot t^{\beta_N - \beta_{k+1}} \to \beta_{k+1} \neq 0$, so from (2.3.4) we get $t^{b_{k+1}-1} \to 0$, that is $b_{k+1} > 1$, in contradiction with the uniqueness hypothesis $b_{k+1} \leq 1$.

Let us note explicitly that the uniqueness proof permits also to affirm that the decomposition is well defined in ${}^\bullet\mathbb{R}$, i.e. that if $x = y$ in ${}^\bullet\mathbb{R}$, then the decomposition of x and the decomposition of y are equal.

■

On the basis of this theorem we introduce two notations: the first one emphasizing the potential nature of an infinitesimal $x \in {}^\bullet\mathbb{R}$, and the second one emphasizing its actual nature.

Definition 2.3.4. *If $x \in {}^\bullet\mathbb{R}$, we say that*

$$x = r + \sum_{i=1}^{k} \alpha_i \cdot t^{a_i} \text{ is the potential decomposition (of } x\text{)} \tag{2.3.5}$$

iff conditions 1., 2., and 3. of theorem 2.3.2 are verified. Of course it is implicit that the symbol of equality in (2.3.5) has to be understood in ${}^\bullet\mathbb{R}$.

For example $x = 1 + t^{1/3} + t^{1/2} + t$ is a decomposition because we have increasing powers of t. The only decomposition of a standard real $r \in \mathbb{R}$ is the void one, i.e. that with $k = 0$ and $\alpha = a : \emptyset \longrightarrow \mathbb{R}$; indeed to see that this is the case, it suffices to go along the existence proof again with this case $x = r \in \mathbb{R}$ (or to prove it directly, e.g. by contradiction).

Definition 2.3.5. *Considering that $t^{a_i} = \mathrm{d}t_{1/a_i}$ we can also use the following notation, emphasizing more the fact that $x \in {}^\bullet\mathbb{R}$ is an actual infinitesimal:*

$$x = {}^\circ x + \sum_{i=1}^{k} {}^\circ x_i \cdot \mathrm{d}t_{b_i} \tag{2.3.6}$$

where we have used the notation ${}^\circ x_i := \alpha_i$ and $b_i := 1/a_i$, so that the condition that uniquely identifies all b_i is $b_1 > b_2 > \cdots > b_k \geq 1$. We call (2.3.6) the actual decomposition *of x or simply the* decomposition *of x. We will also use the notation $\mathrm{d}^i x := {}^\circ x_i \cdot \mathrm{d}t_{b_i}$ (and simply $\mathrm{d}x := \mathrm{d}^1 x$) and we will call ${}^\circ x_i$ the* i-th standard part of x *and $\mathrm{d}^i x$ the* i-th infinitesimal part of x *or the i-th* differential *of x. So let us note that we can also write*

$$x = {}^\circ x + \sum_i \mathrm{d}^i x$$

and in this notation all the addenda are uniquely determined (the number of them too). Finally, if $k \geq 1$ that is if $x \in {}^\bullet\mathbb{R} \setminus \mathbb{R}$, we set $\omega(x) := b_1$ and $\omega_i(x) := b_i$. The real number $\omega(x) = b_1$ is the greatest order in the actual decomposition (2.3.6), corresponding to the smallest in the potential decomposition (2.3.5), and is called the order *of the Fermat real $x \in {}^\bullet\mathbb{R}$. The number $\omega_i(x) = b_i$ is called the i-th order of x. If $x \in \mathbb{R}$ we set $\omega(x) := 0$ and $\mathrm{d}^i x := 0$. Observe that in general $\omega(x) = \omega(\mathrm{d}x)$, $\mathrm{d}(\mathrm{d}x) = \mathrm{d}x$ and that, using the notations of the potential decomposition (2.3.4), we have $\omega(x) = 1/a_1$.*

Example. If $x = 1 + t^{1/3} + t^{1/2} + t$, then ${}^\circ x = 1$, $\mathrm{d}x = \mathrm{d}t_3$ and hence x is a third order infinitesimal, i.e. $\omega(x) = 3$, $\mathrm{d}^2 x = \mathrm{d}t_2$ and $\mathrm{d}^3 x = \mathrm{d}t$; finally all the standard parts are ${}^\circ x_i = 1$.

Remark 2.3.6. To avoid misunderstanding, it is important to underline that there is an opposite meaning of the word "order" in standard analysis and in the previous definition. Indeed, in standard analysis if we say that the infinitesimal function (for $t \to 0^+$) $t \mapsto x(t)$ is of order greater than the function $t \mapsto y(t)$, we mean that

$$\lim_{t \to 0^+} \frac{x(t)}{y(t)} = 0.$$

Intuitively this implies that we have to think $x(t)$ smaller than $y(t)$, at least for sufficiently small $t \in (0, \delta)$. Because the connection between the definition of order given in Definition 2.3.5 and the standard definition of order (with respect to the standard infinitesimal $t \mapsto t$) is given by $\omega(x) = 1/a_1$, for Fermat reals the meaning will be the opposite one: if $x, y \in D_\infty$ are two infinitesimals, having every standard part positive ${}^\circ x_i, {}^\circ y_j > 0$, and with $\omega(x) > \omega(y)$, then we have to think at $x \in {}^\bullet\mathbb{R}$ as a *bigger number* with respect to $y \in {}^\bullet\mathbb{R}$. More formally, in the next section we will see that this will correspond to say that if $a, b \in \mathbb{N}$ are such that $x^a \neq 0$ and $x^{a+1} = 0$, $y^b \neq 0$ and $y^{b+1} = 0$, then $a \geq b$. When we will introduce the order relation in ${}^\bullet\mathbb{R}$ (see 4), we will see that if for two infinitesimals we have $\omega(x) > \omega(y)$, then $x > y$ iff ${}^\circ x_1 > 0$. Recalling the Remark 2.3.3 we can remember this difference between classical and actual order, recalling that $\mathrm{d}t_a > \mathrm{d}t_b$ if $a > b$ and that the smallest non zero infinitesimal is $\mathrm{d}t_1 = \mathrm{d}t$, because $\mathrm{d}t_a = 0$ if $a < 1$.

2.4 The ideals D_k

In this section we will introduce the sets of nilpotent infinitesimals corresponding to a k-th order neighborhood of 0. Every smooth function restricted to this neighborhood becomes a polynomial of order k, obviously given by its k-th order Taylor's formula (without remainder). We start with a theorem characterizing infinitesimals of order less than k.

Theorem 2.4.1. *If $x \in {}^\bullet\mathbb{R}$ and $k \in \mathbb{N}_{>1}$, then $x^k = 0$ in ${}^\bullet\mathbb{R}$ if and only if ${}^\circ x = 0$ and $\omega(x) < k$.*

Proof: If $x^k = 0$, then taking the standard part map of both sides, we have ${}^\circ(x^k) = ({}^\circ x)^k = 0$ and hence ${}^\circ x = 0$. Moreover $x^k = 0$ means $x_t^k = o(t)$ and hence $\left(\frac{x_t}{t^{1/k}}\right)^k \to 0$ and $\frac{x_t}{t^{1/k}} \to 0$. We rewrite this condition using the potential decomposition $x = \sum_{i=1}^k \alpha_i \cdot t^{a_i}$ of x (note that in this way we have $\omega(x) = \frac{1}{a_1}$) obtaining

$$\lim_{t\to 0^+} \sum_i \alpha_i \cdot t^{a_i - \frac{1}{k}} = 0 = \lim_{t\to 0^+} t^{a_1 - \frac{1}{k}} \cdot \left[\alpha_1 + \alpha_2 \cdot t^{a_2 - a_1} + \cdots + \alpha_k \cdot t^{a_k - a_1}\right]$$

But $\alpha_1 + \alpha_2 \cdot t^{a_2-a_1} + \cdots + \alpha_k \cdot t^{a_k - a_1} \to \alpha_1 \neq 0$, hence we must have that $t^{a_1 - \frac{1}{k}} \to 0$, and so $a_1 > \frac{1}{k}$, that is $\omega(x) < k$.
Vice versa if ${}^\circ x = 0$ and $\omega(x) < k$, then $x = \sum_{i=1}^k \alpha_i \cdot t^{a_i} + o(t)$, and

$$\lim_{t\to 0^+} \frac{x_t}{t^{1/k}} = \lim_{t\to 0^+} \sum_i \alpha_i \cdot t^{a_i - \frac{1}{k}} + \lim_{t\to 0^+} \frac{o(t)}{t} \cdot t^{1-\frac{1}{k}}$$

But $t^{1-\frac{1}{k}} \to 0$ because $k > 1$ and $t^{a_i - \frac{1}{k}} \to 0^+$ because $\frac{1}{a_i} \leq \frac{1}{a_1} = \omega(x) < k$ and hence $x^k = 0$ in ${}^\bullet\mathbb{R}$.

■

If we want that in a k-th order infinitesimal neighborhood a smooth function is equal to its k-th Taylor's formula, i.e.

$$\forall h \in D_k : \; f(x+h) = \sum_{i=0}^k \frac{h^i}{i!} \cdot f^{(i)}(x)$$

we need to take infinitesimals which are able to delete the remainder, that is, such that $h^{k+1} = 0$. The previous theorem permits to extend the definition of the ideal D_k to real number subscripts instead of natural numbers k only.

Definition 2.4.2. *If $a \in \mathbb{R}_{>0} \cup \{\infty\}$, then*

$$D_a := \{x \in {}^\bullet\mathbb{R} \,|\, {}^\circ x = 0,\ \omega(x) < a+1\}$$

Moreover we will simply denote D_1 by D.

1. If $x = \mathrm{d}t_3$, then $\omega(x) = 3$ and $x \in D_3$. More in general $\mathrm{d}t_k \in D_a$ if and only if $\omega(\mathrm{d}t_k) = k < a+1$. E.g. $\mathrm{d}t_k \in D$ if and only if $1 \le k < 2$.
2. $D_\infty = \bigcup_a D_a = \{x \in {}^\bullet\mathbb{R} \,|\, {}^\circ x = 0\}$ is the set of all the infinitesimals of ${}^\bullet\mathbb{R}$.
3. $D_0 = \{0\}$ because the only infinitesimal having order strictly less than 1 is, by definition of order, $x = 0$ (see the Definition 2.3.5).

The following theorem gathers several expected properties of the sets D_a and of the order of an infinitesimal $\omega(x)$:

Theorem 2.4.3. *Let $a,\ b \in \mathbb{R}_{>0}$ and $x,\ y \in D_\infty$, then*

1. $a \le b \quad\Longrightarrow\quad D_a \subseteq D_b$
2. $x \in D_{\omega(x)}$
3. $a \in \mathbb{N} \quad\Longrightarrow\quad D_a = \{x \in {}^\bullet\mathbb{R} \,|\, x^{a+1} = 0\}$
4. $x \in D_a \quad\Longrightarrow\quad x^{\lceil a \rceil + 1} = 0$
5. $x \in D_\infty \setminus \{0\}$ *and* $k = [\omega(x)] \quad\Longrightarrow\quad x \in D_k \setminus D_{k-1}$
6. $\mathrm{d}(x \cdot y) = \mathrm{d}x \cdot \mathrm{d}y$
7. $x \cdot y \neq 0 \quad\Longrightarrow\quad \dfrac{1}{\omega(x \cdot y)} = \dfrac{1}{\omega(x)} + \dfrac{1}{\omega(y)}$
8. $x + y \neq 0 \quad\Longrightarrow\quad \omega(x+y) = \omega(x) \vee \omega(y)$
9. D_a *is an ideal*

In this statement if $r \in \mathbb{R}$, then $\lceil r \rceil$ is the *ceiling* of the real r, i.e. the unique integer $\lceil r \rceil \in \mathbb{Z}$ such that $\lceil r \rceil - 1 < r \le \lceil r \rceil$. Moreover if $r,\ s \in \mathbb{R}$, then $r \vee s := \max(r, s)$.

Proof: Property *1.* and *2.* follow directly from Definition 2.4.2 of D_a, whereas property *3.* follows from Theorem 2.4.1. From *1.* and *3.* property *4.* follows: in fact $x \in D_a \subseteq D_{\lceil a \rceil}$ because $a \le \lceil a \rceil$, hence $x^{\lceil a \rceil + 1} = 0$ from

property *3*. To prove property *5*., if $k = [\omega(x)]$, then $k \le \omega(x) < k+1$, hence directly from Definition 2.4.2 the conclusion follows.
To prove *6*. let

$$x = \sum_{i=1}^{k} {}^\circ x_i \cdot \mathrm{d}t_{a_i} \quad \text{and} \quad y = \sum_{j=1}^{N} {}^\circ y_j \cdot \mathrm{d}t_{b_j} \tag{2.4.1}$$

be the decompositions of x and y (considering that they are infinitesimals, so that ${}^\circ x = {}^\circ y = 0$). Recall that $\mathrm{d}x = {}^\circ x_1 \cdot \mathrm{d}t_{a_1}$ and $\mathrm{d}y = {}^\circ y_1 \cdot \mathrm{d}t_{b_1}$. From (2.4.1) we have

$$x \cdot y = \sum_{i=1}^{k} \sum_{j=1}^{N} {}^\circ x_i {}^\circ y_j \, \mathrm{d}t_{a_i} \, \mathrm{d}t_{b_j} = \sum_{i=1}^{k} \sum_{j=1}^{N} {}^\circ x_i {}^\circ y_j \, \mathrm{d}t_{\frac{a_i b_j}{a_i + b_j}} \tag{2.4.2}$$

where we have used the Remark *2.3.3*. But $\omega(x) = a_1 \ge a_i$ and $\omega(y) = b_1 \ge b_j$ from the Definition 2.3.5 of decomposition. Hence

$$\frac{1}{a_1} + \frac{1}{b_1} \le \frac{1}{a_i} + \frac{1}{b_j}$$

$$\frac{a_1 b_1}{a_1 + b_1} \ge \frac{a_i b_j}{a_i + b_j}$$

so that the greatest infinitesimal in the product (2.4.2) is

$$\mathrm{d}(x \cdot y) = {}^\circ x_1 {}^\circ y_1 \, \mathrm{d}t_{a_1} \, \mathrm{d}t_{b_1} = \mathrm{d}x \cdot \mathrm{d}y$$

From this proof, property *7*. follows, because $x \cdot y \ne 0$ by hypothesis, and hence its order is given by

$$\omega(x \cdot y) = \frac{a_1 b_1}{a_1 + b_1} = \left(\frac{1}{a_1} + \frac{1}{b_1} \right)^{-1} = \left(\frac{1}{\omega(x)} + \frac{1}{\omega(y)} \right)^{-1}$$

From the decompositions (2.4.1) we also have

$$x + y = \sum_{i=1}^{k} {}^\circ x_i \, \mathrm{d}t_{a_i} + \sum_{j=1}^{N} {}^\circ y_j \, \mathrm{d}t_{b_j}$$

and therefore, because by hypothesis $x + y \ne 0$, its order is given by the greatest infinitesimal in this sum, that is

$$\omega(x + y) = a_1 \vee b_1 = \omega(x) \vee \omega(y)$$

It remains to prove property *9*. First of all $\omega(0) = 0 < a + 1$, hence $0 \in D_a$. If $x, \ y \in D_a$, then $\omega(x)$ and $\omega(y)$ are strictly less than $a + 1$ and hence $x + y \in D_a$ follows from property *8*. Finally if $x \in D_a$ and $y \in {}^\bullet\mathbb{R}$, then

$x \cdot y = x \cdot {}^\circ y + x \cdot (y - {}^\circ y)$, so $\omega(x \cdot y) = \omega(x \cdot {}^\circ y) \vee \omega(x \cdot (y - {}^\circ y)) = \omega(x) \vee \omega(x \cdot z)$, where $z := y - {}^\circ y \in D_\infty$ is an infinitesimal. If $x \cdot z = 0$, we have $\omega(x \cdot y) = \omega(x) < a + 1$, otherwise from property *7.*

$$\frac{1}{\omega(x \cdot z)} = \frac{1}{\omega(x)} + \frac{1}{\omega(z)} \geq \frac{1}{\omega(x)}$$

and hence $\omega(x \cdot y) \leq \omega(x) < a + 1$; in any case the conclusion $x \cdot y \in D_a$ follows.

■

Property *4.* of this theorem cannot be proved substituting the ceiling $\lceil a \rceil$ with the integer part $[a]$. In fact if $a = 1.2$ and $x = \mathrm{d}t_{2.1}$, then $\omega(x) = 2.1$ and $[a] + 1 = 2$ so that $x^{[a]+1} = x^2 = \mathrm{d}t_{\frac{2.1}{2}} \neq 0$ in ${}^\bullet\mathbb{R}$, whereas $\lceil a \rceil + 1 = 3$ and $x^3 = \mathrm{d}t_{\frac{2.1}{3}} = 0$.

Finally let us note the increasing sequence of ideals/neighborhoods of zero:

$$\{0\} = D_0 \subset D = D_1 \subset D_2 \subset \cdots \subset D_k \subset \cdots \subset D_\infty. \tag{2.4.3}$$

Because of (2.4.3) and of the property $\mathrm{d}t_a = 0$ if $a < 1$, we can say that $\mathrm{d}t$ is the smallest infinitesimals and $\mathrm{d}t_2$, $\mathrm{d}t_3$, etc. are greater infinitesimals. As we mentioned in the Remark 2.3.6, after the introduction of the order relation in ${}^\bullet\mathbb{R}$, we will see that this "algebraic" idea of order of magnitude will correspond to a property of this order relation, so that we will also have $\mathrm{d}t < \mathrm{d}t_2 < \mathrm{d}t_3 < \ldots$. Moreover, from the properties *1.* and *5.* of the previous theorem it follows that if $x^a \neq 0$ and $x^{a+1} = 0$, then $a = [\omega(x)]$, so that if also $y^b \neq 0$, $y^{b+1} = 0$ and $\omega(x) > \omega(y)$, then $a \geq b$. This proves what has been stated in Remark 2.3.6.

2.5 Products of powers of nilpotent infinitesimals

In this section we will introduce several simple instruments that will be very useful to decide whether a product of the form $h_1^{i_1} \cdot \ldots \cdot h_n^{i_n}$, with $h_k \in D_\infty \setminus \{0\}$, is zero or whether it belongs to some D_k.

Theorem 2.5.1. *Let $h_1, \ldots, h_n \in D_\infty \setminus \{0\}$ and $i_1, \ldots, i_n \in \mathbb{N}$, then*

1. $h_1^{i_1} \cdot \ldots \cdot h_n^{i_n} = 0 \iff \displaystyle\sum_{k=1}^{n} \frac{i_k}{\omega(h_k)} > 1$

2. $h_1^{i_1} \cdot \ldots \cdot h_n^{i_n} \neq 0 \implies \dfrac{1}{\omega(h_1^{i_1} \cdot \ldots \cdot h_n^{i_n})} = \displaystyle\sum_{k=1}^{n} \frac{i_k}{\omega(h_k)}$

Proof: Let

$$h_k = \sum_{r=1}^{N_k} \alpha_{kr} t^{a_{kr}} \tag{2.5.1}$$

be the potential decomposition of h_k for $k = 1, \dots, n$. Then by definition 2.3.4 of potential decomposition and the definition 2.3.5 of order, we have $0 < a_{k1} < a_{k2} < \dots < a_{kN_k} \le 1$ and $j_k := \omega(h_k) = \frac{1}{a_{k1}}$, hence $\frac{1}{j_k} \le a_{kr}$ for every $r = 1, \dots, N_k$. Therefore from (2.5.1), collecting the terms containing t^{1/j_k} we have

$$h_k = t^{1/j_k} \cdot \left(\alpha_{k1} + \alpha_{k2} t^{a_{k2}-1/j_k} + \dots + \alpha_{kN_k} t^{a_{kN_k}-1/j_k} \right)$$

and hence

$$\begin{aligned} h_1^{i_1} \cdot \ldots \cdot h_n^{i_n} = t^{\frac{i_1}{j_1}+\dots+\frac{i_n}{j_n}} \cdot \left(\alpha_{11} + \alpha_{12} t^{a_{12}-\frac{1}{j_1}} + \dots + \alpha_{1N_1} t^{a_{1N_1}-\frac{1}{j_1}} \right)^{i_1} \cdot \ldots \\ \ldots \cdot \left(\alpha_{n1} + \alpha_{n2} t^{a_{n2}-\frac{1}{j_n}} + \dots + \alpha_{nN_n} t^{a_{nN_n}-\frac{1}{j_n}} \right)^{i_n} \end{aligned} \tag{2.5.2}$$

Hence if $\sum_k \frac{i_k}{j_k} > 1$ we have that $t^{\frac{i_1}{j_1}+\dots+\frac{i_n}{j_n}} = 0$ in ${}^\bullet\mathbb{R}$, so also $h_1^{i_1} \cdot \ldots \cdot h_n^{i_n} = 0$. Vice versa if $h_1^{i_1} \cdot \ldots \cdot h_n^{i_n} = 0$, then the right hand side of (2.5.2) is a $o(t)$ as $t \to 0^+$, that is

$$\begin{aligned} t^{\frac{i_1}{j_1}+\dots+\frac{i_n}{j_n}-1} \cdot \left(\alpha_{11} + \alpha_{12} t^{a_{12}-\frac{1}{j_1}} + \dots + \alpha_{1N_1} t^{a_{1N_1}-\frac{1}{j_1}} \right)^{i_1} \cdot \ldots \\ \ldots \cdot \left(\alpha_{n1} + \alpha_{n2} t^{a_{n2}-\frac{1}{j_n}} + \dots + \alpha_{nN_n} t^{a_{nN_n}-\frac{1}{j_n}} \right)^{i_n} \to 0 \end{aligned}$$

But each term $\left(\alpha_{k1} + \alpha_{k2} t^{a_{k2}-\frac{1}{j_k}} + \dots + \alpha_{kN_k} t^{a_{kN_k}-\frac{1}{j_k}} \right)^{i_k} \to \alpha_k^{i_k} \neq 0$ so, necessarily, we must have $\frac{i_1}{j_1} + \dots + \frac{i_n}{j_n} - 1 > 0$, and this concludes the proof of *1*.
To prove *2*. it suffices to apply recursively property *7*. of Theorem (2.4.3), in fact

$$\begin{aligned} \frac{1}{\omega(h_1^{i_1} \cdot \ldots \cdot h_n^{i_n})} &= \frac{1}{\omega(h_1^{i_1})} + \frac{1}{\omega(h_2^{i_2} \cdot \ldots \cdot h_n^{i_n})} = \\ &\frac{1}{\omega(h_1 \cdot \overset{i_1}{\ldots} \cdot h_1)} + \frac{1}{\omega(h_2^{i_2} \cdot \ldots \cdot h_n^{i_n})} = \dots \\ &= \frac{i_1}{\omega(h_1)} + \frac{1}{\omega(h_2^{i_2} \cdot \ldots \cdot h_n^{i_n})} = \frac{i_1}{\omega(h_1)} + \dots + \frac{i_n}{\omega(h_n)} \end{aligned}$$

and this concludes the proof. ■

Example 2.5.2. $\omega(\mathrm{d}t_{a_1}^{i_1} \cdot \ldots \cdot \mathrm{d}t_{a_n}^{i_n})^{-1} = \sum_k \frac{i_k}{\omega(\mathrm{d}t_{a_k})} = \sum_k \frac{i_k}{a_k}$ and $\mathrm{d}t_{a_1}^{i_1} \cdot \ldots \cdot \mathrm{d}t_{a_n}^{i_n} = 0$ if and only if $\sum_k \frac{i_k}{a_k} > 1$, so e.g. $\mathrm{d}t \cdot h = 0$ for every $h \in D_\infty$.

From this theorem we can derive four simple corollaries that will be useful in the course of the present work. Some of these corollaries are useful because they give properties of powers like $h_1^{i_1} \cdot \ldots \cdot h_n^{i_n}$ in cases where exact values of the orders $\omega(h_k)$ are unknown. The first corollary gives a necessary and sufficient condition to have $h_1^{i_1} \cdot \ldots \cdot h_n^{i_n} \in D_p \setminus \{0\}$.

Corollary 2.5.3. *In the hypotheses of the previous Theorem 2.5.1 let $p \in \mathbb{R}_{>0}$, then we have*

$$h_1^{i_1} \cdot \ldots \cdot h_n^{i_n} \in D_p \setminus \{0\} \qquad \iff \qquad \frac{1}{p+1} < \sum_{k=1}^{n} \frac{i_k}{\omega(h_k)} \leq 1$$

Proof: This follows almost directly from Theorem 2.5.1. In fact if $h_1^{i_1} \cdot \ldots \cdot h_n^{i_n} \in D_p \setminus \{0\}$, then its order is given by $\omega(h_1^{i_1} \cdot \ldots \cdot h_n^{i_n}) = \left[\sum_k \frac{i_k}{\omega(h_k)}\right]^{-1} =: a$ and moreover $a \geq 1$ because $h_1^{i_1} \cdot \ldots \cdot h_n^{i_n} \neq 0$. Furthermore, $h_1^{i_1} \cdot \ldots \cdot h_n^{i_n}$ being an element of D_p, we also have $a < p+1$, from which the conclusion $\frac{1}{p+1} < \frac{1}{a} \leq 1$ follows.

Vice versa if $\frac{1}{p+1} < \frac{1}{a} := \sum_k \frac{i_k}{\omega(h_k)} \leq 1$, then from Theorem 2.5.1 we have $h_1^{i_1} \cdot \ldots \cdot h_n^{i_n} \neq 0$ and $\omega(h_1^{i_1} \cdot \ldots \cdot h_n^{i_n}) = a$; but $a < p+1$ by hypothesis, hence $h_1^{i_1} \cdot \ldots \cdot h_n^{i_n} \in D_p$.

$\blacksquare$

Now we will prove a sufficient condition to have $h_1^{i_1} \cdot \ldots \cdot h_n^{i_n} = 0$, starting from the hypotheses $h_k \in D_{j_k}$ only, that is $\omega(h_k) < j_k + 1$. The typical situation where this applies is for $j_k = [\omega(h_k)] \in \mathbb{N}$.

Corollary 2.5.4. *Let $h_k \in D_{j_k}$ for $k = 1, \ldots, n$ and $i_1, \ldots, i_n \in \mathbb{N}$, then*

$$\sum_{k=1}^{n} \frac{i_k}{j_k + 1} \geq 1 \qquad \Longrightarrow \qquad h_1^{i_1} \cdot \ldots \cdot h_n^{i_n} = 0$$

In fact $\sum_{k=1}^{n} \frac{i_k}{\omega(h_k)} > \sum_{k=1}^{n} \frac{i_k}{j_k+1} \geq 1$ because $\omega(h_k) < j_k + 1$, hence the conclusion follows from Theorem 2.5.1.

Let h, $k \in D$; we want to see if $h \cdot k = 0$. Because in this case $\sum_k \frac{i_k}{j_k+1} = \frac{1}{2} + \frac{1}{2} = 1$ we always have

$$h \cdot k = 0. \tag{2.5.3}$$

We will see that this is a great conceptual difference between Fermat reals and the ring of SDG, where, not necessarily, the product of two first order infinitesimal is zero. The consequences of this property of Fermat reals arrive very deeply in the development of the theory of Fermat reals, forcing us, e.g., to develop several new concepts if we want to generalize the derivation formula (2.1.2) to functions defined on infinitesimal domains, like $f : D \longrightarrow {}^\bullet\mathbb{R}$ (see 3). We will return more extensively to this difference between Fermat reals and SDG in Chapter 4 about order relation on ${}^\bullet\mathbb{R}$. We only mention here that looking at the simple Definition 2.3.1, the equality (2.5.3) has an intuitively clear meaning, and it is to preserve this intuition that we keep this equality instead of changing completely the theory toward a less intuitive one.

The next corollary solves the same problem of the previous one, but starting from the hypotheses $h_k^{j_k} = 0$:

Corollary 2.5.5. *If $h_1, \dots, h_n \in D_\infty$ and $h_k^{j_k} = 0$ for $j_1, \dots, j_n \in \mathbb{N}$, then if $i_1, \dots, i_n \in \mathbb{N}$, we have*

$$\sum_{k=1}^{n} \frac{i_k}{j_k} \geq 1 \quad \Longrightarrow \quad h_1^{i_1} \cdot \ldots \cdot h_n^{i_n} = 0$$

In fact if $h_k^{j_k} = 0$, then $j_k > 0$ and $h_k \in D_{j_k - 1}$ by Theorem 2.4.3, so the conclusion follows from the previous corollary.
Finally, the latter corollary permits e.g. to passfrom a formula like

$$\forall h \in D_p^n : \quad f(h) = \sum_{\substack{i \in \mathbb{N}^n \\ |i| \leq p}} h^i \cdot a_i$$

to a formula like

$$\forall h \in D_q^n : \quad f(h) = \sum_{\substack{i \in \mathbb{N}^n \\ |i| \leq q}} h^i \cdot a_i$$

where $q < p$. In the previous formulas $D_a^n = D_a \times \overset{n}{\ldots} \times D_a$ and we have used the classical multi-indexes notations, e.g. $h^i = h_1^{i_1} \cdot \ldots \cdot h_n^{i_n}$ and $|i| = \sum_{k=1}^{n} i_k$.

Corollary 2.5.6. *Let $p \in \mathbb{N}_{>0}$ and $h_k \in D_p$ for each $k = 1, \dots, n$; $i \in \mathbb{N}^n$ and $h \in D_\infty^n$, then*

$$|i| > p \quad \Longrightarrow \quad h^i = 0.$$

To prove it, we only have to apply Corollary 2.5.4:

$$\sum_{k=1}^{n} \frac{i_k}{p+1} = \frac{\sum_k i_k}{p+1} = \frac{|i|}{p+1} \geq \frac{p+1}{p+1} = 1.$$

Let us note explicitly that the possibility to prove all these results about products of powers of nilpotent infinitesimals is essentially tied with the choice of little-oh polynomials in the definition of the equivalence relation $\sim$ in Definition 2.1.2. Equally effective and useful results are not provable for the more general family of nilpotent functions (see e.g. Giordano [2004]).

2.6 Identity principle for polynomials

In this section we want to prove that if a polynomial $a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ of ${}^\bullet\mathbb{R}$ is identically zero, then $a_k = 0$ for all $k = 0, \dots, n$. To prove this conclusion, it suffices to mean "identically zero" as "equal to zero for every x belonging to the extension of an open subset of $\mathbb{R}$". Therefore we firstly define what this extension is.

Definition 2.6.1. *If U is an open subset of $\mathbb{R}^n$, then ${}^\bullet U := \{x \in {}^\bullet\mathbb{R}^n \mid {}^\circ x \in U\}$. Here with the symbol ${}^\bullet\mathbb{R}^n$ we mean ${}^\bullet\mathbb{R}^n := {}^\bullet\mathbb{R} \times \overset{n}{\ldots} \times {}^\bullet\mathbb{R}$.*

We shall give further the general definition of the extension functor ${}^\bullet(-)$; in these first chapters we only want to examine some elementary properties of the ring ${}^\bullet\mathbb{R}$ that will be used later.

The identity principle for polynomials can now be stated in the following way:

Theorem 2.6.2. *Let $a_0, \ldots, a_n \in {}^\bullet\mathbb{R}$ and U be an open neighborhood of 0 in $\mathbb{R}$ such that*

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0 \quad \textit{in} \quad {}^\bullet\mathbb{R} \quad \forall x \in {}^\bullet U \tag{2.6.1}$$

Then

$$a_0 = a_1 = \cdots = a_n = 0 \quad \textit{in} \quad {}^\bullet\mathbb{R}$$

Proof: Because U is an open neighborhood of 0 in $\mathbb{R}$, we can always find $x_1, \ldots, x_{n+1} \in U$ such that $x_i \neq x_j$ for i, $j = 1, \ldots, n+1$ with $i \neq j$. Hence from hypothesis (2.6.1) we have

$$a_n x_k^n + \cdots + a_1 x_k + a_0 = 0 \quad \text{in} \quad {}^\bullet\mathbb{R} \quad \forall k = 1, \ldots, n+1$$

That is, in vectorial form

$$(a_n, \ldots, a_0) \cdot \begin{bmatrix} x_1^n & x_2^n & \ldots & x_{n+1}^n \\ x_1^{n-1} & x_2^{n-1} & \ldots & x_{n+1}^{n-1} \\ \vdots & & & \\ x_1 & x_2 & \ldots & x_{n+1} \\ 1 & 1 & \ldots & 1 \end{bmatrix} = 0 \quad \text{in} \quad {}^\bullet\mathbb{R}$$

This matrix V is a Vandermonde matrix, hence it is invertible

$$(a_n, \ldots, a_0) \cdot V = \underline{0} \quad \text{in} \quad {}^\bullet\mathbb{R}^{n+1}$$

$$(a_n, \ldots, a_0) \cdot V \cdot V^{-1} = \underline{0} \quad \text{in} \quad {}^\bullet\mathbb{R}^{n+1}$$

hence $a_k = 0$ in ${}^\bullet\mathbb{R}$ for every $k = 0, \ldots, n$.

■

This theorem can be extended to polynomials with more than one variable using recursively the previous theorem, one variable per time:

Corollary 2.6.3. *Let $a_i \in {}^\bullet\mathbb{R}$ for every $i \in \mathbb{N}^n$ with $|i| \leq d$. Let U be an open neighborhood of $\underline{0}$ in $\mathbb{R}^n$ such that*

$$\sum_{\substack{i \in \mathbb{N}^n \\ |i| \leq d}} a_i x^i = 0 \quad \forall x \in {}^\bullet U$$

Then

$$a_i = 0 \quad \forall i \in \mathbb{N}^n : \quad |i| \leq d$$

2.7 Invertible Fermat reals

We can see more formally that to prove (2.1.1) we cannot embed the reals $\mathbb{R}$ into a field but only into a ring, necessarily containing nilpotent element. In fact, applying (2.1.1) to the function $f(h) = h^2$ for $h \in D$, where $D \subseteq {}^\bullet\mathbb{R}$ is a given subset of ${}^\bullet\mathbb{R}$, we have

$$f(h) = h^2 = f(0) + h \cdot f'(0) = 0 \quad \forall h \in D.$$

Where we have supposed the preservation of the equality $f'(0) = 0$ from $\mathbb{R}$ to ${}^\bullet\mathbb{R}$. In other words, if D and $f(h) = h^2$ verify (2.1.1), then necessarily each element $h \in D$ must be a new type of number whose square is zero. Of course in a field the only subset D verifying this property is $D = \{0\}$.

Because we cannot have property (2.1.1) and a field at the same time, we need a sufficiently good family of cancellation laws as substitutes. We will dedicate a full chapter of this work to this problem, developing the notion of equality up to a k-th order infinitesimal (see Section 3). At present to prove the uniqueness of (2.1.2) we need the following simplest form of these cancellation laws:

Theorem 2.7.1. *If $x \in {}^\bullet\mathbb{R}$ is a Fermat real and r, $s \in \mathbb{R}$ are standard real numbers, then*

$$x \cdot r = x \cdot s \text{ in } {}^\bullet\mathbb{R} \quad \text{and} \quad x \neq 0 \quad \Longrightarrow \quad r = s$$

Remark. As a consequence of this result, we can always cancel a non zero Fermat real in an equality of the form $x \cdot r = x \cdot s$ where r, s are *standard* reals. This is obviously tied with the uniqueness part of (2.1.2) and implies that formula (2.1.2) uniquely identifies the first derivative in case it is a standard real number.

Proof: From the Definition 2.3.1 of equality in ${}^\bullet\mathbb{R}$ and from $x \cdot r = x \cdot s$ we have

$$\lim_{t \to 0^+} \frac{x_t \cdot (r - s)}{t} = 0.$$

But if we had $r \neq s$ this would implies $\lim_{t \to 0^+} \frac{x_t}{t} = 0$, that is $x = 0$ in ${}^\bullet\mathbb{R}$ and this contradicts the hypothesis $x \neq 0$.

■

The last result of this section takes its ideas from similar situations of formal power series and gives also a formula to compute the inverse of an invertible Fermat real.

Theorem 2.7.2. *Let $x = {}^\circ x + \sum_{i=1}^n {}^\circ x_i \cdot \mathrm{d}t_{a_i}$ be the decomposition of a Fermat real $x \in {}^\bullet\mathbb{R}$. Then*

$$x \text{ is invertible}$$

if and only if $^\circ x \neq 0$*, and in this case*

$$\frac{1}{x} = \frac{1}{^\circ x} \cdot \sum_{j=0}^{+\infty} (-1)^j \cdot \left(\sum_{i=1}^{n} \frac{^\circ x_i}{^\circ x} \cdot \mathrm{d}t_{a_i} \right)^j \tag{2.7.1}$$

In the formula (2.7.1) we have to note that the series is actually a finite sum because any $\mathrm{d}t_{a_i}$ is nilpotent.

1. $(1+\mathrm{d}t_2)^{-1} = 1 - \mathrm{d}t_2 + \mathrm{d}t_2^2 - \mathrm{d}t_2^3 + \cdots = 1 - \mathrm{d}t_2 + \mathrm{d}t$ because $\mathrm{d}t_2^3 = 0$
2. $(1+\mathrm{d}t_3)^{-1} = 1 - \mathrm{d}t_3 + \mathrm{d}t_3^2 - \mathrm{d}t_3^3 + \mathrm{d}t_3^4 - \cdots = 1 - \mathrm{d}t_3 + \mathrm{d}t_3^2 - \mathrm{d}t$

Proof: If $x \cdot y = 1$ for some $y \in {}^\bullet\mathbb{R}$, then, taking the standard parts of each side we have $^\circ x \cdot {}^\circ y = 1$ and hence $^\circ x \neq 0$. Vice versa the idea is to start from the series

$$\frac{1}{1+r} = \sum_{j=0}^{+\infty} (-1)^j \cdot r^j \quad \forall r \in \mathbb{R} : \quad |r| < 1$$

and, intuitively, to define

$$\begin{aligned}\left(^\circ x + \sum_i {}^\circ x_i \, \mathrm{d}t_{a_i} \right)^{-1} &= {}^\circ x^{-1} \cdot \left(1 + \sum_i \frac{^\circ x_i}{^\circ x} \, \mathrm{d}t_{a_i} \right)^{-1} \\ &= {}^\circ x^{-1} \cdot \sum_{j=0}^{+\infty} (-1)^j \cdot \left(\sum_i \frac{^\circ x_i}{^\circ x} \, \mathrm{d}t_{a_i} \right)^j\end{aligned}$$

So let $y := {}^\circ x^{-1} \cdot \sum_{j=0}^{+\infty} (-1)^j \cdot \left(\sum_i \frac{^\circ x_i}{^\circ x} \, \mathrm{d}t_{a_i} \right)^j$ and $h := x - {}^\circ x = \sum_i {}^\circ x_i \, \mathrm{d}t_{a_i} \in D_\infty$ so that we can also write

$$y = {}^\circ x^{-1} \cdot \sum_{j=0}^{+\infty} (-1)^j \cdot \frac{h^j}{^\circ x^j}$$

But $h \in {}^\bullet\mathbb{R}$ is a little-oh polynomial with $h(0) = 0$, so it is also continuous, hence for a sufficiently small $\delta > 0$ we have

$$\forall t \in (-\delta, \delta) : \quad \left| \frac{h_t}{^\circ x} \right| < 1$$

Therefore

$$\forall t \in (-\delta, \delta) : \quad y_t = \frac{1}{^\circ x} \cdot \left(1 + \frac{h_t}{^\circ x} \right)^{-1} = \frac{1}{^\circ x + h_t} = \frac{1}{x_t}$$

From this equality and from Definition 2.3.1 it follows $x \cdot y = 1$ in ${}^\bullet\mathbb{R}$.

■

2.8 The derivation formula

Even if, in the following of this work, we will see several generalizations of the derivation formula (2.1.2), we want to give here a proof of (2.1.2) because it has been the principal motivation for the construction of the ring of Fermat reals ${}^\bullet\mathbb{R}$. Anyhow, before considering the proof of the derivation formula, we have to understand how to extend a given smooth function $f:\mathbb{R}\longrightarrow\mathbb{R}$ to a certain function ${}^\bullet f:{}^\bullet\mathbb{R}\longrightarrow{}^\bullet\mathbb{R}$.

Definition 2.8.1. *Let A be an open subset of $\mathbb{R}^n$, $f:A\longrightarrow\mathbb{R}$ a smooth function and $x\in{}^\bullet A$ then we define*

$$^\bullet f(x):=f\circ x.$$

This definition is correct because we have seen (see Subsection 2.2) that little-oh polynomials are preserved by smooth functions, and because the function f is locally Lipschitz, so

$$\left|\frac{f(x_t)-f(y_t)}{t}\right|\le K\cdot\left|\frac{x_t-y_t}{t}\right|\qquad\forall t\in(-\delta,\delta)$$

for a sufficiently small δ and some constant K, and hence if $x=y$ in ${}^\bullet\mathbb{R}$, then also ${}^\bullet f(x)={}^\bullet f(y)$ in ${}^\bullet\mathbb{R}$.

The function ${}^\bullet f$ is an extension of f, that is

$$^\bullet f(r)=f(r)\quad\text{in}\quad{}^\bullet\mathbb{R}\quad\forall r\in\mathbb{R},$$

as it follows directly from the definition of equality in ${}^\bullet\mathbb{R}$ (i.e. Definition 2.3.1), thus we can still use the symbol $f(x)$ both for $x\in{}^\bullet\mathbb{R}$ and $x\in\mathbb{R}$ without confusion. After the introduction of the extension of smooth functions, we can also state the following useful *elementary transfer theorem* for equalities, whose proof follows directly from the previous definitions:

Theorem 2.8.2. *Let A be an open subset of $\mathbb{R}^n$, and τ, $\sigma:A\longrightarrow\mathbb{R}$ be smooth functions. Then*

$$\forall x\in{}^\bullet A:\quad{}^\bullet\tau(x)={}^\bullet\sigma(x)$$

iff

$$\forall r\in A:\quad\tau(r)=\sigma(r).$$

Now we will prove the derivation formula (2.1.2).

Theorem 2.8.3. *Let A be an open set in $\mathbb{R}$, $x\in A$ and $f:A\longrightarrow\mathbb{R}$ a smooth function, then*

$$\exists!\,m\in\mathbb{R}\;\forall h\in D:\quad f(x+h)=f(x)+h\cdot m.\tag{2.8.1}$$

In this case we have $m=f'(x)$, where $f'(x)$ is the usual derivative of f at x.

Proof: Uniqueness follows from the previous cancellation law Theorem 2.7.1, indeed if $m_1 \in \mathbb{R}$ and $m_2 \in \mathbb{R}$ both verify (2.8.1), then $h \cdot m_1 = h \cdot m_2$ for every $h \in D$. But there exists a non zero first order infinitesimal, e.g. $\mathrm{d}t \in D$, so from Theorem (2.7.1) it follows $m_1 = m_2$.

To prove the existence part, take $h \in D$, so that $h^2 = 0$ in ${}^\bullet\mathbb{R}$, i.e. $h_t^2 = o(t)$ for $t \to 0^+$. But f is smooth, hence from its second order Taylor's formula we have

$$f(x+h_t) = f(x) + h_t \cdot f'(x) + \frac{h_t^2}{2} \cdot f''(x) + o(h_t^2)$$

But

$$\frac{o(h_t^2)}{t} = \frac{o(h_t^2)}{h_t^2} \cdot \frac{h_t^2}{t} \to 0 \quad \text{for} \quad t \to 0^+$$

so

$$\frac{h_t^2}{2} \cdot f''(x) + o(h_t^2) = o_1(t) \quad \text{for} \quad t \to 0^+$$

and we can write

$$f(x+h_t) = f(x) + h_t \cdot f'(x) + o_1(t) \quad \text{for} \quad t \to 0^+$$

that is

$$f(x+h) = f(x) + h \cdot f'(x) \quad \text{in} \quad {}^\bullet\mathbb{R}$$

and this proves the existence part because $f'(x) \in \mathbb{R}$.

■

For example $e^h = 1 + h$, $\sin(h) = h$ and $\cos(h) = 1$ for every $h \in D$.

Analogously we can prove the following infinitesimal Taylor's formula; in its statement we use the usual multi-indexes notations (see e.g. Prodi [1987], Silov [1978b]) and the notation $D_n^d := D_n \times \overset{d}{\ldots} \times D_n$.

Lemma 2.8.4. *Let A be an open set in $\mathbb{R}^d$, $x \in A$, $n \in \mathbb{N}_{>0}$ and $f : A \longrightarrow \mathbb{R}$ a smooth function, then*

$$\forall h \in D_n^d : \quad f(x+h) = \sum_{\substack{j \in \mathbb{N}^d \\ |j| \le n}} \frac{h^j}{j!} \cdot \frac{\partial^{|j|} f}{\partial x^j}(x)$$

For example $\sin(h) = h - \frac{h^3}{6}$ if $h \in D_3$ so that $h^4 = 0$.

It is possible to generalize several results of the present work to functions of class $\mathcal{C}^n$ only, instead of smooth ones. However it is an explicit purpose of this work to simplify statements of results, definitions and notations, even if, as a result of this searching for simplicity, its applicability will only hold for a more restricted class of functions. Some more general results, stated for $\mathcal{C}^n$ functions, but less simple can be found in Giordano [2004].

Note that $m = f'(x) \in \mathbb{R}$, i.e. the slope is a standard real number, and that we can use the previous formula with standard real numbers x only, and not with a generic $x \in {}^\bullet\mathbb{R}$, but we shall remove this limitation in a subsequent chapter.

In other words we can say that the derivation formula (2.1.2) allows us to differentiate the usual differentiable functions using a language with infinitesimal numbers and to obtain from this an ordinary function.

If we apply this theorem to the smooth function $p(r) := \int_x^{x+r} f(t)\,\mathrm{d}t$, for f smooth, then we immediately obtain the following

Corollary 2.8.5. *Let A be open in $\mathbb{R}$, $x \in A$ and $f : A \longrightarrow \mathbb{R}$ smooth. Then*

$$\forall h \in D : \quad \int_x^{x+h} f(t)\,\mathrm{d}t = h \cdot f(x).$$

Moreover $f(x) \in \mathbb{R}$ is uniquely determined by this equality.

We close this section by introducing a very simple notation useful to emphasize some equalities: if $h, k \in {}^\bullet\mathbb{R}$ then we say that $\exists h/k$ iff $\exists! r \in \mathbb{R} \ : \ h = r \cdot k$, and obviously we denote this $r \in \mathbb{R}$ with h/k. Therefore we can say, e.g., that

$$\left.\begin{aligned} f'(x) &= \frac{f(x+h)-f(x)}{h} \\ f(x) &= \frac{1}{h} \cdot \int_x^{x+h} f(t)\,\mathrm{d}t. \end{aligned}\right. \qquad \forall h \in D_{\neq 0}$$

Moreover we can prove some natural properties of this "ratio", like the following one

$$\exists \frac{u}{v}, \frac{x}{y} \quad \text{and} \quad vy \neq 0 \Longrightarrow \frac{u}{v} + \frac{x}{y} = \frac{uy+vx}{vy}$$

Example 2.8.6. Consider e.g. $x = 1 + 2\,\mathrm{d}t_3 + \mathrm{d}t_2 + 5\,\mathrm{d}t_{4/3}$, then using the previous ratio we can find a formula to calculate all the coefficients of this decomposition. Indeed, let us consider first the term $2\,\mathrm{d}t_3$: if we multiply both sides by $\mathrm{d}t_{3/2}$, where

$$\frac{3}{2} = \frac{1}{1 - \frac{1}{\omega(\mathrm{d}t_3)}}$$

we obtain

$$(x - {}^\circ x) \cdot \mathrm{d}t_{3/2} = 2\,\mathrm{d}t_3\,\mathrm{d}t_{3/2} + \mathrm{d}t_2\,\mathrm{d}t_{3/2} + 5\,\mathrm{d}t_{4/3}\,\mathrm{d}t_{3/2}$$

but $\mathrm{d}t_3\,\mathrm{d}t_{3/2} = \mathrm{d}t$ whereas $\mathrm{d}t_a\,\mathrm{d}t_{3/2} = 0$ if $a < 3$, so

$$\frac{(x - {}^\circ x)\,\mathrm{d}t_{3/2}}{\mathrm{d}t} = 2.$$

Analogously we have

$$\frac{(x - {}^\circ x - 2\,\mathrm{d}t_3)\,\mathrm{d}t_2}{\mathrm{d}t} = 1 \qquad \text{and} \qquad \frac{(x - {}^\circ x - 2\,\mathrm{d}t_3 - \mathrm{d}t_2)\,\mathrm{d}t_4}{\mathrm{d}t} = 5$$

where

$$2 = \frac{1}{1 - \frac{1}{\omega(\mathrm{d}t_2)}} \qquad \text{and} \qquad 4 = \frac{1}{1 - \frac{1}{\omega(\mathrm{d}t_{4/3})}}.$$

Using the same idea we can prove the recursive formula

$$\alpha_{i+1} = \frac{1}{1 - \frac{1}{\omega_{i+1}(x)}} \quad \Longrightarrow \quad \frac{\left(x - {}^\circ x - \sum_{k=1}^{i} x_i\,\mathrm{d}t_{\omega_i(x)}\right) \cdot \mathrm{d}t_{\alpha_{i+1}}}{\mathrm{d}t} = x_{i+1}.$$

Finally, directly from the definition of decomposition it follows

$$\alpha \neq \frac{1}{1 - \frac{1}{\omega_{i+1}(x)}}\ \forall i \quad \Longrightarrow \quad \frac{\left(x - {}^\circ x - \sum_{k=1}^{i} x_i\,\mathrm{d}t_{\omega_i(x)}\right) \cdot \mathrm{d}t_{\alpha}}{\mathrm{d}t} = 0$$

$$\frac{\left(x - {}^\circ x - \sum_{k=1}^{i} x_i\,\mathrm{d}t_{\omega_i(x)}\right) \cdot \mathrm{d}t_{\alpha}}{\mathrm{d}t} \neq 0 \quad \Longrightarrow \quad \alpha = \frac{1}{1 - \frac{1}{\omega_{i+1}(x)}}$$

so that all the terms of the decomposition of a Fermat real are uniquely determined by these recursive formulas.

Chapter 3

Equality up to k-th order infinitesimals

3.1 Introduction

As proved in Theorem 2.8.3, the derivation formula has several limitations that we are forced to avoid if we want to obtain results like Stokes's theorem in the space $\mathbf{Man}(M;N)$ of smooth functions between two smooth manifolds M, N. Let us analyze the hypotheses of Theorem 2.8.3 so as to motivate some generalizations:

1. "*The point $x \in A$ is a standard real*". This is the hypothesis that can be more easily generalized. Indeed we can consider that any general Fermat real $x \in {}^\bullet\mathbb{R}$ can be written as the sum of its standard part ${}^\circ x \in \mathbb{R}$, and of its infinitesimal part $k_x := x - {}^\circ x \in D_\infty$. The infinitesimal part k_x is of course nilpotent and hence, for $h \in D$, we can compute $f(x+h) = f({}^\circ x + k_x + h)$ using the usual infinitesimal Taylor's formula or arbitrary order (see Theorem 2.8.4). We will follow this idea in this chapter, but another solution is included in the generalization of the following hypothesis.

2. "*The function $f : A \longrightarrow \mathbb{R}$ is a standard smooth function*". As we already mentioned, not every function we are interested in is of type ${}^\bullet f$, i.e. is the extension of a classical smooth function. We already mentioned, as a simple example, the function $t \in \mathbb{R}_{\geq 0} \mapsto \sin(h{\cdot}t) \in {}^\bullet\mathbb{R}$, where $h \in D_k$ is an infinitesimal. More generally any function of type $x \in {}^\bullet\mathbb{R} \mapsto {}^\bullet g(h, x) \in {}^\bullet\mathbb{R}$, where $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a given smooth function and $h \in {}^\bullet\mathbb{R} \setminus \mathbb{R}$ is a non standard Fermat real, is not of type ${}^\bullet f$ for some f, because it can happen that ${}^\bullet g(h, r) \in {}^\bullet\mathbb{R} \setminus \mathbb{R}$ for a standard $r \in \mathbb{R}$ (whereas, of course, ${}^\bullet f(r) = f(r) \in \mathbb{R}$ for every $r \in \mathbb{R}$). This implies that, on the one hand, we need a more general notion of smooth function, surely including domains and codomains of

type $\varphi : {}^\bullet U \longrightarrow {}^\bullet\mathbb{R}$, where $U \subseteq \mathbb{R}$ is open; on the other hand we have to define a notion of derivative for this new type of smooth function. We will solve this problem introducing the *smooth incremental ratio* (an idea that is mainly due to G. Reyes, see Kock [1981]) , i.e. for every $x \in {}^\bullet U$, a function $h \in {}^\bullet(-\delta_x, \delta_x) \mapsto \varphi[x, h] \in {}^\bullet\mathbb{R}$ verifying

$$\varphi(x + h) = \varphi(x) + \varphi[x, h] \cdot h \quad \forall h \in {}^\bullet(-\delta_x, \delta_x) \tag{3.1.1}$$

and formalizing Fermat's method: $\varphi'(x) := \varphi[x, 0]$. These results are not usable for functions of the type $\varphi : D_k \longrightarrow {}^\bullet\mathbb{R}$ which are not defined on the extension of a standard open set. This problem is tied with the next hypothesis analyzed in this list.

3. "*The domain of the smooth function $f : A \longrightarrow \mathbb{R}$ is an open set*". Especially considering spaces like spaces of functions, more general than locally flat spaces, sometimes the more general results can be stated only in infinitesimal domains like the above D_k. Examples are the existence and uniqueness of the flux corresponding to a given vector field or the existence and uniqueness of the exterior derivative of an n-form. For this reason we will have to define some notion of derivative for functions of type $\psi : D_k \longrightarrow {}^\bullet\mathbb{R}$. At first sight, the definition of derivative for this type of function may seem an easy goal. In fact, intuitively, a function of this type can be thought as some type of polynomial of degree $k \in \mathbb{N}$. The problem is due to the fact that, in our setting, the derivation formula does not determine uniquely the coefficients of this polynomial. Indeed we know that in ${}^\bullet\mathbb{R}$ we have $h \cdot k = 0$ for every first order infinitesimal h, $k \in D$, so both the coefficients $m_1 = 1 + k$, for a fixed $k \in D$, and $m_2 = 1$, verify, for every $h \in D$, the derivation formula $f(h) = f(0) + h \cdot m_i$ if $f(h) = h$. We want to underline, even if it will be formally clear only later in this work, that here we do not have a problem of existence, but of uniqueness only. In other words, e.g. for a function of type $\psi : D \longrightarrow {}^\bullet\mathbb{R}$ there always exists an $m \in {}^\bullet\mathbb{R}$ such that $\psi(h) = \psi(0) + h \cdot m$ for every $h \in D$, but this coefficient $m \in {}^\bullet\mathbb{R}$ is not uniquely determined by this formula. We can tackle this problem in several ways. For example we can try to find another formula that uniquely identifies what we intuitively think of as the derivative of $\psi : D_k \longrightarrow {}^\bullet\mathbb{R}$ at 0. The idea of the smooth incremental ratio (3.1.1) goes in this direction. Anyhow, in this work we followed another idea: because we only have a uniqueness and not an existence problem, we shall try to define precisely "*what is the simplest $m \in {}^\bullet\mathbb{R}$ that verifies the derivation formula, and we will call "derivative" this simplest coefficient*". E.g. among $m_1 = 1 + k$ and $m_2 = 1$ in the previous example, the simplest one will surely be $m_2 = 1$ and hence we shall have $\psi'(0) = 1$. This chapter is devoted to the development of these ideas. Indeed $m_2 = 1$

is simpler than $m_1 = 1 + k$ in the sense that m_2 is m_1 up to second order infinitesimals.

Let us start from the hypothesis

$$m \in {}^\bullet\mathbb{R} \quad \text{and} \quad \forall h \in D : \quad h \cdot m = 0$$

and try to derive some necessary condition on $m \in {}^\bullet\mathbb{R}$ based on the idea that "because we have a product with $h \in D$, some infinitesimal in the decomposition of m will give zero if multiplied by h, so not every infinitesimal in the decomposition of m is really useful to obtain the final value of the product $h \cdot m$". In fact let

$$m = {}^\circ m + \sum_{i=1}^{N} {}^\circ m_i \cdot \mathrm{d}t_{a_i} \tag{3.1.2}$$

be the decomposition of m, and

$$h = \sum_{j=1}^{k} {}^\circ h_j \cdot \mathrm{d}t_{b_j} \tag{3.1.3}$$

be the decomposition of a generic $h \in D$. Then

$$h \cdot m = \sum_{j=1}^{k} {}^\circ m {}^\circ h_j \, \mathrm{d}t_{b_j} + \sum_{j=1}^{k} \sum_{i=1}^{N} {}^\circ h_j {}^\circ m_i \, \mathrm{d}t_{\frac{a_i b_j}{a_i + b_j}} \tag{3.1.4}$$

But $h \in D$, hence $b_j \le \omega(h) = b_1 < 2$ so that if we had $a_i \le 2$ we would have

$$\frac{1}{b_j} + \frac{1}{a_i} > \frac{1}{2} + \frac{1}{2} = 1$$

and hence $\frac{a_i b_j}{a_i + b_j} < 1$ and $\mathrm{d}t_{\frac{a_i b_j}{a_i + b_j}} = 0$. Therefore we can write

$$\begin{aligned}\forall h \in D : \quad h \cdot m &= \sum_{j=1}^{k} {}^\circ m {}^\circ h_j \, \mathrm{d}t_{b_j} + \sum_{j=1}^{k} \sum_{\substack{i=1 \\ a_i > 2}}^{N} {}^\circ h_j {}^\circ m_i \, \mathrm{d}t_{\frac{a_i b_j}{a_i + b_j}} = \\ &= h \cdot \left({}^\circ m + \sum_{\substack{i=1 \\ a_i > 2}}^{N} {}^\circ m_i \cdot \mathrm{d}t_{a_i} \right) \end{aligned} \tag{3.1.5}$$

Looking at (3.1.5), we can say that "in a product of type $h \cdot m$, with $h \in D$, only sufficiently big infinitesimals ($a_i > 2$) in the decomposition of m will survive". In other words, "all the infinitesimals of order less or equal 2 are useless to define the value of the product $h \cdot m$".

The quantity

$$\iota_2(m) := {}^\circ m + \sum_{\substack{i=1 \\ a_i > 2}}^{N} {}^\circ m_i \cdot \mathrm{d}t_{a_i}$$

is exactly the number m up to second order infinitesimals[1], in the sense that it is obtained from m neglecting all the "small" infinitesimals $\mathrm{d}^i m = {}^\circ m_i \, \mathrm{d}t_{a_i}$ of order $\omega({}^\circ m_i \, \mathrm{d}t_{a_i}) = a_i \leq 2$. In the example mentioned at the item 3., where $k \in D$, we have $\iota_2(m_1) = \iota_2(1+k) = 1$ and, indeed, among all the Fermat reals $m \in {}^\bullet\mathbb{R}$ that verify the derivation formula, $\iota_2(m)$ will be our candidate for the definition of "the simplest Fermat real that verifies the derivation formula". In fact, the formula (3.1.5) can be written as

$$\forall h \in D : \quad h \cdot m = h \cdot \iota_2(m)$$

and it can be interpreted intuitively saying "among all the numbers m that gives the same value of the product $h \cdot m$, the number $\iota_2(m)$ is the simplest one because it contains the minimal information, neglecting all the useless infinitesimals, i.e. not useful to define the value of the product $h \cdot m$".

3.2 Equality up to k-th order infinitesimals

The considerations of the previous section give us sufficient heuristic motivations to define:

Definition 3.2.1. *Let $m = {}^\circ m + \sum_{i=1}^{N} {}^\circ m_i \cdot \mathrm{d}t_{\omega_i(m)}$ be the decomposition of $m \in {}^\bullet\mathbb{R}$ and $k \in \mathbb{R}_{\geq 0} \cup \{\infty\}$, then*

1. $\iota_k m := \iota_k(m) := {}^\circ m + \sum_{\substack{i=1 \\ \omega_i(m) > k}}^{N} {}^\circ m_i \cdot \mathrm{d}t_{\omega_i(m)}$

2. ${}^\bullet\mathbb{R}_k := \{\iota_k m \,|\, m \in {}^\bullet\mathbb{R}\}$.

Finally if x, $y \in {}^\bullet\mathbb{R}$, we will say $x =_k y$ iff $\iota_k x = \iota_k y$ in ${}^\bullet\mathbb{R}$, and we will read it as x is equal to y up to k-th order infinitesimals.

Remark 3.2.2. Firstly note that if $0 \leq k < 1$ then the condition $\omega_i(m) > k$ is trivial because we always have that $\omega_i(m) \geq 1$. Hence

$$\iota_0 m = m \quad \text{and} \quad {}^\bullet\mathbb{R}_0 := {}^\bullet\mathbb{R}$$

Moreover $\iota_\infty m = {}^\circ m$ and ${}^\bullet\mathbb{R}_\infty := \mathbb{R}$.

The first simple property we can note about ι_k is that $\iota_j(\iota_k x) = \iota_{j \vee k}(x)$ (recalling that $j \vee k := \max(j,k)$) so that we have, e.g., $\iota_j(\iota_k x) = \iota_k(\iota_j x)$

[1] Remember that for Fermat reals, the greater is the order and the bigger the infinitesimal has to be thought, see Remark 2.3.6.

and $\iota_k x = x$ for every $x \in {}^\bullet\mathbb{R}_k$. Moreover we have the following chain of inclusions

$$\mathbb{R} = {}^\bullet\mathbb{R}_\infty \subseteq \ldots \subseteq {}^\bullet\mathbb{R}_3 \subseteq {}^\bullet\mathbb{R}_2 \subseteq {}^\bullet\mathbb{R}_1 \subseteq {}^\bullet\mathbb{R}_0 = {}^\bullet\mathbb{R} \tag{3.2.1}$$

In fact if $z \in {}^\bullet\mathbb{R}_k$, we can write $z = \iota_k m$ for some $m \in {}^\bullet\mathbb{R}$; but $\iota_j(\iota_k m) = \iota_k(m) = z$ if $j \leq k$ and hence z is also of type $\iota_j(m')$ for $m' = \iota_k m$ and so $z \in {}^\bullet\mathbb{R}_j$. The intuitive meaning of (3.2.1) is clear: the more infinitesimals we neglect from a Fermat real $m \in {}^\bullet\mathbb{R}$ and the less terms will remain in the decomposition of m; continuing in this way, only the standard part ${}^\circ m$ remains.
We start the study of ι_k considering the relationships between this operation and the algebraic operations on ${}^\bullet\mathbb{R}$.

Theorem 3.2.3. *Let x, $y \in {}^\bullet\mathbb{R}$ and $k \in \mathbb{R}_{\geq 1}$, then*

1. *If $x = r + \sum_{h=1}^M \gamma_h \cdot \mathrm{d}t_{c_h}$ in ${}^\bullet\mathbb{R}$ (not necessarily the decomposition of x), with r, $\gamma_h \in \mathbb{R}$ and $c_h \in \mathbb{R}_{\geq 1}$, then $\iota_k x = r + \sum_{h:c_h>k} \gamma_h \cdot \mathrm{d}t_{c_h}$*
2. *$\iota_k(x+y) = \iota_k x + \iota_k y$*
3. *$\iota_k 0 = 0$*
4. *$\iota_k(r \cdot x) = r \cdot \iota_k x \quad \forall r \in \mathbb{R}$*
5. *$\iota_k(x \cdot y) = \iota_k(\iota_k x \cdot \iota_k y)$, that is $x \cdot y =_k \iota_k x \cdot \iota_k y$*
6. *The relation $=_k$ is an equivalence relation and the quotient set ${}^\bullet\mathbb{R}_{=_k} := {}^\bullet\mathbb{R}/ =_k$ is a ring with respect to pointwise operations*

Proof: To prove *1*. we can consider that if $x = r + \sum_{h=1}^M \gamma_h \cdot \mathrm{d}t_{c_h}$, then

$$\begin{aligned} x &= r + \sum_{h=1}^M \gamma_h \cdot \mathrm{d}t_{c_h} = \\ &= {}^\circ x + \sum_{q:q\in\{c_j \,|\, j=1,\ldots,M\}} \mathrm{d}t_q \cdot \sum\{\gamma_h \,|\, h=1,\ldots,M\,,\, c_h = q\} \end{aligned}$$

where we have summed all the addends $\gamma_h \, \mathrm{d}t_{c_h}$ having the same order $c_h = q$. Now call $\{q_1, \ldots, q_P\} := \{c_j \,|\, j = 1, \ldots, M\}$ the distinct elements of the set of all the c_j, and $\bar{\gamma}_a := \sum\{\gamma_h \,|\, h = 1, \ldots, M\,,\, c_h = q_a\}$. Hence $x = {}^\circ x + \sum_{a=1}^P \bar{\gamma}_a \, \mathrm{d}t_{q_a}$, and we can suppose that every $\bar{\gamma}_a \neq 0$. Recalling the construction of the decomposition of a Fermat real (see the existence proof of Theorem 2.3.2) we can state that $P = N$, where N is the number of addends in the decomposition of x, and that permuting the addends of this sum we obtain the decomposition of x, i.e. for a suitable permutation σ of $\{1, \ldots, N\}$ we have

$$q_{\sigma(i)} = \omega_i(x) \qquad \text{and} \qquad \bar{\gamma}_{\sigma(i)} = {}^\circ x_i$$

or, in other words, we can say that

$$x = {}^\circ x + \sum_{i=1}^{N} \bar{\gamma}_{\sigma(i)} \, \mathrm{d}t_{q_{\sigma(i)}} \quad \text{is the decomposition of } x.$$

Therefore, by Definition 3.2.1

$$\begin{aligned}
\iota_k x &= {}^\circ x + \sum_{\substack{i=1 \\ q_{\sigma(i)}>k}}^{N} \bar{\gamma}_{\sigma(i)} \, \mathrm{d}t_{q_{\sigma(i)}} = {}^\circ x + \sum_{\substack{a=1 \\ q_a>k}}^{N} \bar{\gamma}_a \, \mathrm{d}t_{q_a} = \\
&= {}^\circ x + \sum_{\substack{a=1 \\ q_a>k}}^{N} \mathrm{d}t_{q_a} \sum \{\gamma_h \mid h = 1, \dots, M \, , \, c_h = q_a\} = \\
&= {}^\circ x + \sum_{\substack{q : q \in \{c_j \mid j=1,\dots,M\} \\ q>k}} \mathrm{d}t_q \cdot \sum \{\gamma_h \mid h = 1, \dots, M \, , \, c_h = q\} = \\
&= r + \sum_{\substack{h=1 \\ c_h>k}}^{M} \gamma_h \, \mathrm{d}t_{c_h}.
\end{aligned}$$

2.) We consider the decompositions of x and y, so that we have that

$$\iota_k x + \iota_k y = {}^\circ x + \sum_{i:\omega_i(x)>k} {}^\circ x_i \cdot \mathrm{d}t_{\omega_i(x)} + {}^\circ y + \sum_{j:\omega_j(y)>k} {}^\circ y_j \cdot \mathrm{d}t_{\omega_j(y)} \qquad (3.2.2)$$

On the other hand we have

$$x + y = {}^\circ x + {}^\circ y + \sum_i {}^\circ x_i \, \mathrm{d}t_{\omega_i(x)} + \sum_j {}^\circ y_j \, \mathrm{d}t_{\omega_j(y)}$$

From this and from the previous result *1.* we have that

$$\iota_k(x + y) = {}^\circ x + {}^\circ y + \sum_{i:\omega_i(x)>k} {}^\circ x_i \, \mathrm{d}t_{\omega_i(x)} + \sum_{j:\omega_j(y)>k} {}^\circ y_j \, \mathrm{d}t_{\omega_j(y)} = \iota_k x + \iota_k y$$

Property *3.* is a general consequence of *2.* for $x = y = 0$.
4.) We multiply x by $r \in \mathbb{R}$ obtaining

$$r \cdot x = r \cdot {}^\circ x + \sum_{i=1}^{N} r \cdot {}^\circ x_i \, \mathrm{d}t_{\omega_i(x)}$$

so that, once again from *1.*, we have

$$\iota_k(r \cdot x) = r \cdot {}^\circ x + \sum_{i:\omega_i(x)>k} r \cdot {}^\circ x_i \, \mathrm{d}t_{\omega_i(x)} = r \cdot \iota_k x$$

5.) Let us consider the product of the decompositions of x and y and let $a_i := \omega_i(x)$, $b_j := \omega_j(y)$ for simplicity, then we have

$$x \cdot y = {}^\circ x \, {}^\circ y + \sum_j {}^\circ x \, {}^\circ y_j \, \mathrm{d}t_{b_j} + \sum_i {}^\circ y \, {}^\circ x_i \, \mathrm{d}t_{a_i} + \sum_{i,j} {}^\circ x_i \, {}^\circ y_j \, \mathrm{d}t_{\frac{a_i b_j}{a_i + b_j}}$$

Hence from *1.* we have

$$\begin{aligned} \iota_k(x \cdot y) = {}^\circ x \, {}^\circ y + \sum_{j : b_j > k} {}^\circ x \, {}^\circ y_j \, \mathrm{d}t_{b_j} + \sum_{i : a_i > k} {}^\circ y \, {}^\circ x_i \, \mathrm{d}t_{a_i} + \\ + \sum \left\{ {}^\circ x_i \, {}^\circ y_j \, \mathrm{d}t_{\frac{a_i b_j}{a_i + b_j}} \mid \frac{a_i b_j}{a_i + b_j} > k \right\} \end{aligned}$$

On the other hand we have

$$\begin{aligned} \iota_k x \cdot \iota_k y = {}^\circ x \, {}^\circ y + \sum_{j : b_j > k} {}^\circ x \, {}^\circ y_j \, \mathrm{d}t_{b_j} + \sum_{i : a_i > k} {}^\circ y \, {}^\circ x_i \, \mathrm{d}t_{a_i} + \\ + \sum_{\substack{i : a_i > k \\ j : b_j > k}} {}^\circ x_i \, {}^\circ y_j \, \mathrm{d}t_{\frac{a_i b_j}{a_i + b_j}} \end{aligned} \tag{3.2.3}$$

and applying *1.* to (3.2.3) we get

$$\begin{aligned} \iota_k(\iota_k x \cdot \iota_k y) = {}^\circ x \, {}^\circ y + \sum_{j : b_j > k} {}^\circ x \, {}^\circ y_j \, \mathrm{d}t_{b_j} + \sum_{i : a_i > k} {}^\circ y \, {}^\circ x_i \, \mathrm{d}t_{a_i} + \\ + \sum \left\{ {}^\circ x_i \, {}^\circ y_j \, \mathrm{d}t_{\frac{a_i b_j}{a_i + b_j}} \mid a_i > k, \, b_j > k, \, \frac{a_i b_j}{a_i + b_j} > k \right\} \end{aligned} \tag{3.2.4}$$

So it suffices to prove that the set of Fermat reals in the third summation sign both in (3.2.3) and (3.2.4) are equal. But immediately we can see that the set in (3.2.4) is a subset of the set of numbers in (3.2.3). For the opposite inclusion we have

$$\frac{a_i b_j}{a_i + b_j} > k \quad \Longrightarrow \quad \frac{1}{a_i} + \frac{1}{b_j} < \frac{1}{k} \tag{3.2.5}$$

but $\frac{1}{a_i} < \frac{1}{a_i} + \frac{1}{b_j}$ because $b_j = \omega_j(y) > 0$ and hence from (3.2.5) we obtain $a_i > k$. Analogously we can prove that $b_j > k$ so that the two sets of Fermat reals are equal.

6.) We have only to prove that the ring operations on the quotient set ${}^\bullet\mathbb{R}/ =_k$, are well defined, i.e. that

$$x =_k x' \quad \text{and} \quad y =_k y' \quad \Longrightarrow \quad x + x' =_k y + y' \tag{3.2.6}$$

$$x =_k x' \quad \text{and} \quad y =_k y' \quad \Longrightarrow \quad x \cdot x' =_k y \cdot y' \tag{3.2.7}$$

Indeed if $x =_k x'$ and $y =_k y'$, then $\iota_k x = \iota_k x'$ and $\iota_k y = \iota_k y'$ (obviously these equalities have to be understood in ${}^\bullet\mathbb{R}$), so $\iota_k(x)\cdot\iota_k(y) = \iota_k(x')\cdot\iota_k(y')$. Applying ι_k to both sides we obtain $\iota_k\left(\iota_k(x)\cdot\iota_k(y)\right) = \iota_k\left(\iota_k(x')\cdot\iota_k(y')\right)$ so that from *5*. we have $\iota_k(x\cdot y) = \iota_k(x'\cdot y')$, that is $x\cdot y =_k x'\cdot y'$. Analogously, using *2*., we can prove (3.2.6).

■

Remark 3.2.4. If $m \in {}^\bullet\mathbb{R}_k$ and $m =_k 0$ then we can write $m = \iota_k n$ for some $n \in {}^\bullet\mathbb{R}$; but, on the other hand, $\iota_k m = \iota_k 0 = 0$ in ${}^\bullet\mathbb{R}$ because $m =_k 0$. But $\iota_k m = \iota_k(\iota_k n) = \iota_k n = m$, so we can finally deduce that m must be zero in ${}^\bullet\mathbb{R}$, i.e. $m = 0$. Therefore:

$$m \in {}^\bullet\mathbb{R}_k \quad \text{and} \quad m =_k 0 \quad \Longrightarrow \quad m = 0 \quad \text{in} \quad {}^\bullet\mathbb{R}$$

This can also be restated saying that the notion of equality up to k-th order infinitesimals, i.e. the equivalence relation $=_k$, is trivial in ${}^\bullet\mathbb{R}_k$, i.e. if m, $n \in {}^\bullet\mathbb{R}_k$ and $m =_k n$, then $m = n$ in ${}^\bullet\mathbb{R}$.
Moreover we can also state (3.2.6) and (3.2.7) saying that if we work with equality up to k-th order infinitesimals, that is with the equivalence relation $=_k$, we can always use ring operations sum and product of ${}^\bullet\mathbb{R}$ and this will preserve the equality $=_k$.

Example. Whereas property *2*. says that ${}^\bullet\mathbb{R}_k$ is closed with respect to sums, in general it is not closed with respect to products. Indeed let

$$x = \mathrm{d}t_3 = y$$

then $\iota_2 x = \iota_2 y = \mathrm{d}t_3$ and $\iota_2 x \cdot \iota_2 y = \left(\mathrm{d}t_3\right)^2 = \mathrm{d}t_{3/2}$. On the other hand $x\cdot y = \mathrm{d}t_{3/2}$ and $\iota_2(x\cdot y) = 0$, so $\iota_2(x\cdot y) \neq \iota_2(x)\cdot\iota_2(y)$. This counterexample exhibits why we stated the relationships between ι_k and the product as in *5* of Theorem 3.2.3.

We close this section with a theorem that states some properties of the order of $\iota_k x - \iota_j x$. The starting idea is roughly the following: with $\iota_k x$ we "delete" in the decomposition of x all the infinitesimals of order less or equal to k; we do the same with $\iota_j x$, so if $j > k$ in the difference $\iota_k x - \iota_j x$ there will remain only infinitesimals of order between k and j.

Theorem 3.2.5. *Let $x \in {}^\bullet\mathbb{R}$ and j, $k \in \mathbb{R}_{\geq 1}$, with $j > k$, then*

1. $k < \omega(\iota_k x - \iota_j x) \leq j$ *and hence* $\iota_k x - \iota_j x \in D_j$
2. $k < \omega(\iota_k x)$
3. $\omega(x - \iota_j x) \leq j$
4. $\forall h \in D_{\frac{1}{j-1}} : \quad h \cdot (\iota_k x - \iota_j x) = 0$

Proof: To prove 1. let $x = r + \sum_{i=1}^{N} \alpha_i \,\mathrm{d}t_{a_i}$ be the decomposition of x, then

$$\begin{aligned}
\iota_k x - \iota_j x = r + \sum_{i:a_i>k} \alpha_i \,\mathrm{d}t_{a_i} - r - \sum_{i:a_i>j} \alpha_i \,\mathrm{d}t_{a_i} = \\
= \sum_{i:k<a_i\le j} \alpha_i \,\mathrm{d}t_{a_i} + \sum_{i:a_i>j} \alpha_i \,\mathrm{d}t_{a_i} - \sum_{i:a_i>j} \alpha_i \,\mathrm{d}t_{a_i} = \\
= \sum_{i:k<a_i\le j} \alpha_i \,\mathrm{d}t_{a_i}
\end{aligned} \tag{3.2.8}$$

So $\sum_{i:k<a_i\le j} \alpha_i \,\mathrm{d}t_{a_i}$ is the decomposition of $\iota_k x - \iota_j x$ and its order is given by $\omega(\iota_k x - \iota_j x) = a_p$, where p is the smallest index $i = 1, \dots, N$ in the decomposition (3.2.8), i.e. $p := \min\{i = 1, \dots, N \mid k < a_i \le j\}$. Therefore $k < \omega(\iota_k x - \iota_j x) = a_p \le j$.
Property *2.* can be proved exactly as the previous *1.* but with $j = +\infty$.
Property *3.* is simply property *1.* with $k = 0$.
4.) If $h \in D_{\frac{1}{j-1}}$, then $\omega(h) < \frac{1}{j-1} + 1 = \frac{j}{j-1}$. Let us analyze the product $h \cdot (\iota_k x - \iota_j x)$:

$$\frac{1}{\omega(h)} + \frac{1}{\omega(\iota_k x - \iota_j x)} > \frac{j-1}{j} + \frac{1}{j} = 1$$

Hence from 2.5.1 the conclusion follows.

■

As a consequence of the previous property *2* of Theorem 3.2.5 we have the following simple cancellation law.

Corollary 3.2.6. *Let $m \in {}^\bullet\mathbb{R}_k$, with $k \ge 1$, and $h \in D_\infty$ with $k + \omega(h) \le k \cdot \omega(h)$, then*

$$m \cdot h = 0 \quad \Longrightarrow \quad m = 0$$

Proof: First we note that $h \ne 0$, because otherwise we had $\omega(h) = \omega(0) = 0$ and $k \le 0$ from the hypothesis $k + \omega(h) \le k \cdot \omega(h)$. Secondly, from the hypothesis $m \cdot h = 0$ we immediately have ${}^\circ m = 0$, so that $m \in D_\infty$. Ad absurdum, if we had $m \ne 0$, then we would also have

$$\frac{1}{\omega(m)} + \frac{1}{\omega(h)} < \frac{1}{k} + \frac{1}{\omega(h)}$$

indeed $\omega(m) > k$ because $m \in {}^\bullet\mathbb{R}_k$ and the previous Theorem 3.2.5. But $\frac{1}{k} + \frac{1}{\omega(h)} = \frac{k+\omega(h)}{k\cdot\omega(h)} \le 1$ by hypothesis and therefore $m \cdot h \ne 0$ by Theorem 2.5.1, in contradiction with the hypothesis.

■

For example if we take m as above and consider the infinitesimal $h = \mathrm{d}t_j$ with $2 \le j \le k$, then

$$m \cdot \mathrm{d}t_j = 0 \quad \Longrightarrow \quad m = 0$$

indeed $\frac{1}{k} + \frac{1}{\omega(\mathrm{d}t_j)} = \frac{1}{k} + \frac{1}{j} \le \frac{2}{j} \le 1$, i.e. $k + \omega(\mathrm{d}t_j) \le k \cdot \omega(\mathrm{d}t_j)$.

3.3 Cancellation laws up to k-th order infinitesimals

The goal of this section is to find for what infinitesimals $h \in D_\infty$ and for what power $j \in \mathbb{N}$ and order $k \in \mathbb{R}_{\geq 1}$ we have $h^j \cdot m = h^j \cdot \iota_k m$. We recall that we started this chapter motivating the definition of $\iota_k x$ starting from the property

$$\forall h \in D : \quad h \cdot m = h \cdot \iota_2 m$$

In this section we want to generalize this property. We will see that, as a consequence of this generalization, we will obtain a cancellation law up to k-th order infinitesimals of the form

$$\begin{aligned} \text{If} \quad & \forall h \in D_{\alpha_1} \times \cdots \times D_{\alpha_n} : \quad h^j \cdot m = 0 \\ \text{then} \quad & m =_k 0 \end{aligned} \tag{3.3.1}$$

and hence a general Taylor's formula for smooth functions of the type $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ with *independent infinitesimals increments*, that is a formula useful to compute with a polynomial a term like $f(x_1 + h_1, \ldots, x_n + h_n)$, with $(h_1, \ldots, h_n) \in D_{\alpha_1} \times \cdots \times D_{\alpha_n}$, i.e. with infinitesimal increments in general of different orders.

We shall use the classical multi-indexes notations (see e.g. Prodi [1987]) frequently used in the study of several variables functions. E.g. in (3.3.1) we already used $h^j := h_1^{j_1} \cdot \ldots \cdot h_n^{j_n}$.
We start proving one simple lemma that will be useful in the following.

Lemma 3.3.1. *Let $m \in {}^\bullet\mathbb{R}$, $k \in \mathbb{R}_{\geq 1}$ and $h \in D_\infty$ such that*

$$\frac{1}{k} + \frac{1}{\omega(h)} > 1 \tag{3.3.2}$$

then

$$h \cdot m = h \cdot \iota_k m \tag{3.3.3}$$

Condition (3.3.2) is not difficult to foresee if we want to obtain (3.3.3), because it implies, as we will see in the following proof, that all the infinitesimals, in the decomposition of m, having an order which is less than or equal to k, multiplied by h will give 0 (compare property (3.3.2) with Theorem 2.5.1).

Proof: Let $h = \sum_{p=1}^N \beta_p \, \mathrm{d}t_{b_p}$ resp. $m = r + \sum_{i=1}^M \alpha_i \, \mathrm{d}t_{a_i}$ be the decompositions of h and m. Then

$$h \cdot m = \sum_{p=1}^{n} r\beta_p \, \mathrm{d}t_{b_p} + \sum_{i,p} \alpha_i \beta_p \, \mathrm{d}t_{\frac{a_i b_p}{a_i + b_p}} \tag{3.3.4}$$

But if $a_i \le k$, then

$$\frac{1}{a_i}+\frac{1}{b_p}\ge\frac{1}{k}+\frac{1}{\omega(h)}>1$$

So if $a_i \le k$, then $\frac{a_i b_p}{a_i+b_p}<1$ and $\mathrm{d}t_{\frac{a_i b_p}{a_i+b_p}}=0$, hence we can write (3.3.4) as

$$\begin{aligned} h\cdot m &= \sum_{p=1}^{n} r\beta_p\,\mathrm{d}t_{b_p}+\sum_p\sum_{i:a_i>k}\alpha_i\beta_p\,\mathrm{d}t_{\frac{a_i b_p}{a_i+b_p}} = \\ &= h\cdot\left(r+\sum_{i:a_i>k}\alpha_i\,\mathrm{d}t_{a_i}\right)=h\cdot\iota_k m \end{aligned}$$

■

In the proof of (3.3.1) the exponents $j\in\mathbb{N}^n$ will be tied with the ideals D_{α_i} through the following term:

Definition 3.3.2. *If $j\in\mathbb{N}^n$, with $n\in\mathbb{N}_{>0}$, and $\alpha\in(\mathbb{R}_{>0}\cup\{\infty\})^n$, then we set by definition*

$$\frac{j}{\alpha+1}:=\sum_{i=1}^{n}\frac{j_i}{\alpha_i+1}$$

Let us note that in the notation $\frac{j}{\alpha+1}$, the variables j and α are n-tuples. In the particular case $n=1$, we have that j and α are real numbers and the notation $\frac{j}{\alpha+1}$ has the usual meaning of a fraction. If $\alpha_i=\infty$, then we define $\frac{j_i}{\infty+1}:=0$. Now we can state and prove the main theorem of this section

Theorem 3.3.3. *Let $m\in{}^\bullet\mathbb{R}$, $n\in\mathbb{N}_{>0}$, $j\in\mathbb{N}^n\setminus\{\underline{0}\}$ and $\alpha\in\mathbb{R}^n_{>0}$. Moreover let us consider $k\in\mathbb{R}$ defined by*

$$\frac{1}{k}+\frac{j}{\alpha+1}=1 \tag{3.3.5}$$

then

1. *$\forall h\in D_{\alpha_1}\times\cdots\times D_{\alpha_n}:\quad h^j\cdot m=h^j\cdot\iota_k m$*
2. *$\omega(m)>k \implies \exists h\in D_{\alpha_1}\times\cdots\times D_{\alpha_n}:\ \frac{1}{\omega(m)}+\frac{1}{\omega(h^j)}=1$*
3. *If $h^j\cdot m=0$ for every $h\in D_{\alpha_1}\times\cdots\times D_{\alpha_n}$, then $m=_k 0$*

The idea of the cancellation law *3.* is that if we have $h_1^{j_1}\cdot\ldots\cdot h_n^{j_n}\cdot m=0$ for every $(h_1,\ldots,h_n)\in D_{\alpha_1}\times\cdots\times D_{\alpha_n}$, then condition (3.3.5) permits to find the best $k\ge 1$ such that $m=_k 0$. Note that there is no limitation neither on the exponents $j\in\mathbb{N}^n\setminus\{\underline{0}\}$ nor on the ideals D_{α_i}, so we can call *3.* the *general cancellation law.*

Proof of Theorem 3.3.3:
1.) By the definition of k it follows

$$\frac{1}{k}+\frac{1}{\omega(h^j)}=1-\frac{j}{\alpha+1}+\frac{1}{\omega(h^j)}=1-\sum_{i=1}^{n}\frac{j_i}{\alpha_i+1}+\sum_{i=1}^{n}\frac{j_i}{\omega(h_i)} \tag{3.3.6}$$

where we have supposed $h^j \neq 0$, otherwise the conclusion is trivial, and we have applied Theorem 2.5.1.
But $\omega(h_i) < \alpha_i + 1$ because $h_i \in D_{\alpha_i}$, so

$$\sum_{i=1}^{n}\frac{j_i}{\omega(h_i)}-\sum_{i=1}^{n}\frac{j_i}{\alpha_i+1}>0 \tag{3.3.7}$$

Hence from (3.3.6) and (3.3.7) we have $\frac{1}{k}+\frac{1}{\omega(h^j)}>1$ and the conclusion follows from Lemma 3.3.1.
2.) For simplicity let $x_i := (\alpha_i+1)\cdot\frac{j}{\alpha+1}$, and $\frac{1}{a} := \omega(m)$, then $\frac{1}{a} > k$ from the hypothesis $\omega(m) > k$ and

$$\frac{1-a}{x_i}>\frac{1}{x_i}\left(1-\frac{1}{k}\right)=\frac{1}{x_i}\cdot\frac{j}{\alpha+1}=\frac{1}{\alpha_i+1}$$

so[2] $\mathrm{d}t_{\frac{x_i}{1-a}} = t^{\frac{1-a}{x_i}} \in D_{\alpha_i}$, and if we set

$$h := \left(\mathrm{d}t_{\frac{x_1}{1-a}},\dots,\mathrm{d}t_{\frac{x_n}{1-a}}\right) \in D_{\alpha_1}\times\dots\times D_{\alpha_n}$$

we have

$$h^j = t^{\frac{1-a}{x_1}j_1}\cdot\ldots\cdot t^{\frac{1-a}{x_n}j_n} = t^{(1-a)\cdot\sum_i\frac{j_i}{x_i}}$$

But

$$\sum_{i=1}^{n}\frac{j_i}{x_i}=\sum_{i=1}^{n}\frac{j_i}{(\alpha_i+1)\cdot\sum_{k=1}^{n}\frac{j_k}{\alpha_k+1}}=1$$

hence $h^j = t^{1-a}$ and

$$\frac{1}{\omega(m)}+\frac{1}{\omega(h^j)}=a+(1-a)=1$$

3.) This part is essentially the contrapositive of *2*. Indeed from *2*. we have

$$\left(\forall h\in D_{\alpha_1}\times\dots\times D_{\alpha_n}:\quad \frac{1}{\omega(m)}+\frac{1}{\omega(h^j)}\neq 1\right)\quad\Longrightarrow\quad \omega(m)\le k \tag{3.3.8}$$

[2] Here we are using the usual abuse of notation that consists in indicating the Fermat real (equivalence class modulo $\sim$, see 2.3.1) $[t\in\mathbb{R}_{\ge 0}\mapsto t^b]_\sim$ simply by t^b.

so if we assume that $h^j \cdot m = 0$ for every $h \in D_{\alpha_1} \times \cdots \times D_{\alpha_n}$, then it immediately follows $^\circ m = 0$ and from Theorem 2.5.1 the equality $h^j \cdot m = 0$ becomes equivalent to

$$\frac{1}{\omega(m)} + \frac{1}{\omega(h^j)} > 1$$

Therefore (3.3.8) is actually stronger than the hypothesis of *3*.
From (3.3.8) it follows $\omega(m) \le k$ and hence $m =_k 0$.

■

For example suppose we want to obtain $m =_2 0$ from a product of the type $h \cdot m = 0$ for every $h \in D_\alpha$. What kind of infinitesimals D_α do we have to choose? We have $k = 2$, $j = 1$ and $n = 1$, hence we must have

$$\frac{1}{k} + \frac{j}{\alpha+1} = \frac{1}{2} + \frac{1}{\alpha+1} = 1$$

hence $\alpha = 1$ and from the general cancellation law we have

$$(\forall h \in D : \quad h \cdot m = 0) \quad \Longrightarrow \quad m =_2 0$$

Analogously if we want $n = 2$, then we must have

$$\frac{1}{2} = 1 - \frac{1}{\alpha_1 + 1} - \frac{1}{\alpha_2 + 1}$$

so that we must choose $\alpha = (\alpha_1, \alpha_2)$ so that $\frac{1}{\alpha_1+1} + \frac{1}{\alpha_2+1} = \frac{1}{2}$, e.g. $\alpha = (3, 3)$, i.e.

$$(\forall h, k \in D_3 : \quad h \cdot k \cdot m = 0) \quad \Longrightarrow \quad m =_2 0$$

Vice versa now suppose to have D_α, with $\alpha \in \mathbb{N}_{>0}$ and we want to find k:

$$\frac{1}{k} = 1 - \frac{1}{\alpha+1}$$

hence $k = \frac{\alpha+1}{\alpha}$ and we obtain

$$(\forall h \in D_\alpha : \quad h \cdot m = 0) \quad \Longrightarrow \quad m =_{\frac{\alpha+1}{\alpha}} 0$$

and

$$(\forall h \in D_\infty : \quad h \cdot m = 0) \quad \Longrightarrow \quad m =_1 0$$

Let us note explicitly that the best we can obtain from the general cancellation law is that m is equal to zero up to first order infinitesimals. As an immediate consequence of the definition of equality in ${}^\bullet\mathbb{R}$, it follows that $h \cdot \mathrm{d}t = 0$ for every infinitesimal $h \in D_\infty$, and because $\mathrm{d}t =_1 0$ but $\mathrm{d}t \neq 0$, this exhibits that a better result cannot be obtained from this type of cancellation law.

A counterexample

The idea to have a cancellation law like the general one *3.* of Theorem 3.3.3 comes from SDG. The particularity of this law is that it is not of the form

"if *a given number* $h \in {}^\bullet\mathbb{R}$ has the property $\mathcal{P}(h)$ (e.g. h is invertible), and $h \cdot m = 0$, then $m = 0$",

as usual, but it is of the form, e.g.

"if $h \cdot m = 0$ *for every* $h \in D$, then $m =_2 0$".

We can foresee that these differences will not cause any problem each time we will use infinitesimal Taylor's formulae. Indeed, as we will see concretely later in the present work, typically these formulae are used for *generic* infinitesimal increments $h \in D^n_\alpha$, i.e. usually we will be able to prove our equalities derived from Taylor's formulae *for every* $h \in D^n_\alpha$.

Finally, these cancellation laws do not guarantee a strict equality but an equality up to infinitesimals of a suitable order k. As we will see, this correspond to have Taylor's formulae with uniqueness up to some order k. If we use these formulae to define derivatives, this implies that we will have derivatives identified up to infinitesimals of some order k. Roughly speaking, even if this is unusual for derivatives of smooth functions, it is very common in mathematics; think e.g. to definite integrals or Radon-Nikodym derivatives, where certain operators are defined up to a suitable notion of equality (i.e. an equivalence relation) like "up to a constant" or "up to a set of measure zero". In the same way, e.g., we will define first derivatives of smooth functions of the type $f : D^n_\alpha \longrightarrow {}^\bullet\mathbb{R}$ up to second order infinitesimals. Exactly as for the Radon-Nikodym derivative, the meaningful properties will be only those "up to second order infinitesimals".

Now we want to see that it is not possible to avoid the quantifier "*for every h*" in the cancellation law. More precisely let us suppose to have an infinitesimal $h \in D_\infty$ with the property of being deleted from every product, i.e. such that

$$\forall m \in {}^\bullet\mathbb{R} : \quad h \cdot m = 0 \quad \Longrightarrow \quad m = 0 \tag{3.3.9}$$

Does such an infinitesimal exist?

Of course $h \neq 0$ and hence $\omega(h) \geq 1$, but it cannot be that $\omega(h) = 1$ because otherwise

$$\frac{1}{\omega(h)} + \frac{1}{\omega(\,\mathrm{d}t)} = 2$$

and hence $h \cdot \mathrm{d}t = 0$ even if $\mathrm{d}t \neq 0$, in contradiction with (3.3.9). Hence it must be that $\omega(h) > 1$.

Now we want to find $k \geq 1$ such that $\frac{1}{k} + \frac{1}{\omega(h)} > 1$, that is

$$\frac{1}{k} > 1 - \frac{1}{\omega(h)} = \frac{\omega(h) - 1}{\omega(h)} > 0$$

the latter inequality being due to $\omega(h) > 1$. Therefore the number k we are searching for must satisfy

$$1 \le k < \frac{\omega(h)}{\omega(h)-1} \tag{3.3.10}$$

A k in this interval exists always because $\omega(h) > 1$, and if we set $m := \mathrm{d}t_k$, then $\frac{1}{k} + \frac{1}{\omega(h)} > 1$, and from Lemma 3.3.1 we have

$$h \cdot m = h \cdot \iota_k m = h \cdot \iota_k(\mathrm{d}t_k) = 0$$

but $m = \mathrm{d}t_k \neq 0$. For these reasons we can affirm that an infinitesimal $h \in D_\infty$ with the property of being deleted form every product, i.e. such that (3.3.9) holds, does not exist.

3.4 Applications to Taylor's formulae

General forms of uniqueness in Taylor's formulae

Corollary 3.4.1. *Let $n \in \mathbb{N}_{>0}$, $\alpha \in \mathbb{R}^n_{>0}$, and for every $j \in \mathbb{N}^n$ with $0 < \frac{j}{\alpha+1} < 1$, let $m_j \in {}^\bullet\mathbb{R}$ and set $k_j \in \mathbb{R}$ such that*

$$\frac{1}{k_j} + \frac{j}{\alpha+1} = 1 \quad , \quad k_{\underline{0}} := 0$$

Then there exists one and only one

$$\bar{m} : \left\{ j \in \mathbb{N}^n \mid \frac{j}{\alpha+1} < 1 \right\} \longrightarrow {}^\bullet\mathbb{R}$$

such that

1. *$\bar{m}_j \in {}^\bullet\mathbb{R}_{k_j}$ for every $j \in \mathbb{N}^n$ such that $\frac{j}{\alpha+1} < 1$*
2. *$\forall h \in D_{\alpha_1} \times \cdots \times D_{\alpha_n}$:* $$\sum_{\substack{j \in \mathbb{N}^n \\ \frac{j}{\alpha+1} < 1}} \frac{h^j}{j!} \cdot m_j = \sum_{\substack{j \in \mathbb{N}^n \\ \frac{j}{\alpha+1} < 1}} \frac{h^j}{j!} \cdot \bar{m}_j$$

Moreover the unique $\bar{m}_j$ is given by $\bar{m}_j = \iota_{k_j} m_j$.

To motivate the statement let us observe that if $h \in D_{\alpha_1} \times \cdots \times D_{\alpha_n}$ and $\frac{j}{\alpha+1} \ge 1$, then $\omega(h_i) < \alpha_i + 1$, and so

$$\frac{j_i}{\omega(h_i)} > \frac{j_i}{\alpha_i + 1}$$

and hence also

$$\sum_{i=1}^n \frac{j_i}{\omega(h_i)} > \sum_{i=1}^n \frac{j_i}{\alpha_i + 1} = \frac{j}{\alpha+1} \ge 1$$

thus $h^j = h_1^{j_1} \cdot \ldots \cdot h_n^{j_n} = 0$. For this reason the general Taylor's formula is restricted to $j \in \mathbb{N}^n$ such that $\frac{j}{\alpha+1} < 1$.

The meaning of this corollary is that if we have an infinitesimal Taylor's formula like

$$f(h) = f(0) + \sum_{j=1}^{n} \frac{h^j}{j!} \cdot m_j \quad \forall h \in D_n$$

then we can substitute the coefficients $m_j \in {}^\bullet\mathbb{R}$ by $\bar{m}_j = \iota_{k_j}(m_j) \in {}^\bullet\mathbb{R}_{k_j}$, that is with m_j up to infinitesimals of order k_j, and the formula remains unchanged

$$f(h) = f(0) + \sum_{j=1}^{n} \frac{h^j}{j!} \cdot \bar{m}_j \quad \forall h \in D_n \tag{3.4.1}$$

But now the new coefficients $\bar{m}_j \in {}^\bullet\mathbb{R}_{k_j}$ are uniquely determined by (3.4.1).

E.g. this will permit to prove that if $f : D \longrightarrow {}^\bullet\mathbb{R}$, then there exist one and only one pair

$$a \in {}^\bullet\mathbb{R}$$
$$b \in {}^\bullet\mathbb{R}_2$$

such that $f(h) = a + h \cdot b$ for every $h \in D$.

Proof of Corollary 3.4.1:

Existence: Let $\bar{m}_j := \iota_{k_j}(m_j)$ for every $j \in \mathbb{N}^n$ such that $\frac{j}{\alpha+1} < 1$. Note that if $j = \underline{0}$, then $\bar{m}_j = m_j = m_{\underline{0}}$, because $k_{\underline{0}} := 0$. Moreover if $j \neq \underline{0}$, then $0 < \frac{j}{\alpha+1} < 1$ and hence $k_j > 1$.

We have $\bar{m}_j \in {}^\bullet\mathbb{R}_{k_j}$ and, from Theorem 3.3.3 for every j we obtain

$$\forall h \in D_{\alpha_1} \times \cdots \times D_{\alpha_n} : \quad h^j \cdot m_j = h^j \cdot \bar{m}_j$$

and hence also the conclusion

$$\forall h \in D_{\alpha_1} \times \cdots \times D_{\alpha_n} : \quad \sum_{\substack{j \in \mathbb{N}^n \\ \frac{j}{\alpha+1} < 1}} \frac{h^j}{j!} \cdot m_j = \sum_{\substack{j \in \mathbb{N}^n \\ \frac{j}{\alpha+1} < 1}} \frac{h^j}{j!} \cdot \bar{m}_j$$

Uniqueness: Let us consider $\hat{m}_j \in {}^\bullet\mathbb{R}_{k_j}$ that verify the identity *2*., we shall use the identity principle for polynomials (Theorem 2.6.2). Indeed for each fixed $h \in D_{\alpha_1} \times \cdots \times D_{\alpha_n}$ and every $r \in {}^\bullet(-1, 1)$ we have $r \cdot h \in D_{\alpha_1} \times \cdots \times D_{\alpha_n}$, hence

$$\sum_{\substack{j \in \mathbb{N}^n \\ \frac{j}{\alpha+1} < 1}} r^j \cdot \frac{h^j}{j!} \cdot (\bar{m}_j - \hat{m}_j) = 0 \quad \forall r \in {}^\bullet(-1, 1)$$

From the identity principle of polynomials every coefficient of this polynomial in r is zero, i.e.

$$\forall j : \quad \frac{h^j}{j!} \cdot (\bar{m}_j - \hat{m}_j) = 0$$

These equalities are also true for every $h \in D_{\alpha_1} \times \cdots \times D_{\alpha_n}$, therefore from Theorem 3.3.3 we obtain $\bar{m}_j =_{k_j} \hat{m}_j$, that is $\bar{m}_j = \hat{m}_j$ because $\bar{m}_j$, $\hat{m}_j \in {}^\bullet\mathbb{R}_{k_j}$ (see Remark 3.2.4).

∎

Using the equalities up to k_j-th order infinitesimals we can state this uniqueness in another equivalent form:

Corollary 3.4.2. *In the hypotheses of the previous Corollary 3.4.1, if $p_j \in {}^\bullet\mathbb{R}$ for every $j \in \mathbb{N}^n$ with $\frac{j}{\alpha+1} < 1$ are such that*

$$\forall h \in D_{\alpha_1} \times \cdots \times D_{\alpha_n} : \quad \sum_{\substack{j \in \mathbb{N}^n \\ \frac{j}{\alpha+1} < 1}} \frac{h^j}{j!} \cdot m_j = \sum_{\substack{j \in \mathbb{N}^n \\ \frac{j}{\alpha+1} < 1}} \frac{h^j}{j!} \cdot p_j$$

then $m_j =_{k_j} p_j$ for every j.

Proof: In fact we can apply the previous Corollary 3.4.1 both with $(m_j)_j$ and with $(p_j)_j$ obtaining that the unique $(\bar{m}_j)_j$ is given by $\bar{m}_j = \iota_{k_j}(m_j) = \iota_{k_j}(p_j)$, so $m_j =_{k_j} p_j$ for every j.

∎

Existence in Taylor's formulae for ordinary smooth functions

The following theorem is a *very simple* evidence that a suitable and meaningful mathematical language can be useful to extend even well known classical results. Indeed, using the language of actual nilpotent infinitesimals we shall see that it is possible to extend the Taylor's formula for $f(x + h)$ to generic infinitesimal increments $h \in D_{\alpha_1} \times \cdots \times D_{\alpha_d}$ (the classical formulation being for $\alpha_1 = \cdots = \alpha_d$):

Theorem 3.4.3. *Let $f : U \longrightarrow \mathbb{R}^u$ be a smooth function, with U open in $\mathbb{R}^d$. Take a standard point $x \in U$ and $\alpha_1, \ldots, \alpha_d \in \mathbb{R}_{>0}$, then there exist one and only one*

$$m : \left\{ j \in \mathbb{N}^d \mid \frac{j}{\alpha + 1} < 1 \right\} \longrightarrow \mathbb{R}^u$$

such that

$$\forall h \in D_{\alpha_1} \times \cdots \times D_{\alpha_d} : \quad f(x + h) = \sum_{\substack{j \in \mathbb{N}^d \\ \frac{j}{\alpha+1} < 1}} \frac{h^j}{j!} \cdot m_j$$

Proof: For simplicity let

$$I := \left\{ j \in \mathbb{N}^d \,|\, \frac{j}{\alpha+1} < 1 \right\} \quad , \quad n := \max \{ |j| \in \mathbb{N} \,|\, j \in I \}$$

where, of course, $|j| := j_1 + \ldots + j_n$. Let us take the infinitesimal Taylor's formula of f of order n (see Theorem 2.8.4):

$$\forall h \in D_n^d : \quad f(x+h) = \sum_{\substack{j \in \mathbb{N}^d \\ |j| \le n}} \frac{h^j}{j!} \cdot m_j \tag{3.4.2}$$

where $m_j := \frac{\partial^{|j|} f}{\partial x^j}(x) \in \mathbb{R}^u$. Now if we take $h \in D_{\alpha_1} \times \cdots \times D_{\alpha_d}$, then $h_i \in D_{\alpha_i}$ and hence $\omega(h_i) < \alpha_i + 1$. We want to apply (3.4.2) with this h, so we have to prove that $h_i \in D_n$, i.e. that $\omega(h_i) < n + 1$. But if we set $j := (0, \overset{i-1}{\ldots\ldots}, 0, \alpha_i, 0, \ldots, 0)$, then

$$\frac{j}{\alpha+1} = \sum_{k=1}^{d} \frac{j_k}{\alpha_k + 1} = \frac{\alpha_i}{\alpha_i + 1} < 1$$

because $\alpha_i > 0$, so $j \in I$ and hence $n \ge |j| = \alpha_i$. Therefore $\omega(h_i) < \alpha_i + 1 \le n + 1$, that is $h_i \in D_n$ and we can apply (3.4.2) obtaining:

$$f(x+h) = \sum_{\substack{j \in \mathbb{N}^d \\ |j| \le n}} \frac{h^j}{j!} \cdot m_j \tag{3.4.3}$$

But we know that if $\frac{j}{\alpha+1} \ge 1$, then $h^j = 0$, so the sum in (3.4.3) is extended to $j \in I$ only. This proves the existence part. Uniqueness follows from Corollary 3.4.1.

■

At present the previous version of the Taylor's formula can be applied to ordinary smooth functions and to standard points $x \in U$ only. In the following results we will remove the limitation that the base point x has to be standard.

Lemma 3.4.4. *Let A be an open set in $\mathbb{R}^d$, $x \in {}^\bullet A$, $n \in \mathbb{N}_{>0}$ and $f : A \longrightarrow \mathbb{R}$ a smooth function, then*

$$\forall h \in D_n^d : \quad f(x+h) = \sum_{\substack{j \in \mathbb{N}^d \\ |j| \le n}} \frac{h^j}{j!} \cdot \frac{\partial^{|j|} f}{\partial x^j}(x) \tag{3.4.4}$$

Note that in (3.4.4) we do not have the problem to define the derivatives of the function f at the non standard point $x \in {}^\bullet A$, because we have to mean $\frac{\partial^{|j|} f}{\partial x^j}(x)$ as

$$\frac{\partial^{|j|} f}{\partial x^j}(x) = {}^\bullet\!\left(\frac{\partial^{|j|} f}{\partial x^j} \right)(x),$$

that is as the Fermat extension of the smooth function $\frac{\partial^{|j|}f}{\partial x^j}(x)$ applied to the non standard point x.

Proof: We prove the result for $d = 1$ only; the proof for the multivariable case is analogous using the suitable multi-indexes notations. Let $k := x - {}^\circ x$ be the nilpotent part of $x \in {}^\bullet A$, then $f(x+h) = f({}^\circ x + k + h)$, and we can use the infinitesimal Taylor's formula (Theorem 2.8.4) for f at the standard point ${}^\circ x$ and with infinitesimal increment $k+h$. Let us firstly suppose $k + h \neq 0$. Then the order of this sum is $\omega(k+h) = \omega(k) \vee \omega(h)$ (see Theorem 2.4.3) and we can write

$$
\begin{aligned}
f(x+h) = f({}^\circ x + k + h) &= \sum_{b=0}^{\omega(h)\vee\omega(k)} \frac{(k+h)^b}{b!} \cdot f^{(b)}({}^\circ x) = \\
&= \sum_{b=0}^{\omega(h)\vee\omega(k)} \sum_{a=0}^{b} \binom{b}{a} \frac{k^a h^{b-a}}{b!} \cdot f^{(b)}({}^\circ x) = \\
&= \sum_{b=0}^{\omega(h)\vee\omega(k)} \sum_{a=0}^{b} \frac{k^a h^{b-a}}{a!(b-a)!} \cdot f^{(b)}({}^\circ x).
\end{aligned}
$$

But $k^a = 0$ if $a > \omega(k)$, hence

$$
\begin{aligned}
f(x+h) &= \sum_{b=0}^{\omega(h)\vee\omega(k)} \sum_{a=0}^{b\wedge\omega(k)} \frac{k^a h^{b-a}}{a!(b-a)!} \cdot f^{(b)}({}^\circ x) = \\
&= \sum_{s\in I} s.
\end{aligned}
$$

where

$$
I := \left\{ \frac{k^a h^{b-a}}{a!(b-a)!} \cdot f^{(b)}({}^\circ x) \;\middle|\; {\scriptstyle b=0,\ldots,\omega(h)\vee\omega(k) \atop a=0,\ldots,b\wedge\omega(k)} \right\}. \tag{3.4.5}
$$

On the other hand we have

$$
\begin{aligned}
\sum_{j=0}^{\omega(h)} \frac{h^j}{j!} \cdot f^{(j)}(x) &= \sum_{j=0}^{\omega(h)} \frac{h^j}{j!} \cdot f^{(j)}({}^\circ x + k) = \\
&= \sum_{j=0}^{\omega(h)} \sum_{i=0}^{\omega(k)} \frac{h^j}{j!} \frac{k^i}{i!} \cdot f^{(j+i)}({}^\circ x) = \\
&= \sum_{t\in J} t
\end{aligned}
$$

where

$$
J := \left\{ \frac{h^j}{j!} \frac{k^i}{i!} \cdot f^{(j+i)}({}^\circ x) \;\middle|\; \begin{array}{c} \scriptstyle j=0,\ldots,\omega(h) \\ \scriptstyle i=0,\ldots,\omega(k) \\ \scriptstyle \frac{i}{\omega(k)}+\frac{j}{\omega(h)}\le 1 \end{array} \right\} \tag{3.4.6}
$$

in fact $h^j k^i = 0$ if $\frac{i}{\omega(k)} + \frac{j}{\omega(h)} > 1$ (see Theorem 2.5.1). Now we can prove that the two sets of addends I and J are equal or they differ at most for zero, i.e. $I \cup \{0\} = J \cup \{0\}$. Indeed, take an element t from J

$$\begin{aligned} t &= \frac{h^j k^i}{j!i!} \cdot f^{(j+i)}({}^\circ x) \\ j &= 0, \dots, \omega(h) \\ i &= 0, \dots, \omega(k) \\ \frac{i}{\omega(k)} &+ \frac{j}{\omega(h)} \le 1, \end{aligned} \tag{3.4.7}$$

then setting $a := i$, $b := a + j = i + j$ we have

$$t = \frac{h^{b-a} k^a}{(b-a)!a!} \cdot f^{(b)}({}^\circ x).$$

Moreover, we have that $a = i \le \omega(k)$ and $a = b - j \le b$, so $a = 0, \dots, b \wedge \omega(k)$. From 3.4.7 we have $i \le \omega(k) - j\frac{\omega(k)}{\omega(h)}$ and hence $i + j \le \omega(k) + j \cdot \left(1 - \frac{\omega(k)}{\omega(h)}\right) = \omega(k) + j \cdot \frac{\omega(h) - \omega(k)}{\omega(h)}$. If $\omega(k) \ge \omega(h)$, then $i + j \le \omega(k) + j \cdot \frac{\omega(h)-\omega(k)}{\omega(h)} \le \omega(k) = \omega(h) \vee \omega(k)$. Vice versa if $\omega(k) < \omega(h)$, then since $\frac{j}{\omega(h)} \le 1$ we have that $\omega(k) + j \cdot \frac{\omega(h)-\omega(k)}{\omega(h)} \le \omega(k) + \omega(h) - \omega(k) = \omega(h) = \omega(h) \vee \omega(k)$. In any case we have proved that $b = i + j \le \omega(h) \vee \omega(k)$, so the addend t is indeed an element of J.

Vice versa, let us consider

$$\begin{aligned} s &= \frac{h^{b-a} k^a}{(b-a)!a!} \cdot f^{(b)}({}^\circ x) \in J \\ b &= 0, \dots, \omega(h) \vee \omega(k) \\ a &= 0, \dots, b \wedge \omega(k), \end{aligned}$$

and set $i := a$, $j = b - a$, then

$$s = \frac{h^j k^i}{j!i!} \cdot f^{(j+i)}({}^\circ x).$$

Moreover, $i \le b \wedge \omega(k) \le \omega(k)$ and $j \le \omega(h)$ or, in the opposite case, we have $h^{b-a} = h^j = 0 = s$. Analogously we have $\frac{i}{\omega(k)} + \frac{j}{\omega(h)} \le 1$ or, in the opposite case, we have $h^{b-a}k^a = h^j k^i = 0 = s$. At the end we have proved that $s \in I \cup \{0\}$.

It remains to prove the case $h + k = 0$. But with the previous deduction we have proved that

$$\sum_{b=0}^{\omega(h)\vee\omega(k)} \frac{(k+h)^b}{b!} \cdot f^{(b)}({}^\circ x) = \sum_{j=0}^{\omega(h)} \frac{h^j}{j!} \cdot f^{(j)}({}^\circ x + k). \tag{3.4.8}$$

If $h+k=0$ the right hand side of (3.4.8) gives $f(^\circ x) = f(^\circ x + k + h) = f(x+h)$. ∎

Using this lemma, and the general uniqueness of Corollary 3.4.1, we can repeat equal the proof of Theorem 3.4.3 obtaining its generalization to a non standard base point $x \in {}^\bullet U$:

Theorem 3.4.5. *Let $f : U \longrightarrow \mathbb{R}^u$ be a smooth function, with U open in $\mathbb{R}^d$. Take a point $x \in {}^\bullet U$ and $\alpha_1, \ldots, \alpha_d \in \mathbb{R}_{>0}$, then there exists one and only one*

$$m : \left\{ j \in \mathbb{N}^d \mid \frac{j}{\alpha+1} < 1 \right\} \longrightarrow {}^\bullet\mathbb{R}$$

such that

1. $\bar{m}_j \in {}^\bullet\mathbb{R}_{k_j}$ *for every* $j \in \mathbb{N}^d$ *such that* $\frac{j}{\alpha+1} < 1$
2. $\forall h \in D_{\alpha_1} \times \cdots \times D_{\alpha_d} : \quad f(x+h) = \sum_{\substack{j \in \mathbb{N}^d \\ \frac{j}{\alpha+1} < 1}} \frac{h^j}{j!} \cdot m_j$

In the following chapters we shall see how to generalize these theorems to more general functions with respect to ${}^\bullet f$, i.e. extension of standard functions. We have an example of a function which is not of this type, considering e.g. $f = {}^\bullet g(p, -)$ for $p \in {}^\bullet\mathbb{R}$ and $g \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$. In this case we will see that in general the coefficients of the corresponding Taylor's formulae will be generic elements $m_j \in {}^\bullet\mathbb{R}_{k_j}$ and not standard reals only.

Example. We want to find the Taylor's formula of

$$f(h,k) = \frac{\sin(h)}{\cos(k)} \tag{3.4.9}$$

for $h \in D_3$ and $k \in D_4$. We can note, using the previous theorem, that the sum in this Taylor's formula is extended to all pair $(i,j) \in \mathbb{N}^2$ such that

$$\frac{i}{3+1} + \frac{j}{4+1} < 1$$

that is such that $5i + 4j < 20$. But to find this Taylor's formula it is simpler to substitute in (3.4.9) the Taylor's formulae of $\sin(h) = h - \frac{h^3}{6}$ for $h \in D_3$ and of $\cos(k) = 1 - \frac{k^2}{2} + \frac{k^4}{24}$ for $k \in D_4$ and to apply the *algebraic* calculus of nilpotent infinitesimals we have developed until now:

$$\begin{aligned} f(h,k) &= \frac{h - \frac{h^3}{6}}{1 - \frac{k^2}{2} + \frac{k^4}{24}} = \left(h - \frac{h^3}{6}\right) \cdot \left(1 + \frac{k^2}{2} - \frac{k^4}{24} + \left[\frac{k^4}{24} - \frac{k^2}{2}\right]^2\right) = \\ &= h + \frac{1}{2}hk^2 - \frac{h^3}{6} \quad \forall h \in D_3 \ \forall k \in D_4 \end{aligned} \tag{3.4.10}$$

For example to obtain this result we have used the equalities $hk^4 = 0$ and $h^3k^2 = 0$, easily deducible from Corollary 2.5.4.

From (3.4.10) and from the previous Theorem 3.4.3 we have

$$\frac{\partial f}{\partial h}(0,0) = 1 \quad \frac{\partial^3 f}{\partial h \partial k^2}(0,0) = 1 \quad \frac{\partial^3 f}{\partial h^3}(0,0) = -1$$

and for all the other indexes i, $j \in \mathbb{N}$ such that $5i + 4j < 20$ we have

$$\frac{\partial^{i+j} f}{\partial h^i \partial k^j}(0,0) = 0$$

Of course this is *only an elementary example*, similar to several exercises one can find in elementary courses of Calculus. The only meaningful difference is that we have not used directly neither the concept of limit nor any rest in the form of suitable little-oh functions. An easy to use algebraic language of nilpotent infinitesimals have been used instead. It can also be useful to note that, in comparison with SDG, for Fermat reals it is very easy to decide if products of type $h^3 k^2$, with $h \in D_3$ and $k \in D_4$, are zero or not; the same easiness is not possible in SDG where starting only from the belonging to some D_k it is not possible to decide products of this type (see e.g. Kock [1981] for more details).

3.5 Extension of some results to D_∞

In this section we want to extend some of the results of the previous sections to the ideal D_∞ of all the infinitesimals (see Definition 2.4.2).

Corollary 3.5.1. *Let $m \in {}^\bullet\mathbb{R}$, $n \in \mathbb{N}_{>0}$, $j \in \mathbb{N}^n \setminus \{\underline{0}\}$ then*

1. $\forall h \in D_\infty^n : \quad h^j \cdot m = h^j \cdot \iota_1 m$
2. *If $h^j \cdot m = 0$ for every $h \in D_\infty^n$, then $m =_1 0$, that is $m = \alpha \, \mathrm{d}t$ for some $\alpha \in \mathbb{R}$.*

Proof: We recall that

$$\iota_1 m = {}^\circ m + \sum_{i:\omega_i(m)>1}^{N} {}^\circ m_i \, \mathrm{d}t_{\omega_i(m)}$$

and because $\omega_i(m) \geq 1$ for every Fermat real m and every $i = 1, \ldots, N$ we can write

$$m = \alpha \, \mathrm{d}t + \iota_1 m$$

where $\alpha := {}^\circ m_{\bar{\imath}}$ if $\omega_{\bar{\imath}}(m) = 1$ for some $\bar{\imath} = 1, \ldots, N$, otherwise $\alpha := 0$. Therefore if $h \in D_\infty^n$, we have $h^j \cdot m = h^j \cdot \alpha \, \mathrm{d}t + h^j \cdot \iota_1 m = h^j \cdot \iota_1 m$ because $k \, \mathrm{d}t = 0$ for every infinitesimal $k \in D_\infty$. This proves *1*.
2.) From the hypothesis $h^j \cdot m = 0$ for every $h \in D_\infty^n$, because $D_a \subset D_\infty$, it follows

$$\forall h \in D_a^n : \quad h^j \cdot m = 0 \tag{3.5.1}$$

where $a \geq 1$. So we can apply Theorem 3.3.3 for each one of these a. We set k_a with

$$\frac{1}{k_a} + \frac{j}{(a, \overset{n}{\ldots}, a) + 1} = 1$$

that is

$$\frac{1}{k_a} = 1 - \sum_{i=1}^{n} \frac{j_i}{a+1} = 1 - \frac{|j|}{a+1}$$

$$k_a = \frac{a+1}{a+1-|j|}$$

and hence from (3.5.1) and Theorem 3.3.3 we have that $m =_{k_a} 0$ for every $a \geq 1$, that is

$$^\circ m + \sum_{\omega_i(m) > k_a} {^\circ m_i} \, \mathrm{d}t_{\omega_i(m)} = 0 \quad \forall a \geq 1 \tag{3.5.2}$$

This implies $^\circ m = 0$ and for each $a \geq 1$ the sum in (3.5.2) does not have addends, i.e.

$$\forall a \geq 1 \, \nexists i = 1, \ldots, N : \omega_i(m) > k_a \tag{3.5.3}$$

But $\lim_{a \to +\infty} k_a = 1^+$, so if we had

$$\exists \bar{\imath} = 1, \ldots, N : \quad \omega_{\bar{\imath}}(m) > 1$$

then we could find a suitable $\bar{a} \geq 1$ such that $\omega_{\bar{\imath}}(m) > k_{\bar{a}} \geq 1$, in contradiction with (3.5.3). Therefore $\omega_i(m) \leq 1$ for each $i = 1, \ldots, N$, i.e. $m =_1 0$ and hence $m = \alpha \, \mathrm{d}t$ for $\alpha = {^\circ m_1}$ or $\alpha = 0$ if $m = 0$.

∎

Using exactly the same ideas used in the proof of the previous corollary we can also generalize Taylor's formulae to the case of D_∞. First the uniqueness:

Corollary 3.5.2. *Let $(m_j)_{j \in \mathbb{N}^d \setminus \{\underline{0}\}}$ be a sequence of $^\bullet\mathbb{R}$, then there exists one and only one*

$$(m_j)_{j \in \mathbb{N}^d \setminus \{\underline{0}\}} \text{ sequence of } {^\bullet\mathbb{R}_1}$$

such that

$$\forall h \in D_\infty^d : \quad \sum_{\substack{j \in \mathbb{N}^d \\ j \neq \underline{0}}} \frac{h^j}{j!} \cdot m_j = \sum_{\substack{j \in \mathbb{N}^d \\ j \neq \underline{0}}} \frac{h^j}{j!} \cdot \bar{m}_j .$$

∎

Secondly, it is also easy to derive the Taylor's formula for standard functions:

Corollary 3.5.3. *Let $f : U \longrightarrow \mathbb{R}^u$ be a smooth function, with U open in $\mathbb{R}^d$. Take $x \in {^\bullet U}$, then there exist one and only one*

$$m : \mathbb{N}^d \longrightarrow \mathbb{R}^u$$

such that

$$\forall h \in D_\infty^d : \quad f(x+h) = \sum_{j \in \mathbb{N}^d} \frac{h^j}{j!} \cdot m_j \tag{3.5.4}$$

■

Let us point out that in formulae like (3.5.4) we do not have a series but a finite sum because every $h_i \in D_\infty$ is nilpotent.

3.6 Some elementary examples

The elementary examples presented in this section want to show, in a few rows, the simplicity of the algebraic calculus of nilpotent infinitesimals. Here "simplicity" means that the dialectic with the corresponding informal calculations, used e.g. in engineering or in physics, is really faithful. The importance of this dialectic can be glimpsed both as a proof of the flexibility of the new language, but also for researches in artificial intelligence like automatic differentiation theories (see e.g. Griewank [2000] and references therein). Last but not least, it may also be important for didactic or historical researches.

1. **Commutation of differentiation and integration.** This example derives from Kock [1981], Lavendhomme [1996]. Suppose we want *to discover* the derivative of the function

$$g(x) := \int_{\alpha(x)}^{\beta(x)} f(x,t)\,\mathrm{d}t \qquad \forall x \in \mathbb{R}$$

where α, β and f are smooth functions. We can see g as a composition of smooth functions, hence we can apply the derivation formula, i.e. Theorem 2.8.3:

$$\begin{aligned} g(x+h) = & \int_{\alpha(x+h)}^{\beta(x+h)} f(x+h,t)\,\mathrm{d}t = \\ & = \int_{\alpha(x)+h\alpha'(x)}^{\alpha(x)} f(x,t)\,\mathrm{d}t + h \cdot \int_{\alpha(x)+h\alpha'(x)}^{\alpha(x)} \frac{\partial f}{\partial x}(x,t)\,\mathrm{d}t + \\ & \quad + \int_{\alpha(x)}^{\beta(x)} f(x,t)\,\mathrm{d}t + h \cdot \int_{\alpha(x)}^{\beta(x)} \frac{\partial f}{\partial x}(x,t)\,\mathrm{d}t + \\ & \quad + \int_{\beta(x)}^{\beta(x)+h\beta'(x)} f(x,t)\,\mathrm{d}t + h \cdot \int_{\beta(x)}^{\beta(x)+h\beta'(x)} \frac{\partial f}{\partial x}(x,t)\,\mathrm{d}t. \end{aligned}$$

Now we use $h^2 = 0$ to obtain e.g. (see Corollary 2.8.5):

$$h \cdot \int_{\alpha(x)+h\alpha'(x)}^{\alpha(x)} \frac{\partial f}{\partial x}(x,t)\,\mathrm{d}t = -h^2 \cdot \alpha'(x) \cdot \frac{\partial f}{\partial x}(\alpha(x),t) = 0$$

and
$$\int_{\alpha(x)+h\alpha'(x)}^{\alpha(x)} f(x,t)\,\mathrm{d}t = -h \cdot \alpha'(x) \cdot f(\alpha(x), t).$$
Calculating in an analogous way similar terms we finally obtain the well known conclusion. Note that the final formula comes out by itself so that we have *"discovered"* it and not simply we have proved it. From the point of view of artificial intelligence or from the didactic point of view, surely this discovering is not a trivial result.

2. **Circle of curvature.** A simple application of the infinitesimal Taylor's formula is the parametric equation for the circle of curvature, that is the circle with second order osculation with a curve $\gamma : [0,1] \longrightarrow \mathbb{R}^3$. In fact if $r \in (0,1)$ and $\dot{\gamma}_r$ is a unit vector, from the second order infinitesimal Taylor's formula we have

$$\forall h \in D_2 : \quad \gamma(r+h) = \gamma_r + h\,\dot{\gamma}_r + \frac{h^2}{2}\,\ddot{\gamma}_r = \gamma_r + h\,\vec{t}_r + \frac{h^2}{2} c_r\,\vec{n}_r \quad (3.6.1)$$

where $\vec{n}$ is the unit normal vector, $\vec{t}$ is the tangent one and c_r the curvature. But once again from Taylor's formula we have $\sin(ch) = ch$ and $\cos(ch) = 1 - \frac{c^2h^2}{2}$. Now it suffices to substitute h and $\frac{h^2}{2}$ from these formulas into (3.6.1) to obtain the conclusion

$$\forall h \in D_2 : \quad \gamma(r+h) = \left(\gamma_r + \frac{\vec{n}_r}{c_r}\right) + \frac{1}{c_r} \cdot \left[\sin(c_r h)\vec{t}_r - \cos(c_r h)\vec{n}_r\right].$$

In a similar way we can prove that any $f \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R})$ can be written $\forall h \in D_k$ as

$$f(h) = \sum_{n=0}^{k} a_n \cdot \cos(nh) + \sum_{n=0}^{k} b_n \cdot \sin(nh),$$

so that now the idea of the Fourier series comes out in a natural way.

3. **Schwarz's theorem.** Using nilpotent infinitesimals we can obtain a simple and meaningful proof of Schwarz's theorem. This simple example aims to show how to manage some differences between our setting and Synthetic Differential Geometry (see Kock [1981], Lavendhomme [1996], Moerdijk and Reyes [1991]). Let $f : V \longrightarrow E$ be a $\mathcal{C}^2$ function between spaces of type $V = \mathbb{R}^m$, $E = \mathbb{R}^n$ (in subsequent chapters we will see that the same proof is still valid for Banach spaces too) and $a \in V$, we want to prove that $\mathrm{d}^2 f(a) : V \times V \longrightarrow E$ is symmetric. Take

$$\begin{aligned} &k \in D_2 \\ &h, j \in \mathcal{D}_\infty \\ &jkh \in D_{\neq 0} \end{aligned}$$

(e.g. we can take $k_t = \mathrm{d}t_2, h_t = j_t = \mathrm{d}t_4$ so that $jkh = \mathrm{d}t$, see also Theorem 2.5.1). Using $k \in D_2$ and Lemma 3.4.4, we have

$$\begin{aligned} j \cdot f(x + hu + kv) = \\ &= j \cdot \left[f(x+hu) + k\, \partial_v f(x+hu) + \frac{k^2}{2} \partial_v^2 f(x+hu) \right] \\ &= j \cdot f(x+hu) + jk \cdot \partial_v f(x+hu) \end{aligned} \tag{3.6.2}$$

where we used the fact that $k^2 \in D$ and j infinitesimal imply $jk^2 = 0$. Now we consider that $jkh \in D$ so that any product of type $jkhi$ is zero for every $i \in D_\infty$, so we obtain

$$jk \cdot \partial_v f(x+hu) = jk \cdot \partial_v f(x) + jkh \cdot \partial_u(\partial_v f)(x). \tag{3.6.3}$$

But $k \in D_2$ and $jk^2 = 0$ hence

$$j \cdot f(x+kv) - j \cdot f(x) = jk \cdot \partial_v f(x).$$

Substituting this in (3.6.3) and hence in (3.6.2) we obtain

$$\begin{aligned} & j \cdot [f(x+hu+kv) - f(x+hu) - f(x+kv) + f(x)] = \\ & = jkh \cdot \partial_u(\partial_v f)(x). \end{aligned} \tag{3.6.4}$$

The left hand side of this equality is symmetric in u, v, hence changing them we have

$$jkh \cdot \partial_u(\partial_v f)(x) = jkh \cdot \partial_v(\partial_u f)(x)$$

and thus we obtain the conclusion because $jkh \neq 0$ and $\partial_u(\partial_v f)(x)$, $\partial_v(\partial_u f)(x) \in E$. From (3.6.4) it follows directly the classical limit relation

$$\lim_{t \to 0^+} \frac{f(x + h_t u + k_t v) - f(x + h_t u) - f(x + k_t v) + f(x)}{h_t k_t} = \partial_u \partial_v f(x)$$

4. **Electric dipole**. In elementary physics, an electric dipole is usually defined as "*a pair of charges with opposite sign placed at a distance d very less than the distance r from the observer*".

   Conditions like $r \gg d$ are frequently used in Physics and very often we obtain a correct formalization if we ask $d \in {}^\bullet\mathbb{R}$ infinitesimal but $r \in \mathbb{R} \setminus \{0\}$, i.e. r finite. Thus we can define an electric dipole as a pair (p_1, p_2) of electric particles, with charges of equal intensity but with opposite sign such that their mutual distance at every time t is a first order infinitesimal:

$$\forall t: \quad |p_1(t) - p_2(t)| =: |\vec{d_t}| =: d_t \in D. \tag{3.6.5}$$

In this way we can calculate the potential at the point x using the properties of D and using the hypothesis that r is finite and not zero. In fact we have

$$\varphi(x) = \frac{q}{4\pi\epsilon_0} \cdot \left(\frac{1}{r_1} - \frac{1}{r_2}\right) \qquad\qquad \vec{r_i} := x - p_i$$

and if $\vec{r} := \vec{r}_2 - \frac{\vec{d}}{2}$ then

$$\frac{1}{r_2} = \left(r^2 + \frac{d^2}{4} + \vec{r} \cdot \vec{d}\right)^{-1/2} = r^{-1} \cdot \left(1 + \frac{\vec{r} \cdot \vec{d}}{r^2}\right)^{-1/2}$$

because for (3.6.5) $d^2 = 0$. For our hypotheses on d and r we have that $\frac{\vec{r} \cdot \vec{d}}{r^2} \in D$ hence from the derivation formula

$$\left(1 + \frac{\vec{r} \cdot \vec{d}}{r^2}\right)^{-1/2} = 1 - \frac{\vec{r} \cdot \vec{d}}{2r^2}$$

In the same way we can proceed for $1/r_1$, hence:

$$\begin{aligned}\varphi(x) &= \frac{q}{4\pi\epsilon_0} \cdot \frac{1}{r} \cdot \left(1 + \frac{\vec{r} \cdot \vec{d}}{2r^2} - 1 + \frac{\vec{r} \cdot \vec{d}}{2r^2}\right) = \\ &= \frac{q}{4\pi\epsilon_0} \cdot \frac{\vec{r} \cdot \vec{d}}{r^3}\end{aligned}$$

The property $d^2 = 0$ is also used in the calculus of the electric field and for the moment of momentum.

5. **Newtonian limit in Relativity.** Another example in which we can formalize a condition like $r \gg d$ using the previous ideas is the Newtonian limit in Relativity; in it we can suppose to have

   - $\forall t: \quad v_t \in D_2 \quad$ and $\quad c \in \mathbb{R}$
   - $\forall x \in M_4: \quad g_{ij}(x) = \eta_{ij} + h_{ij}(x) \quad$ with $\quad h_{ij}(x) \in D$.

   where $(\eta_{ij})_{ij}$ is the matrix of the Minkowski's metric. This conditions can be interpreted as $v_t \ll c$ and $h_{ij}(x) \ll 1$ (low speed with respect to the speed of light and weak gravitational field). In this way we have, e.g. the equalities:

$$\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = 1 + \frac{v^2}{2c^2} \qquad \text{and} \qquad \sqrt{1 - h_{44}(x)} = 1 - \frac{1}{2} h_{44}(x).$$

6. **Linear differential equations.** Let

$$L(y) := A_0 \frac{\mathrm{d}^N y}{\mathrm{d}t^N} + \ldots + A_{N-1} \frac{\mathrm{d}y}{\mathrm{d}t} + A_N \cdot y = 0$$

be a linear differential equation with constant coefficients. Once again we want *to discover* independent solutions in case the characteristic polynomial has multiple roots e.g.

$$(r - r_1)^2 \cdot (r - r_3) \cdot \ldots \cdot (r - r_N) = 0.$$

The idea is that in ${}^\bullet\mathbb{R}$ we have $(r-r_1)^2 = 0$ also if $r = r_1+h$ with $h \in D$. Thus $y(t) = \mathrm{e}^{(r_1+h)t}$ is a solution too. But $\mathrm{e}^{(r_1+h)t} = \mathrm{e}^{r_1 t} + ht \cdot \mathrm{e}^{r_1 t}$, hence

$$\begin{aligned} L\left[\mathrm{e}^{(r_1+h)t}\right] &= 0 \\ &= L\left[\mathrm{e}^{r_1 t} + ht \cdot \mathrm{e}^{r_1 t}\right] \\ &= L\left[\mathrm{e}^{r_1 t}\right] + h \cdot L\left[t \cdot \mathrm{e}^{r_1 t}\right] \end{aligned}$$

We obtain $L\left[t \cdot \mathrm{e}^{r_1 t}\right] = 0$, that is $y_1(t) = t \cdot \mathrm{e}^{r_1 t}$ must be a solution. Using k-th order infinitesimals we can deal with other multiple roots in a similar way.

We think that these elementary examples are able to show that some results that frequently may appear as unnatural in a standard context, using Fermat reals may be even discovered, even by suitably designed algorithm.

Chapter 4

Order relation

4.1 Infinitesimals and order properties

Like in other disciplines, also in mathematics the layout of a work reflects the personal philosophical ideas of the authors. In particular the present work is based on the idea that a good mathematical theory is able to construct a good dialectic between formal properties, proved in the theory, and their informal interpretations. The dialectic has to be, as far as possible, in both directions: theorems proved in the theory should have a clear and useful intuitive interpretation and, on the other hand, the intuition corresponding to the theory has to be able to suggest true sentences, i.e. conjectures or sketch of proofs that can then be converted into rigorous proofs.

In a theory of new numbers, like the present one about Fermat reals, the introduction of an order relation can be a hard test of the excellence of this dialectic between formal properties and their informal interpretations. Indeed if we introduce a new ring of numbers (like ${}^\bullet\mathbb{R}$) extending the real field $\mathbb{R}$, we want that the new order relation, defined on the new ring, will extend the standard one on $\mathbb{R}$. This extension naturally leads to the wish of findings a geometrical representation of the new numbers, in accord with the above principle of having a good formal/informal dialectic.

For example, on the one hand in NSA the order relation on ${}^*\mathbb{R}$ has the best formal properties among all the theories of actual infinitesimals. On the other hand, the dialectic of these properties with the informal interpretations is not always good, due to the use of, e.g., an ultrafilter in the construction of ${}^*\mathbb{R}$. Indeed, in an ultrafilter on $\mathbb{N}$ we can always find a highly non constructive set $A \subset \mathbb{N}$; any sequence of reals $x : \mathbb{N} \longrightarrow \mathbb{R}$ which is constant to 1 on A is strictly greater than 0 in ${}^*\mathbb{R}$, but it seems not easy to give neither an intuitive interpretation nor a clear and meaningful geometric representation of the relation $x > 0$ in ${}^*\mathbb{R}$. In fact, it is also for motivations of this type that some approaches to give a constructive definition of a field similar to ${}^*\mathbb{R}$ have been attempted (see e.g. Palmgren [1995, 1997, 1998] and references therein).

In SDG we have a preorder relation (i.e. a reflexive and transitive relation, which is not necessarily anti-symmetric) with very poor properties only. Nevertheless, the works developed in SDG (see e.g. Lavendhomme [1996]) exhibits that meaningful geometric results can be obtained in infinite dimensional spaces, even if the order properties of the ground base ring are not so rich. Once again, the dialectic between formal properties and their intuitive interpretations represents a hard test for SDG too. E.g. it seems not so easy to interpret intuitively that every infinitesimal h in SDG verifies both $h \geq 0$ and $h \leq 0$. The lack of a total order, i.e. of the trichotomy law

$$x < y \quad \text{or} \quad y < x \quad \text{or} \quad x = y \tag{4.1.1}$$

makes really difficult, or even impossible, to have a geometrical representation of the infinitesimals of SDG.

We want to start this section showing that in our setting there is a strong connection between some order properties and some algebraic properties. In particular, we will show that it is not possible to have good order properties and at the same time a uniqueness without limitations in the derivation formula (see the discussion starting Chapter 3). We know that in ${}^{\bullet}\mathbb{R}$ the product of any two first order infinitesimals $h, k \in D$ is always zero: $h \cdot k = 0$, and a consequence of this property is that we have some limitations in the uniqueness of the derivation formula, and for these reasons we introduce the notion of equality up to k-th order infinitesimals (see Chapter 3). In the following theorem we can see that the property $h \cdot k = 0$ is a general consequence if we suppose to have a total order on D. The idea of this theorem can be glimpsed at from the Figure 4.1, where it is represented that if we neglect h^2 and k^2 because we consider them zero, then we have strong reasons to expect that also $h \cdot k$ will be zero

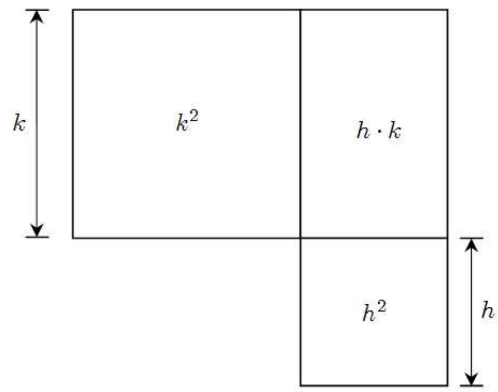

Figure 4.1: How to guess that $h \cdot k = 0$ for two first order infinitesimals $h, k \in D$

From this picture comes the idea to find a formal demonstration based on the implication

$$h, k \geq 0 \quad , \quad h \leq k \quad \Longrightarrow \quad 0 \leq hk \leq k^2 = 0$$

All these ideas conduct toward the following theorem.

Theorem 4.1.1. *Let $(R, \leq)$ be a generic ordered ring and $D \subseteq R$ a subset of this ring, such that*

1. $0 \in D$
2. $\forall h \in D : \quad h^2 = 0$ *and* $-h \in D$
3. $(D, \leq)$ *is a total order*

then

$$\forall h, k \in D : \quad h \cdot k = 0 \tag{4.1.2}$$

This theorem implies that if we want a total order in our theory of infinitesimal numbers, and if in this theory we consider $D = \{h \,|\, h^2 = 0\}$, then we must accept that the product of any two elements of D must be zero. For example, if we think that a geometric representation of infinitesimals cannot be possible if we do not have, at least, the trichotomy law, then in this theory we must also have that the product of two first order infinitesimals is zero. Finally, because in SDG property (4.1.2) is false, this theorem also implies that in SDG it is not possible to define a total order (and not only a preorder) on the set D of first order infinitesimals compatible with the ring operations.

Proof of Theorem 4.1.1:
Let h, $k \in D$ be two elements of the subset D. By hypotheses 0, $-h$, $-k \in D$, hence all these elements are comparable with respect to the order relation $\leq$, because, by hypotheses this relation is total (i.e. (4.1.1) is true). E.g.

$$h \leq k \quad \text{or} \quad k \leq h$$

We will consider only the case $h \leq k$, because analogously we can deal with the case $k \leq h$, simply exchanging everywhere h with k and vice versa.

First sub-case: $k \geq 0$. By multiplying both sides of $h \leq k$ by $k \geq 0$ we obtain

$$hk \leq k^2 \tag{4.1.3}$$

If $h \geq 0$ then, multiplying by $k \geq 0$ we have $0 \leq hk$, so from (4.1.3) we have $0 \leq hk \leq k^2 = 0$, and hence $hk = 0$.

If $h \leq 0$ then, multiplying by $k \geq 0$ we have

$$hk \leq 0 \tag{4.1.4}$$

If, furthermore, $h \geq -k$, then multiplying by $k \geq 0$ we have $hk \geq -k^2$, hence form (4.1.4) $0 \geq hk \geq -k^2 = 0$, hence $hk = 0$.
If, otherwise, $h \leq -k$, then multiplying by $-h \geq 0$ we have $-h^2 = 0 \leq hk \leq 0$ from (4.1.4), hence $hk = 0$. This concludes the discussion of the case $k \geq 0$.

Second sub-case: $k \leq 0$. In this case we have $h \leq k \leq 0$. Multiplying both inequalities by $h \leq 0$ we obtain $h^2 = 0 \geq hk \geq 0$ and hence $hk = 0$.

■

Property (4.1.2) is incompatible with the uniqueness in a possible derivation formula like

$$\exists!\, m \in R : \quad \forall h \in D : \quad f(h) = f(0) + h \cdot m \tag{4.1.5}$$

framed in the ring R of Theorem 4.1.1. In fact, if $a, b \in D$ are two elements of the subset $D \subseteq R$, then both a and b play the role of $m \in R$ in (4.1.5) for the linear function

$$f : h \in D \mapsto h \cdot a = 0 \in R$$

So, if the derivation formula (4.1.5) applies to linear functions (or less, to constant functions), the uniqueness part of this formula cannot hold in the ring R.

In the next section we will introduce a natural and meaningful total order relation on ${}^\bullet\mathbb{R}$. Therefore, the previous Theorem 4.1.1 strongly motivate that for the ring of Fermat reals ${}^\bullet\mathbb{R}$ we must have that the product of two first order infinitesimals must be zero and hence, that for the derivation formula in ${}^\bullet\mathbb{R}$ the uniqueness cannot hold in its strongest form. Since we will also see that the order relation permits to have a geometric representation of Fermat reals, we can summarize the conclusions of this section saying that the uniqueness in the derivation formula is incompatible with a natural geometric interpretation of Fermat reals and hence with a good dialectic between formal properties and informal interpretations in this theory.

4.2 Order relation

From the previous sections one can draw the conclusion that the ring of Fermat reals ${}^\bullet\mathbb{R}$ is essentially "the little-oh" calculus. But, on the other hand the Fermat reals give us more flexibility than this calculus: working with ${}^\bullet\mathbb{R}$ we do not have to bother ourselves with remainders made of "little-oh", but we can neglect them and use the useful algebraic calculus with nilpotent infinitesimals. But thinking the elements of ${}^\bullet\mathbb{R}$ as new numbers, and not simply as "little-oh functions", permits to treat them in a different and new way, for example to define on them an order relation with a clear geometrical interpretation[1].

First of all, let us introduce the useful notation

$$\forall^0 t \geq 0 : \quad \mathcal{P}(t)$$

and we will read the quantifier $\forall^0 t \geq 0$ saying *"for every $t \geq 0$ (sufficiently) small"*, to indicate that the property $\mathcal{P}(t)$ is true for all t in some right[2] neighborhood of $t = 0$, i.e.

$$\exists \delta > 0 : \quad \forall t \in [0, \delta) : \quad \mathcal{P}(t)$$

The first heuristic idea to define an order relation is the following

$$x \leq y \iff x - y \leq 0 \iff \exists z : \quad z = 0 \quad \text{in} \quad {}^\bullet\mathbb{R} \quad \text{and} \quad x - y \leq z$$

More precisely, if x, $y \in {}^\bullet\mathbb{R}$ are two little-oh polynomials, we want to ask locally that[3] x_t is less than or equal to y_t, but up to a $o(t)$ for $t \to 0^+$, where the little-oh function $o(t)$ depends on x and y. Formally:

Definition 4.2.1. *Let x, $y \in {}^\bullet\mathbb{R}$, then we say*

$$x \leq y$$

iff we can find $z \in {}^\bullet\mathbb{R}$ such that $z = 0$ in ${}^\bullet\mathbb{R}$ and

$$\forall^0 t \geq 0 : \quad x_t \leq y_t + z_t$$

Recall that $z = 0$ in ${}^\bullet\mathbb{R}$ is equivalent to $z_t = o(t)$ for $t \to 0^+$. It is immediate to see that we can equivalently define $x \leq y$ if and only if we can find $x' = x$ and $y' = y$ in ${}^\bullet\mathbb{R}$ such that $x_t \leq y_t$ for every t sufficiently small. From this it also follows that the relation $\leq$ is well defined on ${}^\bullet\mathbb{R}$, i.e. if $x' = x$ and $y' = y$ in ${}^\bullet\mathbb{R}$ and $x \leq y$, then $x' \leq y'$. As usual we will use the notation $x < y$ for $x \leq y$ and $x \neq y$.

Theorem 4.2.2. *The relation $\leq$ is an order, i.e. is reflexive, transitive and anti-symmetric; it extends the order relation of $\mathbb{R}$ and with it $({}^\bullet\mathbb{R}, \leq)$ is an ordered ring. Finally the following sentences are equivalent:*

[1] We will see that this order relation is different from the order of infinite or infinitesimal originally introduced by P. Du Bois-Reymond (see Hardy [1910]).

[2] We recall that, by Definition 2.1.2, our little-oh polynomials are always defined on $\mathbb{R}_{\geq 0}$

[3] We recall that, to simplify the notations, we do not use equivalence classes as elements of ${}^\bullet\mathbb{R}$ but directly little-oh functions. The only notion of equality between little-oh functions is, of course, the equivalence relation defined in Definition 2.3.1 and, as usual, we must always prove that our relations between little-oh polynomials are well defined.

1. *$h \in D_\infty$, i.e. h is an infinitesimal*

2. $\forall r \in \mathbb{R}_{>0} : \quad -r < h < r$

Hence an infinitesimal can be thought of as a number with standard part zero, or as a number smaller than every standard positive real number and greater than every standard negative real number (thus it has in this sense the same property as an infinitesimal both in NSA and in SDG (in the latter case with real numbers of type $\frac{1}{n}$ $(n \in \mathbb{N}_{>0})$ only).

Proof: It is immediate to prove that the relation is reflexive. To prove transitivity, if $x \le y$ and $y \le w$, then we have

$$\forall^0 t \ge 0 : \quad x_t \le y_t + z_t \quad \text{and} \quad \forall^0 t \ge 0 : \quad y_t \le w_t + z'_t$$

and these imply

$$\forall^0 t \ge 0 : \quad x_t \le y_t + z_t \le w_t + z_t + z'_t$$

showing that $x \le w$. To prove that it is also anti-symmetric, take $x \le y$ and $y \le x$, then we have

$$x_t \le y_t + z_t \quad \forall t \in [0, \delta_1) \tag{4.2.1}$$
$$y_t \le x_t + z'_t \quad \forall t \in [0, \delta_2) \tag{4.2.2}$$

$$\lim_{t \to 0^+} \frac{z_t}{t} = 0 \quad \text{and} \quad \lim_{t \to 0^+} \frac{z'_t}{t} = 0$$

because z and z' are equal to zero in ${}^\bullet\mathbb{R}$, that is are $o(t)$ for $t \to 0^+$. Hence from (4.2.1) and (4.2.2) for $\delta := \min\{\delta_1, \delta_2\}$ we have

$$-\frac{z'_t}{t} \le \frac{x_t - y_t}{t} \le \frac{z_t}{t} \quad \forall t \in [0, \delta)$$

and hence $\lim_{t \to 0^+} \frac{x_t - y_t}{t} = 0$, that is $x = y$ in ${}^\bullet\mathbb{R}$.

If r, $s \in \mathbb{R}$ and $r \le s$ as real numbers, then it suffices to take $z_t = 0$ for every $t \ge 0$ in the Definition 4.2.1 to obtain that $r \le s$ in ${}^\bullet\mathbb{R}$ too. Vice versa if $r \le s$ in ${}^\bullet\mathbb{R}$, then for some $z = 0$ in ${}^\bullet\mathbb{R}$ we have

$$\forall^0 t \ge 0 : \quad r \le s + z_t$$

and hence for $t = 0$ we have $r \le s$ in $\mathbb{R}$ because $z = 0$ and hence $z_0 = 0$. This proves that the order relation $\le$ defined in ${}^\bullet\mathbb{R}$ extends the order relation on $\mathbb{R}$.

The relationships between the ring operations and the order relation can be stated as

$$
\begin{aligned}
&x \leq y \quad \Longrightarrow \quad x + w \leq y + w \\
&x \leq y \quad \Longrightarrow \quad -x \geq -y \\
&x \leq y \quad \text{and} \quad w \geq 0 \quad \Longrightarrow \quad x \cdot w \leq y \cdot w
\end{aligned}
$$

The first two are immediate consequences of the Definition 4.2.1. To prove the last one, let us suppose that

$$
\begin{aligned}
&x_t \leq y_t + z_t \quad \forall^0 t \geq 0 \\
&w_t \geq z'_t \quad \forall^0 t \geq 0
\end{aligned}
\tag{4.2.3}
$$

then $w_t - z'_t \geq 0$ for every t small and hence from (4.2.3)

$$x_t \cdot (w_t - z'_t) \leq y_t \cdot (w_t - z'_t) + z_t \cdot (w_t - z'_t) \quad \forall^0 t \geq 0$$

from which it follows

$$x_t \cdot w_t \leq y_t \cdot w_t + (-x_t z'_t - y_t z'_t + z_t w_t - z_t z'_t) \quad \forall^0 t \geq 0$$

But $-xz' - yz' + zw - zz' = 0$ in ${}^\bullet\mathbb{R}$ because $z = 0$ and $z' = 0$ and hence the conclusion follows.

Finally we know (see Definition 2.4.2) that $h \in D_\infty$ if and only if ${}^\circ h = 0$ and this is equivalent to

$$\forall r \in \mathbb{R}_{>0} : \quad -r < {}^\circ h < r \tag{4.2.4}$$

But if, e.g., ${}^\circ h < r$, then

$$\forall^0 t \geq 0 : \quad h_t \leq r$$

because the function $t \to h_t$ is continuous, and hence we also have $h \leq r$ in ${}^\bullet\mathbb{R}$. Analogously, from (4.2.4)we can prove that $-r \leq h$ for all $r \in \mathbb{R}_{>0}$. Of course $r \notin D_\infty$ if $r \in \mathbb{R}$, so it cannot be that $h = r$.

Vice versa if

$$\forall r \in \mathbb{R}_{>0} : \quad -r < h < r$$

then, e.g., $h_t \leq r + z_t$ for t small. Hence, for $t = 0$ we have $-r \leq {}^\circ h = h_0 \leq r$ for every $r > 0$, and so ${}^\circ h = 0$.

$\blacksquare$

Example. We have e.g. $\mathrm{d}t > 0$ and $\mathrm{d}t_2 - 3\,\mathrm{d}t > 0$ because for $t \geq 0$ sufficiently small $t^{1/2} > 3t$ and hence

$$t^{1/2} - 3t > 0 \quad \forall^0 t \geq 0$$

From examples like these ones we can guess that our little-oh polynomials are always locally comparable with respect to pointwise order relation, and this is the first step to prove that for our order relation the trichotomy law holds. In the following statement we will use the notation $\forall^0 t > 0 : \mathcal{P}(t)$, that naturally means

$$\forall^0 t \geq 0 : \quad t \neq 0 \quad \Longrightarrow \quad \mathcal{P}(t)$$

where $\mathcal{P}(t)$ is a generic property depending on t.

Lemma 4.2.3. *Let $x, y \in {}^\bullet\mathbb{R}$, then*

1. $^\circ x < {}^\circ y \quad \Longrightarrow \quad \forall^0 t \geq 0 : \quad x_t < y_t$
2. *If $^\circ x = {}^\circ y$, then*

$$\left(\forall^0 t > 0 : \quad x_t < y_t\right) \quad or \quad \left(\forall^0 t > 0 : \quad x_t > y_t\right) \quad or \quad \left(x = y \quad in \quad {}^\bullet\mathbb{R}\right)$$

Proof:
1.) Let us suppose that $^\circ x < {}^\circ y$, then the continuous function $t \geq 0 \mapsto y_t - x_t \in \mathbb{R}$ assumes the value $y_0 - x_0 > 0$ hence is locally positive, i.e.

$$\forall^0 t \geq 0 : \quad x_t < y_t$$

2.) Now let us suppose that $^\circ x = {}^\circ y$, and introduce a notation for the potential decompositions of x and y (see Definition 2.3.4). From the definition of equality in ${}^\bullet\mathbb{R}$, we can always write

$$x_t = {}^\circ x + \sum_{i=1}^{N} \alpha_i \cdot t^{a_i} + z_t \quad \forall t \geq 0$$

$$y_t = {}^\circ y + \sum_{j=1}^{M} \beta_j \cdot t^{b_j} + w_t \quad \forall t \geq 0$$

where $x = {}^\circ x + \sum_{i=1}^{N} \alpha_i \cdot t^{a_i}$ and $y = {}^\circ y + \sum_{j=1}^{M} \beta_j \cdot t^{b_j}$ are the potential decompositions of x and y (hence $0 < \alpha_i < \alpha_{i+1} \leq 1$ and $0 < \beta_j < \beta_{j+1} \leq 1$), whereas w and z are little-oh polynomials such that $z_t = o(t)$ and $w_t = o(t)$ for $t \to 0^+$.

Case: $a_1 < b_1$ In this case the least power in the two decompositions is $\alpha_1 \cdot t^{a_1}$, and hence we expect that the second alternative of the conclusion is the true one if $\alpha_1 > 0$, otherwise the first alternative will be the true one if $\alpha_1 < 0$ (recall that always $\alpha_i \neq 0$ in a decomposition). Indeed, let us analyze, for $t > 0$, the condition $x_t < y_t$: the following formulae are all equivalent to it

$$\sum_{i=1}^{N} \alpha_i \cdot t^{a_i} < \sum_{j=1}^{N} \beta_j \cdot t^{b_j} + w_t - z_t$$

$$t^{a_1} \cdot \left[\alpha_1 + \sum_{i=2}^{N} \alpha_i \cdot t^{a_i - a_1}\right] < \ t^{a_1} \cdot \left[\sum_{j=1}^{N} \beta_j \cdot t^{b_j - a_1} + (w_t - z_t) \cdot t^{-a_1}\right]$$

$$\alpha_1 + \sum_{i=2}^{N} \alpha_i \cdot t^{a_i - a_1} < \sum_{j=1}^{N} \beta_j \cdot t^{b_j - a_1} + (w_t - z_t) \cdot t^{-a_1}$$

Therefore, let us consider the function

$$f(t) := \sum_{j=1}^{N} \beta_j \cdot t^{b_j - a_1} + (w_t - z_t) \cdot t^{-a_1} - \alpha_1 - \sum_{i=2}^{N} \alpha_i \cdot t^{a_i - a_1} \quad \forall t \geq 0$$

We can write

$$(w_t - z_t) \cdot t^{-a_1} = \frac{w_t - z_t}{t} \cdot t^{1-a_1}$$

and $\frac{w_t - z_t}{t} \to 0$ as $t \to 0^+$ because $w_t = o(t)$ and $z_t = o(t)$. Furthermore, $a_1 \leq 1$ hence t^{1-a_1} is bounded in a right neighborhood of $t = 0$. Therefore, $(w_t - z_t) \cdot t^{-a_1} \to 0$ and the function f is continuous at $t = 0$ too, because $a_i < a_i$ and $a_1 < b_1 < b_j$. By continuity, the function f is locally strictly positive if and only if $f(0) = -\alpha_1 > 0$, hence

$$\left(\forall^0 t > 0 : \quad x_t < y_t\right) \iff \alpha_1 < 0$$
$$\left(\forall^0 t > 0 : \quad x_t > y_t\right) \iff \alpha_1 > 0$$

Case: $a_1 > b_1$ We can argue in an analogous way with b_1 and β_1 instead of a_1 and α_1.

Case: $a_1 = b_1$ We shall exploit the same idea used above and analyze the condition $x_t < y_t$. The following are equivalent ways to express this condition

$$t^{a_1} \cdot \left[\alpha_1 + \sum_{i=2}^{N} \alpha_i \cdot t^{a_i - a_1}\right] < t^{a_1} \cdot \left[\beta_1 + \sum_{j=2}^{N} \beta_j \cdot t^{b_j - a_1} + (w_t - z_t) \cdot t^{-a_1}\right]$$

$$\alpha_1 + \sum_{i=2}^{N} \alpha_i \cdot t^{a_i - a_1} < \beta_1 + \sum_{j=2}^{N} \beta_j \cdot t^{b_j - a_1} + (w_t - z_t) \cdot t^{-a_1}$$

Hence, exactly as we have demonstrated above, we can state that

$$\alpha_1 < \beta_1 \quad \Longrightarrow \quad \forall^0 t > 0 : \quad x_t < y_t$$
$$\alpha_1 > \beta_1 \quad \Longrightarrow \quad \forall^0 t > 0 : \quad x_t > y_t$$

Otherwise $\alpha_1 = \beta_1$ and we can restart with the same reasoning using a_2, b_2, α_2, β_2, etc. If $N = M$, the number of addends in the decompositions, using this procedure we can prove that

$$\forall t \geq 0 : \quad x_t = y_t + w_t - z_t,$$

that is $x = y$ in ${}^\bullet\mathbb{R}$.s

It remains to consider the case, e.g., $N < M$. In this hypotheses, using the previous procedure we would arrive at the following analysis of the condition $x_t < y_t$:

$$0 < \sum_{j>N} \beta_j \cdot t^{b_j} + w_t - z_t$$

$$0 < t^{b_{N+1}} \cdot \left[\beta_{N+1} + \sum_{j>N+1} \beta_j \cdot t^{b_j - b_{N+1}} + (w_t - z_t) \cdot t^{-b_{N+1}} \right]$$

$$0 < \beta_{N+1} + \sum_{j>N+1} \beta_j \cdot t^{b_j - b_{N+1}} + (w_t - z_t) \cdot t^{-b_{N+1}}$$

Hence

$$\beta_{N+1} > 0 \implies \forall^0 t > 0 : \quad x_t < y_t$$

$$\beta_{N+1} < 0 \implies \forall^0 t > 0 : \quad x_t > y_t$$

■

This lemma can be used to find an equivalent formulation of the order relation.

Theorem 4.2.4. *Let $x, y \in {}^\bullet\mathbb{R}$, then*

1. $x \le y \iff \left(\forall^0 t > 0 : \quad x_t < y_t\right)$ *or* $(x = y$ *in* ${}^\bullet\mathbb{R})$
2. $x < y \iff \left(\forall^0 t > 0 : \quad x_t < y_t\right)$ *and* $(x \neq y$ *in* ${}^\bullet\mathbb{R})$

Proof:

1.) $\Rightarrow$ If $^\circ x < {}^\circ y$ then, from the previous Lemma 4.2.3 we can derive that the first alternative is true. If $^\circ x = {}^\circ y$, then from Lemma 4.2.3 we have

$$\left(\forall^0 t > 0 : \quad x_t < y_t\right) \quad \text{or} \quad (x = y \quad \text{in} \quad {}^\bullet\mathbb{R}) \quad \text{or} \quad \left(\forall^0 t > 0 : \quad x_t > y_t\right) \tag{4.2.5}$$

In the first two cases we have the conclusion. In the third case, from $x \le y$ we obtain

$$\forall^0 t \ge 0 : \quad x_t \le y_t + z_t \tag{4.2.6}$$

with $z_t = o(t)$. Hence from the third alternative of (4.2.5) we have

$$0 < x_t - y_t \le z_t \quad \forall^0 t > 0$$

and hence $\lim_{t \to 0^+} \frac{x_t - y_t}{t} = 0$, i.e. $x = y$ in ${}^\bullet\mathbb{R}$.

1.) $\Leftarrow$ This follows immediately from the reflexive property of $\le$ or from the Definition 4.2.1.

2.) $\Rightarrow$ From $x < y$ we have $x \le y$ and $x \neq y$, so the conclusion follows from the previous *1*.

2.) $\Leftarrow$ From $\forall^0 t > 0 : x_t < y_t$ and from *1.* it follows $x \le y$ and hence $x < y$ from the hypotheses $x \ne y$.

∎

Now we can prove that our order is total

Corollary 4.2.5. *Let $x, y \in {}^\bullet\mathbb{R}$, then in ${}^\bullet\mathbb{R}$ we have*

1. $x \le y$ *or* $y \le x$ *or* $x = y$
2. $x < y$ *or* $y < x$ *or* $x = y$

Proof:
1.) If ${}^\circ x < {}^\circ y$, then from Lemma 4.2.3 we have $x_t < y_t$ for $t \ge 0$ sufficiently small. Hence from Theorem 4.2.4 we have $x \le y$. We can argue in the same way if ${}^\circ x > {}^\circ y$. Also the case ${}^\circ x = {}^\circ y$ can be handled in the same way using *2.* of Lemma 4.2.3.

2.) This part is a general consequence of the previous one. Indeed, if we have $x = y$, then we have the conclusion. Otherwise we have $x \ne y$, and using the previous *1.* we can deduce strict inequalities from inequalities because $x \ne y$.

∎

From the proof of Lemma 4.2.3 and from Theorem 4.2.4 we can deduce the following

Theorem 4.2.6. *Let $x, y \in {}^\bullet\mathbb{R}$. If ${}^\circ x \ne {}^\circ y$, then*

$$x < y \iff {}^\circ x < {}^\circ y$$

Otherwise, if ${}^\circ x = {}^\circ y$, then

1. *If $\omega(x) > \omega(y)$, then $x > y$ iff ${}^\circ x_1 > 0$*
2. *If $\omega(x) = \omega(y)$, then*

$$\begin{aligned} {}^\circ x_1 > {}^\circ y_1 \quad &\Longrightarrow \quad x > y \\ {}^\circ x_1 < {}^\circ y_1 \quad &\Longrightarrow \quad x < y \end{aligned}$$

This Theorem proves also some sentences about the order relation anticipated in the Remark 2.3.6.

Example. The previous Theorem gives an effective criterion to decide whether $x < y$ or not. Indeed, if the two standard parts are different, then the order relation can be decided on the basis of these standard parts only. E.g. $2 + \mathrm{d}t_2 > 3\,\mathrm{d}t$ and $1 + \mathrm{d}t_2 < 3 + \mathrm{d}t$.

Otherwise, if the standard parts are equal, we firstly have to look at the order and at the first standard parts, i.e. $^\circ x_1$ and $^\circ y_1$, which are the coefficients of the biggest infinitesimals in the decompositions of x and y. E.g. $3\,\mathrm{d}t_2 > 5\,\mathrm{d}t$, and $\mathrm{d}t_2 > a\,\mathrm{d}t$ for every $a \in \mathbb{R}$, and $\mathrm{d}t < \mathrm{d}t_2 < \mathrm{d}t_3 < \ldots < \mathrm{d}t_k$ for every $k > 3$, and $\mathrm{d}t_k > 0$.
If the orders are equal we have to compare the first standard parts. E.g. $3\,\mathrm{d}t_5 > 2\,\mathrm{d}t_5$.
The other cases fall within the previous ones, because of the properties of the ordered ring ${}^\bullet\mathbb{R}$. E.g. we have that $\mathrm{d}t_5 - 2\,\mathrm{d}t_3 + 3\,\mathrm{d}t < \mathrm{d}t_5 - 2\,\mathrm{d}t_3 + \mathrm{d}t_{3/2}$ if and only if $3\,\mathrm{d}t < \mathrm{d}t_{3/2}$, which is true because $\omega(\mathrm{d}t) = 1 < \omega(\mathrm{d}t_{3/2}) = \frac{3}{2}$. Finally $\mathrm{d}t_5 - 2\,\mathrm{d}t_3 + 3\,\mathrm{d}t > \mathrm{d}t_5 - 2\,\mathrm{d}t_3 - \mathrm{d}t$ because $3\,\mathrm{d}t > -\mathrm{d}t$.

4.2.1 Absolute value

Having a total order we can define the absolute value

Definition 4.2.7. *Let $x \in {}^\bullet\mathbb{R}$, then*

$$|x| := \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Exactly like for the real field $\mathbb{R}$ we can prove the usual properties of the absolute value:

$$\begin{aligned} &|x| \geq 0 \\ &|x+y| \leq |x| + |y| \\ &-|x| \leq x \leq |x| \\ &||x| - |y|| \leq |x - y| \\ &|x| = 0 \iff x = 0 \end{aligned}$$

Moreover, also the following cancellation law is provable.

Theorem 4.2.8. *Let $h \in {}^\bullet\mathbb{R} \setminus \{0\}$ and $r,\ s \in \mathbb{R}$, then*

$$|h| \cdot r \leq |h| \cdot s \quad \Longrightarrow \quad r \leq s$$

Proof: In fact if $|h| \cdot r \leq |h| \cdot s$ then from Theorem 4.2.4 we obtain that either

$$\forall^0 t > 0 : \quad |h_t| \cdot r \leq |h_t| \cdot s \tag{4.2.7}$$

or $|h| \cdot r = |h| \cdot s$. But $h \neq 0$ so

$$\big(\forall^0 t > 0 : \quad h_t > 0\big) \quad \text{or} \quad \big(\forall^0 t > 0 : \quad h_t < 0\big)$$

hence we can always find a $\bar{t} > 0$ such that $|h_{\bar{t}}| \neq 0$ and to which (4.2.7) is applicable. Therefore, in the first case we must have $r \leq s$. In the second one we have

$$|h| \cdot r = |h| \cdot s$$

but $h \neq 0$, hence $|h| \neq 0$ and so the conclusion follows from Theorem 2.7.1. ■

4.3 Powers and logarithms

In this section we will tackle definition and properties of powers x^y and logarithms $\log_x y$. Due to the presence of nilpotent elements in ${}^\bullet\mathbb{R}$, we cannot define these operations without any limitation. E.g. we cannot define the square root having the usual properties, like

$$x \in {}^\bullet\mathbb{R} \implies \sqrt{x} \in {}^\bullet\mathbb{R} \tag{4.3.1}$$

$$x = y \quad \text{in} \quad {}^\bullet\mathbb{R} \implies \sqrt{x} = \sqrt{y} \quad \text{in} \quad {}^\bullet\mathbb{R} \tag{4.3.2}$$

$$\sqrt{x^2} = |x|$$

because they are incompatible with the existence of $h \in D$ such that $h^2 = 0$, but $h \neq 0$. Indeed, the general property stated in the Subsection 2.2 permits to obtain a property like (4.3.1) (i.e. the closure of ${}^\bullet\mathbb{R}$ with respect to a given operation) only for smooth functions. Moreover, the Definition 2.8.1 states that to obtain a well defined operation we need a locally Lipschitz function. For these reasons, we will limit x^y to $x > 0$ and x invertible only, and $\log_x y$ to x, $y > 0$ and both x, y invertible.

Definition 4.3.1. *Let x, $y \in {}^\bullet\mathbb{R}$, with x strictly positive and invertible, then*

1. $x^y := [t \geq 0 \mapsto x_t^{y_t}]_{=}$ *in* ${}^\bullet\mathbb{R}$

2. *If $y > 0$ and y is invertible, then* $\log_x y := [t \geq 0 \mapsto log_{x_t} y_t]_{=}$ *in* ${}^\bullet\mathbb{R}$

Because of Theorem 4.2.4 from $x > 0$ we have

$$\forall^0 t > 0 : \quad x_t > 0$$

so that, exactly as we proved in Subsection 2.2 and in Definition 2.8.1, the previous operations are well defined in ${}^\bullet\mathbb{R}$ because ${}^\circ x \neq 0 \neq {}^\circ y$.
From the elementary transfer theorem 2.8.2 the usual properties follow:

$$\begin{aligned}
(x^y)^z &= x^{y \cdot z} \\
x^y \cdot x^z &= x^{y+z} \\
x^n &= x \cdot \overset{n}{\ldots\ldots} \cdot x \quad \text{if} \quad n \in \mathbb{N} \\
\log_x (x^y) &= y \\
x^{\log_x y} &= y \\
\log(x \cdot y) &= \log x + \log y \\
\log_x (y^z) &= z \cdot \log_x y \\
x^{\log y} &= y^{\log x}
\end{aligned}$$

About the monotonicity properties, it suffices to use Theorem 4.2.4 to prove immediately the usual properties (where x, y and w are invertible)

$$\begin{array}{rcccc} z>0 & \text{and} & x\ge y>0 & \Longrightarrow & x^z \ge y^z \\ z<0 & \text{and} & x\ge y>0 & \Longrightarrow & x^z \le y^z \\ w>1 & \text{and} & x\ge y>0 & \Longrightarrow & \log_w x \ge \log_w y \\ 0<w<1 & \text{and} & x\ge y>0 & \Longrightarrow & \log_w x \le \log_w y \end{array}$$

Analogous implications, but with strict equalities, are true if we suppose $x > y$.

Finally, it can be useful to state here the *elementary transfer theorem for inequalities*, whose proof follows immediately from the definition of $\le$ and from Theorem 4.2.4:

Theorem 4.3.2. *Let A be an open subset of $\mathbb{R}^n$, and τ, $\sigma : A \longrightarrow \mathbb{R}$ be smooth functions. Then*

$$\forall x \in {}^\bullet A : \quad {}^\bullet\tau(x) \le {}^\bullet\sigma(x)$$

iff

$$\forall r \in A : \quad \tau(r) \le \sigma(r).$$

4.4 Geometrical representation of Fermat reals

At the beginning of this chapter we argued that one of the conducting idea in the construction of Fermat reals is to maintain always a clear intuitive meaning. More precisely, we always tried, and we will always try, to keep a good dialectic between provable formal properties and their intuitive meaning. In this direction we can see the possibility to find a geometrical representation of Fermat reals.

The idea is that to any Fermat real $x \in {}^\bullet\mathbb{R}$ we can associate the function

$$t \in \mathbb{R}_{\ge 0} \mapsto {}^\circ x + \sum_{i=1}^{N} {}^\circ x_i \cdot t^{1/\omega_i(x)} \in \mathbb{R} \tag{4.4.1}$$

where N is, of course, the number of addends in the decomposition of x. Therefore, a geometric representation of this function is also a geometric representation of the number x, because different Fermat reals have different decompositions, see 2.3.2. Finally, we can guess that, because the notion of equality in ${}^\bullet\mathbb{R}$ depends only on the germ generated by each little-oh polynomial (see Definition 2.3.1), we can represent each $x \in {}^\bullet\mathbb{R}$ with only the first small part of the function (4.4.1).

Definition 4.4.1. *If* $x \in {}^\bullet\mathbb{R}$ *and* $\delta \in \mathbb{R}_{>0}$*, then*

$$\mathrm{graph}_\delta(x) := \left\{ ({}^\circ x + \sum_{i=1}^{N} {}^\circ x_i \cdot t^{1/\omega_i(x)}, t) \,|\, 0 \le t < \delta \right\}$$

where N *is the number of addends in the decomposition of* x*.*

Note that the value of the function are placed in the abscissa position, so that the correct representation of $\mathrm{graph}_\delta(x)$ is given by the Figure 4.2. This inversion of abscissa and ordinate in the $\mathrm{graph}_\delta(x)$ permits to represent this graph as a line tangent to the classical straight line $\mathbb{R}$ and hence to have a better graphical picture (see the following Figures). Finally, note that if $x \in \mathbb{R}$ is a standard real, then $N = 0$ and the $\mathrm{graph}_\delta(x)$ is a vertical line passing through ${}^\circ x = x$.

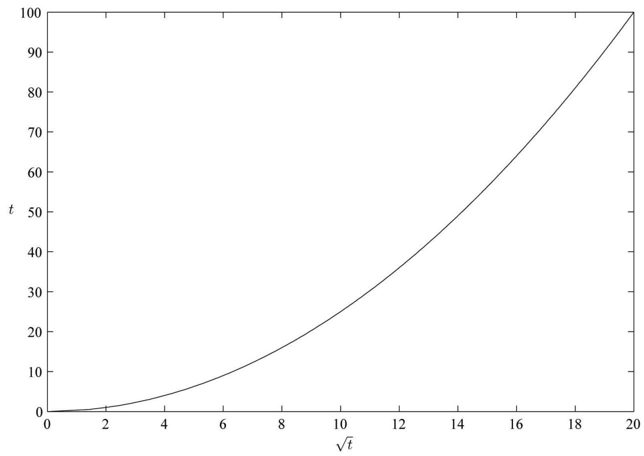

Figure 4.2: The function representing the Fermat real $\mathrm{d}t_2 \in D_3$

The following theorem permits to represent geometrically the Fermat reals

Theorem 4.4.2. *If* $\delta \in \mathbb{R}_{>0}$*, then the function*

$$x \in {}^\bullet\mathbb{R} \mapsto \mathrm{graph}_\delta(x) \subset \mathbb{R}^2$$

is injective. Moreover if $x, y \in {}^\bullet\mathbb{R}$*, then we can find* $\delta \in \mathbb{R}_{>0}$ *(depending on* x *and* y*) such that*

$$x < y$$

if and only if

$$\forall p, q, t : \quad (p, t) \in \mathrm{graph}_\delta(x) \quad , \quad (q, t) \in \mathrm{graph}_\delta(y) \quad \Longrightarrow \quad p < q \quad (4.4.2)$$

Proof: The application $\rho(x) := \mathrm{graph}_\delta(x)$ for $x \in {}^\bullet\mathbb{R}$ is well defined because it depends on the terms ${}^\circ x$, ${}^\circ x_i$ and $\omega_i(x)$ of the decomposition of x (see Theorem 2.3.2 and Definition 2.3.5). Now, suppose that $\mathrm{graph}_\delta(x) = \mathrm{graph}_\delta(y)$, then

$$\forall t \in [0,\delta): \quad {}^\circ x + \sum_{i=1}^N {}^\circ x_i \cdot t^{1/\omega_i(x)} = {}^\circ y + \sum_{j=1}^M {}^\circ y_j \cdot t^{1/\omega_j(y)} \tag{4.4.3}$$

Let us consider the Fermat reals generated by these functions, i.e.

$$x' := \left[t \ge 0 \mapsto {}^\circ x + \sum_{i=1}^N {}^\circ x_i \cdot t^{1/\omega_i(x)} \right]_{= \text{ in } {}^\bullet\mathbb{R}}$$
$$y' := \left[t \ge 0 \mapsto {}^\circ y + \sum_{j=1}^M {}^\circ y_j \cdot t^{1/\omega_j(y)} \right]_{= \text{ in } {}^\bullet\mathbb{R}}$$

then the decompositions of x' and y' are exactly the decompositions of x and y

$$x' = {}^\circ x + \sum_{i=1}^N {}^\circ x_i \, \mathrm{d}t_{\omega_i(x)} = x \tag{4.4.4}$$
$$y' = {}^\circ y + \sum_{j=1}^M {}^\circ y_j \, \mathrm{d}t_{\omega_j(y)} = y \tag{4.4.5}$$

But from (4.4.3) it follows $x' = y'$ in ${}^\bullet\mathbb{R}$, and hence also $x = y$ from (4.4.4) and (4.4.5).

Now suppose that $x < y$, then, using the same notations of the previous part of this proof, we have also $x' = x$ and $y' = y$ and hence

$$x' = {}^\circ x + \sum_{i=1}^N {}^\circ x_i \cdot t^{1/\omega_i(x)} < {}^\circ y + \sum_{j=1}^M {}^\circ y_j \cdot t^{1/\omega_j(y)} = y'$$

We apply Theorem 4.2.4 obtaining that locally $x'_t < y'_t$, i.e.

$$\exists \delta > 0: \quad \forall^0 t \ge 0: \quad {}^\circ x + \sum_{i=1}^N {}^\circ x_i \cdot t^{1/\omega_i(x)} < {}^\circ y + \sum_{j=1}^M {}^\circ y_j \cdot t^{1/\omega_j(y)}$$

This is an equivalent formulation of (4.4.2), and, because of Theorem 4.2.4 it is equivalent to $x' = x < y' = y$.

∎

Example. In Figure 4.3 we have the representation of some first order infinitesimals.

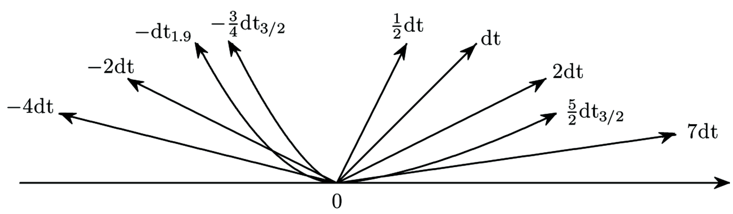

Figure 4.3: Some first order infinitesimals

The arrows are justified by the fact that the representing function (4.4.1) is defined on $\mathbb{R}_{\geq 0}$ and hence has a clear first point and a direction. The smaller is $\alpha \in (0,1)$ and the nearer is the representation of the product $\alpha\,\mathrm{d}t$, to the vertical line passing through zero, which is the representation of the standard real $x = 0$. Finally, recall that $\mathrm{d}t_k \in D$ if and only if $1 \leq k < 2$.
If we multiply two infinitesimals we obtain a smaller number, hence one whose representation is nearer to the vertical line passing through zero, as represented in the Figure 4.4

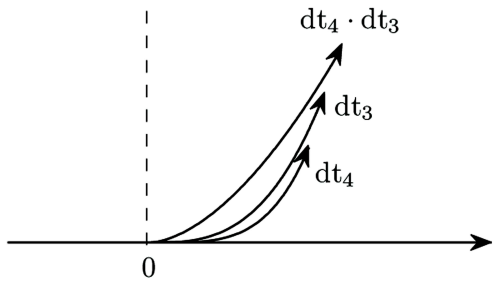

Figure 4.4: The product of two infinitesimals

In Figure 4.5 we have a representation of some infinitesimals of order greater than 1. We can see that the greater is the infinitesimal $h \in D_a$ (with respect to the order relation $\leq$ defined in ${}^\bullet\mathbb{R}$) and the higher is the order of intersection of the corresponding line $\mathrm{graph}_\delta(h)$.
Finally, in Figure 4.6 we represent the order relation on the basis of Theorem 4.4.2. Intuitively, the method to see if $x < y$ is to look at a suitably small neighborhood (i.e. at a suitably small $\delta > 0$) at $t = 0$ of their representing lines $\mathrm{graph}_\delta(x)$ and $\mathrm{graph}_\delta(y)$: if, with respect to the horizontal directed straight line, the curve $\mathrm{graph}_\delta(x)$ comes before the curve $\mathrm{graph}_\delta(y)$, then x is less than y.

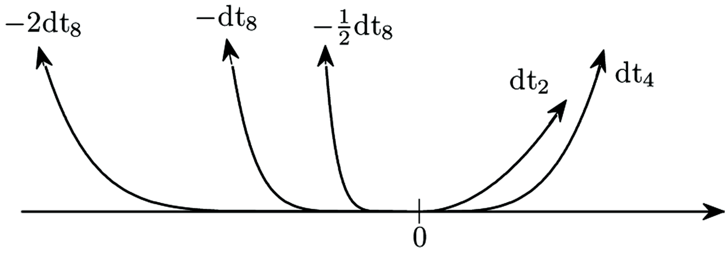

Figure 4.5: Some higher order infinitesimals

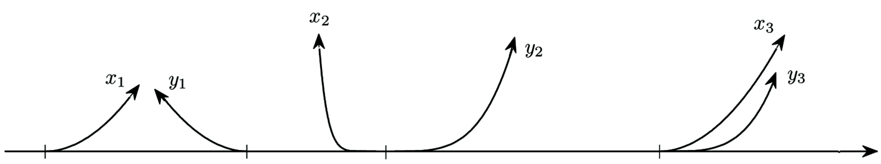

Figure 4.6: Different cases in which $x_i < y_i$

Part II

Infinite dimensional spaces

Chapter 5

Approaches to differential geometry of infinite dimensional spaces

5.1 Introduction

In this section we want to list some of the most important, i.e. well-established, approaches that are used to define geometrical structures in infinite dimensional spaces. One of the most important example we have in mind is the set $\mathbf{Man}(M, N)$ of all the smooth applications between two finite dimensional manifolds M and N. For the aims of the present section, we are interested to list some of the most studied structures on $\mathbf{Man}(M, N)$, and its subspaces, that permit to develop at least a tangency theory, i.e. the notion of tangent functor and the notion of differentiability of maps between this type of infinite dimensional spaces, and have sufficiently good categorical properties. This is not a trivial goal because, for example, an important example we can cite is the group $\mathrm{Diff}(M)$ of all the diffeomorphisms of a manifold M. Flows in a compact manifold M can be considered as 1-parameter subgroups of $\mathrm{Diff}(M)$, and it would seem useful to express the smoothness of a flow by means of a suitable differentiable structure on $\mathrm{Diff}(M)$, which should also behave like a classical Lie group with respect to this structure.

A typical restriction to distinguish among different approaches to infinite dimensional spaces is the hypotheses of compactness of the domain M, assumed to obtain some desired property: is this a necessary hypotheses or are we forced to assume it due to some restrictions of the chosen approach?

Another interesting property is the possibility to extend the classical notion of manifold to a more general type of space, so as to get better categorical properties, like the existence of infinite products or co-products or a cartesian closed category[1].

Finally, several authors had to tackle the following problem: suppose we have a new notion of smooth space able to include the space $\mathbf{Man}(M, N)$, at least for M compact and finite dimensional, and to embed faithfully (i.e. injectively, see Appendix A) the category of smooth finite dimensional manifolds. Even if the extension of the notion of finite dimensional manifold is faithful, usually the category $\mathcal{C}$ of these new smooth spaces includes spaces which are too much general, so that it seems really hard to generalize for these spaces meaningful results of differential geometry of finite dimensional manifolds. For this reason, several authors (see e.g. Kriegl and Michor [1997], Frölicher and Kriegl [1988], Lavendhomme [1996], Moerdijk and Reyes [1991]) try to select, among all their new smooth spaces in $\mathcal{C}$, the best ones having some new more restrictive properties. In this way the category $\mathcal{C}$ acts as a universe, usually closed with respect to strong categorical operations (like arbitrary limits, colimits and cartesian closedness), and the restricted class of smooth spaces works as a true generalization of the notion of manifold.

For example, in Kriegl and Michor [1997] the category of Frölicher spaces acts as a universe, but indeed the monograph is about manifolds modeled in convenient vector spaces instead of classical Banach spaces (see subsection *The convenient vectors spaces settings* on page 88). This permits to Kriegl and Michor [1997] to generalize as far as possible to infinite dimensional manifolds the results of finite dimensional spaces, but as a consequence the class of manifolds modeled in convenient vector spaces loses some desired categorical properties.

Analogously, in SDG (see e.g. Lavendhomme [1996], Moerdijk and Reyes [1991], Kock [1981]) the class of restricted smooth spaces is introduced with the notion of microlinear space and the universe is a suitable topos, i.e. a whole model for intuitionistic set theory. In this approach, the infinitesimals are used to define the properties of this class of restricted, better behaved, spaces.

Of course, this is not possible in theories that have not an explicit language of actual infinitesimals, like in the case of diffeological spaces (see Iglesias-Zemmour [2008]). For them we can proceed either as in convenient vector spaces theory considering the notion of vector space in the category of smooth diffeological spaces (i.e. smooth diffeological spaces that are also vector spaces with smooth operations, see Iglesias-Zemmour [2008]) and considering manifolds modeled in diffeological vector spaces, or we can try to develop directly for a generic diffeological space some notion of differential geometry (see e.g. Iglesias-Zemmour [2008], Laubinger [2008, 2006], Hector and Macías-Virgós [2002], Hector [1995], Souriau [1984, 1981]). In the fol-

[1] For a short introduction, mainly motivated to fix common notations, of the few notions of category theory used in the present work, see Appendix A.

lowing subsections we will return to this problem giving some more precise definitions.

To understand better some differences between the approaches we are going to describe shortly in this section, we want to motivate the notion of cartesian closure, because is one of the basic choice shared by several authors like Bastiani [1963], Bell [1998], Brown [1961, 1963, 1964], Chen [1982], Colombeau [1973], Frölicher and Bucher [1966], Frölicher and Kriegl [1988], Kock [1981], Kriegl and Michor [1997], Lavendhomme [1996], Lawvere [1979], Lawvere et al. [1981], Moerdijk and Reyes [1991], Seip [1981], Souriau [1981], Steenrod [1967], Vogt [1971]. We firstly fix the notations for the notions of *adjoint* of a map.

Definition 5.1.1. *If X, Y, Z are sets and $f : X \longrightarrow Z^Y$, $g : X \times Y \longrightarrow Z$ are maps, then*

$$\forall (x,y) \in X \times Y : \quad f^\vee(x,y) := [f(x)]\,(y) \in Z$$
$$\forall x \in X : \quad g^\wedge(x) := g(x,-) \in Z^Y$$

hence

$$f^\vee : X \times Y \longrightarrow Z$$
$$g^\wedge : X \longrightarrow Z^Y$$

The map $f^\vee$ is called the adjoint *of f and the map $g^\wedge$ is called the* adjoint[2] *of g.*

Let us note that $(f^\vee)^\wedge = f$ and $(g^\wedge)^\vee = g$, that is the two applications

$$(-)^\vee : \left(Z^Y\right)^X \longrightarrow Z^{X\times Y}$$
$$(-)^\wedge : Z^{X\times Y} \longrightarrow \left(Z^Y\right)^X$$

are one the inverse of the other and hence represent in explicit form the bijection of sets $\left(Z^Y\right)^X \simeq Z^{X\times Y}$ i.e. $\mathbf{Set}(X, \mathbf{Set}(Y,Z)) \simeq \mathbf{Set}(X \times Y, Z)$.

One of the main aim of the second part of the present work is to generalize the notions of smooth manifold and of smooth map between two manifolds so as to obtain a new category “with good properties” that will be denoted by $\mathcal{C}^\infty$; if we call *smooth maps* the morphisms of $\mathcal{C}^\infty$ and *smooth spaces* its objects, then this category must be *cartesian closed*, i.e. it has to verify the following properties for every pair of smooth space X, $Y \in \mathcal{C}^\infty$:

1. $\mathcal{C}^\infty(X,Y)$ is a smooth space, i.e. $\mathcal{C}^\infty(X,Y) \in \mathcal{C}^\infty$
2. The maps $(-)^\vee$ and $(-)^\wedge$ are smooth, i.e. they realize in the category $\mathcal{C}^\infty$ the bijection $\mathcal{C}^\infty(X, \mathcal{C}^\infty(Y,Z)) \simeq \mathcal{C}^\infty(X \times Y, Z)$

[2] Here we are using the notations of Adamek et al. [1990], but some authors, e.g. Kriegl and Michor [1997], used opposite notations for the adjoint maps.

Property 1. is another way to state that the category we want to construct must contain as objects the space of all the smooth maps between two generic objects X, $Y \in \boldsymbol{\mathcal{C}}^\infty$

$$\begin{aligned}\boldsymbol{\mathcal{C}}^\infty(X,Y) &= \{f \mid X \xrightarrow{f} Y \text{ is smooth}\} = \\ &= \{f \mid X \xrightarrow{f} Y \text{ is a morphism of } \boldsymbol{\mathcal{C}}^\infty\}.\end{aligned}$$

Moreover, let us note that as a consequence of 2. we have that

$$X \xrightarrow{f} \boldsymbol{\mathcal{C}}^\infty(Y,Z) \text{ is smooth} \iff X \times Y \xrightarrow{f^\vee} Z \text{ is smooth} \tag{5.1.1}$$

$$X \times Y \xrightarrow{g} Z \text{ is smooth} \iff X \xrightarrow{g^\wedge} \boldsymbol{\mathcal{C}}^\infty(Y,Z) \text{ is smooth.} \tag{5.1.2}$$

The importance of (5.1.1) and (5.1.2) can be explained saying that if we want to study a smooth map having values in the space $\boldsymbol{\mathcal{C}}^\infty(Y,Z)$, then it suffices to study its adjoint map $f^\vee$. If, e.g., the spaces X, Y and Z are finite dimensional manifolds, then $\boldsymbol{\mathcal{C}}^\infty(Y,Z)$ is infinite-dimensional, but $f^\vee : X \times Y \longrightarrow Z$ is a standard smooth map between finite dimensional manifolds, and hence we have a strong simplification. Conversely, if $g : X \times Y \longrightarrow Z$ is a smooth map, then it generates a smooth map with values in $\boldsymbol{\mathcal{C}}^\infty(Y,Z)$, and all the smooth maps with values in this type of spaces can be generated in this way. Of course, this idea is frequently used, even if informally, in the calculus of variations. Let us note explicitly that the cartesian closure of the category $\boldsymbol{\mathcal{C}}^\infty$, i.e. properties 1. and 2., does not say anything about smooth maps with a *domain* of the form $\boldsymbol{\mathcal{C}}^\infty(Y,Z)$, but it reformulates in a convenient way the problem of smoothness of maps with *codomain* of this type. For a more abstract notion of cartesian closed category, see e.g. Mac Lane [1971], Borceux [1994], Arbib and Manes. [1975], Adamek et al. [1990].

We also want to see a different motivation drawn from Frölicher and Kriegl [1988]. Let us suppose to have a smooth function $g : \mathbb{R} \times I \longrightarrow \mathbb{R}$, where $I = [a,b]$, and define the integral function

$$f(t) := \int_a^b g(t,s)\,\mathrm{d}s \quad \forall t \in \mathbb{R}.$$

Then we can look at the function f as the composition of two applications

$$f : t \in \mathbb{R} \mapsto g(t,-) \mapsto \int_a^b g(t,-) \in \mathbb{R}.$$

Hence, if we denote

$$i : h \in \mathcal{C}^\infty(I,\mathbb{R}) \mapsto \int_a^b h \in \mathbb{R},$$

then

$$f = i \circ g^\wedge \qquad \text{i.e.} \qquad f(t) = i\left(g^\wedge(t,-)\right) \quad \forall t \in \mathbb{R}.$$

In this way, it is natural to try a proof of the formula for the derivation under the integral sign in the following way:

$$\frac{\mathrm{d}f}{\mathrm{d}t}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(i \circ g^\wedge\right) = \mathrm{d}i\left(g^\wedge(t)\right)\left[\frac{\mathrm{d}g^\wedge}{\mathrm{d}t}(t)\right] = \\ = i\left[\partial_1 g(t,-)\right] = \int_a^b \partial_1 g(t,s)\,\mathrm{d}s. \quad (5.1.3)$$

Here we have supposed that the following properties hold:

- $g^\wedge : \mathbb{R} \longrightarrow \mathcal{C}^\infty(I,\mathbb{R})$ is smooth,
- $i : \mathcal{C}^\infty(I,\mathbb{R}) \longrightarrow \mathbb{R}$ is smooth,
- the chain rule for the derivative of the composition of two functions,
- the differential of the function i is given by $\mathrm{d}i(h) = i$ for every $h \in \mathcal{C}^\infty(I,\mathbb{R})$, because i is linear,
- $\frac{\mathrm{d}g^\wedge}{\mathrm{d}t}(t) = \partial_1 g(t,-)$.

Let us note explicitly that the space $\mathcal{C}^\infty(I,\mathbb{R})$ is infinite dimensional.

Even if in the present work we will be able to prove all these properties, the aim of (5.1.3) is not to suggest a new proof, but to hint that a theory where we can consider the previous properties seems to be very flexible and powerful.

5.2 Banach manifolds and locally convex vector spaces

Banach manifolds is the more natural generalization of finite-dimensional manifolds if one takes Banach spaces as local model spaces. Even if, as we will see more precisely in this section, this theory does not satisfy our condition to present in this chapter only generalized notions of manifolds able to develop at least a tangency theory and having sufficiently good categorical properties, Banach manifolds are the most studied concept in infinite dimensional differential geometry. Some well known references on Banach manifolds are Lang [1999], Abraham et al. [1988]. Among the most important theorems in this framework we can cite the implicit and inverse function theorems and the existence and uniqueness of solutions of Lipschitz ordinary differential equations on such spaces. The use of charts to prove these fundamental results is indispensable, so it is not easy to generalize them to more general contexts where we cannot use the notion of chart having values in some modeling space with sufficiently good properties.

For the purposes of the present analysis, a typical example of infinite-dimensional Banach space is the space $\mathcal{C}^r(M,E)$ of $\mathcal{C}^r$-maps, where M is a compact manifold and E is a Banach space. The vector space $\mathcal{C}^r(M,E)$ is a Banach space with respect to the norm

$$\|f\|_r := \max_{1\le i\le r} \sup_{m\in M} \left\| \mathrm{d}^i f(m)\right\|, \tag{5.2.1}$$

but the theory fails for the space $\mathcal{C}^\infty(M,E) := \bigcap_{r=1}^{+\infty}\mathcal{C}^r(M,E)$ of smooth mappings defined in M and with values in E. On the one hand, even if it is not a formal motivation, but it remains very important in the real development of mathematics, the hypotheses of considering $r < +\infty$ and M compact in the previous definition 5.2.1 are not intrinsic to the problem but are motivated solely by the limitations of the instrument we are trying to implement, i.e. a norm in the space $\mathcal{C}^r(M,E)$. On the other hand, more formally, any two different norms $\|-\|_r$ and $\|-\|_s$ are not equivalent, and hence the space $\mathcal{C}^\infty(M;,E)$ is not normable with a norm generating the same topology generated by the family of norms $(\|-\|_r)_{r=1}^{+\infty}$ (for details, see e.g. Friedman [1963]; in the following, saying that the space $\mathcal{C}^\infty(M,E)$ is not normable, we will always mean with respect to this topology).

Moreover, $\mathcal{C}^\infty(M,E)$ is not a Banach manifold: indeed, it is separable and metric (see Friedman [1963]), hence if it were a Banach manifold, then it would be embeddable as an open subset of an Hilbert space (see Henderson [1970]), and hence it would be normable.

Therefore, the category of Banach manifolds and smooth maps **Ban** is not cartesian closed because it is not closed with respect to exponential objects $\mathbf{Ban}(M,E) = \mathcal{C}^\infty(M,E)$, see condition 1. in the previous definition of cartesian closed category, section 5.1.

This also proves that the category of Banach manifolds **Ban** and smooth maps does not have arbitrary limits: in fact if it had infinite products (a particular case of limit in a category, see Appendix A), then we would have

$$\prod_{m\in M} E = \mathbf{Ban}(M,E) = \mathcal{C}^\infty(M,E),$$

but we had already seen that this space is not a Banach manifold.

These important counter-examples can conduct us toward the idea of considering spaces equipped with a family of norms, like $(\|-\|_r)_{r=1}^{+\infty}$, or, more generally, of seminorms, i.e. toward the theory of locally convex vector spaces (see e.g. Jarchow [1981]). But any locally convex topology on the space $\mathcal{C}^\infty(M,E)$ is incompatible with cartesian closure, as stated in the following

Theorem 5.2.1. *Let F be a locally convex vector space contained in a sub-category $\mathcal{T}$ of the category* **Top** *of topological spaces and continuous functions such that $\mathcal{T}(F,\mathbb{R})$ always contains all the linear continuous functionals* *on the space F*

$$\mathrm{Lin}(F,\mathbb{R}) \subseteq \mathcal{T}(F,\mathbb{R}).$$

Then we have the following implication

$$\mathcal{T} \text{ is cartesian closed} \quad\Longrightarrow\quad F \text{ is normable.}$$

Hence the category **Ban** *is not cartesian closed because the space*

$$F = \mathcal{C}^\infty(\mathbb{R},\mathbb{R})$$

is not normable.

Proof: We can argue as in Kriegl and Michor [1997]: because $\mathcal{T}$ is cartesian closed, every evaluation

$$\mathrm{ev}_{XY}(x,f) := f(x) \quad \forall x \in X\, \forall f \in \mathcal{T}(X,Y)$$

is an arrow of $\mathcal{T}$ (this is a general result in every cartesian closed category, see e.g. Mac Lane [1971]) and hence it is also a continuous function, because $\mathcal{T}$ is a subcategory of **Top** by hypotheses. In this case, we also have that the restriction of $\mathrm{ev}_{F\mathbb{R}}$ to the subspace $F^* := \mathrm{Lin}(F,\mathbb{R}) \subseteq \mathcal{T}(F,\mathbb{R})$ of linear continuous functionals on the space F would also be (jointly) continuous:

$$\varepsilon := \mathrm{ev}_{F\mathbb{R}}|_{F\times F^*} : F \times F^* \longrightarrow \mathbb{R}.$$

Then we can find neighborhoods $U \subseteq F$ and $V \subseteq F^*$ of zero such that $\varepsilon(U \times V) \subseteq [-1,1]$, that is

$$U \subseteq \{u \in F \mid \forall f \in V : \ |f(u)| \leq 1\}.$$

But then, taking a generic functional we can always find $\lambda \in \mathbb{R}_{\neq 0}$ such that $\lambda g \in V$, and hence $|g(u)| \leq 1/\lambda$ for every $u \in U$. Any continuous functional is thus bounded on U, so the neighborhood U itself is bounded (see e.g. Jarchow [1981], Kriegl and Michor [1997]). But any locally convex vector space with a bounded neighborhood of zero is normable (see e.g. Jarchow [1981], Donoghue and Smith [1952]).

■

This theorem also asserts that notions like Fréchet manifolds (manifolds modeled in locally convex metrizable and complete vector spaces) are incompatible with cartesian closedness too.

For a more detailed study about cartesian closedness and Banach manifolds, see Brown [1961, 1963, 1964]; for a more detailed study about the relationships between the topology on spaces of continuous linear functionals $\mathrm{Lin}(F,E)$ and normable spaces, see Keller [1965], Maissen [1963].

Because one of our aim is to obtain a category $\boldsymbol{\mathcal{C}}^\infty$ of "smooth" (and hence topological) spaces embedding the category **Ban**, a direct consequence of Theorem 5.2.1 is that, in general, we will not have a locally convex topology on spaces of functions like $\mathcal{C}^\infty(M,\mathbb{R})$. Nevertheless, in $\mathcal{C}^\infty$ we always have that every arrow (i.e. every smooth function in a generalized sense) is also continuous and every evaluation is smooth.

Finally, another important problem in the theory of Banach manifolds is tied with infinite dimensional Lie groups. As it is well known, they appear in several connections in physics, like in the study of both compressible and incompressible fluids, in magnetohydrodynamics, in plasma-dynamics or in electrodynamics (see e.g. Abraham et al. [1988] and references therein). The fundamental results of Omori [1978] (see also Omori and de la Harpe [1972], Omori [1997]) show that a Banach Lie group G acting smoothly, transitively and effectively on a compact manifold M must necessary be finite dimensional. This result strongly underlines that the space of all the diffeomorphisms $G = \mathrm{Diff}(M)$ of a compact manifold in itself cannot be a Banach Lie group.

It is important to note that the present work is not in contrast with the theory of Banach manifolds, but rather it tries to complement it overpassing some of its defects, like the absence of a calculus of actual infinitesimals and the lacking of spaces of mappings. On the one hand, a first aim of the present work is to obtain a category $\mathcal{C}^\infty$ of smooth spaces with better categorical properties (e.g. we will see that the category $\mathcal{C}^\infty$ is cartesian closed and possesses arbitrary limits and colimits, e.g. infinite products, infinite disjoint sums or quotient spaces). On the other hand, of course we aim at exploiting the language of nilpotent infinitesimals. We will see that the category **Ban** of smooth Banach manifolds is faithfully embedded in our category $\mathcal{C}^\infty$ of smooth spaces.

5.3 The convenient vector spaces settings

It is very interesting to note that the original idea to define the differential of functions $f:\mathbb{R}^n \longrightarrow \mathbb{R}^m$ reducing it to the composition $f\circ c$ with differentiable curves $c:\mathbb{R}\longrightarrow\mathbb{R}^n$ goes back (for didactic reasons!) to Hadamard [1923]: in this work a function $f:\mathbb{R}^2\longrightarrow\mathbb{R}$ was called differentiable if all the compositions $f\circ c$ with differentiable curves $c:\mathbb{R}\longrightarrow\mathbb{R}^2$ are again differentiable and satisfy the chain rule. Later (see Michal [1938]) this notion has been extended to mapping $f:E\longrightarrow F$ between generic topological vector spaces: f is defined to be differentiable at $x\in E$ if there exists a continuous linear mapping $l:E\longrightarrow F$ such that $f\circ c:\mathbb{R}\longrightarrow F$ is differentiable at 0 with derivative $(l\circ c')(0)$ for each everywhere differentiable curve $c:\mathbb{R}\longrightarrow E$ with $c(0)=x$. This notion of differentiable function is really more restrictive that the usual one, but it is equivalent to the standard notion of smooth function if in it we replace the word “differentiable” with “smooth”. More generally if we replace “differentiable” with “of class $\mathcal{C}^k$

and with locally Lipschitz k-th derivative", we obtain an equivalence with the classical notion. These results have been proved by Boman [1967] and all the theory of convenient vector spaces depends strongly on these non trivial results.

Several theories which detach from the theory of Banach manifolds, like the convenient vector spaces setting or the following diffeological spaces, are grounded on generalization of this idea (not necessarily knowing the cited article Hadamard [1923]). In particular, the theory of convenient vector spaces is probably the most developed theory of infinite dimensional manifolds ables to overpass several problems of Banach manifolds. Presently, the most complete reference is Kriegl and Michor [1997], even if the theory started with Frölicher and Bucher [1966] and Frölicher and Kriegl [1988].

Only to mention few results, in the convenient vector spaces setting the hard implicit function theorem of Nash and Moser (see Hamilton [1982], Kriegl and Michor [1997]) can be proved, very good results can also be obtained for both holomorphic and real analytic calculus, the theorem of De Rham can be proved and the theory of infinite dimensional Lie groups can be well developed.

Because in the present work we will show that any manifold modeled in convenient vector spaces can be embedded in our category $\boldsymbol{\mathcal{C}}^\infty$, we present very briefly one of the possible equivalent definitions of this type of spaces and some few notions about smooth manifolds modeled in convenient vector spaces.

Definition 5.3.1. *We say that E is a* convenient vector space *iff E is a locally convex vector space where every smooth curve has a primitive, i.e.*

$$\forall c \in \mathcal{C}^\infty(\mathbb{R}, E)\, \exists p \in \mathcal{C}^\infty(\mathbb{R}, E) : \; p' = c$$

Considering the Cauchy-Bochner integral, any Banach space is hence a convenient vector space, but several non trivial example directly comes from the cartesian closedness of the category of all the convenient vector spaces (see Kriegl and Michor [1997]).

As mentioned above what type of topology can be considered in a convenient vector space, due to the cartesian closedness of the related category, is a non trivial point. The idea to reduce, as far as possible, any possible notion to the corresponding notion for smooth curves, can carry us toward the natural idea to consider the final topology for which any smooth curve is also continuous, i.e. the following

Definition 5.3.2. *Let E be a convenient vector space, then we say that*

$$U \textit{ is } c^\infty\textit{-open in } E$$

iff

$$\forall c \in \mathcal{C}^\infty(\mathbb{R}, E) : \; c^{-1}(U) \textit{ is open in } \mathbb{R}$$

The category of convenient vector spaces is cartesian closed so that, e.g. $\mathcal{C}^\infty(\mathbb{R},\mathbb{R})$ is again a convenient vector space. We can now define as usual the notion of chart modeled in a c^∞-open set of a convenient vector space and hence the corresponding notion of smooth manifold and of smooth map between two manifolds. So as to avoid confusion with our category $\boldsymbol{\mathcal{C}^\infty}$, in the following we will denote with $\mathcal{C}^\infty_{\text{cvs}}$ the category of smooth manifolds modeled in convenient vector spaces. Using suitable generalizations of Boman's theorem (Boman [1967]), it is hence possible to prove the following (see Kriegl and Michor [1997])

Theorem 5.3.3. *Let M, N be manifolds modeled on convenient vector spaces, then we have that $f : M \longrightarrow N$ is smooth iff*

$$\forall c \in \mathcal{C}^\infty_{\text{cvs}}(\mathbb{R}, M) : \quad f \circ c \in \mathcal{C}^\infty_{\text{cvs}}(\mathbb{R}, N).$$

Using the notion of c^∞-open subset of a convenient vector space and the notion of chart is possible to define a topology on every manifold considering the final topology in which every chart is continuous. We have hence the expected result that W is open in this topology on M if and only if $c^{-1}(W)$ is open in $\mathbb{R}$ for every smooth curve $c \in \mathcal{C}^\infty_{\text{cvs}}(\mathbb{R}, M)$ (see Kriegl and Michor [1997]).

The notion of Frölicher space provides the possibility to construct a category with very good properties acting as a universe for the class of manifolds modeled in convenient vector spaces. We cite here the definition of Frölicher space only to underline the analogies with our smooth spaces in $\boldsymbol{\mathcal{C}^\infty}$:

Definition 5.3.4. *A Frölicher space is a triple $(X, \mathcal{C}_X, \mathcal{F}_X)$ consisting of a set X, a subset $\mathcal{C}_X \subseteq X^{\mathbb{R}}$ of curves on this set, and a subset $\mathcal{F}_X \subseteq \mathbb{R}^X$ of real valued functions defined on X, with the following properties:*

1. $\forall f : \quad f \in \mathcal{F}_X \iff [\forall c \in \mathcal{C}_X : \ f \circ c \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R})]$
2. $\forall c : \quad c \in \mathcal{C}_X \iff [\forall f \in \mathcal{F}_X : \ f \circ c \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R})]$

The category of Frölicher spaces is cartesian closed and possesses arbitrary limits and colimits. A locally convex vector space E is a convenient vector space if and only if it is a Frölicher space with respect to curves and functions defined as $\mathcal{C}_X := \mathcal{C}^\infty_{\text{cvs}}(\mathbb{R}, E)$ and $\mathcal{F}_X := \mathcal{C}^\infty_{\text{cvs}}(E, \mathbb{R})$. Finally, because of cartesian closedness, it is possible to define a unique structure of Frölicher space on the set $Y := \mathcal{C}^\infty(M, N)$ of all the smooth maps between two manifolds given by

$$\mathcal{C}_Y := \left\{ c : \mathbb{R} \longrightarrow N^M \,|\, c^\vee : \mathbb{R} \times N \longrightarrow M \text{ is smooth} \right\}$$

and

$$\mathcal{F}_Y := \left\{ f : N^M \longrightarrow \mathbb{R} \,|\, \forall c \in \mathcal{C}_Y : \ f \circ c \in \mathcal{C}^\infty(\mathbb{R};,\mathbb{R}) \right\}.$$

In the following we will use again the symbol $\mathcal{C}^\infty(M,N)$ to indicate this structure of Frölicher space.

As mentioned at the beginning of this chapter, the notion of manifold modeled in convenient vector spaces permits to include several infinite dimensional spaces non ascribable into Banach manifold theory, but, at the same time, forces us to lose some good categorical property. In particular the space of all smooth mappings $\mathcal{C}^\infty(M,N)$ between two manifolds has a manifold structure only for M and N finite dimensional (see Kriegl and Michor [1997], Chapter IX). Moreover, if $\mathfrak{C}^\infty(M,N)$ is this manifold structure[3] on the set $\mathcal{C}^\infty(M,N)$, then the exponential law

$$\mathcal{C}^\infty(M,\mathfrak{C}^\infty(N,P)) \simeq \mathcal{C}^\infty(M\times N,P)$$

holds if and only if N is compact (see Kriegl and Michor [1997], Theorem 42.14).

Using an intuitive interpretation introduced by Lawvere [1979] we can say that in the convenient vector spaces settings the fundamental figure of our spaces is the curve and every notion is reduced to a corresponding notion about curves. We will use several times later this intuitive, and fruitfully, interpretations also for other types of figures. In the notion of Frölicher space there is a particular stress in the symmetry between curves and functions, but this symmetry has not been adopted by other authors, like in the following approach about diffeological spaces.

We will see that both Frölicher spaces and manifolds modeled in convenient vector spaces are embedded in our category $\boldsymbol{\mathcal{C}}^\infty$ of smooth spaces, so that our approach can supply a language of actual infinitesimals also to these settings.

5.4 Diffeological spaces

Using the language of the "fundamental figures" given on a general space X introduced by Lawvere [1979], we can describe diffeological spaces as a natural generalization of the previously seen idea to take as fundamental figures all the smooth curves $c:\mathbb{R}\longrightarrow X$ on the space X. To define the concept of diffeological space, we first denote with

$$\mathrm{Op} := \{U \mid \exists n\in\mathbb{N} : U \text{ is open in } \mathbb{R}^n\}$$

the set of all the domains of our new figures in the space X. In informal words, the idea of a diffeological space is to say that the structure on the space X is specified if we give all the smooth figures $p:U\longrightarrow X$, for $U\in\mathrm{Op}$. More formally, we have

[3]Note that, e.g. if $M=N=\mathbb{R}$, this structure is different from the structure of convenient vector space (and Frölicher space) $\mathcal{C}^\infty(\mathbb{R},\mathbb{R})$; for this reason the authors of Kriegl and Michor [1997] use a different symbol $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R})$.

Definition 5.4.1. *We say that* $(\mathcal{D}, X)$ *is a* diffeological space *iff* X *is a set and* $\mathcal{D} = \{\mathcal{D}_U\}_{U \in \mathrm{Op}}$ *is a family of sets of functions*

$$\mathcal{D}_U \subseteq \mathbf{Set}(U, X) \quad \forall U \in \mathrm{Op}$$

The functions $p \in \mathcal{D}_U$ *are called* parametrizations *or* plots *or* figures on X of type U. *The family* $\mathcal{D}$ *has to satisfies the following conditions:*

1. *Every point of* X *is a figure, i.e. for every* $U \in \mathrm{Op}$ *and every constant map* $p : U \longrightarrow X$*, we must have that* $p \in \mathcal{D}_U$*.*

2. *Every set of figures* $\mathcal{D}_U$ *is closed with respect to re-parametrization, i.e. if* $p : U \longrightarrow X$ *is a figure in* $\mathcal{D}_U$*, and* $f \in \mathcal{C}^\infty(V, U)$*, where* $V \in \mathrm{Op}$*, then* $p \circ f \in \mathcal{D}_V$*.*

3. *The family* $\mathcal{D} = \{\mathcal{D}_U\}_{U \in \mathrm{Op}}$ *verifies a sheaf property, i.e. let* $V \in \mathrm{Op}$*,* $(U_i)_{i \in I}$ *be an open cover of* V *and* $p : V \longrightarrow X$ *a map such that* $p|_{U_i} \in \mathcal{D}_{U_i}$*, then* $p \in \mathcal{D}_V$*. In other words, to be locally a figure implies to be a figure globally too.*

Finally a map $f : X \longrightarrow Y$ *between two diffeological spaces* $(X, \mathcal{D}^X)$ *and* $(Y, \mathcal{D}^Y)$ *is said to be* smooth *if it takes figures of the domain space in figures of the codomain space, i.e. if*

$$\forall U \in \mathrm{Op}\ \forall p \in \mathcal{D}_U^X : \quad f \circ p \in \mathcal{D}_U^Y$$

If compared with Frölicher spaces, in Diffeology (i.e. the study of diffeological spaces, see Iglesias-Zemmour [2008]) the principal differences are in the generalization of the types of figures, in the losing of the symmetry between figures and corresponding functions (i.e. maps of type $f : X \longrightarrow U$ for $U \in \mathrm{Op}$) and in the fundamental sheaf property. For example, the generalization to figures of arbitrary dimension instead of curves only, permits to prove the cartesian closure of the category of diffeological spaces very easily and without the use of the non trivial Boman's theorem (see Frölicher and Kriegl [1988], Kriegl and Michor [1997], Boman [1967]). The original idea to consider figures of general dimension instead of curves only, and the fundamental sheaf condition date back to Chen [1977, 1982]; the definition of diffeological space, essentially in the form given above, is originally of Souriau [1981, 1984].
The category of diffeological spaces has very good categorical properties, with arbitrary limits (subspaces, products, pullbacks, etc.) and colimits (quotient spaces, sums, pushforwards, etc.) and cartesian closedness (so that set theoretical compositions and evaluations are always smooth). Classical Fréchet manifolds are fully and faithfully embedded in this category (see Losik [1992]).

We can now define a *diffeological vector space* (over $\mathbb{R}$) any diffeological space $(E, \mathcal{D})$, where E is a vector space (over $\mathbb{R}$), and such that the addiction and the multiplication by a scalar

$$(u, v) \in E \times E \mapsto u + v \in E \quad \text{and} \quad (r, u) \in \mathbb{R} \times E \mapsto ru \in E$$

are smooth (with respect to the suitable product diffeologies on the domains) and, as usual, the notion of *smooth manifolds modeled on diffeological vector spaces*.

Anyway, differential geometry on generic diffeological spaces can be developed surprisingly far as showed e.g. by Iglesias-Zemmour [2008]: homotopy theory, exterior differential calculus, differential forms, Lie derivatives, integration on chains and Stokes formula, de Rham cohomology, Cartan formula, generalization of symplectic geometry to diffeological spaces, etc. As said in Iglesias-Zemmour [2008]:

> Thanks to the strong stability of diffeology under the most important categorical operations [...] every general construction relating to this theory applies to spaces of functions, differential forms, fiber bundles, homotopy, etc. without leaving the strict framework of diffeology. This makes the development of differential geometry much more easier, much more natural, than usually.

It is also interesting to note that some of these generalizations (like Stokes formula) are general consequences of this type of extension of the notion of manifolds, as proved by Losik [1994], and hence are not peculiar of Diffeology.

From the point of view of the present work, Diffeology is surely formally clear, but sometimes lacks from the point of view of the intuitive geometrical interpretation. To illustrate this assertion, we can consider the notion of tangent vector as formulated in Iglesias-Zemmour [2008]. In the following we will assume that $(X, \mathcal{D})$ is a diffeological space and $x \in X$ is a point in the space X. The first idea is that the figures $q : U \longrightarrow X$ of type $U \subseteq \mathbb{R}^n$ of the space X permit to define the notion of smooth p-form without having the notion of tangent vector, but abstracting the properties of the pullback q^* of the figure $q \in \mathcal{D}_U$. In other words, let us suppose that we have already defined what is a differential p-form on X, then we would be able to define the pullback q^* of q as a map that associates to each point $u \in U \subseteq \mathbb{R}^n$ a p-form in $\Lambda^p(\mathbb{R}^n)$. The idea is hence to define directly a p-form as this action on figures through pullback, and asking the natural condition of composition of pullbacks in case we take a parametrization $f \in \mathcal{C}^\infty(V, U)$ of the domain of the figure q:

Definition 5.4.2. *A* differential p-form defined on X *is a family of maps* $(\alpha_U)_{U \in \mathrm{Op}}$. *Each* α_U*, for* U *open in* $\mathbb{R}^n$*, associates to each figure* $q \in \mathcal{D}_U$ *a* *smooth p-form* $\alpha_U(q) : U \longrightarrow \Lambda^p(\mathbb{R}^n)$ *, i.e.*

$$\alpha_U : \mathcal{D}_U \longrightarrow \mathcal{C}^\infty(U, \Lambda^p(\mathbb{R}^n)),$$

and it has to satisfies the condition

$$\alpha_V(q \circ f) = f^*(\alpha_U(q))$$

for every plot $q \in \mathcal{D}_U$ *and for every smooth parametrization* $f \in \mathcal{C}^\infty(V, U)$ *defined on the open set* $V \in \mathrm{Op}$*. The set of all the differential p-forms defined on* X *will be denoted by* $\Omega^p(X)$*.*

The method used to arrive at this definition is the (frequently used in mathematics) "inversion of the effect with the cause" in case of bijection between effects and causes. Indeed, if $X =$ is an open set of $\mathbb{R}^d$, then it is possible to prove that we have a natural isomorphism between the new definition and the classical notion of smooth p-form, i.e. $\Omega^p(U) \simeq \mathcal{C}^\infty(U, \Lambda^p(U))$, in other words pullbacks of p-forms uniquely determine the p-forms themselves.
The previous definition satisfy all the properties one needs from it, like the possibility to define a diffeology on $\Omega^p(X)$, vector space structure, pullbacks, exterior differential, exterior product, a natural notion of germ generated by a p-form so that two forms are equal if and only if they generate the same germ (that if they are "locally" equal), etc.

The first intuitive drawback of the definition of $\Omega^p(X)$ is that there is no mention to spaces $\Lambda^p_x(X)$ of p-forms associated to each point $x \in X$ and of the relationships between these spaces and the whole $\Omega^p(X)$. Therefore, to understand better the following definitions, we introduce the following

Definition 5.4.3. *We say that two forms* α, $\beta \in \Omega^p(X)$ have the same value at x*, and we write* $\alpha \sim_x \beta$*, if and only if for every figure* $q \in \mathcal{D}_U$ *such that*

$$0 \in U \quad \textit{and} \quad q(0) = x$$

(in this case we will say that q is centered at x*) we have that*

$$\alpha(q)(0) = \beta(q)(0).$$

Equivalence classes of p-forms by means of the equivalence relation $\sim_x$ *are called* values of α at x *and we will denote with* $\Lambda^p_x(X) := \Omega^p(X)/\sim_x$ this quotient set.

Using these values of 1-forms we can define tangent vectors. Firstly we introduce the paths on X and the values of a 1-form on each path with the following

Definition 5.4.4. *Let us introduce the space of all the paths on* X*, i.e.*

$$\mathrm{Paths}(X) := \mathcal{C}^\infty(\mathbb{R}, X)$$

and for each path $q \in \mathrm{Paths}(X)$*, the map* $j(q) : \Omega^1(X) \longrightarrow \mathbb{R}$ *evaluating each 1-form at zero*

$$j(q) : \alpha \in \Omega^1(X) \mapsto \alpha(q)(0) \in \mathbb{R}.$$

The map $j(q)$ *is linear and smooth (because it is an evaluation), hence*

$$j : \mathrm{Paths}(X) \longrightarrow L^\infty(\Omega^1(X), \mathbb{R}),$$

where $L^\infty(\Omega^1(X), \mathbb{R})$ *is the space of all the linear smooth functionals defined on the space of 1-forms of* X*.*

Secondly we say that the set of all these values $j(q)$ generates the whole tangent space. The set of these generators is introduced in the following

Definition 5.4.5. *The space* $C_x^\wedge(X)$ *is the image of all the paths passing through* x *under the map* j*:*

$$C_x^\wedge(X) := \{j(q) \,|\, q \in \mathrm{Paths}(X) \text{ and } q(0) = x\} \subseteq L^\infty(\Omega^1(X), \mathbb{R})$$

In the space $C_x^\wedge(X)$ is naturally defined a multiplication by a scalar $r \in \mathbb{R}$ that formalizes the idea to increase the speed of going through a given path $q \in \mathrm{Paths}(X)$:

$$r \cdot j(q) = j\left[q(r \cdot (-))\right],$$

where $q(r \cdot (-))$ is the path $q(r \cdot (-)) : s \in \mathbb{R} \longrightarrow q(r \cdot s) \in X$. But the space $C_x^\wedge(X)$ is not necessarily a vector space because is not closed with respect to addiction of these values $j(q)$ of 1-forms on paths q centered at x, hence we finally define

Definition 5.4.6. *A tangent vector* $v \in T_x(X)$ *is a linear combination of elements of* $C_x^\wedge(X)$*, i.e.*

$$v = \sum_{i=1}^n s_i v_i$$

for some

$$\begin{aligned} &n \in \mathbb{N} \\ &(v_i)_{i=1}^n \text{ sequence of } C_x^\wedge(X) \\ &(s_i)_{i=1}^n \text{ sequence of } \mathbb{R}. \end{aligned}$$

As we said, even if the definitions we have just introduced are formally correct, their intuitive geometric meaning remains obscure. In classical manifolds theory, the definition of tangent vector through 1-forms is not geometrically intrinsic unless of Riemannian manifolds, so it is not clear why passing to a more general space we are able to obtain this identification in an intrinsic way. Secondly, diffeological spaces include also spaces with singular points, like $X = \{(x,y) \in \mathbb{R}^2 \mid x \cdot y = 0\}$. At the origin $x = (0,0) \in X$ there is no way to define in a geometrically meaningful way the sum of the two tangent vectors corresponding to $\boldsymbol{i} = (1,0)$ and $\boldsymbol{j} = (0,1)$ (without using the superspace $\mathbb{R}^2$). This is the principal motivation that conducts SDG to introduce the notion of *microlinear space* as the spaces where to each pair of tangent vectors it is possible to associate an infinitesimal parallelogram, fully contained in the space itself, whose diagonal represents the sum of these two tangent vectors. The previous space X is not microlinear exactly at the origin.

As we will see, our category $\boldsymbol{\mathcal{C}}^\infty$ is exactly the category of diffeological spaces and concretely we will only generalize the definition of diffeological space so as to obtain a more flexible instruments that will permit us to define the category ${}^\bullet\boldsymbol{\mathcal{C}}^\infty$ of spaces extended with the new infinitesimal points. E.g. we will have that $\mathbb{R} \in \boldsymbol{\mathcal{C}}^\infty$ and ${}^\bullet\mathbb{R} \in {}^\bullet\boldsymbol{\mathcal{C}}^\infty$. Hence, the theory of Fermat reals naturally includes diffeological spaces and also provides to them a language of actual infinitesimals. The use of these infinitesimals opens the possibility to simplify and clarify some concepts already developed in the framework of diffeological spaces, e.g. gaining a more clear geometrical meaning. We will also see that using these infinitesimal we will also arrive to new results, like the existence of infinitesimals flows corresponding to a given smooth vector field.

5.5 Synthetic differential geometry

The fundamental ideas upon which SGD[4] born, originate from the work of Ehresmann [1951], Weil [1953] and A. Grothendieck (see Artin et al. [1972]). Ehresmann [1951] introduced the concept of k-jet at a point p in a manifold M as an important geometric structure determined by the k-th order Taylor's formula of real valued functions f defined in a neighborhood of $p \in M$. As said by Mac Lane [1980]:

> [...] the study of jets can be seen as a development of the earlier idea of studying the "infinitely nearby" points on algebraic curves on manifolds. Presumably it was Ehresmann's initiative which stimulated the paper by Weil [1953].

In this work A. Weil introduced the idea to formalize nilpotent infinitesimals using algebraic methods, more precisely using quotient rings like $\mathbb{R}[x]/(x^2)$ or $\mathbb{R}[x,y]/(x^2,y^2)$, in general formal power series in n variables $\mathbb{R}[[x_1,\dots,x_n]]$ modulo the $(k+1)$-th power of a given ideal $I = (i_1,\dots,i_m)$ of series $i_1,\dots,i_m \in \mathbb{R}[[x_1,\dots,x_n]]$ with zeros constant term, i.e. such that $i_j(\underline{0}) = 0$ for every $j = 1,\dots,m$. These type of objects are now called *Weil algebras*, and C. Ehresmann's jets are also special cases of Weil algebras. Very

[4]Frequently SDG is also called smooth infinitesimal analysis.

roughly, we can guess the fundamental idea of A. Weil saying that, e.g., an element $p \in \mathbb{R}[x]/(x^2)$ can be written as $p = a + x \cdot b$, with a, $b \in \mathbb{R}$, with addiction given in the more obvious way and multiplication given by $(a + x \cdot b) \cdot (\alpha + x \cdot \beta) = a\alpha + x \cdot (a\beta + b\alpha)$, that is the same result we would obtain if we multiply the two polynomials $a + x \cdot b$ and $\alpha + x \cdot \beta$ with the formal rules $x^2 = 0$. At the end, with a construction as simple as the definition of the field of complex numbers, we have extended the real field into a ring with a non-zero element x having zero square, i.e. a first order infinitesimal (but in this ring there are not infinitesimals of greater order). Using the same idea, we can see that with the Weil algebra $\mathbb{R}[x, y]/(x^2, y^2)$ we have extended the real field with two first order infinitesimals x, y whose product is not zero[5] $x \cdot y \neq 0$. Suitably generalized to algebras of germs of smooth functions defined on manifolds, these two examples, i.e. $\mathbb{R}[x]/(x^2)$ and $\mathbb{R}[x, y]/(x^2, y^2)$, correspond isomorphically to the first and second tangent bundle respectively (see e.g. Weil [1953], Kriegl and Michor [1997, 1996], Kock [1981], Lavendhomme [1996], Moerdijk and Reyes [1991], Bertram [2008] for more details). The next fundamental step to obtain a single framework where all these types of nilpotent infinitesimals are available, has been performed by A. Grothendieck who tried to use nilpotent infinitesimals in his theory of schemes to treat infinitesimal structures in algebraic geometry. The basic idea was to study an algebraic locus like $S^1 = \left\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\right\}$, not only as a subset of points in the plane, but as the functor $S^1_{\mathrm{F}} : \mathbf{CRing} \longrightarrow \mathbf{Set}$ from the category $\mathbf{CRing}$ of commutative rings with 1 to the category of sets defined as

$$S^1_{\mathrm{F}}(A) := \left\{(a, b) \in A^2 \mid a^2 + b^2 = 0\right\}$$
$$S^1_{\mathrm{F}}(A \xrightarrow{f} B) := (f \times f)|_{S^1_{\mathrm{F}(A)}} : S^1_{\mathrm{F}}(A) \longrightarrow S^1_{\mathrm{F}}(B)$$

(where $f : A \longrightarrow B$ is a ring homomorphism and $f \times f : (a, b) \in A^2 \mapsto (f(a), f(b)) \in B^2$). Using this approach algebraic geometers started to understand that the functor corresponding to the trivial locus $\{x \in \mathbb{R} \mid x = x\} = \mathbb{R}$, i.e. the functor $R(A) := \{a \in A \mid a = a\} = A =$ the underlying set of the ring A, behaves like a set of scalars containing infinitesimals. E.g. $D(A) := \left\{a \in A \mid a^2 = 0\right\}$ is a subfunctor of this functor R and plays the role of the space of first order infinitesimals. Being a subfunctor, D "behaves" like a subset[6] of R. These ideas conducted to the notion of Grothendieck topos. Lawvere found that in the Grothendieck topos, and in other similar categories that later will originate the general notion of topos (see Gray [1971]), an intuitionistic set-theoretic language can be directly interpreted

[5] We recall Section 4.1 to underline an important difference with our approach.

[6] In the sense that each Topos is a model of intuitionistic set theory, so that it is possible to define a formal language for intuitionistic set theory where sentences like $D \subseteq R$ are rigorous and true in the model (see Moerdijk and Reyes [1991], Kock [1981] for more details).

in any topos. In Lawvere [1979] he proposes a way to generalize these construction of algebraic geometry to smooth manifolds theory, and to use this generalization as a foundation for infinitesimal reasoning valid both for finite and infinite dimensional manifolds. This proposal was part of a big project whose objective is to establish an intrinsic axiomatizaton for continuum mechanics. The inclusion of infinite dimensional spaces like functions spaces is a natural consequence of the cartesian closedness of every topos.

The construction of a model for SDG which embeds the category of smooth finite dimensional manifolds is not a simple task. Classical references are Moerdijk and Reyes [1991], Kock [1981]. Here we only want to sketch some of the fundamental ideas, first of all to underline the conceptual differences between SDG and the above mentioned approaches to infinite dimensional differential geometry.
The first idea to generalize from the context of algebraic geometry to manifolds theory is to find a corresponding of the category of **CRing** of commutative rings, i.e. to pass from a context of polynomial operations to more general smooth functions. Indeed, that category is replaced by that of $\mathcal{C}^\infty$-rings:

Definition 5.5.1. *A $\mathcal{C}^\infty$-ring $(A, +, \cdot, \iota)$ is a ring $(A, +, \cdot)$ together with an interpretation $\iota(f)$ of each possible smooth map $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, that is a map*

$$i(f) : A^n \longrightarrow A^m$$

such that ι preserves projections, compositions and identity maps, i.e.:

1. *If $p : \mathbb{R}^m \longrightarrow \mathbb{R}$ is a projection, then $\iota(p) : A^m \longrightarrow A$ is a projection.*
2. *If $\mathbb{R}^d \xrightarrow{g} \mathbb{R}^n \xrightarrow{f} \mathbb{R}^m$ are smooth, then $\iota(f \circ g) = \iota(f) \circ \iota(g)$.*
3. *If $1_{\mathbb{R}^n} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is the identity map, then $\iota(1_{\mathbb{R}^n}) = 1_{\iota(A^n)}$.*

A homomorphism of $\mathcal{C}^\infty$-rings *is a ring homomorphism which preserves the interpretation of smooth maps, that is such that*

$$\begin{tikzcd} A^n \arrow[r, "\varphi^n"] \arrow[d, "A(f)"'] & B^n \arrow[d, "B(f)"] \\ A^m \arrow[r, "\varphi^m"] & B^m \end{tikzcd}$$

We may define a $\mathcal{C}^\infty$-ring in an equivalent but more concise way: let C^∞ denote the category whose objects are the spaces $\mathbb{R}^d$, $d \geq 0$, and with smooth functions as arrows, then a $\mathcal{C}^\infty$-ring is a finite product preserving functor $A : \mathcal{C}^\infty \longrightarrow \mathbf{Set}$, and a $\mathcal{C}^\infty$-homomorphism is just a natural transformation

$\varphi : A \longrightarrow B$. Indeed, given such a functor, the set $A(\mathbb{R})$ has the structure of a commutative ring $(A(\mathbb{R}), +_A, \cdot_A)$ given by $+_A := A(\mathbb{R} \times \mathbb{R} \xrightarrow{+} \mathbb{R})$ and $\cdot_A := A(\mathbb{R} \times \mathbb{R} \xrightarrow{\cdot} \mathbb{R})$, where $+ : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ and $\cdot : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ are the ring operations on $\mathbb{R}$.
Here are some examples of $\mathcal{C}^\infty$-rings

Example 5.5.2. The ring $\mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R})$ of real valued smooth functions $a : \mathbb{R}^d \longrightarrow \mathbb{R}$, with pointwise ring operations, is a $\mathcal{C}^\infty$-ring. Usually it is denoted simply with $\mathcal{C}^\infty(\mathbb{R}^d)$. The smooth function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is interpreted in the following way. Let $(h_1, \dots, h_n) \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R})^n$, be n elements of the ring $\mathcal{C}^\infty(\mathbb{R}^d)$. Their product

$$(h_1, \dots, h_n) : x \in \mathbb{R}^d \mapsto (h_1(x), \dots, h_n(x)) \in \mathbb{R}^n$$

can be can be composed with $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and projected into its m components obtaining

$$\iota(f) := (p_1 \circ f \circ (h_1, \dots, h_n), \dots, p_m \circ f \circ (h_1, \dots, h_n)) \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R}),$$

where $p_i : \mathbb{R}^m \longrightarrow \mathbb{R}$ are the projections.

Example 5.5.3. If M is a smooth manifold, the ring of real valued functions defined on M, i.e. $\mathcal{C}^\infty(M, \mathbb{R})$, is a $\mathcal{C}^\infty$-ring. Here a smooth function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is interpreted using composition, similarly to the previous example. This ring is also denoted by $\mathcal{C}^\infty(M)$. Moreover, it is well known that

$$\mathcal{C}^\infty(M) = \mathcal{C}^\infty(N) \quad \Longrightarrow \quad M = N.$$

If $g : N \longrightarrow M$ is a smooth map between manifolds, then the $\mathcal{C}^\infty$-homomorphism given by

$$\mathcal{C}^\infty(g) : a \in \mathcal{C}^\infty(M, \mathbb{R}) \mapsto a \circ g \in \mathcal{C}^\infty(N, \mathbb{R})$$

verifies the analogous embedding property:

$$\mathcal{C}^\infty(g) = \mathcal{C}^\infty(h) \quad \Longrightarrow \quad g = h.$$

This means that manifolds can be faithfully considered as $\mathcal{C}^\infty$-rings.

Example 5.5.4. Let A be a $\mathcal{C}^\infty$-ring and I an ideal of A, then the quotient ring A/I is also a $\mathcal{C}^\infty$-ring. Indeed, if $A(f) : A^n \longrightarrow A^m$ is the interpretation of $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, we can define the interpretation $(A/I)(f) : (A/I)^n \longrightarrow (A/I)^m$ as

$$\begin{aligned}(A/I)(f)([a_1]_I, \dots, [a_n]_I) &:= \\ &= ([p_1(A(f)(a_1, \dots a_n))]_I, [p_m(A(f)(a_1, \dots a_n))]_I),\end{aligned}$$

where $[a_i]_I \in A/I$ denotes the equivalent classes of the quotient ring, and $p_j : A^m \longrightarrow A$ are the projections (see e.g. Moerdijk and Reyes [1991] for more details). Examples included in this case are the analogous of the above mentioned $D_k := \mathcal{C}^\infty(\mathbb{R})/(x^{k+1})$ and $D(2) := \mathcal{C}^\infty(\mathbb{R})/(x^2, y^2)$, or the ring $\triangle := \mathcal{C}_0^\infty(\mathbb{R}^n) = \mathcal{C}^\infty(\mathbb{R}^n)/m^g_{\{0\}}$, where $m^g_{\{0\}}$ is the ideal of smooth functions having zero germs at $0 \in \mathbb{R}^n$ and finally $\mathbb{I} := \mathcal{C}_0^\infty(\mathbb{R}^n \setminus \{0\})$. These $\mathcal{C}^\infty$-rings will play the role, in the final model, respectively of *infinitesimals of k-th order D_k*, of *pairs of infinitesimals of first order whose product is not necessarily zero* $D(2)$, of the set of all the infinitesimals $\triangle$ and of the set of all the *invertible infinitesimals* $\mathbb{I}$.

For each subset $X \subseteq \mathbb{R}^n$, a function $f : X \longrightarrow \mathbb{R}$ is said to be *smooth* if there is an open superset $U \supseteq X$ and a smooth function $g : U \longrightarrow \mathbb{R}$ which extends f, i.e. $g|_X = f$. We can proceed as in the previous example using composition to define the $\mathcal{C}^\infty$-ring $\mathcal{C}^\infty(X)$ of real valued functions defined on X. An important example that uses this generalization and the previous example is $\mathcal{C}^\infty(\mathbb{N})/K$, where $\mathcal{C}^\infty(\mathbb{N})$ is the ring of smooth functions on the natural numbers, and K is the ideal of eventually vanishing functions. This ring will act, in the final model, as the set of *infinitely large natural numbers*.

Example 5.5.5. A $\mathcal{C}^\infty$-ring A is called *finitely generated* if it is isomorphic to one of the form $\mathcal{C}^\infty(\mathbb{R}^n)/I$, for some $n \in \mathbb{N}$ and some finitely generated ideal $I = (i_1, \ldots, i_m)$. For example, given an open subset $U \subseteq \mathbb{R}^n$ we can find a smooth function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that $f(x) \neq 0$ if and only if $x \neq U$. So U is diffeomorphic to the closed set $\hat{U} = \{(x, y) \mid y \cdot f(x) = 1\} \subseteq \mathbb{R}^{n+1}$. Hence we have the isomorphism of $\mathcal{C}^\infty$-rings

$$\mathcal{C}^\infty(U) \simeq \mathcal{C}^\infty(\mathbb{R}^{n+1})/(y \cdot f(x) - 1).$$

This proves that $\mathcal{C}^\infty(U)$ is finitely generated. Using this result and Whitney's embedding theorem it is possible to prove that for a manifold M, the $\mathcal{C}^\infty$-ring $\mathcal{C}^\infty(M)$ is finitely generated too (see Moerdijk and Reyes [1991], Kock [1981]).

Therefore, the category $\mathbb{L}$ of finitely generated $\mathcal{C}^\infty$-rings seems a good step toward the goal to embed finite dimensional manifolds in a category with infinitesimal objects. However, function spaces can in general not be constructed in $\mathbb{L}$. In order to have these function spaces, the first step is to extend the category $\mathbb{L}$ in the category $\mathbf{Set}^{\mathbb{L}^{\mathrm{op}}}$ of presheaves on $\mathbb{L}$, i.e. of functors $F : \mathbb{L} \longrightarrow \mathbf{Set}$:

$$\mathbf{Man} \subseteq \mathbb{L} \subseteq \mathbf{Set}^{\mathbb{L}^{\mathrm{op}}}.$$

This is a natural step in this context because the embedding $\mathbb{L} \subseteq \mathbf{Set}^{\mathbb{L}^{\mathrm{op}}}$ is a well know result in category theory (see Yoneda embedding in Appendix

A), and because the category $\mathbf{Set}^{\mathbb{L}^{\mathrm{op}}}$ is a topos. So we concretely see the possibility to embed the category of smooth manifolds in a topos containing infinitesimal objects too. Let us note that manifolds are directly embedded in $\mathbf{Set}^{\mathbb{G}^{\mathrm{op}}}$ without "an extension with new infinitesimal points", so the approach is very different with respect, e.g., to NSA or to the present work.

So, what is the ring of scalars representing the geometric line in the topos $\mathbf{Set}^{\mathbb{L}^{\mathrm{op}}}$? If A, $B \in \mathbb{L}$ are finitely generated $\mathcal{C}^\infty$-rings, and $f : A \longrightarrow B$ is a $\mathcal{C}^\infty$-homomorphism, this geometric line is represented by the functor

$$R(A) = \mathbb{L}(A, \mathcal{C}^\infty(\mathbb{R})) \tag{5.5.1}$$

$$R(A \xrightarrow{f} B) : g \in R(A) \mapsto g \circ f \in R(B) \tag{5.5.2}$$

corresponding, via the Yoneda embedding, to the $\mathcal{C}^\infty$-ring $\mathcal{C}^\infty(\mathbb{R})$. The set of first order infinitesimal D corresponds in the topos $\mathbf{Set}^{\mathbb{L}^{\mathrm{op}}}$ to the functor

$$D(A) = \mathbb{L}(A, \mathcal{C}^\infty(\mathbb{R})/(x^2)) \tag{5.5.3}$$

$$D(A \xrightarrow{f} B) : g \in D(A) \mapsto g \circ f \in D(B). \tag{5.5.4}$$

Indeed, the topos $\mathbf{Set}^{\mathbb{L}^{\mathrm{op}}}$ is not the final model of SDG for several reasons. Among these, we can cite that in the topos $\mathbf{Set}^{\mathbb{L}^{\mathrm{op}}}$ are not provable properties like $1 \neq 0$ or $\forall r \in \mathbb{R}(x \text{ is invertible } \vee (1-x) \text{ is invertible})$, and this is essentially due because the embedding $\mathbf{Man} \subseteq \mathbf{Set}^{\mathbb{L}^{\mathrm{op}}}$ does not preserve open covers. A description of the final models is outside the scopes of the present work. For more details see e.g. Moerdijk and Reyes [1991] and references therein. In the light of the examples (5.5.1), (5.5.2) and (5.5.3), (5.5.4) we can quote Moerdijk and Reyes [1991]:

> In recent years, several alternative solutions to the problem of generalizing manifolds to include function spaces and spaces with singularities have been proposed in the literature. A particularly appealing one is the theory of convenient vector spaces [...]. These structures are in a way simpler than the sheaves considered in this book, but one should notice that the theory of convenient vector spaces does not include an attempt to develop an appropriate framework for infinitesimal structures, which is one of the main motivations of our approach

The present work tries to go exactly in the direction to have a simple generalization of manifolds (indeed, simpler than convenient vector spaces and as simple as diffeological spaces) and at the same time infinitesimals structures.

Hence, it is in the opinion of the researchers in SDG that these topos models are not sufficiently simple, even if, at the same time, they are very rich and formally powerful. For these reasons smooth infinitesimal analysis is usually presented in an "axiomatic" way, in the framework of a naive intuitionistic set theory[7], but with explicit introduction of particular axioms useful to deal with smooth spaces (i.e. objects of $\mathbf{Set}^{\mathbb{L}^{\mathrm{op}}}$ or a better model) and smooth functions (i.e. arrows of $\mathbf{Set}^{\mathbb{L}^{\mathrm{op}}}$ or a better model). This possibility is due to the above mentioned internal language for a set theory that can be defined in every topos (that represents its intuitionistic semantics). For example a basic assumption is the so-called Kock-Lawvere axiom:

Assumption 5.5.6. *R is a ring and we define $D := \left\{h \in R \,|\, h^2 = 0\right\}$, called the set of first order infinitesimal. They satisfy:*

$$\forall f : D \longrightarrow R \; \exists! m \in R : \; \forall h \in D : \quad f(h) = f(0) + h \cdot m. \tag{5.5.5}$$

The universal quantifier "for every function $f : D \longrightarrow R$" really means "for every set theoretical function from D to R", but definable using intuitionistic logic. In semantical terms, this corresponds to "for every arrow in the model $\mathbf{Set}^{\mathbb{L}^{\mathrm{op}}}$", i.e. for every smooth natural transformation between the functor D (see (5.5.3) and (5.5.4)) and the functor R (see (5.5.1) and (5.5.2)). It is not surprising to assert that (5.5.5) is incompatible with classical logic: putting

$$f(h) = \begin{cases} 1 \text{ if } & h \neq 0 \\ 0 \text{ if } & h = 0 \end{cases} \tag{5.5.6}$$

then applying the Kock-Lawvere axiom (5.5.5) with this function and considering the hypothesis $\exists h_0 \in D : \; h_0 \neq 0$, we obtain

$$1 = 0 + h_0 \cdot m.$$

Squaring this equality we obtain $1 = 0$. Considering this incompatibility with classical logic a motivation to consider intuitionistic logic, is a natural passage only in a context of topos theory and only if one already is thinking to the existence of models like $\mathbf{Set}^{\mathbb{L}^{\mathrm{op}}}$. But in another context we think that the more natural idea is to criticize (5.5.5) asking some kind of limitation on the functions to which it can be really applied. Indeed, this was one of the first motivation to start the present work. Indeed, we will take strong inspiration from SDG in this work, but we can affirm that these two theories are very different. Our attention to stress the intuitive meaning of the new infinitesimals numbers does not find a correspondence in SDG, where infinitesimal of very different types can be defined, but sometimes loosing the corresponding intuitive meaning. About this point of view we can quote Conway [1999]:

[7] Exactly as almost every mathematician works in naive (classical) set theory. On the other hand to work in SDG, one has to learn to work in intuitionistic logic, i.e. avoiding the law of the excluded middle, the proofs by reduction ad absurdum ending with a double negation, the full De Morgan laws, the equivalence between double negation and affirmation, the full equivalence between universal and existential quantifiers through negation, the axiom of choice, etc.

> I think I should point out that [SDG] isn't really trying to be a candidate for setting up infinitesimal analysis. It's just a formal algebraic technique for "working up to any given order in some small variable s" - for instance if you want to work up to second order in s, you just declare that $s^3 = 0$.

Even if we do not completely agree with this strong affirmation, it represents an authoritative opinion that underlines the differences between SDG and our approach.

Finally we cite that the work of Weil [1953] has been the base for several other research tempting to formalize in some way nilpotent infinitesimal methods (but without getting all the difficulties of SDG). In this direction we can cite Weil functors (see Kriegl and Michor [1997], Kolár et al. [1993], Kriegl and Michor [1996]) and the recent Bertram [2008].

Chapter 6

The cartesian closure of a category of figures

6.1 Motivations and basic hypotheses

In this section we shall define the basic constructions which will lead us to the category of $\mathcal{C}^n$ *spaces* and $\mathcal{C}^n$ *functions*; we will realize these constructions for a generic $n \in \mathbb{N}_{>0} \cup \{+\infty\}$, even if in the next chapters concerning calculus and differential geometry we will consider the case $n = +\infty$ only. Any $\mathcal{C}^n$ manifold is a $\mathcal{C}^n$ space too, and the category $\mathcal{C}^n$ of all $\mathcal{C}^n$ spaces is cartesian closed (see Section 5.1), hence it contains several infinite-dimensional spaces, the first of which we are interested in is $\mathcal{C}^n(M, N)$, i.e. the space of all the usual $\mathcal{C}^n$ functions between two manifolds M and N. It is important to note that, exactly as in Kriegl and Michor [1997] and in Moerdijk and Reyes [1991], the category $\mathcal{C}^n$ contains many "pathological" spaces; actually $\mathcal{C}^n$ works as a "cartesian closed universe" and we will see that, like in Kock [1981], Lavendhomme [1996], Moerdijk and Reyes [1991], the particular *inf-linear* $\mathcal{C}^n$ *spaces* have the best properties, and will work as a good substitute of manifolds (we have already made some comments about this way of proceeding in Section 5.1).

The ideas used in this section arise from analogous ideas about diffeological spaces and Frölicher spaces (see Section 5.3), in particular our first references are Chen [1982] and Frölicher and Kriegl [1988]; actually $\mathcal{C}^\infty$ is the category of diffeological spaces (see Section 5.4). For these reasons, in this section we will not present the proofs of the most elementary facts; these can be indeed easily generalized from analogous proofs of Chen [1982], Frölicher and Kriegl [1988], Kriegl and Michor [1997] or Iglesias-Zemmour [2008]. The results presented in this and the following chapter have been already published in Giordano [2004].

We present the definition of cartesian closure starting from a concrete category $\mathcal{F}$ of topological spaces (satisfying few conditions) and embedding it in a cartesian closed category $\bar{\mathcal{F}}$. We will call $\bar{\mathcal{F}}$ the *cartesian closure of* $\mathcal{F}$. We need this generality because we shall use it to define both domain and codomain of the extension functor ${}^\bullet(-) : \boldsymbol{\mathcal{C}}^\infty \longrightarrow {}^\bullet\boldsymbol{\mathcal{C}}^\infty$, that generalizes the construction $\mathbb{R} \mapsto {}^\bullet\mathbb{R}$ associating to each smooth space $M \in \boldsymbol{\mathcal{C}}^\infty$ its extension with our infinitesimal points ${}^\bullet M \in {}^\bullet\boldsymbol{\mathcal{C}}^\infty$. Indeed, the categories acting as the domain and the codomain of this functor will be defined starting from two different categories $\mathcal{F}$ and applying the cartesian closure.

The problem to generalize the definition of ${}^\bullet\mathbb{R}$ to a functor ${}^\bullet(-)$ can also be seen from the following point of view: at this stage of the present work, it is natural to define a tangent vector to a manifold M as a map

$$t : D \longrightarrow {}^\bullet M.$$

But we have to note that the map t has to be "regular" in some sense, hence we need some kind of geometric structure both on the domain of first order infinitesimals D and on the codomain ${}^\bullet M$. On the other hand, it is natural to expect that the ideal D is not of type ${}^\bullet N$ for some manifold N because the only standard real number in D is 0. We shall define suitable structures on D and ${}^\bullet M$ so that they will become objects of the category ${}^\bullet\boldsymbol{\mathcal{C}}^\infty$ of extended smooth spaces, i.e. so that D, ${}^\bullet M \in {}^\bullet\boldsymbol{\mathcal{C}}^\infty$. Subsequently we shall define the concept of tangent vector so that $t \in {}^\bullet\boldsymbol{\mathcal{C}}^\infty(D, {}^\bullet M)$, i.e. t will be an arrow of the category ${}^\bullet\boldsymbol{\mathcal{C}}^\infty$ of smooth extended spaces and smooth extended functions.

In this chapter we will assume the following hypotheses on the category $\mathcal{F}$:

Assumption 6.1.1.

1. *$\mathcal{F}$ is a subcategory of the category of topological spaces* **Top***, and contains all the constant maps $c : H \longrightarrow X$ and all the open subspaces $U \subseteq H$ (with the induced topology) of every object $H \in \mathcal{F}$. The corresponding inclusion $i : U \hookrightarrow H$ is also an arrow of $\mathcal{F}$, i.e. $i \in \mathcal{F}_{UH} := \mathcal{F}(U, H)$.*

In the following we will denote by $|-| : \mathcal{F} \longrightarrow$ **Set** the forgetful functor which associates to any $H \in F$ its support set $|H| \in$ **Set**. Moreover with $\mathcal{T}_H$ we will denote the topology of H and with $(U \prec H)$ the topological subspace of H induced on the open set $U \in \mathcal{T}_H$. The remaining assumptions on $\mathcal{F}$ are the following:

2. *The category $\mathcal{F}$ is closed with respect to restrictions to open sets, that is if $f \in \mathcal{F}_{HK}$ and U, V are open sets in H, K resp. and finally $f(U) \subseteq V$, then $f|_U \in \mathcal{F}(U \prec H, V \prec K)$;*

3. *Every topological space $H \in \mathcal{F}$ has the following "sheaf property": let H, $K \in \mathcal{F}$ be two objects of $\mathcal{F}$, $(H_i)_{i\in I}$ an open cover of H and $f : |H| \longrightarrow |K|$ a map such that*

$$\forall i \in I : f|_{H_i} \in \mathcal{F}(H_i \prec H, K),$$

then $f \in \mathcal{F}_{HK}$.

For the construction of the domain of the extension functor $^\bullet(-)$: $\boldsymbol{\mathcal{C}}^\infty \longrightarrow {}^\bullet\boldsymbol{\mathcal{C}}^\infty$ we want to consider a category $\mathcal{F}$ which permits to embed finite dimensional manifolds in $\boldsymbol{\mathcal{C}}^n$. For this aim we will set $\mathcal{F} = \mathbf{O}\mathbb{R}^n$, the category having as objects open sets $U \subseteq \mathbb{R}^u$ (with the induced topology), for some $u \in \mathbb{N}$ depending on U, and with hom-set the usual space $\mathcal{C}^n(U,V)$ of $\mathcal{C}^n$ functions between the open sets $U \subseteq \mathbb{R}^u$ and $V \subseteq \mathbb{R}^v$. Thus, $\boldsymbol{\mathcal{C}}^n := \overline{\mathbf{O}\mathbb{R}^n}$, i.e. $\boldsymbol{\mathcal{C}}^n$ is the cartesian closure of the category $\mathbf{O}\mathbb{R}^n$.

In general, what type of category $\mathcal{F}$ we have to choose depends on the setting we need: e.g. in case we want to consider manifolds with boundary we have to take the analogous of the above mentioned category $\mathbf{O}\mathbb{R}^n$ but having as objects sets of type $U \subseteq \mathbb{R}^u_+ = \{x \in \mathbb{R}^u \,|\, x_u \geq 0\}$.

6.2 The cartesian closure and its first properties

The basic idea to define a $\boldsymbol{\mathcal{C}}^n$ space X (which faithfully generalizes the notion of manifold) is to substitute the notion of chart by a family of mappings $d : H \longrightarrow X$ with $H \in \mathcal{F}$. Indeed, for $\mathcal{F} = \mathbf{O}\mathbb{R}^n$ these mappings are of type $d : U \longrightarrow X$ with U open in some $\mathbb{R}^u$, thus they can be thought of as u-dimensional figures on X (see also Sections 5.4 and 5.3). Hence, a $\boldsymbol{\mathcal{C}}^n$ space can be thought as a support set together with the specification of all the finite-dimensional figures on the space itself. Generally speaking we can think of $\mathcal{F}$ as a category of *types of figures* (see Lawvere [1979] for this interpretation). Always considering the case $\mathcal{F} = \mathbf{O}\mathbb{R}^n$, we can also think $\mathcal{F}$ as a category which represents *a well known notion of regular space and regular function*: with the cartesian closure $\bar{\mathcal{F}}$ we want to extend this notion to a more general type of spaces (e.g. spaces of mappings). These are the ideas we have already seen in Section 5.4 in the case of diffeological spaces, only suitably generalized to a category of topological spaces $\mathcal{F}$ instead of $\mathcal{F} = \mathbf{O}\mathbb{R}^\infty$, which is the case of diffeology. This generalization permits to obtain in an easy way the cartesian closedness of $\bar{\mathcal{F}}$, and thus to have at our disposal a general instrument $\mathcal{F} \mapsto \bar{\mathcal{F}}$ very useful in the construction of the codomain of the extension functor $^\bullet(-)$, where we will choose a different category of types of figures $\mathcal{F}$.

Definition 6.2.1. *In the sequel we will frequently use the notation $f \cdot g := g \circ f$ for the composition of maps so as to facilitate the lecture of diagrams, but we will continue to evaluate functions "on the right" hence $(f \cdot g)(x) = g(f(x))$.*

Objects and arrows of $\bar{\mathcal{F}}$ generalize the same notions of the diffeological setting (see Section 5.4).

Definition 6.2.2. *If X is a set, then we say that $(\mathcal{D}, X)$ is an object of $\bar{\mathcal{F}}$ (or simply an $\bar{\mathcal{F}}$-object) if $\mathcal{D} = \{\mathcal{D}_H\}_{H \in \mathcal{F}}$ is a family with*

$$\mathcal{D}_H \subseteq \mathbf{Set}(|H|, X) \quad \forall H \in \mathcal{F}.$$

We indicate by the notation $\mathcal{F}_{JH} \cdot \mathcal{D}_H$ the set of all the compositions $f \cdot d$ of functions $f \in \mathcal{F}_{JH}$ and $d \in \mathcal{D}_H$. The family $\mathcal{D}$ has finally to satisfy the following conditions:

1. *$\mathcal{F}_{JH} \cdot \mathcal{D}_H \subseteq \mathcal{D}_J$.*
2. *$\mathcal{D}_H$ contains all the constant maps $d : |H| \longrightarrow X$.*
3. *Let $H \in \mathcal{F}$, $(H_i)_{i \in I}$ an open cover of H and $d : |H| \longrightarrow X$ a map such that $d|_{H_i} \in \mathcal{D}_{(H_i \prec H)}$, then $d \in \mathcal{D}_H$.*

Finally, we set $|(\mathcal{D}, X)| := X$ to denote the underlying set of the space $(\mathcal{D}, X)$.

Because of condition *1.* we can think of $\mathcal{D}_H$ as the set of *all* the regular functions defined on the "well known" object $H \in \mathcal{F}$ and with values in the new space X; in fact this condition says that the set of figures $\mathcal{D}_H$ is closed with respect to re-parametrizations with a generic $f \in \mathcal{F}_{JH}$. Condition 3. is the above mentioned sheaf property and asserts that the property of being a figure $d \in \mathcal{D}_H$ has a local character depending on $\mathcal{F}$.

We will frequently write $d \in_H X$ to indicate that $d \in \mathcal{D}_H$ and we can read it[1] saying that *d is a figure of X of type H* or *d belong to X at the level H* or *d is a generalized element of X of type H*.

The definition of arrow $f : X \longrightarrow Y$ (also called *smooth function in $\bar{\mathcal{F}}$*) between two spaces $X, Y \in \bar{\mathcal{F}}$ is the usual one for diffeological spaces, that is f takes, through composition, generalized elements $d \in_H X$ of type H in the domain X to generalized elements of the same type in the codomain Y

Definition 6.2.3. *Let X, Y be $\bar{\mathcal{F}}$-objects, then we will write*

$$f : X \longrightarrow Y$$

or, more precisely if needed[2]

$$\bar{\mathcal{F}} \vDash f : X \longrightarrow Y$$

[1] The following are common terminologies used in topos theory, see Lawvere [1979], Kock [1981], Moerdijk and Reyes [1991]

[2] We shall frequently use notations of type $\mathbb{C} \vDash f : A \longrightarrow B$ if we need to specify better the category $\mathbb{C}$ we are considering (see Appendix A).

iff f maps the support set of X into the support set of Y:

$$f : |X| \longrightarrow |Y|$$

and

$$d \cdot f \in_{H} Y$$

for every type of figure $H \in \mathcal{F}$ and for every figure d of X of that type, i.e. $d \in_{H} X$. In this case, we will also use the notation $f(d) := d \cdot f$.

Note that we have $f : X \longrightarrow Y$ in $\bar{\mathcal{F}}$ iff

$$\forall H \in \mathcal{F}\ \forall x \in_{H} X : \quad f(x) \in_{H} Y,$$

moreover $X = Y$ iff

$$\forall H \in \mathcal{F}\ \forall d : \quad d \in_{H} X \iff d \in_{H} Y.$$

These and many other properties justify the notation $\in_{H}$ and the name "generalized elements".

With these definitions $\bar{\mathcal{F}}$ becomes a category. Note that it is, in general, in the second Grothendieck universe (see Artin et al. [1972], Adamek et al. [1990]) because $\mathcal{D}$ is a family indexed in the set of objects of $\mathcal{F}$ (this is not the case for $\mathcal{F} = \mathbf{O}\mathbb{R}^n$, which is a set and not a class).
The simplest $\bar{\mathcal{F}}$-object is $\bar{K} := (\mathcal{F}_{(-)K}, |K|)$ for $K \in \mathcal{F}$, where we recall that $\mathcal{F}_{HK} = \mathcal{F}(H, K) = \left\{ f \,|\, H \xrightarrow{f} K \text{ in } \mathcal{F} \right\}$. For the space $\bar{K} \in \bar{\mathcal{F}}$ we have that

$$\bar{\mathcal{F}} \vDash \bar{K} \xrightarrow{d} X \qquad \iff \qquad d \in_{K} X.$$

Moreover, $\mathcal{F}(H, K) = \bar{\mathcal{F}}(\bar{H}, \bar{K})$. Therefore $\mathcal{F}$ is fully embedded in $\bar{\mathcal{F}}$ if $\bar{H} = \bar{K}$ implies $H = K$; e.g. this is true if the given category $\mathcal{F}$ verifies the following hypothesis

$$|H| = |K| = S \ \text{ and } \ H \xrightarrow{1_S} K \xrightarrow{1_S} H \quad \Longrightarrow \quad H = K.$$

E.g. this is true for $\mathcal{F} = \mathbf{O}\mathbb{R}^n$.

Moreover, let us note that the composition of two smooth functions in $\bar{\mathcal{F}}$ of type $d : \bar{H} \longrightarrow X$ and $f : X \longrightarrow \bar{K}$ for $H, K \in \mathcal{F}$, gives $d \cdot f \in \bar{\mathcal{F}}(\bar{H}, \bar{K}) = \mathcal{F}(H, K)$, which is an arrow in the old category of types of figures $\mathcal{F}$.

Another way to construct an object of $\bar{\mathcal{F}}$ on a given support set X is to generate it starting from a given family $\mathcal{D}^0 = (\mathcal{D}^0_H)_H$, with $\mathcal{D}^0_H \subseteq \mathbf{Set}(|H|, X)$ for any $H \in \mathcal{F}$, closed with respect to constant functions, i.e. such that

$$\forall H \in \mathcal{F}\ \forall d : |H| \longrightarrow X \text{ is constant} \quad \Longrightarrow \quad d \in \mathcal{D}^0_H.$$

We will indicate this space by $(\mathcal{F} \cdot \mathcal{D}^0, X)$. Its figures are, locally, compositions $f \cdot d$ with $f \in \mathcal{F}_{HK}$ and $d \in \mathcal{D}^0_K$. More precisely $\delta \in_{H} (\mathcal{F} \cdot \mathcal{D}^0, X)$ iff

$\delta : |H| \longrightarrow X$ and for every $h \in |H|$ there exist an open neighborhood U of h in H, a space $K \in \mathcal{F}$, a figure $d \in \mathcal{D}^0_K$ and $f : (U \prec H) \longrightarrow K$ in $\mathcal{F}$ such that $\delta|_U = f \cdot d$. Diagrammatically we have:

$$\begin{tikzcd} H \arrow[r, "\delta"] & X \\ h \in U \arrow[u, hook] \arrow[ur, "\delta|_U"] \arrow[r, "f"'] & K \arrow[u, "d"'] \end{tikzcd}$$

On each space $X \in \bar{\mathcal{F}}$ we can put the final topology $\mathcal{T}_X$ for which any figure $d \in_H X$ is continuous, that is

Definition 6.2.4. *If $X \in \bar{\mathcal{F}}$, then we say that a subset $U \subseteq |X|$ is* open *in X, and we will write $U \in \mathcal{T}_X$ iff $d^{-1}(U) \in \mathcal{T}_H$ for any $H \in \mathcal{F}$ and any $d \in_H X$.*

With respect to this topology any arrow of $\bar{\mathcal{F}}$ is continuous and we still have the initial $\mathcal{T}_H$ in the space $\bar{H}$, that is $\mathcal{T}_H = \mathcal{T}_{\bar{H}}$ (recall that, because of the fundamental hypotheses 6.1.1, every type of figure $H \in \mathcal{F}$ is a topological space).

Recalling that in the case $\mathcal{F} = \mathbf{O}\mathbb{R}^\infty$ we obtain that the cartesian closure $\bar{\mathcal{F}}$ is the category of diffeological spaces, it can be useful to cite here Iglesias-Zemmour [2008]:

> Even if diffeology is a theory which avoids topology on purpose, topology is not completely absent from its content. But, in contrary to some approach of standard differential geometry, here the topology is a byproduct of the main structure, that is diffeology. Locality, through local smooth maps, or local diffeomorphisms, is introduced without referring to any topology a priori but will suggest the definition of a topology a posteriori [i.e. $\mathcal{T}_X$].

Ultimately, this choice is due to the necessity to obtain a cartesian closed category. In fact, if we do not start from a primitive notion of topology in the definition of $\bar{\mathcal{F}}$-space, we can obtain cartesian closedness without having the problem to define a topology in the set of maps $\bar{\mathcal{F}}(X, Y)$. Indeed, this is not an easy problem, and classical solutions like the compact-open topology (see e.g. Dugundji [1966], Kriegl and Michor [1997] and references therein) is not applicable to the smooth case. In fact, the compact-open topology, which essentially coincides with the topology of uniform convergence, is well suited for continuous maps $f : X \longrightarrow Y$ between locally compact Haussdorff topological spaces X and Y (indeed, the category of these topological spaces is cartesian closed, see Mac Lane [1971]). It can be generalized to the case of $\mathcal{C}^k$-regularity using k-jets ($k \in \mathbb{N}_{>0}$), i.e. using Taylor's formulae up to k-th order (see e.g. Kriegl and Michor [1997]), but a generalization including the smooth case $\mathcal{C}^\infty$ even for a compact domain X fails. In fact, for X compact and Y a Banach space, the space $\mathcal{C}^k(X,Y)$ with the $\mathcal{C}^k$ compact-open topology is normable, but the space $\mathcal{C}^\infty(\mathbb{R},\mathbb{R})$ is not normable, so its topology cannot be the compact-open one (see also Section 5.2 for more details).

The study of the relationships between different topologies on the space of maps $\mathcal{C}^\infty(M,N)$ for M, N manifolds, is not completely solved (see again Kriegl and Michor [1997] for some results in this direction).

6.3 Categorical properties of the cartesian closure

We shall now examine subobjects in $\bar{\mathcal{F}}$ and their relationships with restrictions of functions; after this we will analyze completeness, co-completeness and cartesian closure of $\bar{\mathcal{F}}$.

Definition 6.3.1. *Let $X \in \bar{\mathcal{F}}$ be a space in the cartesian closure of $\mathcal{F}$, and $S \subseteq |X|$ a subset, then we define*

$$(S \prec X) := (\mathcal{D}, S)$$

where, for every type of figure $H \in \mathcal{F}$, we have set

$$d \in \mathcal{D}_H \quad :\Longleftrightarrow \quad d : |H| \longrightarrow S \quad \text{and} \quad d \cdot i \in_H X.$$

Here $i : S \hookrightarrow |X|$ is the inclusion map. In other words, we have a figure d of type H in the subspace S iff composing d with the inclusion map i we obtain a figure of the same type in the superspace X. We will call $(S \prec X)$ the subspace induced on S by X.

Using this definition only it is very easy to prove that $(S \prec X) \in \bar{\mathcal{F}}$ and that its topology $\mathcal{T}_{(S \prec X)}$ contains the induced topology by $\mathcal{T}_X$ on the subset S. Moreover we have that $\mathcal{T}_{(S \prec X)} \subseteq \mathcal{T}_X$ if S is open in X, hence in this case we have on $(S \prec X)$ exactly the induced topology.

Finally we can prove that these subspaces have good relationships with restrictions of maps:

Theorem 6.3.2. *Let $f : X \longrightarrow Y$ be an arrow of $\bar{\mathcal{F}}$ and U, V be subsets of $|X|$ and $|Y|$ respectively, such that $f(U) \subseteq V$, then*

$$(U \prec X) \xrightarrow{f|_U} (V \prec Y) \quad \text{in} \quad \bar{\mathcal{F}}.$$

∎

Using our notation for subobjects we can prove the following useful and natural properties directly from definition 6.3.1.

- $(U \prec \bar{H}) = \overline{(U \prec H)}$ for U open in $H \in \mathcal{F}$ (recall the definition of $\bar{H} \in \bar{\mathcal{F}}$, for $H \in \mathcal{F}$, given in Section 6.2 and also recall that, because of the Hypotheses 6.1.1 the subspace $(U \prec H)$ is a type of figure, i.e. $(U \prec H) \in \mathcal{F}$, and we can thus apply the operator $(\bar{-}) : \mathcal{F} \longrightarrow \bar{\mathcal{F}}$ of inclusion of the types of figures $\mathcal{F}$ into the cartesian closure $\bar{\mathcal{F}}$).
- $i : (S \prec X) \hookrightarrow X$ is the lifting[3] of the inclusion $i : S \hookrightarrow |X|$ from **Set** to $\bar{\mathcal{F}}$
- $(|X| \prec X) = X$
- $(S \prec (T \prec X)) = (S \prec X) \quad \text{if} \quad S \subseteq T \subseteq |X|$
- $(S \prec X) \times (T \prec Y) = (S \times T \prec X \times Y)$.

These properties imply that the relation $X \subseteq Y$ iff $|X| \subseteq |Y|$ and $(|X| \prec Y) = X$ is a partial order. Note that this relation is stronger than saying that the inclusion is an arrow, because it asserts that X and the inclusion verify the universal property of $(|X| \prec Y)$, that is X is a subobject of Y. A trivial but useful property of this subobjects notation is the following

Corollary 6.3.3. *Let $S \subseteq |X'|$ and $X' \subseteq X$ in $\bar{\mathcal{F}}$, then*

$$(S \prec X') = (S \prec X),$$

that is in the operator $(S \prec -)$ we can change the superspace X with any one of its subspaces $X' \subseteq X$ containing S.

Proof: In fact $X' \subseteq X$ means $X' = (|X'| \prec X)$ and hence $(S \prec X') = (S \prec (|X'| \prec X)) = (S \prec X)$ because of the previous properties of the operator $(- \prec -)$.

∎

An expected property that transfers from $\mathcal{F}$ to $\bar{\mathcal{F}}$ is the sheaf property; in other words it states that the property of being a smooth arrow of the cartesian closure $\bar{\mathcal{F}}$ is a local property.

Theorem 6.3.4. *Let $X, Y \in \bar{\mathcal{F}}$ be spaces in the cartesian closure, $(U_i)_{i \in I}$ an open cover of X and $f : |X| \longrightarrow |Y|$ a map from the support set of X to that of Y such that*

$$\bar{\mathcal{F}} \vDash (U_i \prec X) \xrightarrow{f|_{U_i}} Y \quad \forall i \in I.$$

Then

$$\bar{\mathcal{F}} \vDash X \xrightarrow{f} Y.$$

∎

[3] For the notion of *lifting* and *co-lifting* see Definition A.3.3

Completeness and co-completeness are analyzed in the following theorem. For its standard proof see e.g. Frölicher and Kriegl [1988] for a similar theorem.

Theorem 6.3.5. *Let $(X_i)_{i\in I}$ be a family of objects in $\bar{\mathcal{F}}$ and $p_i : |X| \longrightarrow |X_i|$ maps for every $i \in I$. Let us define*

$$d \in_H X \quad :\Longleftrightarrow \quad d : |H| \longrightarrow |X| \quad \text{and} \quad \forall\, i \in I : \quad d \cdot p_i \in_H X_i$$

then $(X \xrightarrow{p_i} X_i)_{i\in I}$ is a lifting of $(|X| \xrightarrow{p_i} |X_i|)_{i\in I}$ in $\bar{\mathcal{F}}$.
Moreover, let $j_i : |X_i| \longrightarrow |X|$ be maps for every $i \in I$, and let us suppose that

$$\forall\, x \in |X|\ \exists\, i \in I\ \exists\, x_i \in X_i : \quad x = j_i(x_i).$$

Let us define $d \in_H X$ iff $d : |H| \longrightarrow |X|$ and for every $h \in |H|$ there exist an open neighborhood U of h in H, an index $i \in I$ and a figure $\delta \in_U X_i$ such that $d|_U = \delta \cdot j_i$; then we have that $(X_i \xrightarrow{j_i} X)_{i\in I}$ is a co-lifting of $(|X_i| \xrightarrow{j_i} |X|)_{i\in I}$ in $\bar{\mathcal{F}}$. ■

The category of $\bar{\mathcal{F}}$ spaces is thus complete and co-complete and we can hence consider spaces like quotient spaces $X/\sim$, disjoint sums $\sum_{i\in I} X_i$, arbitrary products $\prod_{i\in I} X_i$, equalizers, etc. (see Theorem A.3.4 for further details about the connections between limits, co-limits, lifting and co-lifting).

Directly from the definitions of lifting and co-lifting, it is easy to prove that on quotient spaces we exactly have the quotient topology and that on any product we have a topology stronger than the product topology. We can write this assertion in the following symbolic way:

$$\mathcal{T}_{X/\sim} = \mathcal{T}_X/\sim \tag{6.3.1}$$

$$\mathcal{T}_X \times \mathcal{T}_Y \subseteq \mathcal{T}_{X\times Y}, \tag{6.3.2}$$

where: X and Y are $\bar{\mathcal{F}}$ spaces, $\sim$ is an equivalence relation on $|X|$, $(X/\sim) \in \bar{\mathcal{F}}$ is the quotient space, $\mathcal{T}_X/\sim$ is the quotient topology, and $\mathcal{T}_X \times \mathcal{T}_Y$ is the product topology. Analogously, let $j_i : X_i \longrightarrow \sum_{i\in I} X_i$ be the canonical injections in the disjoint sum of the family of $\bar{\mathcal{F}}$-spaces $(X_i)_{i\in I}$, i.e. $j_i(x) = (x, i)$. Then we can prove that A is open in $\sum_{i\in I} X_i$ if and only if

$$\forall i \in I : \quad j_i^{-1}(A) \in \mathcal{T}_{X_i}, \tag{6.3.3}$$

that is on the disjoint sum we have exactly the colimit topology. Because any colimit can be obtain as a lifting from **Set** of quotient spaces and disjoint sums (see Mac Lane [1971]), we have the general result that the topology on the colimit of $\bar{\mathcal{F}}$-spaces is exactly the colimit topology. In symbolic notations we can write

$$\mathcal{T}\left(\operatorname*{colim}_{i\in I} X_i\right) = \operatorname*{colim}_{i\in I} \mathcal{T}_{X_i}.$$

Finally if we define

$$\mathcal{D}_H := \{d : |H| \longrightarrow \bar{\mathcal{F}}(X,Y) \mid \bar{H} \times X \xrightarrow{d^\vee} Y \quad \text{in} \quad \bar{\mathcal{F}}\} \quad \forall H \in \mathcal{F}$$

(we recall that we use the notations $d^\vee(h,x) := d(h)(x)$ and $\mu^\wedge(x)(y) := \mu(x,y)$, see Section 5.1) then $\langle \mathcal{D}, \bar{\mathcal{F}}(X,Y) \rangle =: Y^X$ is an object of $\bar{\mathcal{F}}$. With this definition, see e.g. Chen [1982] or Frölicher and Kriegl [1988], it is easy to prove that $\bar{\mathcal{F}}$ is cartesian closed, i.e. that the $\bar{\mathcal{F}}$-isomorphism $(-)^\vee$ realizes

$$(Y^X)^Z \simeq Y^{Z \times X}.$$

Chapter 7

The category $\mathcal{C}^n$

7.1 Observables on $\mathcal{C}^n$ spaces and separated spaces

If our aim is to embed the category of $\mathcal{C}^n$ manifolds into a cartesian closed category, the most natural way to apply the results of the previous Chapter 6 is to take as category $\mathcal{F}$ of types of figures $\mathcal{F} = \mathbf{Man}^n$, that is to consider directly the cartesian closure of the category of finite dimensional $\mathcal{C}^n$ manifolds[1]. We shall not follow this idea for several reasons; as we have already mentioned, we will consider instead $\mathcal{C}^n := \overline{\mathbf{O}\mathbb{R}^n}$, that is the cartesian closure of the category $\mathbf{O}\mathbb{R}^n$ of open sets and $\mathcal{C}^n$ maps. For $n = \infty$ this gives exactly diffeological spaces. Indeed, as we noted in the previous Chapter 6, $\overline{\mathbf{Man}^n}$ is in the second Grothendieck universe and, essentially for simplicity, from this point of view the choice $\mathcal{F} = \mathbf{O}\mathbb{R}^n$ is better. In spite of this choice, it is natural to expect, and in fact we will prove it, that the categories of both finite and infinite-dimensional manifolds are faithfully embedded in the previous $\mathcal{C}^n = \overline{\mathbf{O}\mathbb{R}^n}$. Another reason to choose our definition of $\mathcal{C}^n$ is that in this way the category $\mathcal{C}^n$ is more natural to accept against $\overline{\mathbf{Man}^n}$; hence, ones again we are opting for a reason of simplicity. We will see that manifolds modelled in convenient vector spaces (see Chapter 5) are faithfully embedded in $\mathcal{C}^n$, hence our choice to take finite dimensional objects in the definition of $\mathcal{C}^n = \overline{\mathbf{O}\mathbb{R}^n}$ is not restrictive from this point of view.

Now we pay attention to another type of maps which go "in the opposite direction" with respect to figures $d : K \longrightarrow X$. They are important also because we shall use them to introduce new infinitesimal points for any $X \in \mathcal{C}^n$. We will introduce these notions for a generic cartesian closure $\bar{\mathcal{F}}$ of a given category if figures $\mathcal{F}$, because we will use them e.g. also in the category ${}^\bullet\mathcal{C}^n$ of extended spaces. So, in the following $\bar{\mathcal{F}}$ will be a category of figures (see Hypothesis 6.1.1).

[1] We shall not formally assume any hypothesis on the topology of a manifold because we will never need it in the following; moreover if not differently specified, with the word "manifold" we will always mean "finite dimensional manifold".

Definition 7.1.1. *Let X be an $\bar{\mathcal{F}}$ space, then we say that*

$$UK \text{ is a zone (in } X)$$

iff $U \in \mathcal{T}_X$, i.e. U is open in X, and $K \in \mathcal{F}$ is a type of figure. Moreover we say that

$$c \text{ is an observable on } UK \quad \textit{and we will write} \quad c \in^{UK} X$$

iff $c : (U \prec X) \longrightarrow \bar{K}$ is a map of the cartesian closure $\bar{\mathcal{F}}$.

So, observables of a $\mathcal{C}^n$ space X are simply maps of class $\mathcal{C}^n$ (i.e. are arrows of this category) defined on an open set of X and with values in an open set $K \subseteq \mathbb{R}^d$ for some $d \in \mathbb{N}$. Recall (see Section 6.2) that for any open set $K \in \mathbf{O}\mathbb{R}^n$ in the $\mathcal{C}^n$ space $\bar{K}$ we take as figures of type $H \in \mathbf{O}\mathbb{R}^n$ all the ordinary $\mathcal{C}^n$-maps $\mathcal{C}^n(H, K)$, i.e. we have

$$\bar{K} = (\mathcal{C}^n(-, K), K).$$

Therefore, the composition of figures $d \in_H X$ with observables $c \in^{UK} X$ gives ordinary $\mathcal{C}^n$ maps:

$$d|_S \cdot c \in \mathcal{C}^n(S, K), \quad \text{where} \quad S := d^{-1}(U),$$

$$\mathcal{C}^n \vDash \overline{(S \prec H)} \xrightarrow{d|_S} (U \prec X) \xrightarrow{c} \bar{K}.$$

From our previous theorems of Chapter 6, it follows that $\mathcal{C}^n$ functions $f : X \longrightarrow Y$ take observables on the codomain to observables on the domain i.e.:

$$c \in^{UK} Y \quad \Longrightarrow \quad f|_S \cdot c \in^{SK} X, \tag{7.1.1}$$

where $S := f^{-1}(U)$:

$$\begin{tikzcd}
(S \prec X) \arrow[r, "f|_S"] \arrow[dr, "f|_S \cdot c"'] & (U \prec Y) \arrow[d, "c"] \\
& \bar{K}
\end{tikzcd}$$

Therefore isomorphic $\mathcal{C}^n$ spaces have isomorphic sets of figures and observables and the isomorphisms are given by suitable simple compositions.

Generalizing, through observables, the equivalence relation of Definition 2.3.1 to generic $\mathcal{C}^n$ spaces, we will have to study the following condition, which is also connected with the faithfulness of the extension functor.

Definition 7.1.2. *If $X \in \mathcal{C}^n$ is a $\mathcal{C}^n$ space and x, $y \in |X|$ are two points, then we write*

$$x \asymp y$$

iff for every zone UK and every observable $c \in^{UK} X$ we have

1. $x \in U \iff y \in U$
2. $x \in U \quad \Longrightarrow \quad c(x) = c(y)$.

In this case we will read the relation $x \asymp y$ saying x and *y* are identified *in X. Moreover we say that X* is separated *iff $x \asymp y$ implies $x = y$ for any x, $y \in |X|$.*

We point out that if two points are identified in X, then a generic open set $U \in \mathcal{T}_X$ contains the first if and only if it contains the second too (take a constant observable $c : U \longrightarrow \mathbb{R}$). Furthermore, from (7.1.1) it follows that $\mathcal{C}^n$ functions $f : X \longrightarrow Y$ preserve the relation $\asymp$:

$$x \asymp y \text{ in } X \quad \Longrightarrow \quad f(x) \asymp f(y) \text{ in } Y \qquad \forall x, y \in |X|.$$

Trivial examples of separated spaces can be obtained considering the objects $\bar{U} \in \mathcal{C}^n$ with $U \in \mathbf{O}\mathbb{R}^n$ (here $\overline{(-)} : \mathbf{O}\mathbb{R}^n \longrightarrow \mathcal{C}^n$ is the embedding of the types of figures $\mathbf{O}\mathbb{R}^n$ into $\mathcal{C}^n$, see 6.2) or taking subobjects of separated spaces. But the full subcategory of separated $\mathcal{C}^n$ spaces has sufficiently good properties, as proved in the following

Theorem 7.1.3. *The category of separated $\mathcal{C}^n$ spaces is complete and admits co-products. Moreover if X, Y are separated then Y^X is separated too, and hence separated spaces form a cartesian closed category.*

Sketch of the proof: We only do some considerations about co-products, because from the definition of lifting (see Theorem 6.3.5) it can be directly proved that products and equalizers of separated spaces are separated too. Let us consider a family $(\mathcal{X}_i)_{i \in I}$ of separated spaces with support sets $X_i := |\mathcal{X}_i|$. Constructing their sum in **Set**

$$X := \sum_{i \in I} X_i$$

$$j_i \ : \ x \in X_i \longmapsto (x, i) \in X,$$

from the completeness of $\mathcal{C}^n$ we can lift this co-product of sets into a co-product $(\mathcal{X}_i \xrightarrow{j_i} \mathcal{X})_{i \in I}$ in $\mathcal{C}^n$. To prove that $\mathcal{X}$ is separated we take two points x, $y \in X = |\mathcal{X}|$ identified in $\mathcal{X}$. These points are of the form $x = (x_r, r)$ and $y = (y_s, s)$, with $x_r \in X_r, y_s \in X_s$ and $r, s \in I$. We want to prove that r and s are necessarily equal. In fact, from (6.3.3), for a generic $A \subseteq \mathcal{X}$ we have that

$$A \in \mathcal{T}_{\mathcal{X}} \quad \iff \quad \forall\, i \in I : \ \ j_i^{-1}(A) \in \mathcal{T}_{\mathcal{X}_i}.$$

and hence $X_r \times \{r\}$ is open in $\mathcal{X}$ and $x = (x_r, r) \asymp y = (y_s, s)$ implies

$$(x_r, r) \in X_r \times \{r\} \iff (y_s, s) \in X_r \times \{r\} \qquad \text{hence} \qquad r = s.$$

Thus $x = y$ iff x_r and $y_s = y_r$ are identified in $\mathcal{X}_r$ and this is a consequence of the following facts:

1. if U is open in $\mathcal{X}_r$ then $U \times \{r\}$ is open in $\mathcal{X}$;
2. if $c \in^{UK} \mathcal{X}_r$, then $\gamma(x, r) := c(x)\ \forall\, x \in U$ is an observable of $\mathcal{X}$ defined on $U \times \{r\}$.

Now let us consider exponential objects. If $f,\ g \in |Y^X|$ are identified, to prove that they are equal is equivalent to prove that $f(x)$ and $g(x)$ are identified in Y for any x. To obtain this conclusion is sufficient to consider that the evaluation in x i.e. the application $\varepsilon_x : \varphi \in |Y^X| \longmapsto \varphi(x) \in |Y|$ is a $\mathcal{C}^n$ map and hence from any observable $c \in^{UK} Y$ we can always obtain the observable $\varepsilon_x|_{U'} \cdot c \in^{U'K} Y^X$ where $U' := \varepsilon_x^{-1}(U)$.

■

Finally let us consider two $\mathcal{C}^n$ spaces such that the topology $\mathcal{T}_{X\times Y}$ is equal to the product of the topologies $\mathcal{T}_X$ and $\mathcal{T}_Y$ (recall that in general we have $\mathcal{T}_X \times \mathcal{T}_Y \subseteq \mathcal{T}_{X\times Y}$). Then if $x,\ x' \in |X|$ and $y,\ y' \in |Y|$ directly from the definition it is possible to prove that $x \asymp x'$ in X and $y \asymp y'$ in Y if and only if $(x, y) \asymp (x', y')$ in $X \times Y$.

7.2 Manifolds as objects of $\mathcal{C}^n$

We can associate in a very natural way a $\mathcal{C}^n$ space $\bar{M}$ to any manifold $M \in$ **Man**^{n} (the category of $\mathcal{C}^n$ manifolds and $\mathcal{C}^n$ functions) with the following

Definition 7.2.1. *The underlying set of $\bar{M}$ is the underlying set of the manifold, i.e. $|\bar{M}| := |M|$, and for every $H \in \mathbf{O}\mathbb{R}^n$ the figures $d : H \longrightarrow M$ of type H are all the ordinary $\mathcal{C}^n$ maps from H to the manifold M, i.e.*

$$d \in_H \bar{M} \qquad :\iff \qquad d \in \mathbf{Man}^n(H, M).$$

This definition is only the trivial generalization from the smooth case to $\mathcal{C}^n$ of the embedding of manifolds into the category of diffeological spaces (see e.g. Iglesias-Zemmour [2008]).
With $\bar{M}$ we obtain a $\mathcal{C}^n$ space with the same topology of the starting manifold. Moreover the observables of $\bar{M}$ are the most natural ones we could expect. In fact, as a consequence of the Definition 7.2.1 it follows that

$$c \in^{UK} \bar{M} \qquad \iff \qquad c \in \mathbf{Man}^n(U, K). \tag{7.2.1}$$

Hence it is clear that the space $\bar{M}$ is separated, because from (7.2.1) we get that charts are observables of the space. The following theorem says that the application $M \mapsto \bar{M}$ from $\mathbf{Man}^n$ to $\mathcal{C}^n$ we are considering is a full embedding, and therefore it also says that the notion of $\mathcal{C}^n$-space is a non-trivial generalization of the notion of manifold which includes infinite-dimensional spaces too.

Theorem 7.2.2. *Let M and N be $\mathcal{C}^n$ manifolds, then*

1. $\bar{M} = \bar{N} \quad \Longrightarrow \quad M = N$
2. $\mathcal{C}^n \vDash \bar{M} \xrightarrow{\ f\ } \bar{N} \quad \Longleftrightarrow \quad \mathbf{Man}^n \vDash M \xrightarrow{\ f\ } N.$

Hence $\mathbf{Man}^n$ *is fully embedded in* $\mathcal{C}^n$.

Proof:
1) If (U, φ) is a chart on M and $A := \varphi(U)$, then $\varphi^{-1}|_A : A \longrightarrow M$ is a figure of $\bar{M}$, that is $\varphi^{-1}|_A \in_A \bar{M} = \bar{N}$. But if $\psi : U \longrightarrow \psi(U) \subseteq \mathbb{R}^k$ is a chart of N, then it is also an observable of $\bar{N}$. We have hence obtained a figure $\varphi^{-1}|_A \in_A \bar{N}$ and an observable $\psi \in^{U\psi(U)} \bar{N}$ of the space $\bar{N}$. But composition of figures and observables gives ordinary $\mathcal{C}^n$ maps, that is the atlases of M and N are compatible.

2) For the implication $\Rightarrow$ we use the same ideas as above and furthermore that $\varphi^{-1}|_A \in_A \bar{M}$ implies $\varphi^{-1}|_A \cdot f \in_A \bar{N}$. Finally we can compose this A-figure of $\bar{N}$ with a chart (observable) of N obtaining an ordinary $\mathcal{C}^n$ map. The implication $\Leftarrow$ follows directly from the Definition 7.2.1.

■

Directly from these definitions we can prove that for two manifolds we also have

$$\overline{M \times N} = \bar{M} \times \bar{N}.$$

This property is useful to prove the properties stated in the following examples.

7.3 Examples of $\mathcal{C}^n$ spaces and functions

1. Let M be a $\mathcal{C}^\infty$ manifold modelled on convenient vector spaces (see Section 5.3). We can define $\bar{M}$ analogously as above, saying that $d \in_H \bar{M}$ iff $d : H \longrightarrow M$ is a smooth map from H (open in some $\mathbb{R}^h$) to the manifold M. In this way smooth curves on M are exactly the figures $c \in_{\mathbb{R}} \bar{M}$ of type $\mathbb{R}$ in $\bar{M}$. On M we obviously think of the natural topology, that is the identification topology with respect to some smooth atlas, which is also the final topology with respect to all smooth curves and hence is also the final topology $\tau_{\bar{M}}$ with respect to all figures of $\bar{M}$. More easily with respect to the previous case of finite dimensional manifolds (due to the results available for manifolds modelled on convenient vector spaces, see Section 5.3), it is possible to

study observables, obtaining that $c \in^{UK} \bar{M}$ if and only if $c : U \longrightarrow K$ is smooth as a map between manifolds modelled on convenient vector spaces. Moreover if (U, φ) is a chart of M on the convenient vector space E, then $\varphi : (U \prec \bar{M}) \longrightarrow (\varphi(U) \prec \bar{E})$ is $\boldsymbol{\mathcal{C}}^\infty$. Using these results it is easy to prove the analogous of Theorem 7.2.2 for the category of manifolds modelled on convenient vector spaces. Hence also classical smooth manifolds modelled on Banach spaces are embedded in $\boldsymbol{\mathcal{C}}^\infty$.

2. It is not difficult to prove that the following applications, frequently used e.g. in calculus of variations, are smooth, that is they are arrows of $\boldsymbol{\mathcal{C}}^\infty$.

   (a) The operator of derivation:

   $$\partial_i : u \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^k) \longmapsto \frac{\partial u}{\partial x_i} \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^k)$$

   To prove that this operator is smooth, i.e. it is an arrow of the category $\boldsymbol{\mathcal{C}}^\infty$, we have to show that it takes figures of type $H \in \mathbf{O}\mathbb{R}^\infty$ on its domain to figures of the same type on the codomain. Figures of type H of the space $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^k)$ are maps of type $d : H \longrightarrow \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^k)$, so that we have to consider the composition $d \cdot \partial_i$. Using cartesian closedness we get that $d^\vee : H \times \mathbb{R}^n \longrightarrow \mathbb{R}^k$ is an ordinary smooth map. But, always due to cartesian closedeness, the composition $d \cdot \partial_i : H \longrightarrow \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^k)$ is a figure if and only if its adjoint $(d \cdot \partial_i)^\vee : H \times \mathbb{R}^n \longrightarrow \mathbb{R}^k$ is an ordinary smooth map, and by a direct calculation we get that $(d \cdot \partial_i)^\vee = \partial_{u+i} d^\vee$, where $u \in \mathbb{N}$ is the dimension of $H \subseteq \mathbb{R}^u$. In fact

   $$\begin{aligned}
   (d \cdot \partial_i)^\vee (h, r) = \partial_i(d(h))(r) &= \frac{\partial d(h)}{\partial x_i}(r) = \\
   &= \lim_{\delta \to 0} \frac{d(h)(r + \delta \vec{e_i}) - d(h)(r)}{\delta} = \\
   &= \lim_{\delta \to 0} \frac{d^\vee(h, r + \delta \vec{e_i}) - d^\vee(h, r)}{\delta} = \\
   &= \partial_{u+i} d^\vee(h, r)
   \end{aligned}$$

   where $\vec{e_i} = (0, \overset{i-1}{\ldots\ldots}, 0, 1, 0, \ldots, 0) \in \mathbb{R}^n$. This equality proves that $d \cdot \partial_i$ is a figure and hence that the operator ∂_i is smooth.

   (b) We can proceed in an analogous way (but here we have to use the derivation under the integral sign) to prove that the integral operator:

   $$\begin{aligned}
   i : \; \mathcal{C}^\infty(\mathbb{R}^2, \mathbb{R}) &\longrightarrow \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}) \\
   u &\longmapsto \int_a^b u(-, s)\, \mathrm{d}s
   \end{aligned}$$

is smooth.

3. Because of cartesian closedness set-theoretical operations like the following are examples of $\mathcal{C}^n$ arrows (see e.g. Adamek et al. [1990]):

    - composition:
    $$(f, g) \in B^A \times C^B \quad \mapsto \quad g \circ f \in C^A$$
    - evaluation:
    $$(f, x) \in Y^X \times X \quad \mapsto \quad f(x) \in Y$$
    - insertion:
    $$x \in X \quad \mapsto \quad (x, -) \in (X \times Y)^Y$$

4. Using the smoothness of the previous set-theoretical operations and the smoothness of the derivation and integral operators, we can easily prove that the classical operator of the calculus of variations is smooth
$$\mathcal{I}(u)(t) := \int_a^b F[u(t,s), \partial_2 u(t,s), s]\,\mathrm{d}s$$
$$\mathcal{I} : \mathcal{C}^\infty(\mathbb{R}^2, \mathbb{R}^k) \longrightarrow \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}),$$
where the function $F : \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R} \longrightarrow \mathbb{R}$ is smooth.

5. Inversion between smooth manifolds modelled on Banach spaces
$$(-)^{-1} : f \in \mathrm{Diff}(N, M) \quad \mapsto \quad f^{-1} \in \mathrm{Diff}(M, N)$$
is a smooth mapping, where $\mathrm{Diff}(M, N)$ is the subspace of $N^M = \boldsymbol{\mathcal{C}}^\infty(\bar{M}, \bar{N})$ given by the diffeomorphisms between M and N.
So $(\mathrm{Diff}(M, M), \circ)$ is a (generalized) Lie group. To prove that $(-)^{-1}$ is smooth let us consider a figure $d \in_U \mathrm{Diff}(N, M)$, then, using cartesian closedness, the map $f := (d \cdot i)^\vee : U \times N \longrightarrow M$, where $i : \mathrm{Diff}(N, M) \hookrightarrow M^N$ is the inclusion, is an ordinary smooth function between Banach manifolds. We have to prove that $g := [d \cdot (-)^{-1} \cdot j]^\vee : U \times M \longrightarrow N$ is smooth, where $j : \mathrm{Diff}(M, N) \hookrightarrow N^M$ is the inclusion. But $f[u, g(u, m)] = m$ and $\mathbf{D}_2 f(u, n) = \mathbf{D}[d(u)](n)$ hence the conclusion follows from the implicit function theorem because $d(u) \in \mathrm{Diff}(N, M)$.

6. Since the category $\boldsymbol{\mathcal{C}}^n$ is complete, we can also have $\boldsymbol{\mathcal{C}}^n$ spaces with singular points like e.g. the equalizer[2] $\{x \in X \mid f(x) = g(x)\}$. In this way, any algebraic curve is a $\boldsymbol{\mathcal{C}}^\infty$ separated space too.

[2] See the Appendix A for the notion of equalizer.

7. Another type of space with singular points is the following. Let $\varphi \in \mathcal{C}^n(\mathbb{R}^k, \mathbb{R}^m)$ and consider the subspace $([0,1]^k \prec \mathbb{R}^k)$, then $(\varphi([0,1]^k) \prec \mathbb{R}^m) \in \mathcal{C}^n$ is a deformation in $\mathbb{R}^m$ of the hypercube $[0,1]^k$.

8. Let C be a continuum body, I the interval for time, and $\mathcal{E}$ the 3-dimensional Euclidean space. We can define on C a natural structure of $\mathcal{C}^\infty$-space. In fact, for any point $p \in C$ let $p_r(t) \in \mathcal{E}$ be the position of p at time t in the frame of reference r; we define figures of type U on C ($U \in \mathbf{O}\mathbb{R}^n$) the functions $d : U \longrightarrow C$ for which the following application

$$\begin{aligned} \tilde{d} : U \times I &\longrightarrow \mathcal{E} \\ (u,t) &\longmapsto d(u)_r(t) \end{aligned}$$

is smooth. For example if $U = \mathbb{R}$ then we can think of $d : \mathbb{R} \longrightarrow C$ as a curve traced on the body and parametrized by $u \in \mathbb{R}$. Hence we are requiring that the position $d(u)_r(t)$ of the particle $d(u) \in C$ in the frame of reference r varies smoothly with the parameter u and the time t. This is a generalization of the continuity of motion of any point of the body (take d constant). This smooth (that is diffeological) space will be separated, as an object of $\mathcal{C}^\infty$, if different points of the body cannot have the same motion:

$$p_r(-) = q_r(-) \quad \Longrightarrow \quad p = q \qquad \forall p, q \in C.$$

The configuration space of C can be viewed (see Wang [1970]) as a space of type

$$M := \sum_{t \in I} M_t \qquad \text{where} \qquad M_t \subseteq \mathcal{E}^C$$

and so, for the categorical properties of $\mathcal{C}^\infty$ the spaces $\mathcal{E}^C$, M_t (no matter how we choose these subspaces M_t) and M are always objects of $\mathcal{C}^\infty$ as well. With this structure the motion of C in the frame r:

$$\begin{aligned} \mu_r : C \times I &\longrightarrow \mathcal{E} \\ (p,t) &\longmapsto p_r(t) \end{aligned}$$

is a smooth map. Note that to obtain these results we need neither M_t nor C to be manifolds, but only the possibility to associate to any point p of C a motion $p_r(-) : I \longrightarrow \mathcal{E}$. If we had the possibility to develop a differential geometry for these spaces too we would have the possibility to obtain many results of continuum mechanics for bodies which cannot be naturally represented using a manifold or having an infinite-dimensional configuration space. Moreover in the next chapter we will see how to extend any $\mathcal{C}^\infty$ space with infinitesimal points, so that we can also consider infinitesimal sub-bodies of C.

Chapter 8

Extending smooth spaces with infinitesimals

8.1 Introduction

The main aim of this chapter is to extend any $\mathcal{C}^\infty$ space and any $\mathcal{C}^\infty$ function by means of our "infinitesimal points". First of all, we will extend to a generic space $X \in \mathcal{C}^\infty$ the notion of nilpotent path and of little-oh polynomial. The sets of these paths will be denoted by $\mathcal{N}_X$ and $X_o[t]$ respectively[1]. Afterward, we shall use the observables φ of the space X to generalize the equivalence relation $\sim$ (i.e. the equality in ${}^\bullet\mathbb{R}$, see Definition 2.3.1) using the following idea

$$\varphi(x_t) = \varphi(y_t) + \mathrm{o}(t) \qquad \text{with} \qquad \varphi \in^{UK} X.$$

Using this equivalence relation we will define ${}^\bullet X := X_o[t]/\sim$, which will be the generalization of the Definition ${}^\bullet\mathbb{R} := \mathbb{R}_o[t]/\sim$. Following this idea, the main problem is to understand how to relate the little-oh polynomials x, y with the domain U of φ. The second problem is that with this definition, ${}^\bullet X$ is a set only, without any kind of structure. Indeed, we will tackle the problem to define a meaningful category ${}^\bullet\mathcal{C}^\infty$ and a suitable structure on ${}^\bullet X$ so that ${}^\bullet X \in {}^\bullet\mathcal{C}^\infty$. In the subsequent sections we will also prove some results that will permit us to prove that the extension functor ${}^\bullet(-) : \mathcal{C}^\infty \longrightarrow {}^\bullet\mathcal{C}^\infty$ preserves the product of manifolds, i.e.

$${}^\bullet(M \times N) \simeq {}^\bullet M \times {}^\bullet N$$

for M, N manifolds. The fact that this useful theorem is not proved for generic $\mathcal{C}^\infty$ spaces is due to the fact that the topology on a product between $\mathcal{C}^\infty$ spaces is generally stronger than the product topology (see (6.3.2), but recall the final considerations of Section 6.2).

[1] See Definition 2.1.1 and Definition 2.1.2 for the case $X = \mathbb{R}$.

8.1.1 Nilpotent paths

If X is a $\mathcal{C}^\infty$ space, then using the topology τ_X we can define the set $\mathcal{C}_0(X)$ of all the maps $x : \mathbb{R}_{\ge 0} \longrightarrow X$ which are continuous at the origin $t = 0$. We want to simplify the notations avoiding the use of germs of continuous functions as equivalent classes (see Bourbaki [1989]), but, at the same time, we will keep attention to consider only local properties $\mathcal{P}(x)$ when we will treat paths $x \in \mathcal{C}_0(X)$ continuous at the origin, i.e. we will always verify that

$$\left(x, y \in \mathcal{C}_0(X) \quad \text{and} \quad x|_{[0,\varepsilon)} = y|_{[0,\varepsilon)} \quad \text{and} \quad \mathcal{P}(x)\right) \quad \Longrightarrow \quad \mathcal{P}(y). \quad (8.1.1)$$

Following this constraint, it is not important how we extend[2] to the whole $\mathbb{R}_{\ge 0}$ a locally defined function $x : [0, \varepsilon) \longrightarrow X$.

Because any $\mathcal{C}^\infty$ function $f : X \longrightarrow Y$ is continuous with respect to the topologies τ_X and τ_Y, we have that $f \circ x \in \mathcal{C}_0(Y)$ if $x \in \mathcal{C}_0(X)$. More locally, if U is open in X and $x(0) \in U$, then on the subspace $(U \prec X)$ we have the induced topology and from this it follows that $\varphi \circ x \in \mathcal{C}_0(K)$ if $\varphi \in^{UK} X$ is an observable of the space X. Let us note explicitly that this is a local property, and hence, on the one hand, with the notation $\varphi \circ x$ we have to mean a function $\varphi \circ x : \mathbb{R}_{\ge 0} \longrightarrow X$ (because $\varphi \circ x$ is an element of $\mathcal{C}_0(X)$). On the other hand, for this function the only important property is that

$$\exists\, \varepsilon > 0 : \quad [0, \varepsilon) \subseteq \{t \in \mathbb{R}_{\ge 0} \,|\, t \in \mathrm{dom}(x) \quad \text{and} \quad x(t) \in \mathrm{dom}(\varphi)\},$$

i.e. that the set of $t \in \mathbb{R}_{\ge 0}$ for which the composition $\varphi(x(t))$ is defined, contains a right neighborhood of the origin.

As many other concepts we will introduce in this chapter, the notion of nilpotent map is defined by means of the composition with a generic observable and by a suitable logical implication to relate the starting value $x(0)$ of a given path $x \in \mathcal{C}_0(X)$ with the domain of the observable[3].

Definition 8.1.1. *Let X be a $\mathcal{C}^\infty$ space and let $x \in \mathcal{C}_0(X)$ a path continuous at the origin, then we say that x* is nilpotent (rel. X) *iff for every zone UK of X and every observable $\varphi \in^{UK} X$ we have that the following implication is true*

$$x(0) \in U \quad \Longrightarrow \quad \exists k \in \mathbb{N} : \|\varphi(x_t) - \varphi(x_0)\|^k = \mathrm{o}(t).$$

[2] To be really rigorous, one has to fix, once and for all, a function $E = E_X^{U,\varepsilon}$ to perform such an extension, but taking into consideration the fact that the whole construction does not depend on this extension function. This function is defined on the set $\mathcal{C}_0([0, \varepsilon), U)$ of function $x : [0, \varepsilon) \longrightarrow U$ continuous at $t = 0^+$ and with values in the subset $U \subseteq X$, i.e it is of the type $E : \mathcal{C}_0([0, \varepsilon), U) \longrightarrow \mathcal{C}_0(X)$, and has the property $E(x)|_{[0,\varepsilon)} = x$.

[3] Recall that, as usual, we will also use the notation x_t for the evaluation of $x \in Cc_0(X)$ at $t \in \mathrm{dom}(x)$ and that our little-oh functions (always for $t \to 0^+$) are always continuous at the origin (see Remark 2.1.3).

Moreover we define

$$\mathcal{N}_X := \mathcal{N}(X) := \{x \in \mathcal{C}_0(X) \mid x \text{ is nilpotent}\}.$$

A direct verification proves that the property of a path to be nilpotent is a local property. Moreover, we will prove later that this definition generalizes the particular notion expressed in Definition 2.1.1.

Because every $f \in \boldsymbol{\mathcal{C}}^\infty(X, Y)$ preserves the observables (see property (7.1.1)), if $x \in \mathcal{N}_X$ then $f \circ x \in \mathcal{N}_Y$, that is $\boldsymbol{\mathcal{C}}^\infty$ functions preserve nilpotent maps too. In case of a manifold M (identified with its embedding $M = \bar{M} \in \boldsymbol{\mathcal{C}}^\infty$) we can state the property of being nilpotent with an existential quantifier instead of an implication

Theorem 8.1.2. *Let M be a $\mathcal{C}^\infty$ manifold and let us consider a map $x : \mathbb{R}_{\geq 0} \longrightarrow |M|$, then x is nilpotent iff we can find a chart (U, φ) on x_0 such that $\|\varphi(x_t) - \varphi(x_0)\|^k = \mathrm{o}(t)$ for some $k \in \mathbb{N}$.*

Proof: If we start from the hypothesis $x \in \mathcal{N}_M$, then it suffices to take any chart on x_0 and to use the property that charts are observables of M to get the conclusion formulated in the statement.

To prove the opposite implication, let us take an observable $\psi \in^{VK} \bar{M}$, where K is open in $\mathbb{R}^p$ and with $x_0 \in V$. Recalling (7.2.1) we get that $\psi \in \mathbf{Man}(V, K)$, i.e. ψ is an ordinary $\mathcal{C}^\infty$ function. The idea is to use the equality

$$\forall^0 t \geq 0 : \quad \psi(x_t) = \psi\left[\varphi^{-1}(\varphi(x_t))\right],$$

which is locally true[4], and the Lipschitz property of $\psi \circ \varphi^{-1}$. Diagrammatically, in the category **Man** of smooth manifolds, our situation is the following

$$\begin{tikzcd}
U \cap V \arrow[r, "\varphi|_{U\cap V}", "\sim"'] \arrow[d, "\psi|_{U\cap V}"] & \varphi(U \cap V) \\
K &
\end{tikzcd}$$

Therefore

$$\gamma := (\varphi|_{U\cap V})^{-1} \cdot \psi|_{U\cap V} \in \mathbf{Man}(\varphi(U \cap V), K),$$

and hence γ is locally Lipschitz with respect to some constant $C > 0$. But $x \in \mathcal{C}_0(\bar{M})$ and $U \cap V \in \mathcal{T}_M$, hence

$$\forall^0 t \geq 0 : \quad x_t \in U \cap V,$$

[4]Recall the definition of $\forall^0 t \geq 0$ given in Section 4.2.

and we can write

$$\begin{aligned}\|\psi(x_t)-\psi(x_0)\|^k &= \|\gamma\left[\varphi(x_t)\right]-\gamma\left[\varphi(x_0)\right]\|^k \leq \\ &\leq C^k\cdot\|\varphi(x_t)-\varphi(x_0)\|^k=\mathrm{o}(t)\end{aligned}$$

■

This will be a typical idea in several definitions of the present work: working with generic $\mathcal{C}^\infty$ spaces we do not have the possibility to consider charts on every point, so we require a condition *for every observable* that *potentially* (i.e. by means of a logical implication) contains the starting point of a given path. We have already used this idea in the Definition 7.1.2 of "x is identified with y", i.e. of the relation $x \asymp y$. Usually, in case we have charts, like in the previous Theorem 8.1.2 we will be able to transform in an equivalent statement this type of implications using an existential quantifier. This theorem also proves that the previous Definition 2.1.1 is a generalization of the old Definition 8.1.1.

Finally we consider the relations between the product of two manifolds M, N and nilpotent paths in the following

Theorem 8.1.3. *Let M, N be smooth manifolds and $x:\mathbb{R}_{\geq 0}\longrightarrow |M|$, $y:\mathbb{R}_{\geq 0}\longrightarrow |N|$ be two maps, then*

$$x\in\mathcal{N}_{\bar{M}}\quad\text{and}\quad y\in\mathcal{N}_{\bar{N}}\qquad\Longleftrightarrow\qquad(x,y)\in\mathcal{N}_{\bar{M}\times\bar{N}},$$

where we set $(x,y)_t:=(x_t,y_t)$.

Proof:

$\Leftarrow$: If $(x,y)\in\mathcal{N}_{\bar{M}\times\bar{N}}=\mathcal{N}_{\overline{M\times N}}$, by the previous Theorem 8.1.2 we get the existence of two charts (U,φ) of M and (V,ψ) of N with $x_0\in U$ and $y_0\in V$ and such that[5]

$$\exists\, k\in\mathbb{N}:\quad \|(\varphi x_t,\psi y_t)-(\varphi x_0 t,\psi y_0)\|^k=\mathrm{o}(t).$$

Therefore, we also have $\{\|\varphi x_t-\varphi x_0\|+\|\psi y_t-\psi y_0\|\}^k=\mathrm{o}(t)$.
But $\|\varphi x_t-\varphi x_0\|\leq\|\varphi x_t-\varphi x_0\|+\|\psi y_t-\psi y_0\|$ and hence

$$\|\varphi x_t-\varphi x_0\|^k\leq\{\|\varphi x_t-\varphi x_0\|+\|\psi y_t-\psi y_0\|\}^k=\mathrm{o}(t).$$

Therefore also $\|\varphi x_t-\varphi x_0\|^k=\mathrm{o}(t)$, that is $x\in\mathcal{N}_M$. Analogously we can proceed for y.

$\Rightarrow$: From the hypotheses $x\in\mathcal{N}_{\bar{M}}$ and Theorem 8.1.2 we get a chart (U,φ) of M on x_0 and a $k\in\mathbb{N}$ such that

$$\|\varphi x_t-\varphi x_0\|^k=\mathrm{o}(t). \tag{8.1.2}$$

[5]If it will be clear from the context, we will sometimes omit the parenthesis in compositions like $fg(x)=f(g(x))$.

Analogously, from $y \in \mathcal{N}_{\bar{N}}$ we obtain a chart (V, ψ) on y_0 and a $k' \in \mathbb{N}$ such that

$$\|\psi y_t - \psi y_0\|^{k'} = \mathrm{o}(t). \tag{8.1.3}$$

We can suppose $k = k'$. Therefore $(U \times V, \varphi \times \psi)$ is a chart of $M \times N$ on (x_0, y_0). Let us try to compute the term

$$\begin{aligned}\|(\varphi \times \psi)(x_t, y_t) - (\varphi \times \psi)(x_0, y_0)\|^k &= \|(\varphi x_t - \varphi x_0, \psi y_t - \psi y_0)\|^k = \\ &= \sum_{i=0}^{k} \binom{k}{i} \|\varphi x_t - \varphi x_0\|^i \cdot \|\psi y_t - \psi y_0\|^{k-i}. \end{aligned} \tag{8.1.4}$$

But

$$\begin{aligned}&\frac{\|\varphi x_t - \varphi x_0\|^i \cdot \|\psi y_t - \psi y_0\|^{k-i}}{t} = \\ &\quad = \frac{\left\{\|\varphi x_t - \varphi x_0\|^k\right\}^{\frac{i}{k}}}{t^{\frac{i}{k}}} \cdot \frac{\left\{\|\psi y_t - \psi y_0\|^k\right\}^{\frac{k-i}{k}}}{t^{\frac{k-i}{k}}} = \\ &\quad\quad = \left\{\frac{\|\varphi x_t - \varphi x_0\|^k}{t}\right\}^{\frac{i}{k}} \cdot \left\{\frac{\|\psi y_t - \psi y_0\|^k}{t}\right\}^{\frac{k-i}{k}}.\end{aligned}$$

Each factor of this product goes to zero for $t \to 0^+$ because of (8.1.2) and (8.1.3). Hence also (8.1.4) goes to zero and this, because of Theorem 8.1.2, proves that

$$(x, y) \in \mathcal{N}_{\overline{M \times N}} = \mathcal{N}_{\bar{M} \times \bar{N}}.$$

■

8.1.2 Little-oh polynomials in $\mathcal{C}^\infty$

We can proceed in a similar way with respect to the generalization of the notion of little-oh polynomial: at first we will define what is a little-oh polynomial in $\mathbb{R}^d$, and secondly we will generalize this notion to a generic space $X \in \mathcal{C}^\infty$ using observables.

Definition 8.1.4. *We say that x is a* little-oh polynomial in $\mathbb{R}^d$*, and we write $x \in \mathbb{R}^d_o[t]$, iff*

1. $x : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}^d$
2. *We can write*

$$x_t = r + \sum_{i=1}^{k} \alpha_i \cdot t^{a_i} + o(t) \quad \textit{as} \quad t \to 0^+$$

for suitable

$$k \in \mathbb{N}$$

$$r, \alpha_1, \ldots, \alpha_k \in \mathbb{R}^d$$

$$a_1, \ldots, a_k \in \mathbb{R}_{\geq 0}.$$

Now let $X \in \boldsymbol{\mathcal{C}}^\infty$ and $x \in \mathcal{C}_0(X)$, then we say that x is a little-oh polynomial (of X) *iff for every zone UK of X, with $K \subseteq \mathbb{R}^{\mathsf{k}}$, and every observable $\varphi \in^{UK} X$ we have*

$$x_0 \in U \quad \Longrightarrow \quad \varphi \circ x \in \mathbb{R}^{\mathsf{k}}_o[t].$$

Moreover

$$X_o[t] := X_o := \{x \in \mathcal{C}_0(X) \,|\, x \textit{ is a little-oh polynomial of } X\}\,.$$

Let us note that for $d = 1$ we have exactly the old Definition 2.1.2. A direct verification proves that being a little-oh polynomial is a local property. Moreover, we will prove later that the two parts of this definition (i.e. that of $X_o[t]$ and that of $\mathbb{R}^d_o[t]$ are equivalent if $X = \mathbb{R}^d$).

Now we have to prove the analogous for little-oh polynomials of the previous results stated for nilpotent paths. Once again, because every $f \in \boldsymbol{\mathcal{C}}^\infty(X, Y)$ preserves the observables, we have that $\boldsymbol{\mathcal{C}}^\infty$ functions preserve little-oh polynomials too

$$x \in X_0[t] \quad \Longrightarrow \quad f \circ x \in Y_o[t].$$

The other results we want to prove relate the notion of little-oh polynomial with that of manifold: at first, as usual, we want to reformulate the Definition 8.1.4 for manifolds; secondly we want to make clear the relationships between little-oh polynomials and the product of manifolds. For these results we need the following Lemmas.

Lemma 8.1.5. *Let $x : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}^m$ and $y : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}^n$ be two maps, then*

$$x \in \mathbb{R}^m_o[t] \quad \textit{and} \quad y \in \mathbb{R}^n_o[t] \quad \Longleftrightarrow \quad (x, y) \in \mathbb{R}^{m+n}_o[t].$$

Proof:
$\Rightarrow$: Let us fix the notations for the little-oh polynomials x and y:

$$x_t = r + \sum_{i=1}^{K} \alpha_i \cdot t^{a_i} + o_1(t)$$

$$y_t = s + \sum_{j=1}^{N} \beta_j \cdot t^{b_j} + o_2(t),$$

where $r, \alpha_1, \ldots, \alpha_K \in \mathbb{R}^m$ and $s, \beta_1, \ldots, \beta_N \in \mathbb{R}^n$. Define $u := (r, s) \in \mathbb{R}^{m+n}$, $\gamma_i := (\alpha_i, \underline{0}) \in \mathbb{R}^{m+n}$, $\gamma_{j+K} := (\underline{0}, \beta_j) \in \mathbb{R}^{m+n}$, $c_i := a_i$ and $c_{j+K} := b_j$, then

$$
\begin{aligned}
(x_t, y_t) &= \left(r + \sum_{i=1}^{K} \alpha_i \cdot t^{a_i} + o_1(t), s + \sum_{j=1}^{N} \beta_j \cdot t^{b_j} + o_2(t) \right) = \\
&= (r, s) + \sum_{i=1}^{K} (\alpha_i, \underline{0}) \cdot t^{a_i} + (o_1(t), \underline{0}) + \\
&\quad + \sum_{j=1}^{N} (\underline{0}, \beta_j) \cdot t^{b_j} + (\underline{0}, o_2(t)) = \\
&= u + \sum_{i=1}^{K} \gamma_i \cdot t^{c_i} + \sum_{i=K+1}^{K+N} \gamma_i \cdot t^{c_i} + (o_1(t), o_2(t)),
\end{aligned}
$$

and this proves the conclusion because $(o_1(t), o_2(t)) = o(t)$.

$\Leftarrow$: By hypotheses we can write

$$
(x_t, y_t) = u + \sum_{k=1}^{H} \gamma_k \cdot t^{c_k} + o(t).
$$

We only have to reverse the previous ideas defining:

$$
\begin{aligned}
r &:= (u_1, \ldots, u_m) & s &:= (u_{m+1}, \ldots, u_{m+n}) \\
\alpha_k &:= (\gamma_k^1, \ldots, \gamma_k^m) & \beta_k &:= (\gamma_k^{m+1}, \ldots, \gamma_k^{m+n}) \\
o_1(t) &:= (o^1(t), \ldots, o^m(t)) & o_2(t) &:= (o^{m+1}(t), \ldots, o^{m+n}(t)) \\
a_i &:= c_i & b_k &:= c_k
\end{aligned}
$$

where we have used the notations

$$
\begin{aligned}
\gamma_k &= (\gamma_k^1, \ldots, \gamma_k^{m+n}) \\
o(t) &= (o^1(t), \ldots, o^{m+n}(t))
\end{aligned}
$$

for the components. Then

$$
\begin{aligned}
(x_t, y_t) &= (r, \underline{0}) + \sum_{k=1}^{H} (\alpha_k, \underline{0}) \cdot t^{c_k} + (o_1(t), \underline{0}) + \\
&\quad (\underline{0}, s) + \sum_{k=1}^{H} (\underline{0}, \beta_k) \cdot t^{c_k} + (\underline{0}, o_2(t)) = \\
&= \left(r + \sum_{i=1}^{H} \alpha_i \cdot t^{c_i} + o_1(t), s + \sum_{j=1}^{H} \beta_j \cdot t^{c_j} + o_2(t) \right),
\end{aligned}
$$

and hence the conclusion follows.

■

From this lemma, if $x \in \mathbb{R}_o^d[t]$, then each component is a 1-dimensional little-oh polynomial $x_i \in \mathbb{R}_o[t]$ for $i = 1, \ldots, d$. But we know (see Section 2.2) that each one of these polynomial is nilpotent, i.e. $x_i \in \mathcal{N}$. Therefore, from Theorem 8.1.3 it follows that $x \in \mathcal{N}_{\mathbb{R}^d}$, i.e.

$$\mathbb{R}_o^d[t] \subseteq \mathcal{N}_{\mathbb{R}^d};$$

from this it also follows that $X_o[t] \subseteq \mathcal{N}_X$.

Lemma 8.1.6. *Let $x \in \mathbb{R}_o^d[t]$ and $f \in \mathcal{C}^\infty(A, \mathbb{R}^p)$, with A open in $\mathbb{R}^d$ and such that, locally, the path x has values in A:*

$$\forall^0 t \geq 0: \quad x_t \in A.$$

Then $f \circ x \in \mathbb{R}_o^p[t]$.

Proof: Let us fix some notations:

$$x_t = r + \sum_{i=1}^{k} \alpha_i \cdot t^{a_i} + w(t) \quad \text{with} \quad w(t) = o(t)$$

$$h(t) := x(t) - x(0) \quad \forall t \in \mathbb{R}_{\geq 0}$$

hence $x_t = x(0) + h_t = r + h_t$. The function $t \mapsto h(t) = \sum_{i=1}^k \alpha_i \cdot t^{a_i} + w(t)$ belongs to $\mathbb{R}_o^d[t] \subseteq \mathcal{N}_{\mathbb{R}^d}$, so we can write $\|h_t\|^N = o(t)$ for some $N \in \mathbb{N}$ if we take as observable of $\mathbb{R}^d$ the identity. From Taylor's's formula we have

$$f(x_t) = f(r + h_t) = f(r) + \sum_{\substack{i \in \mathbb{N}^d \\ |i| \leq N}} \frac{\partial^i f}{\partial x^i}(r) \cdot \frac{h_t^i}{i!} + o\left(\|h_t\|^N\right). \tag{8.1.5}$$

But

$$\frac{o\left(\|h_t\|^N\right)}{|t|} = \frac{o\left(\|h_t\|^N\right)}{\|h_t^N\|} \cdot \frac{\|h_t^N\|}{|t|} \to 0$$

hence $o\left(\|h_t\|^N\right) = o(t) \in \mathbb{R}_o^p[t]$. Now we have to note that, for a multi-index $i \in \mathbb{N}^d$, it results that $h_t^i = h_1^{i_1}(t) \cdot \ldots \cdot h_d^{i_d}(t) \in \mathbb{R}_o[t]$ because, from the previous Lemma 8.1.5, each function $h_j(t) \in \mathbb{R}_o[t]$ and because $\mathbb{R}_o[t]$ is an algebra. Moreover, if $\beta \in \mathbb{R}^p$ and $h \in \mathbb{R}_o[t]$, then $\beta \cdot h \in \mathbb{R}_o^p[t]$, so each addend $\frac{\partial^i f}{\partial x^i}(r) \cdot \frac{h_t^i}{i!}$ is a little-oh polynomial of $\mathbb{R}_o^p[t]$. From (8.1.5) and the closure of little-oh polynomials $\mathbb{R}_o^p[t]$ with respect to linear operations, the conclusion $f \circ x \in \mathbb{R}_o^p[t]$ follows.

■

Using these lemmas we can prove the above cited results about little-oh polynomials in manifolds.

Theorem 8.1.7. *If M is a $\mathcal{C}^\infty$ manifold and $x : \mathbb{R}_{\ge 0} \longrightarrow |M|$ is a map, then we have that $x \in \bar{M}_o[t]$ if and only if there exists a chart (U, φ) of M such that:*

1. $x(0) \in U$
2. $\varphi \circ x \in \mathbb{R}^d_o[t]$*, where* $d := \dim(M)$.

Proof: To prove that the hypotheses $x \in \bar{M}_o[t]$ implies conditions 1. and 2. it suffices to take any chart on x_0 and to use the property that charts are observables of $\bar{M}$.

For the opposite implication we start considering that, by the Definition 8.1.4 we have that $\varphi \circ x \in \mathbb{R}^d_o[t]$ is continuous at $t = 0^+$, and hence also x is continuous at $t = 0^+$, i.e $x \in \mathcal{C}_0(M)$. Now, take a generic observable $\psi \in^{VK} \bar{M}$, where K is open in $\mathbb{R}^p$ and $x_0 \in V$. We have

$$\mathbb{R}^d \supseteq \varphi(U \cap V) \xrightarrow[\sim]{(\varphi|_{U\cap V})^{-1}} U \cap V \xrightarrow{\psi|_{U\cap V}} K \subseteq \mathbb{R}^p \ ,$$

and hence

$$(\varphi|_{U\cap V})^{-1} \cdot \psi|_{U\cap V} \in \mathcal{C}^\infty(\varphi(U \cap V, K). \tag{8.1.6}$$

From $x_0 \in U \cap V$ and from the continuity of the path x at $t = 0^+$ we get

$$\forall^0 t \ge 0 : \quad x_t \in U \cap V.$$

From this, from (8.1.6), from the hypotheses $\varphi \circ x \in \mathbb{R}^d_o[t]$ and from Lemma 8.1.6 the conclusion $(\varphi|_{U\cap V})^{-1} \cdot \psi|_{U\cap V} \circ \varphi \circ x = \psi \circ x \in \mathbb{R}^p_o[t]$ follows.

∎

Theorem 8.1.8. *Let M, N be $\mathcal{C}^\infty$ manifolds and $x : \mathbb{R}_{\ge 0} \longrightarrow |M|$, $y : \mathbb{R}_{\ge 0} \longrightarrow |M|$ two maps. Then*

$$x \in \bar{M}_o[t] \quad \textit{and} \quad y \in \bar{N}_o[t] \quad \Longleftrightarrow \quad (x, y) \in \left(\bar{M} \times \bar{N}\right)_o[t].$$

Proof: The proof is an almost purely logical consequence of Theorem 8.1.7 and of Lemma 8.1.5.

$\Leftarrow$: By hypotheses $(x, y) \in \left(\bar{M} \times \bar{N}\right)_o[t] = \left(\overline{M \times N}\right)_o[t]$. Because $M \times N$ is a manifold, from Theorem 8.1.7 we get the existence of charts (U, φ) of M and (V, ψ) of N on x_0 and y_0 resp. such that

$$(\varphi \times \psi) \circ (x, y) = (\varphi \circ x, \psi \circ y) \in \mathbb{R}^{m+n}_o[t].$$

Hence $\varphi \circ x \in \mathbb{R}^m_o[t]$ and $\psi \circ y \in \mathbb{R}^n_o[t]$ from Lemma 8.1.5.

$\Rightarrow$: Analogously, if $x \in \bar{M}_o[t]$ and $y \in \bar{N}_o[t]$, then we can find charts as above, but with $\varphi \circ x \in \mathbb{R}^m_o[t]$ and $\psi \circ y \in \mathbb{R}^n_o[t]$. Once again from Lemma 8.1.5 we obtain

$$(\varphi \circ x, \psi \circ y) = (\varphi \times \psi) \circ (x, y) \in \mathbb{R}^{m+n}_o[t],$$

and hence the conclusion follows from Theorem 8.1.7.

∎

8.2 The Fermat extension of spaces and functions

Considering the previous definitions of nilpotent and little-oh paths and the Definition 7.1.2 it is now clear how to generalize the definition of equality in ${}^\bullet\mathbb{R}$ (see Definition 2.3.1) to a generic $X \in \boldsymbol{\mathcal{C}}^\infty$:

Definition 8.2.1. *Let X be a $\boldsymbol{\mathcal{C}}^\infty$ space and let x, $y \in X_o[t]$ be two little-oh polynomials, then we say that*

$$x \sim y \quad \text{in} \quad X \qquad \text{or simply} \qquad x = y \quad \text{in} \quad {}^\bullet X$$

iff for every zone UK of X and every observable $\varphi \in^{UK} X$ we have

1. $x_0 \in U \iff y_0 \in U$
2. $x_0 \in U \implies \varphi(x_t) = \varphi(y_t) + o(t)$

Obviously we will write ${}^\bullet X := X_o[t]/\sim$ and ${}^\bullet f(x) := f \circ x$ if $f \in \boldsymbol{\mathcal{C}}^\infty(X, Y)$ and $x \in {}^\bullet X$ and we will call them the *Fermat extension of X and of f* respectively. As usual, we will also define the *standard part of $x \in {}^\bullet X$* as ${}^\circ x := x(0) \in X$.

We prove the correctness of the definition of ${}^\bullet f$ in the following:

Theorem 8.2.2. *If $f \in \boldsymbol{\mathcal{C}}^\infty(X, Y)$ and $x = y$ in ${}^\bullet X$ then ${}^\bullet f(x) = {}^\bullet f(y)$ in ${}^\bullet Y$.*

Proof: Take a zone VK in Y and an observable $\psi \in^{VK} Y$, then from the continuity of f, we have $U := f^{-1}(V) \in \mathcal{T}_X$. We can thus apply hypothesis $x = y$ in ${}^\bullet X$ with the zone UK and the observable $\varphi := f|_U \cdot \psi \in^{UK} X$. From this the conclusion follows considering that $f \circ x, f \circ y \in Y_o[t]$ and $x_0 \in U = f^{-1}(V)$ iff $f(x_0) \in V$.

$\blacksquare$

Using the continuity of $\varphi \circ x$ we can note that $x = y$ in ${}^\bullet X$ implies that x_0 and y_0 are identified in X (see Definition 7.1.2) and thus using constant maps $\hat{x}(t) := x$, for $x \in X$, we obtain an injection $\hat{(-)} : |X| \longrightarrow {}^\bullet X$ if the space X is separated. Therefore, if Y is separated too, ${}^\bullet f$ is really an extension of f. Finally, note that the application ${}^\bullet(-)$ preserves compositions and identities.

Using ideas very similar to the ones used above for similar theorems, we can prove that if $X = M$ is a $\mathcal{C}^\infty$ manifold then we have that $x = y$ in ${}^\bullet M$ iff there exists a chart (U, φ) of M such that

1. x_0, $y_0 \in U$
2. $\varphi(x_t) = \varphi(y_t) + \mathrm{o}(t)$.

Moreover the previous conditions do not depend on the chart (U, φ). In particular if $X = U$ is an open set in $\mathbb{R}^k$, then $x = y$ in ${}^\bullet U$ is simply equivalent to the limit relation $x(t) = y(t) + \mathrm{o}(t)$ as $t \to 0^+$; hence if $i : U \hookrightarrow \mathbb{R}^k$ is the inclusion map, it's easy to prove that its Fermat extension ${}^\bullet i : {}^\bullet U \longrightarrow {}^\bullet\mathbb{R}^k$ is injective. We will always identify ${}^\bullet U$ with ${}^\bullet i({}^\bullet U)$, so we simply write ${}^\bullet U \subseteq {}^\bullet\mathbb{R}^k$. According to this identification, if U is open in $\mathbb{R}^k$, we can also prove that

$$ {}^\bullet U = \{x \in {}^\bullet\mathbb{R}^k \mid {}^\circ x \in U\}. \tag{8.2.1}$$

This property says that the preliminary definition of ${}^\bullet U$ given in Definition 2.6.1 is equivalent to the previous, more general, Definition 8.2.1 of extension. Using the previous equivalent way to express the relation $\sim$ on manifolds, we see that $(x, y) = (x', y')$ in ${}^\bullet(M \times N)$ iff $x = x'$ in ${}^\bullet M$ and $y = y'$ in ${}^\bullet N$. From this conclusion and from Theorem 8.1.8 we prove that the following applications

$$\alpha_{MN} := \alpha \ : \ ([x]_\sim, [y]_\sim) \in {}^\bullet M \times {}^\bullet N \longmapsto [(x, y)]_\sim \in {}^\bullet(M \times N) \tag{8.2.2}$$

$$\beta_{MN} := \beta \ : \ [z]_\sim \in {}^\bullet(M \times N) \longmapsto ([z \cdot p_M]_\sim, [z \cdot p_N]_\sim) \in {}^\bullet M \times {}^\bullet N \tag{8.2.3}$$

(for clarity we have used the notation with the equivalence classes) are well-defined bijections with $\alpha^{-1} = \beta$ (obviously p_M, p_N are the projections). We will use the first one of them in the following section with the temporary notation $\langle p, x\rangle := \alpha(p, x)$, hence $f\langle p, x\rangle = f(\alpha(p, x))$ for $f : {}^\bullet(M \times N) \longrightarrow Y$. This simplifies our notations but permits to avoid the identification of ${}^\bullet M \times {}^\bullet N$ with ${}^\bullet(M \times N)$ until we will have proved that α and β are arrows of the category ${}^\bullet\mathcal{C}^\infty$.

8.3 The category of Fermat spaces

Up to now every ${}^\bullet X$ is a simple set only. Now we want to use the general passage from a category of the types of figures $\mathcal{F}$ to its cartesian closure $\bar{\mathcal{F}}$ so as to put on any ${}^\bullet X$ a useful structure of $\bar{\mathcal{F}}$ space. Our aim is to obtain in this way a new cartesian closed category $\bar{\mathcal{F}} =: {}^\bullet\mathcal{C}^\infty$, called the *category of Fermat spaces*, and a functor ${}^\bullet(-) : \mathcal{C}^\infty \longrightarrow {}^\bullet\mathcal{C}^\infty$, called the *Fermat functor*. Therefore we have to choose $\mathcal{F}$, that is we have to understand what can be the types of figures of ${}^\bullet X$. It may seem very natural to take ${}^\bullet g : {}^\bullet U \longrightarrow {}^\bullet V$ as arrow in $\mathcal{F}$ if $g : U \longrightarrow V$ is in $\mathbf{O}\mathbb{R}^\infty$ (in Giordano [2001] we followed this way). The first problem in this idea is that, e.g.

$$ {}^\bullet\mathbb{R} \xrightarrow{\ {}^\bullet f\ } {}^\bullet\mathbb{R} \quad \Longrightarrow \quad {}^\bullet f(0) = f(0) \in \mathbb{R},$$

hence there cannot exist a constant function of the type ${}^\bullet f$ to a non-standard value, and so we cannot satisfy the closure of $\mathcal{F}$ with respect to generic constant functions (see the hypotheses about the types of figures $\mathcal{F}$ in Section 6.1). But we can make further considerations about this problem so as to better motivate the choice of $\mathcal{F}$. The first one is that we surely want to have the possibility to lift maps[6] as simple as the sum between Fermat reals:

$$s : (p, q) \in {}^\bullet\mathbb{R} \times {}^\bullet\mathbb{R} \longrightarrow p + q \in {}^\bullet\mathbb{R}.$$

Therefore, we have to choose $\mathcal{F}$ so that the map $s^\wedge(p) : q \in {}^\bullet\mathbb{R} \longrightarrow p+q \in {}^\bullet\mathbb{R}$ is an arrow of ${}^\bullet\boldsymbol{\mathcal{C}}^\infty$. Note that this map is neither constant nor of the type ${}^\bullet f$ because $s^\wedge(p)(0) = p$ and p could be a non standard Fermat real.
The second consideration is about the map α defined in (8.2.2): if we want α to be an arrow of ${}^\bullet\boldsymbol{\mathcal{C}}^\infty$, then in the following situation we have to obtain a ${}^\bullet\boldsymbol{\mathcal{C}}^\infty$ arrow

$$\begin{array}{ccccccc} {}^\bullet\mathbb{R} \times {}^\bullet\mathbb{R} & \xrightarrow{p \times 1_{{}^\bullet\mathbb{R}}} & {}^\bullet\mathbb{R} \times {}^\bullet\mathbb{R} & \xrightarrow{\alpha} & {}^\bullet(\mathbb{R} \times \mathbb{R}) & \xrightarrow{{}^\bullet g} & {}^\bullet\mathbb{R} \\ (r, s) & \longmapsto & (p, s) & \longmapsto & \langle p, s\rangle & \longmapsto & {}^\bullet g\langle p, s\rangle, \end{array}$$

where $p \in {}^\bullet\mathbb{R}$ and $g \in \mathcal{C}^\infty(\mathbb{R}^2, \mathbb{R})$. The idea we shall follow is exactly to take as arrows of $\mathcal{F}$ all the maps that locally are of the form $\delta(s) = {}^\bullet g\langle p, s\rangle$, where $p \in {}^\bullet(\mathbb{R}^{\mathsf{p}})$ works as a parameter of ${}^\bullet g\langle -, -\rangle$. Obviously, in this way δ could also be a constant map to a non standard value (take as g a projection). Frequently one can find maps of the form ${}^\bullet g\langle p, -\rangle$ in informal calculations in physics or geometry. Actually, they simply are $\mathcal{C}^\infty$ maps with some fixed parameter p, which could be an infinitesimal distance (e.g. in the potential of the electric dipole, see below), an infinitesimal coefficient associated to a metric (like, e.g., in Einstein's formula (1.0.1)), or a side $l := s(a, -)$ of an infinitesimal surface $s : [a, b] \times [c, d] \longrightarrow {}^\bullet\mathbb{R}$, where $[a, b], [c, d] \subseteq D_k$.
Note the importance of the map α to perform passages like the following

$$\begin{array}{lll} M \times N \xrightarrow{f} Y & \text{in} & \boldsymbol{\mathcal{C}}^\infty \\ {}^\bullet(M \times N) \xrightarrow{{}^\bullet f} {}^\bullet Y & \text{in} & {}^\bullet\boldsymbol{\mathcal{C}}^\infty \\ {}^\bullet M \times {}^\bullet N \xrightarrow{{}^\bullet f} {}^\bullet Y & \text{in} & {}^\bullet\boldsymbol{\mathcal{C}}^\infty \text{ (identification via } \alpha) \\ {}^\bullet N \xrightarrow{{}^\bullet f^\wedge} {}^\bullet Y^{{}^\bullet M} & & \text{using cartesian closedness.} \end{array}$$

This motivates the choice of arrows in $\mathcal{F}$, but there is a second problem about the choice of the objects of the category $\mathcal{F}$. Take a manifold M and an arrow $t : D \longrightarrow {}^\bullet M$ in ${}^\bullet\boldsymbol{\mathcal{C}}^\infty$. Even if we have not still defined formally what is the meaning of this "arrow", we want to think t as a tangent vector applied either to a standard point $t(0) \in M$ or to a non standard one, $t(0) \in {}^\bullet M \setminus M$. Roughly speaking, this is the case if we can write $t(h) = {}^\bullet g\langle p, h\rangle$ for every

[6] I.e. to consider their adjoint function using cartesian closedness.

$h \in D$ and for some g, p. If we want to obtain this equality it is useful to have two properties: the first one is that the identity map over D, i.e. 1_D, is a figure of D, i.e. $1_D \in_D D$. In this way, from the property $t : D \longrightarrow {}^\bullet M$ of being an arrow of ${}^\bullet\mathcal{C}^\infty$ we can deduce that t is a figure of ${}^\bullet M$ of the type D, i.e. $t \in_D {}^\bullet M$. The second property we would like to obtain is to have maps of the form ${}^\bullet g\langle p, -\rangle : D \longrightarrow {}^\bullet M$ as figures of ${}^\bullet M$. Of course, we can thus say that necessarily $t = {}^\bullet g\langle p, -\rangle$ for some $g \in \mathcal{C}^\infty(\mathbb{R}^{\mathsf{p}}, \mathbb{R})$ and $p \in \mathbb{R}^{\mathsf{p}}$. Therefore, to obtain these properties, it would be useful to have D as an object of $\mathcal{F}$. But D is not the extension of a standard subset of $\mathbb{R}$, thus what will be the objects of $\mathcal{F}$? We will take generic subsets S of ${}^\bullet(\mathbb{R}^{\mathsf{s}})$ with the topology $\mathcal{T}_S$ generated by $\mathcal{U} = {}^\bullet U \cap S$, for U open in $\mathbb{R}^{\mathsf{s}}$ (in this case we will say that the open set $\mathcal{U}$ *is defined by* U *in* S). In other words $A \in \mathcal{T}_S$ if and only if

$$A = \bigcup \{ {}^\bullet U \cap S \subseteq A \,|\, U \text{ is open in } \mathbb{R}^{\mathsf{s}} \}. \tag{8.3.1}$$

These are the motivations to introduce the category of the types of figures $\mathcal{F}$ by means of the following

Definition 8.3.1. *We call* $\mathbf{S}^\bullet\mathbb{R}^\infty$ *the category whose objects are topological spaces* $(S, \mathcal{T}_S)$*, with* $S \subseteq {}^\bullet(\mathbb{R}^{\mathsf{s}})$ *for some* $\mathsf{s} \in \mathbb{N}$ *which depends on* S*, and with the previous topology* $\mathcal{T}_S$*. In the following we will frequently use the simplified notation* S *instead of the complete* $(S, \mathcal{T}_S)$*.*

If $S \subseteq {}^\bullet(\mathbb{R}^{\mathsf{s}})$ *and* $T \subseteq {}^\bullet(\mathbb{R}^{\mathsf{t}})$ *then we say that*

$$S \xrightarrow{\ f\ } T \quad \textit{in} \quad \mathbf{S}^\bullet\mathbb{R}^\infty$$

iff f *maps* S *in* T *and for every* $s \in S$ *we can write*

$$f(x) = {}^\bullet g\langle p, x\rangle \quad \forall x \in {}^\bullet V \cap S \tag{8.3.2}$$

for some

$$\begin{aligned}&V \textit{ open in } \mathbb{R}^{\mathsf{s}} \textit{ such that } s \in {}^\bullet V\\ &p \in {}^\bullet U, \textit{ where } U \textit{ is open in } \mathbb{R}^{\mathsf{p}}\\ &g \in \mathcal{C}^\infty(U \times V, \mathbb{R}^{\mathsf{t}}).\end{aligned}$$

Moreover we will consider on $\mathbf{S}^\bullet\mathbb{R}^\infty$ *the forgetful functor given by the inclusion* $|-| : \mathbf{S}^\bullet\mathbb{R}^\infty \hookrightarrow$ **Set***, i.e.* $|(S, \mathcal{T}_S)| := S$*. The category* $\mathbf{S}^\bullet\mathbb{R}^\infty$ *will be called* the category of subsets of ${}^\bullet\mathbb{R}^\infty$ *(but note that here* ∞ *indicates the class of regularity of the functions we are considering).*

Remark.

1. In other words *locally* a $\boldsymbol{\mathcal{C}}^\infty$ function $f : S \longrightarrow T$ between two types of figures $S \subseteq {}^\bullet(\mathbb{R}^{\mathsf{s}})$ and $T \subseteq {}^\bullet(\mathbb{R}^{\mathsf{t}})$ is constructed in the following way:

(a) start with an ordinary standard function $g \in \mathcal{C}^\infty(U \times V, \mathbb{R}^{\mathsf{t}})$, with U open in $\mathbb{R}^{\mathsf{p}}$ and V open in $\mathbb{R}^{\mathsf{s}}$. The space $\mathbb{R}^{\mathsf{p}}$ has to be thought as a space of parameters for the function g;

(b) consider its Fermat extension obtaining ${}^\bullet g : {}^\bullet(U \times V) \longrightarrow {}^\bullet(\mathbb{R}^{\mathsf{t}})$;

(c) consider the composition ${}^\bullet g \circ \langle -, - \rangle : {}^\bullet U \times {}^\bullet V \longrightarrow {}^\bullet(\mathbb{R}^{\mathsf{t}})$, where $\langle -, - \rangle$ is the map α given by (8.2.2);

(d) fix a parameter $p \in {}^\bullet U$ as a first variable of the previous composition, i.e. consider ${}^\bullet g\langle p, - \rangle : {}^\bullet V \longrightarrow {}^\bullet(\mathbb{R}^{\mathsf{t}})$. Locally, the map f is of this form: $f = {}^\bullet g\langle p, - \rangle$.

2. Because in the Definition 8.3.1 we ask $s \in {}^\bullet V$ we have that $\mathcal{V} := {}^\bullet V \cap S$ is a neighborhood of s defined by V in S (see (8.3.1)). Analogously ${}^\bullet U$ is a neighborhood of the parameter p.

To simplify the presentation, in case the context will be sufficiently clear, we shall consider the coupling of variables[7] (S, s), (T, t), (p, p), (q, q) etc. in properties of the form $S \subseteq \mathbb{R}^{\mathsf{s}}$, $T \subseteq \mathbb{R}^{\mathsf{t}}$, $p \in {}^\bullet(\mathbb{R}^{\mathsf{p}})$ or $q \in {}^\bullet(\mathbb{R}^{\mathsf{q}})$ respectively. In fact, in these cases we have that the second variable in the pairing, e.g. the number $\mathsf{s} \in \mathbb{N}$ in the pairing (S, s), is uniquely determined by the first variable S. E.g. the number $\mathsf{p} \in \mathbb{N}$ is uniquely determined by the point $p \in {}^\bullet(\mathbb{R}^{\mathsf{p}})$. Therefore, if we denote by $\sigma(V) \in \mathbb{N}$ the unique $\mathsf{v} \in \mathbb{N}$ in a pairing (V, v), then any formula of the form $\mathcal{P}(V, \mathsf{v})$ can be interpreted as

$$\mathsf{v} = \sigma(V) \implies \mathcal{P}(V, \mathsf{v}).$$

Now we have to prove that $\mathbf{S}^\bullet\mathbb{R}^\infty$ verifies the hypothesis of Section 6.1 about the category of the types of figures. Firstly, we prove that $\mathbf{S}^\bullet\mathbb{R}^\infty$ is indeed a category. In the following proofs we will frequently use the properties

$$x \in {}^\bullet U \iff {}^\circ x \in U$$

$$^\circ\left({}^\bullet g\langle p, x \rangle\right) = g({}^\circ p, {}^\circ x).$$

The first one follows from (8.2.1), and the second one can be proved directly:

$$^\circ\left({}^\bullet g\langle p, x \rangle\right) = \left(g(p_t, x_t)\right)|_{t=0} = g(p_0, x_0) = g({}^\circ p, {}^\circ x).$$

Theorem 8.3.2. $\mathbf{S}^\bullet\mathbb{R}^\infty$ *is a category*

Proof: In this proof we will consider the coupling of variables (S, s), (T, t), (R, r), (p, p) and (q, q). If we consider any $p \in {}^\bullet\mathbb{R}$ and the projection $g : (r, s) \in \mathbb{R} \times \mathbb{R}^{\mathsf{s}} \mapsto s \in \mathbb{R}^{\mathsf{s}}$, then we have that ${}^\bullet g\langle p, s \rangle_t = g(p_t, s_t) = s_t$, hence ${}^\bullet g\langle p, s \rangle = s$ and this suffices to prove that the identity 1_S for $S \in \mathbf{S}^\bullet\mathbb{R}^n$ is always an arrow of $\mathbf{S}^\bullet\mathbb{R}^\infty$.

[7] Note the use of a different font for the second variable in the pairing, so that it will be easier to identify such pairings.

Now let us consider

$$S \xrightarrow{f} T \xrightarrow{g} R \quad \text{in} \quad \mathbf{S}^\bullet\mathbb{R}^\infty$$

and a point $s \in S$. We have to prove that $f \circ g$ is again an arrow of $\mathbf{S}^\bullet\mathbb{R}^\infty$. Using self-evident notations we can assert that

$$f(x) = {}^\bullet h\langle p, x\rangle \quad \forall x \in {}^\bullet V_s \cap S \ni s \tag{8.3.3}$$

$$g(y) = {}^\bullet k\langle q, y\rangle \quad \forall y \in {}^\bullet V_{fs} \cap T \ni f(s), \tag{8.3.4}$$

where $h \in \mathcal{C}^\infty(U_p \times V_s, \mathbb{R}^{\mathsf{t}})$ and $k \in \mathcal{C}^\infty(U_q \times V_{fs}, \mathbb{R}^{\mathsf{r}})$. Hence $U_q \times h^{-1}(V_{fs})$ is open in $\mathbb{R}^{\mathsf{q}} \times \mathbb{R}^{\mathsf{p}} \times \mathbb{R}^{\mathsf{s}}$. But ${}^\circ f(s) = h({}^\circ p, {}^\circ s) \in V_{fs}$ because $f(s) \in {}^\bullet V_{fs}$, and ${}^\circ q \in U_q$, so

$$({}^\circ q, {}^\circ p, {}^\circ s) \in U_q \times h^{-1}(V_{fs}). \tag{8.3.5}$$

Hence, we can find three open sets $A \subseteq \mathbb{R}^{\mathsf{q}}$, $B \subseteq \mathbb{R}^{\mathsf{p}}$ and $C \subseteq \mathbb{R}^{\mathsf{s}}$ such that$({}^\circ q, {}^\circ p, {}^\circ s) \in A \times B \times C \subseteq U_q \times h^{-1}(V_{fs})$ and we can correctly define

$$\rho : (x_1, x_2, y) \in A \times B \times C \mapsto k\left[x_1, h(x_2, y)\right] \in \mathbb{R}^{\mathsf{r}},$$

obtaining a map $\rho \in \mathcal{C}^\infty(A \times B \times C, \mathbb{R}^{\mathsf{r}})$; this is the first step to prove that locally the composition $f(g(-))$ is of the form (8.3.2). The parameter corresponding to this local form is $\langle q, p\rangle \in {}^\bullet(A \times B)$ because of (8.2.2) and (8.3.5). The neighborhood we are searching for this local equality is ${}^\bullet C \cap S \ni s$, in fact let us take a generic $x \in {}^\bullet C \cap S$, then

$$({}^\circ p, {}^\circ x) \in B \times C \subseteq h^{-1}(V_{fs}) \subseteq U_p \times V_s. \tag{8.3.6}$$

Therefore, ${}^\circ x \in V_s$ and hence $x \in {}^\bullet V_s \cap S$ so that we can use (8.3.3) obtaining $f(x) = {}^\bullet h\langle p, x\rangle$. We can continue, saying that then ${}^\circ f(x) = h({}^\circ p, {}^\circ x) \in V_{fs}$, because of (8.3.6), and hence $f(x) \in {}^\bullet V_{fs} \cap T$. Now we can apply (8.3.4) with $y = f(x)$ obtaining

$$g(f(x)) = {}^\bullet k\langle q, fx\rangle = {}^\bullet k\langle q, {}^\bullet h\langle p, x\rangle\rangle = {}^\bullet \rho\langle q, p, x\rangle.$$

■

To prove that $\mathbf{S}^\bullet\mathbb{R}^\infty$ is a subcategory of the category **Top** of topological spaces, we need the following

Theorem 8.3.3. *If $f : S \longrightarrow T$ in $\mathbf{S}^\bullet\mathbb{R}^\infty$, then f is continuous with respect to the topologies $\mathcal{T}_S$ and $\mathcal{T}_T$.*

Proof: Take A open in T and $s \in f^{-1}(A)$; we have to prove that, for some W_s open in $\mathbb{R}^{\mathsf{s}}$, we have $s \in {}^\bullet W_s \cap S \subseteq f^{-1}(A)$ (see (8.3.1)). From $f(s) \in A \in \mathcal{T}_T$ we have that $f(s) \in {}^\bullet V_{fs} \cap T \subseteq A$ for some open set $V_{fs} \subseteq \mathbb{R}^{\mathsf{t}}$. On the other hand, from $s \in f^{-1}(A) \subseteq S$ and the Definition 8.3.1 of arrow in $\mathbf{S}^\bullet\mathbb{R}^\infty$, it follows that in a neighborhood $\mathcal{V}_s := {}^\bullet V_s \cap S$ of s we can write the function f as $f = {}^\bullet g\langle p, -\rangle$, where $p \in {}^\bullet U \subseteq {}^\bullet(\mathbb{R}^{\mathsf{p}})$ is the usual parameter. Diagrammatically the situation is as follow:

$$\begin{tikzcd}
S \arrow[r, "f"] & T & V_{fs} \ni f(s) \arrow[l, hook'] \\
s \in \mathcal{V}_s \arrow[u, hook] \arrow[ur, "{}^\bullet g\langle p,-\rangle"'] \arrow[r, hook] & {}^\bullet(\mathbb{R}^{\mathsf{s}})
\end{tikzcd}$$

Intuitively, the idea is to consider the standard part of f and to define the open set W_s we searched for using the counter image, along this standard part, of the open set V_{fs}. In fact, let us define

$$W_s := \left[g({}^\circ p, -)\right]^{-1}(V_{fs}),$$

then W_s is open in $\mathbb{R}^{\mathsf{s}}$ and we have

$$\begin{aligned}
s \in {}^\bullet W_s &\iff {}^\circ s \in W_s \\
&\iff g({}^\circ p, {}^\circ s) \in V_{fs} \\
&\iff {}^\bullet g\langle p, s\rangle \in {}^\bullet V_{fs} \\
&\iff f(s) \in {}^\bullet V_{fs}.
\end{aligned} \tag{8.3.7}$$

The latter property $f(s) \in {}^\bullet V_{fs}$ is true, so we have that $s \in {}^\bullet W_s \cap S$. It remains to prove that ${}^\bullet W_s \cap S \subseteq f^{-1}(A)$. Let us take a point $x \in {}^\bullet W_s \cap S$, then ${}^\circ x \in W_s \subseteq V_s$, and hence $g({}^\circ p, {}^\circ x) \in V_{fs}$. So, $x \in {}^\bullet V_s$ and $f(x) = {}^\bullet g\langle p, x\rangle \in {}^\bullet V_{fs}$. But $f(x) \in T$, so $f(x) \in {}^\bullet V_{fs} \cap T \subseteq A$.

■

In the following theorem we prove that the category $\mathbf{S}^\bullet\mathbb{R}^\infty$ is closed with respect to subspaces (with the induced topology) and the corresponding inclusion:

Theorem 8.3.4. *Let $S \subseteq {}^\bullet(\mathbb{R}^{\mathsf{s}})$, and $U \in \mathcal{T}_S$ be an open set, with $i : U \hookrightarrow S$ the corresponding inclusion. Then we have:*

1. *$(U \prec \mathcal{T}_S) \in \mathbf{S}^\bullet\mathbb{R}^\infty$, that is the topology $\mathcal{T}_U$ defined by (8.3.1) coincides with the induced topology $\mathcal{T}_{(U \prec S)}$.*
2. *The inclusion $i : U \longrightarrow S$ is an arrow of $\mathbf{S}^\bullet\mathbb{R}^\infty$.*

Proof : By (8.3.1), if $A \in \mathcal{T}_U$ we have that A is the union of ${}^\bullet V \cap U \subseteq A$ for V open in $\mathbb{R}^{\mathsf{s}}$. But ${}^\bullet V \cap U = ({}^\bullet V \cap S) \cap U$ because $U \subseteq S$. Therefore, A is the union of sets of the form $W \cap U$ with $W \in \mathcal{T}_S$, because $W := {}^\bullet V \cap S \in \mathcal{T}_S$, i.e. A is open in the subspace $(U \prec \mathcal{T}_S)$. Vice versa if we can write $A = B \cap U$, where B is open in $\mathcal{T}_S$, then by (8.3.1)

$$\forall s \in A \, \exists V \text{ open in } \mathbb{R}^{\mathsf{s}} : \quad s \in {}^\bullet V \cap S \subseteq B,$$

so $s \in A \subseteq U$ and hence $s \in {}^\bullet V \cap U \subseteq B \cap U = A$, and this proves that $A \in \mathcal{T}_U$, thus $\mathcal{T}_U = \mathcal{T}_{(U \prec S)}$.

Property 2. can be proved following ideas similar to those used in Theorem 8.3.2 to prove that the identities 1_S are always arrows of the category $\mathbf{S}^\bullet\mathbb{R}^\infty$.

$\blacksquare$

Now we will prove the closure of $\mathbf{S}^\bullet\mathbb{R}^\infty$ with respect to restriction to open sets (see Hypothesis 6.1.1):

Theorem 8.3.5. *Let $f : S \longrightarrow T$ in $\mathbf{S}^\bullet\mathbb{R}^\infty$, U an open set in S and V an open set in T, with $f(U) \subseteq V$. Then*

$$f|_U : (U \prec S) \longrightarrow (V \prec T) \quad \textit{in} \quad \mathbf{S}^\bullet\mathbb{R}^\infty .$$

Proof: Recalling the Definition 8.3.1 of an arrow in $\mathbf{S}^\bullet\mathbb{R}^\infty$, and using the fact that, by hypotheses we already have that $f : S \longrightarrow T$ in $\mathbf{S}^\bullet\mathbb{R}^\infty$, we only have to prove that equalities of the form (8.3.2) hold locally also with respect to the topology of $(U \prec S)$. Because of the previous Theorem 8.3.4 we can work with $\mathcal{T}_U$ instead of $\mathcal{T}_{(U \prec S)}$. Take $s \in U$, since $U \subseteq S$ and $f : S \longrightarrow T$ in $\mathbf{S}^\bullet\mathbb{R}^\infty$, using the usual notations we can write

$$f(x) = {}^\bullet g\langle p, x\rangle \tag{8.3.8}$$

for every $x \in {}^\bullet V_s \cap S \ni s$ and where $g \in \mathcal{C}^\infty(U_p \times V_s)$. Hence $s \in {}^\bullet V_s \cap U$, and because ${}^\bullet V_s \cap U \subseteq {}^\bullet V_s \cap S$ we have again the equality (8.3.8) in the neighborhood ${}^\bullet V_s \cap U$ of s in $\mathcal{T}_U$, and this proves the conclusion.

$\blacksquare$

Since it is trivial to prove that $\mathbf{S}^\bullet\mathbb{R}^\infty$ contains all the constant maps (it suffices to take $g(p, x) := p$), to prove that $\mathbf{S}^\bullet\mathbb{R}^\infty$ is a category of the types of figures, it remain to prove the sheaf property:

Theorem 8.3.6. *Let H, $K \in \mathbf{S}^\bullet\mathbb{R}^\infty$, and $(H_i)_{i\in I}$ be an open cover of H such that the map $f : H \longrightarrow K$ verifies*

$$\forall i \in I : \quad f|_{H_i} \in \mathbf{S}^\bullet\mathbb{R}^\infty(H_i, K). \tag{8.3.9}$$

Then

$$f : H \longrightarrow K \quad \textit{in} \quad \mathbf{S}^\bullet\mathbb{R}^\infty .$$

Proof: Take $s \in H$, then $s \in H_i$ for some $i \in I$, and from (8.3.9) it follows that we can write

$$\left(f|_{H_i}\right)(x) = f(x) = {}^\bullet g\langle p, x\rangle \quad \forall x \in {}^\bullet V_s \cap H_i.$$

But H_i is open in H so that we can also say that $s \in {}^\bullet V_s' \cap H \subseteq H_i$ for some open set V_s' of $\mathbb{R}^{\mathsf{h}}$. The new neighborhood ${}^\bullet(V_s \cap V_s') \cap H$ of s and the restriction $g|_{U_p \times (V_s \cap V_s')}$ verify that the function f is locally of the form

$f = {}^\bullet g\langle p, -\rangle$ in a neighborhood of s in H.

$\blacksquare$

We have proved that $\mathbf{S}^\bullet\mathbb{R}^\infty$ and the forgetful functor $|-|$ verify the hypotheses of Section 6.1 about the category of the types of figures and hence we can define

$$^\bullet\boldsymbol{\mathcal{C}}^\infty := \overline{\mathbf{S}^\bullet\mathbb{R}^\infty}.$$

Each object of $^\bullet\boldsymbol{\mathcal{C}}^\infty$ will be called a *Fermat space*.

We close this section with the following simple but useful result that permits to obtain functions in $\mathbf{S}^\bullet\mathbb{R}^\infty$ starting from ordinary $\mathcal{C}^\infty$ functions.

Theorem 8.3.7. *Let $f \in \mathcal{C}^\infty(\mathbb{R}^{\mathsf{k}}, \mathbb{R}^{\mathsf{h}})$ be a standard $\mathcal{C}^\infty$ function and $H \subseteq {}^\bullet(\mathbb{R}^{\mathsf{h}})$ and $K \subseteq {}^\bullet(\mathbb{R}^{\mathsf{k}})$ be subsets of Fermat reals. If the function f verifies ${}^\bullet f|_K(K) \subseteq H$, then*

$$^\bullet f|_K : K \longrightarrow H \quad in \quad \mathbf{S}^\bullet\mathbb{R}^\infty .$$

Proof: It suffices to define $g(x, y) := f(y)$ for $x \in \mathbb{R}$ and $y \in \mathbb{R}^{\mathsf{k}}$ to obtain that

$$^\bullet g\langle 0, k\rangle_t = g(0, k_t) = f(k_t) = {}^\bullet f(k)_t,$$

that is $({}^\bullet f|_K) = {}^\bullet g\langle 0, k\rangle \in H$ for every $k \in K$.

$\blacksquare$

Chapter 9

The Fermat functor

9.1 Putting a structure on the sets ${}^\bullet X$

Now the problem is: what Fermat space could we associate to sets like ${}^\bullet X$ or D?

Definition 9.1.1. *Let $X \in \mathcal{C}^\infty$, then for any subset $Z \subseteq {}^\bullet X$ we call ${}^\bullet(ZX)$ the extended space generated on Z (see Section 6.2) by the following set of figures $d : T \longrightarrow Z$ (where $T \subseteq {}^\bullet(\mathbb{R}^{\mathrm{t}})$ is a type of figure in $\mathbf{S}{}^\bullet\mathbb{R}^\infty$)*

$$d \in \mathcal{D}^0_T(Z) \quad :\Longleftrightarrow \quad \begin{array}{l} d \text{ is constant or we can write} \\ d = {}^\bullet h|_T \text{ for some } h \in_V X \text{ such that } T \subseteq {}^\bullet V. \end{array} \tag{9.1.1}$$

Thus in the non-trivial case we start from a standard figure $h \in_V X$ of type $V \in \mathbf{O}\mathbb{R}^\infty$ such that ${}^\bullet V \supseteq T$; we extend this figure obtaining ${}^\bullet h : {}^\bullet V \longrightarrow {}^\bullet X$, and finally the restriction ${}^\bullet h|_T$ is a generating figure if it maps T in Z. This choice is very natural, and the adding of the alternative "d is constant" in the previous disjunction is due to the need to have all constant figures in a family of generating figures.

Using this definition of ${}^\bullet(ZX)$, we set (with some abuses of language)

$$\begin{gathered} {}^\bullet X := {}^\bullet({}^\bullet X X) \\ D := {}^\bullet(D\mathbb{R}) \\ {}^\bullet\mathbb{R} := {}^\bullet({}^\bullet\mathbb{R}\mathbb{R}) \\ {}^\bullet\mathbb{R}^k := {}^\bullet({}^\bullet(\mathbb{R}^k)\mathbb{R}^k) \\ D_k := {}^\bullet(D_k\mathbb{R}^k). \end{gathered}$$

We will call ${}^\bullet(ZX)$ *the Fermat space induced on Z by $X \in \mathcal{C}^\infty$*. We can now study the extension functor:

Theorem 9.1.2. *Let $f \in \mathcal{C}^\infty(X, Y)$ and Z a subset of ${}^\bullet X$ with ${}^\bullet f(Z) \subseteq W \subseteq {}^\bullet Y$, then in ${}^\bullet\mathcal{C}^\infty$ we have that*

$$ {}^\bullet(ZX) \xrightarrow{{}^\bullet f|_Z} {}^\bullet(WY). $$

Therefore ${}^\bullet(-) : \mathcal{C}^\infty \longrightarrow {}^\bullet\mathcal{C}^\infty$ is a functor, called the Fermat functor.

Proof: Take a figure $\delta \in_S {}^\bullet(ZX)$ of type $S \in \mathbf{S}^\bullet\mathbb{R}^\infty$ in the domain. We have to prove that $\delta \cdot {}^\bullet f|_Z$ locally factors through $\mathbf{S}^\bullet\mathbb{R}^\infty$ and $\mathcal{D}^0(W)$ (see in Section 6.2 the definition of space generated by a family of figures). Hence taking $s \in S$, since $\delta \in_S {}^\bullet(ZX)$, we can write $\delta|_U = f_1 \cdot d$, where U is an open neighborhood of s, $f_1 \in \mathbf{S}^\bullet\mathbb{R}^\infty(U \prec S, T)$ and $d \in \mathcal{D}^0_T(Z)$:

$$\begin{tikzcd} S \arrow[rr, "\delta"] & & Z \\ & & \\ s \in U \arrow[uu, hook] \arrow[uurr, "\delta|_U"] \arrow[rr, "f_1"'] & & T \arrow[uu, "d"'] \end{tikzcd}$$

We omit the trivial case d constant, hence we can suppose, using the same notations used in the Definition 9.1.1, to have $d = {}^\bullet h|_T : T \longrightarrow Z$ with $h \in_V X$. Therefore

$$ (\delta \cdot {}^\bullet f|_Z)|_U = \delta|_U \cdot {}^\bullet f|_Z = f_1 \cdot d \cdot {}^\bullet f|_Z = f_1 \cdot {}^\bullet h|_T \cdot {}^\bullet f|_Z = f_1 \cdot {}^\bullet(hf)|_T. $$

But $hf \in_V Y$ since $f \in \mathcal{C}^\infty(X, Y)$ and $h \in_V X$, so $(\delta \cdot {}^\bullet f|_Z)|_U = f_1 \cdot d_1$, where $d_1 := {}^\bullet(hf)|_T \in \mathcal{D}^0_T(W)$, which is the conclusion. The other functorial properties, i.e. ${}^\bullet(1_X) = 1_{{}^\bullet X}$ and ${}^\bullet(f \cdot g) = {}^\bullet f \cdot {}^\bullet g$, follow directly from the definition of the Fermat extension ${}^\bullet f$ of $f \in \mathcal{C}^\infty(X, Y)$. ■

9.2 The Fermat functor preserves product of manifolds

We want to prove that the bijective applications α defined in 8.2.2 and β defined in 8.2.3, i.e.

$$ \alpha_{MN} : ([x]_\sim, [y]_\sim) \in {}^\bullet M \times {}^\bullet N \longmapsto [(x, y)]_\sim \in {}^\bullet(M \times N) \tag{9.2.1} $$

$$ \beta_{MN} : [z]_\sim \in {}^\bullet(M \times N) \longmapsto ([z \cdot p_M]_\sim, [z \cdot p_N]_\sim) \in {}^\bullet M \times {}^\bullet N \tag{9.2.2} $$

are arrows of $ECInfty$. Where it will be clear from the context, we shall use the simplified notations $\alpha := \alpha_{MN}$ and $\beta := \beta_{MN}$. To simplify the proof we will use the following preliminary results. The first one is a general property of the cartesian closure $\bar{\mathcal{F}}$ of a category of figures $\mathcal{F}$ (see Chapter 6).

Lemma 9.2.1. *Suppose that $\mathcal{F}$ admits finite products $K \times J$ for every objects K, $J \in \mathcal{F}$, and an isomorphism*[1]

$$\gamma_{KJ} : \overline{K \times J} \xrightarrow[\sim]{} \bar{K} \times \bar{J} \quad \text{in} \quad \bar{\mathcal{F}}.$$

Moreover, let Z, X, $Y \in \bar{\mathcal{F}}$ with X and Y generated by $\mathcal{D}^X$ and $\mathcal{D}^Y$ respectively. Then we have

$$X \times Y \xrightarrow{f} Z \quad \text{in} \quad \bar{\mathcal{F}}$$

if and only if for any K, $J \in \mathcal{F}$ and $d \in \mathcal{D}^X_K$, $\delta \in \mathcal{D}^Y_J$ we have

$$\gamma_{KJ} \cdot (d \times \delta) \cdot f \in_{K \times J} Z.$$

The second Lemma asserts that the category of figures $\mathcal{F} = \mathbf{S}^\bullet\mathbb{R}^\infty$ verifies the hypotheses of the previous one.

Lemma 9.2.2. *The category $\mathbf{S}^\bullet\mathbb{R}^\infty$ admits finite products and the above mentioned isomorphisms γ_{KJ}. For $K \subseteq {}^\bullet(\mathbb{R}^\mathsf{k})$ and $J \subseteq {}^\bullet(\mathbb{R}^\mathsf{j})$ these are given by*

$$K \times J = \left\{ \langle x, y \rangle \in {}^\bullet(\mathbb{R}^{\mathsf{k}+\mathsf{j}}) \mid x \in K \ , \ j \in J \right\}$$

$$\gamma_{KJ} : \langle x, y \rangle \in K \times J \longmapsto (x, y) \in K \times_{\mathrm{s}} J,$$

where we recall that $\langle x, y \rangle = \alpha_{\mathbb{R}^\mathsf{k}\mathbb{R}^\mathsf{j}}(x, y) = [t \mapsto (x_t, y_t)]_\sim$ and where $K \times_{\mathrm{s}} J$ is the set theoretical product of the subsets K and J.
Moreover let M, N be $\mathcal{C}^\infty$ manifolds, and $h \in_V M$, $l \in_{V'} N$ with $K \subseteq {}^\bullet V$ and $J \subseteq {}^\bullet V'$, then

$$\gamma_{KJ} \cdot ({}^\bullet h|_K \times {}^\bullet l|_J) \cdot \alpha_{MN} = {}^\bullet(h \times l)|_{K \times J}.$$

The proofs of these lemmas are direct consequences of the given definitions.

Theorem 9.2.3. *Let M, N be $\mathcal{C}^\infty$ manifolds, then in ${}^\bullet\boldsymbol{\mathcal{C}}^\infty$ we have the isomorphism*

$${}^\bullet(M \times N) \simeq {}^\bullet M \times {}^\bullet N.$$

Proof: Note that in the statement each manifold is identified with the corresponding $\boldsymbol{\mathcal{C}}^\infty$ space $\bar{M}$. Hence we mean ${}^\bullet M = {}^\bullet\bar{M} = {}^\bullet({}^\bullet M\bar{M})$ (see Definition 9.1.1 for the notation ${}^\bullet(ZX)$). To prove that α is a ${}^\bullet\boldsymbol{\mathcal{C}}^\infty$ arrow we can use Lemma 9.2.1, because of Lemma 9.2.2 and considering that ${}^\bullet M$ and ${}^\bullet N$ are generated by $\mathcal{D}^0({}^\bullet M)$ and $\mathcal{D}^0({}^\bullet N)$. Since these generating sets are defined using a disjunction (see (9.1.1)) we have to check four cases depending on $d \in \mathcal{D}^0_K({}^\bullet M)$ and $\delta \in \mathcal{D}^0_J({}^\bullet N)$. In the first case we have

[1]Recall the definition of the embedding $(\bar{-}) : \mathcal{F} \longrightarrow \bar{\mathcal{F}}$ given in Section 6.2.

$d = {}^\bullet h|_K \in \mathcal{D}^0_K({}^\bullet M)$ and $\delta = {}^\bullet l|_J \in \mathcal{D}^0_J({}^\bullet N)$ (we are using the same notations of the previous Lemma 9.2.2). Thus

$$\gamma_{KJ} \cdot (d \times \delta) \cdot \alpha = \gamma_{KJ} \cdot ({}^\bullet h|_K \times {}^\bullet l|_J) \cdot \alpha = {}^\bullet (h \times l)|_{K \times J}.$$

That is $\gamma_{KJ} \cdot (d \times \delta) \cdot \alpha$ is a generating element in ${}^\bullet(M \times N)$, and so it is also a figure. In the second case let us suppose δ constant with value $n \in {}^\bullet N$ and $d = {}^\bullet h|_K \in \mathcal{D}^0_K({}^\bullet M)$. Take a chart $l : \mathbb{R}^{\mathsf{p}} \longrightarrow U$ on ${}^\circ n = n_0 \in U \subseteq N$ and let $W := {}^\bullet(\mathbb{R}^{\mathsf{p}})$, $p := {}^\bullet l^{-1}(n) \in W$. Note that ${}^\bullet l(p) = n = \delta(-)$. We have to prove that $\gamma_{KJ} \cdot (d \times \delta) \cdot \alpha \in_{K \times J} {}^\bullet(M \times N)$, so let us start to calculate the map $\gamma_{KJ} \cdot (d \times \delta) \cdot \alpha$ at a generic element $\langle k, j\rangle \in K \times J$. We have

$$\begin{aligned}
\alpha\{(d \times \delta)[\gamma_{KJ}(\langle k, j\rangle)]\} &= \alpha\,[(d \times \delta)(k, j)] \\
&= \alpha\,[d(k), n] \\
&= \alpha[{}^\bullet h(k), {}^\bullet l(p)] \\
&= \{\gamma_{KW} \cdot [{}^\bullet h|_K \times {}^\bullet l|_J] \cdot \alpha\}\langle k, p\rangle \\
&= {}^\bullet(h \times l)|_{K \times W}\langle k, p\rangle,
\end{aligned} \tag{9.2.3}$$

where we have used once again the equality of Lemma 9.2.2. Thus let us call τ the map $\tau : \langle k, j\rangle \in |K \times J| \mapsto \langle k, p\rangle \in |K \times W|$, so that we can write (9.2.3) as

$$\gamma_{KJ} \cdot (d \times \delta) \cdot \alpha = \tau \cdot {}^\bullet(h \times l)|_{K \times W}.$$

But ${}^\bullet(h \times l)|_{K \times W}$ is a generating figure of ${}^\bullet(M \times N)$ and τ is an arrow of $\mathbf{S}{}^\bullet\mathbb{R}^\infty$, and this proves that $\gamma_{KJ} \cdot (d \times \delta) \cdot \alpha \in_{K \times J} {}^\bullet(M \times N)$. The remaining cases are either trivial (both d and δ constant) or analogous to the latter one.

To prove that the map β_{MN} is an arrow of ${}^\bullet\mathcal{C}^\infty$ is simpler. Indeed, take $d \in_H {}^\bullet(M \times N)$ to prove that $d \cdot \beta_{MN} \in_H {}^\bullet M \times {}^\bullet N$. Due to the universal property of the product ${}^\bullet M \times {}^\bullet N$, it suffices to consider the composition of this map $d \cdot \beta_{MN}$ with the projections of this product. But, if $p_M : M \times N \longrightarrow M$ is the projection on M, then

$${}^\bullet M \times {}^\bullet N \xrightarrow{\alpha_{MN}} {}^\bullet(M \times N) \xrightarrow{{}^\bullet p_M} {}^\bullet M$$

and ${}^\bullet p_M(\alpha_{MN}(x, y)) = {}^\bullet p_M(\langle x, y\rangle) = x$, so $\alpha_{MN} \cdot {}^\bullet p_M$ is the projection of the product ${}^\bullet M \times {}^\bullet N$ on ${}^\bullet M$. Therefore the conclusion $d \cdot \beta_{MN} \in_H {}^\bullet M \times {}^\bullet N$ is equivalent to

$$d \cdot \beta_{MN} \cdot \alpha_{MN} {}^\bullet p_M = d \cdot {}^\bullet p_M \in_H {}^\bullet M$$

$$d \cdot \beta_{MN} \cdot \alpha_{MN} {}^\bullet p_N = d \cdot {}^\bullet p_N \in_H {}^\bullet M$$

which are true since ${}^\bullet p_M$ and ${}^\bullet p_N$ are arrows of ${}^\bullet\mathcal{C}^\infty$. ■

In the following we shall always use the isomorphism α to identify these spaces, hence we write ${}^\bullet M \times {}^\bullet N = {}^\bullet(M \times N)$, e.g. ${}^\bullet(\mathbb{R}^d) = ({}^\bullet\mathbb{R})^d =: {}^\bullet\mathbb{R}^d$.

9.2.1 Figures of Fermat spaces

In this section we want to understand better the figures of the Fermat space ${}^\bullet(ZX)$; we will use these results later, for example when we will study the embedding of **Man** into ${}^\bullet\boldsymbol{\mathcal{C}}^\infty$, or to prove some logical properties of the Fermat functor.

From the general definition of $\bar{\mathcal{F}}$–space generated by a family of figures $\mathcal{D}^0$ (see Section 6.2), a figure $\delta \in_S {}^\bullet(ZX)$, for $S \in \mathbf{S}^\bullet\mathbb{R}^\infty$, can be locally factored as $\delta|_V = f \cdot d$ through an arrow $f \in \mathbf{S}^\bullet\mathbb{R}^\infty(V,T)$ and a generating function $d \in \mathcal{D}^0_T(Z)$; here $V = V(s)$ is an open neighborhood of the considered point $s \in S$, so that we can always suppose V to be of the form $V = {}^\bullet B \cap S$ (see in (8.3.1) the definition of topology for S). Hence, either $\delta|_V$ is constant (if d is constant) or we can write $d = {}^\bullet h|_T$ and $f = {}^\bullet g(p,-)$ so that

$$\delta(x) = d[f(x)] = {}^\bullet h[{}^\bullet g(p,x)] = {}^\bullet(gh)(p,x) \quad \forall x \in V = {}^\bullet B \cap S,$$

where $A \times B$ is an open neighborhood of $({}^\circ p, {}^\circ s)$. Therefore we can write

$$\delta(x) = {}^\bullet\gamma(p,x) \quad \forall x \in {}^\bullet B \cap S,$$

with $\gamma := g|_{A\times B} \cdot h \in \boldsymbol{\mathcal{C}}^\infty(A \times B, X)$. Thus figures of ${}^\bullet(ZX)$ are locally necessarily either constant maps or a natural generalization of the maps of $\mathbf{S}^\bullet\mathbb{R}^\infty$, that is "extended $\boldsymbol{\mathcal{C}}^\infty$ arrows ${}^\bullet\gamma(-,-)$ with a fixed parameter ${}^\bullet\gamma(p,-)$". Using the properties of ${}^\bullet\boldsymbol{\mathcal{C}}^\infty$ and of its arrow $\alpha_{\mathbb{R}^\mathsf{p}\mathbb{R}^\mathsf{s}}$ it is easy to prove that these conditions are sufficient too. Moreover if $X = M$ is a manifold, the condition "$\delta|_V$ constant" can be omitted. In fact if $\delta|_V$ is constant with value $m \in Z \subseteq {}^\bullet M$, then taking a chart φ on ${}^\circ m \in M$ we can write $\delta(x) = m = {}^\bullet\gamma(p,x)$, where $p = {}^\bullet\varphi(m)$ and $\gamma(x,y) = \varphi^{-1}(x)$. We have proved the following

Theorem 9.2.4. *Let $X \in \boldsymbol{\mathcal{C}}^\infty$, $Z \subseteq {}^\bullet X$, $S \subseteq {}^\bullet\mathbb{R}^\mathsf{s}$ and $\delta : S \longrightarrow Z$. Then we have*

$$\delta \in_S {}^\bullet(ZX)$$

iff for every point $s \in S$ there exist an open set B in $\mathbb{R}^\mathsf{s}$ such that $s \in {}^\bullet B$ and such that either

$$\delta|_{{}^\bullet B\cap S} \textit{ is constant,} \tag{9.2.4}$$

or we can write

$$\delta(x) = {}^\bullet\gamma\langle p,x\rangle \quad \forall x \in {}^\bullet B \cap S$$

for some

$$p \in {}^\bullet A, \textit{ where } A \textit{ is open in } \mathbb{R}^\mathsf{p}$$
$$\gamma \in \mathcal{C}^\infty(A \times B, X).$$

Moreover if $X = M$ is a manifold, condition (9.2.4) can be omitted and there remains only the second alternative. ∎

Using this result we can prove several useful properties of the Fermat functor. The following ones say that we can arrive at the same Fermat space starting from several different constructions.

Theorem 9.2.5. *The Fermat functor has the following properties:*

1. *If $X \in \boldsymbol{\mathcal{C}}^\infty$ and $Z \subseteq |{}^\bullet X|$, then ${}^\bullet(ZX) = (Z \prec {}^\bullet X)$.*

2. *If $S \subseteq |{}^\bullet\mathbb{R}^{\mathsf{s}}|$, then $\bar{S} = {}^\bullet(S\mathbb{R}^{\mathsf{s}}) = (S \prec {}^\bullet\mathbb{R}^{\mathsf{s}})$.*

E.g. if $f : {}^\bullet\mathbb{R}^{\mathsf{s}} \longrightarrow {}^\bullet X$ is a ${}^\bullet\boldsymbol{\mathcal{C}}^\infty$ arrow, then we also have $f : \overline{{}^\bullet\mathbb{R}^{\mathsf{s}}} \longrightarrow {}^\bullet X$ because ${}^\bullet\mathbb{R}^{\mathsf{s}} = {}^\bullet({}^\bullet(\mathbb{R}^{\mathsf{s}})\mathbb{R}^{\mathsf{s}}) = \overline{{}^\bullet\mathbb{R}^{\mathsf{s}}}$ and the previous property 2 holds. Therefore, $f \in_{{}^\bullet\mathbb{R}^{\mathsf{s}}} {}^\bullet X$ and locally we can write f either as a constant function or, with the usual notations, as $f(x) = {}^\bullet\gamma(p, x)$. For functions $f : I \longrightarrow {}^\bullet X$ defined on some set $I \subseteq D_\infty$ of infinitesimals which contains $0 \in I$, these two alternatives globally holds instead of only locally, because the set of infinitesimals I is contained in any open neighborhood of 0.

Proof: To prove 1. let us consider a figure $\delta \in_S (Z \prec {}^\bullet X)$ of type $S \in \mathbf{S}{}^\bullet\mathbb{R}^\infty$ and let $i : Z \hookrightarrow |{}^\bullet X|$ be the inclusion. We have to prove that $\delta \in_S {}^\bullet(ZX)$, and we will prove it locally, that is using the sheaf property of the space ${}^\bullet(ZX)$. By the definition of subspace, we have that $\delta \cdot i = \delta \in_S {}^\bullet X = {}^\bullet(|{}^\bullet X|X)$, so that for every $s \in S$ we can locally factor the figure δ through $\mathbf{S}{}^\bullet\mathbb{R}^\infty$ and a generating figure $d \in \mathcal{D}^0_K(|{}^\bullet X|)$, i.e. $\delta|_U = f \cdot d$ for some open neighborhood U of s and some $f : (U \prec S) \longrightarrow K$ in $\mathbf{S}{}^\bullet\mathbb{R}^\infty$. If d is constant, then so is $\delta|_U$ and hence $\delta|_U \in_U {}^\bullet(ZX)$. Otherwise, we can write $d = {}^\bullet h|_K$ for some $h \in_V X$, with V open in $\mathbb{R}^{\mathsf{k}}$ such that $K \subseteq {}^\bullet V$ (see Definition 9.1.1). To prove that $\delta|_U \in_U {}^\bullet(ZX)$ we exactly need to prove that the map $\delta|_U$ factors in the same way, but with a generating figure d' having values in Z and not in the bigger $|{}^\bullet X|$ (like d does). For this reason we change the subset K with the smaller $K' := f(U) \subseteq K \subseteq {}^\bullet V \subseteq {}^\bullet\mathbb{R}^{\mathsf{k}}$, so $K' \in \mathbf{S}{}^\bullet\mathbb{R}^\infty$, and we set $d' := {}^\bullet h|_{K'}$. The map d' has values in Z, in fact for $x \in K' = f(U)$ we have $x = f(u)$ for some $u \in U$, and

$$d'(x) = {}^\bullet h(x) = {}^\bullet h(f(u)) = d(f(u)) = \delta(u) \in Z.$$

Hence $d' \in \mathcal{D}^0_K(Z)$ and $\delta|_U(u) = d(f(u)) = d'(f(u))$ for every $u \in U$, so $\delta|_U \in_U {}^\bullet(ZX)$. We have proved that

$$\forall s \in S\, \exists U \text{ open neighborhood of } s \text{ in } S : \quad \delta|_U \in_U {}^\bullet(ZX),$$

hence $\delta \in_S {}^\bullet(ZX)$ from the sheaf property of the space ${}^\bullet(ZX) \in {}^\bullet\boldsymbol{\mathcal{C}}^\infty$. For the opposite inclusion we only have to make the opposite passage: from $d' : K' \longrightarrow Z$ with values in Z to $d := d' : K' \longrightarrow |{}^\bullet X|$ with values in the bigger $|{}^\bullet X|$, but this is trivial.

Because of the just proved property 1., to prove 2. we have to verify only the equality $\bar{S} = {}^\bullet(S\mathbb{R}^{\mathsf{s}})$, so take a figure $\delta \in_T \bar{S} = (\mathbf{S}{}^\bullet\mathbb{R}^\infty(-, S), S)$

and then $\delta \in \mathbf{S}^\bullet\mathbb{R}^\infty(T,S)$. But $1_{\mathbb{R}^\mathsf{s}} \in_{\mathbb{R}^\mathsf{s}} \mathbb{R}^\mathsf{s}$, so ${}^\bullet(1_{\mathbb{R}^\mathsf{s}})|_S = 1_{{}^\bullet\mathbb{R}^\mathsf{s}}|_S = 1_S$, so $1_S \in \mathcal{D}^0_S(S)$ and $\delta = \delta \cdot 1_S$ factors through a map of $\mathbf{S}^\bullet\mathbb{R}^\infty(T,S)$ (δ itself) and a generating figure of $\mathcal{D}^0_S(S)$, i.e. $\delta \in_T {}^\bullet(S\mathbb{R}^\mathsf{s})$. To prove the opposite inclusion, let us take $\delta \in_T {}^\bullet(S\mathbb{R}^\mathsf{s})$, then from Theorem 9.2.4 we have that in a suitable neighborhood U of a given generic point $s \in T$ we have that either $\delta|_U$ is constant, or we can write $\delta|_U = {}^\bullet\gamma\langle p,-\rangle|_U$ for some $\gamma \in \mathcal{C}^\infty(A \times B, \mathbb{R}^\mathsf{s})$. In both cases we have that $\delta|_U \in \mathbf{S}^\bullet\mathbb{R}^\infty(U,S)$, so $\delta|_U \in_U \bar{S}$, and the conclusion follows from the sheaf property of $\bar{S}$. ■

9.3 The embedding of manifolds in ${}^\bullet\mathcal{C}^\infty$

If we consider a $\mathcal{C}^\infty$ space X, we have just seen that we have the possibility to associate a Fermat space to any subset $Z \subseteq |{}^\bullet X|$. Thus if X is separated we can put a structure of ${}^\bullet\mathcal{C}^\infty$ space on the set $|X|$ of standard points of X, by means of $\bar{X} := {}^\bullet(|X|X) = (|X| \prec {}^\bullet X)$. Intuitively X and $\bar{X}$ seem very similar, and in fact we have

Theorem 9.3.1. *Let X, Y be $\mathcal{C}^\infty$ separated spaces, then*

1. $\bar{X} = \bar{Y} \quad \Longrightarrow \quad X = Y$
2. $\bar{X} \xrightarrow{\ f\ } \bar{Y}$ *in* ${}^\bullet\mathcal{C}^\infty \quad \Longleftrightarrow \quad X \xrightarrow{\ f\ } Y$ *in* $\mathcal{C}^\infty$.

Hence $\mathcal{C}^\infty$ separated spaces are fully embedded in ${}^\bullet\mathcal{C}^\infty$, and so is **Man**.

Proof: The equality $\bar{X} = \bar{Y}$ implies the equality of the support sets $|X| = |Y|$. We consider now a figure $d \in_H X$ of type H, where H is an open set of $\mathbb{R}^\mathsf{h}$. Taking the extension of d and then the restriction to standard points we obtain

$$(H \prec {}^\bullet\bar{H}) \xrightarrow{{}^\bullet d|_H} (|X| \prec {}^\bullet X) = \bar{X} = \bar{Y}. \qquad (9.3.1)$$

But from Theorem 9.2.5 we have $(H \prec {}^\bullet\bar{H}) = (H \prec {}^\bullet\mathbb{R}^\mathsf{h}) = {}^\bullet(H\mathbb{R}^\mathsf{h}) = \bar{H}$, hence

$${}^\bullet d|_H = d : \bar{H} \longrightarrow \bar{Y} \quad \text{in} \quad {}^\bullet\mathcal{C}^\infty$$

and so $d \in_H \bar{Y}$. Therefore for every $s \in H$ either d is constant in some open neighborhood V of s, or, using the usual notations, we can write

$$d(x) = {}^\bullet\gamma(p,x) \quad \forall x \in {}^\bullet B \cap H = B \cap H, \qquad (9.3.2)$$

where ${}^\bullet B \cap H = B \cap H$ because $H \subseteq \mathbb{R}^\mathsf{h}$ is made of standard point only. Let us note that the equality in (9.3.2) has to be understood in the space ${}^\bullet Y$. Hence for every $x \in B \cap H$ we have that ${}^\circ d(x) \asymp {}^\circ[\gamma(p,x)]$ in Y, and so we can write $d(x) = \gamma(p_0, x)$ because Y is separated and $x \in B \cap H \subseteq \mathbb{R}^\mathsf{h}$ is standard. Therefore $d|_{B\cap H}$ is a Y-valued arrow of $\mathcal{C}^\infty$ defined in a neighborhood of the fixed s. The conclusion $d \in_H Y$ thus follows from the sheaf property of Y. Analogously we can prove the opposite inclusion, so $X = Y$.

If we suppose that $f : \bar{X} \longrightarrow \bar{Y}$ in ${}^\bullet\mathcal{C}^\infty$, then from the proof of *1.* we have seen that if $d \in_{_H} X$ then $d \in_{_H} \bar{X}$. Hence $f(d) \in_{_H} \bar{Y}$. But once again from the previous proof of *1.* we have seen that this implies that $f(d) \in_{_H} Y$, and so $f : X \longrightarrow Y$ in $\mathcal{C}^\infty$.

To prove the opposite implication it suffices to extend f so that ${}^\bullet f : {}^\bullet X \longrightarrow {}^\bullet Y$, to restrict it to standard points only so that

$$ {}^\bullet f|_{|X|} : (|X| \prec {}^\bullet X) = \bar{X} \longrightarrow (|Y| \prec {}^\bullet Y) = \bar{Y}, $$

and finally to consider that our spaces are separated so that ${}^\bullet f|_{|X|} = f$. ■

An immediate corollary of this theorem is that the extension functor is another full embedding for separated spaces.

Corollary 9.3.2. *Let X, Y be $\mathcal{C}^\infty$ separated spaces, then*

1. ${}^\bullet X = {}^\bullet Y \quad \Longrightarrow \quad X = Y$
2. *If* ${}^\bullet X \xrightarrow{\ f\ } {}^\bullet Y$ *in* ${}^\bullet\mathcal{C}^\infty$ *and* $f(|X|) \subseteq |Y|$ *then*
$$ X \xrightarrow{\ f|_{|X|}\ } Y \quad \textit{in} \quad \mathcal{C}^\infty $$
3. ${}^\bullet X \xrightarrow{\ {}^\bullet f\ } {}^\bullet Y \quad \textit{in} \quad {}^\bullet\mathcal{C}^\infty \quad \Longleftrightarrow \quad X \xrightarrow{\ f\ } Y \quad \textit{in} \quad \mathcal{C}^\infty$
4. *If* $f,\ g : X \longrightarrow Y$ *are* $\mathcal{C}^\infty$ *functions, then*
$$ {}^\bullet f = {}^\bullet g \quad \Longrightarrow \quad f = g. $$

Proof: To prove *1.* we start to prove that the support sets of X and Y are equal. Indeed, if we take standard parts, since ${}^\bullet X = {}^\bullet Y$, we have

$$ \{{}^\circ x \mid x \in {}^\bullet X\} = |X| = \{{}^\circ x \mid x \in {}^\bullet Y\} = |Y|. $$

Hence $\bar{X} = (|X| \prec {}^\bullet X) = (|Y| \prec {}^\bullet Y) = \bar{Y}$ and the conclusion follows from *1.* of Theorem 9.3.1.

To prove *2.* let us take the restriction of f to $|X| \subseteq |{}^\bullet X|$, then $f|_{|X|} : \bar{X} = (|X| \prec {}^\bullet X) \longrightarrow (|Y| \prec {}^\bullet Y) = \bar{Y}$ in ${}^\bullet\mathcal{C}^\infty$, so the conclusion follows from *2.* of Theorem 9.3.1. Property *3.* follows from the just proved *2.* considering that ${}^\bullet f|_{|X|} = f$ and using Theorem 9.1.2. The same idea of considering restrictions can be used to prove *4.* ■

9.4 The standard part functor cannot exist

It is very natural to ask if it is possible to define a standard part functor, that is a way to associate to *every* Fermat space $X \in {}^\bullet\mathcal{C}^\infty$ a space ${}^\circ X \in \mathcal{C}^\infty$ intuitively corresponding to its "standard points" only. This application

$^\circ(-) : {}^\bullet\mathcal{C}^\infty \longrightarrow \mathcal{C}^\infty$ has to satisfy some expected properties, some example of which are functoriality, its support set has to be included in the original space, i.e. $|{}^\circ X| \subseteq |X|$, and we must also have examples like ${}^\circ D = \{0\}$ and ${}^\circ({}^\bullet\mathbb{R}) = \mathbb{R}$. Because, intuitively, the Fermat extension ${}^\bullet X \in {}^\bullet\mathcal{C}^\infty$ appears to be some kind of completion of the standard space $X \in \mathcal{C}^\infty$, we also expect that the Fermat functor is the left adjoint of the standard part functor, ${}^\bullet(-) \dashv {}^\circ(-)$. Indeed, it is natural to expect that this adjunction is related to the following equivalence[2]

$$\frac{\mathcal{C}^\infty \models X \xrightarrow{\ {}^\circ f\ } {}^\circ Y}{{}^\bullet\mathcal{C}^\infty \models {}^\bullet X \xrightarrow{\ f\ } Y} \tag{9.4.1}$$

If one tries to define this standard part space (and the corresponding standard part map acting on arrows, i.e. $f \in {}^\bullet\mathcal{C}^\infty(X,Y) \mapsto {}^\circ f \in \mathcal{C}^\infty({}^\circ X, {}^\circ Y)$), then several difficulties arise.

For example, the first trivial point that has to be noted in the searching for the definition of ${}^\circ X$, is that we want to have $|{}^\circ X| \subseteq |X|$, that is the standard points have to be searched in the same Fermat space $X \in {}^\bullet\mathcal{C}^\infty$ from which we have started. For a generic space $X \in {}^\bullet\mathcal{C}^\infty$, that is in general not a space of the form $X = {}^\bullet Y$, we do not have an easy way to associate to each point $x \in X$ another point $s \in X$ making the role of its standard part. Because, on the contrary, the definition of standard part is a trivial problem in numerical spaces of the form ${}^\bullet\mathbb{R}^d$, the natural idea seems to use, as it has been done several times in past definitions, observables like $X \supseteq U \xrightarrow{\ \varphi\ } {}^\bullet\mathbb{R}^d$ and to reduce the problem from the space X to the numerical space ${}^\bullet\mathbb{R}^d$. But this idea naturally leads to the problem of how it is possible to return back from ${}^\bullet\mathbb{R}^d$ to X. Unfortunately, this seems solvable only for spaces X sufficiently similar to manifolds, where charts are invertible observables (thus not for generic spaces X).

Moreover, we also have to consider examples like $X = \{\mathrm{d}t\} \subseteq D \setminus \{0\}$, where it seems natural to expect that ${}^\circ X = \emptyset$, so the searched map $x \mapsto {}^\circ x = s$ in general cannot be defined and we have to restrict our aim to prove, whether this would be possible, that for every $x \in X$ there exists at most one $s \in X$ corresponding to its standard part.

Another idea could be to identify the standard points $s \in X$ as those points that can be obtained as standard values of figures of the form $\delta : {}^\bullet U \longrightarrow X$, i.e. of point of the form $s = \delta(r)$ for $r \in U$. But the case of constant figures having non standard values, like $\delta(u) = \mathrm{d}t$, represent a counter example to this intuition.

These are only few examples of unsuccessful attempts that can be tried if one would like to define a standard part functor. The confirmation that

[2] The horizontal line indicating the logical equivalence between the formula above and the formula below, similar to the notations in the logical calculus of Gentzen, but where the line indicates logical deduction of the formula below from the formula above.

this is not a trivial goal is given by the following impossibility results. For their proof we need some preliminary lemmas.

9.4.1 Smooth functions with standard values

The following result state that a function defined on the Fermat reals and having standard values only, i.e. of the form $f : {}^\bullet\mathbb{R} \longrightarrow \mathbb{R}$, is necessarily the Fermat extension of its restriction $f|_\mathbb{R}$ to the standard points only.

Lemma 9.4.1. *If $f : {}^\bullet\mathbb{R} \longrightarrow \mathbb{R}$ is smooth (i.e. it is an arrow of ${}^\bullet\mathcal{C}^\infty$), then*

1. *$f|_\mathbb{R} : \mathbb{R} \longrightarrow \mathbb{R}$ is smooth in $\mathcal{C}^\infty$*
2. *$f = {}^\bullet(f|_\mathbb{R})$.*

Proof: To prove *1.* we only have to consider the general Theorem 6.3.2 about the restriction of maps. Indeed, since the map f has only values in $\mathbb{R}$, we have $f(\mathbb{R}) \subseteq \mathbb{R}$ and hence since $f : {}^\bullet\mathbb{R} \longrightarrow \mathbb{R}$ in ${}^\bullet\mathcal{C}^\infty$, we have

$$f|_\mathbb{R} : (\mathbb{R} \prec {}^\bullet\mathbb{R}) = \bar{\mathbb{R}} \longrightarrow (\mathbb{R} \prec \mathbb{R}) = \bar{\mathbb{R}} \quad \text{in} \quad {}^\bullet\mathcal{C}^\infty,$$

from which the conclusion *1.* follows thanks to Theorem 9.3.1.

To prove *2.* we will use Theorem 9.2.4. In fact, for every $x \in {}^\bullet\mathbb{R}$ we can write $f(y) = \gamma(p, y)$ for every $y \in \mathcal{V}$ in an open neighborhood $\mathcal{V}$ of x. Possibly considering the composition with a translation, we can suppose ${}^\circ p = \underline{0}$ and hence $p \in D_n^{\mathsf{p}} \subseteq {}^\bullet\mathbb{R}^{\mathsf{p}}$ for some order $n \in \mathbb{N}_{>0}$. Considering the infinitesimal Taylor's formula of γ of order n with respect to the variable $p \in {}^\bullet\mathbb{R}^{\mathsf{p}}$, we have

$$f(y) = \gamma(\underline{0} + p, y) = \sum_{|\alpha| \le n} \frac{p^\alpha}{\alpha!} \cdot \partial_1^\alpha \gamma(\underline{0}, y) \quad \forall y \in \mathcal{V}, \tag{9.4.2}$$

where ∂_1 indicates the derivation with respect to the first slot in $\gamma(-,-)$. But $f(y) \in \mathbb{R}$ and hence ${}^\circ(f(y)) = f(y)$, so the infinitesimal part of $f(y)$ is zero. From (9.4.2) we thus obtain

$$\sum_{\substack{|\alpha| \le n \\ \alpha \ne \underline{0}}} \frac{p^\alpha}{\alpha!} \cdot \partial_1^\alpha \gamma(\underline{0}, y) = 0.$$

Therefore $f(y) = \gamma(\underline{0}, y)$ for every $y \in \mathcal{V}$ and hence $f(x) = \gamma(\underline{0}, x) = {}^\bullet[\gamma(\underline{0}, -)](x) = {}^\bullet f|_\mathbb{R}(x)$. ∎

From this lemma we obtain the following expected result:

Corollary 9.4.2. *If $f : {}^\bullet\mathbb{R} \longrightarrow \mathbb{R}$ is smooth, then f is constant.*

Proof: From the previous Lemma 9.4.1, if $g := f|_{\mathbb{R}}$, then $f = {}^\bullet g$, hence using the derivation formula with g we have

$$\forall x \in \mathbb{R}\, \forall h \in D : \quad f(x+h) = f(x) + h \cdot g'(x). \tag{9.4.3}$$

But $f(x+h) \in \mathbb{R}$ hence $f(x+h) = {}^\circ f(x+h) = {}^\circ\left[f(x) + h \cdot g'(x)\right] = {}^\circ f(x) = g(x)$, so from (9.4.3) we obtain $f(x+h) = g(x) = g(x) + h \cdot g'(x)$ and hence $g'(x) = 0$ and so g is constant because from Lemma 9.4.1 we have that $g : \mathbb{R} \longrightarrow \mathbb{R}$ is smooth. ∎

The most natural example of a function defined on ${}^\bullet\mathbb{R}$ but with standard values is the standard part map ${}^\circ(-) : {}^\bullet\mathbb{R} \longrightarrow \mathbb{R}$, which of course is not constant, so we have the following

Corollary 9.4.3. *The standard part map* ${}^\circ(-) : {}^\bullet\mathbb{R} \longrightarrow \mathbb{R}$ *is not smooth.* ∎

As a consequence of this corollary we have that the standard part functor cannot exists. We will prove this assertion in two ways:

Theorem 9.4.4. *Let* $\bar{(-)} : \mathcal{C}^\infty_{\text{sep}} \longrightarrow {}^\bullet\mathcal{C}^\infty$ *be the embedding of separated* $\mathcal{C}^\infty$*-spaces into the category* ${}^\bullet\mathcal{C}^\infty$ *of Fermat spaces (see Section 9.3). Then, there does not exist a functor*

$${}^\circ(-) : {}^\bullet\mathcal{C}^\infty \longrightarrow \mathcal{C}^\infty$$

with the following properties:

1. *There exists a universal arrow of the form* $(\eta, {}^\bullet\mathbb{R}) : \mathbb{R} \xrightarrow{\eta} {}^\circ({}^\bullet\mathbb{R})$.
2. *In* $\mathcal{C}^\infty$ *we have the isomorphism* ${}^\circ\bar{\mathbb{R}} \simeq \mathbb{R}$.
3. *The functor* ${}^\circ(-)$ *preserves terminal objects.*

Therefore, there does not exists a right adjoint of the Fermat functor that satisfies the isomorphism ${}^\circ\bar{\mathbb{R}} \simeq \mathbb{R}$ *in* $\mathcal{C}^\infty$ *and preserves terminal objects.*

Proof: We proceed by reduction to the absurd, recalling (see Appendix A) that such a universal arrow has to verify

$${}^\bullet\mathbb{R} \in {}^\bullet\mathcal{C}^\infty \quad \text{(trivial)}$$

$$\mathcal{C}^\infty \vDash \mathbb{R} \xrightarrow{\quad\eta\quad} {}^\circ({}^\bullet\mathbb{R}),$$

and has to be the co-simplest arrow among all arrows satisfying this property, i.e. for every pair (μ, A) that verifies

$$A \in {}^\bullet\mathcal{C}^\infty \tag{9.4.4}$$

$$\mathcal{C}^\infty \vDash \mathbb{R} \xrightarrow{\quad\mu\quad} {}^\circ A, \tag{9.4.5}$$

there exists one and only one arrow φ such that

$$^{\bullet}\mathcal{C}^{\infty} \vDash {}^{\bullet}\mathbb{R} \xrightarrow{\varphi} A \tag{9.4.6}$$

$$\mathcal{C}^{\infty} \vDash \quad \begin{tikzcd} \mathbb{R} \arrow[r, "\eta"] \arrow[dr, "\mu"'] & {}^{\circ}({}^{\bullet}\mathbb{R}) \arrow[d, "{}^{\circ}\varphi"] \\ & {}^{\circ}A \end{tikzcd} \tag{9.4.7}$$

Let us set $A = \bar{\mathbb{R}} \in {}^{\bullet}\mathcal{C}^{\infty}$ in (9.4.4) and (9.4.5) and let $\mu : \mathbb{R} \longrightarrow {}^{\circ}\bar{\mathbb{R}}$ be the $\mathcal{C}^{\infty}$-isomorphism of the hypothesis ${}^{\circ}\bar{\mathbb{R}} \simeq \mathbb{R}$, then by (9.4.6) and (9.4.7) we obtain that $\varphi : {}^{\bullet}\mathbb{R} \longrightarrow \bar{\mathbb{R}}$ in ${}^{\bullet}\mathcal{C}^{\infty}$ and $\eta \cdot {}^{\circ}\varphi = \mu$. From Corollary 9.4.2 we have that φ must be constant, that is there exist a value $r \in \mathbb{R}$ such that

$$\begin{tikzcd} {}^{\bullet}\mathbb{R} \arrow[r, "\varphi"] \arrow[dr, "t"'] & \bar{\mathbb{R}} \\ & \mathbf{1} \arrow[u, "r"'] \end{tikzcd}$$

where $\mathbf{1} \in {}^{\bullet}\mathcal{C}^{\infty}$ is the terminal object. Therefore $\varphi = t \cdot r$ and hence ${}^{\circ}\varphi = {}^{\circ}t \cdot {}^{\circ}r$ since ${}^{\circ}(-)$ is supposed to be a functor. So the map ${}^{\circ}\varphi$ factors through ${}^{\circ}\mathbf{1}$ which, by hypothesis, is the terminal object of $\mathcal{C}^{\infty}$, hence ${}^{\circ}\varphi$ is constant too. But this is impossible because $\eta \cdot {}^{\circ}\varphi = \mu$ and $\mu : \mathbb{R} \longrightarrow {}^{\circ}\bar{\mathbb{R}}$ is an isomorphism. ■

Finally, we want to prove a similar conclusion starting from the equivalence (9.4.1)

Theorem 9.4.5. *The equivalence* (9.4.1) *is false for* $n = \infty$, $X = Y = \mathbb{R}$ *and* $f = {}^{\circ}(-) : {}^{\bullet}\mathbb{R} \longrightarrow \mathbb{R}$ *the standard part map if* ${}^{\circ}\bar{\mathbb{R}} = \mathbb{R}$ *and* ${}^{\circ}({}^{\circ}(-)) = {}^{\circ}(-)$.

Proof: Indeed from Corollary 9.4.3 we know that the standard part map f is not smooth, that is the property ${}^{\bullet}\mathcal{C}^{\infty} \vDash {}^{\bullet}\mathbb{R} \xrightarrow{f} \mathbb{R}$ is false. On the other hand, we have that for $X = Y = \mathbb{R}$ and $f = {}^{\circ}(-)$ the property $\mathcal{C}^{\infty} \vDash X \xrightarrow{{}^{\circ}f} {}^{\circ}Y$ becomes

$$\mathcal{C}^{\infty} \vDash \mathbb{R} \xrightarrow{{}^{\circ}({}^{\circ}(-))} {}^{\circ}\bar{\mathbb{R}}$$

that is, by the assumed hypotheses

$$\mathcal{C}^{\infty} \vDash \mathbb{R} \xrightarrow{{}^{\circ}(-)} \mathbb{R},$$

which is true because the standard part map is the identity on $\mathbb{R}$. ∎

Analyzing the proofs of these theorems, we can see that the only possibility to avoid this impossibility result is to change radically the definition of the category of Fermat spaces ${}^\bullet\mathcal{C}^\infty$ so as to include non constant maps of the form $f : {}^\bullet\mathbb{R} \longrightarrow \mathbb{R}$. This seems possible thanks to the flexibility of the cartesian closure construction (Chapter 6), but this idea has not been developed in the present work.

Chapter 10

Logical properties of the Fermat functor

In this section we want to investigate some logical properties of the Fermat functor, with the aim to arrive to a general transfer theorem. We will see that there are strict connections between the Fermat functor and intuitionistic logic.

10.1 Basic logical properties of the Fermat functor

In this section we will start to investigate some basic logical properties of the Fermat functor, i.e. the relationships between a given logical operator (i.e. a propositional connective or a quantifier) and the related preservation of the Fermat functor of that operator.

The first theorem establishes the relationships between the Fermat functor and the preservation of implication.

Theorem 10.1.1. *Let X, $Y \in \boldsymbol{\mathcal{C}}^\infty$ with $|X|$ is open in Y and such that $X \subseteq Y$ in $\boldsymbol{\mathcal{C}}^\infty$ (see Section 6.3), then ${}^\bullet X \subseteq {}^\bullet Y$ in ${}^\bullet\boldsymbol{\mathcal{C}}^\infty$.*

In other words, the Fermat functor preserves implication if the antecedent is a property represented by an open set.

Proof: Let us first assume that $X \subseteq Y$ and recall that $X \subseteq Y$ means $|X| \subseteq |Y|$ and $X = (|X| \prec Y)$, i.e. the space X has exactly the structure induced by the superspace Y on one of its subsets. This is equivalent to the following two properties:

$$\forall \delta : \quad \delta \in_H X \quad \Longrightarrow \quad \delta \in_H Y \tag{10.1.1}$$

$$\forall \delta : \quad \delta : |H| \longrightarrow |X| \quad , \quad \delta \cdot i \in_H Y \quad \Longrightarrow \quad \delta \in_H X, \tag{10.1.2}$$

where $i : |X| \hookrightarrow |Y|$ is the inclusion. Using the Fermat functor we have that ${}^\bullet i : {}^\bullet X \longrightarrow {}^\bullet Y$ in ${}^\bullet\boldsymbol{\mathcal{C}}^\infty$. How does the map ${}^\bullet i$ act? If, to be more clear, we use the notation $[x]_X := [(x_t)_t]_\sim$ with explicit use of equivalence classes, then we have

$$\forall x : \quad [x]_X \in |{}^\bullet X| \quad \Longrightarrow \quad {}^\bullet i\left([x]_X\right) = [x \cdot i]_Y = [x]_Y \in |{}^\bullet Y|,$$

hence ${}^\bullet i : [x]_X \mapsto [x]_Y$. We want to prove that this map is injective. In fact, let us take $[x]_X,\ [y]_X \in {}^\bullet X$ such that $[x]_Y = [y]_Y$ and an observable $\psi : (V \prec X) \longrightarrow K$ defined on the open set $V \in \mathcal{T}_X$. From the results of Section 6.3 it follows that $(V \prec X) = (V \prec (|X| \prec Y)) = (V \prec Y)$ and also that $V \in \mathcal{T}_Y$ because, by hypothesis, $|X|$ is open in Y. Therefore VK is a zone of Y too and hence $\psi : (V \prec Y) \longrightarrow K$ is an observable of Y. From the equality $[x]_Y = [y]_Y$ it follows

$$x_0 \in V \iff y_0 \in V$$

$$x_0 \in V \quad \Longrightarrow \quad \psi(x_t) = \psi(y_t) + o(t)$$

which proves that $[x]_X = [y]_X$, that is the map ${}^\bullet i$ is injective. This injection is exactly the generalization of the identification that permits to write ${}^\bullet U \subseteq {}^\bullet\mathbb{R}^k$ if U is open in $\mathbb{R}^k$ (see Section 8.2). For these reasons we simply write $|{}^\bullet X| \subseteq |{}^\bullet Y|$ identifying $|{}^\bullet X|$ with ${}^\bullet i(|{}^\bullet X|) \subseteq |{}^\bullet Y|$. Now we have to prove that ${}^\bullet X \subseteq {}^\bullet Y$, i.e. ${}^\bullet X = (|{}^\bullet X| \prec {}^\bullet Y)$, i.e. ${}^\bullet(|{}^\bullet X| X) = (|{}^\bullet X| \prec {}^\bullet Y)$ since ${}^\bullet X = {}^\bullet(|{}^\bullet X| X)$ by the Definition 9.1.1 of Fermat functor. So, let us first consider a generic figure $\delta \in_S {}^\bullet X$ of type $S \in \mathbf{S}{}^\bullet\mathbb{R}^\infty$; using Theorem 9.2.4 we have that for every $s \in S$ there exists an open neighborhood $V = {}^\bullet B \cap S$ of s in S such that either $\delta|_V$ is constant or we can write $\delta|_V = {}^\bullet\gamma(p,-)|_V$ for some $\gamma \in \boldsymbol{\mathcal{C}}^\infty(A \times B, X)$. In the first case, trivially $\delta|_V \in_S (|{}^\bullet X| \prec {}^\bullet Y)$, because any space always contains all constant figures. In the second case, since $i \in \boldsymbol{\mathcal{C}}^\infty(X,Y)$ we have $\gamma \cdot i = \gamma \in \boldsymbol{\mathcal{C}}^\infty(A \times B, Y)$ and, once again from Theorem 9.2.4, we obtain that $\delta|_U \in_U (|{}^\bullet X| \prec {}^\bullet Y)$. From the sheaf property of the space $(|{}^\bullet X| \prec {}^\bullet Y)$ the conclusion $\delta \in_S (|{}^\bullet X| \prec {}^\bullet Y)$ follows. Vice versa if $\delta \in_S (|{}^\bullet X| \prec {}^\bullet Y)$, then $\delta \in_S (|{}^\bullet X| {}^\bullet Y)$ by Theorem 9.2.5 so that, using again Theorem 9.2.4 and notations similar to those used above, we have that either $\delta|_V$ is constant or $\delta|_V = {}^\bullet\gamma(p,-)|_V$, but now with $\gamma \in \boldsymbol{\mathcal{C}}^\infty(A \times B, Y)$. The first case is trivial. For the second one, it suffices to restrict γ so as to obtain a function with values in X instead of Y. But $|X|$ is open in Y so $\gamma^{-1}(|X|)$ is open in $A \times B$. Thus, we can find C and D open neighborhood of ${}^\circ p$ and ${}^\circ s$ respectively such that $\mu := \gamma|_{C \times D} \in \boldsymbol{\mathcal{C}}^\infty(C \times D, (|X| \prec Y)) = \boldsymbol{\mathcal{C}}^\infty(C \times D, X)$, the last equality following from $X \subseteq Y$. Of course $\delta|_{{}^\bullet D \cap S} = {}^\bullet\mu(p,-)|_{{}^\bullet D \cap S}$ and hence $\delta \in_S {}^\bullet X$.

∎

The following theorem says that the Fermat functor takes open sets to open sets.

Theorem 10.1.2. *If $X \in \boldsymbol{\mathcal{C}}^\infty$ and U is open in X, then ${}^\bullet U$ is open in ${}^\bullet X$*

Proof: From the previous theorem we know that $|{}^\bullet U| \subseteq |{}^\bullet X|$. Let us take a figure $d \in_S {}^\bullet X$ of type $S \subseteq {}^\bullet\mathbb{R}^{\mathsf{s}}$; to prove that ${}^\bullet U$ is open in ${}^\bullet X$ we have to prove that $d^{-1}({}^\bullet U)$ is open in S, that is we have to prove that $d^{-1}({}^\bullet U)$ is generated by sets of the form ${}^\bullet C \cap S$ for C open in $\mathbb{R}^{\mathsf{s}}$. So, let us take a point $s \in d^{-1}({}^\bullet U)$, once again from the characterization of the figures of ${}^\bullet X$ (Theorem 9.2.4), we have the existence of an open neighborhood $V = {}^\bullet B \cap S$ of s in S such that either $d|_V$ is constant, or we can write $d|_V = {}^\bullet\gamma(p, -)|_V$, for $\gamma \in \boldsymbol{\mathcal{C}}^\infty(A \times B, X)$ and $p_0 \in A$ open in $\mathbb{R}^{\mathsf{p}}$. In the first trivial case we can take $C := \mathbb{R}^{\mathsf{s}}$, so we can consider the second one only. Because $d(s) \in {}^\bullet U$, we have that ${}^\circ d(s) = \gamma(p_0, s_0) \in U$. Since U is open in X, we have that $\gamma^{-1}(U)$ is open in $A \times B$, so from $(p_0, s_0) \in \gamma^{-1}(U)$ we get the existence of two open sets D and C, respectively in $A \subseteq \mathbb{R}^{\mathsf{p}}$ and $B \subseteq \mathbb{R}^{\mathsf{s}}$, such that $(p_0, s_0) \in D \times C \subseteq \gamma^{-1}(U)$. From this we obtain that $s \in {}^\bullet C \cap S$, which is the first part of our conclusion. But C is open in B, so ${}^\bullet C \subseteq {}^\bullet B$ from the previous theorem and hence ${}^\bullet C \cap S \subseteq {}^\bullet B \cap S = V$, and we can write $d(x) = {}^\bullet\gamma(p, x)$ for every $x \in {}^\bullet C \cap S$. Therefore ${}^\circ d(x) = \gamma(p_0, x_0) \in U$ because $(p_0, x_0) \in D \times C \subseteq \gamma^{-1}(U)$. From ${}^\circ d(x) \in U$ we hence get $d(x) \in {}^\bullet U$ because U is open, and hence we have also proved that $x \in d^{-1}(U)$ for every $x \in {}^\bullet C \cap S$, which is the final part of our conclusion.

■

From this theorem we also obtain the important conclusion that the Fermat functor preserves open covers, i.e. if $(U_\alpha)_{\alpha \in A}$ is an open cover of $X \in \boldsymbol{\mathcal{C}}^\infty$, then $({}^\bullet U_\alpha)_{\alpha \in A}$ is an open cover of ${}^\bullet X$.

The following theorem is the converse of the previous 10.1.1 in the case where the spaces are separated.

Theorem 10.1.3. *In the hypothesis of Theorem 10.1.1, if X and Y are separated, then ${}^\bullet X \subseteq {}^\bullet Y$ in ${}^\bullet\boldsymbol{\mathcal{C}}^\infty$ implies $X \subseteq Y$ in $\boldsymbol{\mathcal{C}}^\infty$.*

Proof: If $\delta \in_U X$ is figure, then ${}^\bullet\delta : {}^\bullet U \longrightarrow {}^\bullet X$ in ${}^\bullet\boldsymbol{\mathcal{C}}^\infty$ and hence ${}^\bullet\delta \in_{{}^\bullet U} {}^\bullet X$. but ${}^\bullet X \subseteq {}^\bullet Y$, so ${}^\bullet\delta \in_{{}^\bullet U} {}^\bullet Y$. From Corollary 9.3.2 we thus have $\delta \in_U Y$. It remains to prove condition (10.1.2). If $\delta \cdot i \in_H Y$, then ${}^\bullet\delta \cdot {}^\bullet i \in_{{}^\bullet H} {}^\bullet Y$. Recalling that ${}^\bullet X$ is always identified with ${}^\bullet i({}^\bullet X)$, we can set $j : {}^\bullet i({}^\bullet X) \hookrightarrow {}^\bullet Y$ the inclusion so that ${}^\bullet\delta \cdot {}^\bullet i = {}^\bullet\delta \cdot {}^\bullet i \cdot j \in_{{}^\bullet H} {}^\bullet Y$ and hence ${}^\bullet\delta \cdot {}^\bullet i \in_{{}^\bullet H} {}^\bullet i({}^\bullet X)$ since ${}^\bullet i({}^\bullet X) \subseteq {}^\bullet Y$. Using the identification

$$
{}^\bullet\boldsymbol{\mathcal{C}}^\infty \vDash {}^\bullet i : \;\; {}^\bullet X \xrightarrow[\sim]{\quad} {}^\bullet i({}^\bullet X)
$$

this means that ${}^\bullet\delta \in_{{}^\bullet H} {}^\bullet X$ and hence $\delta \in_H X$ from Corollary 9.3.2.

■

From the preservation of the inclusion we can prove that if X is an open subspace of Y, then the operators $(- \prec {}^\bullet X)$ and $(- \prec {}^\bullet Y)$ conduct to the same subspaces, i.e. we can change the superspace Y with any other open superspace X.

Corollary 10.1.4. *If $X \subseteq Y$ in $\mathcal{C}^\infty$, $|X|$ is open in Y and $Z \subseteq |{}^\bullet X|$ then $(Z \prec {}^\bullet X) = (Z \prec {}^\bullet Y)$.*

Proof: This is a trivial consequence of Corollary 6.3.3. In fact since from Theorem 10.1.1, we have that ${}^\bullet X \subseteq {}^\bullet Y$ in ${}^\bullet\mathcal{C}^\infty$ and hence we can apply the cited Corollary 6.3.3. ∎

From this result we can prove that the Fermat functor preserves also counter images of open sets through $\mathcal{C}^\infty$ functions.

Theorem 10.1.5. *Let $f : X \longrightarrow Y$ and $Z \subseteq Y$ in $\mathcal{C}^\infty$, with $|Z|$ open in Y. Moreover define the spaces ${}^\bullet f^{-1}({}^\bullet Z) := ({}^\bullet f^{-1}(|{}^\bullet Z|) \prec {}^\bullet X) \in {}^\bullet\mathcal{C}^\infty$ and $f^{-1}(Z) := (f^{-1}(|Z|) \prec X) \in \mathcal{C}^\infty$. Then we have the equality*

$$^\bullet[f^{-1}(Z)] = {}^\bullet f^{-1}({}^\bullet Z)$$

as Fermat spaces.

Proof: Let us start from the support sets of the two spaces:

$$\begin{aligned} x \in {}^\bullet f^{-1}(|{}^\bullet Z|) = ({}^\bullet f)^{-1}\left(|{}^\bullet Z|\right) &\iff {}^\bullet f(x) \in |{}^\bullet Z| \\ &\iff \forall^0 t: \quad f(x_t) \in |Z| \end{aligned}$$

On the other hand we have

$$\begin{aligned} x \in {}^\bullet \left[f^{-1}(|Z|)\right] &\iff \forall^0 t: \quad x_t \in f^{-1}(|Z|) \\ &\iff \forall^0 t: \quad f(x_t) \in |Z| \end{aligned}$$

Hence the support sets are equal. Now we have

$$^\bullet\left[f^{-1}(Z)\right] = {}^\bullet\left({}^\bullet\left[f^{-1}(|Z|)\right] f^{-1}(Z)\right) = \left({}^\bullet\left[f^{-1}(|Z|)\right] \prec {}^\bullet\left[f^{-1}(Z)\right]\right), \tag{10.1.3}$$

the first equality following from the Definition 9.1.1 of Fermat functor, and the second one from Theorem 9.2.5. But $f^{-1}(|Z|)$ is open in X because $|Z|$ is open in Y, hence from the previous Corollary 10.1.4 we can change in (10.1.3) the superspace ${}^\bullet\left[f^{-1}(Z)\right]$ with ${}^\bullet X \supseteq {}^\bullet\left[f^{-1}(Z)\right]$, hence

$$^\bullet\left[f^{-1}(Z)\right] = \left({}^\bullet\left[f^{-1}(|Z|)\right] \prec {}^\bullet X\right) = \left({}^\bullet f^{-1}(|{}^\bullet Z|) \prec {}^\bullet X\right) = {}^\bullet f^{-1}({}^\bullet Z),$$

where we have used the equality of support sets, i.e.

$$^\bullet\left[f^{-1}(|Z|)\right] = {}^\bullet f^{-1}(|{}^\bullet Z|)$$

and the definition of the space ${}^\bullet f^{-1}({}^\bullet Z)$.

∎

Now we consider the relationships between the Fermat functor and the other propositional connectives.

Theorem 10.1.6. *The Fermat functor preserves intersections and unions of open sets and the intuitionistic negations, i.e.*

1. *If $A \subseteq X$ and $B \subseteq X$ in $\mathcal{C}^\infty$ and $|A|$, $|B|$ are open in X, then*

$$^\bullet(A \cap_X B) = {}^\bullet A \cap_{^\bullet X} {}^\bullet B$$

and

$$^\bullet(A \cup_X B) = {}^\bullet A \cup {}^\bullet B$$

where, e.g. $A \cup_X B := (|A| \cup |B| \prec X)$, ${}^\bullet A \cap_{^\bullet X} {}^\bullet B := (|{}^\bullet A| \cap |{}^\bullet B| \prec {}^\bullet X)$, etc.

2. *If $X \subseteq Y$ in $\mathcal{C}^\infty$ and $|X|$ is open in Y, then*

$$^\bullet\left[\mathrm{int}_Y(Y \setminus X)\right] \subseteq \mathrm{int}_{^\bullet Y}({}^\bullet Y \setminus {}^\bullet X),$$

where $\mathrm{int}_T(S)$ is the interior of the set S in the topological space T.

3. *In the hypotheses of the previous item, if X and Y are separated and the topology of ${}^\bullet Y$ is generated by open subsets of the form ${}^\bullet B$ with B open in Y, i.e. $A = \bigcup\{{}^\bullet B \subseteq A \mid B \in \tau_Y\}$ for every $A \in \tau_{^\bullet Y}$, then*

$$^\bullet\left[\mathrm{int}_Y(Y \setminus X)\right] = \mathrm{int}_{^\bullet Y}({}^\bullet Y \setminus {}^\bullet X),$$

i.e. in this case the Fermat functor preserves intuitionistic negations.

When the topology of a Fermat space of the form ${}^\bullet Y$ is generated by open subsets of the form ${}^\bullet B$ with B open in Y, we will say that *the topology of ${}^\bullet Y$ is ${}^\bullet(-)$-generated.*

Proof:

1. We start proving that the space $A \cap_X B$ is the infimum of the spaces A and B with respect to the partial order of inclusion between $\mathcal{C}^\infty$ spaces. In fact, because of Corollary 10.1.4 we have

$$A \cap_X B = (|A| \cap |B| \prec X) = (|A| \cap |B| \prec A) = (|A| \cap |B| \prec B),$$

that is, $A \cap_X B \subseteq A$ and $A \cap_X B \subseteq B$. Now, let us consider a space $C \in \mathcal{C}^\infty$ such that $C \subseteq A$ and $C \subseteq B$, then $|C| \subseteq |A| \cap |B|$ and, e.g., $C = (|C| \prec A) = (|C| \prec (|A| \prec X)) = (|C| \prec X)$. But $|A \cap_X B|$ is open in X and we can hence apply Corollary 10.1.4 again, obtaining $C = (|C| \prec A \cap_X B)$, i.e. $C \subseteq A \cap_X B$. Analogously we can prove that $A \cup_X B$ is the supremum of the spaces A and B, or the analogous properties in the category ${}^\bullet\mathcal{C}^\infty$.

Therefore, from $A \cap_X B \subseteq A$ and $A \cap_X B \subseteq B$ we obtain ${}^\bullet(A \cap_X B) \subseteq {}^\bullet A$ and ${}^\bullet(A \cap_X B) \subseteq {}^\bullet B$ and hence ${}^\bullet(A \cap_X B) \subseteq {}^\bullet A \cap_{^\bullet X} {}^\bullet B$ because of the greatest lower bound property. Vice versa, if $\delta \in_S {}^\bullet A \cap_{^\bullet X} {}^\bullet B$ is a figure of type $S \subseteq {}^\bullet\mathbb{R}^{\mathsf{s}}$, then using the characterization of the figures of a Fermat space, i.e. Theorem 9.2.4, we can say that for every $s \in S$ there exists an open neighborhood $V = {}^\bullet C \cap S$ of s such that either $\delta|_V$ is constant or we can write $\delta|_V = {}^\bullet\gamma(p,-)|_V$ for $\gamma \in \mathcal{C}^\infty(C \times D, X)$. In the first case $\delta|_V \in_V {}^\bullet(A \cap_X B)$; in the second one ${}^\circ\delta(s) = \gamma(p_0, s_0) \in A \cap_X B$, therefore we can find a sufficiently small neighborhood $E \times F$ of (p_0, s_0) such that for $U := {}^\bullet F \cap S$ we have $\delta|_U = {}^\bullet\gamma(p,-)|_U : U \longrightarrow {}^\bullet(A \cap_X B)$, so that $\delta|_U \in_U {}^\bullet(A \cap_X B)$. The conclusion $\delta \in_S {}^\bullet(A \cap_X B)$ follows from the sheaf property of the space ${}^\bullet(A \cap_X B)$. Analogously we can prove that the Fermat functor preserves unions of $\mathcal{C}^\infty$ spaces.

2. Let us start proving that $\mathrm{int}_Y(Y \setminus X)$ verifies the expected lattice properties. Being defined as a subspace of Y, we have

$$\mathcal{C}^\infty \vDash \mathrm{int}_Y(Y \setminus X) \subseteq Y. \tag{10.1.4}$$

Moreover, because $|\mathrm{int}_Y(Y \setminus X)| \subseteq |Y| \setminus |X|$, we have that $|X| \cap |\mathrm{int}_Y(Y \setminus X)| = \emptyset$, so

$$\mathcal{C}^\infty \vDash X \cap \mathrm{int}_Y(Y \setminus X) = \emptyset. \tag{10.1.5}$$

Now, we can prove that among the open subspaces of the space Y, the subspace $\mathrm{int}_Y(Y \setminus X)$ is the greatest one verifying the previous properties (10.1.4) and (10.1.5). Indeed if $A \in \mathcal{C}^\infty$ is open in Y, i.e. $|A| \in \mathcal{T}_Y$, and $A \subseteq Y$, $X \cap A = \emptyset$, considering its support set we have $|A| \subseteq |Y| \setminus |X|$ and hence $|A| \subseteq |\mathrm{int}_Y(Y \setminus X)|$ because $|A|$ is open in Y. From $A \subseteq Y$, and using Corollary 10.1.4 we also get

$$A = (|A| \prec Y) = (|A| \prec \mathrm{int}_Y(Y \setminus X)),$$

that is $A \subseteq \mathrm{int}_Y(Y \setminus X)$.

Applying the Fermat functor to the properties (10.1.4) and (10.1.5) we obtain ${}^\bullet\left[\mathrm{int}_Y(Y \setminus X)\right] \subseteq {}^\bullet Y$ and ${}^\bullet X \cap {}^\bullet\left[\mathrm{int}_Y(Y \setminus X)\right] = \emptyset$, and hence

$${}^\bullet\left[\mathrm{int}_Y(Y \setminus X)\right] \subseteq \mathrm{int}_{{}^\bullet Y}({}^\bullet Y \setminus {}^\bullet X)$$

3. To prove the opposite inclusion, let us take a figure $\delta \in_S \mathrm{int}_{{}^\bullet Y}({}^\bullet Y \setminus {}^\bullet X)$ of type $S \subseteq {}^\bullet\mathbb{R}^{\mathsf{s}}$. Then, for every $s \in S$ we have $\delta(s) \in \mathrm{int}_{{}^\bullet Y}({}^\bullet Y \setminus {}^\bullet X)$, so that $\delta(s) \in A \subseteq |{}^\bullet Y| \setminus |{}^\bullet X|$, with A open in ${}^\bullet Y$. But, by hypothesis, we can find an open set $B \in \mathcal{T}_Y$ such that $\delta(s) \in {}^\bullet B \subseteq A \subseteq |{}^\bullet Y| \setminus |{}^\bullet X|$, and hence $B \subseteq \mathrm{int}_Y(Y \setminus X)$ because $B \subseteq {}^\bullet B$ and $|X| \subseteq |{}^\bullet X|$ (all the spaces and their subspaces are separated by hypothesis). Now, we can proceed in the usual way using the characterization of the figures of a Fermat space (Theorem 9.2.4), from which we get the existence of an open neighborhood $V = {}^\bullet C \cap S$ of s such that either $\delta|_V$ is constant or we can write $\delta|_V = {}^\bullet\gamma(p,-)|_V$ for $\gamma \in \mathcal{C}^\infty(C \times D, Y)$. In the first case $\delta|_V \in_V {}^\bullet\left[\mathrm{int}_Y(Y \setminus X)\right]$; in the second one ${}^\circ\delta(s) = \gamma(p_0, s_0) \in B$, therefore we can find a sufficiently small neighborhood $E \times F$ of (p_0, s_0) such that for $U := {}^\bullet F \cap S$ we have $\delta|_U = {}^\bullet\gamma(p,-)|_U : U \longrightarrow {}^\bullet B$, so that $\delta|_U \in_U {}^\bullet B \subseteq {}^\bullet\left[\mathrm{int}_Y(Y \setminus X)\right]$. The conclusion $\delta \in_S {}^\bullet[\mathrm{int}_Y(Y \setminus X)]$ follows from the sheaf property of the space ${}^\bullet[\mathrm{int}_Y(Y \setminus X)]$.

■

Definition 10.1.7. *If X, $Y \in \mathcal{C}^\infty$ are separated space and $|X|$ is open in Y, then we will use the notation*

$$\neg_Y X := (\mathrm{int}_Y(Y \setminus X) \prec Y)$$

$$\neg_{{}^\bullet Y}{}^\bullet X := (\mathrm{int}_{{}^\bullet Y}({}^\bullet Y \setminus {}^\bullet X) \prec {}^\bullet Y)\,.$$

Moreover, if A, B are open in Y, then we also set

$$A \Rightarrow_Y B := \neg_Y A \cup_Y B.$$

Therefore, from the previous theorem we can say that

$${}^\bullet(\neg_Y X) = \neg_{{}^\bullet Y}{}^\bullet X$$

$${}^\bullet(A \Rightarrow_Y B) = ({}^\bullet A \Rightarrow_{{}^\bullet Y} {}^\bullet B)$$

Let us note that the hypotheses of *3*. in the previous theorem are surely verified for X, Y manifolds.

Finally, we have to consider the relationships between the Fermat functor and the logical quantifiers.

Definition 10.1.8. *Let $\mathcal{F}$ be a category of types of figures and $f : X \longrightarrow Y$ be an arrow of the cartesian closure $\bar{\mathcal{F}}$. Then for $Z \subseteq |X|$ we set*

$$\exists_f(Z) := (f(Z) \prec Y) \tag{10.1.6}$$

$$\forall_f(Z) := (\mathrm{int}_Y\{y \in |Y| \,|\, f^{-1}(\{y\}) \subseteq Z\} \prec Y) \tag{10.1.7}$$

Theorem 10.1.9. *Let $f : X \longrightarrow Y$ be a $\mathcal{C}^\infty$-map. Moreover, let us suppose that*

1. *Z is open in X,*
2. *f is open with respect to the topologies τ_X and τ_Y,*
3. *$f|_Z : (Z \prec X) \longrightarrow (f(Z) \prec Y)$ has a left[1] inverse in $\mathcal{C}^\infty$,*
4. *X, Y are separated.*

Then we have

$${}^\bullet(\exists_f(Z)) = \exists_{{}^\bullet f}({}^\bullet Z)$$

i.e., in these hypotheses, the Fermat functor preserves existential quantifiers.

[1] Let us note that here the word "left" is with respect to the composition of functions represented by the symbol $(f \cdot g)(x) = g(f(x))$ (that permits an easier reading of diagrams), so that it corresponds to "right" with respect to the notation with $(f \circ g)(x) = f(g(x))$.

Theorem 10.1.10. *Let $f : X \longrightarrow Y$ be a $\mathcal{C}^\infty$-map. Moreover, let us suppose that*

1. *Z is open in X,*
2. *the topology of ${}^\bullet Y$ is ${}^\bullet(-)$-generated,*
3. *X, Y are separated.*

Then we have

$$^\bullet(\forall(Z)) = \forall_{^\bullet f}(^\bullet Z)$$

i.e., in these hypotheses, the Fermat functor preserves existential quantifiers.

To motivate the definitions (10.1.6) and (10.1.7) we can consider as f a projection $p : A \times B \longrightarrow B$ of a product, then for $Z \subseteq |A \times B|$ we have

$$|\exists_p(Z)| = p(Z) = \{b \mid \exists x \in Z : \ b = p(x)\} = \{b \in B \mid \exists a \in A : \ Z(a, b)\},$$

where we used $Z(a, b)$ for $(a, b) \in Z$. This justifies the definition of $\exists_f$ as a generalization of this $\exists_p$.

Taking the difference $|Y| \setminus |\exists_f(Z)|$ we obtain

$$\begin{aligned}
|Y| \setminus |\exists_f(Z)| &= |Y| \setminus f(Z) = \{y \in Y \mid \neg(\exists x \in Z : \ y = f(x))\} = \\
&= \{y \mid \forall x \in X : \ y = f(x) \Rightarrow x \notin Z\} = \\
&= \{y \mid f^{-1}(\{y\}) \subseteq X \setminus Z\} = \\
&= |\forall_f(X \setminus Z)|
\end{aligned}$$

This justifies fully the definition of $\forall_f$ in the case of classical logic. For example, in the case of a projection $p : A \times B \longrightarrow B$, for $Z \subseteq |A \times B|$ we have

$$|\forall_p(Z)| = \{b \mid \forall x \in X : \ b = p(x) \Rightarrow x \in Z\} = \{b \in B \mid \forall a \in A : \ Z(a, b)\}$$

In an intuitionistic context [2] the interpretation of a formula in a topological space must always result in an open set (We recall that like the classical logic can be interpreted in any boolean algebra of generic subsets of a given superset, the intuitionistic logic can be interpreted in the Heyting algebra of the open sets of any topological space (see e.g. Rasiowa and Sikorski [1963], Scott [1968]) and this motivates the use of the interior operator int_Y in the definition (10.1.7). Finally, we recall that the projection of a product is always an open map if on the product space $A \times B$ we have the product

[2] We recall that in intuitionistic logic a quantifier cannot be defined starting from the other one; the best result that it is possible to obtain is that $[\forall x : \neg\varphi(x)] \iff [\neg\exists x : \varphi(x)]$, where it is important to recall that, in general, $\neg\neg\varphi(x)$ is not equivalent to $\varphi(x)$ in intuitionistic logic (as it can be guessed using topological considerations, because of the interior operator, starting from our Definition 10.1.7 of negation).

topology, like in our context if A and B are manifolds (see Section 9.2, (6.3.2) and the final discussion in Section 6.3). Moreover if $a \in A$, then $g : b \in B \longrightarrow (a,b) \in A \times B$ is a left inverse of class $\mathcal{C}^\infty$ of the projection p, so the map p verifies all the hypotheses of Theorem 10.1.9.

To prove this theorem we need the following two lemmas, which repeat in our context well known results (see e.g. Taylor [1999]).

Lemma 10.1.11. *If $\mathcal{F}$ is a category of types of figures, and $f : X \longrightarrow Y$ in $\bar{\mathcal{F}}$, then we have:*

1. *If A, A' are subspaces of X (not necessarily open) with $A \subseteq A'$, then $\exists_f(A) \subseteq \exists_f(A')$.*
2. *If $A \subseteq X$ and $B \subseteq Y$, then in the category $\bar{\mathcal{F}}$ we have the equivalence*

$$\frac{A \subseteq f^{-1}(B)}{\exists_f(A) \subseteq B} \tag{10.1.8}$$

that is $\exists_f \dashv f^{-1}$ with respect to the order relation $\subseteq$ between subspaces.

Lemma 10.1.12. *If $\mathcal{F}$ is a category of types of figures, and $f : X \longrightarrow Y$ in $\bar{\mathcal{F}}$, then we have:*

1. *If A, A' are subspaces of X (not necessarily open) with $A \subseteq A'$, then $\forall_f(A) \subseteq \forall_f(A')$.*
2. *if $A \subseteq X$ and $B \subseteq Y$, then in the category $\bar{\mathcal{F}}$ we have the equivalence*

$$\frac{f^{-1}(B) \subseteq A}{B \subseteq \forall_f(A)} \tag{10.1.9}$$

that is $f^{-1} \dashv \forall_f$ with respect to the order relation $\subseteq$ between subspaces.

Lemma 10.1.13. *If $f : X \longrightarrow Y$ in $\mathcal{C}^\infty$ and $Z \subseteq |X|$, then*

$$^\bullet(f|_Z) = {}^\bullet f|_{^\bullet Z}$$

Proof: Both the functions are defined in $^\bullet Z = {}^\bullet(Z \prec X)$, so let $x \in {}^\bullet Z$, we have $^\bullet(f|_Z)(x) = (f(x_t))_{t \geq 0} = ({}^\bullet f|_{^\bullet Z})(x)$.

∎

Proof of Lemmas 10.1.11 and 10.1.12: let us assume that A and A' are subspaces of X with $A \subseteq A'$. We recall that $\exists_f(A) = (f(A) \prec Y)$ and $\exists_f(A') = (f(A') \prec Y)$; but $|f(A)| \subseteq |f(A')|$, we can hence apply Corollary 6.3.3 to change in $\exists_f(A)$ the superspace Y with the superspace $(f(A') \prec Y) \subseteq Y$, obtaining

$$\exists_f(A) = (f(A) \prec (f(A') \prec Y)) = (f(A) \prec \exists_f(A')),$$

that is $\exists_f(A) \subseteq \exists_f(A')$.

Now let us assume that $A \subseteq f^{-1}(B)$, then $|f(A)| \subseteq B$ as sets so that, applying once again Corollary 6.3.3 we can change the superspace Y in $\exists_f(A) = (f(A) \prec Y)$ with the superspace B obtaining $\exists_f(A) = (f(A) \prec B)$, that is the conclusion $\exists_f(A) \subseteq B$. Reversing this deduction we can obtain a proof for the opposite implication. In a similar way we can also prove the analogous properties of the universal quantifier.

■

Proof of Theorem 10.1.9: The first idea is to use the uniqueness of the adjoints of f^{-1}, that is the property that the spaces $\exists_f(A)$ and $\forall_f(A)$ are uniquely determined by the equivalences (10.1.8) and (10.1.9) respectively, and to use the preservation of the relation $X \subseteq Y$ by the Fermat functor. Indeed, if we suppose that ${}^\bullet Z \subseteq {}^\bullet f^{-1}({}^\bullet W)$, then we also have ${}^\bullet Z \subseteq {}^\bullet\left(f^{-1}(W)\right)$ by the preservation of counter images. By Theorem 10.1.3 this implies $Z \subseteq f^{-1}(W)$ and hence $\exists_f(Z) \subseteq W$ by Lemma 10.1.11 and so ${}^\bullet\left(\exists_f(Z)\right) \subseteq {}^\bullet W$ applying the preservation of implications. At the same time, the hypothesis ${}^\bullet Z \subseteq {}^\bullet f^{-1}({}^\bullet W)$ implies $\exists_{{}^\bullet f}({}^\bullet Z) \subseteq {}^\bullet W$ since Lemma 10.1.11 is true for the category ${}^\bullet\mathcal{C}^\infty$ too. All these implications can be reversed in a direct way using Theorem 10.1.3 and our hypothesis that the spaces X and Y (and hence all their subspaces) are separated. Therefore, we have the equivalences

$$\frac{\dfrac{{}^\bullet Z \subseteq {}^\bullet f^{-1}({}^\bullet W)}{\exists_{{}^\bullet f}({}^\bullet Z) \subseteq {}^\bullet W}}{{}^\bullet\left(\exists_f(Z)\right) \subseteq {}^\bullet W} \tag{10.1.10}$$

In them, if we set $W := \exists_f(Z)$, then the third one is trivially true, and from the second one we obtain

$$\exists_{{}^\bullet f}({}^\bullet Z) \subseteq {}^\bullet\left(\exists_f(Z)\right). \tag{10.1.11}$$

This part of the deduction cannot be reversed because, e.g., in (10.1.10) instead of a generic subspace of ${}^\bullet Y$ we have a subspace of the form ${}^\bullet W$ only. So, let us first recall that

$$ {}^\bullet(\exists_f(Z)) = {}^\bullet(f(Z) \prec Y)$$

$$\exists_{{}^\bullet f}({}^\bullet Z) = ({}^\bullet f({}^\bullet Z) \prec {}^\bullet Y)$$

To prove the opposite relations of (10.1.11) we need to assume the existence of a left inverse g of the restriction $f|_Z$, i.e. a $\mathcal{C}^\infty$-map $g : (f(Z) \prec Y) \longrightarrow (Z \prec X)$ such that $g \cdot f|_Z = 1_{f(Z)}$. Let us take a figure $\delta \in_S {}^\bullet(\exists_f(Z))$ of type $S \subseteq {}^\bullet\mathbb{R}^{\mathsf{s}}$. Then

$${}^\bullet\mathcal{C}^\infty \vDash S \xrightarrow{\quad\delta\quad} {}^\bullet(f(Z) \prec Y) \xrightarrow{\quad{}^\bullet g\quad} {}^\bullet(Z \prec X)\ ,$$

and hence $\delta\cdot{}^\bullet g\in_S{}^\bullet(Z\prec X)$. Composing this map with the restriction ${}^\bullet(f|_Z)={}^\bullet f|_{{}^\bullet Z}:{}^\bullet(Z\prec X)\longrightarrow({}^\bullet f({}^\bullet Z)\prec{}^\bullet Y)$ we obtain

$$\delta\cdot{}^\bullet g\cdot{}^\bullet f|_{{}^\bullet Z}=\delta\cdot{}^\bullet(g\cdot f|_Z)=\delta\in_S({}^\bullet f({}^\bullet Z)\prec{}^\bullet Y)=\exists_{{}^\bullet f}({}^\bullet Z).$$

We have hence proved the first condition (10.1.1) to prove that ${}^\bullet\left(\exists_f(Z)\right)\subseteq\exists_{{}^\bullet f}({}^\bullet Z)$. This part of the deduction also proves that we have the relation $|{}^\bullet(\exists_f(Z))|\subseteq|\exists_{{}^\bullet f}({}^\bullet Z)|$ between the corresponding support sets. Hence we can now prove the second condition (10.1.2); let us consider a map $\delta:S\longrightarrow|{}^\bullet(\exists_f(Z))|$ such that $\delta\cdot i\in_S\exists_{{}^\bullet f}({}^\bullet Z)$, where $i:|{}^\bullet(\exists_f(Z))|\hookrightarrow|\exists_{{}^\bullet f}({}^\bullet Z)|$ is the inclusion map. So we have $\delta\cdot i=\delta\in_S\exists_{{}^\bullet f}({}^\bullet Z)$ and hence also $\delta\in_S{}^\bullet\left(\exists_f(Z)\right)$ since (10.1.11). This easily proves also the second condition (10.1.2) and hence $\exists_{{}^\bullet f}({}^\bullet Z)={}^\bullet\left(\exists_f(Z)\right)$.

■

Proof of Theorem 10.1.10: Analogously to how we did in the previous proof, we can proceed for the universal quantifier obtaining the equivalences

$$\frac{\dfrac{{}^\bullet f^{-1}({}^\bullet W)\subseteq{}^\bullet Z}{{}^\bullet W\subseteq\forall_{{}^\bullet f}({}^\bullet Z)}}{{}^\bullet W\subseteq{}^\bullet\left(\forall_f(Z)\right)}\tag{10.1.12}$$

from which we obtain

$${}^\bullet\left(\forall_f(Z)\right)\subseteq\forall_{{}^\bullet f}({}^\bullet Z).\tag{10.1.13}$$

Now, let us consider the opposite inclusion, recalling that

$${}^\bullet(\forall_f(Z))={}^\bullet\left(\left[\mathrm{int}_Y\left\{y\,|\,f^{-1}\{y\}\subseteq Z\right\}\right]\prec Y\right)$$

$$\forall_{{}^\bullet f}({}^\bullet Z)=\left(\mathrm{int}_{{}^\bullet Y}\left\{y\,|\,{}^\bullet f^{-1}\{y\}\subseteq{}^\bullet Z\right\}\prec{}^\bullet Y\right).$$

So let us consider a figure $\delta\in_S\forall_{{}^\bullet f}({}^\bullet Z)$ and a point $s\in S$, then

$$\delta(s)\in\left(\mathrm{int}_{{}^\bullet Y}\left\{y\,|\,{}^\bullet f^{-1}\{y\}\subseteq{}^\bullet Z\right\}\prec{}^\bullet Y\right).$$

Because, by hypothesis, the topology of ${}^\bullet Y$ is generated by open sets of the form ${}^\bullet U$, $U\in\mathcal{T}_X$, by the definition of interior we obtain

$$\exists U\in\mathcal{T}_X:\ \ \delta(s)\in{}^\bullet U\subseteq\left\{y\,|\,{}^\bullet f^{-1}\{y\}\subseteq{}^\bullet Z\right\}.\tag{10.1.14}$$

It is natural to expect that the property ${}^\bullet f^{-1}\{y\}\subseteq{}^\bullet Z$ can be extended to the whole set ${}^\bullet U$, indeed

$$\begin{aligned}\forall x\in{}^\bullet f^{-1}({}^\bullet U):\ &{}^\bullet f(x)\in{}^\bullet U\\ &{}^\bullet f^{-1}\{{}^\bullet fx\}\subseteq{}^\bullet Z\quad\text{ by (10.1.14)}\\ &\text{but } x\in{}^\bullet f^{-1}\{{}^\bullet fx\}\\ &\text{hence } x\in{}^\bullet Z.\end{aligned}$$

Therefore we have ${}^\bullet f^{-1}({}^\bullet U) \subseteq {}^\bullet Z$, that is ${}^\bullet\left(f^{-1}(U)\right) \subseteq {}^\bullet Z$, and hence $f^{-1}(U) \subseteq Z$ because we are considering separated spaces, and so $U \subseteq \forall_f(Z)$. But $\delta(s) \in {}^\bullet U$, and setting $V := \delta^{-1}({}^\bullet U)$ we obtain an open neighborhood of s such that

$$\delta|_V : V \longrightarrow {}^\bullet U.$$

Therefore $\delta|_V \in_V {}^\bullet U \subseteq {}^\bullet(\forall_f(Z))$. The conclusion $\delta \in_S {}^\bullet(\forall_f(Z))$ follows from the sheaf property of the space ${}^\bullet(\forall_f(Z))$. The second condition (10.1.2) can be proved analogously to what we already did above for the existential quantifier.

∎

10.2 The general transfer theorem

In this section, for simplicity of notations, every arrow f of the categories $\mathcal{C}^\infty$ and ${}^\bullet\mathcal{C}^\infty$ is supposed to have unique domain and codomain (they will be denoted by $\mathrm{dom}(f)$ and $\mathrm{cod}(f)$ respectively; see Appendix A for more details about this hypothesis, which at a first reading may seem trivial).

In the previous section, it has been underlined that the logical operators defined above, like $A \cap_Y B$ or $\neg_Y B$, or $\forall_f(A)$ take subspaces of a given space Y to subspaces of the same or of another space (like e.g. $f^{-1}(A) \subseteq X$ if $A \subseteq Y$). Therefore, we have now the possibility to compose these operators to construct new spaces, like e.g. the following

$$S := \forall_\varepsilon\left(A \Rightarrow_Z \exists_\delta\left(B \cap_Y \forall_x\left(C \Rightarrow_X D\right)\right)\right) \in \mathcal{C}^\infty, \tag{10.2.1}$$

where e.g.

$$X \xrightarrow{\quad x \quad} Y \xrightarrow{\quad \delta \quad} Z \xrightarrow{\quad \varepsilon \quad} W \tag{10.2.2}$$

$$C, D \subseteq X \tag{10.2.3}$$

$$B \subseteq Y \tag{10.2.4}$$

$$A \subseteq Z \tag{10.2.5}$$

In this section, we want to

1. define the family of formulae, like that used in (10.2.1) to define S, that permit to define spaces in $\mathcal{C}^\infty$ or in ${}^\bullet\mathcal{C}^\infty$ by means of logical operators;

2. show that to each formula there corresponds a suitable operator that maps subspaces of $\mathcal{C}^\infty$ into new subspaces of the same category;

3. define a ${}^\bullet(-)$-transform ${}^\bullet\xi$ of a formula ξ, called the *Fermat transform of ξ*. To the Fermat transform ${}^\bullet\xi$ corresponds an operator acting on spaces of the category ${}^\bullet\mathcal{C}^\infty$;

4. find a way to associate to every formula φ, a set of conditions like (10.2.2), (10.2.3), (10.2.4), (10.2.5) and other suitable hypotheses that will permit to apply all the theorems of the previous Section 10.1. Indeed, in the general transfer theorem we have to assume on superspaces, subspaces and maps, all the hypotheses of the theorems of the previous section, if we want that the Fermat functor preserves all the logical operations;

5. prove that the operator corresponding to ${}^\bullet\xi$ is the Fermat transform of the operator corresponding to the formula ξ, that is the general transfer theorem.

We will also include, in our formulae, the symbol of product because in case of manifolds the Fermat functor preserves also this operation (see Theorem 9.2.3).

Definition 10.2.1. *Let*

$$\mathcal{S} := \left\{\ulcorner\times\urcorner, \ulcorner\neg\urcorner, \ulcorner\Rightarrow\urcorner, \ulcorner\cap\urcorner, \ulcorner\cup\urcorner, \ulcorner\exists\urcorner, \ulcorner\forall\urcorner, \ulcorner{}^{-1}\urcorner, \ulcorner(\urcorner, \ulcorner)\urcorner\right\}$$

be a set of distinct elements called symbols. *An* expression in $\mathcal{C}^\infty$ *is a finite sequence of symbols in $\mathcal{S}$, objects or arrows of $\mathcal{C}^\infty$. Sequences of length 0 are admitted, but those of length 1 are identified with the element itself. For example the following*

$$\ulcorner\neg_Y A\urcorner := (\ulcorner\neg\urcorner, Y, A)$$

$$\ulcorner\exists_f(A)\urcorner := (\ulcorner\exists\urcorner, f, \ulcorner(\urcorner, A, \ulcorner)\urcorner)$$

are examples of expressions. We will use similar abbreviations for other expressions like, e.g., $\ulcorner A \Rightarrow_Y B\urcorner := (A, \ulcorner\Rightarrow\urcorner, Y, B)$.
If φ and ψ are expressions, then with the symbol $\ulcorner(\varphi \cap_\chi \psi)\urcorner$ *we mean the n-tuple* $(\ulcorner(\urcorner, \varphi, \ulcorner\cap\urcorner, \chi, \psi, \ulcorner)\urcorner)$. *We will use similar notations to construct expressions, like e.g.*

$$\ulcorner\exists_f(\varphi)\urcorner := (\ulcorner\exists\urcorner, f, \ulcorner(\urcorner, \varphi, \ulcorner)\urcorner).$$

We will denote with $\mathcal{L}^+(\mathcal{C}^\infty)$ *the intersection of all the classes L of expressions verifying*

1. *If* $A \in \mathcal{C}^\infty$, *then* $A \in L$

2. *If* φ, χ, $\psi \in L$, *then*

$$\ulcorner(\varphi \times \psi)\urcorner\ ,\ \ulcorner\neg_\chi\varphi\urcorner\ ,\ \ulcorner(\varphi \Rightarrow_\chi \psi)\urcorner\ ,\ \ulcorner(\varphi \cap_\chi \psi)\urcorner\ ,\ \ulcorner(\varphi \cup_\chi \psi)\urcorner \in L \tag{10.2.6}$$

3. *If f is an arrow of $\mathcal{C}^\infty$ and $\varphi \in L$, then*

$$\ulcorner\exists_f(\varphi)\urcorner \quad , \quad \ulcorner\forall_f(\varphi)\urcorner \quad , \quad \ulcorner f^{-1}(\varphi)\urcorner \in L$$

An analogous definition can be stated in the category ${}^\bullet\mathcal{C}^\infty$, *and the related class of expressions will be denoted by* $\mathcal{L}^+({}^\bullet\mathcal{C}^\infty)$.

As usual, see e.g. Monk [1976], we can prove the following

Theorem 10.2.2. *If* $\xi \in \mathcal{L}^+(\mathcal{C}^\infty)$, *then one and only one of the following holds:*

1. $\xi = A$ *for some object* $A \in \mathcal{C}^\infty$ *(expression of length 1);*
2. $\xi = \ulcorner(\varphi \times \psi)\urcorner$ *for some* φ, $\psi \in \mathcal{L}^+(\mathcal{C}^\infty)$;
3. $\xi = \ulcorner\neg_\chi\psi\urcorner$ *for some* χ, $\psi \in \mathcal{L}^+(\mathcal{C}^\infty)$;
4. $\xi = \ulcorner(\varphi \Rightarrow_\chi \psi)\urcorner$ *for some* φ, χ, $\psi \in \mathcal{L}^+(\mathcal{C}^\infty)$;
5. $\xi = \ulcorner(\varphi \cap_\chi \psi)\urcorner$ *for some* φ, χ, $\psi \in \mathcal{L}^+(\mathcal{C}^\infty)$;
6. $\xi = \ulcorner(\varphi \cup_\chi \psi)\urcorner$ *for some* φ, χ, $\psi \in \mathcal{L}^+(\mathcal{C}^\infty)$;
7. $\xi = \ulcorner\exists_f(\varphi)\urcorner$ *for some* $\varphi \in \mathcal{L}^+(\mathcal{C}^\infty)$ *and some arrow* f *of* $\mathcal{C}^\infty$;
8. $\xi = \ulcorner\forall_f(\varphi)\urcorner$ *for some* $\varphi \in \mathcal{L}^+(\mathcal{C}^\infty)$ *and some arrow* f *of* $\mathcal{C}^\infty$;
9. $\xi = \ulcorner f^{-1}(\varphi)\urcorner$ *for some* $\varphi \in \mathcal{L}^+(\mathcal{C}^\infty)$ *and some arrow* f *of* $\mathcal{C}^\infty$.

Moreover, the expressions φ, ψ, χ, *the object* A *and the arrow* f *asserted to exist are uniquely determined by* ξ.

Actually, the expressions of $\mathcal{L}^+(\mathcal{C}^\infty)$ are not well formed formulae because we can consider in the set $\mathcal{L}^+(\mathcal{C}^\infty)$ expressions like $\ulcorner A \cap_X B\urcorner$, but with A and B that are not subspaces of $X \in \mathcal{C}^\infty$. Analogously, an expression of the form $\ulcorner\exists_f(A)\urcorner$ is a formula only if $f : X \longrightarrow Y$ and $A \subseteq X$. This means that we are dealing with a typed language and, e.g., the previous $\ulcorner\exists_f(A)\urcorner$ is a formula only if A is of the form "subsets of the domain of f". In the following definition we will define what is this type.

Definition 10.2.3. *If* $\xi \in \mathcal{L}^+(\mathcal{C}^\infty)$, *then the type* $\tau(\xi)$ *is defined recursively by the following conditions:*

1. *If* $\xi = A$ *for some object* $A \in \mathcal{C}^\infty$, *then* $\tau(\xi) := A$.
2. *If* $\xi = \ulcorner(\varphi \times \psi)\urcorner$ *for some* φ, $\psi \in \mathcal{L}^+(\mathcal{C}^\infty)$, *then* $\tau(\xi) := \tau(\varphi) \times \tau(\psi)$.
3. *If* $\xi = \ulcorner\neg_\chi\psi\urcorner$ *for some* χ, $\psi \in \mathcal{L}^+(\mathcal{C}^\infty)$, *and if* $\tau(\psi) \subseteq \tau(\chi)$ *in* $\mathcal{C}^\infty$, *then*
$$\tau(\xi) := \neg_{\tau(\chi)}\tau(\psi)$$
4. *If* $\xi = \ulcorner(\varphi \Rightarrow_\chi \psi)\urcorner$ *for some* φ, χ, $\psi \in \mathcal{L}^+(\mathcal{C}^\infty)$, *and if* $\tau(\varphi) \subseteq \tau(\chi)$ *and* $\tau(\psi) \subseteq \tau(\chi)$, *then*
$$\tau(\xi) := \tau(\varphi) \Rightarrow_{\tau(\chi)} \tau(\psi)$$

5. *If* $\xi = \ulcorner(\varphi \cap_\chi \psi)\urcorner$ *for some* φ, χ, $\psi \in \mathcal{L}^+(\boldsymbol{\mathcal{C}}^\infty)$, *and if* $\tau(\varphi) \subseteq \tau(\chi)$ *and* $\tau(\psi) \subseteq \tau(\chi)$, *then*
$$\tau(\xi) := \tau(\varphi) \cap_{\tau(\chi)} \tau(\psi)$$

6. *If* $\xi = \ulcorner(\varphi \cup_\chi \psi)\urcorner$ *for some* φ, χ, $\psi \in \mathcal{L}^+(\boldsymbol{\mathcal{C}}^\infty)$, *and if* $\tau(\varphi) \subseteq \tau(\chi)$ *and* $\tau(\psi) \subseteq \tau(\chi)$, *then*
$$\tau(\xi) := \tau(\varphi) \cup_{\tau(\chi)} \tau(\psi)$$

7. *If* $\xi = \ulcorner\exists_f(\varphi)\urcorner$ *for some* $\varphi \in \mathcal{L}^+(\boldsymbol{\mathcal{C}}^\infty)$ *and some arrow* $f : X \longrightarrow Y$ *of* $\boldsymbol{\mathcal{C}}^\infty$, *and if* $\tau(\varphi) \subseteq X$, *then*
$$\tau(\xi) := \exists_f(\tau(\varphi))$$

8. *If* $\xi = \ulcorner\forall_f(\varphi)\urcorner$ *for some* $\varphi \in \mathcal{L}^+(\boldsymbol{\mathcal{C}}^\infty)$ *and some arrow* $f : X \longrightarrow Y$ *of* $\boldsymbol{\mathcal{C}}^\infty$, *and if* $\tau(\varphi) \subseteq X$, *then*
$$\tau(\xi) := \forall_f(\tau(\varphi))$$

9. *If* $\xi = \ulcorner f^{-1}(\varphi)\urcorner$ *for some* $\varphi \in \mathcal{L}^+(\boldsymbol{\mathcal{C}}^\infty)$ *and some arrow* $f : X \longrightarrow Y$ *of* $\boldsymbol{\mathcal{C}}^\infty$, *and if* $\tau(\varphi) \subseteq Y$, *then*
$$\tau(\xi) := f^{-1}(\tau(\varphi))$$

In all the other cases the type $\tau(\xi)$ *is not defined. Analogously we can define* ${}^\bullet\tau(\xi)$, *the type of expressions* $\xi \in \mathcal{L}^+({}^\bullet\boldsymbol{\mathcal{C}}^\infty)$ *in the category of Fermat spaces.*

Let us note that e.g. when we say "If $\xi = \ulcorner(\varphi \times \psi)\urcorner$ for some φ, $\psi \in \mathcal{L}^+(\boldsymbol{\mathcal{C}}^\infty)$, then $\tau(\xi) := \tau(\varphi) \times \tau(\psi)$", we implicitly mean "If $\xi = \ulcorner(\varphi \times \psi)\urcorner$ for some φ, $\psi \in \mathcal{L}^+(\boldsymbol{\mathcal{C}}^\infty)$, then $\tau(\varphi)$ *and* $\tau(\psi)$ *are defined and* $\tau(\xi) := \tau(\varphi) \times \tau(\psi)$".
Now we can define the formulae of $\boldsymbol{\mathcal{C}}^\infty$ as the expressions ξ in $\mathcal{L}^+(\boldsymbol{\mathcal{C}}^\infty)$ for which the type $\tau(\xi)$ is defined:

Definition 10.2.4. *The set* $\mathcal{L}(\boldsymbol{\mathcal{C}}^\infty)$ *of* formulae in $\boldsymbol{\mathcal{C}}^\infty$ *is defined recursively by the following condition:* $\xi \in \mathcal{L}(\boldsymbol{\mathcal{C}}^\infty)$ *if and only if one of the following alternatives is true:*

1. $\xi = A$ *for some object* $A \in \boldsymbol{\mathcal{C}}^\infty$;

2. $\xi = \ulcorner(\varphi \times \psi)\urcorner$ *for some* φ, $\psi \in \mathcal{L}(\boldsymbol{\mathcal{C}}^\infty)$;

3. *If* $\xi = \ulcorner\neg_\chi\psi\urcorner$ *for some* χ, $\psi \in \mathcal{L}(\boldsymbol{\mathcal{C}}^\infty)$, *then* $\tau(\psi) \subseteq \tau(\chi)$ *in* $\boldsymbol{\mathcal{C}}^\infty$;

4. *If* $\xi = \ulcorner(\varphi \Rightarrow_\chi \psi)\urcorner$ *for some* φ, χ, $\psi \in \mathcal{L}(\boldsymbol{\mathcal{C}}^\infty)$, *then* $\tau(\varphi) \subseteq \tau(\chi)$ *and* $\tau(\psi) \subseteq \tau(\chi)$;

5. *If* $\xi = \ulcorner(\varphi \cap_\chi \psi)\urcorner$ *for some* φ, χ, $\psi \in \mathcal{L}(\mathcal{C}^\infty)$, *then* $\tau(\varphi) \subseteq \tau(\chi)$ *and* $\tau(\psi) \subseteq \tau(\chi)$;

6. *If* $\xi = \ulcorner(\varphi \cup_\chi \psi)\urcorner$ *for some* φ, χ, $\psi \in \mathcal{L}(\mathcal{C}^\infty)$, *then* $\tau(\varphi) \subseteq \tau(\chi)$ *and* $\tau(\psi) \subseteq \tau(\chi)$;

7. *If* $\xi = \ulcorner\exists_f(\varphi)\urcorner$ *for some* $\varphi \in \mathcal{L}(\mathcal{C}^\infty)$ *and some arrow* $f : X \longrightarrow Y$ *of* $\mathcal{C}^\infty$, *then* $\tau(\varphi) \subseteq X$;

8. *If* $\xi = \ulcorner\forall_f(\varphi)\urcorner$ *for some* $\varphi \in \mathcal{L}(\mathcal{C}^\infty)$ *and some arrow* $f : X \longrightarrow Y$ *of* $\mathcal{C}^\infty$, *then* $\tau(\varphi) \subseteq X$;

9. *If* $\xi = \ulcorner f^{-1}(\varphi)\urcorner$ *for some* $\varphi \in \mathcal{L}(\mathcal{C}^\infty)$ *and some arrow* $f : X \longrightarrow Y$ *of* $\mathcal{C}^\infty$, *then* $\tau(\varphi) \subseteq Y$.

Therefore, if $\xi \in \mathcal{L}(\mathcal{C}^\infty)$ is a formula, then the type $\tau(\xi)$ is defined, and hence the type τ is an application

$$\tau : \mathcal{L}(\mathcal{C}^\infty) \longrightarrow \mathrm{Obj}(\mathcal{C}^\infty),$$

where $\mathrm{Obj}(\mathcal{C}^\infty)$ is the class of all the objects of the category $\mathcal{C}^\infty$. An analogous property can be stated for ${}^\bullet\mathcal{C}^\infty$. As usual, we can say that φ is a *subformula* of ξ if both ξ and φ are formulae and $\xi = (\chi, \varphi, \psi)$ for some expressions χ and ψ.

The condition that the type $\tau(\xi)$ is defined is exactly the minimal condition for the formula ξ of being meaningful. E.g. for the formula

$$\xi := \ulcorner\forall_\varepsilon\,(A \Rightarrow_Z \exists_\delta\,(B \cap_Y \forall_x\,(C \Rightarrow_X D)))\urcorner \tag{10.2.7}$$

we have that the type $\tau(\xi)$ is defined if and only if all the following conditions are true:

$$\begin{gathered}
C \subseteq X \\
D \subseteq X \\
(C \Rightarrow_X D) \subseteq \mathrm{dom}(x) \\
B \subseteq Y \\
\forall_x(C \Rightarrow_X D) \subseteq Y \\
B \cap_Y \forall_x(C \Rightarrow_X D) \subseteq \mathrm{dom}(\delta) \\
A \subseteq Z \\
\exists_\delta(B \cap_Y \forall_x(C \Rightarrow_X D)) \subseteq Z \\
[A \Rightarrow_Z \exists_\delta(B \cap_Y \forall_x(C \Rightarrow_X D))] \subseteq \mathrm{dom}(\varepsilon).
\end{gathered}$$

They are obviously more complicated, but more general, than conditions 10.2.2, 10.2.3, 10.2.4 and 10.2.5. Nevertheless, the hypothesis that the type $\tau(\xi)$ is defined (which, by Definition 10.2.4, is a consequence of the condition that ξ is a formula) is not everything we need to apply all the theorems of Section 10.1. For example, to the previously listed conditions related to the formula ξ of (10.2.7), we have to add hypotheses like: "the spaces X, Y, Z are separated and the topology of their Fermat extension is ${}^\bullet(-)$-generated", "the arrows x, δ, ε, are open and with left inverse" and "all the subspaces appearing in the previous list of conditions are open in the corresponding superspace". We will introduce these types of hypotheses directly in the statement of the general transfer theorem.

Now we can define the list of objects and arrows occurring in a formula φ. They are formally different from the free variables defined for a logical formula, because they have to be thought of as all the elements of the category $\mathcal{C}^\infty$ (or ${}^\bullet\mathcal{C}^\infty$) occurring in the formula φ. These objects and arrows will be the elements that have to be ${}^\bullet(-)$-transformed in the general transfer theorem, so e.g. in the formula $\ulcorner\exists_f(A)\urcorner$ the only object is A and the only arrow is f (whereas in a logical formula of the form $\exists\, f\,(A)$ the variable f is not free).

Definition 10.2.5. *Let $\xi \in \mathcal{L}(\mathcal{C}^\infty)$ be a formula, then the* list of objects $\text{ob}(\xi)$ *and the* list of arrows $\text{ar}(\xi)$ *are expressions defined recursively by the following conditions:*

1. *If $\xi = \ulcorner A\urcorner$ for some object $A \in \mathcal{C}^\infty$, then*

$$\begin{aligned}\text{ob}(\xi) &:= A\\ \text{ar}(\xi) &:= \emptyset.\end{aligned}$$

2. *If $\xi = \ulcorner(\varphi \times \psi)\urcorner$ for some φ, $\psi \in \mathcal{L}(\mathcal{C}^\infty)$, then*

$$\begin{aligned}\text{ob}(\xi) &:= (\text{ob}(\chi), \text{ob}(\psi))\\ \text{ar}(\xi) &:= (\text{ar}(\chi), \text{ar}(\psi)).\end{aligned}$$

3. *If $\xi = \ulcorner\neg_\chi\psi\urcorner$ for some χ, $\psi \in \mathcal{L}(\mathcal{C}^\infty)$, then*

$$\begin{aligned}\text{ob}(\xi) &:= (\text{ob}(\chi), \text{ob}(\psi))\\ \text{ar}(\xi) &:= (\text{ar}(\chi), \text{ar}(\psi)).\end{aligned}$$

4. *If $\xi = \ulcorner(\varphi \Rightarrow_\chi \psi)\urcorner$ for some φ, χ, $\psi \in \mathcal{L}(\mathcal{C}^\infty)$, then*

$$\begin{aligned}\text{ob}(\xi) &:= (\text{ob}(\varphi), \text{ob}(\chi), \text{ob}(\psi))\\ \text{ar}(\xi) &:= (\text{ar}(\varphi), \text{ar}(\chi), \text{ar}(\psi)).\end{aligned}$$

5. *If $\xi = \ulcorner(\varphi \cap_\chi \psi)\urcorner$ for some φ, χ, $\psi \in \mathcal{L}(\mathcal{C}^\infty)$, then*

$$\begin{aligned}\text{ob}(\xi) &:= (\text{ob}(\varphi), \text{ob}(\chi), \text{ob}(\psi))\\ \text{ar}(\xi) &:= (\text{ar}(\varphi), \text{ar}(\chi), \text{ar}(\psi)).\end{aligned}$$

6. *If* $\xi = \ulcorner(\varphi \cup_\chi \psi)\urcorner$ *for some* φ, χ, $\psi \in \mathcal{L}(\boldsymbol{\mathcal{C}}^\infty)$, *then*

$$\mathrm{ob}(\xi) := (\mathrm{ob}(\varphi), \mathrm{ob}(\chi), \mathrm{ob}(\psi))$$
$$\mathrm{ar}(\xi) := (\mathrm{ar}(\varphi), \mathrm{ar}(\chi), \mathrm{ar}(\psi)).$$

7. *If* $\xi = \ulcorner\exists_f(\varphi)\urcorner$ *for some* $\varphi \in \mathcal{L}(\boldsymbol{\mathcal{C}}^\infty)$ *and some arrow* $f : X \longrightarrow Y$ *of* $\boldsymbol{\mathcal{C}}^\infty$, *then*

$$\mathrm{ob}(\xi) := \mathrm{ob}(\varphi)$$
$$\mathrm{ar}(\xi) := (f, \mathrm{ar}(\varphi)).$$

8. *If* $\xi = \ulcorner\forall_f(\varphi)\urcorner$ *for some* $\varphi \in \mathcal{L}(\boldsymbol{\mathcal{C}}^\infty)$ *and some arrow* $f : X \longrightarrow Y$ *of* $\boldsymbol{\mathcal{C}}^\infty$, *then*

$$\mathrm{ob}(\xi) := \mathrm{ob}(\varphi)$$
$$\mathrm{ar}(\xi) := (f, \mathrm{ar}(\varphi)).$$

9. *If* $\xi = \ulcorner f^{-1}(\varphi)\urcorner$ *for some* $\varphi \in \mathcal{L}(\boldsymbol{\mathcal{C}}^\infty)$ *and some arrow* $f : X \longrightarrow Y$ *of* $\boldsymbol{\mathcal{C}}^\infty$, *then*

$$\mathrm{ob}(\xi) := \mathrm{ob}(\varphi)$$
$$\mathrm{ar}(\xi) := (f, \mathrm{ar}(\varphi)).$$

Now we can define the operator corresponding to a given formula $\xi \in \mathcal{L}(\boldsymbol{\mathcal{C}}^\infty)$ simply as the type $\tau(\xi)$ of the formula with the explicit indication of objects and arrows occurring in the formula itself.

Definition 10.2.6. *If* $\xi \in \mathcal{L}(\boldsymbol{\mathcal{C}}^\infty)$ *is a formula and* $\mathrm{ob}(\xi) =: (A_1 \ldots, A_n)$, $\mathrm{ar}(\xi) =: (f_1, \ldots, f_m)$ *are the lists of objects and arrows occurring in* ξ, *then*

$$\omega_\xi(A_1, \ldots, A_n, f_1, \ldots, f_m) := \tau(\xi)$$

Finally, we can define the Fermat transform of a formula.

Definition 10.2.7. *Let* $\xi \in \mathcal{L}(\boldsymbol{\mathcal{C}}^\infty)$ *be a formula, then the* Fermat transform $^\bullet\xi$ *is defined recursively by the following conditions:*

1. *If* $\xi = \ulcorner A\urcorner$ *for some object* $A \in \boldsymbol{\mathcal{C}}^\infty$, *then*

$$^\bullet\xi := {}^\bullet A$$

2. *If* $\xi = \ulcorner(\varphi \times \psi)\urcorner$ *for some* φ, $\psi \in \mathcal{L}(\boldsymbol{\mathcal{C}}^\infty)$, *then*

$$^\bullet\xi := \ulcorner({}^\bullet\varphi \times {}^\bullet\psi)\urcorner$$

3. *If* $\xi = \ulcorner \neg_\chi \psi \urcorner$ *for some* χ, $\psi \in \mathcal{L}(\mathcal{C}^\infty)$, *then*

$$^\bullet\xi := \ulcorner \neg_{^\bullet\chi} {}^\bullet\varphi \urcorner$$

4. *If* $\xi = \ulcorner (\varphi \Rightarrow_\chi \psi) \urcorner$ *for some* φ, χ, $\psi \in \mathcal{L}(\mathcal{C}^\infty)$, *then*

$$^\bullet\xi := \ulcorner ({}^\bullet\varphi \Rightarrow_{^\bullet\chi} {}^\bullet\psi) \urcorner$$

5. *If* $\xi = \ulcorner (\varphi \cap_\chi \psi) \urcorner$ *for some* φ, $\psi \in \mathcal{L}(\mathcal{C}^\infty)$, *then*

$$^\bullet\xi := \ulcorner ({}^\bullet\varphi \cap_{^\bullet\chi} {}^\bullet\psi) \urcorner$$

6. *If* $\xi = \ulcorner (\varphi \cup_\chi \psi) \urcorner$ *for some* φ, $\psi \in \mathcal{L}(\mathcal{C}^\infty)$, *then*

$$^\bullet\xi := \ulcorner ({}^\bullet\varphi \cup_{^\bullet\chi} {}^\bullet\psi) \urcorner$$

7. *If* $\xi = \ulcorner \exists_f(\varphi) \urcorner$ *for some* $\varphi \in \mathcal{L}(\mathcal{C}^\infty)$ *and some arrow* $f : X \longrightarrow Y$ *of* $\mathcal{C}^\infty$, *then*
$$^\bullet\xi := \ulcorner \exists_{^\bullet f}({}^\bullet\varphi) \urcorner$$

8. *If* $\xi = \ulcorner \forall_f(\varphi) \urcorner$ *for some* $\varphi \in \mathcal{L}(\mathcal{C}^\infty)$ *and some arrow* $f : X \longrightarrow Y$ *of* $\mathcal{C}^\infty$, *then*
$$^\bullet\xi := \ulcorner \forall_{^\bullet f}({}^\bullet\varphi) \urcorner$$

9. *If* $\xi = \ulcorner f^{-1}(\varphi) \urcorner$ *for some* $\varphi \in \mathcal{L}(\mathcal{C}^\infty)$ *and some arrow* $f : X \longrightarrow Y$ *of* $\mathcal{C}^\infty$, *then*
$$^\bullet\xi := \ulcorner {}^\bullet f^{-1}({}^\bullet\varphi) \urcorner$$

We can now state the general transfer theorem:

Theorem 10.2.8. *Let* $\xi \in \mathcal{L}(\mathcal{C}^\infty)$ *be a formula in* $\mathcal{C}^\infty$ *without occurrences of* $\ulcorner \times \urcorner$, *and let* $\mathrm{ob}(\xi) =: (A_1, \dots, A_n)$, $\mathrm{ar}(\xi) =: (f_1, \dots, f_m)$ *be objects and arrows occurring in the formula* ξ. *Let us suppose that for every* $i = 1, \dots, m$ *and every* j, $k = 1, \dots, n$*:*

1. $f_i : X_i \longrightarrow Y_i$ *is open and with left inverse.*

2. *Let* φ *and* ψ *be subformulae of* ξ *and* Z *be any space in the list* $\tau(\psi)$, $X_1, \dots, X_n$, $Y_1, \dots, Y_n$, *then*

$$\tau(\varphi) \subseteq Z \quad \Longrightarrow \quad |\tau(\varphi)| \text{ is open in } Z.$$

3. *Let* φ *be a subformula of* ξ, *then the topology of* $^\bullet\tau(\varphi)$ *is* $^\bullet(-)$*-generated.*

4. *All the spaces* X_i *are separated and the topology of their Fermat extension is* $^\bullet(-)$*-generated.*

Then we have:

$$^\bullet[\omega_\xi(A_1,\dots,A_n,f_1,\dots,f_m)] = \omega_{^\bullet\xi}(^\bullet A_1,\dots,{}^\bullet A_n,{}^\bullet f_1,\dots,{}^\bullet f_m)$$

For manifolds we can also include the product:

Theorem 10.2.9. *Let $\xi \in \mathcal{L}(\boldsymbol{\mathcal{C}}^\infty)$ be a generic formula in $\boldsymbol{\mathcal{C}}^\infty$, and let $\mathrm{ob}(\xi) =: (A_1,\dots,A_n)$, $\mathrm{ar}(\xi) =: (f_1,\dots,f_m)$ be objects and arrows occurring in the formula ξ. Let us suppose that for every $i = 1,\dots,m$ and every j, $k = 1,\dots,n$:*

1. *$f_i : X_i \longrightarrow Y_i$ is open and with left inverse.*
2. *Let φ and ψ be subformulae of ξ and Z be any space in the list $\tau(\psi)$, $X_1,\dots,X_n$, $Y_1,\dots,Y_n$, then*
$$\tau(\varphi) \subseteq Z \quad\Longrightarrow\quad |\tau(\varphi)| \text{ is open in } Z.$$
3. *Let φ be a subformula of ξ, then the topology of $^\bullet\tau(\varphi)$ is $^\bullet(-)$-generated.*
4. *All the spaces X_i are separated and the topology of their Fermat extension is $^\bullet(-)$-generated.*
5. *If $\ulcorner(\varphi\times\psi)\urcorner$ is a subformula of ξ, then $\tau(\varphi)$ and $\tau(\psi)$ are manifolds.*

Then we have:

$$^\bullet[\omega_\xi(A_1,\dots,A_n,f_1,\dots,f_m)] = \omega_{^\bullet\xi}(^\bullet A_1,\dots,{}^\bullet A_n,{}^\bullet f_1,\dots,{}^\bullet f_m) \tag{10.2.8}$$

Proof of Theorem 10.2.8 and Theorem 10.2.9: We proceed by induction on the length of the formula ξ. If ξ is made of one object only, i.e. $\xi = A$, then $\omega_\xi(A_1,\dots,A_n,f_1,\dots,f_m) = \tau(\xi) = A$, $n = 1$, $A_1 = A$, $m = 0$. Analogously

$$\omega_{^\bullet\xi}(^\bullet A_1,\dots,{}^\bullet A_n,{}^\bullet f_1,\dots,{}^\bullet f_m) = {}^\bullet A$$

since $^\bullet\xi = {}^\bullet A$ and $^\bullet\tau(^\bullet\xi) = {}^\bullet A$; the conclusion is hence trivial.

Now suppose that the equality (10.2.8) is true for every formula of length less than $N > 0$ and that in the formula ξ occur N symbols. Using the Definition (10.2.4) we have to consider several cases depending on the form of ξ. We will proceed for the case $\xi = \ulcorner(\varphi \Rightarrow_\chi \psi)\urcorner$, $\xi = \ulcorner\exists_{f_i}(\varphi)\urcorner$ and $\xi = \ulcorner f_i^{-1}(\varphi)\urcorner$, the other ones being analogous.

In the case $\xi = \ulcorner\exists_{f_i}(\varphi)\urcorner$, from the Definition (10.2.4) we get $\tau(\varphi) \subseteq X_i$. Moreover, φ is a subformula of ξ, hence from the hypothesis *2.* we obtain that $|\tau(\varphi)|$ is open in X_i. We can thus apply Theorem 10.1.9 since f_i is open by hypotheses *1.*, and we obtain that

$$^\bullet[\tau(\xi)] = {}^\bullet[\exists_{f_i}(\tau(\varphi)] = \exists_{^\bullet f_i}(^\bullet[\tau(\varphi)]). \tag{10.2.9}$$

By induction hypotheses, we get

$$^\bullet[\tau(\varphi)] = {}^\bullet\tau({}^\bullet\varphi) = \omega_{{}^\bullet\varphi}({}^\bullet A_{r_1}, \dots, {}^\bullet A_{r_a}, {}^\bullet f_{s_1}, \dots, {}^\bullet f_{s_b}) \tag{10.2.10}$$

where $\mathrm{ob}(\varphi) = (A_{r_1}, \dots, A_{r_a})$ and $\mathrm{ar}(\varphi) = (f_{s_1}, \dots, f_{s_b})$ are objects and arrows occurring in φ, hence $\{r_1, \dots, r_a\} \subseteq \{1, \dots, n\}$ and $\{s_1, \dots, s_b\} \subseteq \{1, \dots, m\}$. On the other hand, ${}^\bullet\xi = \ulcorner \exists_{{}^\bullet f_i}({}^\bullet\varphi)\urcorner$ and hence

$$^\bullet\tau({}^\bullet\xi) = \exists_{{}^\bullet f_i}({}^\bullet\tau({}^\bullet\varphi)). \tag{10.2.11}$$

The conclusion for this case follows from (10.2.9), (10.2.10) and (10.2.11), indeed:

$$\begin{aligned} {}^\bullet\,[\omega_\xi(A_1, \dots, A_n, f_1, \dots, f_m)] &= {}^\bullet[\tau(\xi)] \\ &= \exists_{{}^\bullet f_i}({}^\bullet[\tau(\varphi)]) \\ &= \exists_{{}^\bullet f_i}(\omega_{{}^\bullet\varphi}({}^\bullet A_{r_1}, \dots, {}^\bullet A_{r_a}, {}^\bullet f_{s_1}, \dots, {}^\bullet f_{s_b})) \\ &= {}^\bullet\tau({}^\bullet\xi) \\ &= \omega_{{}^\bullet\xi}({}^\bullet A_1, \dots, {}^\bullet A_n, {}^\bullet f_1, \dots, {}^\bullet f_m). \end{aligned}$$

Let us note that the hypotheses that the topology of all the spaces ${}^\bullet X_i$ is ${}^\bullet(-)$-generated must be used in the case $\xi = \ulcorner \forall_{f_i}(\varphi)\urcorner$.

In the case $\xi = \ulcorner(\varphi \Rightarrow_\chi \psi)\urcorner$, from the Definition (10.2.4) we get $\tau(\varphi) \subseteq \tau(\chi)$ and $\tau(\psi) \subseteq \tau(\chi)$. Moreover, φ, χ and ψ are subformulae of ξ, hence from the hypothesis *2.* we obtain that both $|\tau(\varphi)|$ and $|\tau(\psi)|$ are open in $\tau(\chi)$, and from the hypothesis *3.* we get that the topology of ${}^\bullet\tau(\chi)$ is ${}^\bullet(-)$-generated. We can hence apply Theorem 10.1.6 obtaining that

$$^\bullet[\tau(\xi)] = {}^\bullet[\tau(\varphi) \Rightarrow_{\tau(\chi)} \tau(\psi)] = {}^\bullet[\tau(\varphi)] \Rightarrow_{{}^\bullet[\tau(\chi)]} {}^\bullet[\tau(\psi)]. \tag{10.2.12}$$

But, by induction hypotheses we get equalities like (10.2.10), i.e.:

$$^\bullet[\tau(\varphi)] = {}^\bullet\tau({}^\bullet\varphi) = \omega_{{}^\bullet\varphi}({}^\bullet A_{r_1}, \dots, {}^\bullet A_{r_a}, {}^\bullet f_{s_1}, \dots, {}^\bullet f_{s_b}) \tag{10.2.13}$$

$$^\bullet[\tau(\chi)] = {}^\bullet\tau({}^\bullet\chi) = \omega_{{}^\bullet\chi}({}^\bullet A_{t_1}, \dots, {}^\bullet A_{t_c}, {}^\bullet f_{u_1}, \dots, {}^\bullet f_{u_d}) \tag{10.2.14}$$

$$^\bullet[\tau(\psi)] = {}^\bullet\tau({}^\bullet\psi) = \omega_{{}^\bullet\psi}({}^\bullet A_{v_1}, \dots, {}^\bullet A_{v_e}, {}^\bullet f_{w_1}, \dots, {}^\bullet f_{w_h}), \tag{10.2.15}$$

On the other hand, ${}^\bullet\xi = \ulcorner {}^\bullet\varphi \Rightarrow_{{}^\bullet\chi} {}^\bullet\psi\urcorner$ and hence

$$^\bullet\tau({}^\bullet\xi) = {}^\bullet\tau({}^\bullet\varphi) \Rightarrow_{{}^\bullet\tau({}^\bullet\chi)} {}^\bullet\tau({}^\bullet\psi). \tag{10.2.16}$$

The conclusion for the first case follows from (10.2.12), (10.2.13), (10.2.14), (10.2.15) and (10.2.11).

Finally, let us note that in the case $\xi = \ulcorner f_i^{-1}(\varphi)\urcorner$ we have to use the hypotheses *2.* to prove that $|\tau(\varphi)|$ is open in Y_i, but we do not need any other hypotheses on the codomain space Y_i. For this reason condition *4.* is stated for the domain spaces X_i only.

For Theorem 10.2.9 we can proceed in a similar way, using Theorem 9.2.3 in case of formulae of type $\xi = \ulcorner(\varphi \times \psi)\urcorner$.

$\blacksquare$

It is natural to expect that there would be some relationship between our transfer theorem and a transfer theorem more similar to those of NSA. The principal difference is that our transfer theorem, even if it concerns formulae, it is used to construct spaces and the theorem itself states an equality between spaces of ${}^\bullet\boldsymbol{\mathcal{C}}^\infty$. On the contrary, the transfer theorem of NSA asserts an equivalence between two sentences. Nevertheless, it seems possible to follow the following scheme:

1. Define the meaning of the sentence "*the formula ξ is intuitionistically true in $\boldsymbol{\mathcal{C}}^\infty$*" using the intuitionistic interpretation of the propositional connectives and quantifiers in this category. An analogous definition of intuitionistic validity can be done in the category ${}^\bullet\boldsymbol{\mathcal{C}}^\infty$.

2. Define the ${}^\bullet(-)$-transform of a given formula ξ.

3. Prove that ξ is intuitionistically true in $\boldsymbol{\mathcal{C}}^\infty$ if and only if ${}^\bullet\xi$ is intuitionistically true in ${}^\bullet\boldsymbol{\mathcal{C}}^\infty$.

This work is planned in future projects.

A specification is adequate here. Though the theory of Fermat reals is compatible with classical logic, the previous theorems state that the Fermat functor behaves really better if the logical formulae are interpreted in open sets. This may seem in contraddiction with the thread of the present work (see Section 1). Indeed, we remember that one of the main aims of the present work is to develop a sufficiently powerful theory of infinitesimal *without forcing the reader to learn a strong formal control of the mathematics he/she is doing*, e.g. forcing the reader to learn to work in intuitionistic logic. Of course, this is not incompatible with the *possibility* to gain more if one is interested to have this type of strong formal control, e.g. if one is already able to work in intuitionistic logic, and the results of this section go exactly in this direction.

Part III

The beginning of a new theory

Chapter 11

Calculus on open domains

11.1 Introduction

We have defined and studied plenty of instruments that can be useful to develop the differential and integral calculus of functions defined on infinitesimal domains like D_k or on bigger sets like the extension ${}^\bullet(a,b)$ of a real interval. We can then start the development of infinitesimal differential geometry, following, where possible, the lines of SDG. But further development can be glimpsed in the calculus of variations, because of cartesian closedness of our categories, because of the possibility to use infinitesimal methods and because of the properties of diffeological maps that, e.g., do not require any compactness hypothesis on the domain of our functions[1]. Of course, this could also open the possibility of several applications, e.g. in general relativity or in continuum mechanics. Indeed, Fermat reals can be considered as the first theory of infinitesimals having a good intuitive interpretation and without the need to possess a non trivial background of knowledge in formal logic to be understood (see Appendix B), and this characteristic can be very useful for its diffusion among physicists, engineers and even mathematicians.

But, exactly as SDG required tens of years to be developed, we have to expect a comparable amount of time for the full development of applications to the geometry of the approach we introduced here. At the same time, Fermat reals seems sufficiently stable and with good properties to permit us to state that such a development can be achieved.

In this chapter we want to introduce the basic theorems and ideas that permits this further development. We shall prove all the theorems which are useful for the development of the calculus both for ${}^\bullet\mathcal{C}^\infty$ functions of the form $f : {}^\bullet U \longrightarrow {}^\bullet\mathbb{R}^d$, where U is open in ${}^\bullet\mathbb{R}^n$, and for functions defined on

[1] We can say that compactness assumptions are only required because of the non adequacy of a tool like normed space (as our Chapter 7 and Section 6.2 prove), in the sense that nothing in the problem of defining smooth spaces and maps forces us to introduce a norm.

infinitesimal sets, like $g : D_n^d \longrightarrow {}^\bullet X$. Subsequently we shall present a first development of infinitesimal differential geometry, primarily in manifolds and in spaces of smooth functions of the form ${}^\bullet N^{{}^\bullet M}$.

Using the Taylor's formula as stated in Theorem 3.4.5, we have a powerful instrument to manage derivatives of functions ${}^\bullet f$ obtained as extensions of ordinary smooth functions $f : \mathbb{R}^d \longrightarrow \mathbb{R}^u$. But this is not the case if $f : {}^\bullet\mathbb{R}^d \longrightarrow {}^\bullet\mathbb{R}^u$ is a generic ${}^\bullet\mathcal{C}^\infty$ arrow, that is if we can write locally $f(x) = {}^\bullet\alpha(p, x)$, where $p \in {}^\bullet\mathbb{R}^n$ and g is smooth, because generally speaking f does not have standard derivatives $\partial_j f(x) \in {}^\bullet\mathbb{R}^u \setminus \mathbb{R}^u$. Therefore, the problem arises how to define the derivatives of this type of functions in our setting. On the one hand, we would like to set e.g. $f'(x) := {}^\bullet(\partial\alpha/\partial x)(p, x)$ (if $d = u = 1$, for simplicity), and so the problem would become the independence in this definition from both the function g and the non standard parameter p. For example, for functions defined on an infinitesimal domain we can see that this problem of independence is not trivial. Let us consider two first order infinitesimals p, $p' \in D$, $p \neq p'$. Because the product of first order infinitesimals is always zero, we have that the null function $f(x) = 0$, for $x \in D$, can be written both as $f(x) = p \cdot x =: {}^\bullet\alpha(p, x)$ and as $f(x) = p' \cdot x = {}^\bullet\alpha(p', x)$. But ${}^\bullet(\partial\alpha/\partial x)(p, x) = p \neq p' = {}^\bullet(\partial\alpha/\partial x)(p', x)$. For functions defined on an open set, this independence can be established, using the method originally used by Fermat and studied by G.E. Reyes (see Moerdijk and Reyes [1991]; see also Bertram [2008] and Shamseddine [1999] for analogous ideas in a context different from that of SDG).

In all this section we will use the notation for intervals as subsets of ${}^\bullet\mathbb{R}$, e.g. $[a, b) := \{x \in {}^\bullet\mathbb{R} \mid a \leq x < b\}$. Notations of the type

$$[a, b)_\mathbb{R} := \{x \in \mathbb{R} \mid a \leq x < b\}$$

will be used to specify that the interval has to be understood as a subset of $\mathbb{R}$.

11.2 The Fermat-Reyes method

The method used by Fermat to calculate derivatives is to assume $h \neq 0$, to construct the incremental ratio

$$\frac{f(x + h) - f(x)}{h},$$

and then to set $h = 0$ in the final result. This idea, which sounds as inconsistent, can be perfectly understood if we think that the incremental ratio can be extended with continuity at $h = 0$ if the function f is differentiable at x. In our smooth context, we need a theorem confirming the existence of a "smooth version" of the incremental ratio. We firstly introduce the notion of segment in an n-dimensional space ${}^\bullet\mathbb{R}^n$, that, as we will prove later, for $n = 1$ coincide with the notion of interval in ${}^\bullet\mathbb{R}$.

Definition 11.2.1. *If a, $b \in {}^\bullet\mathbb{R}^n$, then*

$$\overrightarrow{[a, b]} := \{a + s \cdot (b - a) \mid s \in [0, 1]\}$$

is the segment of ${}^\bullet\mathbb{R}^n$ going from $a \in {}^\bullet\mathbb{R}^n$ to $b \in {}^\bullet\mathbb{R}^n$.

Theorem 11.2.2. *Let U be an open set of $\mathbb{R}$, and $f : {}^\bullet U \longrightarrow {}^\bullet\mathbb{R}$ be a ${}^\bullet\boldsymbol{\mathcal{C}}^\infty$ function. Let us define the* thickening of ${}^\bullet U$ along the x-axis *by*

$$\widetilde{{}^\bullet U} := \left\{(x, h) \mid \overrightarrow{[x, x + h]} \subseteq {}^\bullet U\right\},$$

Then $\widetilde{{}^\bullet U}$ is open in ${}^\bullet\mathbb{R}^2$ and there exists one and only one ${}^\bullet\boldsymbol{\mathcal{C}}^\infty$ map $r : \widetilde{{}^\bullet U} \longrightarrow {}^\bullet\mathbb{R}$ such that

$$f(x + h) = f(x) + h \cdot r(x, h) \quad \forall (x, h) \in \widetilde{{}^\bullet U}.$$

Hence we define $f'(x) := r(x, 0) \in {}^\bullet\mathbb{R}$ for every $x \in {}^\bullet U$.
Moreover if $f(x) = {}^\bullet\alpha(p, x)$, $\forall x \in \mathcal{V} \subseteq {}^\bullet U$ with $\alpha \in \mathcal{C}^\infty(A \times B, \mathbb{R})$, then

$$f'(x) = {}^\bullet\left(\frac{\partial\alpha}{\partial x}\right)(p, x).$$

We anticipate the proof of this theorem by the following lemmas

Lemma 11.2.3. *Let U be an open set of $\mathbb{R}^n$ and $v \in {}^\bullet\mathbb{R}^n$, then the* thickening of ${}^\bullet U$ along v *defined as*

$$\widetilde{{}^\bullet U_v} := \left\{(x, h) \in {}^\bullet\mathbb{R}^n \times {}^\bullet\mathbb{R} \mid \overrightarrow{[x, x + hv]} \subseteq {}^\bullet U\right\} \tag{11.2.1}$$

is open in ${}^\bullet\mathbb{R}^n \times {}^\bullet\mathbb{R}$.

Proof: Let us take a generic point $(x, h) \in \widetilde{{}^\bullet U_v}$; we want to prove that $(x, h) \in {}^\bullet(A \times B) \subseteq \widetilde{{}^\bullet U_v}$ for some subsets A of ${}^\bullet\mathbb{R}^n$ and B of ${}^\bullet\mathbb{R}$. Because the point (x, h) is in the thickening, we have that

$$\forall s \in [0, 1] : \quad x + s \cdot hv \in {}^\bullet U.$$

Taking the standard parts we obtain

$$\forall s \in [0, 1]_{\mathbb{R}} : \quad {}^\circ x + s \cdot {}^\circ h \cdot {}^\circ v =: \varphi(s) \in U.$$

The function $\varphi : [0, 1]_{\mathbb{R}} \longrightarrow U$ is continuous and thus

$$\varphi\left([0, 1)_{\mathbb{R}}\right) = \overrightarrow{[{}^\circ x, {}^\circ x + {}^\circ h {}^\circ v]} =: K$$

is compact in $\mathbb{R}^n$. But $K \subseteq U$ and U is open, so the distance of K from the complement $\mathbb{R}^n \setminus U$ is strictly positive; let us call $2a := d\left(K, \mathbb{R}^n \setminus U\right) > 0$ this distance, so that for every $c \in K$ we have that

$$B_a(c) := \{x \in \mathbb{R}^n \mid d(x, c) < a\} \subseteq U.$$

Now, set $A := B_{a/2}(^\circ x)$ and $B := B_b(^\circ h)$, where we have fixed $b \in \mathbb{R}_{>0}$ such that $b \cdot \|^\circ v\| \le \frac{a}{2}$. We have $x \in {}^\bullet A$ because $^\circ x \in A$ and A is open; analogously $h \in {}^\bullet B$ and thus $(x, h) \in {}^\bullet A \times {}^\bullet B = {}^\bullet(A \times B)$. We have finally to prove that taking a generic point $(y, k) \in {}^\bullet(A \times B)$, the whole segment $\overrightarrow{[y, y + kv]}$ is contained in ${}^\bullet U$; so, let us take also a Fermat number $0 \le s \le 1$. Since U is open, to prove that $y + skv \in {}^\bullet U$ is equivalent to prove that the standard part $y + skv$ is in U, i.e. that $^\circ y + {}^\circ s {}^\circ k {}^\circ v \in U$. For, let us observe that

$$\|^\circ y + {}^\circ s {}^\circ k {}^\circ v - {}^\circ x - {}^\circ s {}^\circ h {}^\circ v\| \le \|^\circ y - {}^\circ x\| + |^\circ s| \cdot \|^\circ v\| \cdot |^\circ k - {}^\circ h| \le \frac{a}{2} + 1 \cdot \|^\circ v\| \cdot b \le a.$$

Therefore, $^\circ y + {}^\circ s {}^\circ k {}^\circ v \in B_a(c) \subseteq U$, where $c = {}^\circ x + {}^\circ s {}^\circ h {}^\circ v \in K$ from our definition of the compact set K.

■

Lemma 11.2.4. *If a, $b \in {}^\bullet\mathbb{R}$, then*

$$\begin{aligned} a < b \quad &\Longrightarrow \quad \overrightarrow{[a, b]} = [a, b] \\ b \le a \quad &\Longrightarrow \quad \overrightarrow{[a, b]} = [b, a]. \end{aligned}$$

Proof: We will prove the first implication, the second being a simple consequence of the first one. To prove the inclusion $\overrightarrow{[a, b]} \subseteq [a, b]$ take $x = a + s \cdot (b - a)$ with $0 \le s \le 1$, then $0 \le s \cdot (b - a) \le b - a$ because $b - a > 0$. Adding a to these inequalities we get $a \le x \le b$. For the proof of the opposite inclusion, let us consider $a \le x \le b$. If we prove the inclusion for $a = 0$ only, we can prove it in general: in fact, $0 \le x - a \le b - a$, so that if $[0, b - a] \subseteq \overrightarrow{[0, b - a]}$ we can derive the existence of $s \in [0, 1]$ such that $x - a = 0 + s \cdot (b - a)$, which is our conclusion. So, let us assume that $a = 0$. If $^\circ b \ne 0$, then b is invertible and it suffices to set $s := \frac{x}{b}$ to have the conclusion. Otherwise, $^\circ b = 0$ and hence also $^\circ x = 0$. Let us consider the decompositions of x and b

$$\begin{aligned} x &= \sum_{i=1}^{k} {}^\circ x_i \cdot \mathrm{d}t_{\omega_i(x)} \\ b &= \sum_{j=1}^{h} {}^\circ b_j \cdot \mathrm{d}t_{\omega_j(b)}. \end{aligned}$$

We have to find a number $s = {}^\circ s + \sum_{n=1}^{N} {}^\circ s_n \cdot \mathrm{d}t_{\omega_n(s)}$ such that $s \cdot b = x$. It is interesting to note that the attempt to find the solution $s \in [0, 1]$ directly from these decompositions and from the property $s\cdot b = x$ is not as easy as to find the solution using directly little-oh polynomials. In fact $\forall^0 t > 0 :\ b_t > 0$ because $b > 0$ and hence for $t > 0$ sufficiently small, we can form the ratio

$$\begin{aligned}\frac{x_t}{b_t} &= \frac{\sum_{i=1}^{k} {}^\circ x_i \cdot t^{\frac{1}{\omega_i(x)}}}{\sum_{j=1}^{h} {}^\circ b_j \cdot t^{\frac{1}{\omega_j(b)}}} \\ &= \frac{t^{\frac{1}{\omega_1(b)}} \cdot \sum_{i=1}^{k} {}^\circ x_i \cdot t^{\frac{1}{\omega_i(x)} - \frac{1}{\omega_1(b)}}}{t^{\frac{1}{\omega_1(b)}} \cdot \sum_{j=1}^{h} {}^\circ b_j \cdot t^{\frac{1}{\omega_j(b)} - \frac{1}{\omega_1(b)}}}. \end{aligned} \tag{11.2.2}$$

Let us note that from Theorem 4.2.6 we can deduce that ${}^\circ x_1 > 0$ since $x > 0$ and hence that $\omega(b) = \omega_1(b) > \omega(x) \geq \omega_i(x)$ because $x < b$. From (11.2.2) we have

$$\begin{aligned}\frac{x_t}{b_t} &= \frac{\sum_{i=1}^{k} {}^\circ x_i \cdot t^{\frac{1}{\omega_i(x)} - \frac{1}{\omega_1(b)}}}{{}^\circ b_1 \cdot \left(1 + \sum_{j=2}^{h} \frac{{}^\circ b_j}{{}^\circ b_1} \cdot t^{\frac{1}{\omega_j(b)} - \frac{1}{\omega_1(b)}}\right)} \\ &= \frac{1}{{}^\circ b_1} \cdot \sum_{i=1}^{k} {}^\circ x_i \cdot t^{\frac{1}{\omega_i(x)} - \frac{1}{\omega_1(b)}} \cdot \sum_{k=0}^{+\infty} (-1)^k \cdot \left(\sum_{j=2}^{h} \frac{{}^\circ b_j}{{}^\circ b_1} \cdot t^{\frac{1}{\omega_j(b)} - \frac{1}{\omega_1(b)}} \right)^k . \end{aligned}$$

Writing, for simplicity, $a \odot b := \frac{a\cdot b}{a+b}$ we can write the previous little-oh polynomial using the common notation with $\mathrm{d}t_a$:

$$s_t := \frac{x_t}{b_t} = \frac{1}{{}^\circ b_1} \cdot \sum_{i=1}^{k} {}^\circ x_i \cdot \mathrm{d}t_{\omega_j(x)\odot\omega_1(b)} \cdot \sum_{k=0}^{+\infty} (-1)^k \cdot \left(\sum_{j=2}^{h} \frac{{}^\circ b_j}{{}^\circ b_1} \cdot \mathrm{d}t_{\omega_j(b)\odot\omega_1(b)} \right)^k . \tag{11.2.3}$$

As usual, the series in this formula is really a finite sum, because D_∞ is an ideal of nilpotent infinitesimals. Going back in these passages, it is quite easy to prove that the previously defined $s \in {}^\bullet\mathbb{R}$ verifies the desired equality $s \cdot b = x$. Moreover, from Theorem 4.2.4 the relations $0 \leq s \leq 1$ follow.

■

It is interesting to make some considerations based on the proof of this lemma. Indeed, we have just proved that in the Fermat reals every equation of the form $a + x\cdot b = c$ with $a < c < a + b$ has a solution[2]. If b is invertible, this is obvious and we have a unique solution. If b is a nilpotent infinitesimal, a possible solution is given by a formula like (11.2.3), but we do not have uniqueness. E.g. if $a = 0$, $c = \mathrm{d}t_2 + \mathrm{d}t$ and $b = \mathrm{d}t_3$, then $x = \mathrm{d}t_6 + \mathrm{d}t_{3/2}$ is

[2] Let us note explicitly, that this is not in contradiction with the non Archimedean property of ${}^\bullet\mathbb{R}$ (let $a = 0$ and $b \in D_\infty$) because of the inequalities that c must verifies to have a solution.

a solution of $a+x\cdot b=c$, but $x+\mathrm{d}t$ is another solution because $\mathrm{d}t\cdot\mathrm{d}t_a=0$ for every $a\ge 1$. Among all the solutions in the case $b\in D_\infty$, we can choose the simplest one, i.e. that "having no useless addends in its decomposition", that is such that

$$\frac{1}{\omega_i(x)}+\frac{1}{\omega(b)}\le 1$$

for every addend $^\circ x_i\cdot\mathrm{d}t_{\omega_i(x)}$ in the decomposition of x. Otherwise, if for some i we have the opposite inequality, we can apply Lemma 3.3.1 with $k:=\omega_i(x)$ to have that $b\cdot x=b\cdot\iota_k x$, i.e. we can delete some "useless addend" considering $\iota_k x$ instead of x. We can thus understand that this algebraic problem is strictly tied with the definition of derivative $f'(x)$, which is the solution of the linear equation $f(x+h)=f(x)+h\cdot f'(x)$: as we give an hint in Chapter 3, if f is defined only on an infinitesimal set like D_n, this equation has not a unique solution and we can define the derivative $f'(x)$ only by considering "the simplest solution", i.e. using a suitable ι_k. We will get back to the problem of defining $f'(x)$, where the function f is defined on an infinitesimal set, in the next Section 12.

The uniqueness of the smooth incremental ratio stated in Theorem 11.2.2 is tied with the following lemma, for the proof of which we decided to introduce nilpotent paths (see Definition 2.1.1) instead of continuous paths at $t=0$, like in [Giordano, 2001]. We will call this lemma *the cancellation law of non-infinitesimal functions.*

Lemma 11.2.5. (Cancellation law of non-infinitesimal functions): *Let U be an open neighborhood of 0 in $\mathbb{R}$, and let*

$$f,\ g:{}^\bullet U\longrightarrow{}^\bullet\mathbb{R}$$

be two ${}^\bullet\mathcal{C}^\infty$ functions such that

$$\forall x\in{}^\bullet U:\ x\text{ is invertible}\quad\Longrightarrow\quad g(x)\text{ is invertible and }g(x)\cdot f(x)=0.$$

Then f is the null function, i.e. $f=0$.

Proof: We have that $f:\overline{{}^\bullet U}\longrightarrow{}^\bullet\mathbb{R}$ and hence $f\in_{{}^\bullet U}{}^\bullet\mathbb{R}$ and we can apply Theorem 9.2.4 at the point $0\in{}^\bullet U$ obtaining that the function f can be written as

$$f(x)={}^\bullet\alpha(p,x)\quad\forall x\in{}^\bullet B\cap{}^\bullet U={}^\bullet(B\cap U)=:\mathcal{V},$$

where $\alpha\in\mathcal{C}^\infty(A\times B,\mathbb{R})$, $p\in{}^\bullet A$, A is an open set of $\mathbb{R}^{\mathsf{p}}$ and B is an open neighborhood of 0 in $\mathbb{R}$. We can always assume that $^\circ p=0$ because, otherwise, we can consider the standard smooth function $(y,x)\mapsto\alpha(y-{}^\circ p,x)$. We can thus write our main hypotheses as

$$\forall x\in\mathcal{V}:\ x\text{ is invertible}\quad\Longrightarrow\quad\lim_{t\to0^+}\frac{g(x)_t\cdot\alpha(p_t,x_t)}{t}=0.\tag{11.2.4}$$

Let us provide some explanation about the notation $g(x)_t$ which is a consequence of our notations concerning quotient sets: we have that $g(x) \in {}^\bullet\mathbb{R} = \mathbb{R}_o[t]/\sim$, hence, avoiding the use of equivalence classes in favor of the new notion of equality $\sim$ in $\mathbb{R}_o[t]$, we have that $g(x)$ is a little-oh polynomial and hence $g(x) : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}$, from which the notation $g(x)_t \in \mathbb{R}$ for $t \in \mathbb{R}_{\geq 0}$ acquires a clear meaning. We firstly want to prove that $\alpha(p_t, x_t) = o(t)$ for every $x \in \mathcal{V}$. Let us take a generic infinitesimal $h \in D_\infty$ and choose a $k \in \mathbb{N}_{>0}$ such that[3]

$$\begin{aligned} h^k &= 0 \quad \text{in} \quad {}^\bullet\mathbb{R} \\ p^k &= \underline{0} \quad \text{in} \quad {}^\bullet\mathbb{R}^{\mathsf{p}}, \end{aligned}$$

and consider a generic non zero $r \in U \cap B \setminus \{0\}$. Then $x := h + r \in {}^\bullet U$, because ${}^\circ x = r \in U$, and x is invertible because its standard part is r and $r \neq 0$. From our hypothesis we have that $g(x)$ is also invertible, i.e. ${}^\circ g(x) = g(x)_0 = \lim_{t \to 0^+} g(x)_t \neq 0$, and hence from (11.2.4) we get

$$\lim_{t \to 0^+} \frac{\alpha(p_t, h_t + r)}{t} = 0. \tag{11.2.5}$$

Because every invertible $x \in \mathcal{V}$ can be written as $x = h + r$ with $r \in \mathbb{R}_{\neq 0}$ and $h \in D_\infty$, we have just proved our conclusion for every $x \in \mathcal{V}$ which is invertible. Now we have to prove (11.2.5) for $r = 0$ too. Let us consider the Taylor's formula of order k with the function α at the point $(\underline{0}, r)$ (which obviously is true for $r = 0$ too):

$$\begin{aligned} \frac{\alpha(\underline{0} + p_t, r + h_t)}{t} = \frac{1}{t} \cdot \Bigg[& \sum_{\substack{q \in \mathbb{N}^{\mathsf{p}+1} \\ |q| \leq k}} \frac{\partial^q \alpha}{\partial (p,x)^q}(\underline{0}, r) \cdot \frac{(p_t, h_t)^q}{q!} + \\ & + \sum_{\substack{q \in \mathbb{N}^{\mathsf{p}+1} \\ |q| = k+1}} \frac{\partial^q \alpha}{\partial (p,x)^q}(\xi_t, \eta_t) \cdot \frac{(p_t, h_t)^q}{q!} \Bigg] \end{aligned} \tag{11.2.6}$$

with $\xi_t \in (\underline{0}, p_t)$ and $\eta_t \in (r, r + h_t)$. But $h^k = 0$ and $p^k = (p_1, \dots, p_{\mathsf{p}})^k = (p_1^k, \dots, p_{\mathsf{p}}^k) = \underline{0}$, hence h, $p_i \in D_k$. Moreover, if $|q| = k + 1$, then

$$\sum_{i=1}^{\mathsf{p}+1} \frac{q_i}{k+1} = \frac{k+1}{k+1} = 1,$$

so that from Corollary 2.5.4 we get

$$(p_t, h_t)^q = p_1(t)^{q_1} \cdot \ldots \cdot p_{\mathsf{p}}(t)^{q_{\mathsf{p}}} \cdot h(t)^{q_{\mathsf{p}+1}} = o(t).$$

[3] This passage is possible exactly because we are considering nilpotent paths as elements of ${}^\bullet\mathbb{R}$.

Therefore, from (11.2.5) and (11.2.6) we obtain

$$\lim_{t\to 0^+} \sum_{\substack{q\in\mathbb{N}^{p+1}\\ |q|\le k}} \frac{\partial^q \alpha}{\partial(p,x)^q}(\underline{0},r)\cdot\frac{1}{q!}\cdot\frac{(p_t,h_t)^q}{t}=0 \quad \forall r\in(U\cap B)_{\ne 0}. \tag{11.2.7}$$

Now, let $\{q_1,\dots,q_N\}$ be an enumeration of all the $q\in\mathbb{N}^{p+1}$ such that $|q|\le k$, and for simplicity set

$$\begin{aligned} b_i(r) &:= \frac{\partial^{q_i}\alpha}{\partial(p,x)^{q_i}}(\underline{0},r)\cdot\frac{1}{q_i!} \quad \forall r\in U\cap B\\ s_i(t) &:= \frac{(p_t,h_t)^{q_i}}{t} \quad \forall t\in\mathbb{R}_{\ge 0}, \end{aligned}$$

so that we can write (11.2.7) as

$$\forall r\in(U\cap B)_{\ne 0}:\quad \lim_{t\to 0^+}\sum_{i=1}^N b_i(r)\cdot s_i(t)=0. \tag{11.2.8}$$

If all the functions b_i are identically zero, then $b_i(\bar r)=b_i(0)$ where $\bar r\in U\cap B\setminus\{0\}$, which always exists because $U\cap B$ is open in $\mathbb{R}$. Therefore, (11.2.8) (and hence also (11.2.7)) is true for $r=0$ too. Otherwise, taking a base of the subspace of $\mathcal{C}^\infty(U\cap B,\mathbb{R})$ generated by the smooth functions $b_1,\dots,b_N$ and expressing all the b_i in this base, we can suppose to have in (11.2.8) only linearly independent functions.

We can now use the following lemma:

Lemma 11.2.6. *Let U be an open neighborhood of 0 in $\mathbb{R}$ and $b_1,\dots,b_N : U\longrightarrow\mathbb{R}$ be linearly independent functions continuous at 0. Then we can find*

$$r_1,\dots,r_N\in U\setminus\{0\}$$

such that

$$\det\begin{bmatrix} b_1(r_1) & \dots & b_N(r_1)\\ \vdots & & \vdots\\ b_1(r_N) & \dots & b_N(r_N)\end{bmatrix}\ne 0.$$

From (11.2.8) we can write

$$\lim_{t\to 0^+}\begin{bmatrix} b_1(r_1) & \dots & b_N(r_1)\\ \vdots & & \vdots\\ b_1(r_N) & \dots & b_N(r_N)\end{bmatrix}\cdot\begin{bmatrix} s_1(t)\\ \vdots\\ s_N(t)\end{bmatrix}=\underline{0}$$

and hence from this lemma we can deduce that $s_i(t)\to 0$ for $t\to 0^+$. Because these limits exist, we can take the limit for $r\to 0$ of (11.2.8) and proceed in the following way

$$\lim_{r\to 0}\lim_{t\to 0^+}\sum_{i=1}^N b_i(r)\cdot s_i(t)=\sum_{i=1}^N \lim_{r\to 0} b_i(r)\cdot\lim_{t\to 0^+} s_i(t)$$
$$=\lim_{t\to 0^+}\sum_{i=1}^N b_i(0)\cdot s_i(t)=0$$

(let us note that we do not exchange the limit signs). This proves that (11.2.8) is true for $r=0$ too. From (11.2.6) for $r=0$ we obtain

$$\lim_{t\to 0^+}\frac{\alpha(p_t,h_t)}{t}=0.$$

This proves that $f(x)=0$ in ${}^\bullet\mathbb{R}$ for every $x\in\mathcal{V}$. Finally, if $x\in{}^\bullet U\setminus\mathcal{V}$ then ${}^\circ x\neq 0$ because otherwise we would have $x\in\mathcal{V}={}^\bullet(B\cap U)$. So x is invertible and hence also $g(x)$ is invertible, so that from $g(x)\cdot f(x)=0$ we can easily deduce $f(x)=0$ also in this case.

∎

Proof of Lemma 11.2.6: We prove the converse by induction on $N\geq 2$, i.e. if all the determinants cited in the statement are zero, then the functions $(b_1,\dots,b_N)$ are linearly dependent. Let us suppose first that $N=2$ and that all these determinants are zero, that is

$$b_1(r)\cdot b_2(s)=b_2(r)\cdot b_1(s)\quad\forall r,s\in U_{\neq 0}.\tag{11.2.9}$$

If the functions b_i, $i=1,2$, are both zero then they are trivially linearly dependent, hence let us suppose, e.g., that $b_1(\bar{s})\neq 0$ for some $\bar{s}\in U$. Due to the continuity of b_1 at 0 we can suppose $\bar{s}\neq 0$, hence from (11.2.9)

$$b_2(r)=b_1(r)\cdot\frac{b_2(\bar{s})}{b_1(\bar{s})}=:b_1(r)\cdot a\quad\forall r\in U_{\neq 0}.$$

From the continuity of b_i at 0 we have that $b_2=b_1\cdot a$, that is (b_1,b_2) are linearly dependent.
Now suppose that the implication is true for any matrix of N functions and we prove the conclusion for matrices of order $N+1$ too. By Laplace's formula with respect to the first row, for every $r_1,\dots,r_{N+1}\in U_{\neq 0}$ we have

$$b_1(r_1)\cdot\begin{vmatrix} b_2(r_2) & \dots & b_{N+1}(r_2)\\ \vdots & & \vdots\\ b_2(r_{N+1}) & \dots & b_{N+1}(r_{N+1})\end{vmatrix}-\dots+$$
$$+(-1)^{N+2}\cdot b_{N+1}(r_1)\cdot\begin{vmatrix} b_1(r_2) & \dots & b_N(r_2)\\ \vdots & & \vdots\\ b_1(r_{N+1}) & \dots & b_N(r_{N+1})\end{vmatrix}=0.\tag{11.2.10}$$

Now we have two cases. Let $\alpha_1(r_2,\dots,r_{N+1})$ denote the first determinant in the previous (11.2.10). If it is zero for any $r_2,\dots,r_{N+1}\in U_{\neq 0}$, then by the induction hypothesis $(b_2,\dots,b_{N+1})$ are linearly dependent, hence the conclusion follows. Otherwise $\bar\alpha_1 := \alpha_1(\bar r_2,\dots,\bar r_{N+1})\neq 0$ for some $\bar r_2,\dots,\bar r_{N+1}\in U_{\neq 0}$. Then from (11.2.10) it follows

$$b_1(r_1) = b_2(r_1)\cdot\frac{\alpha_2}{\bar\alpha_1} - \ldots - (-1)^{N+2}\cdot b_{N+1}(r_1)\cdot\frac{\alpha_{N+1}}{\bar\alpha_1} \quad \forall r_1\in U_{\neq 0},$$

where we used obvious notations for the other determinants in (11.2.10). From the continuity of b_i the previous formula is true for $r_1=0$ too and this proves the conclusion.

$\blacksquare$

Proof of Theorem 11.2.2: We will define the function $r : {}^\bullet\widetilde{U} \longrightarrow {}^\bullet\mathbb{R}$ patching together smooth functions defined on open subsets covering ${}^\bullet\widetilde{U}$. Therefore, we have to take a generic point $(x,h)\in{}^\bullet\widetilde{U}$, to define the function r on some open neighborhood of (x,h) in ${}^\bullet\widetilde{U}$, and to prove that every two of such local functions agree on the intersection of their domains.
As usual, we have that $f\in_{{}^\bullet U}{}^\bullet\mathbb{R}$ and, since $x\in{}^\bullet U$, we can write

$$f|_{\mathcal{V}} = {}^\bullet\alpha(p,-)|_{\mathcal{V}}, \tag{11.2.11}$$

where $\alpha\in\boldsymbol{\mathcal{C}}^\infty(\bar U\times\bar V,\mathbb{R})$, $\mathcal{V}:={}^\bullet\bar V\cap{}^\bullet U={}^\bullet(\bar V\cap U)$ is an open neighborhood of x and ${}^\bullet\bar U$ is an open neighborhood of $p\in{}^\bullet\mathbb{R}^{\mathsf{p}}$ defined by the open subset $\bar U$ of $\mathbb{R}^{\mathsf{p}}$. Because ${}^\bullet\widetilde{U}$ is open in ${}^\bullet\mathbb{R}\times{}^\bullet\mathbb{R}={}^\bullet\mathbb{R}^2$, we can find two open subset A and B of $\mathbb{R}$ such that

$$(x,h)\in{}^\bullet(A\times B)\subseteq{}^\bullet\widetilde{U}$$

and such that

$$a+s\cdot b\in\bar V \quad \forall a\in A,\ b\in B,\ s\in[0,1]_{\mathbb{R}}. \tag{11.2.12}$$

Let us define

$$\gamma(q,a,b):=\int_0^1 \partial_2\alpha(q,a+s\cdot b)\,\mathrm{d}s \quad \forall q\in\bar U,\ a\in A,\ b\in B. \tag{11.2.13}$$

We have that $\gamma\in\boldsymbol{\mathcal{C}}^\infty(\bar U\times A\times B,\mathbb{R})$, so that if we define

$$r(a,b):={}^\bullet\gamma(p,a,b) \quad \forall (a,b)\in{}^\bullet(A\times B),$$

then we have

$$r\in{}^\bullet\boldsymbol{\mathcal{C}}^\infty({}^\bullet(A\times B),\mathbb{R}) \tag{11.2.14}$$

$${}^\bullet(A\times B)\text{ open neighborhood of }(u,h)\text{ in }{}^\bullet\widetilde{U}. \tag{11.2.15}$$

For every $(a,b) \in {}^\bullet(A \times B)$ we have

$$\begin{aligned} b_t \cdot r(a,b)_t &= \int_0^1 \partial_2 \alpha(p_t, a_t + s \cdot b_t) \cdot b_t \, \mathrm{d}s \\ &= \int_{a_t}^{a_t+b_t} \partial_2 \alpha(p_t, y) \, \mathrm{d}y \\ &= \alpha(p_t, a_t + b_t) - \alpha(p_t, a_t). \end{aligned} \tag{11.2.16}$$

But from $(a,b) \in {}^\bullet(A \times B) = {}^\bullet A \times {}^\bullet B$ and (11.2.12) it follows $^\circ a$, $^\circ a + {}^\circ b \in \bar{V}$, and hence also a, $a + b \in {}^\bullet\bar{V}$. From the definition of thickening we also have that a, $a+b \in {}^\bullet U$. We can thus use (11.2.11) at the points a, $b \in \mathcal{V} = {}^\bullet\bar{V} \cap {}^\bullet U$, so that we can write (11.2.16) as

$$\forall (a,b) \in {}^\bullet(A \times B) : \quad b \cdot r(a,b) = f(a+b) - f(a). \tag{11.2.17}$$

We have proved that for every $(x,h) \in \widetilde{{}^\bullet U}$ there exist an open neighborhood ${}^\bullet(A \times B)$ of (x,h) in $\widetilde{{}^\bullet U}$ and a smooth function $r \in {}^\bullet\mathcal{C}^\infty({}^\bullet(A \times B), \mathbb{R})$ such that (11.2.17) holds.

If $\rho \in {}^\bullet\mathcal{C}^\infty \left({}^\bullet(C \times D), {}^\bullet\mathbb{R}\right)$ is another such functions, then

$$\forall (x,h) \in {}^\bullet(C \times D) \cap {}^\bullet(A \times B) : \quad h \cdot [r(x,h) - \rho(x,h)] = 0,$$

so that for every $x \in {}^\bullet C \cap {}^\bullet A$ we have that

$$\forall h \in {}^\bullet(D \times B) : \quad h \cdot [r(x,h) - \rho(x,h)] = 0.$$

For Lemma 11.2.5 applied with $g(h) := h$ and $f(h) := r(x,h) - \rho(x,h)$, we have $r(x,h) = \rho(x,h)$ for every $(x,h) \in {}^\bullet(C \times D) \cap {}^\bullet(A \times B)$, which proves the conclusion for the sheaf property of ${}^\bullet\mathbb{R}$. Finally, let us note that from (11.2.13) for $b = 0$ we obtain $r(a,0) = \partial_2 \alpha(p,a)$, which is the last part of the statement.

∎

Using this theorem, we can develop all the differential calculus for non standard smooth functions of type $f : {}^\bullet\mathbb{R}^n \longrightarrow {}^\bullet\mathbb{R}$. We will see now the first steps of this development, underlining the main differences with respect to Lavendhomme [1996] and Moerdijk and Reyes [1991], to which we refer as a guideline for a complete development.

Definition 11.2.7. *Let U be an open subset of $\mathbb{R}$, and $f : {}^\bullet U \longrightarrow {}^\bullet\mathbb{R}$ a ${}^\bullet\mathcal{C}^\infty$ function. Then*

1. $f'[-] : \widetilde{{}^\bullet U} \longrightarrow {}^\bullet\mathbb{R}$
2. $f(x+h) = f(x) + h \cdot f'[x,h] \quad \forall (x,h) \in \widetilde{{}^\bullet U}$.

Moreover we will also set $f'(x) := f'[x,0]$ for every $x \in {}^\bullet U$.

Let us note that the notation for the smooth incremental ratio as a function uses square brackets like in $f'[-]$. For this reason there is no way to confuse the smooth incremental ratio $f'[-]$ and its values $f'[x,h]$ with the corresponding derivative f' and its values $f'(x)$.

First of all, from property *1.* in the previous definition, it follows that

$$f' : {}^\bullet U \longrightarrow {}^\bullet\mathbb{R}.$$

The following theorem contains the first expected properties of the derivative.

Theorem 11.2.8. *Let U be an open subset of $\mathbb{R}$, and $f,\, g : {}^\bullet U \longrightarrow {}^\bullet\mathbb{R}$ be smooth ${}^\bullet\mathcal{C}^\infty$ functions. Finally, let us consider a Fermat real $r \in {}^\bullet\mathbb{R}$. Then*

1. $(f+g)' = f' + g'$
2. $(r \cdot f)' = r \cdot f'$
3. $(f \cdot g)' = f' \cdot g + f \cdot g'$
4. $(1_{{}^\bullet\mathbb{R}})' = 1$
5. $r' = 0$

Proof: We report the proof essentially as a first example to show how to use precisely the Fermat-Reyes method in our context.

The first step is to prove, e.g., that $f+g$ is smooth in ${}^\bullet\mathcal{C}^\infty$. Looking at the diagram

$$\begin{tikzcd}
 & {}^\bullet\mathbb{R} & & \\
{}^\bullet U \arrow[ur, "f"] \arrow[rr, "\langle f,g\rangle"] \arrow[dr, "g"'] & & {}^\bullet\mathbb{R}^2 \arrow[ul, "p_1"'] \arrow[dl, "p_2"] \arrow[r, "+"] & {}^\bullet\mathbb{R}, \\
 & {}^\bullet\mathbb{R} & &
\end{tikzcd}$$

where $+ : (r,s) \in {}^\bullet\mathbb{R}^2 \mapsto r+s \in {}^\bullet\mathbb{R}$ is the sum of Fermat reals, we can see that $f+g = \langle f,g\rangle \cdot +$ and hence it is smooth because it can be expressed as a composition of smooth functions. The proof that the sum $f+g$ is smooth, even if it is almost trivial, can show us why it is very important to work in a cartesian closed category like ${}^\bullet\mathcal{C}^\infty$. We have, indeed, the possibility to consider very general set theoretical operations like compositions or evaluations.

Now we have only to calculate $(f+g)(x+h)$ using the definition of smooth incremental ratio and its uniqueness

$$\begin{aligned}(f+g)(x+h) &= f(x+h)+g(x+h)\\ &= f(x)+h\cdot f'[x,h]+g(x)+h\cdot g'[x,h]\\ &= (f+g)(x)+h\cdot\left\{f'[x,h]+g'[x,h]\right\} \quad \forall (x,h)\in \widetilde{{}^\bullet U}.\end{aligned}$$

From the uniqueness of the smooth incremental ratio of $f+g$ we obtain $(f+g)'[-]=f'[-]+g'[-]$ and thus the conclusion evaluating these ratios at $h=0$.

As a further simple example, we consider only the derivative of the product. The smoothness of $f\cdot g$ can be proved analogously to what we have just done for the sum. Now, let us evaluate for every $(x,h)\in\widetilde{{}^\bullet U}$

$$\begin{aligned}(f\cdot g)(x+h) &= f(x+h)\cdot g(x+h)\\ &= \left\{f(x)+h\cdot f'[x,h]\right\}\cdot\left\{g(x)+h\cdot g'[x,h]\right\}\\ &= (f\cdot g)(x)+h\cdot\\ &\quad\cdot\left\{f(x)\cdot g'[x,h]+g(x)\cdot f'[x,h]+h^2\cdot f'[x,h]\cdot g'[x,h]\right\}.\end{aligned}$$

From the uniqueness of the smooth incremental ratio of $f\cdot g$ we have thus

$$(f\cdot g)'[x,h]=f(x)\cdot g'[x,h]+g(x)\cdot f'[x,h]+h^2\cdot f'[x,h]\cdot g'[x,h],$$

which gives the conclusion setting $h=0$. The other properties can be proved analogously.

∎

The next expected property that permits a deeper understanding of the Fermat-Reyes method is the chain rule.

Theorem 11.2.9. *If U and V are open subsets of $\mathbb{R}$ and*

$$f : {}^\bullet U \longrightarrow {}^\bullet\mathbb{R}$$

$$g : {}^\bullet V \longrightarrow {}^\bullet U$$

are ${}^\bullet\mathcal{C}^\infty$ functions, then

$$(f\circ g)'=(f'\circ g)\cdot g'.$$

We will give a proof of this theorem with the aim of explaining in a general way the Fermat-Reyes method. We first need the following

Lemma 11.2.10. *Let U be an open subset of $\mathbb{R}^{\mathsf{k}}$, $x\in{}^\bullet U$ and $v\in{}^\bullet\mathbb{R}^{\mathsf{k}}$. Then there exists*

$$r\in\mathbb{R}_{>0}$$

such that

$$\forall h\in(-r,r): \quad (x,h)\in\widetilde{{}^\bullet U_v}.$$

Proof: If ${}^\circ v = \underline{0}$, then for every $s \in [0,1]$ and every $h \in {}^\bullet\mathbb{R}$ we have ${}^\circ(x + shv) = {}^\circ x \in U$, hence $x + shv \in {}^\bullet U$, that is $\overrightarrow{[x, x+hv]} \subseteq {}^\bullet U$. In this case we have thus $(x,h) \in \widetilde{{}^\bullet U_v}$ for every $h \in {}^\bullet\mathbb{R}$.

Otherwise, if ${}^\circ v \neq \underline{0}$ then from ${}^\circ x \in U$ we obtain

$$\exists\, \rho > 0 : \quad B_\rho({}^\circ x) \subseteq U$$

because U is open in $\mathbb{R}^{\mathsf{k}}$. Take as $r \in \mathbb{R}_{>0}$ any real number verifying

$$0 < r < \min\left(\rho, \frac{\rho}{\|{}^\circ v\|}\right).$$

For such an r, if $s \in [0,1]$ and $h \in (-r, r)$, then

$$\begin{aligned} {}^\circ(x + shv) = {}^\circ x + {}^\circ s \cdot {}^\circ h \cdot {}^\circ v \in B_\rho({}^\circ x) &\iff \|{}^\circ s \cdot {}^\circ h \cdot {}^\circ v\| < \rho \\ &\Longleftarrow |{}^\circ h| \cdot \|{}^\circ v\| < \rho \end{aligned} \tag{11.2.18}$$

the last implication is due to the assumption that $s \in [0,1]$. But (11.2.18) holds because $|h| < r$ and hence ${}^\circ|h| = |{}^\circ h| < r$ and $r \cdot \|{}^\circ v\| < \rho$ for the definition of r.

■

The next result works for the Fermat-Reyes methods like a sort of "compactness principle" analogous to the compactness theorem of mathematical logic. It is the generalization to more than just one open set U of the previous lemma.

Theorem 11.2.11. (Compactness principle):
For $i = 1, \ldots, n$, let U^i be open sets of $\mathbb{R}^{\mathsf{k_i}}$, $v \in {}^\bullet\mathbb{R}^{\mathsf{k_i}}$, $x_i \in {}^\bullet U^i$ and finally $a_i \in {}^\bullet\mathbb{R}$. Then there exists

$$r \in \mathbb{R}_{>0}$$

such that

$$\forall i = 1, \ldots, n \ \forall h \in (-r, r) : \quad (x_i, h \cdot a_i) \in \widetilde{{}^\bullet U^i_{v_i}}.$$

Proof: For every $x_i \in U^i$ we apply the previous Lemma 11.2.10 obtaining the existence of $r_i \in \mathbb{R}_{>0}$ such that

$$\forall k \in (-r_i, r_i) : \quad (x_i, k) \in \widetilde{{}^\bullet U^i_{v_i}}. \tag{11.2.19}$$

Now, let us set

$$r := \min_{i : {}^\circ a_i \neq 0} \frac{r_i}{|{}^\circ a_i|} \in \mathbb{R}_{>0},$$

then taking a generic $h \in (-r, r)$ we have

$$-r < {}^\circ h < r. \tag{11.2.20}$$

If ${}^\circ a_i = 0$, then trivially $-r_i < {}^\circ h \cdot {}^\circ a_i < r_i$ and hence $-r_i < h \cdot a_i < r_i$, so that from (11.2.19) we get the conclusion for this first case, i.e. $(x_i, ha_i) \in \widetilde{{}^\bullet U^i_{v_i}}$.

Otherwise, if ${}^\circ a_i \neq 0$, then $r \le \frac{r_i}{|{}^\circ a_i|}$ and from (11.2.20) we get $|{}^\circ h| < r \le \frac{r_i}{|{}^\circ a_i|}$ and hence $-r_i < ha_i < r$, and once again the conclusion follows from (11.2.19).

∎

We can use this theorem in the following way:

1. every time in a proof we need a property of the form

$$(x_i, ha_i) \in {}^\bullet\widetilde{U}_i \tag{11.2.21}$$

   we will assume "to have chosen h so little that (11.2.21) is verified".

2. We derive the conclusion $\mathcal{A}(h)$ under n of such hypothesis, so that we have concretely deduced that

$$\left(\forall i = 1, \ldots, n : \quad (x_i, ha_i) \in {}^\bullet\widetilde{U}_i\right) \quad \Longrightarrow \quad \mathcal{A}(h).$$

3. At this point we can apply the compactness principle obtaining

$$\exists\, r \in \mathbb{R}_{>0}\ \forall h \in (-r, r) : \quad \mathcal{A}(h).$$

4. Usually the property $\mathcal{A}(h)$ is of the form

$$\mathcal{A}(h) \iff h \cdot \tau(h) = h \cdot \sigma(h), \tag{11.2.22}$$

   and hence we can deduce $\tau(h) = \sigma(h)$ for every $h \in (-r, r)$ from the cancellation law of non-infinitesimal functions, and in particular $\tau(0) = \sigma(0)$. If the property $\mathcal{A}$ has the form (11.2.22), then we can also suppose that h is invertible because the cancellation law can be applied also in this case. But at the end we will anyway set $h = 0$, in perfect agreement with the classical description of the Fermat method (see e.g. Bottazzini et al. [1992], Bell [1937], Edwards [1979]).

Let us note that, as mentioned above, conceptually this way to proceed reflects the same idea of the compactness theorem of mathematical logic, because in every proof we can only have a finite number of hypothesis of type (11.2.21). Even if this method does not involve explicitly infinitesimal methods, using it the final proofs are very similar to those we would have if h were an actual infinitesimal, i.e. $h \in D_\infty$.

In the following proof we will concretely use this method.

Proof of Theorem 11.2.9: First of all the composition

$$(-) \circ (-) : {}^\bullet U^{{}^\bullet V} \times {}^\bullet\mathbb{R}^{{}^\bullet U} \longrightarrow {}^\bullet\mathbb{R}^{{}^\bullet V}$$

is a smooth map of ${}^\bullet\mathcal{C}^\infty$ and hence $f \circ g$ is smooth because it can be written as a composition of smooth maps.

For a generic
$$(x,h) \in {}^\bullet\widetilde{V} \tag{11.2.23}$$
we can always write
$$(f \circ g)(x+h) = f\left[g(x+h)\right] = f\left[g(x) + h \cdot g'[x,h]\right]$$
because $x + h \in {}^\bullet V$ and hence $f \circ g$ is defined at $x + h$. Now we would like to use the smooth incremental ratio of f at the point $g(x)$ with increment $h \cdot g'[x,h]$. For this end we assume
$$(g(x), h \cdot g'[x,h]) \in {}^\bullet\widetilde{U} \tag{11.2.24}$$
so that we can write
$$(f \circ g)(x+h) = f(gx) + h \cdot g'[x,h] \cdot f'\left[gx, h \cdot g'[x,h]\right].$$
Using the compactness principle and the cancellation law of non-infinitesimal functions we get
$$\exists\, r \in \mathbb{R}_{>0} : \ \forall h \in (-r,r) : \quad g'[x,h] \cdot f'\left[gx, h \cdot g'[x,h]\right] = (f \circ g)'[x,h],$$
and thus the conclusion for $h = 0$.

■

Let us note that these ideas, that do not use infinitesimal methods, can be repeated in a standard context, with only slight modifications, so that they represent an interesting alternative way to teach a significant part of the calculus with strongly simpler proofs.

To realize a comparison with the Levi-Civita field (see Appendix B) we now prove the inverse function theorem.

Theorem 11.2.12. *Let U be an open subset of $\mathbb{R}$, x a point in ${}^\bullet U$, and*
$$f : {}^\bullet U \longrightarrow {}^\bullet\mathbb{R}$$
a ${}^\bullet\mathcal{C}^\infty$ map such that
$$f'(x) \text{ is invertible.}$$
Then there exist two open subsets X, Y of $\mathbb{R}$ such that

1. *$x \in {}^\bullet X$ and $f(x) \in {}^\bullet Y$, i.e. ${}^\bullet X$ and ${}^\bullet Y$ are open neighborhoods of x and $f(x)$ respectively*
2. *$f|_{{}^\bullet X} : {}^\bullet X \longrightarrow {}^\bullet Y$ is invertible and $(f|_{{}^\bullet X})^{-1} : {}^\bullet Y \longrightarrow {}^\bullet X$ is a ${}^\bullet\mathcal{C}^\infty$ map*
3. *$\left[(f|_{{}^\bullet X})^{-1}\right]'(fx_1) = \dfrac{1}{f'(x_1)}$ for every $x_1 \in {}^\bullet X$*

Proof: Because $x \in {}^\bullet U$ we can write $f|_{\mathcal{V}} = {}^\bullet\alpha(p,-)|_{\mathcal{V}}$, where $\alpha \in \mathcal{C}^\infty(A \times B, \mathbb{R})$, $p \in {}^\bullet A$, A is an open set of $\mathbb{R}^{\mathsf{p}}$ and B is an open subset of $\mathbb{R}$ such that $x \in {}^\bullet B$ and finally $\mathcal{V} := {}^\bullet B \cap {}^\bullet U$. Considering $B \cap U$ instead of B we can assume, for simplicity, that $B \subseteq U$.

We have that[4] $f'(x) = \partial_2\alpha(p,x)$ is invertible, hence its standard part is not zero

$$^\circ f'(x) = \partial_2\alpha(^\circ p, {}^\circ x) \in \mathbb{R}_{\neq 0}.$$

Since α is smooth, we can find a neighborhood $C \times D \subseteq A \times B \subseteq A \times U$ of $(^\circ p, {}^\circ x)$ where $\partial_2\alpha(p_1, x_1) \neq 0$ for every $(p_1, x_1) \in C \times D$. We can also assume to have taken this neighborhood sufficiently small in order to have that

$$\inf_{(p_1,x_1)\in C\times D} |\partial_2\alpha(p_1,x_1)| =: m > 0. \tag{11.2.25}$$

By the standard implicit function theorem, we get an open neighborhood $E \times X \subseteq C \times D$ of $(^\circ p, {}^\circ x)$, an open neighborhood Y of $\alpha(^\circ p, {}^\circ x)$ and a smooth function $\beta \in \mathcal{C}^\infty(E \times Y, X)$ such that

$$\forall p_1 \in E\, \forall x_1 \in X : \quad \alpha(p_1, x_1) \in Y \tag{11.2.26}$$

$$\alpha\left[p_1, \beta(p_1, y_1)\right] = y_1 \quad \forall (p_1, y_1) \in E \times Y \tag{11.2.27}$$

$$\forall p_1 \in E\, \forall y \in Y\ \exists!\, x \in X : \quad \alpha(p_1, x) = y. \tag{11.2.28}$$

We can assume that X is connected. Let us define $g := {}^\bullet\beta(p,-)$, then $g \in \mathcal{C}^\infty({}^\bullet Y, {}^\bullet X)$. Moreover, $x \in {}^\bullet X$ and $f(x) \in {}^\bullet Y$ because $^\circ x \in X$, $^\circ f(x) = \alpha(^\circ p, {}^\circ x) \in Y$ and X, Y are open. From (11.2.26), if $x_1 \in {}^\bullet X$, then $^\circ f(x_1) = \alpha(^\circ p, {}^\circ x_1) \in Y$, hence $f(x_1) \in {}^\bullet Y$, so that f maps ${}^\bullet X$ in ${}^\bullet Y$. From (11.2.27), noting that $p \in {}^\bullet E$, because $^\circ p \in E$, and that $X \subseteq D \subseteq B$, we obtain

$$\forall y \in {}^\bullet Y : \quad f\left(g(y)\right) = \alpha\left[p, g(y)\right] = \alpha\left[p, \beta(p,y)\right] = y.$$

This proves that g is a smooth left[5] inverse of $f|_{{}^\bullet X} : {}^\bullet X \longrightarrow {}^\bullet Y$, which is thus surjective. If we prove that $f|_{{}^\bullet X}$ is injective, this left inverse will also be the right inverse. So, let us suppose that $f(x_1) = f(x_2)$ in ${}^\bullet Y$ for x_1, $x_2 \in {}^\bullet X$, i.e.

$$\lim_{t\to 0^+} \frac{\alpha(p_t, x_{1t}) - \alpha(p_t, x_{2t})}{t} = 0.$$

But we can write

$$\alpha(p_t, x_{1t}) - \alpha(p_t, x_{2t}) = (x_{1t} - x_{2t}) \cdot \partial_2\alpha(p_t, \xi_t) \quad \forall t \in \mathbb{R}_{>0}$$

for a suitable $\xi_t \in (x_{1t}, x_{2t})$. Moreover, from (11.2.25) and from

$$\forall^0 t > 0 : \quad \xi_t \in (x_{1t}, x_{2t}) \subseteq X \subseteq D \quad \text{and} \quad p_t \in E \subseteq C$$

[4]Because it is sufficiently clear from the context, we use here simplified notations like $\partial_2\alpha(p,x)$ instead of ${}^\bullet(\partial_2\alpha)(p,x)$.

[5]With respect to the notation for the composition $(g \cdot f)(y) = f(g(y))$.

(here we are using the assumption that X is connected), we have that $|\partial_2\alpha(p_t,\xi_t)| \ge m$. Therefore

$$\begin{aligned}\lim_{t\to 0^+}\left|\frac{x_{1t}-x_{2t}}{t}\right| &= \lim_{t\to 0^+}\left|\frac{\alpha(p_t,x_{1t})-\alpha(p_t,x_{2t})}{t\cdot\partial_2\alpha(p_t,\xi_t)}\right| \ge \\ &\ge \lim_{t\to 0^+}\left|\frac{\alpha(p_t,x_{1t})-\alpha(p_t,x_{2t})}{t\cdot m}\right| = 0,\end{aligned}$$

and this proves that $x_1 = x_2$ in ${}^\bullet X$ and thus also that $f|_{{}^\bullet X} : {}^\bullet X \longrightarrow {}^\bullet Y$ is invertible with smooth inverse given by $(f|_{{}^\bullet X})^{-1} = g$.

Now we can use the Fermat-Reyes method to prove the formula for the derivative of the inverse function. Let us consider a point $(x_1,h) \in \widetilde{{}^\bullet X}$ in the thickening of ${}^\bullet X$, then $f(x_1+h) = f(x_1) + h\cdot f'[x_1,h]$. Applying $g = (f|_{{}^\bullet X})^{-1}$ to both sides of this formula we obtain

$$x_1 + h = g\left[fx_1 + h\cdot f'[x_1,h]\right].$$

It is natural, at this point, to try to use the smooth incremental ratio of the smooth function g. For this end we have to assume that

$$(fx_1, h\cdot f'[x_1,h]) \in \widetilde{{}^\bullet Y}$$

so that we can write

$$x_1 + h = g\left(f(x_1)\right) + h\cdot f'[x_1,h]\cdot g'\left[fx_1, h\cdot f'[x,h]\right].$$

Because $g\left(f(x_1)\right) = x_1$, we obtain the equality

$$h = h\cdot f'[x_1,h]\cdot g'\left[fx_1, h\cdot f'[x,h]\right].$$

From the compactness principle (Theorem 11.2.11) and the cancellation law of non-infinitesimal functions (Lemma 11.2.5) we obtain

$$1 = f'[x_1,h]\cdot g'\left[fx_1, h\cdot f'[x,h]\right],$$

from which the conclusion follows setting $h = 0$.

■

We have shown, using meaningful examples, that the Fermat-Reyes method can be used to try a generalization of several results of differential calculus to ${}^\bullet\mathcal{C}^\infty$ functions of the form $f : {}^\bullet U \longrightarrow {}^\bullet\mathbb{R}^d$, with U open in $\mathbb{R}^n$.

Indeed, this can be done for several theorems. We only list here the main results that we have already proved, leaving a complete report of them for a subsequent work. For most of them the proofs are very similar to the analogous presented e.g. in Lavendhomme [1996]:

1. the formula for the derivative of $\frac{1}{f(x)}$ if $f(x) \in {}^\bullet\mathbb{R}$ is invertible,

2. the notion of right and left derivatives, i.e. $f'_+(a)$ and $f'_-(b)$ for a ${}^\bullet\mathcal{C}^\infty$ function of the form $f : [a, b] \longrightarrow {}^\bullet\mathbb{R}^d$,

3. definition of higher order derivatives using higher order smooth incremental ratios,

4. 1-dimensional Taylor's formula with integral rest (see the next Section 11.3 about the integral calculus),

5. uniqueness theorem for Taylor's formulas,

6. the functional operation of taking the derivative is smooth, i.e. the map $f \in {}^\bullet\mathbb{R}^{{}^\bullet U} \mapsto f' \in {}^\bullet\mathbb{R}^{{}^\bullet U}$ is ${}^\bullet\mathcal{C}^\infty$,

7. the functional operation of taking the smooth incremental ratio is smooth, i.e. the map $f \in {}^\bullet\mathbb{R}^{{}^\bullet U} \mapsto f'[-] \in {}^\bullet\mathbb{R}^{\widetilde{{}^\bullet U}}$ is ${}^\bullet\mathcal{C}^\infty$,

8. definition of partial derivatives using smooth partial incremental ratio,

9. the functional operation of taking the partial derivative and the smooth partial incremental ratio are smooth,

10. linearity of the map: $v \in {}^\bullet\mathbb{R}^n \mapsto \frac{\partial f}{\partial v}(x) \in {}^\bullet\mathbb{R}^d$,

11. definition of differentials of arbitrary order,

12. Euler-Schwarz theorem (differentials are symmetric),

13. d-dimensional chain rule,

14. several variables Taylor's formula with integral rest,

15. uniqueness of d-dimensional Taylor's formula,

16. majoration of differentials: $\| \mathrm{d}^i f.h^i \| \le M \cdot \|h\|^i$ for every $h \in {}^\bullet\mathbb{R}^d$ and some positive constant M,

17. infinitesimal Taylor's formula for functions of the form $f : {}^\bullet U \longrightarrow {}^\bullet\mathbb{R}^d$ and U open in $\mathbb{R}^n$.

11.3 Integral calculus

It is now natural to study the existence of primitives of generic smooth functions $f : [a, b] \longrightarrow {}^\bullet\mathbb{R}$ and hence the existence of an integration theory. We will tackle this problem firstly for a, $b \in \mathbb{R}$, then for $a = -\infty$ and $b = +\infty$, and finally for a, $b \in {}^\bullet\mathbb{R}$. Like in SDG, the problem is solved proving existence and uniqueness of the simplest Cauchy initial value problem.

We firstly recall our notations for intervals, e.g. $(a,b] := \{x \in {}^\bullet\mathbb{R} \mid a < x \leq b\}$, whereas if $a, b \in \mathbb{R}$, then $(a,b]_{\mathbb{R}} := (a,b] \cap \mathbb{R}$. Using Theorem 4.2.4 it is not hard to prove that if $a, b \in \mathbb{R}$

$$
\begin{gathered}
{}^\bullet\{(a,b)_{\mathbb{R}}\} = (a,b) \\
{}^\bullet\{[a,b]_{\mathbb{R}}\} \subsetneqq [a,b],
\end{gathered}
$$

for example, $x_t := a - t^2$ is equal to a in ${}^\bullet\mathbb{R}$, and hence it belongs to the interval $[a,b]$, but $x \notin {}^\bullet\{[a,b]_{\mathbb{R}}\}$ because x does not map $\mathbb{R}_{\geq 0}$ into $[a,b]_{\mathbb{R}}$. We also recall that there can be any order relationship between a Fermat number $a \in {}^\bullet\mathbb{R}$ and its standard part: e.g. $a = {}^\circ a - \mathrm{d}t < {}^\circ a$ whereas $a = {}^\circ a + \mathrm{d}t > {}^\circ a$. For this reason, a general inclusion relationship between the interval (a,b) and the interval $({}^\circ a, {}^\circ b)$ does not hold, even if ${}^\circ a < {}^\circ x < {}^\circ b$ implies $a < x < b$.

To solve the problem of existence and uniqueness of primitives, we need two preliminary results. The first one is called by Bell [1998] the *constancy principle*.

Lemma 11.3.1. *Let $a, b \in \mathbb{R}$ with $a < b$, and $f : (a,b) \longrightarrow {}^\bullet\mathbb{R}$ a ${}^\bullet\mathcal{C}^\infty$ function such that*

$$f'(x) = 0 \quad \forall x \in (a,b).$$

Then f is constant.

Proof: Let $x, y \in (a,b)$ and $h := y - x$. We can suppose $h > 0$, otherwise we can repeat the proof exchanging the role of x and y. So, we have that $\overrightarrow{[x, x+h]} = \overrightarrow{[x,y]} = [x,y] \subseteq (a,b)$ because $a < x < y < b$, therefore $(x,h) \in \widetilde{(a,b)} = {}^\bullet\widetilde{(a,b)_{\mathbb{R}}}$. Using the smooth incremental ratio (Theorem 11.2.2) we get

$$f(y) = f(x) + h \cdot f'[x,h]. \tag{11.3.1}$$

As proved in Theorem 11.2.2, we can always find a smooth function α and a parameter $p \in {}^\bullet\mathbb{R}^{\mathsf{p}}$ such that

$$f'[x,h]_t = \int_0^1 \partial_2\alpha(p_t, x_t + s \cdot h_t)\, \mathrm{d}s. \tag{11.3.2}$$

But for every $s \in [0,1]_{\mathbb{R}}$ we have that

$$x + s \cdot h = x + s \cdot (y - x) \in [x,y] \subseteq (a,b),$$

so that $f'(x + s \cdot h) = 0$, i.e. $\partial_2\alpha(p, x + s \cdot h) = 0$ in ${}^\bullet\mathbb{R}$. Written in explicit form this means

$$\lim_{t \to 0^+} \frac{\partial_2\alpha(p_t, x_t + s \cdot h_t)}{t} = 0.$$

From (11.3.2) using dominated convergence we have

$$\begin{aligned}\lim_{t\to 0^+}\frac{f'[x,h]_t}{t} &= \lim_{t\to 0^+}\frac{1}{t}\cdot\int_0^1 \partial_2\alpha(p_t,x_t+s\cdot h_t)\;\mathrm{d}s\\ &= \int_0^1 \lim_{t\to 0^+}\frac{\partial_2\alpha(p_t,x_t+s\cdot h_t)}{t}\;\mathrm{d}s = 0,\end{aligned}$$

that is $f'[x,h]=0$ in ${}^\bullet\mathbb{R}$ and hence $f(y)=f(x)$ from (11.3.1).

■

The second preliminary result permits to extend the validity of an equality from an open interval (a,b) to its borders.

Lemma 11.3.2. *Let a, $b\in\mathbb{R}$, with $a<b$, $c\in{}^\bullet\mathbb{R}$, and $f:[a,b]\longrightarrow{}^\bullet\mathbb{R}$ a ${}^\bullet\mathcal{C}^\infty$ function such that*

$$\forall x\in(a,b):\quad f(x)=c.$$

Then

$$f(a)=f(b)=c$$

Proof: We prove that $f(a)=c$, analogously we can proceed for $f(b)=c$. Let us write the function f as the parametrized extension of a smooth function in a neighborhood of $x=a$:

$$f(x)=\alpha(p,x)\quad\forall x\in{}^\bullet V\cap[a,b],\tag{11.3.3}$$

where V is open in $\mathbb{R}$ and $a\in{}^\bullet V$.

Let $\rho:\widetilde{{}^\bullet V}\longrightarrow{}^\bullet\mathbb{R}$ be the incremental ratio of $\alpha(p,-):{}^\bullet V\longrightarrow{}^\bullet\mathbb{R}$:

$$\alpha(p,x+h)=\alpha(p.x)+h\cdot\rho(x,h)\quad\forall(x,h)\in\widetilde{{}^\bullet V}.\tag{11.3.4}$$

Since $a={}^\circ a\in V$ we can find a $\delta\in\mathbb{R}_{>0}$ such that $(a-2\delta,a+2\delta)_\mathbb{R}\subseteq V$ and with $a+\delta<b$. Then $[a,a+\delta]\subseteq(a-2\delta,a+2\delta)={}^\bullet(a-2\delta,a+2\delta)_\mathbb{R}\subseteq{}^\bullet V$, i.e. $(a,\delta)\in\widetilde{{}^\bullet V}$ and we can hence use the previous (11.3.4) and(11.3.3) obtaining

$$\alpha(p,a+\delta)=\alpha(p,a)+\delta\cdot\rho(a,\delta).$$

$$f(a+\delta)=f(a)+\delta\cdot\rho(a,\delta),\tag{11.3.5}$$

because $a+\delta<b$, $[a,a+\delta]\subseteq{}^\bullet V$.

But we know that it is always possible to take the smooth incremental ratio ρ so that

$$\rho(a,\delta)_t=\int_0^1\partial_2\alpha(p_t,a_t+s\cdot\delta)\;\mathrm{d}s.\tag{11.3.6}$$

But f is constant on (a,b), so that for every $s\in[0,1]_\mathbb{R}$ we have $a+s\cdot\delta\in(a,b)$ and hence $f'(a+s\cdot\delta)=\partial_2\alpha(p,a+s\cdot\delta)=0$ in ${}^\bullet\mathbb{R}$, i.e.

$$\lim_{t\to 0^+}\frac{\partial_2\alpha(p_t,a_t+s\cdot\delta)}{t}=0\quad\forall s\in[0,1]_\mathbb{R}.$$

From this and from (11.3.6), using dominated convergence we get

$$\lim_{t\to 0^+} \frac{\rho(a,\delta)_t}{t} = 0,$$

that is $\rho(a,\delta) = 0$ in ${}^\bullet\mathbb{R}$. Finally, from this and from (11.3.5) we obtain the conclusion: $f(a+\delta) = f(a) = c$.

$\blacksquare$

We can now prove existence and uniqueness of primitives in the first case of domains $[a,b]$ with real boundaries.

Theorem 11.3.3. *Let $a,\ b \in \mathbb{R}$ with $a < b$, $f : [a,b] \longrightarrow {}^\bullet\mathbb{R}$ a ${}^\bullet\mathcal{C}^\infty$ function and $u \in [a,b]$. Then there exists one and only one ${}^\bullet\mathcal{C}^\infty$ map*

$$I : [a,b] \longrightarrow {}^\bullet\mathbb{R}$$

such that

$$I'(x) = f(x) \quad \forall x \in (a,b)$$
$$I(u) = 0$$

Proof: We can prove the existence assuming that $u = a$; in fact, if $I' = f$ on (a,b) and $I(a) = 0$, then $J(x) := I(x) - I(u)$ verifies $J' = f$ on (a,b) and $J(u) = 0$.

For every $x \in [a,b]$ we can write

$$f|_{\mathcal{V}_x} = \alpha_x(p,-)|_{\mathcal{V}_x}$$

for suitable $p^x \in {}^\bullet\mathbb{R}^{\mathsf{p}^x}$, U_x open subset of $\mathbb{R}^{\mathsf{p}^x}$ such that $p^x \in {}^\bullet U_x$, V_x open in $\mathbb{R}$ such that $x \in {}^\bullet V_x \cap [a,b] =: \mathcal{V}_x$ and $\alpha_x \in \mathcal{C}^\infty(U_x \times V_x, \mathbb{R})$. We can assume that the open sets V_x are of the form $V_x = (x - \delta_x, x + \delta_x)_\mathbb{R}$ for a suitable $\delta_x > 0$.

The idea is to patch together suitable integrals of the functions $\alpha_x(p^x,-)$. The problem in this idea is that we have to realize the condition $I(a) = 0$, which forces us to patch together integrals which are "each one is the extension of the previous one", i.e. on the intersection of their domains two integrals must have the same value at one point, so that we can prove they are equal on the whole intersection. Moreover, we must use the compactness of the interval $[a,b]_\mathbb{R}$ because, generally speaking, a smooth function can be non integrable in an open set.

We have that $(V_x)_{x\in[a,b]_\mathbb{R}}$ is an open cover of the real interval $[a,b]_\mathbb{R}$, thus we can cover $[a,b]_\mathbb{R}$ with a finite number of V_x, that is we can find $x_1,\ldots,x_n \in [a,b]_\mathbb{R}$ such that $(V_{x_i})_{i=1,\ldots,n}$ is an open cover of $[a,b]_\mathbb{R}$. We will use simplified notations like $V_i := V_{x_i}$, ,$\delta_i := \delta_{x_i}$, $\alpha_i := \alpha_{x_i}$, etc.

We can always suppose to have chosen the indexes $i = 1,\ldots,n$ and the amplitudes $\delta_i > 0$ such that

$$a = x_1 < x_2 < \ldots < x_n = b$$

$$x_i - \delta_i < x_{i+1} - \delta_{i+1} < x_i + \delta_i < x_{i+1} + \delta_{i+1} \quad \forall i = 1, \dots, n-1,$$

in this way the intervals $V_i = (x_i - \delta_i, x_i + \delta_i)_{\mathbb{R}}$ intersect in the sub-interval $(x_{i+1} - \delta_{i+1}, x_i + \delta_i)_{\mathbb{R}}$.

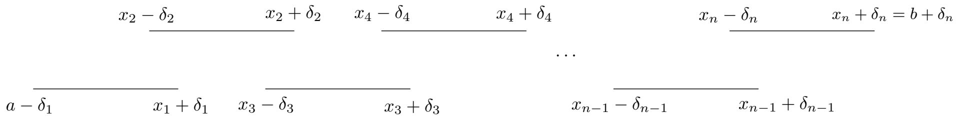

Figure 11.1: Intervals for the recursive definition of a primitive

For any $i = 1, \dots, n$ let us choose a point in this sub interval $\bar{x}_i \in (x_{i+1} - \delta_{i+1}, x_i + \delta_i)_{\mathbb{R}}$ (these are the points in the intersections of the domains of the integrals we are going to define and mentioned in the previous intuitive sketch of the ideas of this proof).

In fact, let us define, recursively:

$$I_1(x)_t := \int_a^{x_t} \alpha_1(p_t^1, s) \, \mathrm{d}s \quad \forall x \in {}^\bullet V_1$$

$$I_{k+1}(x)_t := \int_{\bar{x}_k}^{x_t} \alpha_{k+1}(p_t^{k+1}, s) \, \mathrm{d}s + I_k(\bar{x}_k) \quad \forall x \in {}^\bullet V_{k+1} \; \forall k = 1, \dots, n-1.$$

Every I_k is a ${}^\bullet\mathcal{C}^\infty$ function defined on ${}^\bullet V_k$, and moreover

$$I_k'(x) = \alpha_k(p^k, x) = f(x) \quad \forall x \in {}^\bullet V_k.$$

Therefore, in a generic point in the intersection

$$\begin{aligned} {}^\bullet V_k \cap {}^\bullet V_{k+1} &= {}^\bullet(V_k \cap V_{k+1}) = {}^\bullet \left[(x_{k+1} - \delta_{k+1}, x_k + \delta_k)_{\mathbb{R}} \right] \\ &= (x_{k+1} - \delta_{k+1}, x_k + \delta_k) \end{aligned}$$

we have

$$\begin{gathered} I_k'(x) = f(x) = I_{k+1}'(x) \qquad (11.3.7) \\ (I_k - I_{k+1})'(x) = 0 \\ (I_k - I_{k+1})(\bar{x}_k) = 0, \end{gathered}$$

so, from Theorem 11.3.1 it follows $I_k = I_{k+1}$ on $(x_{k+1} - \delta_{k+1}, x_k + \delta_k)$. We can hence use the sheaf property of the space $[a, b]$ with the open cover $({}^\bullet V_k \cap [a, b])_{i=1,\dots,n}$ to patch together the functions $I_k|_{{}^\bullet V_k \cap [a,b]}$ obtaining the map $I : [a, b] \longrightarrow {}^\bullet\mathbb{R}$. This function satisfies the conclusion of the statement because of (11.3.7) and because of the equalities $I(a) = I_1(a) = 0$.

To prove the uniqueness, let us suppose that J verifies $J' = f$ on (a, b) and $J(u) = 0$, then using again Theorem 11.3.1 we have that $(J - I)|_{(a,b)}$ is constant and equal to zero. Finally, using Lemma 11.3.2, we can extend this constancy to the whole closed interval $[a, b]$.

■

The second case is for domains $[a, b] = {}^\bullet\mathbb{R}$.

Theorem 11.3.4. *If $f : {}^\bullet\mathbb{R} \longrightarrow {}^\bullet\mathbb{R}$ is smooth and $u \in {}^\bullet\mathbb{R}$ then there exists one and only one smooth $I : {}^\bullet\mathbb{R} \longrightarrow {}^\bullet\mathbb{R}$ such that $I' = f$ and $I(u) = 0$.*

Proof: For every $k \in \mathbb{N}_{>0}$ let us define

$$f_k := f|_{[a-k,a+k]},$$

where $a := {}^\circ u$. Moreover, define

$$I_k(x) := \int_u^x f_k \quad \forall x \in (a-k, a+k) \subseteq [a-k, a+k].$$

Therefore, we have that setting $V_k := (a-k, a+k) = {}^\bullet(a-k, a+k)_{\mathbb{R}}$, we obtain that $(V_k)_{k>0}$ is an open cover of ${}^\bullet\mathbb{R}$. Moreover, $I'_k(a) = f_k(x) = f(x)$ for every $x \in V_k$, so that I_k and I_j coincide in $V_k \cap V_j$. From the sheaf property of ${}^\bullet\mathbb{R}$ we get

$$\exists!\, I : {}^\bullet\mathbb{R} \longrightarrow {}^\bullet\mathbb{R} \text{ smooth} : \quad I|_{V_k} = I_k \quad \forall k \in \mathbb{N}_{>0}.$$

Now, let us note that

$$\forall x, h \in {}^\bullet\mathbb{R}\ \exists\, k \in \mathbb{N}_{>0} : \quad \overrightarrow{[x, x+h]} \subseteq V_k,$$

then we also have

$$I(x+h) = I_k(x+h) = I(x) + h \cdot I'[x,h] = I_k(x) + h \cdot I'_k[x,h],$$

so that the smooth incremental ratios of I and I_k are equal, i.e. $I'[x,h] = I'_k[x,h]$. Thus, $I'(x) = I'_k(x) = f(x)$, and finally $I(u) = I_1(u) = 0$.

This proves the existence part. The uniqueness follows from Lemma 11.3.1.

■

To extend Theorem 11.3.3 to non standard boundaries $a, b \in {}^\bullet\mathbb{R}$ we need the following result.

Lemma 11.3.5. *Let $a, b \in {}^\bullet\mathbb{R}$ with ${}^\circ a < {}^\circ b$, and $f : [a,b] \longrightarrow {}^\bullet\mathbb{R}$ a ${}^\bullet\mathcal{C}^\infty$ function. Then there exist $\delta \in \mathbb{R}_{>0}$ and a ${}^\bullet\mathcal{C}^\infty$ function $\bar{f} : (a-\delta, b+\delta) \longrightarrow {}^\bullet\mathbb{R}$ such that*

$$\bar{f}|_{[a,b]} = f$$

Proof: As usual, let us write the function f as the parametrized extension of an ordinary smooth function in a neighborhood of $x = a$:

$$f(x) = \alpha(p, x) \quad \forall x \in {}^\bullet V \cap [a,b], \tag{11.3.8}$$

where V is open in $\mathbb{R}$ and $a \in {}^\bullet V$ so that ${}^\circ a \in V$.

We can make the same in a neighborhood of $x = b$:

$$f(x) = \beta(q, x) \quad \forall x \in {}^\bullet U \cap [a,b], \tag{11.3.9}$$

where U is open in $\mathbb{R}$ and $b \in {}^\bullet U$ so that $^\circ b \in U$.

Because U and V are open subsets of $\mathbb{R}$, we can always suppose to have $U = (^\circ b - \eta, {}^\circ b + \eta)_\mathbb{R}$ and $V = (^\circ a - \eta, {}^\circ a + \eta)_\mathbb{R}$ with $\eta \in \mathbb{R}_{>0}$ such that $^\circ a + \eta < {}^\circ b - \eta$ because $^\circ a < {}^\circ b$. Therefore, we have

$$
\begin{gathered}
{}^\bullet V \cap (^\circ a, {}^\circ b) = (^\circ a - \eta, {}^\circ a + \eta) \cap (^\circ a, {}^\circ b) \subseteq {}^\bullet V \cap [a, b] \\
{}^\bullet U \cap (^\circ a, {}^\circ b) = (^\circ b - \eta, {}^\circ b + \eta) \cap (^\circ a, {}^\circ b) \subseteq {}^\bullet U \cap [a, b] \\
{}^\bullet V \cap {}^\bullet U = (^\circ a - \eta, {}^\circ a + \eta) \cap (^\circ b - \eta, {}^\circ b + \eta) = \emptyset,
\end{gathered}
$$

so that any two of the following smooth functions

$$
\begin{gathered}
\alpha(p, -) : {}^\bullet V \longrightarrow {}^\bullet \mathbb{R} \\
f|_{(^\circ a, {}^\circ b)} : (^\circ a, {}^\circ b) \longrightarrow {}^\bullet \mathbb{R} \\
\beta(q, -) : {}^\bullet U \longrightarrow {}^\bullet \mathbb{R}
\end{gathered}
$$

are equal on the intersection of their domains for (11.3.8) and (11.3.9).

For the sheaf property of $(^\circ a - \eta, {}^\circ a + \eta) \cup (^\circ a, {}^\circ b) \cup (^\circ b - \eta, {}^\circ b + \eta) = (^\circ a - \eta, {}^\circ b + \eta)$ we thus have

$$
\exists!\, g : (^\circ a - \eta, {}^\circ b + \eta) \longrightarrow {}^\bullet \mathbb{R} \text{ smooth}: \quad g|_{(^\circ a, {}^\circ b)} = f.
$$

If we set $\delta := \frac{\eta}{2}$ we have that $^\circ a - \eta < a - \delta < b + \delta < {}^\circ b + \eta$, as we can verify considering the standard parts of all these numbers, and hence $\bar{f} := g|_{(a-\delta, b+\delta)}$ verifies

$$
\bar{f}|_{(^\circ a, {}^\circ b)} = f.
$$

Because $(^\circ a, {}^\circ b) \subseteq [a, b]$ we have to verify that the function $\bar{f}$ and the function f also coincide on $[a, b] \setminus (^\circ a, {}^\circ b)$. We firstly note that $^\circ a \in V \subseteq {}^\bullet V$ and $^\circ b \subseteq U \subseteq {}^\bullet U$, so we can apply (11.3.8) and (11.3.9) at $x = {}^\circ a$ and $x = {}^\circ b$ too. Therefore, we have $\bar{f}|_{[^\circ a, {}^\circ b]} = f$. Secondly, if $x \in [a, b] \setminus (^\circ a, {}^\circ b)$, then either $a \le x \le {}^\circ a$ or $^\circ b \le x \le b$; we will deal with the first case, the second being analogous. From these inequalities, it follows that $^\circ x = {}^\circ a$ so that from the infinitesimal Taylor's formula we get

$$
\begin{aligned}
f(x) = f\left[^\circ x + (x - {}^\circ x)\right] &= \sum_{i=0}^{n} \partial_2^{(i)} \alpha(p, {}^\circ a) \cdot \frac{(x - {}^\circ a)^i}{i!} = \\
&= \bar{f}\left[^\circ x + (x - {}^\circ x)\right] = \bar{f}(x).
\end{aligned}
$$

■

Theorem 11.3.6. *Let $a, b \in {}^\bullet \mathbb{R}$ with $^\circ a < {}^\circ b$, $f : [a, b] \longrightarrow {}^\bullet \mathbb{R}$ a ${}^\bullet \mathcal{C}^\infty$ function and $u \in [a, b]$. Then there exists one and only one ${}^\bullet \mathcal{C}^\infty$ map*

$$
I : [a, b] \longrightarrow {}^\bullet \mathbb{R}
$$

such that

$$
\begin{gathered}
I'(x) = f(x) \quad \forall x \in (a, b) \\
I(u) = 0
\end{gathered}
$$

Proof: From Lemma 11.3.5 there exist a $\delta\in\mathbb{R}_{>0}$ and a smooth function $\bar f:(a-\delta,b+\delta)\longrightarrow{}^\bullet\mathbb{R}$ such that $\bar f|_{[a,b]}=f$.

But $a-\delta<{}^\circ a-\frac{\delta}{2}<{}^\circ b+\frac{\delta}{2}<b+\delta$, so the interval with real boundaries $[\alpha,\beta]:=\left[{}^\circ a-\frac{\delta}{2},{}^\circ b+\frac{\delta}{2}\right]$ is contained in $(a-\delta,b+\delta)$. Finally $\alpha={}^\circ a-\frac{\delta}{2}<a\le u\le b<{}^\circ b+\frac{\delta}{2}=\beta$ and we can thus apply Theorem 11.3.3 obtaining existence and uniqueness of the primitive J of the function $\bar f|_{[\alpha,\beta]}$ such that $J(u)=0$. But $[a,b]\subseteq[\alpha,\beta]$ and hence $I:=J|_{[a,b]}$ verifies the existence part of the conclusion. The uniqueness part follows, in the usual way, from Lemma 11.3.1 and Lemma 11.3.2.

∎

We can now define

Definition 11.3.7. *Let a, $b\in{}^\bullet\mathbb{R}$ with ${}^\circ a<{}^\circ b$. Moreover, let us consider a ${}^\bullet\mathcal{C}^\infty$ function $f:[a,b]\longrightarrow{}^\bullet\mathbb{R}$ and a point $u\in[a,b]$. Then*

1. $\displaystyle\int_u^{(-)} f:=\int_u^{(-)} f(s)\,\mathrm{d}s:[a,b]\longrightarrow{}^\bullet\mathbb{R}$
2. $\displaystyle\int_u^u f=0$
3. $\displaystyle\forall x\in(a,b):\quad\left(\int_u^{(-)} f\right)'(x)=\frac{\mathrm{d}}{\mathrm{d}x}\int_u^x f(s)\,\mathrm{d}s=f(x)$

It is important to note that in this way we obtain a generalization of the usual notion of integral. Indeed, for a, b, $u\in\mathbb{R}$ with $a<u<b$, let us consider a standard smooth function

$$f\in\mathcal{C}^\infty([a,b]_\mathbb{R},{}^\bullet\mathbb{R}).$$

Let us extend smoothly this function on an open interval $(a-\delta,b+\delta)_\mathbb{R}$ with $\delta\in\mathbb{R}_{>0}$, so that outside $[a,b]_\mathbb{R}$ the extension of f is constant. Let us consider the smooth function

$$I:={}^\bullet\left(\int_u^{(-)} f(s)\,\mathrm{d}s\right):(a-\delta,b+\delta)\longrightarrow{}^\bullet\mathbb{R},$$

where here the integral symbol has to be understood as the classical Riemann integral on $\mathbb{R}$. Now we have that

$$[a,b]\subseteq(a-\delta,b+\delta)$$

so that we can consider the restriction $I|_{[a,b]}$. It is not hard to prove that this restriction verifies all the properties of the previous Definition 11.3.7 for the function ${}^\bullet f$, but at the same time, because it is the extension of a classical integral, it also verifies

$$\forall x\in[a,b]_\mathbb{R}:\quad I(x)=\int_u^x f(s)\,\mathrm{d}s\in\mathbb{R}.$$

These theorems can be used to try a generalization of several results of integral (smooth!) calculus to ${}^\bullet\mathcal{C}^\infty$ functions of the form $f : \prod_{i=1}^n [a_i, b_i] \longrightarrow {}^\bullet\mathbb{R}^d$, with ${}^\circ a_i < {}^\circ b_i$.

Indeed, this can be done for several theorems. We only list here the main results that we have already proved, leaving a complete report of them for a subsequent work. For most of them the proofs are very similar to the analogous presented e.g. in Lavendhomme [1996]:

1. property of linearity of integrals,
2. fundamental theorem of calculus,
3. integration by parts formula,
4. formulas of the form $\int_u^v f + \int_v^w f = \int_u^w f$, $\int_u^v f = -\int_v^u f$,
5. integration formula by change of variable,
6. derivation under the integral sign,
7. smoothness of the function $(f, u, v) \in {}^\bullet\mathbb{R}^{[a,b]} \times [a, b]^2 \mapsto \int_u^v f \in {}^\bullet\mathbb{R}$,
8. majorization of integrals: if $|fx| \leq M$ for every $x \in [a, b]$, then $|\int_a^b f| \leq M \cdot (b - a)$,
9. majorization of d-dimensional integrals $\|\int_a^b f\| \leq M \cdot \sqrt{d} \cdot (b - a)$ if $\|f(x)\| \leq M$ for every $x \in [a, b]$ and $f : [a, b] \longrightarrow {}^\bullet\mathbb{R}^d$,
10. Fubini theorem for double integrals.

Example.

1. **Divergence and curl.** Classically the $\mathrm{div}\vec{A}(x)$ is the density of the flux of $\vec{A} \in \mathcal{C}^\infty(U, \mathbb{R}^3)$ through an "infinitesimal parallelepiped" centered at $x \in U \subseteq \mathbb{R}^3$. To formalize this concept we take three vectors $\vec{h}_1$, $\vec{h}_2$, $\vec{h}_3 \in {}^\bullet\mathbb{R}^3$ and express them with respect to a fixed base $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3 \in \mathbb{R}^3$:

   $$\vec{h}_i = k_i^1 \cdot \vec{e}_1 + k_i^2 \cdot \vec{e}_2 + k_i^3 \cdot \vec{e}_3 \qquad \text{where} \qquad k_i^j \in {}^\bullet\mathbb{R}.$$

   We say that $P := (x, \vec{h}_1, \vec{h}_2, \vec{h}_3)$ is a *(first order) infinitesimal parallelepiped* if

   $$\begin{aligned} & x \in \mathbb{R}^3 \\ & \forall i, j, k = 1, 2, 3 : \quad k_i^1 \cdot k_i^2 \cdot k_i^3 \in D. \end{aligned}$$

The flux of the vector field $\vec{A}$ through such a parallelepiped (toward the outer) is by definition the sum of the fluxes through every "face"

$$\int_P \vec{A}\cdot\vec{n}\,\mathrm{d}S := \int_0^1 \mathrm{d}t \int_0^1 \vec{A}(x+t\vec{h}_1+s\vec{h}_2)\cdot\vec{h}_2\times\vec{h}_1\,\mathrm{d}s+$$
$$\int_0^1 \mathrm{d}t \int_0^1 \vec{A}(x+\vec{h}_3+t\vec{h}_1+s\vec{h}_2)\cdot\vec{h}_1\times\vec{h}_2\,\mathrm{d}s+\ldots,$$

where the ... indicate similar terms for the other faces of the parallelepiped. Let us note that e.g. the function $s \mapsto \vec{A}(x+t\vec{h}_1+s\vec{h}_2)$ is a ${}^\bullet\mathcal{C}^\infty$ arrow of type ${}^\bullet\alpha(p,s)$, where here the parameter is $p = (x,t,\vec{h}_1,\vec{h}_2) \in {}^\bullet\mathbb{R}^{10}$. We have hence concrete examples of ${}^\bullet\mathcal{C}^\infty$ functions to which we can apply the results of the previous sections. Now, it is easy to prove that if $\vec{A} \in \mathcal{C}^\infty(U;\mathbb{R}^3)$ and $\mathrm{Vol}(\vec{h}_1,\vec{h}_2,\vec{h}_3)$, i.e. the oriented volume of the infinitesimal parallelepiped $P=(x,\vec{h}_1,\vec{h}_2,\vec{h}_3)$, is not zero, then the following ratio between first order infinitesimals exists and is independent by $\vec{h}_1$, $\vec{h}_2$, $\vec{h}_3$:

$$\mathrm{div}\vec{A}(x) := \frac{1}{\mathrm{Vol}(\vec{h}_1,\vec{h}_2,\vec{h}_3)}\cdot\int_P \vec{A}\cdot\vec{n}\,\mathrm{d}S.$$

To define the curl of a vector field $\vec{A} \in {}^\bullet\mathcal{C}^\infty(U,\mathbb{R}^3)$ we can say that $C := (x,\vec{h}_1,\vec{h}_2)$ is a *(first order) infinitesimal cycle* if

$$x \in U \qquad \text{and} \qquad \forall p,q=1,2,3: \sum_{i,j=1}^{3} |k_i^p\cdot k_j^q| \in D.$$

The circulation of the vector field $\vec{A}$ on this cycle C is defined as the sum of the "line integrals" on every "side":

$$\int_C \vec{A}\cdot\vec{t}\,\mathrm{d}l := \int_0^1 \vec{A}(x+t\vec{h}_1)\cdot\vec{h}_1\,\mathrm{d}t + \int_0^1 \vec{A}(x+\vec{h}_1+t\vec{h}_2)\cdot\vec{h}_2\,\mathrm{d}t+\ldots,$$

where ... indicates similar terms for the other side of the cycle C. Once again, using exactly the calculations frequently done in elementary courses of physics, one can prove that there exists one and only one vector, $\mathrm{curl}\,\vec{A}(x) \in \mathbb{R}^3$, such that

$$\int_C \vec{A}\cdot\vec{t}\,\mathrm{d}l = \mathrm{curl}\vec{A}(x)\cdot\vec{h}_1\times\vec{h}_2$$

for every infinitesimal cycle $C=(x,\vec{h}_1,\vec{h}_2)$, representing thus the (vector) density of the circulation of $\vec{A}$.

2. **Limits in ${}^\bullet\mathbb{R}$.** Because the theory of Fermat reals is not an alternative way for the foundation of calculus, but a rigorous way to have at disposal infinitesimal methods, there is no need to think that the notion

of limit expressed by Weierstrass' $\varepsilon - \delta$'s is conceptually incompatible with our use of infinitesimals. A similar approach is already used, e.g. in the study of the Levi-Civita field (see Appendix B and references therein). Therefore, we can introduce the following:

Definition 11.3.8. *Let $f : U \longrightarrow {}^\bullet\mathbb{R}$ be a ${}^\bullet\mathcal{C}^\infty$ function defined in $U \subseteq {}^\bullet\mathbb{R}$ and l, $\bar{x} \in {}^\bullet\mathbb{R}$ be two Fermat reals. Then we say that l* is the limit of $f(x)$ for $x \to \bar{x}$ *if and only if*

$$\forall \varepsilon \in {}^\bullet\mathbb{R}_{>0}\ \exists \delta \in {}^\bullet\mathbb{R}_{>0} :\ \forall x \in U : \quad 0 < |x - \bar{x}| < \delta \Rightarrow |f(x) - l| < \varepsilon$$

Analogously we can define the right and the left limit.

Using the total order on ${}^\bullet\mathbb{R}$ and replicating the standard proof, we can prove that this limit, if it exists, is unique.

Theorem 11.3.9. *In the hypothesis of the previous Definition 11.3.8, there exists at most one $l \in {}^\bullet\mathbb{R}$ such that l is the limit of $f(x)$ for $x \to \bar{x}$. In this case, we will use the notation $l = \lim_{x\to\bar{x}} f(x)$.*

If $f : (a, b) \longrightarrow {}^\bullet\mathbb{R}$, ${}^\circ a < {}^\circ b$, and ${}^\circ\bar{x} \in ({}^\circ a, {}^\circ b)$, we want to prove that $\lim_{x\to\bar{x}} f(x) = f(\bar{x})$. Let us consider a generic $\varepsilon > 0$, we want to find a $\delta \in D_{>0}$. From the inequalities $0 < |x-\bar{x}| < \delta$ it follows that $x-\bar{x} \in D$ and hence from the first order Taylor's formula

$$|f(x) - f(\bar{x})| = \left|f'(\bar{x}) \cdot (x - \bar{x})\right| \le \left|f'(\bar{x})\right| \cdot \delta.$$

If $f'(\bar{x}) \in D_\infty$, then $|f'(\bar{x})| \cdot \delta = 0 < \varepsilon$ because $\delta \in D$ is a first order infinitesimal. Otherwise, $f'(\bar{x})$ is invertible and it suffices to fix δ such that

$$\delta < \frac{\varepsilon}{|f'(\bar{x})|},$$

e.g.

$$\delta := \min\left\{\mathrm{d}t, \frac{\varepsilon}{2\,|f'(\bar{x})|}\right\} \in D_{>0}.$$

This expected result (even if the topology we considered on the Fermat reals has not been defined as the one induced by the absolute value, but the natural topology induced by the smooth figures of ${}^\bullet\mathbb{R}$; see the Definition 6.2.4) says us that in the context of ${}^\bullet\mathcal{C}^\infty$ functions, the notion of limit is interesting only at the border points $\bar{x} = a$ or $\bar{x} = b$ on which the function f is not defined. From this point of view, lemmas 11.3.2 and 11.3.5 represent possible substitutes of the notion of limit in ${}^\bullet\mathbb{R}$.

Chapter 12

Calculus on infinitesimal domains

It is natural to expect that we cannot restrict our differential calculus to smooth functions defined on open sets, but that we have to extend the notion of derivatives to functions defined on infinitesimal sets, e.g. $0 \in I \subseteq D_\infty$.

As we prompted above, the infinitesimal Taylor's formula does not uniquely identifies the derivatives appearing in its addends, so that we must use the map ι_k to consider the simplest numbers that verify a given Taylor's formula.

12.1 The generalized Taylor's formula

In this section we want to prove the Taylor's formula for functions defined on an infinitesimal domain, like e.g. $f : D_\alpha \longrightarrow {}^\bullet X$, with $\alpha \in \mathbb{R}_{>0}$ and $X \in \boldsymbol{\mathcal{C}}^\infty$. The possibility to prove the following theorems has been the first motivation to choose little-oh polynomials instead of the more general nilpotent functions (like in Giordano [2004]) to define ${}^\bullet\mathbb{R}$. A stronger algebraic control on the properties of nilpotent infinitesimals, and better order properties have been the second motivation.

We start proving some preliminary results that permit to affirm that if $f(0) \in {}^\bullet U$, where U is open in the space X, then $f(h) \in {}^\bullet U$ for every h in the infinitesimal domain of f.

Lemma 12.1.1. *Let X be a $\boldsymbol{\mathcal{C}}^\infty$ space, U an open subset of X and $x \in {}^\bullet X$, then*

$$^\circ x \in U \quad \Longrightarrow \quad x \in {}^\bullet U.$$

Let us note that we have already frequently used the analogous of this result for spaces of the form $X = \mathbb{R}^s$, but in this particular case the notion of little-oh polynomial does not depend on observables but only on the norm of $\mathbb{R}^s$. For this reason, in this particular situation, the passage from $x \in {}^\bullet X$ to $x \in {}^\bullet U$ is trivial.

Proof: Because $^\circ x = x(0) \in U \in \mathcal{T}_U$ and x is continuous at $t = 0$, we have that locally x has values in U, i.e.

$$\forall^0 t : \ x_t \in U.$$

So, because all the properties we are considering are local, we can assume that $x : \mathbb{R}_{\ge 0} \longrightarrow U$, i.e. x has globally values in U. To prove that $x \in {}^\bullet U$ it remains to prove that[1] $x \in U_o[t]$, where we are meaning $U = (U \prec X)$, that is on the subset U the structure induced by the superspace X. So, let us consider a zone VK of $(U \prec X)$ such that $x(0) \in V$ and an observable $\varphi \in_{VK} (U \prec X)$. We have that $V \in \mathcal{T}_{(U \prec X)} \subseteq \mathcal{T}_X$ because U is open in X, and hence VK is a zone of the space X too. Moreover

$$(V \prec (U \prec X)) = (V \prec X) \xrightarrow{\ \varphi\ } K$$

and hence φ is also an observable of X such that $\varphi \in_{VK} X$. But, by hypotheses $x \in {}^\bullet X$, so that $x \in X_o[t]$ and hence $\varphi \circ x \in \mathbb{R}^{\mathsf{k}}_o[t]$, which is the conclusion.

■

The main aim of this section is to prove an infinitesimal Taylor's formula for functions of the form $f : D_\alpha \longrightarrow {}^\bullet X$ through the composition with observables $\varphi \in^{UK} X$. Precisely we want to consider a ${}^\bullet\mathcal{C}^\infty$ function $f : D_\alpha \longrightarrow {}^\bullet X$ with $f(0) \in {}^\bullet U := {}^\bullet(U \prec X)$ (in general, the function f will not be the extension of a classical one, that is f is not necessarily of the form $f = {}^\bullet g|_D$) and we will prove the Taylor's formula for the function ${}^\bullet\varphi(f(-)) : D_\alpha \longrightarrow K \subseteq {}^\bullet\mathbb{R}^{\mathsf{k}}$. First of all, we prove that this composition is well defined, that is the following theorem holds:

Theorem 12.1.2. *Let X be a $\mathcal{C}^n$ space and let $U \in \mathcal{T}_X$ be an open set. Let us consider an infinitesimal set $I \subseteq D^d_\infty$, with $d \in \mathbb{N}_{>0}$ and containing the null vector: $\underline{0} \in I$. Finally, let $f : I \longrightarrow {}^\bullet X$ be a ${}^\bullet\mathcal{C}^\infty$ function with $f(\underline{0}) \in {}^\bullet U$. Then $f(h) \in {}^\bullet U$ for every $h \in I$.*

Proof: From the hypothesis on f it follows that $f \in_I {}^\bullet X$ because $I = {}^\bullet(I\mathbb{R}) = \overline{I}$ (see Theorem 9.2.5). Hence, since $\underline{0} \in I$, by the results of Section 9.2.1, we can globally say that either f is constant, and the proof is trivial, or we can write the equality $f(h) = {}^\bullet\gamma(p, h)$ in ${}^\bullet X$ for every $h \in I$. For the sake of clarity let $y := f(h)$, thus taking standard parts we get

$$^\circ y \asymp {}^\circ[{}^\bullet\gamma(p, h)] = \gamma(p_0, 0) = {}^\circ[{}^\bullet\gamma(p, 0)] \asymp {}^\circ f(0), \tag{12.1.1}$$

that is $^\circ y$ and $^\circ f(0)$ are identified in X (see Definition 7.1.2 for the definition of the relation $x \asymp y$). But $f(0) \in {}^\bullet U$, hence $^\circ f(0) \in U$ and $^\circ y \in U$ from

[1] Let us recall the general definition of the set of all the little-oh polynomials in the space $X \in \mathcal{C}^\infty$, i.e. the Definition 8.1.4.

(12.1.1). Finally, $y = f(h) \in {}^\bullet X$ and hence $y = f(h) \in {}^\bullet U$ because of the previous Lemma 12.1.1.

■

We will state both the 1-dim Taylor's formula and the d-dimensional one, because the first case can be stated in a considerably simpler way.

Theorem 12.1.3. *Let X be a $\mathcal{C}^\infty$ space, $\alpha \in \mathbb{R}_{>0}$, $U \in \tau_X$ be an open set of X and*

$$f : D_\alpha \longrightarrow {}^\bullet X \quad \text{with} \quad f(0) \in {}^\bullet U$$

$$\varphi : {}^\bullet U \longrightarrow {}^\bullet \mathbb{R}^{\mathsf{k}}$$

be ${}^\bullet\mathcal{C}^\infty$ maps. Define $k_j \in \mathbb{R}$ such that

$$k_0 := 0$$

$$\frac{1}{k_j} + \frac{j}{\alpha+1} = 1 \quad \forall j = 1, \ldots, [\alpha] =: n.$$

Then there exists one and only one n-tuple $(m_1, \ldots, m_n)$ such that

1. $m_j \in {}^\bullet\mathbb{R}^{\mathsf{k}}_{k_j}$ *for every* $j = 1, \ldots, n$
2. $\varphi[f(h)] = \sum_{j=0}^{n} \frac{h^j}{j!} \cdot m_j \quad \forall h \in D_\alpha$.

The more general statement, with infinitesimal increments taken in a product of ideals of different order, i.e. $h \in D_{\alpha_1} \times \ldots \times D_{\alpha_d}$, is the following[2]

Theorem 12.1.4. *Let X be a $\mathcal{C}^\infty$ space, $\alpha_1, \ldots, \alpha_d \in \mathbb{R}_{>0}$, $U \in \tau_X$ an open subset of X and*

$$f : D_{\alpha_1} \times \cdots \times D_{\alpha_d} \longrightarrow {}^\bullet X \quad \text{with} \quad f(\underline{0}) \in {}^\bullet U$$

$$\varphi : {}^\bullet U \longrightarrow {}^\bullet \mathbb{R}^{\mathsf{k}}$$

be ${}^\bullet\mathcal{C}^\infty$ maps. Define $k_j \in \mathbb{R}$ such that

$$k_{\underline{0}} := 0$$

$$\frac{1}{k_j} + \frac{j}{\alpha+1} = 1 \quad \forall j \in \mathbb{N}^d : \; 0 < \frac{j}{\alpha+1} < 1.$$

Then there exists one and only one

$$m : \left\{ j \in \mathbb{N}^d \,|\, \frac{j}{\alpha+1} < 1 \right\} \longrightarrow {}^\bullet\mathbb{R}^{\mathsf{k}}$$

such that

[2] Recall the Definition 3.3.2 for the definition of the term $\frac{j}{\alpha+1}$, where α, $j \in \mathbb{N}^d$.

1. *$m_j \in {}^\bullet\mathbb{R}^{\mathsf{k}}_{k_j}$ for every $j \in \mathbb{N}^d$ such that $\frac{j}{\alpha+1} < 1$*

2. *$\varphi\,[f(h)] = \sum_{\substack{j\in\mathbb{N}^d \\ \frac{j}{\alpha+1}<1}} \frac{h^j}{j!} \cdot m_j \quad \forall h \in D_{\alpha_1} \times \cdots \times D_{\alpha_d}$.*

Proof: The domain of our function is

$$\begin{aligned} D_{\alpha_1} \times \cdots \times D_{\alpha_d} &= {}^\bullet(D_{\alpha_1}\mathbb{R}) \times \cdots \times {}^\bullet(D_{\alpha_d}\mathbb{R}) = \\ &= \overline{D_{\alpha_1}} \times \cdots \times \overline{D_{\alpha_d}} = \overline{D_{\alpha_1} \times \cdots \times D_{\alpha_d}} \end{aligned}$$

where we have used Theorem 9.2.5 for the second equality and Lemma 9.2.2 for the latter equality. Setting $I := D_{\alpha_1} \times \cdots \times D_{\alpha_d}$ for the sake of simplicity, we have thus

$$f : \overline{I} \longrightarrow {}^\bullet X \quad \text{and hence} \quad f \in_I {}^\bullet X.$$

From the results of Section 9.2.1, since I is an infinitesimal set containing $\underline{0}$, we get that either f is constant or we can write

$$f = {}^\bullet\gamma(p, -)|_I \tag{12.1.2}$$

for some $p \in {}^\bullet A$, A is open in $\mathbb{R}^{\mathsf{p}}$, and some $\gamma \in \mathcal{C}^\infty(A \times B, X)$ with $I \subset {}^\bullet B$ and B open in $\mathbb{R}^d$. The case f constant is trivial because it suffices to set $m_j := 0$ for $j \neq \underline{0}$, $m_{\underline{0}} := \varphi\,[f(\underline{0})]$ to have the existence part and to apply Corollary 3.4.1 for the uniqueness part. In the second case (12.1.2) our aim is, of course, to use the composition $\varphi \circ {}^\bullet\gamma(p, -)$, so that now we would like to find where this composition is defined and to prove that its domain contains the previous infinitesimal set I. We have that $\eta := {}^\bullet\gamma(p, -) : {}^\bullet B \longrightarrow {}^\bullet X$ in ${}^\bullet\mathcal{C}^\infty$, hence, since ${}^\bullet U \in \mathcal{T}_{{}^\bullet X}$, we also get that $\eta^{-1}({}^\bullet U)$ is open in ${}^\bullet B$ and hence it is also open in ${}^\bullet\mathbb{R}^d$ because B is open in $\mathbb{R}^d$. But we have that $\underline{0} \in \eta^{-1}({}^\bullet U)$ if and only if $\eta(\underline{0}) = {}^\bullet\gamma(p, \underline{0}) = f(\underline{0}) \in {}^\bullet U$ which is true by hypothesis. Thus, since $\eta^{-1}({}^\bullet U)$ is open in ${}^\bullet\mathbb{R}^d$ we obtain that

$$\exists\, B_1 \text{ open in } \mathbb{R}^d : \quad \underline{0} \in {}^\bullet B_1 \subseteq \eta^{-1}({}^\bullet U) \subseteq {}^\bullet B.$$

So we are in the following situation

$${}^\bullet B_1 \xrightarrow{\ \eta|_{{}^\bullet B_1}\ } {}^\bullet U \xrightarrow{\ \varphi\ } {}^\bullet\mathbb{R}^{\mathsf{k}} \ ,$$

and hence the composition $\varphi \circ {}^\bullet\gamma(p, -)|_{{}^\bullet B_1} =: \psi$ is defined in ${}^\bullet B_1$ which, being open and containing $\underline{0}$, it also contains the infinitesimal set I (Lemma 12.1.1). But the idea is to use the Taylor's formula for standard smooth functions, i.e. Theorem 3.4.5, and we do not know whether the function ψ is the extension of an ordinary standard function. So, we have to note that ${}^\bullet B_1 = \overline{{}^\bullet B_1}$ and hence $\psi \in_{{}^\bullet B_1} {}^\bullet\mathbb{R}^{\mathsf{k}}$, so that we can apply once again Theorem 9.2.4 obtaining that locally, in a neighborhood of $\underline{0} \in {}^\bullet B_1$, we can express the figure ψ as $\psi = {}^\bullet\delta(q,-)$ for a suitable $\delta \in \mathcal{C}^\infty(C \times E, \mathbb{R}^{\mathsf{k}})$, with $I \subseteq {}^\bullet E$ (the case ψ constant can be dealt as seen above). Therefore we have

$$\psi(x) = {}^\bullet\delta(q,x) = \varphi\left[{}^\bullet\gamma(p,x)\right] = \varphi\left[f(x)\right] \quad \forall x \in {}^\bullet E \cap {}^\bullet B_1 = {}^\bullet(E \cap B).$$

To the standard smooth function $\delta \in \mathcal{C}^\infty(C \times E, \mathbb{R}^{\mathsf{k}})$ we can apply Theorem 3.4.5 at the non-standard point $(q, \underline{0}) \in {}^\bullet C \times {}^\bullet E = {}^\bullet(C \times E)$ with infinitesimal increment $(q, \underline{0} + h)$, $h \in I$; we obtain

$$\forall h \in I: \quad \psi(h) = \left[{}^\bullet\delta(q,h)\right] = \varphi\left[f(h)\right] = \sum_{\substack{j \in \mathbb{N}^d \\ \frac{j}{\alpha+1} < 1}} \frac{h^j}{j!} \cdot \frac{\partial^{|j|}\delta}{\partial x^j}(\underline{0}).$$

Now it suffices to apply Corollary 3.4.1 to obtain the conclusion.

∎

Analogously we can state and prove a Taylor's formula for functions $f : D^d_\infty \longrightarrow {}^\bullet X$, with coefficients $m_{\underline{0}} \in {}^\bullet\mathbb{R}^{\mathsf{k}}_0$ and $m_j \in {}^\bullet\mathbb{R}^{\mathsf{k}}_1$.

Definition 12.1.5. *In the hypothesis of the previous Theorem 12.1.4, we set*

$$\partial\varphi(f) : \left\{ j \in \mathbb{N}^d \,|\, \frac{j}{\alpha+1} < 1 \right\} \longrightarrow {}^\bullet\mathbb{R}^{\mathsf{k}}$$

such that:

1. $\partial_j\varphi(f) \in {}^\bullet\mathbb{R}^{\mathsf{k}}_{k_j}$ *for every* $j \in \mathbb{N}^d$ *such that* $\frac{j}{\alpha+1} < 1$
2. $\varphi\left[f(h)\right] = \sum_{\substack{j \in \mathbb{N}^d \\ \frac{j}{\alpha+1} < 1}} \frac{h^j}{j!} \cdot \partial_j\varphi(f) \quad \forall h \in D_{\alpha_1} \times \cdots \times D_{\alpha_d}$.

In the case $X = \mathbb{R}^{\mathsf{k}}$ *and* $\varphi = 1_{\mathbb{R}^{\mathsf{k}}}$ *we will use the simplified notations*

$$\partial_j f := \partial_j f(\underline{0}) := \partial_j\varphi(f)$$

$$f^{(n)}(0) := \partial_n f(0) \quad \textit{if} \quad f : D_\alpha \longrightarrow {}^\bullet\mathbb{R} \textit{ and } n < \alpha + 1.$$

Let us note that using these notations we have that $\partial_j\varphi(f) = \partial_j(\varphi \circ f)$. For example if $f : D \longrightarrow {}^\bullet\mathbb{R}$ is smooth, then we have

$$f'(0) \in {}^\bullet\mathbb{R}_2 \quad \text{and} \quad \forall h \in D: \quad f(h) = f(0) + h \cdot f'(0),$$

with $f'(0)$ uniquely determined by this property. Using this notation we have that $f \mapsto f'(0)$ is a derivation up to second order infinitesimals

Theorem 12.1.6. *Let* f, $g : D \longrightarrow {}^\bullet\mathbb{R}$ *and* $r \in {}^\bullet\mathbb{R}$, *then*

1. $(f+g)'(0) = f'(0) + g'(0)$

2. $(r \cdot f)'(0) =_2 r \cdot f'(0)$ *and if* $r \in \mathbb{R}$*, then* $(r \cdot f)'(0) = r \cdot f'(0)$

3. $(f \cdot g)'(0) =_2 f'(0) \cdot g(0) + f(0) \cdot g'(0)$

In other words the map $f \in {}^\bullet\mathbb{R}^D \mapsto {}^\bullet\mathbb{R}_{=_2}$ *is a derivation (see Theorem 3.2.3 for the definition of* ${}^\bullet\mathbb{R}_{=_k}$*).*

Proof: We use the notations of the proof of Theorem 12.1.4 and we prove property *3*., the others being similar. Thus we can write

$$\begin{aligned} f = {}^\bullet\gamma(p,-)|_D \quad &, \quad g = {}^\bullet\eta(q,-)|_D \\ f'(0) = \iota_2\left[\partial_2\gamma(p,0)\right] \quad &, \quad g'(0) = \iota_2\left[\partial_2\eta(q,0)\right], \end{aligned} \tag{12.1.3}$$

where ∂_2 means partial derivative with respect to the second slot. Therefore, recalling Theorem 3.2.3 about the properties of $=_k$, we have

$$\begin{aligned} (f \cdot g)'(0) &=_2 \partial_2\left(\gamma(p,-) \cdot \eta(q,-)\right)(0) =_2 \\ &=_2 \partial_2\gamma(p,0) \cdot \eta(q,0) + \gamma(p,0) \cdot \partial_2\eta(q,0). \end{aligned} \tag{12.1.4}$$

But from (12.1.3) we have that $f'(0) =_2 \partial_2\gamma(p,0)$, $g'(0) =_2 \partial_2\eta(q,0)$ and $=_2$ is a congruence relation with respect to ring operations (Theorem 3.2.3), hence from (12.1.4) we obtain the conclusion.

■

It is important now to make some considerations about the meaning of the derivative $\partial_j f(\underline{0})$, for $f : D^d_n \longrightarrow {}^\bullet\mathbb{R}$, with respect to the order of infinitesimals $n \in \mathbb{N}_{>0}$. We have already hinted in Section 11.2 to the fact that the best properties for derivatives can be proved using the Fermat method for functions $f : V \longrightarrow {}^\bullet\mathbb{R}$ defined in a neighborhood V of the point we are interested to, e.g. $\underline{0} \in {}^\bullet U \subseteq V$, with U open in $\mathbb{R}^d$. But if we start from a function of the form $f : D^d_n \longrightarrow {}^\bullet\mathbb{R}$ defined on an infinitesimal set, then, roughly speaking, the domain D^d_n is "too small to give sufficient information" for the definition of $\partial_j f(\underline{0})$ using the Fermat method. Indeed, we do not have as domain a full neighborhood to uniquely determine the smooth incremental ratio of f. The Taylor's formula determines the derivatives $\partial_j f(\underline{0})$ in the set ${}^\bullet\mathbb{R}^{\mathsf{k}}_{k_j}$ and hence forces us to work up to k_j-th order infinitesimals, i.e. using the congruence[3] $=_{k_j}$. As a further proof of these informal considerations, it seems plausible to expect that the larger is the order $n \in \mathbb{N}_{>0}$ the larger is the "information" we have at disposal. More precisely, the situation we want to analyze is the following: if we take a smooth function $f : D^d_m \longrightarrow {}^\bullet X$ and $n < m$, then what is the relationship between $\partial_j\varphi(f)$ and $\partial_j\varphi\left(f|_{D^d_n}\right)$? The answer is: it results $k_j(n) > k_j(m)$ and $\partial_j\varphi(f)$ is equal to $\partial_j\varphi\left(f|_{D^d_n}\right)$ up to infinitesimals of order $k_j(n)$.

[3]We have to note that k_j, defined in the statement of Theorem 12.1.4, really depends on the order n, thus if we need to distinguish two situations with two orders, we will use the more complete notation $k_j(n)$.

Theorem 12.1.7. *Let $X \in \mathcal{C}^\infty$ be a smooth space. Let us consider n, m, $d \in \mathbb{N}_{>0} \cup \{+\infty\}$, $j \in \mathbb{N}^d$ with $n < m$ and $1 \le |j| \le n$. Moreover, let us consider an open set $U \in \tau_X$ and smooth maps of the form*

$$f : D^d_m \longrightarrow {}^\bullet X \quad \text{with} \quad f(\underline{0}) \in {}^\bullet U$$

$$\varphi : {}^\bullet U \longrightarrow {}^\bullet \mathbb{R}^{\mathsf{k}}.$$

Finally, let

$$\frac{1}{k_j(p)} + \frac{j}{p+1} = 1 \quad \forall p \in \mathbb{N}_{>0} \cup \{+\infty\}.$$

Then we have

1. $k_j(n) > k_j(m)$
2. $\partial_j \varphi(f) =_{k_j(n)} \partial_j \varphi \left(f|_{D^d_n} \right)$
3. $k_j(m) < \omega \left[\partial_j \varphi(f) - \partial_j \varphi \left(f|_{D^d_n} \right) \right] \le k_j(n)$ *and hence*

$$\partial_j \varphi(f) - \partial_j \varphi \left(f|_{D^d_n} \right) \in D_{k_j(n)}$$

Proof: To prove *1*. it suffices to note that $k_j(p) = \frac{p+1}{p+1-|j|}$, because $\frac{j}{p+1} = \frac{|j|}{p+1}$, and that for $a > b$ the real function $x \mapsto \frac{x+a}{x+b}$ has a derivative $\frac{b-a}{(x+b)^2}$ which is negative for every x.
To prove *2*. from Definition 12.1.5 we have

$$\forall h \in D^d_m : \quad \varphi\left[f(h)\right] = \sum_{|j| \le m} \frac{h^j}{j!} \cdot \partial_j \varphi(f).$$

But $n < m$ so $D^d_n \subseteq D^d_m$ and thus from Corollary 2.5.6 we have

$$\forall h \in D^d_n : \quad \varphi\left[f(h)\right] = \sum_{|j| \le n} \frac{h^j}{j!} \cdot \partial_j \varphi(f).$$

Now using Theorem 3.3.3 we obtain $h^j \cdot \partial_j \varphi(f) = h^j \cdot \iota_{k_j(n)} \left[\partial_j \varphi(f)\right]$ for every $h \in D^d_n$ and substituting we get

$$\forall h \in D^d_n : \quad \varphi\left[f(h)\right] = \sum_{|j| \le n} \frac{h^j}{j!} \cdot \iota_{k_j(n)} \left[\partial_j \varphi(f)\right],$$

hence from the uniqueness in Taylor's formula we obtain $\iota_{k_j(n)} \left[\partial_j \varphi(f)\right] = \partial_j \varphi \left(f|_{D^d_n} \right) = \iota_{k_j(n)} \left[\partial_j \varphi \left(f|_{D^d_n} \right) \right]$, because $\iota_k(x) = x$ if $x \in {}^\bullet \mathbb{R}_k$, and this proves *2*.

Property *3*. follows directly from Theorem 3.2.5.

■

Much in the same way as the Fermat method provides a very useful instrument to derive the calculus for functions defined on open sets, the previous theorems show us that the derivatives $\partial_j\varphi(f)$ provides a useful instrument to study functions defined on infinitesimal sets. The following theorem states that equality of derivatives through observables implies identity of the functions:

Theorem 12.1.8. *Let $X \in \mathcal{C}^\infty$ be a smooth space and $n,\, d \in \mathbb{N}_{>0}$. Let us consider two smooth functions*

$$f,\ g : D_n^d \longrightarrow {}^\bullet X \quad \text{with} \quad f(\underline{0}) = g(\underline{0}).$$

Moreover, let us assume that the derivatives of these functions are equal, i.e.

$$\partial_j {}^\bullet\varphi(f) = \partial_j {}^\bullet\varphi(g) \quad \forall j \in \mathbb{N}^d :\ 1 \le |j| \le n$$

for every observable $\varphi : U \longrightarrow \mathbb{R}^{\mathsf{k}}$ of the space X with $f(\underline{0}) \in {}^\bullet U$. Then

$$f = g$$

Let us note that this theorem, which is a consequence of our definition of equality in ${}^\bullet X$ using observables (see Definition 8.2.1), is not trivial, because in our context we do not have charts on our spaces $X \in \mathcal{C}^\infty$.
Proof: Take $h \in D_n^d$, we have to prove that $y := f(h)$ and $z := g(h)$ are equal in ${}^\bullet X$. Using typical notations and neglecting some details, we can say that $y = f(h) = {}^\bullet\gamma(p, h)$ in ${}^\bullet X$ so that

$$\begin{aligned} {}^\circ y = {}^\circ\left[{}^\bullet\gamma(p,h)\right] = \gamma({}^\circ p, \underline{0}) = \\ = {}^\circ\left[{}^\bullet\gamma(p,\underline{0})\right] = {}^\circ f(\underline{0}). \end{aligned}$$

But $f(\underline{0}) = g(\underline{0})$ in ${}^\bullet X$ by hypothesis, so ${}^\circ y = {}^\circ f(\underline{0}) \asymp {}^\circ g(\underline{0}) = {}^\circ z$. Now let us take an observable $\varphi : U \longrightarrow \mathbb{R}^{\mathsf{k}}$ of X, we have to prove that

$$y_0 \in U \iff z_0 \in U \tag{12.1.5}$$

$$y_0 \in U \implies \varphi(y_t) = \varphi(z_t) + o(t). \tag{12.1.6}$$

The first one follows directly from the identification ${}^\circ y \asymp {}^\circ z$. For the second one, if $y_0 \in U$, then $y_0 = {}^\circ y = {}^\circ f(\underline{0})$, thus ${}^\circ f(\underline{0}) \in U$ and hence $f(\underline{0}) \in {}^\bullet U$ from Lemma 12.1.1. We can thus apply our hypotheses to obtain the equality of derivatives $\partial_j {}^\bullet\varphi(f) = \partial_j {}^\bullet\varphi(g)$ for every $j \in \mathbb{N}^d$ with $1 \le |j| \le n$. Using Taylor's formula

$${}^\bullet\varphi(y) = {}^\bullet\varphi\left[f(h)\right] = \sum_{|j|\le n} \frac{h^j}{j!}\cdot\partial_j {}^\bullet\varphi(f) = \sum_{|j|\le n} \frac{h^j}{j!}\cdot\partial_j {}^\bullet\varphi(g) = {}^\bullet\varphi\left[g(h)\right] = {}^\bullet\varphi(z),$$

this equality being in ${}^\bullet\mathbb{R}^{\mathsf{k}}$, i.e. $\varphi(y_t) = \varphi(z_t) + o(t)$, which is the conclusion stated in the theorem.

■

There is the possibility to connect the methods developed for the differential calculus of function defined on open sets (see the previous Section 11) with the differential calculus of smooth functions defined on infinitesimal sets. Indeed, the following results prove that functions of the form $f : S \longrightarrow {}^\bullet\mathbb{R}^n$ can be seen locally as "infinitesimal polynomials with smooth coefficients".

Theorem 12.1.9. *Let $S \subseteq {}^\bullet\mathbb{R}^{\mathsf{s}}$ and $f : S \longrightarrow {}^\bullet\mathbb{R}^n$ a map (in* **Set***). Then it results that*

$$f : S \longrightarrow {}^\bullet\mathbb{R}^n \text{ is smooth in } {}^\bullet\boldsymbol{\mathcal{C}}^\infty \tag{12.1.7}$$

if and only if for every $x \in S$ we can write

$$f(y) = \sum_{\substack{|q|\le k\\ q\in\mathbb{N}^d}} a_q(y)\cdot p^q \quad \forall y \in {}^\bullet V \cap S, \tag{12.1.8}$$

for suitable:

1. *$d,\ k \in \mathbb{N}$*
2. *$p \in D_k^d$*
3. *V open subset of $\mathbb{R}^{\mathsf{s}}$ such that $x \in {}^\bullet V$*
4. *$(a_q)_{\substack{|q|\le k\\ q\in\mathbb{N}^d}}$ family of $\boldsymbol{\mathcal{C}}^\infty(V, \mathbb{R}^n)$.*

In other words, every smooth function $f : S \longrightarrow {}^\bullet\mathbb{R}^n$ can be constructed locally starting from some "infinitesimal parameters"

$$p_1, \ldots, p_d \in D_k$$

and from ordinary smooth functions

$$a_q \in \boldsymbol{\mathcal{C}}^\infty(V, \mathbb{R}^n)$$

and using polynomial operation only with p_1, ..., p_d and with coefficients $a_q(-)$. Roughly speaking, we can say that they are "infinitesimal polynomials with smooth coefficients. The polynomials variables act as parameters only". By the sheaf property, here "locally" means that this construction using infinitesimal polynomials has to be done in a neighborhood of each point $x \in S$, but in such a way to have equal polynomials on intersecting neighborhoods.

If $f : I \longrightarrow {}^\bullet\mathbb{R}^n$ with $\underline{0} \in I \subseteq D^{\mathsf{s}}_\infty$, then for $x = \underline{0}$ we can write (12.1.8) globally as

$$f(h) = \sum_{\substack{|q|\le k\\ q\in\mathbb{N}^d}} a_q(h)\cdot p^q \quad \forall h \in I, \tag{12.1.9}$$

because $I \subseteq {}^\bullet V$. From (12.1.9) we obtain

$$\partial_j f(h_1) = \iota_{k_j}\left(\sum_{\substack{|q|\le k\\ q\in\mathbb{N}^d}} \partial_j a_q(h_1)\cdot p^q \quad \forall h_1 \in I\right),$$

where $j \in \mathbb{N}^d$, $k_j \in \mathbb{R}$ and α_1, ..., $\alpha_d \in \mathbb{N}_{>0}$ are such that $0 < \frac{j}{\alpha+1} < 1$ and $\frac{1}{k_j} + \frac{j}{\alpha+1} = 1$. Therefore from Theorem 3.3.3 we get

$$\forall h \in D_{\alpha_1}\times\cdots\times D_{\alpha_d} : \quad h\cdot\partial_j f(h_1) = h\cdot\sum_{\substack{|q|\le k\\ q\in\mathbb{N}^d}} \partial_j a_q(h_1)\cdot p^q. \tag{12.1.10}$$

All this permits to use the results about the differential calculus of functions like $a_q \in \mathcal{C}^\infty(V, {}^\bullet\mathbb{R}^n)$, defined on open sets, to functions defined on infinitesimal sets. Moreover, equalities of the form (12.1.10) permit to avoid the use of the map $\iota_k : {}^\bullet\mathbb{R}^n \longrightarrow {}^\bullet\mathbb{R}^n_{=k}$.

Proof: The implication (12.1.8)$\Rightarrow$ (12.1.7) follows directly from Theorem 9.2.4. For the opposite implication, let us write $f|_{\mathcal{V}} = {}^\bullet\alpha(\pi,-)|_{\mathcal{V}}$, as usual, in a neighborhood of $x \in {}^\bullet V \cap S$, for $\alpha \in \mathcal{C}^\infty(U\times V, \mathbb{R}^n)$ and where $\pi \in {}^\bullet U \subseteq {}^\bullet\mathbb{R}^d$ works as the usual non standard parameter. Set $r := {}^\circ\pi$ and $p := \pi - r$ so that $p \in D^d_k$ for some $k \in \mathbb{N}_{>0}$. Using the infinitesimal Taylor's formula we get

$$f(y) = \alpha(\pi, y) = \alpha(r+p, y) = \sum_{|q|\le k} \partial_q\alpha(r,y)\cdot\frac{p^q}{q!},$$

from which we have the conclusion setting $a_q := \frac{1}{q!}\cdot\partial_q\alpha(r,-)$.

∎

Taking an enumeration of all these multi-indexes $q \in \mathbb{N}^d$, i.e.

$$\{q_1,\ldots,q_N\} = \left\{q\in\mathbb{N}^d \ :\ |q|\le k\ ,\ p^q \ne 0\right\}\setminus\{\underline{0}\}$$
$$q_i \ne q_j \quad \text{if} \quad i \ne j,$$

then we can write the infinitesimal polynomial (12.1.8) in a simpler way, even if it hide the powers p^q of the infinitesimal parameter $p \in D^d_k$. In fact, using this enumeration we can write

$$f(y) = \sum_{i=1}^N a_{q_i}(y)\cdot p_1^{q_{i1}}\cdot\ldots\cdot p_d^{q_{id}} + a_{\underline{0}}(y).$$

It suffices to set $\pi_i := p_1^{q_{i1}} \cdot \ldots \cdot p_d^{q_{id}}$, $b_i := a_{q_i}$ and $b_0 := a_{\underline{0}}$ to have

$$f(y) = b_0(y) + \sum_{i=1}^{N} b_i(y) \cdot \pi_i \quad \forall y \in {}^\bullet V \cap S. \tag{12.1.11}$$

As usual, both the smooth functions b_i and the infinitesimal parameters π_i are not uniquely determined by formulas of the form (12.1.11).

12.2 Smoothness of derivatives

In our smooth context, it is important that the definition of derivative for our non standard smooth functions always produces a smooth operator. On the other hand, it is natural to expect, exactly as for the standard part map (see Corollary 9.4.3) that every function $\iota_k : {}^\bullet\mathbb{R} \longrightarrow {}^\bullet\mathbb{R}$ mapping a Fermat real $x \in {}^\bullet\mathbb{R}$ to "x up to k-th order infinitesimals", i.e. $\iota_k(x)$, cannot be smooth. If this is so, then also the first derivative cannot be smooth being thought as a function of f, i.e. $f \in {}^\bullet\mathbb{R}^D \mapsto f'(0) \in {}^\bullet\mathbb{R}$. Let us consider, for example, the following function defined on the infinitesimal set D_2 of second order infinitesimals and depending on the parameter $p \in {}^\bullet\mathbb{R}$:

$$f_p(h) := \frac{1}{2}(h+p)^2 \quad \forall h \in D_2.$$

We have $f_p(h) = \frac{1}{2}p^2 + hp + \frac{h^2}{2}$, so that from the Taylor's formula we have

$$\begin{aligned} f_p'(0) = \iota_{\frac{3}{2}}(p) \quad &\text{in fact:} \quad \frac{n+1}{n+1-j} = \frac{2+1}{2+1-1} = \frac{3}{2} \\ f_p''(0) = \iota_3(1) = 1 \quad &\text{in fact:} \quad \frac{n+1}{n+1-j} = \frac{2+1}{2+1-2} = 3. \end{aligned}$$

So, if the map

$$f \in {}^\bullet\mathbb{R}^{D_2} \mapsto f'(0) \in {}^\bullet\mathbb{R}$$

is smooth, also the map

$$p \in {}^\bullet\mathbb{R} \mapsto f_p \in {}^\bullet\mathbb{R}^{D_2} \mapsto f_p'(0) = \iota_{\frac{3}{2}}(p) \in {}^\bullet\mathbb{R}$$

would be smooth. Therefore, the smoothness of the maps $\iota_k : {}^\bullet\mathbb{R} \longrightarrow {}^\bullet\mathbb{R}$ is strictly tied with the smoothness of the derivatives. We will see that these maps are not smooth, as a simple consequence of the following general result.

Theorem 12.2.1. *Let M, N be manifolds, and $f : {}^\bullet N \longrightarrow {}^\bullet M$ be a ${}^\bullet\mathcal{C}^\infty$ function. Then*

$$f(N) \subseteq M \quad \Longrightarrow \quad f = {}^\bullet(f|_N).$$

In other words, a ${}^\bullet\mathcal{C}^\infty$ function between extended manifolds that takes standard points to standard points, can be realized as the extension of an ordinary smooth function.

Proof: Let us consider a generic point $n_1 \in {}^\bullet N$. We want to prove that $f(n_1) = {}^\bullet(f|_N)(n_1)$. Let $n := {}^\circ n_1 \in N$ so $f(n) \in M$ and we can consider a chart (U, φ) on $n \in N$ and another one (V, ψ) on $f(n)$. We can assume, for simplicity, $\varphi(U) = \mathbb{R}^{\mathsf{n}}$, $\psi(V) = \mathbb{R}^{\mathsf{m}}$, $\varphi(n) = \underline{0}$ and the open set U sufficiently small so that $f({}^\bullet U) \subseteq {}^\bullet V$. Diagrammatically, in ${}^\bullet\mathcal{C}^\infty$, the situation is the following

$$
{}^\bullet\mathbb{R}^{\mathsf{n}} \xrightarrow{\ {}^\bullet\varphi^{-1}\ } {}^\bullet U \xrightarrow{\ f|_{{}^\bullet U}\ } {}^\bullet V \xrightarrow{\ {}^\bullet\psi\ } {}^\bullet\mathbb{R}^{\mathsf{m}} .
$$

To the smooth function ${}^\bullet\varphi^{-1} \cdot f|_{{}^\bullet U} \cdot {}^\bullet\psi : {}^\bullet\mathbb{R}^{\mathsf{n}} \longrightarrow {}^\bullet\mathbb{R}^{\mathsf{m}}$ we can apply Theorem 9.2.4 obtaining that in a neighborhood of $\underline{0}$ we can write

$$
\psi\left[f\left(\varphi^{-1}y\right)\right] = {}^\bullet\gamma(p, y),
$$

where $\gamma \in \mathcal{C}^\infty(A \times B, \mathbb{R}^{\mathsf{m}})$, $p \in {}^\bullet A$, A is open in $\mathbb{R}^{\mathsf{p}}$ and B is an open neighborhood of $\underline{0}$ in $\mathbb{R}^{\mathsf{n}}$. Setting $r := {}^\circ p \in \mathbb{R}^{\mathsf{p}}$ and $h := p - r \in D_k^{\mathsf{p}}$ for some $k \in \mathbb{N}_{>0}$, we get

$$
\psi\left[f\left(\varphi^{-1}y\right)\right] = {}^\bullet\gamma(r + h, y) = \sum_{\substack{q \in \mathbb{N}^{\mathsf{p}} \\ |q| \le k}} \partial_1^q \gamma(r, y) \cdot \frac{h^q}{q!},
$$

where ∂_1 means the derivative with respect to the first slot of $\gamma(-,-)$. Setting for simplicity $a_i(y) := \frac{1}{q_i!} \cdot \partial_1^{q_i}\gamma(r, y)$, $h_i := h^{q_i}$, where $\{q_1, \ldots, q_N\} = \{q \in \mathbb{N}^{\mathsf{p}} : |q| \le k\}$, $q_i \ne q_j$ if $i \ne j$, we can write

$$
\psi\left[f\left(\varphi^{-1}y\right)\right] = a_0(y) + \sum_{i=1}^N a_i(y) \cdot h_i \quad \forall y \in {}^\bullet B, \tag{12.2.1}
$$

where the functions $a_i \in \mathcal{C}^\infty(B, \mathbb{R}^{\mathsf{m}})$ are standard smooth maps. We can suppose $(a_1, \ldots, a_N)$ linearly independent in the real vector space $\mathcal{C}^\infty(B, \mathbb{R}^{\mathsf{m}})$, because otherwise we can select among them a basis and express the other functions as a linear combination of the basis. Now, let us evaluate the standard part of (12.2.1) at a generic standard point $r \in B$:

$$
\forall r \in B : \quad {}^\circ\left\{\psi\left[f\left(\varphi^{-1}r\right)\right]\right\} = {}^\circ a_0(r) + \sum_{i=1}^N {}^\circ a_i(r) \cdot {}^\circ h_i = {}^\circ a_0(r), \tag{12.2.2}
$$

because ${}^\circ h_i = 0$ since $h_i = h^{q_i} = (p - {}^\circ p)^{q_i} \in D_\infty$. But $r \in B \subseteq \mathbb{R}^{\mathsf{n}}$, so $\varphi^{-1}(r) \in U \subseteq {}^\bullet U$ and $f\left(\varphi^{-1}r\right) \in M \cap {}^\bullet V = V$ because $f(N) \subseteq M$. Thus, $\psi\left[f\left(\varphi^{-1}r\right)\right] \in \mathbb{R}^{\mathsf{m}}$ and hence from (12.2.2) we get

$$
\psi\left[f\left(\varphi^{-1}r\right)\right] = a_0(r) \quad \forall r \in B. \tag{12.2.3}
$$

Therefore

$$\sum_{i=1}^{N} a_i(r) \cdot h_i = \underline{0} \quad \forall r \in B. \tag{12.2.4}$$

The functions a_i are continuous and linearly independent, so that from Lemma 11.2.6 we can find $r_1, \dots, r_N \in B$ such that

$$\det \begin{bmatrix} a_1(r_1) & \dots & a_N(r_1) \\ \vdots & & \\ a_1(r_N) & \dots & a_N(r_N) \end{bmatrix} \neq 0.$$

We can write (12.2.4) as

$$\begin{bmatrix} a_1(r_1) & \dots & a_N(r_1) \\ \vdots & & \\ a_1(r_N) & \dots & a_N(r_N) \end{bmatrix} \cdot \begin{bmatrix} h_1 \\ \vdots \\ h_N \end{bmatrix} = \underline{0},$$

and therefore we obtain that $h_i = 0$ for every $i = 1, \dots, N$, and

$$\psi \left[f \left(\varphi^{-1} y \right) \right] = a_0(y)$$

for every $y \in {}^\bullet B$ from (12.2.1), which has to be understood as an abbreviation of

$${}^\bullet\psi \left[f \left({}^\bullet\varphi^{-1} y \right) \right] = {}^\bullet a_0(y) \quad \forall y \in {}^\bullet B. \tag{12.2.5}$$

Now $n_1 \asymp n$ because ${}^\circ n_1 = n$, so that ${}^\bullet\varphi(n_1) \asymp \varphi(n) = \underline{0} \in {}^\bullet B$ and thus also ${}^\bullet\varphi(n_1) \in {}^\bullet B$ from the definition of $\asymp$. We can thus apply (12.2.5) with $y = {}^\bullet\varphi(n_1)$ obtaining that ${}^\bullet\psi\left[f(n_1)\right] = {}^\bullet a_0\left({}^\bullet\varphi n_1\right)$, and hence

$$f(n_1) = {}^\bullet\psi^{-1}\left[{}^\bullet a_0\left({}^\bullet\varphi n_1\right)\right] = {}^\bullet\left(\psi^{-1} \circ a_0 \circ \varphi\right)|_{\varphi^{-1}(B)}(n_1). \tag{12.2.6}$$

Finally, we must prove that $\left(\psi^{-1} \circ a_0 \circ \varphi\right)(x) = \left(f|_N\right)(x)$ in an open neighborhood of n, but from (12.2.3) and taking a generic $x \in \varphi^{-1}(B) \subseteq N$ we get

$$\begin{aligned} &\psi\left[f(x)\right] = a_0\left[\varphi(x)\right] \\ &\quad f(x) = \left(f|_N\right)(x) = \psi^{-1}\left[a_0\left(\varphi x\right)\right] = \left(\psi^{-1} \circ a_0 \circ \varphi\right)|_{\varphi^{-1}(B)}(x), \end{aligned}$$

and therefore $f(n_1) = {}^\bullet\left(f|_N\right)(n_1)$ from (12.2.6).

■

From this general result it follows

Corollary 12.2.2. *Let $k \in \mathbb{R}_{\geq 1}$, then the function*

$$\iota_k : {}^\bullet\mathbb{R} \longrightarrow {}^\bullet\mathbb{R} \tag{12.2.7}$$

is not smooth.

Proof: In fact $\iota_k(r) = r$ for every $r \in \mathbb{R}$, so from the previous Theorem 12.2.1 we have

$$\iota_k = {}^{\bullet}(\iota_k|_{\mathbb{R}}) = {}^{\bullet}1_{\mathbb{R}} = 1_{{}^{\bullet}\mathbb{R}}$$

if ι_k is smooth. But this is impossible because, e.g. $\iota_k(\mathrm{d}t_k) = 0 \neq \mathrm{d}t_k = 1_{{}^{\bullet}\mathbb{R}}(\mathrm{d}t_k)$.

■

This negative result will be counteracted in two ways: in the first one we will prove that, in spite of this corollary, any map of the form

$$(f, \varphi, h) \mapsto h^j \cdot \partial_j \varphi(f)$$

is smooth; the second one says that the negative result is due to the choice of a wrong codomain in (12.2.7).

Theorem 12.2.3. *Let $X \in \boldsymbol{\mathcal{C}}^\infty$ be a smooth space, $n \in \mathbb{N}_{>0}$ and $U \in \tau_X$ an open set of X. Let us consider the ${}^{\bullet}\boldsymbol{\mathcal{C}}^\infty$-maps*

$$f : D_n \longrightarrow {}^{\bullet}X \quad \text{with} \quad f(0) \in {}^{\bullet}U$$

$$\varphi : {}^{\bullet}U \longrightarrow {}^{\bullet}\mathbb{R}^{\mathsf{k}}.$$

Finally, let

$$A = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 2 & 2^2 & \dots & 2^n \\ \vdots & & & \\ n & n^2 & \dots & n^n \end{bmatrix}^{-1}$$

be the inverse of the submatrix $V(1;1)$ obtained deleting the first row and the first column of the Vandermonde matrix V determined by $(0, 1, 2, \dots, n)$. Then for every $j = 1, \dots, n$ we have

$$h^j \cdot \partial_j \varphi(f) = \sum_{i=1}^{n} j! \cdot a_{ij} \cdot \{\varphi[f(i \cdot h)] - \varphi[f(0)]\} \quad \forall h \in D_n, \qquad (12.2.8)$$

hence the function

$$(f, \varphi, h) \in {}^{\bullet}U^{D_n} \times \left({}^{\bullet}\mathbb{R}^{\mathsf{k}}\right)^{{}^{\bullet}U} \times D_n \mapsto h^j \cdot \partial_j \varphi(f) \in {}^{\bullet}\mathbb{R}^{\mathsf{k}} \qquad (12.2.9)$$

is smooth in ${}^{\bullet}\boldsymbol{\mathcal{C}}^\infty$.

Proof: From the infinitesimal Taylor's formula, Theorem 12.1.3, we have

$$\varphi[f(h)] = \varphi[f(0)] + \sum_{j=1}^{n} \frac{h^j}{j!} \cdot \partial_j \varphi(f) \quad \forall h \in D_n. \qquad (12.2.10)$$

Let $x_j := \frac{h^j}{j!} \cdot \partial_j \varphi(f)$ for $j = 1, \dots, n$ and for a fixed $h \in D_n$ let h, $2h$, $3h$, ..., $n \cdot h$ in (12.2.10). We obtain

$$\begin{aligned}
\varphi(fh) - \varphi(f0) &= x_1 + \ldots + x_n \\
\varphi[f(2h)] - \varphi(f0) &= 2x_1 + 2^2 x_2 + \ldots + 2^n x_n \\
&\ldots \\
\varphi[f(nh)] - \varphi(f0) &= nx + n^2 x_2 + \ldots + n^n x_n.
\end{aligned}$$

So that we can write this system of equations in $x_1, \dots, x_n$ as

$$\begin{bmatrix} 1 & 1 & \ldots & 1 \\ 2 & 2^2 & \ldots & 2^n \\ \vdots & & & \\ n & n^2 & \ldots & n^n \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \varphi(fh) - \varphi(f0) \\ \vdots \\ \varphi[f(nh)] - \varphi(f0) \end{bmatrix},$$

from which the first part (12.2.8) of the conclusion follows.

The second part follows noting that the right hand side of (12.2.8) gives the function (12.2.9) as a composition of smooth functions (among which we have to consider some evaluations, that in a cartesian closed category are always smooth; see Section 7.3).

■

Analogously, we can prove a corresponding result in the d-dimensional case:

Theorem 12.2.4. *Let $X \in \mathcal{C}^\infty$ be a smooth space, n, $d \in \mathbb{N}_{>0}$ and $U \in \tau_X$ an open set. Let us consider the ${}^\bullet\mathcal{C}^\infty$-maps*

$$f : D_n^d \longrightarrow {}^\bullet X \quad \textit{with} \quad f(\underline{0}) \in {}^\bullet U$$

$$\varphi : {}^\bullet U \longrightarrow {}^\bullet \mathbb{R}^{\mathsf{k}}.$$

Then for every $j \in \mathbb{N}^d$ such that $|j| \le n$, the map

$$(f, \varphi, h) \in {}^\bullet U^{D_n^d} \times \left({}^\bullet\mathbb{R}^{\mathsf{k}}\right)^{{}^\bullet U} \times D_n^d \mapsto h^j \cdot \partial_j \varphi(f) \in {}^\bullet\mathbb{R}^{\mathsf{k}}$$

is a ${}^\bullet\mathcal{C}^\infty$ map.

The second solution of the negative result of Corollary 12.2.2 is to admit that the codomain of the map ι_k is not correct, but we have to change it as follows

$$\iota_k : {}^\bullet\mathbb{R} \longrightarrow {}^\bullet\mathbb{R}_{=_k}.$$

Indeed, we have

Theorem 12.2.5. *Let $k \in \mathbb{R}_{\ge 1}$, then the map*

$$\iota_k : {}^\bullet\mathbb{R} \longrightarrow {}^\bullet\mathbb{R}_{=_k}$$

is a ${}^\bullet\mathcal{C}^\infty$ map. We recall here that ${}^\bullet\mathbb{R}_{=_k}$ is the quotient set ${}^\bullet\mathbb{R}/ =_k$.

Proof: For clarity, we will use the notations with the equivalence classes, so that here the map ι_k as to be understood as defined by

$$\iota_k(m) := \left[{}^\circ m + \sum_{\substack{i=1 \\ \omega_i(m)>k}}^{N} {}^\circ m_i \cdot \mathrm{d}t_{\omega_i(m)} \right]_{=_k} \quad \forall m \in {}^\bullet\mathbb{R}. \tag{12.2.11}$$

But using the notations with the equivalence classes, the equality

$${}^\circ u + \sum_{\substack{i=1 \\ \omega_i(u)>k}}^{N} {}^\circ u_i \cdot \mathrm{d}t_{\omega_i(u)} =_k u \quad \forall u \in {}^\bullet\mathbb{R}$$

now gives

$$\iota_k(u) = [u]_{=_k} \quad \forall u \in {}^\bullet\mathbb{R},$$

and hence the map ι_k defined in (12.2.11) is simply the projection onto the quotient set ${}^\bullet\mathbb{R}_{=_k}$ which is always smooth by the co-completeness of the category ${}^\bullet\mathcal{C}^\infty$ (see Theorem 6.3.5).

∎

To clarify further the relationships between ${}^\bullet\mathbb{R}_k$ and ${}^\bullet\mathbb{R}_{=_k}$ we also prove the following

Theorem 12.2.6. *Let $k \in \mathbb{R}_{\geq 1}$, then the map*

$$i_k : x \in {}^\bullet\mathbb{R}_{=_k} \mapsto \iota_k(x) \in {}^\bullet\mathbb{R}_k$$

is not smooth in ${}^\bullet\mathcal{C}^\infty$.

Even if we have these negative results, the map $\iota_k : {}^\bullet\mathbb{R} \longrightarrow {}^\bullet\mathbb{R}_k$ as defined in the Definition 3.2.1 has not to be forgotten: if we only need algebraic properties like those expressed in results like those of Chapter 3, then we do not need the whole map ι_k but only terms of the form $\iota_k(m)$, and in this case, we can think $\iota_k(m) \in {}^\bullet\mathbb{R}_k$. If, instead, we need to prove smoothness of derivatives, then we need the map ι_k thought with codomain: $\iota_k : {}^\bullet\mathbb{R} \longrightarrow {}^\bullet\mathbb{R}_{=_k}$. This little bit of notational confusion disappears completely once we specify domains and codomains of the map ι_k we are considering.

Proof: Firstly, the map $i_k : {}^\bullet\mathbb{R}_{=_k} \longrightarrow {}^\bullet\mathbb{R}_k$ is well defined because the definition of $x =_k y$ is exactly $\iota_k(x) = \iota_k(y)$ (see Definition 3.2.1).

Now, let us take a figure $\delta \in_H {}^\bullet\mathbb{R}_{=_k}$ on the quotient set ${}^\bullet\mathbb{R}_{=_k}$. By Theorem 6.3.5 this means that for every $h \in H$ we can find a neighborhood U of h in H and a figure $\alpha \in_U {}^\bullet\mathbb{R}$ such that $\delta|_U = \alpha \cdot [-]_{=_k}$. Let us consider the composition $\delta \cdot i_k : H \longrightarrow {}^\bullet\mathbb{R}_k$ in the neighborhood U:

$$i_k\left(\delta(u)\right) = i_k \left\{ [\alpha(u)]_{=_k} \right\} = \iota_k \left[\alpha(u)\right].$$

So, taking $\alpha = 1_{^\bullet\mathbb{R}}$, from this we would obtain that if i_k were smooth, then also $\iota_k(u)$ would be smooth in u. In other words, it would be smooth if considered as a map from ${}^\bullet\mathbb{R}$ to ${}^\bullet\mathbb{R}_k \hookrightarrow {}^\bullet\mathbb{R}$, but we already know that this does not hold from Corollary 12.2.2.

■

Now we have to understand with respect to what variables we have to mean that "derivatives are smooth functions", because we are considering functions defined on infinitesimal sets. The natural answer is given by the following

Definition 12.2.7. *Let $X \in \mathcal{C}^\infty$ be a smooth space, $d \in \mathbb{N}_{>0}$, $n \in \mathbb{N}_{>0} \cup \{+\infty\}$, and $U \in \tau_X$ an open set. Let us consider the ${}^\bullet\mathcal{C}^\infty$ maps*

$$\varphi : {}^\bullet U \longrightarrow {}^\bullet\mathbb{R}^{\mathsf{k}}$$

$$f : V \longrightarrow {}^\bullet X \quad \textit{with} \quad x \in V \subseteq {}^\bullet\mathbb{R}^d \quad \textit{and} \quad f(x) \in {}^\bullet U.$$

Moreover, let us suppose that V verifies

$$\forall h \in D_n^d : \quad x + h \in V,$$

so that we can define $f_x : h \in D_n^d \mapsto f(x+h) \in {}^\bullet X$. Then for every multi-index $j \in \mathbb{N}^d$ with $1 \leq |j| \leq n$ we define

$$\partial_j \varphi(f)_x := \partial_j \varphi\left(f_x\right).$$

As usual, if $X = \mathbb{R}^{\mathsf{k}}$ and $\varphi = 1_{\mathbb{R}^{\mathsf{k}}}$, we will use the simplified notations $\partial_j f_x := \partial_j f(x) := \partial_j \varphi(f)_x$ and $f^{(j)}(x) := \partial_j f(x)$ if $k = 1$ and $1 \leq j \leq n$.

Therefore the derivative $\partial_j \varphi(f)_x$ is characterized by the Taylor's formula

$$\forall h \in D_n^d : \quad \varphi\left[f(x+h)\right] = \sum_{\substack{j \in \mathbb{N}^d \\ |j| \leq n}} \frac{h^j}{j!} \cdot \partial_j \varphi(f)_x$$

and by the conditions $\partial_j \varphi(f)_x \in {}^\bullet\mathbb{R}_{=_{k_j}}$ for every multi-index j. As above, with these notations we have that $\partial_j \varphi(f)_x = \partial_j(\varphi \circ f)_x$.

Example. Let us consider $f : D \longrightarrow {}^\bullet\mathbb{R}$, we want to find $f'(i)$ for $i \in D$. By Definition 12.2.7 we have

$$f(i+h) = f(i) + h \cdot f'(i) \quad \forall h \in D, \tag{12.2.12}$$

with $f'(i) \in {}^\bullet\mathbb{R}_{=_2}$. But $i + h \in D$ so that we also get

$$f(i+h) = f(0) + (i+h) \cdot f'(0) \tag{12.2.13}$$

with $f'(0) \in {}^\bullet\mathbb{R}_{=_2}$. On the other hand from $i \in D$ we also have $f(i) = f(0) + i \cdot f'(0)$, and substituting in (12.2.12) we get $f(i+h) = f(0) + i \cdot f'(0) + h \cdot f'(i)$. From this and from (12.2.13) we finally obtain

$$h \cdot f'(0) = h \cdot f'(i) \quad \forall h \in D,$$

that is (see Theorem 3.3.3) $f'(0) =_2 f'(i)$, i.e. $f'(0) = f'(i)$ because both derivatives are in ${}^\bullet\mathbb{R}_{=_2}$. This confirms an intuitive result, i.e. that every smooth function $f : D \longrightarrow {}^\bullet\mathbb{R}$ is a straight line, and hence it has constant derivative.

Using the notation of the Definition 12.2.7 we can now state the following

Theorem 12.2.8. *Let X, d, n, U, V and j as in the hypothesis of Definition 12.2.7, then the function*

$$\partial_j : (f, \varphi, x) \in {}^\bullet U^V \times \left({}^\bullet\mathbb{R}^{\mathsf{k}}\right)^{{}^\bullet U} \times V \mapsto \partial_j \varphi(f)_x \in {}^\bullet\mathbb{R}^{\mathsf{k}}_{=_{k_j}}$$

is a ${}^\bullet\mathcal{C}^\infty$ map.

Proof: Let us consider figures $\alpha \in_A {}^\bullet U^V$, $\beta \in_B \left({}^\bullet\mathbb{R}^{\mathsf{k}}\right)^{{}^\bullet U}$ and $\gamma \in_C V$, we have to prove that $(\alpha \times \beta \times \gamma) \cdot \partial_j \in_{A \times B \times C} {}^\bullet\mathbb{R}^{\mathsf{k}}_{=_{k_j}}$. Due to cartesian closedness we have that $\alpha^\vee : \bar{A} \times V \longrightarrow {}^\bullet U$ and $\beta^\vee : \bar{B} \times {}^\bullet U \longrightarrow {}^\bullet\mathbb{R}^{\mathsf{k}}$ are smooth. In the following we will always use identifications of type $\bar{A} \times \bar{B} = \overline{A \times B}$, based on the isomorphism of Lemma 9.2.1 and Lemma 9.2.2. We proceed locally, that is using the sheaf property of the space ${}^\bullet\mathbb{R}^{\mathsf{k}}_{=_k}$, so let us fix generic $(a, b, c) \in A \times B \times C$. First of all, we have to note that the function

$$\Theta : (a_1, b_1, x) \in A \times B \times V \mapsto \beta^\vee \left\{b_1, \alpha^\vee [a_1, x]\right\} \in {}^\bullet\mathbb{R}^{\mathsf{k}},$$

is smooth, being the composition of smooth functions. Let, for simplicity, $\mathcal{V} := A \times B \times V$. We have that $\Theta : \overline{A \times B \times V} \longrightarrow {}^\bullet\mathbb{R}^{\mathsf{k}}$ in ${}^\bullet\mathcal{C}^\infty$, and hence we have the figure $\Theta \in_{A \times B \times V} {}^\bullet\mathbb{R}^{\mathsf{k}}$. Let us apply to this figure Theorem 9.2.4 at the point $(a, b, \gamma(c)) \in \mathcal{V}$, obtaining that in an open neighborhood ${}^\bullet\mathcal{U} \cap \mathcal{V}$ of $(a, b, \gamma(c))$ generated by the open set $\mathcal{U}$ of $\mathbb{R}^{\mathsf{a}} \times \mathbb{R}^{\mathsf{b}} \times \mathbb{R}^d$ we can write $\Theta(a_1, b_1, x) = {}^\bullet\delta(p, a_1, b_1, x)$ for every $(a_1, b_1, x) \in {}^\bullet\mathcal{U} \cap \mathcal{V}$, where $\delta \in \mathcal{C}^\infty(\mathcal{A} \times \mathcal{U}, \mathbb{R}^{\mathsf{k}})$, $p \in {}^\bullet\mathcal{A}$ and $\mathcal{A}$ is open in $\mathbb{R}^{\mathsf{p}}$. Note that, being $\mathbb{R}^{\mathsf{k}}$ a manifold, we do not have the classical alternative "Θ is locally constant or $\Theta = {}^\bullet\delta(p, -)$", see the above cited theorem. Roughly speaking, to obtain the map $(\alpha \times \beta \times \gamma) \cdot \partial_j$ of our conclusion we have to derive the function δ with respect to the fourth variable x, to compose the result with the mapping $\iota_{k_j} : {}^\bullet\mathbb{R}^{\mathsf{k}} \longrightarrow {}^\bullet\mathbb{R}^k_{=_{k_j}}$ and finally to compose the final result with the figure $\gamma : C \longrightarrow V$. To formalize this reasoning, we start from the open set $\mathcal{U}$ of $\mathbb{R}^{\mathsf{a}} \times \mathbb{R}^{\mathsf{b}} \times \mathbb{R}^d$. Therefore, $p_3(\mathcal{U})$ is open in $\mathbb{R}^d$, where $p_3 : \mathbb{R}^{\mathsf{a}} \times \mathbb{R}^{\mathsf{b}} \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is the projection onto the third space. Hence $p_3(\mathcal{U}) \cap V$

is open in V and $\gamma^{-1}\left[p_3(\mathcal{U})\cap V\right] =: \mathcal{C}$ is open in C because $\gamma : C \longrightarrow V$, being smooth, is continuous. On the other hand, $p_{12}(\mathcal{U})$ is open in $\mathbb{R}^{\mathsf{a}}\times\mathbb{R}^{\mathsf{b}}$, where $p_{12} : \mathbb{R}^{\mathsf{a}}\times\mathbb{R}^{\mathsf{b}}\times\mathbb{R}^{d} \longrightarrow \mathbb{R}^{\mathsf{a}}\times\mathbb{R}^{\mathsf{b}}$ is the projection onto the first two factors. Therefore, $p_{12}(\mathcal{U})\cap(A\times B) =: \mathcal{D}$ is open in $A\times B$ and hence $\mathcal{D}\times\mathcal{C}$ is open in $A\times B\times C$. In this open set we will realize the above mentioned compositions. Indeed, first of all we have that $(a,b,\gamma(c))\in\mathcal{U}$ and hence $(a,b)\in\mathcal{D}=p_{12}(\mathcal{U})\cap(A\times B)$; moreover, $\gamma(c)\in p_3(\mathcal{U})\cap V$ and hence $(a,b,c)\in\mathcal{D}\times\mathcal{C}$. Now, for a generic $(a_1,b_1,c_1)\in\mathcal{D}\times\mathcal{C}$ we have

$$\begin{aligned}
\left[(\alpha\times\beta\times\gamma)\cdot\partial_j\right](a_1,b_1,c_1) &= \partial_j\beta(b_1)\left(\alpha(a_1)\right)_{\gamma(c_1)}\\
&= \partial_j\left(\beta(b_1)\circ\alpha(a_1)\right)_{\gamma(c_1)}\\
&= \partial_j\left(\beta^{\vee}(b_1,-)\circ\alpha^{\vee}(a_1,-)\right)_{\gamma(c_1)}\\
&= \partial_j\left(\Theta(a_1,b_1,-)\right)_{\gamma(c_1)}\\
&= \iota_{k_j}\left[{}^{\bullet}\left(\partial_4\delta\right)(p,a_1,b_1,\gamma(c_1))\right].
\end{aligned}$$

This proves that we can express $(\alpha\times\beta\times\gamma)\cdot\partial_j$ on the open neighborhood $\mathcal{D}\times\mathcal{C}$ of (a,b,c) as a composition of smooth maps (so, here we are using Theorem 12.2.5 about the smoothness of the map ι_{k_j}) and hence the conclusion follows from the sheaf property of the space ${}^{\bullet}\mathbb{R}^{\mathsf{k}}_{=k_j}\in{}^{\bullet}\boldsymbol{\mathcal{C}}^{\infty}$.

∎

Using this result, or the analogous for derivative of smooth functions defined on open sets, we can easily extend the Taylor's formula to vector spaces of smooth functions of the form ${}^{\bullet}\mathbb{R}^{Z}$.

Theorem 12.2.9. *Let Z be a $\boldsymbol{\mathcal{C}}^{\infty}$ space, $\alpha_1,\dots,\alpha_d\in\mathbb{R}_{>0}$, and*

$$f : D_{\alpha_1}\times\cdots\times D_{\alpha_d} \longrightarrow {}^{\bullet}\mathbb{R}^{Z}$$

be a ${}^{\bullet}\boldsymbol{\mathcal{C}}^{\infty}$ map. Define $k_j\in\mathbb{R}$ such that

$$k_{\underline{0}} := 0$$

$$\frac{1}{k_j}+\frac{j}{\alpha+1}=1\quad\forall j\in J:=\left\{j\in\mathbb{N}^d \mid \frac{j}{\alpha+1}<1\right\}:\quad j\neq\underline{0}.$$

Then there exists one and only one family of smooth functions

$$m : J \longrightarrow \sum_{j\in J}{}^{\bullet}\mathbb{R}^{Z}_{=k_j}$$

such that

1. *$m_j(z)\in{}^{\bullet}\mathbb{R}^{\mathsf{k}}_{=k_j}$ for every $j\in\mathbb{N}^d$ such that $\frac{j}{\alpha+1}<1$ and every $z\in Z$*
2. *$f(h)=\sum_{\substack{j\in\mathbb{N}^d\\ \frac{j}{\alpha+1}<1}}\frac{h^j}{j!}\cdot m_j\quad\forall h\in D_{\alpha_1}\times\cdots\times D_{\alpha_d}$ in the vector space ${}^{\bullet}\mathbb{R}^{Z}$.*

Proof: For cartesian closedness, the adjoint of the map f is smooth:

$$f^\vee : Z \times D_{\alpha_1} \times \cdots \times D_{\alpha_d} \longrightarrow {}^\bullet\mathbb{R}.$$

Let us indicate this map, for simplicity, again with $f(-,-)$. Then it suffices to consider the smooth functions

$$m_j : z \in Z \mapsto f(z,-) \in {}^\bullet\mathbb{R}^{D_{\alpha_1}\times\cdots\times D_{\alpha_d}} \mapsto \partial_j f(z,-) \in {}^\bullet\mathbb{R}_{=_{k_j}}$$

to obtain the desired pointwise equality

$$f(z,h) = \sum_{\substack{j\in\mathbb{N}^d \\ \frac{j}{\alpha+1}<1}} \frac{h^j}{j!} \cdot m_j(z) \quad \forall z \in Z$$

from the infinitesimal Taylor's formula (Theorem 12.1.4). The uniqueness part follows from the corresponding uniqueness of the cited theorem.

■

Using analogous ideas, we can also extend Theorem 12.1.9 to functions of the form $f : S \longrightarrow {}^\bullet\mathbb{R}^Z$, where Z is a generic ${}^\bullet\mathcal{C}^\infty$ space; the corresponding statement can be easily obtain simply replacing in Theorem 12.1.9 the space ${}^\bullet\mathbb{R}^n$ with the space ${}^\bullet\mathbb{R}^Z$.

Chapter 13

Infinitesimal differential geometry

The use of nilpotent infinitesimals permits to develop many concepts of differential geometry in an intrinsic way, without being forced to use coordinates. In this way the use of charts becomes specific of suitable areas of differential geometry, e.g. where one strictly needs some solution in a finite neighborhood and not in an infinitesimal one only (e.g. this is the case for the inverse function theorem).

We recall that we named this kind of intrinsic geometry *infinitesimal differential geometry*.

The possibility to avoid coordinates using infinitesimal neighborhoods instead, permits to perform some generalizations to more abstract spaces, like spaces of mappings. Even if the categories $\mathcal{C}^\infty$ and ${}^\bullet\mathcal{C}^\infty$ are very big and not very much can be said about generic objects, in this section we shall see that the best properties can be formulated for a restricted class of extended spaces, the *infinitesimally linear* ones, to which spaces of mappings between manifolds belong to.

All this section takes strong inspiration from the corresponding part of SDG, in the sense that all the statements of theorems and definitions have a strict analogue in SDG. Our reference for proofs not depending on the model presented in this work but substantially identical to those in SDG is Lavendhomme [1996].

13.1 Tangent spaces and vector fields

We start from the fundamental idea of tangent vector. It is natural to define a tangent vector to a space $X \in {}^\bullet\mathcal{C}^\infty$ as an arrow (in ${}^\bullet\mathcal{C}^\infty$) of type $t : D \longrightarrow X$. Therefore $\mathrm{T}X := X^D = {}^\bullet\mathcal{C}^\infty(D, X)$ with projection $\pi : t \in \mathrm{T}X \mapsto t(0) \in X$ is the tangent bundle of X.

We can also define the differential of an application $f : X \longrightarrow Y$ in ${}^\bullet\mathcal{C}^\infty$

simply by composition

$$\mathrm{d}f : t \in \mathrm{T}X = X^D \longrightarrow f \circ t \in \mathrm{T}Y = Y^D.$$

In the following we will also use the notations

$$\mathrm{T}_x X := (\{t \in \mathrm{T}X \mid t(0) = x\} \prec \mathrm{T}M) \in {}^\bullet\mathcal{C}^\infty$$

$$\mathrm{d}f_x : t \in \mathrm{T}_x X \mapsto \mathrm{d}f_x[t] := f \circ t \in \mathrm{T}_{f(x)} Y$$

for the tangent space at the point $x \in X$ and for the differential of the application $f : X \longrightarrow Y$ at the point x.

Note that using the absolute value it is also possible to consider "boundary tangent vectors" taking $|D| := \{\ |h| : h \in D\}$ instead of D, for example at the initial point of a curve or at a point in the boundary of a closed set. In the following, $M \in \mathbf{Man}$ will always be a d-dimensional smooth manifold and we will use the simplified notation $\mathrm{T}M$ for $\mathrm{T}({}^\bullet M)$.
It is important to note that with this definition of tangent vector we obtain a generalization of the classical notion. In fact, in general we have that $t(0) \in {}^\bullet M$ and $\varphi'(t) := \partial_1\varphi(t) \in {}^\bullet\mathbb{R}^d_{=2}$ if φ is a chart on ${}^\circ t(0) \in M$. In other words, a tangent vector $t : D \longrightarrow {}^\bullet M$ can be applied to a non standard point or have a non standard speed. If we want to study classical tangent vectors we have to consider the following $\mathcal{C}^\infty$ object

Definition 13.1.1. *We call* $\mathrm{T}_{\mathrm{st}}M$ *the* $\mathcal{C}^\infty$ *object with support set*

$$|\mathrm{T}_{\mathrm{st}}M| := \{{}^\bullet f|_D \ : \ f \in \mathcal{C}^\infty(\mathbb{R}, M)\},$$

and with figures of type U *(open in* $\mathbb{R}^{\mathsf{u}}$*) given by the substructure induced by* TM*, i.e.*

$$d \in_U \mathrm{T}_{\mathrm{st}}M \quad :\Longleftrightarrow \quad d : U \longrightarrow |\mathrm{T}_{\mathrm{st}}M| \quad \textit{and} \quad {}^\bullet\mathcal{C}^\infty \models d \cdot i \in_{\bar{U}} \mathrm{T}M,$$

where $i : |\mathrm{T}_{\mathrm{st}}M| \hookrightarrow \mathrm{T}M$ *is the inclusion.*

That is in $\mathrm{T}_{\mathrm{st}}M$ we consider only tangent vectors of the form $t = {}^\bullet f|_D$, i.e. obtained as extension of ordinary smooth functions $f : \mathbb{R} \longrightarrow M$, and we take as figures of type $U \subseteq \mathbb{R}^{\mathsf{u}}$ the functions d with values in $\mathrm{T}_{\mathrm{st}}M$ which in the category ${}^\bullet\mathcal{C}^\infty$ verify $d^\vee : \bar{U} \times D \longrightarrow {}^\bullet M$. Note that, intuitively speaking, d takes a standard element $u \in U \subseteq \mathbb{R}^k$ to the standard element $d(u) \in \mathrm{T}_{\mathrm{st}}M$.

Theorem 13.1.2. *Let* $t \in \mathrm{T}M$ *be a tangent vector and* (U, φ) *a chart of* M *on* ${}^\circ t(0)$*. Then*

$$t \in \mathrm{T}_{\mathrm{st}}M \iff t(0) \in M \quad \textit{and} \quad \varphi'(t) \in \mathbb{R}^d.$$

Proof: If $t = {}^\bullet f|_D \in \mathrm{T}_{\mathrm{st}}M$ then $t(0) = f(0) = {}^\circ t(0) \in M$ and $\varphi'(t) = \iota_2\left[(\varphi \circ f)'(0)\right] \in \mathbb{R}^d$ because

$$\mathcal{C}^\infty \vDash V := f^{-1}(U) \xrightarrow{f|_V} U \xrightarrow{\varphi} \mathbb{R}^d$$

and hence $(\varphi \circ f)'(0) \in \mathbb{R}^d$. Vice versa if $t(0) \in M$ and $\varphi'(t) \in \mathbb{R}^d$, then applying the generalized derivation formula (Theorem 12.1.3) we obtain ${}^\bullet\varphi(t(h)) = {}^\bullet\varphi(t(0)) + h \cdot \varphi'(t)$ for any $h \in D$. But ${}^\bullet\varphi(t(0)) = \varphi(t(0))$ because $t(0) \in M$. Hence setting $a := \varphi(t(0)) \in \mathbb{R}^d$ and $b := \varphi'(t) \in \mathbb{R}^d$ we can define

$$f(s) := \varphi^{-1}(a + s \cdot b) \in M \quad \forall s : \ |s| < r,$$

where $r \in \mathbb{R}_{>0}$ has been taken such that $B_{r \cdot \|b\|}(a) \subseteq U$. The standard smooth function $f : (-r, r) \longrightarrow M$ can be defined on the whole of $\mathbb{R}$ in any way that preserves its smoothness.

We have that

$$t(h) = {}^\bullet\varphi^{-1}({}^\bullet\varphi(t(h))) = {}^\bullet\varphi^{-1}(a + h \cdot b) =: {}^\bullet f|_D(h) \quad \forall h \in D,$$

and this proves that $t \in \mathrm{T}_{\mathrm{st}}M$ is a standard tangent vector.

■

In the following result we prove that the definition of standard tangent vector $t \in \mathrm{T}_{\mathrm{st}}M$ is equivalent to the classical one.

Theorem 13.1.3. *In the category $\mathcal{C}^\infty$ the object $\mathrm{T}_{\mathrm{st}}M$ is isomorphic to the usual tangent bundle of M*

Proof: We have to prove that $\mathrm{T}^m_{\mathrm{st}} := \{t \in \mathrm{T}_{\mathrm{st}}M \,|\, t(0) = m\} \simeq \mathrm{T}_m$ where here $\mathrm{T}_m := \{f \in \mathcal{C}^\infty(\mathbb{R}, M) \,|\, f(0) = m\}/\sim$ is the usual tangent space of M at $m \in M$. Note that $\mathrm{T}_m \in \mathcal{C}^\infty$ because of completeness and co-completeness.
Firstly we prove that

$$\begin{array}{rlcl} \alpha : & [f]_\sim \in \mathrm{T}_m & \mapsto & \dfrac{\mathrm{d}(\varphi \circ f)}{\mathrm{d}t}(0) \in \mathbb{R}^d \\ \alpha^{-1} : & v \in \mathbb{R}^d & \mapsto & [r \mapsto \varphi^{-1}(\varphi m + r \cdot v)]_\sim \in \mathrm{T}_m \end{array} \tag{13.1.1}$$

are arrows of $\mathcal{C}^\infty$, where $\varphi : V \longrightarrow \mathbb{R}^d$ is a chart on m with $\varphi(V) = \mathbb{R}^d$.
Secondly we prove that

$$\begin{array}{rlcl} \beta : & t \in \mathrm{T}^m_{\mathrm{st}} & \mapsto & \varphi'(t) \in \mathbb{R}^d \\ \beta^{-1} : & v \in \mathbb{R}^d & \mapsto & {}^\bullet[r \mapsto \varphi^{-1}(\varphi m + r \cdot v)]|_D \in \mathrm{T}^m_{\mathrm{st}} \end{array}$$

are arrows of $\mathcal{C}^\infty$.

To prove the smoothness of $\alpha : \mathrm{T}_m \longrightarrow \mathbb{R}^d$ in $\mathcal{C}^\infty$, let us take a figure $d \in_H \mathrm{T}_m$, where H is open in $\mathbb{R}^{\mathsf{h}}$. For the sheaf property of T_m to prove that $d \cdot \alpha : H \longrightarrow \mathbb{R}^d$ is smooth we can proceed proving that it is locally smooth.

From the definition of figures of the quotient space T_m (see Theorem 6.3.5), for every $h \in H$ there exist an open neighborhood U of h in H and a smooth function $\delta \in \boldsymbol{\mathcal{C}}^\infty(\mathbb{R}, M^{\mathbb{R}})$ such that

$$\forall u \in U : \quad \delta(u)(0) = m$$

$$d|_U = \delta \cdot [-]_\sim,$$

where $[-]_\sim : \{f \in \mathcal{C}^\infty(\mathbb{R}, M) \,|\, f(0) = m\} \longrightarrow \mathrm{T}_m$ is the canonical projection map of the quotient set T_m.

Thus, we have

$$(d \cdot \alpha)\,|_U = d|_U \cdot \alpha : u \in U \mapsto \alpha\,(du) = \alpha\,[\delta u]_\sim = \frac{\mathrm{d}(\varphi \circ \delta u)}{\mathrm{d}t}(0).$$

But

$$\forall r \in \mathbb{R} : \quad (\varphi \circ \delta u)(r) = \varphi\,[\delta(u)(r)] = \varphi\left[\delta^\vee(u, r)\right],$$

and for the cartesian closedness of $\boldsymbol{\mathcal{C}}^\infty$ we have that $\delta^\vee : U \times \mathbb{R} \longrightarrow M$ is smooth. Therefore

$$(d \cdot \alpha)|_U : u \in U \mapsto \mathrm{d}_m\varphi\left[\partial_2\delta^\vee(u, 0)\right] \in \mathbb{R}^d,$$

where $\mathrm{d}_m\varphi$ is the differential of φ at the point $m \in M$. We thus have that $(d \cdot \alpha)|_U = \mathrm{d}_m\varphi\,[\partial_2\delta^\vee(-, 0)] \in \boldsymbol{\mathcal{C}}^\infty(U, \mathbb{R}^d)$ which proves that $d \cdot \alpha$ is locally smooth.

Now, let us consider the inverse α^{-1} defined in (13.1.1). This map is exactly the composition of the adjoint $\tilde{\alpha}^\wedge$ of the smooth map

$$\tilde{\alpha} : (v, r) \in \mathbb{R}^d \times \mathbb{R} \mapsto \varphi^{-1}\,(\varphi m + r \cdot v) \in M$$

with the canonical projection:

$$\begin{tikzcd}
\mathbb{R}^d \arrow[r, "\tilde{\alpha}^\wedge"] \arrow[dr, "\alpha^{-1}"'] & \left\{f \in M^{\mathbb{R}} \,|\, f(0) = m\right\} \arrow[d, "{[-]_\sim}"] \\
& \mathrm{T}_m
\end{tikzcd}$$

and hence it is smooth because of the type of figures we have on a quotient set, see Theorem 6.3.5.

We now prove that $\beta : \mathrm{T}^m_{\mathrm{st}}M \longrightarrow \mathbb{R}^d$ is smooth. If $d \in_U \mathrm{T}^m_{\mathrm{st}}$ is a figure in ${}^\bullet\boldsymbol{\mathcal{C}}^\infty$ of type U, where U is an open set of $\mathbb{R}^{\mathsf{u}}$, then $d^\vee : \bar{U} \times D \longrightarrow {}^\bullet M$ in ${}^\bullet\boldsymbol{\mathcal{C}}^\infty$. But $\bar{U} \times D = \bar{U} \times \bar{D} = \overline{U \times D}$ hence $d^\vee \in_{U \times D} {}^\bullet M$. Thus, for every $u \in U$ we can locally write $d^{\vee}|_{\mathcal{V}} = {}^{\bullet}\gamma(p,-,-)|_{\mathcal{V}}$ where $\mathcal{V} := {}^{\bullet}(A \times B) \cap (U \times D)$ is an open neighborhood of $(u,0)$ defined by $A \times B$ in $U \times D$, $\gamma \in \mathcal{C}^{\infty}(W \times A \times B, M)$ is an standard smooth function, and $p \in {}^{\bullet}W$, where W is open in ${}^{\bullet}\mathbb{R}^{\mathsf{p}}$. But

$$\mathcal{V} = {}^{\bullet}(A \times B) \cap (U \times D) = ({}^{\bullet}A \cap U) \times ({}^{\bullet}B \cap D) = (A \cap U) \times D$$

because $U \subseteq \mathbb{R}^{\mathsf{u}}$. Now, we have

$$\begin{aligned}\beta[d(x)] &= \varphi'[d(x)] \\ &= \varphi'\left[d^{\vee}(x,-)\right] \\ &= \iota_2 \left\{ \frac{\mathrm{d}}{\mathrm{d}r}\{\varphi[\gamma(p,x,r)]\}|_{r=0} \right\} \\ &= \iota_2 \{ \mathrm{d}_m\varphi\left[\partial_3\gamma(p,x,0)\right]\} \quad \forall x \in A \cap U.\end{aligned}$$

But ${}^{\circ}\{\beta[d(x)]\} = \beta[d(x)]$ because $\beta : \mathrm{T}_{\mathrm{st}} \longrightarrow \mathbb{R}^d$ and hence

$$\begin{aligned}\beta[d(x)] &= {}^{\circ}\left[\iota_2 \{ \mathrm{d}_m\varphi\left[\partial_3\gamma(p,x,0)\right]\}\right] \\ &= {}^{\circ}\left[\mathrm{d}_m\varphi\left[\partial_3\gamma(p,x,0)\right]\right] \\ &= \mathrm{d}_m\varphi\left[\partial_3\gamma(p_0,x,0)\right] \quad \forall x \in A \cap U, \end{aligned} \tag{13.1.2}$$

so that $(d \cdot \beta)|_{A \cap U} = \mathrm{d}_m\varphi\left[\partial_3\gamma(p_0,-,0)\right] \in \mathcal{C}^{\infty}(A \cap U, \mathbb{R}^d)$ is an ordinary smooth function. Note the importance to have as U a standard open set in the last passage of (13.1.2), and this represents a further strong motivation for the definition we gave for $\mathrm{T}_{\mathrm{st}}M$.

To prove the regularity of β^{-1} we consider the map

$$\tilde{\beta} : v \in \mathbb{R}^d \mapsto \left[r \in \mathbb{R} \mapsto \varphi^{-1}(\varphi m + r \cdot v) \in M\right] \in \mathcal{C}^{\infty}(\mathbb{R}, M).$$

Then we have

$$\begin{gathered}\mathcal{C}^{\infty} \models \mathbb{R}^d \xrightarrow{\tilde{\beta}} M^{\mathbb{R}} \\ \mathcal{C}^{\infty} \models \mathbb{R}^d \times \mathbb{R} \xrightarrow{\tilde{\beta}^{\vee}} M \\ {}^{\bullet}\mathcal{C}^{\infty} \models {}^{\bullet}\mathbb{R}^d \times {}^{\bullet}\mathbb{R} \xrightarrow{{}^{\bullet}\tilde{\beta}^{\vee}} {}^{\bullet}M \\ {}^{\bullet}\mathcal{C}^{\infty} \models \mathbb{R}^d \times D \xrightarrow{\bar{\beta}} {}^{\bullet}M \quad \text{where} \quad \bar{\beta} := {}^{\bullet}\left(\tilde{\beta}^{\vee}\right)|_{\mathbb{R}\times D} \\ {}^{\bullet}\mathcal{C}^{\infty} \models \mathbb{R}^d \xrightarrow{\bar{\beta}^{\wedge}} {}^{\bullet}M^D.\end{gathered}$$

This map is actually β^{-1}, in fact

$$\begin{aligned}\forall v \in \mathbb{R}^d : \quad \bar{\beta}^{\wedge}(v) : h \in D \mapsto \bar{\beta}(v,h) &= {}^{\bullet}\left(\tilde{\beta}^{\vee}\right)(v,h) \\ &= \left(\tilde{\beta}^{\vee}(v,h_t)\right)_{t\geq 0} \\ &= \left(\tilde{\beta}(v)(h_t)\right)_{t\geq 0} \\ &= \beta^{-1}(v)(h).\end{aligned}$$

Therefore $\beta^{-1} : \mathbb{R}^d \longrightarrow \mathrm{T}M$ is smooth in ${}^\bullet\mathcal{C}^\infty$. But, finally, β^{-1} is actually with values in $\mathrm{T}^m_{\mathrm{st}}$ because $\beta^{-1}(v)(0) = m$ for every $v \in \mathbb{R}^d$, so that $\beta^{-1} = \beta^{-1} \cdot i \in_{\bar{\mathbb{R}}^d} \mathrm{T}M$ where $i : \mathrm{T}^m_{\mathrm{st}} \hookrightarrow \mathrm{T}M$ is the inclusion. We have thus prove that β^{-1} is a figure of type $\mathbb{R}^d$ of the $\mathcal{C}^\infty$ space $\mathrm{T}^m_{\mathrm{st}}$ and hence it is also smooth in this category, which is the conclusion.

■

For any object $X \in {}^\bullet\mathcal{C}^\infty$ the multiplication of a tangent vector t by a scalar $r \in {}^\bullet\mathbb{R}$ can be defined simply "increasing its speed" by a factor r:

$$(r \cdot t)(h) := t(r \cdot h) \quad \forall h \in D. \tag{13.1.3}$$

But, as we have already noted, in the category ${}^\bullet\mathcal{C}^\infty$ we have spaces with singular points too, like algebraic curves with double points. For this reason, we cannot define the sum of tangent vectors for every smooth space $X \in {}^\bullet\mathcal{C}^\infty$, but we need to introduce a class of objects in which this operation is possible.
The following definition simply states that in these spaces there always exists the infinitesimal parallelogram generated by a finite number of given vectors at the same point m.

Definition 13.1.4. *Let $X \in {}^\bullet\mathcal{C}^\infty$, then we say that X is* infinitesimally linear*, or simply* inf-linear*, at the point $m \in X$ if and only if the following conditions are fulfilled*

1. *for any $k \in \mathbb{N}_{>1}$ and for any $t_1, \dots, t_k \in \mathrm{T}_m X$, there exists one and only one $p : D^k \longrightarrow X$ in ${}^\bullet\mathcal{C}^\infty$ such that*

$$\forall i = 1, \dots, k : \quad p(0, \overset{i-1}{\dots\dots}, 0, h, 0, \dots, 0) = t_i(h) \quad \forall h \in D.$$

   We will call the map p the infinitesimal parallelogram generated by *$t_1, \dots, t_k$.*

2. *The application*

$$(-) +_m \dots +_m (-) : (t_1, \dots, t_k) \in (T_m X)^k \mapsto p \in X^{D^k}$$

   that associates to the k tangent vectors at $m \in X$ the infinitesimal parallelogram p, is ${}^\bullet\mathcal{C}^\infty$-smooth.

Moreover, we will simply say that X is inf-linear *if it is inf-linear at each point $m \in X$ and if the application*

$$m \in X \mapsto (-) +_m \dots +_m (-) \in \sum_{m \in X} \left(X^{D^k}\right)^{(T_m X)^k}$$

is ${}^\bullet\mathcal{C}^\infty$-smooth.

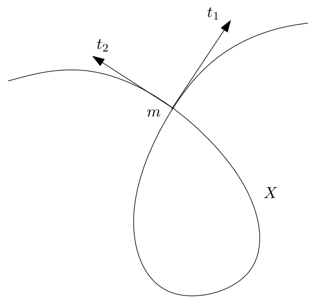

Figure 13.1: An example of space which is not inf-linear at $m \in X$.

The following theorem gives meaningful examples of inf-linear objects.

Theorem 13.1.5. *The extension of any manifold ${}^\bullet M$ is inf-linear at every point $m \in {}^\bullet M$. If $M_i \in$ **Man** for $i = 1, \ldots, s$ then the exponential object*

$$ {}^\bullet M_1^{{}^\bullet M_2^{\cdot^{\cdot^{{}^\bullet M_s}}}} \simeq {}^\bullet M_1^{{}^\bullet (M_2 \times \cdots \times M_s)} \tag{13.1.4}$$

is also inf-linear at every point.

The importance of the isomorphism (13.1.4) lies in the fact that complex spaces like

$$ {}^\bullet M_1^{{}^\bullet M_2^{\cdot^{\cdot^{{}^\bullet M_s}}}} = {}^\bullet \mathcal{C}^\infty({}^\bullet M_s, {}^\bullet \mathcal{C}^\infty({}^\bullet M_{s-1}, \cdots, {}^\bullet \mathcal{C}^\infty({}^\bullet M_2, {}^\bullet M_1) \cdots) $$

are now no more difficult to handle than classical spaces of mappings like ${}^\bullet M^{{}^\bullet N} = {}^\bullet \mathcal{C}^\infty({}^\bullet N, {}^\bullet M)$. Let us note explicitly that this isomorphism is a consequence of cartesian closedness and of the preservation of products of manifolds of the Fermat functor.

Proof: Given any chart (U, φ) on ${}^\circ m \in M$ we can define the infinitesimal parallelogram p as

$$ p(h_1, \ldots, h_k) := {}^\bullet \varphi^{-1} \left({}^\bullet \varphi(m) + \sum_{i=1}^k h_i \cdot \varphi'(t_i) \right) \quad \forall h_1, \ldots, h_k \in D. \tag{13.1.5}$$

If fact if $\tau(h) := p(0, \overset{i-1}{\ldots\ldots}, 0, h, 0, \ldots, 0)$ then $\varphi(\tau(h)) = \varphi(m) + h \cdot \varphi'(t_i)$ for every $h \in D$; this implies that $t(0) = \tau(0)$ and $\varphi'(\tau) = \varphi'(t_i)$, hence $t_i = \tau$. To prove the uniqueness of the parallelogram generated by $t_1, \ldots, t_k \in \mathrm{T}_m M$, let us consider that if $p : D^k \longrightarrow {}^\bullet M$ is such that $p(0, \overset{i-1}{\ldots\ldots}, 0, h, 0, \ldots, 0) = t_i(h)$ for every tangent vector t_i and every $h \in D$, then

$$ \varphi[p(0, \overset{i-1}{\ldots\ldots}, 0, h, 0, \ldots, 0)] = \varphi[t_i(h)] = \varphi(m) + h \cdot \varphi'(t_i) $$

and so

$$ \varphi[p(h_1, \ldots, p_k)] = \varphi(m) + \sum_{i=1}^k h_i \cdot \varphi'(t_i) $$

from the first order infinitesimal Taylor's formula, so that we obtain again the definition (13.1.5), and this proves the uniqueness part.

Considering the exponential object, because

$$ {}^\bullet M_1^{{}^\bullet M_2^{\cdot^{\cdot^{{}^\bullet M_s}}}} \simeq {}^\bullet M_1^{{}^\bullet M_2 \times \cdots \times {}^\bullet M_s} \simeq {}^\bullet M_1^{{}^\bullet (M_2 \times \cdots \times M_s)}, $$

it suffices to prove the conclusion for $s = 2$. First of all we note that, because of the previously proved uniqueness, the definition 13.1.5 of the infinitesimal parallelogram does not depend on the chart φ on ${}^\circ m$. Now let $t_1, \dots, t_k$ be k tangent vectors at $f \in {}^\bullet N^{{}^\bullet M}$. We shall define their parallelogram[1] $p : {}^\bullet M \longrightarrow {}^\bullet N^{D^k}$ patching together smooth functions defined on open subsets, and using the sheaf property of ${}^\bullet N^{D^k}$. Indeed, for every $m \in {}^\bullet M$ we can find a chart (U_m, φ_m) of N on ${}^\circ f(m)$ with $\varphi_m(U_m) = \mathbb{R}^n$. Now $m \in V_m := f^{-1}({}^\bullet U_m) \in \tau_{{}^\bullet M}$ and for every $x \in V_m$ we have $t_i^\vee(0, x) = f(x) \in {}^\bullet U_m$. Hence $t_i^\vee(h, x) \in {}^\bullet U_m$ for any $h \in D$ by Theorem 12.1.2. Therefore we can define

$$ p_m^\vee(x, h) := \varphi_m^{-1} \left\{ \sum_{i=1}^{k} \varphi_m[t_i^\vee(h^i, x)] - (k-1) \cdot \varphi_m(fx) \right\} \quad \forall x \in V_m, \forall h \in D^k \tag{13.1.6} $$

and we have that $p_m^\vee : (V_m \prec {}^\bullet M) \times D^k \longrightarrow {}^\bullet N$ is smooth, because it is a composition of smooth functions. We claim that if $x \in V_m \cap V_{m'}$ then $p_m^\vee(x, -) = p_{m'}^\vee(x, -)$, in fact from the generalized Taylor's formula we have $\varphi_m[t_i^\vee(h^i, x)] = \varphi_m(fx) + h^i \cdot \varphi'_m[t_i^\vee(-, x)]$ and hence substituting in (13.1.6) we can write

$$ p_m^\vee(x, h) = \varphi_m^{-1} \left\{ k\varphi_m(fx) + \sum_{i=1}^{k} h^i \cdot \varphi'_m[t_i^\vee(-, x)] - k\varphi_m(fx) + \varphi_m(fx) \right\} \tag{13.1.7} $$

$$ = \varphi_m^{-1} \left\{ \varphi_m(fx) + \sum_{i=1}^{k} h^i \cdot \varphi'_m[t_i^\vee(-, x)] \right\} \quad \forall x \in V_m, \forall h \in D^k. $$

But (U_m, φ_m) is a chart on ${}^\circ f(x)$, so $p_m^\vee(x, -)$ is the infinitesimal parallelogram generated by the tangent vectors $t_i^\vee(-, x)$ at $f(x)$, and we know that (13.1.7) does not depend on φ_m, so $p_m = p_{m'}$. For the sheaf property of ${}^\bullet N^{D^k}$ we thus have the existence of a smooth $p : {}^\bullet M \longrightarrow {}^\bullet N^{D^k}$ such that

$$ \forall m \in {}^\bullet M : \quad p|_{V_m} = p_m. $$

From this and from (13.1.7) it is also easy to prove that $p : D^k \longrightarrow {}^\bullet N^{{}^\bullet M}$ verifies the desired properties. Uniqueness follows noting that $p^\vee(m, -)$ is the infinitesimal parallelogram generated by $t_i^\vee(-, m)$. From (13.1.6) it also

[1] For simplicity, in this proof we will use implicitly the cartesian closedness property.

follow easily that the map $(m, t_1, \dots, t_k) \mapsto p$ is smooth because it is given by the composition of smooth maps.

■

Another important family of inf-linear spaces is given by the following

Theorem 13.1.6. *Let X be an inf-linear space and $Z \in {}^\bullet\mathcal{C}^\infty$ be another smooth space. Then the space*

$$X^Z$$

is inf-linear.

Proof: Let $t_1, \dots, t_k : D \longrightarrow X^Z$ be $k \in \mathbb{N}_{>1}$ tangent vectors at the point $m \in X^Z$. Because of cartesian closedness the adjoint maps $t_i^\vee : Z \times D \longrightarrow X$ are smooth; we will simply denote them with the initial symbol t_i again. Finally, for every $z \in Z$, because X is inf-linear, we know that the map

$$(-) +_{m(z)} \dots +_{m(z)} (-) : \left(\mathrm{T}_{m(z)}X\right)^k \longrightarrow X^{D^k}$$

is smooth in $z \in Z$ because it is composed by smooth functions.

Then the adjoint of the map:

$$p(z, h_1, \dots, h_k) := \left[t_1(z, -) +_{m(z)} \dots +_{m(z)} t_k(z, -)\right](h_1, \dots, h_k)$$

verifies the desired properties.

■

If X is inf-linear at $x \in X$ then we can define the sum of tangent vectors $t_1, t_2 \in \mathrm{T}_x X$ simply taking the diagonal of the parallelogram p generated by these vectors

$$(t_1 + t_2)(h) := p(h, h) \quad \forall h \in D. \tag{13.1.8}$$

With these operations $\mathrm{T}_x X$ becomes a ${}^\bullet\mathbb{R}$-module:

Theorem 13.1.7. *If X is inf-linear at the point $x \in X$, then with respect to the sum defined in (13.1.8) and the product by scalar defined by (13.1.3), the tangent space $T_x X$ is a ${}^\bullet\mathbb{R}$-module.*

Proof: We only prove that the sum is associative. Analogously, one can prove the other axioms of module, . Let us consider the tangent vectors t_1, t_2, $t_3 \in \mathrm{T}_x X$, and denote by p_{12} the infinitesimal parallelogram generated by t_1 and t_2, by $p_{12,3}$ the parallelogram generated by $t_1 + t_2$ and by t_3, and analogously for the symbols p_{23} and $p_{1,23}$. Then $p_{12,3}$ is characterized by the properties

$$p_{12,3} : D^2 \longrightarrow X$$

$$p_{12,3}(h, 0) = (t_1 + t_2)(h) = p_{12}(h, h) \quad \forall h \in D$$

$$p_{12,3}(0, h) = t_3(h) \quad \forall h \in D.$$

Now, let $l : D^3 \longrightarrow X$ be the parallelogram generated by all the three vectors. Then the map $l(-,-,0)$ verifies

$$l(-,-,0) : D^2 \longrightarrow X$$

$$l(h,0,0) = t_1(h) \quad \text{and} \quad l(0,h,0) = t_2(h) \quad \forall h \in D,$$

so $l(-,-,0) = p_{12}$. Now let us consider the application

$$\lambda : (h,k) \in D^2 \mapsto l(h,h,k) \in X.$$

It is smooth as a composition of smooth maps and verifies

$$\lambda(h,0) = l(h,h,0) = p_{12}(h,h) = (t_1 + t_2)(h) \quad \forall h \in D$$

$$\lambda(0,k) = l(0,0,k) = t_3(k) \quad \forall k \in D.$$

Therefore, $p_{12,3} = \lambda$ and $((t_1 + t_2) + t_3)(h) = p_{12,3}(h,h) = \lambda(h,h) = l(h,h,h)$. Analogously we can prove that $(t_1 + (t_2 + t_3))(h) = l(h,h,h)$, that is, we get the conclusion.

■

It is now quite easy to prove that the differential at a point is linear

Theorem 13.1.8. *If $f : X \longrightarrow Y$ is ${}^\bullet\mathcal{C}^\infty$ smooth, the space X is inf-linear at the point $x \in X$, and the space Y is inf-linear at the point $f(x) \in Y$, then the differential*

$$\mathrm{d}f_x : T_x X \longrightarrow T_{f(x)} Y$$

is linear.

Proof: Let $r \in {}^\bullet\mathbb{R}$ and $t \in \mathrm{T}_x X$, we first prove homogeneity

$$\mathrm{d}f_x[r \cdot t](h) = f\left((r \cdot t)(h)\right) = f\left(t(r \cdot h)\right) \quad \forall h \in D.$$

On the other hand

$$\left(r \cdot \mathrm{d}f_x[t]\right)(h) = \mathrm{d}f_x(r \cdot h) = f\left(t(r \cdot h)\right) \quad \forall h \in D,$$

and therefore $\mathrm{d}f_x[r \cdot t] = r \cdot \mathrm{d}f_x[t]$.

To prove additivity, let p be the infinitesimal parallelogram generated by t_1, $t_2 \in \mathrm{T}_x X$ and l the parallelogram generated by $\mathrm{d}f_x[t_1]$, $\mathrm{d}f_x[t_2] \in \mathrm{T}_{fx} Y$. We have

$$\mathrm{d}f_x[t_1 + t_2](h) = f\left((t_1 + t_2)(h)\right) = f\left(p(h,h)\right) \quad \forall h \in D. \tag{13.1.9}$$

On the other hand, we obviously have

$$\left(\mathrm{d}f_x[t_1] + \mathrm{d}f_x[t_2]\right)(h) = l(h,h) \quad \forall h \in D. \tag{13.1.10}$$

But the smooth map

$$f\left(p(-,-)\right) : D^2 \longrightarrow X$$

verifies

$$\begin{aligned} f\left(p(h,0)\right) &= f\left(t_1(h)\right) = \mathrm{d}f_x[t_1](h) \\ f\left(p(0,h)\right) &= f\left(t_2(h)\right) = \mathrm{d}f_x[t_2](h), \end{aligned}$$

and therefore $l = f\left(p(-,-)\right)$. From this and (13.1.9), (13.1.10) we get the conclusion.

■

In the case $X = {}^\bullet\mathbb{R}^d$ and $Y = {}^\bullet\mathbb{R}^n$ we have

$$\begin{aligned} \mathrm{d}f_x[t](h) &= f\left(t(h)\right) \\ &= f\big(t(0) + h \cdot t'(0)\big) \\ &= f\left(t(0)\right) + h \cdot t'(0) \cdot f'\left(t(0)\right) \\ &= f(x) + h \cdot t'(0) \cdot f'(x). \end{aligned}$$

The differential $\mathrm{d}f_x[t] \in \mathrm{T}_{f(x)}Y$ is thus uniquely determined by the linear function $h \in D \mapsto h \cdot t'(0) \cdot f'(x)$ and hence it is uniquely determined by the vector of Fermat reals $\iota_2\left[f'(x)\right] \in {}^\bullet\mathbb{R}^n_2$, as expected.

Vector fields on a generic object $X \in {}^\bullet\mathcal{C}^n$ are naturally defined as ${}^\bullet\mathcal{C}^\infty$ maps of the form

$$V : X \longrightarrow \mathrm{T}X \quad \text{such that} \quad V(x)(0) = x \quad \forall x \in X.$$

In the case of manifolds, $X = {}^\bullet M$, this implies that $V(m)(0) \in M$ for every $m \in M$, we therefore introduce the following condition to characterize the standard vector fields:

Definition 13.1.9. *If $X \in {}^\bullet\mathcal{C}^\infty$ is a Fermat space and $t \in T_xX$ is a tangent vector at $x \in X$, then we say that t* has standard speed *if and only if for every observable $\varphi \in^{UK} X$ with $x \in U$ and $K \subseteq {}^\bullet\mathbb{R}^{\mathsf{k}}$ we have*

$$(\varphi \circ t)'(0) \in \mathbb{R}^{\mathsf{k}}. \tag{13.1.11}$$

As usual, if $X = {}^\bullet M$ is a manifold, this condition is equivalent to saying that there exists a chart on the point ${}^\circ x \in M$ such that condition (13.1.11) holds. Using Theorem (13.1.2) we have the following equivalence:

$$\forall m \in M : \quad V(m) \text{ has standard speed} \tag{13.1.12}$$

if and only if

$$V|_M : (M \prec {}^\bullet M) \longrightarrow (\{{}^\bullet f|_D : f \in \mathcal{C}^n(\mathbb{R}, M)\} \prec \mathrm{T}M).$$

From this, using the definition of arrow in $\mathcal{C}^\infty$ and the embedding Theorem 9.3.1, it follows that (13.1.12) holds if and only if

$$V|_M : M \longrightarrow \mathrm{T}_{\mathrm{st}}(M) \text{ in } \mathcal{C}^n,$$

that is we obtain the standard notion of vector field on M because of Theorem 13.1.2.

Vice versa if we have

$$W : M \longrightarrow \mathrm{T}_{\mathrm{st}}(M) \text{ in } \boldsymbol{\mathcal{C}}^n$$

then we can extend it to ${}^\bullet M$ obtaining a vector field verifying condition (13.1.12). In fact, for fixed $m \in {}^\bullet M$ and $h \in D$ we can choose a chart (U, x) on ${}^\circ m$ and we can write

$$W|_U = \sum_{i=1}^{d} A_i \cdot \frac{\partial}{\partial x_i},$$

with $A_i \in \boldsymbol{\mathcal{C}}^\infty(U, \mathbb{R})$. But $m \in {}^\bullet U$ because ${}^\circ m \in U$ and hence we can define

$$\tilde{W}(m, h) := \sum_{i=1}^{d} {}^\bullet A_i(m) \cdot \frac{\partial}{\partial x_i}(m)(h) \quad \forall h \in D.$$

This definition does not depend on the chart (U, x) and, because of the sheaf property of ${}^\bullet M$ it provides a ${}^\bullet\boldsymbol{\mathcal{C}}^\infty$ function

$$\tilde{W} : {}^\bullet M \times D \longrightarrow {}^\bullet M \quad \text{such that} \quad \tilde{W}(m, 0) = m$$

and with $(\tilde{W}^\wedge)|_M = V$, that is verifying condition (13.1.12) of standard speed.

Finally we can easily see that any vector field can be identified equivalently with an infinitesimal transformation of the space into itself. In fact, using cartesian closedness we have

$$V \in (X^D)^X \simeq X^{X \times D} \simeq X^{D \times X} \simeq (X^X)^D.$$

If W corresponds to V in this isomorphism then $W : D \longrightarrow X^X$ and $V(x)(0) = x$ is equivalent to say that $W(0) = 1_X$, that is W is the tangent vector at 1_X to the space of transformations X^X, that is an infinitesimal path traced from 1_X.

13.2 Infinitesimal integral curves

To the notion of vector field there is naturally associated the notion of integral curve. In our context we are interested to define this concept in infinitesimal terms, i.e. for curves defined on an infinitesimal set.

Definition 13.2.1. *Let $X \in {}^\bullet\boldsymbol{\mathcal{C}}^\infty$ be a smooth space, $V : X \longrightarrow TX$ a vector field on X and $x \in X$ a point in it. Then we say that γ* is the (inf-)integral curve of V at x *if and only if*

1. *$\gamma : D_\infty \longrightarrow X$ is smooth*

2. *$\gamma(0) = x$*

3. *$\gamma(t+h) = V\left[\gamma(t)\right](h)$ for every $t \in D_\infty$ and every $h \in D$.*

Moreover, we say that the vector field V is inf-complete *if and only if*

1. *$\forall x \in X\, \exists!\, \gamma_x \in X^{D_\infty} :\ \gamma_x$ is the integral curve of V at x*

2. *The map associating to each point $x \in X$ the corresponding integral curve γ_x*

$$x \in X \mapsto \gamma_x \in X^{D_\infty}$$

   is smooth.

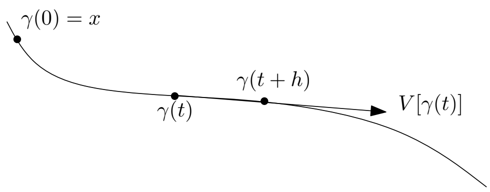

Figure 13.2: Explanation of the definition of integral curve

Let us note explicitly the methodological analogy among the Definition 13.1.4 of inf-linear space and the previous definition of inf-complete vector field. These definitions are indeed divided into two parts: in the first one we have that the predicate we are defining depends on some parameter (the point m in the definition of inf-linear at m and the point x in the definition of integral curve). To each value of this parameter there corresponds a unique smooth function defined on an infinitesimal object (D^k in the definition of inf-linearity and D_∞ in the definition of integral curve). Let us note that this uniqueness is possible only because the object is defined on an infinitesimal space. In the second part of the definition we extend the predicate to a global object (the whole space X in the definition of inf-linearity and the vector field in the definition of inf-completeness) universally quantifying over these parameters, that is requiring that the first predicate holds for every possible value of the parameters. To every universal quantification there corresponds a further smoothness condition about the function that to each parameter assigns the corresponding unique infinitesimal function. The same requirement has been used in the definition of inf-linearity at a given point m, where the universal quantification is over every k-tuples of tangent vectors. This method, which in some sense is implicit in SDG where every function defined in intuitionistic logic is smooth, can be used to transpose several definitions of SDG to our infinitesimal differential geometry.

First of all, we have to prove that the notion of inf-integral curve generalizes, in some way, the classical notion. For simplicity let $X = {}^\bullet\mathbb{R}^d$, the same reasoning can be applied to the case of more general manifolds, because all the notions we are using are local. So, let $V : {}^\bullet\mathbb{R}^d \longrightarrow \left({}^\bullet\mathbb{R}^d\right)^D$ be a standard vector field, then from what we have just seen above, we know that we can find a smooth function

$$\bar{V} : {}^\bullet\mathbb{R}^d \longrightarrow {}^\bullet\mathbb{R}^d$$

such that

$$V(x)(h) = x + h \cdot \bar{V}(x) \quad \forall x \in {}^\bullet\mathbb{R}^d \; \forall h \in D.$$

Therefore, if $\gamma : D_\infty \longrightarrow {}^\bullet\mathbb{R}^d$ is an integral curve of V at $x \in {}^\bullet\mathbb{R}^d$, by the Definition 13.2.1 we get

$$\gamma(t+h) = V\left[\gamma(t)\right](h)$$

$$\gamma(t) + h \cdot \gamma'(t) = \gamma(t) + h \cdot \bar{V}\left[\gamma(t)\right]$$

$$h \cdot \gamma'(t) = h \cdot \bar{V}\left[\gamma(t)\right] \quad \forall t \in D_\infty \; \forall h \in D.$$

This implies

$$\gamma'(t) =_2 \bar{V}\left[\gamma(t)\right] \quad \forall t \in D_\infty.$$

So we have the classical notion of integral curve up to second order infinitesimals. Now, let $\eta : (-\delta, +\delta)_\mathbb{R} \longrightarrow \mathbb{R}^d$, $\delta \in \mathbb{R}_{>0}$, be a standard integral curve of V', i.e.

$$\eta'(t) = \bar{V}\left[\eta(t)\right] \quad \forall t \in (-\delta, +\delta)_\mathbb{R}. \tag{13.2.1}$$

Then extending η to $(-\delta, +\delta) \subseteq {}^\bullet\mathbb{R}$ and using the elementary transfer theorem (Theorem 2.8.2) we obtain that the equality (13.2.1) holds also for every $t \in (-\delta, +\delta)$, and hence it holds also in D_∞:

$$\eta'(t) = \bar{V}\left[\eta(t)\right] \quad \forall t \in D_\infty,$$

and thus $\eta|_{D_\infty} : D_\infty \longrightarrow {}^\bullet\mathbb{R}^d$ is an inf-integral curve of the vector field V. Any two of these standard integral curves, let us say η_1 and η_2, agree in some neighborhood $\mathcal{U}$ of $t = 0$ if $\eta_1(0) = \eta_2(0)$. Therefore, the corresponding inf-integral curves coincide on the whole $D_\infty \subseteq \mathcal{U}$:

$$\eta_1|_{D_\infty} = \eta_2|_{D_\infty}.$$

For this reason in Definition 13.2.1, we say that γ is *the* inf-integral curve of V at the point x.

The next step is to prove that spaces of mappings between manifolds always verify the just introduced definition.

Theorem 13.2.2. *Every vector field V in spaces of the form $X = {}^\bullet M$ or $X = {}^\bullet M^{{}^\bullet N}$, where M and N are manifolds and where N admits partitions of unity, is inf-complete.*

Proof: The first part of the statement, i.e. the case $X = {}^\bullet M$, is really a particular case of the second one where one takes as $N = \{*\}$ any 0-dimensional manifold. So, let us prove only the second part of the statement. Moreover, to simplify the notations, we will simply use the symbols M and N to indicate the extensions ${}^\bullet M$ and ${}^\bullet N$.

Our vector field is a smooth map of the form

$$V : M^N \longrightarrow \left(M^N\right)^D.$$

Moreover, let us consider a point $\mu \in M^N$. We have to prove that there exists one and only one inf-integral curve $\gamma : D_\infty \longrightarrow M^N$ passing from μ at $t = 0$. We will construct γ using the sheaf property of the space N.

Because N admits partitions of unity, we can consider a standard open cover $(U_n)_{n\in N}$ of N such that the closure $\bar{U}_n =: K_n$ is compact and such that the partition of unity $(\rho_n)_{n\in N}$ is subordinate to the open cover $(U_n)_{n\in N}$. From cartesian closedness we can think of V as a map of the form

$$V : M^N \times D \times N \longrightarrow M.$$

Using the partition of unity $(\rho_n)_{n\in N}$ every smooth map $f : K_n \longrightarrow M$ can be extended to a smooth map defined on the whole N. Moreover, this extension, which essentially is the multiplication by a cut-off function, can be defined as a smooth application

$$\chi_n : M^{K_n} \longrightarrow M^N \quad \forall n \in N$$

such that

$$\chi_n(f)\,|_{K_n} = f \quad \forall f \in M^{K_n}\ \forall n \in N.$$

For each $n \in N$ we can hence define

$$W_n : (f, h, x) \in M^{K_n} \times D \times K_n \mapsto V\left[\chi_n(f), h, x\right] \in M$$

obtaining a family $(W_n)_{n\in N}$ of smooth functions.

From cartesian closedness, these functions can be thought as

$$W_n : M^{K_n} \times D \longrightarrow M^{K_n}.$$

But here M^{K_n} is a Banach manifold because K_n is compact, and hence we can apply the standard local existence of integral curves for the vector field W_n in Banach spaces obtaining the existence of a smooth map $\gamma_n : D_\infty \longrightarrow M^{K_n}$ such that

$$\begin{cases} \gamma_n(t+h) = W_n\left[\gamma_n(t), h\right] & \forall t \in D_\infty\ \forall h \in D \\ \gamma_n(0) = \mu|_{K_n} \end{cases} \tag{13.2.2}$$

It is not hard to prove that γ_n and γ_m agree on $K_n \cap K_m$ because they verify the same initial value problem.

From the sheaf property of the space N there exist one and only one smooth function $\gamma : D_\infty \times N \longrightarrow M$ such that

$$\gamma|_{D_\infty \times U_n} = \gamma_n|_{U_n} \quad \forall n \in N.$$

From (13.2.2) it hence follows that γ is the integral curve of V at μ we searched for. Condition *2*. of the definition of inf-completeness of V follows from the classical theorem of smooth dependence from the initial conditions (see e.g. Abraham et al. [1988]).

■

13.3 Ideas for the calculus of variations

In this section we want to show the flexibility of our theory proving a very general form of the Euler-Lagrange equation. Even if the result holds for lagrangians defined on very general spaces, the proof uses infinitesimal methods and, when specified in the space ${}^\bullet\mathbb{R}$, is essentially identical to the one sometimes presented in classical courses of physics using informal infinitesimal argumentations.

We start with the notion of minimum of a functional

Definition 13.3.1. *Let $Y \in {}^\bullet\mathcal{C}^\infty$ be a Fermat space, $\mu \in Y$ a point in it, and $J : Y \longrightarrow {}^\bullet\mathbb{R}$ a ${}^\bullet\mathcal{C}^\infty$ function.*

Then we say that J has a minimum at μ *if and only if*

$$\forall \tau \in T_\mu Y : \quad J[\tau(h)] \geq J(\mu) \quad \forall h \in D.$$

In other words, the value $J(\mu)$ has to be minimum along every tangent vector of Y starting from μ.

The first positive characteristic of our approach is that in this definition of minimum we have used tangent vectors $\tau : D \longrightarrow Y$ at $\mu \in Y$ instead of some notion of neighborhood of $\mu \in Y$ (like in the classical approach, see e.g. Gelfand and Fomin [1963]).

The total order of ${}^\bullet\mathbb{R}$ seems essential in the proof of the following

Theorem 13.3.2. *Let $Y \in {}^\bullet\mathcal{C}^\infty$ be a Fermat space, $\mu \in Y$ a point in it, and $J : Y \longrightarrow {}^\bullet\mathbb{R}$ a ${}^\bullet\mathcal{C}^\infty$ function. Moreover, let us suppose that*

$$J \text{ has a minimum at } \mu.$$

Then

$$\forall \tau \in T_\mu Y : \quad \mathrm{d}J_\mu[\tau] = \underline{0} \tag{13.3.1}$$

Proof: Firstly, let us note that

$$D \xrightarrow{\ \tau\ } Y \xrightarrow{\ J\ } {}^\bullet\mathbb{R} \ ,$$

so that $J\circ\tau$ is smooth and we can apply the Taylor's formula (Theorem 12.1.3)

$$\forall h\in D:\quad \mathrm{d}J_\mu[\tau]=J\left[\tau(h)\right]=J\left[\tau(0)\right]+h\cdot(J\circ\tau)'(0)=J(\mu)+h\cdot(J\circ\tau)'(0), \tag{13.3.2}$$

where $(J\circ\tau)'(0)\in{}^\bullet\mathbb{R}_2$ so that its order verifies

$$\omega\left[(J\circ\tau)'(0)\right]=:b>2.$$

If $(J\circ\tau)'(0)>0$, then we could set $a:=\frac{4b}{3b-2}$ and $h:=-\mathrm{d}t_a$. It is easy to check that $1\le a<2$ because $b>2$, so $h\in D_{<0}$ and we have $h\cdot(J\circ\tau)'(0)\le 0$. But

$$\frac{1}{\omega(h)}+\frac{1}{\omega\left[(J\circ\tau)'(0)\right]}=\frac{1}{a}+\frac{1}{b}=\frac{3b+2}{4b}<1$$

because $b>2$, so it is $h\cdot(J\circ\tau)'(0)<0$. But then, from (13.3.2) we would have $J\left[\tau(h)\right]<J(\mu)$ in contradiction with the hypothesis that J has a minimum at μ. Analogously, we can prove that it cannot be that $(J\circ\tau)'(0)<0$ and thus we obtain

$$(J\circ\tau)'(0)=0$$

from the trichotomy law. From (13.3.2) it follows that $\mathrm{d}J[\tau]=J(\mu)$, that is $\mathrm{d}J[\tau]$ is the null tangent vector.

■

Let us note explicitly the importance, in the previous proof, of the possibility to construct an infinitesimal $h\in D$ having the desired properties with respect to the order relation, e.g. $h<0$, and of a suitable order $\omega(h)$ so that to assure that the product $h\cdot(J\circ\tau)'(0)$ is not zero.

The functionals we are interested in are of the form

$$J(\eta)=\int_a^b L\left[t,\eta(t),\mathrm{d}\eta_t\right]\,\mathrm{d}t\quad \forall\eta\in X^{[a,b]}=:Y, \tag{13.3.3}$$

where

$$a,b\in\mathbb{R}\quad\text{with}\quad {}^\circ a<{}^\circ b$$

$$L:X\times \mathrm{T}X\longrightarrow{}^\bullet\mathbb{R}\quad\text{in}\quad {}^\bullet\mathcal{C}^\infty$$

and where we recall that $\mathrm{d}\eta_t$ is the differential of $\eta:[a,b]\longrightarrow X$ at the point $t\in[a,b]$, i.e. the map $\mathrm{d}\eta_t:\tau\in\mathrm{T}_t[a,b]\longrightarrow\mathrm{d}\eta_t[\tau]=\tau\cdot\eta\in\mathrm{T}_{\eta(t)}X$; moreover, we recall that $\mathrm{T}_yY=(\{t\in\mathrm{T}Y\mid t(0)=y\}\prec\mathrm{T}Y)$ and that $\mathrm{T}Y=Y^D$.

Concretely, the proof works if we can apply a Taylor's formula to $J\left[\tau(h)\right]$ and if we also have a vector space structure on the tangent space T_xX (for the derivation by parts formula), so that interesting cases are $X={}^\bullet\mathbb{R}^d$ or, more generally, any inf-linear vector space of the form $X = {}^\bullet\mathbb{R}^{{}^\bullet M}$, M being a generic smooth manifold (see Theorem 12.2.9 and Theorem 13.1.6).

Let us note that, due to cartesian closedeness of ${}^\bullet\mathcal{C}^\infty$, the notion of smooth map both for the Lagrangian L and for the functional J does not present any problem even for spaces of functions like

$$X = {}^\bullet\mathbb{R}^{{}^\bullet M_1^{\cdots {}^\bullet M_s}} \simeq {}^\bullet\mathbb{R}^{{}^\bullet(M_1\times\cdots\times M_s)}.$$

For these reasons, in the following we will assume

$$X = {}^\bullet\mathbb{R}^{{}^\bullet M} \quad , \quad M \text{ manifold}$$

so that our functional (13.3.3) if a map of the form $J : Y \longrightarrow {}^\bullet\mathbb{R}$ where[2]

$$Y := \left({}^\bullet\mathbb{R}^{{}^\bullet M}\right)^{[a,b]} = {}^\bullet\mathbb{R}^{[a,b]\times{}^\bullet M}$$

We want to prove the Euler-Lagrange equations for a standard Lagrangian at a standard point $\mu \in Y$, so let us firstly assume that J has a minimum at a standard function

$$\mu : [a,b] \longrightarrow {}^\bullet\mathbb{R}^{{}^\bullet M},$$

$$\forall t \in (a,b)_{\mathbb{R}} : \quad \mu(M) \subseteq \mathbb{R}, \tag{13.3.4}$$

(recalling Theorem 12.2.1).

Secondly, let us assume that the Lagrangian L gets standard values at μ, i.e.

$$\forall t \in (a,b)_{\mathbb{R}} : \quad L\left[t, \mu(t), \mathrm{d}\mu_t\right] \in \mathbb{R}. \tag{13.3.5}$$

To prove the Euler-Lagrange equations in a space of the form $X = {}^\bullet\mathbb{R}^{{}^\bullet M}$ (that, we recall, in general is not a Banach space) we will use infinitesimal methods, ensuing the following thread of thoughts.

Let us start considering a tangent vector $\tau \in \mathrm{T}_\mu Y$, i.e. a function

$$\tau : D \longrightarrow {}^\bullet\mathbb{R}^{[a,b]\times{}^\bullet M}.$$

Because of cartesian closedness, we can think of τ as a map from $[a,b] \times D$ into ${}^\bullet\mathbb{R}^{{}^\bullet M}$. Using Taylor's formula in the space ${}^\bullet\mathbb{R}^{{}^\bullet M}$ (see Theorem 12.2.9) we can write

$$\forall t \in [a,b]\, \forall h \in D : \quad \tau(t,h) = \mu(t) + h \cdot \nu(t), \tag{13.3.6}$$

where $\nu := \tau'(0) : [a,b] \longrightarrow {}^\bullet\mathbb{R}^{{}^\bullet M}_{=2}$. Because Euler-Lagrange equations are a necessary condition that follows from (13.3.1), let us assume that the

[2]In the following we will use implicitly the cartesian closedeness, without changing notation from a map to its adjoint.

derivative ν of our tangent vector τ is a standard smooth function, i.e. let us assume that

$$\begin{gathered}\nu : (a,b) \longrightarrow {}^\bullet\mathbb{R}^{\bullet M} = X \\ \forall t \in (a,b)_{\mathbb{R}} : \quad \nu(M) \subseteq \mathbb{R}\end{gathered} \tag{13.3.7}$$

and that verifies (13.3.6) on the open set (a,b).

For a generic $h \in D$, let us calculate

$$\begin{aligned}J[\tau(h)] &= \int_a^b L\left[t, \tau(h,t), \frac{\partial \tau}{\partial t}(h,t)\right] \mathrm{d}t \\ &= \int_a^b L\left[t, \mu(t) + h \cdot \nu(t), \mu'(t) + h \cdot \nu'(t)\right] \mathrm{d}t.\end{aligned}$$

We use the first order Taylor's formula firstly with respect to the second variable and after with respect to the third variable (traditionally indicated with q and $\dot{q}$ respectively) obtaining

$$\begin{aligned}J[\tau(h)] = \int_a^b &\left\{L\left[t, \mu(t), \mu'(t) + h \cdot \nu'(t)\right] + \right. \\ &\left. + h \cdot \mathrm{d}_2 L\left[t, \mu(t), \mu'(t) + h \cdot \nu'(t)\right] . \nu(t)\,\mathrm{d}t\right\} = \\ = \int_a^b &\left\{L\left[t, \mu(t), \mu'(t)\right] + h \cdot \mathrm{d}_3 L\left[t, \mu(t), \mu'(t)\right] . \nu'(t) + \right. \\ &\left. + h \cdot \mathrm{d}_2 L\left[t, \mu(t), \mu'(t)\right] . \nu(t) + h^2 \cdot T\right\} \mathrm{d}t,\end{aligned}$$

where we have used the notation $\mathrm{d}_i L[t, q, \dot{q}] . v$ for the differential of the Lagrangian with respect to its i-th argument at the point $(t, q, \dot{q})$ and applied to the tangent vector v, and where T is a term containing the second derivative of L, but non influencing our calculation because it is multiplied by $h^2 = 0$. Therefore, we have

$$\begin{aligned}J[\tau(h)] = J[\tau(0)] + h \cdot \int_a^b &\left\{\mathrm{d}_3 L\left[t, \mu(t), \mu'(t)\right] . \nu'(t) + ,\right. \\ &\left. + \mathrm{d}_2 L\left[\mu(t), \mu'(t)\right] . \nu(t)\,\mathrm{d}t\right\}\end{aligned}$$

that is

$$\mathrm{d}J_\mu[\tau](h) = h \cdot \int_a^b \left\{\mathrm{d}_3 L\left[t, \mu(t), \mu'(t)\right] . \nu'(t) + \mathrm{d}_2 L\left[t, \mu(t), \mu'(t)\right] . \nu(t)\right\} \mathrm{d}t.$$

But for Theorem 13.3.2 we have $\mathrm{d}J_\mu[\tau] = \underline{0}$, that is

$$h \cdot \int_a^b \left\{\mathrm{d}_3 L\left[t, \mu(t), \mu'(t)\right] . \nu'(t) + \mathrm{d}_2 L\left[t, \mu(t), \mu'(t)\right] . \nu(t)\right\} \mathrm{d}t = 0.$$

We will not delete now the factor $h \in D$ from this equation because this would imply that the integral is equal to zero only up to second order infinitesimals, but we will continue to take this factor for another step, where we will use the hypothesis about the standard nature of both functions μ, ν and of the Lagrangian L (see equations (13.3.4), (13.3.5) and (13.3.7)).

Now we can apply the integration by part formula to the term

$$\mathrm{d}_3 L[t, \mu(t), \mu'(t)]. \nu'(t)$$

and with the bilinear form $\beta(\delta, v) := \delta. v$, where δ is a smooth linear functional, i.e. $\delta \in \mathrm{Lin}({}^\bullet\mathbb{R}^{\bullet M}, {}^\bullet\mathbb{R})$, and where $v \in {}^\bullet\mathbb{R}^{\bullet M}$. We obtain

$$\begin{aligned} 0 = h \cdot & \left[\mathrm{d}_3 L \left[t, \mu(t), \mu'(t)\right] . \nu(t) \right]_a^b - \\ & - h \cdot \int_a^b \left\{ \mathrm{d}_2 L \left[t, \mu(t), \mu'(t)\right] - \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{d}_3 L \left[t, \mu(t), \mu'(t)\right] \right\} . \nu(t) \, \mathrm{d}t \end{aligned} \tag{13.3.8}$$

Restricting to the case where

$$\nu(a) = \nu(b) = \underline{0} \in {}^\bullet\mathbb{R}^{\bullet M} \tag{13.3.9}$$

we obtain that necessarily

$$h \cdot \int_a^b \left\{ \mathrm{d}_2 L \left[t, \mu(t), \mu'(t)\right] - \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{d}_3 L \left[t, \mu(t), \mu'(t)\right] \right\} . \nu(t) \, \mathrm{d}t = 0$$

holds, because $\mathrm{d}_3 L[x].(-)$ is linear. In this equality we note that the integrated function is a standard function, because of our hypothesis (13.3.4), (13.3.5) and (13.3.7), so the integral itself is a standard real and we can delete the first order infinitesimal factor h because of Theorem 2.7.1:

$$\int_a^b \left\{ \mathrm{d}_2 L \left[t, \mu(t), \mu'(t)\right] - \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{d}_3 L \left[t, \mu(t), \mu'(t)\right] \right\} . \nu(t) \, \mathrm{d}t = 0.$$

The usual proof of the so called fundamental lemma of calculus of variations, which uses a continuous function for ν and not a smooth one, can be easily substituted by a formally identical argumentation, but with a smooth function of the form

$$\beta(\bar{t}, \bar{h}, \delta_1, \delta_2, x) := b\left(\frac{x - \bar{t} + \delta_1}{\delta_1}\right) \cdot b\left(\frac{\bar{t} + \bar{h} + \delta_2 - x}{\delta_2}\right) \quad \forall x \in \mathbb{R}.$$

where $b \in \boldsymbol{\mathcal{C}}^\infty(\mathbb{R}, \mathbb{R})$ is a standard smooth bump function, i.e.

$$b(t) = 0 \quad \forall t \leq 0 \quad \text{and} \quad b(s) = 1 \quad \forall s \geq 1, \tag{13.3.10}$$

and where $(\bar{t}, \bar{h}, \delta_1, \delta_2)$ are real parameters.

From the smooth version of the fundamental lemma and from (13.3.8) we obtain the conclusion:

$$\mathrm{d}_2 L \left[t, \mu(t), \mu'(t)\right] = \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{d}_3 L \left[t, \mu(t), \mu'(t)\right] \quad \forall \bar{t} \in (a, b)_{\mathbb{R}}.$$

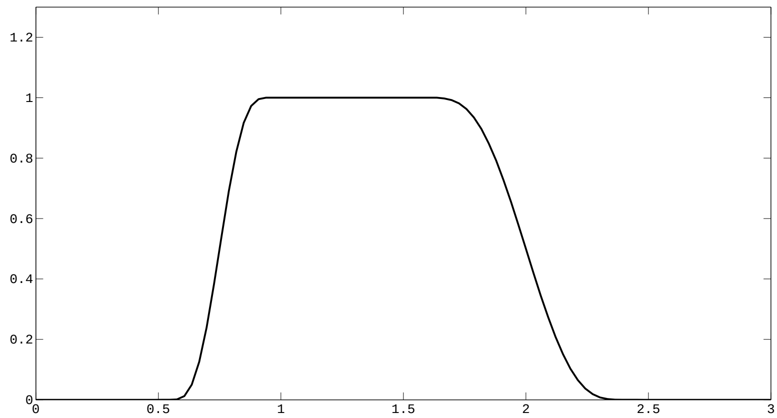

Figure 13.3: An example of function $\beta(\bar{t}, h, \delta_1, \delta_2, -)$ for $\bar{t} - \delta_1 = 0.5$, $\bar{t} = 1$, $\bar{t} + h = 1.5$, $\bar{t} + h + \delta_2 = 2.5$.

Theorem 13.3.3. *Let M be a manifold and set for simplicity $X = {}^\bullet\mathbb{R}^{\bullet M}$. Let us consider a smooth map*

$$L : X \times TX \longrightarrow {}^\bullet\mathbb{R}$$

and an interval $[a, b]$ with $^\circ a < {}^\circ b$. Let us define the functional

$$J(\eta) = \int_a^b L\left[t, \eta(t),\, \mathrm{d}\eta_t\right]\,\mathrm{d}t \quad \forall \eta \in X^{[a,b]} \tag{13.3.11}$$

and assume that J has a minimum at the point $\mu \in X^{[a,b]}$ such that

$$\forall t \in (a, b)_{\mathbb{R}} : \quad \mu(M) \subseteq \mathbb{R}$$

$$\forall t \in (a, b)_{\mathbb{R}} : \quad L\left[t, \mu(t),\, \mathrm{d}\mu_t\right] \in \mathbb{R}.$$

Then we have

$$\mathrm{d}_2 L\left[t, \mu(t), \mu'(t)\right] = \frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{d}_3 L\left[t, \mu(t), \mu'(t)\right] \quad \forall t \in (a, b)_{\mathbb{R}}.$$

■

We have to admit that the proof we gave of the Euler-Lagrange equation in the space $X = {}^\bullet\mathbb{R}^{\bullet M}$ could be elaborated further and presented in a more clear way, e.g. clarifying better some passages, like the identification of the tangent space $\mathrm{T}X$ with the space of the derivatives of the form $\mu'(r) \in {}^\bullet\mathbb{R}^{\bullet M}$ (i.e. the identification of the differential $\mathrm{d}\mu_r[\tau](h) = \mu(r) + h \cdot \tau'(0) \cdot \mu'(r) \in {}^\bullet\mathbb{R}^{\bullet M}$ with the element $\mu'(r)$ of the ${}^\bullet\mathbb{R}$-module ${}^\bullet\mathbb{R}^{\bullet M}$). However, in our opinion already in the present form it has positive features:

1. The general notion of differential $\mathrm{d}J$ of a function $J : Y \longrightarrow Z$ between two inf-linear spaces $Y,\ Z \in {}^\bullet\boldsymbol{\mathcal{C}}^\infty$ can be used to define the notion of minimum of a functional, without any need to define norms on function spaces.

2. Functionals of the form (13.3.11) are smooth even if the domain can be of the form $X^{[a,b]}$, with

$$X = {}^\bullet\mathbb{R}^{{}^\bullet M_1^{\cdot^{\cdot^{\cdot\,{}^\bullet M_s}}}}$$

and without any compactness hypothesis on the manifolds $M_1, \ldots, M_s$.

3. The proof is formally the usual one used in the situation where $X = \mathbb{R}^d$, but our smooth framework is more appropriate, e.g. because of cartesian closedness and completeness and co-completeness.

Chapter 14

Further developments

Several ideas can be developed starting from this foundations of the theory of Fermat reals we provided in the present work. Some are systematic, with high feasibility; some other are, at the present stage, only sketches of ideas. In the next sections we should present some of them, with no aim to be exhaustive in their presentation.

14.1 First order infinitesimals whose product is not zero

We have seen (see Theorem 4.1.1 and the related discussion) that it is impossible to have good properties for the order relation of the ground ring and at the same time to have the existence of two first order infinitesimals whose product is not zero. On the other hand we have had to develop the notion of equality up to k-th order infinitesimals (Chapter 3) to bypass this algebraic problem, first of all in connection with its relationships with Taylor's formula for functions defined on infinitesimal domains (Section 12). In the present work we have seen that a total order can be very useful. For example, our geometrical representation of Fermat reals is strongly based on the trichotomy law, and we have also seen that the possibility to have a total order can be very useful in some proofs (see Section 13.3). On the other hand, the possibility to have two first order infinitesimals whose product is not zero, opens, like in SDG, the possibility to prove a general cancellation law of the form

$$(\forall h \in D : \ h \cdot m = h \cdot n) \quad \Longrightarrow \quad m = n,$$

and hence to avoid the use of the equality up to a k-th order infinitesimal.

The ideal solution would be to keep all the results we have shown in the present work and, at the same time, to have the possibility to consider pairs of first order infinitesimals whose product is not necessarily zero. An idea, inspired by rings like

$$\mathbb{R}[t,s]/\langle t^2=0, s^2=0\rangle,$$

we can try to explore, can be roughly stated saying that "two first order infinitesimals $(h_t)_t$ and $(k_s)_s$ have a non zero product $(h_t \cdot k_s)_{t,s}$ if they depend on two independent variables t and s". A possible formalization of this idea can be sketched in the following way.

Firstly let us fix a way to embed a space of type $\mathbb{R}^n$ into $\mathbb{R}^m$ if $n<m$, e.g.

$$(r_1,\dots,r_n)\in\mathbb{R}^n \mapsto (r_1,\dots,r_n,0,\overset{m-n}{\ldots\ldots},0)\in\mathbb{R}^m. \tag{14.1.1}$$

Then, instead of little-oh polynomials $x:\mathbb{R}_{\ge 0}\longrightarrow\mathbb{R}$, let us consider maps of the form

$$y: r\in\mathbb{R}^n_{\ge 0}\mapsto x\left(t_i^n(r)\right)\in\mathbb{R},$$

where $x\in\mathbb{R}_0[t]$ is a usual little-oh polynomial and where $t_i^n : r\in\mathbb{R}^n_{\ge 0}\mapsto r_i\in\mathbb{R}_{\ge 0}$ is the projection onto the i-th component. In this case we say that *y depends on the variable t_i^n* or, where there is no confusion, simply on the variable t_i. Therefore, our map y can now be written as

$$x_{t_i} = r+\sum_{j=1}^{k}\alpha_j\cdot t_i^{a_j}+o(t_i) \quad \text{as} \quad t_i\to 0^+,$$

where the limit has to be understood along the directed set

$$(\mathbb{R}^n,\le)$$

$$(r_1,\dots,r_n)\le(s_1,\dots,s_n) \quad :\Longleftrightarrow \quad r_i\le s_i.$$

But if we sum this map y with a map z that depends on the variable t_j^m, what do we obtain? Intuitively, a map which is a function of the two variables t_i^m and t_j^m if we firstly embed $\mathbb{R}^n$ into $\mathbb{R}^m$ using (14.1.1). Thus, more generally, we have to consider maps of the form

$$x_{t_{i_1}\cdots t_{i_v}} = r+\sum_{j=1}^{k}\alpha_j\cdot t_{i_1}^{a_{1j}}\cdot\ldots\cdot t_{i_v}^{a_{vj}}+o(t_{i_1})+\ldots+o(t_{i_v}), \tag{14.1.2}$$

$t_{i_1},\dots,t_{i_v}$ being all the variables from which the map x depends on. In (14.1.2) the limit has to be mean along the directed set

$$(\mathbb{R}^m,\le)$$

$$(r_1,\dots,r_m)\le(s_1,\dots,s_m) \quad :\Longleftrightarrow \quad r_{i_1}\le s_{i_1},\dots,r_{i_v}\le s_{i_v}.$$

More precisely, with a writing like

$$\mathcal{P}(o(\varphi_1),\dots,o(\varphi_n))$$

where $\mathcal{P}$ is a generic property and $\varphi_1, \ldots, \varphi_n$ are free variables in $\mathcal{P}$ for functions in the space $\mathbb{R}^{\mathbb{R}^m_{\ge 0}}$, we mean

$$\exists w_1, \ldots, w_n \in \mathbb{R}^{\mathbb{R}^m_{\ge 0}} : \quad \begin{cases} \forall^0 t : \quad \mathcal{P}\left(w_1(t), \ldots, w_n(t)\right) \\ w_i = o(\varphi_i) \quad \forall i = 1, \ldots, n \end{cases}$$

And an "equality" of the type $w = o(\varphi)$, as usual in our context, means

$$\exists \lim_{t \to 0^+} \frac{w(t)}{\varphi(t)} \in \mathbb{R} \text{ and } w(0) = 0.$$

The analogue of the equality in ${}^\bullet\mathbb{R}$ (i.e. the equivalence relation introduced in Definition 2.3.1) is now that $x \sim y$ if and only if

$$x_{t_{i_1} \ldots t_{i_v}} = y_{t_{i_1} \ldots t_{i_v}} + o(t_{i_1}) + \ldots + o(t_{i_v}) \text{ as } t_{i_k} \to 0^+ \ \forall k,$$

where $t_{i_1}, \ldots, t_{i_v}$ are all the variables from which the maps x and y depend on.

This idea seems positive for two reasons: firstly, if we define a new Fermat reals ring in this way, considering only the subring of all the maps $\mathbb{R}_o[t_i]$ which only depend on one variable t_i, we obtain a ring ${}^\bullet\mathbb{R}[t_i]$ isomorphic to the present ${}^\bullet\mathbb{R}$. This means that we are not loosing all the results we have proved in the present work.

Secondly, let us consider $h_{t_1} := t_1$ and $k_{t_2} := t_2$, then we have that $h^2 \sim 0$ and $k^2 \sim 0$, but if we were to have $h \cdot k \sim 0$, then we would get

$$t_1 \cdot t_2 = o(t_1) + o(t_2)$$

$$t_2 = \frac{o(t_1)}{t_1} + \frac{o(t_2)}{t_1}.$$

But the left hand side of this equality goes to zero for $t_1 \to 0^+$ and $t_2 \to 0^+$, whereas on the right hand side the limit

$$\lim_{\substack{t_1 \to 0^+ \\ t_2 \to 0^+}} \frac{o(t_2)}{t_1}$$

does not exist. We therefore have indeed an example of two first order infinitesimals whose product is not zero.

Of course, from Theorem 4.1.1 it follows that every subring ${}^\bullet\mathbb{R}[t_i]$ is totally ordered, but the whole ring cannot be totally ordered.

14.2 Relationships with Topos theory

It is possible to define a meaningful notion of powerset diffeology (see Iglesias-Zemmour [2008]) defined on the powerset $\mathcal{P}(X)$ of any diffeological space X.

Let us recall that any diffeological space is also a $\mathcal{C}^\infty$ space. Therefore, we can try to see whether there is some relation between this powerset diffeology and the powerset object as defined in Topos theory. In case of a positive answer, this would imply that our category $\mathcal{C}^\infty$ is a Topos. It would start thus the possibility to consider its internal language, almost surely in intuitionistic logic, to describe the objects of $\mathcal{C}^\infty$. Independently from the results related to the powerset diffeology, we can try to see whether an axiomatic approach to $\mathcal{C}^\infty$ and ${}^\bullet\mathcal{C}^\infty$ can be developed. This could be useful for those readers who are interested in the study of infinitesimal differential geometry without being forced to consider the whole construction of $\mathcal{C}^\infty$ and ${}^\bullet\mathcal{C}^\infty$. Almost surely, this axiomatic description can be introduced in classical logic instead of intuitionistic logic. Indeed, such an axiomatic description can be developed without considering the internal language of a Topos, and hence without assuming that all our objects and maps are smooth. Of course, we would need an axiom that permits to construct a general family of smooth functions starting from smooth functions, and a "starting point" for this construction, like the assumption that all the standard smooth functions are arrows of the category $\mathcal{C}^\infty$.

14.3 A transfer theorem for sentences

We have seen the proof of a transfer theorem for the construction of ${}^\bullet\mathcal{C}^\infty$ spaces using logical formulas and the preservation properties of the Fermat functor ${}^\bullet(-)$ (see Chapters 9 and 10). As already stated at the end of Chapter 10, differently from our situation, the transfer theorem of NSA asserts an equivalence between two sentences. Nevertheless, it seems possible to follow the following scheme:

1. Define the meaning of the sentence "*the formula ξ is intuitionistically true in $\mathcal{C}^\infty$*" using the intuitionistic interpretation of the propositional connectives and quantifiers in this category. An analogous definition of intuitionistic validity can be done in the category ${}^\bullet\mathcal{C}^\infty$.

2. Define the ${}^\bullet(-)$-transform of a given formula ξ.

3. Prove that ξ is intuitionistically true in $\mathcal{C}^\infty$ if and only if ${}^\bullet\xi$ is intuitionistically true in ${}^\bullet\mathcal{C}^\infty$.

14.4 Two general theorems for two very used techniques

We used several times two techniques in our proofs. The first one is usually a way to speed up several proofs saying "the considered function is smooth because it can be expressed as a composition of smooth functions". Among these functions we have also to consider set theoretical operations like those listed in Section 7.3 or those related to cartesian closedness. It would be useful to define generally which logical terms can be obtained in this way and to prove a general theorem that roughly states that every function given by local formulas "smooth in each variable" is indeed smooth in ${}^\bullet\mathcal{C}^\infty$. This theorem can be assumed as an axiom in the above mentioned axiomatic description of $\mathcal{C}^\infty$ and ${}^\bullet\mathcal{C}^\infty$ and it would substitute very well the global hypotheses to work in intuitionistic logic (where every function can be assumed to be smooth). In other words, instead of saying: "because we are working in a Topos of smooth spaces and functions, every space and every functions we will define using intuitionistic logic is smooth", we can say: "because of the categorical property of our categories and because the considered function f is locally smooth in every variable, the considered space X and the function f are smooth".

Another very useful technique we have used is based on the local form of figures of Fermat spaces (see Theorem 9.2.4). It would be useful, even if it seems not easy to find the corresponding statement, to prove a general theorem that permits to transfer a property that is "locally true and valid for smooth function of the form ${}^\bullet\alpha(p,-)$" to a property that is "globally true" for function that are locally of the form ${}^\bullet\alpha(p,-)$.

14.5 Infinitesimal differential geometry

After a verification of the idea presented in Section 14.1, it would be natural to present a development of infinitesimal differential geometry along the lines already presented in SDG (see e.g. Lavendhomme [1996]). As we have already said several times, frequently the proofs and the definitions given in SDG can be easily reformulated in the context of Fermat spaces, so that the development of this idea sometimes coincides with the formal repetition in our context of those proofs. On the other hand, the property that the product of two first order infinitesimal is always zero, which is one of the most important differences between our theory and SDG, forces us to find a completely new thread of ideas. In contrast to SDG, in our context the study of the relationships between classical results on manifolds and our infinitesimal version is usually a not hard task, whereas in SDG these relationships must always pass through the construction of a suitable topos and a corresponding non trivial embedding of a class of standard smooth manifolds (see Section 5.5 and e.g. Moerdijk and Reyes [1991] for more details).

14.6 Automatic differentiation

Like in the Levi-Civita field (see Section B.5) using Fermat reals we have all the instrument to try a computer implementation of algorithms for automatic differentiation. Even if in the present work we have concentrated ourselves in developing a "smooth framework", it is not hard to prove the following result

Theorem 14.6.1. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$, $x \in \mathbb{R}$ and $n \in \mathbb{N}_{>0}$, then*

$$f \text{ is of class } \mathcal{C}^n \text{ at the point } x$$

if and only if the following conditions are verified

1. *f is locally Lipschitz in a neighborhood of x*
2. $\exists\, m_1, \ldots, m_n \in \mathbb{R}\ \forall h \in D_n : \quad f(x+h) = \sum_{j=0}^{n} \frac{h^j}{j!} \cdot m_j$

This permits to reproduce in the context of Fermat reals several applications of the Levi-Civita field in the frame of automatic differentiation theory (see Shamseddine [1999], Berz et al. [1996], Berz [1992] and Section B.5).

14.7 Calculus of variations

We sketched in Section 13.3 some ideas that our framework can give in the context of the calculus of variations. In SDG this topic has been approached in [Bunge and Heggie, 1984] and [Nishimura, 1999]. It is thus natural to try to reformulate in our context these results and in general to study whether the possibility to consider exponential spaces in the category ${}^\bullet\mathcal{C}^\infty$ can lead to some more general results, or at least to have a more natural approach to some classical results. Indeed, we have shown that the use of infinitesimal methods can be useful both to define well-known notion of calculus of variations without being forced to introduce a norm, and hence without assuming a corresponding compactness hypotheses. On the other hand, we have also shown that these infinitesimal methods can also be very useful to generalize in spaces of mappings the Euler-Lagrange equations. What other results are generalizable in this type of spaces? What other notions can be defined using tangent vectors like in Definition 13.3.1 without considering a neighborhood generated by a norm instead?

14.8 Infinitesimal calculus with distributions

In the present work, every space and function we have considered is smooth. This can be useful in a context like infinitesimal differential geometry, but it is obviously a limitation if one needs to apply infinitesimal methods in contexts with non smooth functions. A possibility is to extend the theory developing an infinitesimal calculus for distributions. Definitions in our framework of the space of all the distributions given by families of smooth functions with a suitable equivalence relation (like in [Antosik et al., 1973] or in [Colombeau, 1992], where non linear polynomial operations on distributions can also be considered) are the most promising ones for this type of generalization of the Fermat reals to a non smooth context.

14.9 Stochastic infinitesimals

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let us consider stochastic processes indexed by $t \in \mathbb{R}_{\geq 0}$. With the symbol $\xrightarrow{P}$ we will denote the convergence in probability.

Using the notion of little-oh relation for stochastic processes, i.e.

$$X_t = o_P(Y_t) \quad :\Longleftrightarrow \quad \exists\, (Q_t)_t \text{ stochastic process} : \begin{cases} X_t = Y_t \cdot Q_t \\ Q_t \xrightarrow{P} 0 \end{cases}$$

we can try to consider suitable stochastic processes $(X_t)_{t\geq 0}$ instead of little-oh polynomials used in the present definition of ${}^\bullet\mathbb{R}$. What will we obtain in this way? Does the corresponding algebraic structure permit to prove in a rigorous and formally identical way informal equalities like

$$\mathrm{d}B(t) = \sqrt{\mathrm{d}t}$$

for a Brownian motion B? Let us note that the square root is not smooth at the origin, and hence the term $\sqrt{\mathrm{d}t}$ has to be understood in a suitable way. For example we can denote by $\sqrt{k}$, for $k \in D_\infty$ infinitesimal, the simplest $h \in D_{2\omega(k)}$ such that $h^2 = k$. Here "simplest" means that the decomposition of h does not contains first order infinitesimals, i.e. $h \in {}^\bullet\mathbb{R}_1$. Let us immediately note that this notion of square root does not verifies $\sqrt{h^2} = |h|$. E.g. if $h^2 = 0$, then $\sqrt{h^2} = 0$ (the simplest number whose square is zero is the zero itself), whereas it can be $|h| \neq 0$, so $\sqrt{h^2} \neq |h|$.

In this context it is possible to conceive the possibility to develop a differential geometry extending a manifold using such stochastic infinitesimals.

From the point of view of cartesian closedness, this possibility is tied with the one of defining interesting probability measurea on the space $\Omega_1^{\Omega_2}$ of measurable mappings between two given probability spaces, without any particular assumption about the topology[1] of the spaces Ω_i. For this a combination of ideas of integrals in infinite dimensional spaces (see e.g. Schwartz

[1] Usually this problem is solved in the case of complete separable metric spaces or locally compact linear spaces or in case of spaces which are Borel-isomorphic to a Borel subset of $\mathbb{R}$. Recall that, due to cartesian closedness, each one of the spaces Ω_i can be itself an exponential object of the form $\Omega_3^{\Omega_4}$ and and so on, so that it is not natural to make strong topological assumptions (see Chapter 5).

[1974] and Itô [1987] and references therein) and generalized Riemann integral (see e.g. [Muldowney, 1987, 2000, Kurtz and Swartz, 2004] and references therein) could be useful.

14.10 Infinite numbers and nilpotent infinitesimal

In every field the property $h^2 = 0$ implies $h = 0$, so its seems impossible to make infinities and nilpotent infinitesimals to coexist. But with the usual properties, also the existence of the square root would be incompatible with the existence of non zero nilpotent infinitesimals, but we have just seen in fact that some meaningful notion of square root is indeed possible. Of course, not all the usual property of this square root can be maintained in the extension from the real field $\mathbb{R}$ to the Fermat ring ${}^\bullet\mathbb{R}$. On the other hand, infinities and nilpotent infinitesimals do coexist in standard analysis, and in our theory we have a good dialectic between potential infinitesimals and actual infinitesimals in ${}^\bullet\mathbb{R}$. These are the motivations to try to make coexist these two types of extended numbers in the same structure. The problem is what property cannot be extended from $\mathbb{R}$ to ${}^\bullet\mathbb{R}$? Is the corresponding formalism sufficiently natural to work with? Let us present some more concrete ideas in this direction.

If we wish to introduce infinities in the ring ${}^\bullet\mathbb{R}$, we will have the problem of the meaning of products of the form $h \cdot H$, where h is an infinitesimals and H is an infinite number. But, unlike NSA where the solution is only formal in case of non convergent sequences $(h_n \cdot H_n)_{n\in\mathbb{N}}$, here we want to follow the way used in standard analysis: "a product of the form $0 \cdot \infty$ can be anything: 0, ∞, $r \neq 0$ or nothing in case it does not converge". Based on this informal motivation, we can understand that the property we have to criticize is

$$\left(x = x' \quad \text{and} \quad y = y'\right) \quad \Longrightarrow \quad x \cdot y = x' \cdot y',$$

because, if we want to have infinitesimals and infinities in ${}^\bullet\mathbb{R}$, we cannot multiply freely two numbers in this ring and to hope to always obtain a meaningful result. E.g. we can try to obtain sufficient conditions of the form: we can multiply x and y in case both are finite or if $x - x'$ goes to zero more quickly than the order with which y goes to infinite and vice versa". E.g. if we define

$$x =_n y \qquad :\Longleftrightarrow \qquad x_t = y_t + o\left(t^n\right) \quad \text{as} \quad t \to 0^+,$$

then we have not one equality only, but a family of equalities, one for each $n \in \mathbb{N}_{>0}$. On the one hand, this is positive because the subring of finite numbers with the equality $=_1$ is exactly the present ring of Fermat reals. On the other hand we can prove the following:

Theorem 14.10.1. *Let x, y, x', $y' : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}$ be maps, and let p, m, $n \in \mathbb{N}_{>0}$, a, $b \in \mathbb{R}$ be numbers that verify*

$$p \leq n \quad \text{and} \quad p \leq m$$

$$\forall^0 t: \quad \left|x_t \cdot t^{m-p}\right| < a \quad \text{and} \quad \left|y'_t \cdot t^{n-p}\right| < b. \tag{14.10.1}$$

Then we have

$$x =_n x' \quad \text{and} \quad y =_m y' \quad \Longrightarrow \quad x \cdot y =_p x' \cdot y'$$

Proof: We have that

$$\begin{aligned} xy - x'y' &= xy - xy' + xy' - x'y' \\ &= x \cdot (y - y') + (x - x') \cdot y' \\ &= x \cdot o_1(t^m) + o_2(t^n)\, y'. \end{aligned}$$

sBut

$$\frac{x_t \cdot o_1(t^m)}{t^p} = \frac{x_t \cdot t^{m-p} \cdot o_1(t^m)}{t^m} \to 0$$

because, by hypotheses, $|x_t \cdot t^{m-p}|$ is bounded from above. Analogously we can deal with the term $o_2(t^n) \cdot y'$ and hence we have the conclusion.

■

Condition (14.10.1) says that the numbers x and y cannot be infinities "too large", and hence includes the case where both x and y are finite. But if we have $x =_1 y$ and z infinite, then $z =_m z$ for every $m \in \mathbb{N}_{>0}$, but $\left|z_t \cdot t^{1-1}\right| = |z_t|$ which is unbounded and hence we cannot use the previous theorem to deduce that $x \cdot z = y \cdot z$. In other words, in this structure we cannot multiply an equality of the form $=_1$ with an infinite number. This make it possible to have the coexistence of $h^2 =_1 0$ with the existence of the inverse of the nilpotent h, i.e. a number k such that $k \cdot h =_m 1$ for every $m \in \mathbb{N}_{>0}$.

As mentioned above, the feasibility of this simple idea is tied with the possibility to create a sufficiently flexible formalism to deal with nilpotent infinitesimals and infinite numbers at the same time, using a family of equalities $=_m$. The first aims to test this construction are of course tied with the possibility to describe Riemann integral sums using our infinitesimals and infinities and the possibility to define at least some δ Dirac like distributions.

Part IV

Appendices

Appendix A

Some notions of category theory

This appendix recalls those (more or less) standard definitions and basic results which are used in the present work. It also aims at clarifying the notations of category theory we use in this work, but it is not meant as an introduction to the subject. For this reason, no proofs and no intuitive interpretations, nor a sufficient amount of examples, are given; they can be found in several standard textbooks on category theory (see e.g. Adamek et al. [1990], Arbib and Manes. [1975], Mac Lane [1971]).

All the definitions and theorems we will state are framed in the set theory **NBG** of von Neumann-Bernay-Gödel, where, in some cases, we can add the axiom about the existence of Grothendieck universes.

A.1 Categories

Definition A.1.1. *A category* $\mathbf{C}$ *is a structure of the form*

$$\mathbf{C} = \left((-) \xrightarrow{(-)} (-), 1_{(-)}, \cdot, \mathcal{O}, \mathcal{A} \right),$$

where $\mathcal{O}$ *and* $\mathcal{A}$ *are classes, called respectively the class of* objects *and the class of* arrows or morphisms *of* $\mathbf{C}$*. The relation*

$$(-) \xrightarrow{(-)} (-) \subseteq \mathcal{O} \times \mathcal{A} \times \mathcal{O}$$

is called the arrow relation of $\mathbf{C}$*. The function*

$$1_{(-)} : \mathcal{O} \longrightarrow \mathcal{A}$$

assigns an arrow 1_A*, called the* identity *of the object* A*. Finally the function*

$$\cdot : \{(f, g) \in \mathcal{A} \times \mathcal{A} \,|\, \exists\, f \cdot g\} \longrightarrow \mathcal{A}$$

is called the composition of the arrows f and g with respect to the objects A, B and C. *The predicate* $\exists\, f \cdot g$ *will be defined by the following conditions, which hold for every object* A, B, C, $D \in \mathcal{O}$ *and every arrow* f, g, $h \in \mathcal{A}$*:*

1. $\exists\, f \cdot g \iff \exists\, A, B, C : \quad A \xrightarrow{f} B \xrightarrow{g} C$, *i.e. the composition* $f \cdot g$ *is defined if the arrow* f *takes some object* A *into* B *and the arrow* g *takes* B *into* C.
2. $A \xrightarrow{f} B \xrightarrow{g} C \quad \Longrightarrow \quad A \xrightarrow{f \cdot g} C$
3. $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \quad \Longrightarrow \quad f \cdot (g \cdot h) = (f \cdot g) \cdot h$, *i.e. the composition is associative*
4. $A \in \mathcal{O} \quad \Longrightarrow \quad A \xrightarrow{1_A} A$
5. $A \xrightarrow{f} B \xrightarrow{g} C \quad \Longrightarrow \quad f \cdot 1_B = f \quad$ *and* $\quad 1_B \cdot g = g$

The following notation

$$\mathbf{C}(A, B) := \left\{ f \in \mathcal{A} \,|\, A \xrightarrow{f} B \right\}$$

is also very used. Let us note that generally speaking $\mathbf{C}(A, B)$, usually called *hom-set*, is a class and not a set. In case it is a set and not a proper class, then the category is said to be *locally small.* If the classes of objects and that of arrows of a category are sets and not proper classes, then the category is called *small.*

It is possible to prove that for every object A of $\mathbf{C}$ there exists one and only one arrow u such that

$$A \xrightarrow{u} A$$

$$B \xrightarrow{f} A \xrightarrow{g} C \quad \Longrightarrow \quad f \cdot u = f \quad \text{and} \quad u \cdot g = g.$$

From the definition of category, this arrow is $u = 1_A$, so that the notion of identity can be defined starting from the arrow relation and the composition map. For this reason, in defining a category we have no need to specify the definition of the identity 1_A.

In the present work, unless it is differently specified, we will not assume that if $A \xrightarrow{f} B$, then the objects A and B are uniquely determined by the arrow f. On the contrary, if the property

$$A \xrightarrow{f} B \quad \text{and} \quad A' \xrightarrow{f} B' \quad \Longrightarrow \quad A = A' \quad \text{and} \quad B = B'$$

holds, then we say that the category $\mathbf{C}$ *has domains and codomains* and we can define the domain and codomain maps:

$$\mathrm{dom} : \mathcal{A} \longrightarrow \mathcal{O}$$

$$\mathrm{cod} : \mathcal{A} \longrightarrow \mathcal{O}$$
$$\mathrm{dom}(f) \xrightarrow{\quad f \quad} \mathrm{cod}(f) \quad \forall f \in \mathcal{A}.$$

In case we have to consider more than one category, we will use notations like

$$\mathbf{C} \models A \xrightarrow{\quad f \quad} B$$
$$\mathbf{C} \models f \cdot g = h.$$

Moreover, we will also use the notations

$$\mathrm{Obj}(\mathbf{C}) := \mathcal{O} \quad , \quad \mathrm{Arr}(\mathbf{C}) := \mathcal{A}$$
$$A \in \mathbf{C} \qquad :\Longleftrightarrow \qquad A \in \mathrm{Obj}(\mathbf{C})$$

In almost all the examples of categories considered in the present work, the objects are sets with some additional structure and the morphisms are maps between the underlying sets that preserve this structure. So we have the category **Set** of all sets, the category **Grp** of all groups, the category **Man** of smooth manifolds, etc. Let us note that every set is a category with only identity arrows.

An example used in this work that is not a category of sets with a structure is given by the category corresponding to a preorder. Indeed, let $(P, \leq)$ be a preordered set; let us fix any element $* \in \mathbf{Set}$ (it is not important what element concretely we choose, e.g. it can be $* = 0 \in \mathbb{R}$) and define

$$\mathcal{O} := P \quad , \quad \mathcal{A} := \{*\}$$
$$x \xrightarrow{\quad f \quad} y \qquad :\Longleftrightarrow \qquad x, y \in P \quad , \quad x \leq y \quad , \quad f = *$$
$$x \xrightarrow{\quad f \quad} y \xrightarrow{\quad g \quad} z \qquad \Longrightarrow \qquad f \cdot g := *.$$

It is easy to prove that in this way we obtain a category.

Definition A.1.2. *Let* $\mathbf{C}$ *be a category, then* $\mathbf{C}^{op}$ *is the category obtained "reversing the direction of all the arrows", i.e.*

$$\mathrm{Obj}\left(\mathbf{C}^{op}\right) := \mathrm{Obj}(\mathbf{C}) \quad , \quad \mathrm{Arr}\left(\mathbf{C}^{op}\right) := \mathrm{Arr}(\mathbf{C})$$
$$\mathbf{C}^{\mathrm{op}} \models A \xrightarrow{\quad f \quad} B \qquad :\Longleftrightarrow \qquad \mathbf{C} \models B \xrightarrow{\quad f \quad} A$$
$$\mathbf{C} \models f \cdot g = h \qquad \Longrightarrow \qquad \mathbf{C}^{\mathrm{op}} \models g \cdot f := h$$

Moreover, if $\mathbf{D}$ *is another category, we say that* $\mathbf{D}$ is a subcategory of $\mathbf{C}$ *if and only if the following conditions hold:*

1. $\mathrm{Obj}(\mathbf{D}) \subseteq \mathrm{Obj}(\mathbf{C})$
2. $\mathbf{D} \models A \xrightarrow{\quad f \quad} B \quad \Longrightarrow \quad \mathbf{C} \models A \xrightarrow{\quad f \quad} B \quad \forall A, B, f$

3. $\mathbf{D} \vDash f \cdot g = h \quad \Longrightarrow \quad \mathbf{C} \vDash f \cdot g = h.$

For two categories, the *product category* is defined by

$$\begin{aligned} \mathrm{Obj}(\mathbf{C} \times \mathbf{D}) &:= \mathrm{Obj}(\mathbf{C}) \times \mathrm{Obj}(\mathbf{D}) \\ \mathrm{Arr}(\mathbf{C} \times \mathbf{D}) &:= \mathrm{Arr}(\mathbf{C}) \times \mathrm{Arr}(\mathbf{D}) \end{aligned}$$

$$\mathbf{C} \times \mathbf{D} \vDash (C_1, D_1) \xrightarrow{(f,g)} (C_2, D_2) \quad :\Longleftrightarrow \quad \begin{cases} \mathbf{C} \vDash C_1 \xrightarrow{f} C_2 \\ \mathbf{D} \vDash D_1 \xrightarrow{g} D_2 \end{cases}$$

$$\mathbf{C} \times \mathbf{D} \vDash (c, d) \cdot (\gamma, \delta) = (f, g) \quad :\Longleftrightarrow \quad \begin{cases} \mathbf{C} \vDash c \cdot \gamma = f \\ \mathbf{D} \vDash d \cdot \delta = g \end{cases}$$

In Chapter 6 we mention at the notion of *Grothendieck universe*, which is defined as follows.

Definition A.1.3. *We say that the class $\mathcal{U}$ is a* Grothendieck universe *if and only if the following conditions hold:*

1. $x \in \mathcal{U}$ *and* $y \in x \quad \Longrightarrow \quad y \in \mathcal{U}$
2. $x, y \in \mathcal{U} \quad \Longrightarrow \quad \{x, y\} \in \mathcal{U}$ *and* $(x, y) \in \mathcal{U}$
3. $x \in \mathcal{U} \quad \Longrightarrow \quad \{y \mid y \subseteq x\} \in \mathcal{U}$
4. *If $(x_i)_{i \in I}$ is a family of elements of $\mathcal{U}$ and if $I \in \mathcal{U}$, then $\bigcup_{i \in I} x_i \in \mathcal{U}$*
5. *x, $y \in \mathcal{U}$ and $f : x \longrightarrow y$ is a map between these sets, then $f \in \mathcal{U}$*
6. *$\mathbb{N} \in \mathcal{U}$, i.e. the set of natural numbers belongs to the universe $\mathcal{U}$*

In other words in a Grothendieck universe all the usual constructions of set theory are possible. A supplementary axiom of set theory that one may need when using category theory is

$$\forall x \, \exists \mathcal{U} \text{ Grothendieck universe} : \quad x \in \mathcal{U}, \tag{A.1.1}$$

that is, every class is an element of a suitable universe. The theory **NBG** changes radically if we assume this axiom. E.g. all our categories can be defined in a given fixed universe (obtaining in this way classes of that universe), but if we need to consider e.g. $\mathbf{Man}_{\mathcal{U}}$ as an element of another class, then we can consider another Grothendieck universe $\mathcal{U}_2$ that contains $\mathbf{Man}_{\mathcal{U}}$ as an element. In this way $\mathbf{Man}_{\mathcal{U}}$ is now a set, and not a proper class, in the universe $\mathcal{U}_2$. All our construction do not depend on the axiom (A.1.1).

A.2 Functors

Definition A.2.1. *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be two categories, then a functor*

$$F : \mathbf{C} \longrightarrow \mathbf{D}$$

is a pair $F = (F_o, F_a)$ *of maps*

$$\begin{aligned} F_o &: \mathrm{Obj}(\mathbf{C}) \longrightarrow \mathrm{Obj}(\mathbf{D}) \\ F_a &: \mathcal{AR}(\mathbf{C}) \longrightarrow \mathrm{Arr}(\mathbf{D}), \end{aligned}$$

where

$$\mathcal{AR}(\mathbf{C}) := \left\{ (A, f, B) \,|\, \mathbf{C} \vDash A \xrightarrow{\ f\ } B \right\}.$$

Because we will always use different symbols for objects and arrows, and because it should be from the context what domain and codomain we are considering, we will simply use the notations

$$\begin{aligned} F(A) &:= F_o(A) \quad \forall A \in \mathrm{Obj}(\mathbf{C}) \\ F(f) &:= F_a(A, f, B) \quad \forall (A, f, B) \in \mathcal{AR}(\mathbf{C}). \end{aligned}$$

Moreover, the following conditions must hold:

1. $F(1_A) = 1_{F(A)}$ *for every object* $A \in \mathbf{C}$*, i.e. the functor preserves the identity maps.*

2. $\mathbf{C} \vDash A \xrightarrow{\ f\ } B \implies \mathbf{D} \vDash F(A) \xrightarrow{\ F(f)\ } F(B)$*, i.e. the functor preserves the arrow relation.*

3. $\mathbf{C} \vDash A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C \implies \mathbf{D} \vDash F(f \cdot g) = F(f) \cdot F(g)$*, i.e. the functor preserves the composition of arrows.*

Finally, a functor of the form

$$F : \mathbf{C}^{\mathrm{op}} \longrightarrow \mathbf{D}$$

is called a contravariant *functor.*

Example. Let $\mathbf{P}$ and $\mathbf{Q}$ be the categories induced by two preordered sets $(P, \leq)$ and $(Q, \preceq)$ respectively. Then, only the preservation of the arrow relation is non trivial in this case, and a functor $f : \mathbf{P} \longrightarrow \mathbf{Q}$ preserves this relation if and only if

$$x \leq y \implies f(x) \preceq f(y) \quad \forall x, y \in P,$$

that is the functors correspond to order preserving morphisms.

If the category $\mathbf{C}$ is locally small, then we can consider the functor

$$\mathbf{C}(-,-) : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \longrightarrow \mathbf{Set}$$

called the *hom-functor* of $\mathbf{C}$ defined on the objects (A, B) as the hom-set $\mathbf{C}(A, B) \in \mathbf{Set}$, and on arrows $(A, B) \xrightarrow{(f,g)} (C, D)$ as

$$\mathbf{C}(f, g) : h \in \mathbf{C}(A, B) \mapsto f \cdot h \cdot g \in \mathbf{C}(C, D).$$

Functors $F : \mathbf{C} \longrightarrow \mathbf{D}$ and $G : \mathbf{D} \longrightarrow \mathbf{E}$ can be composed by considering the composition of the corresponding maps acting on objects and arrows.

Definition A.2.2. *A functor $F : \mathbf{C} \longrightarrow \mathbf{D}$ is called* faithful *(resp.* full*) if and only if for any two objects A, $B \in \mathbf{C}$, the mapping*

$$f \in \mathbf{C}(A, B) \mapsto F(f) \in \mathbf{D}(FA, FB) \tag{A.2.1}$$

is injective (resp. surjective). A full and faithful functor is called an embedding.

A category $\mathbf{C}$ with a faithful functor $F : \mathbf{C} \longrightarrow \mathbf{D}$ is called a concrete category based on $\mathbf{D}$.

All the categories of sets with a suitable structure and the corresponding morphisms are concrete categories based on $\mathbf{Set}$. The corresponding faithful functor associate to each pair $(S, \mathcal{S})$ made of a set S with the structure $\mathcal{S}$ the underlying set $S \in \mathbf{Set}$, and to each morphisms the corresponding map between the underlying sets.

Definition A.2.3. *Given two functors F, $G : \mathbf{C} \longrightarrow \mathbf{D}$ taking the same domain category $\mathbf{C}$ to the same codomain category $\mathbf{D}$, we say that $\tau : F \longrightarrow D$* is a natural transformation *if and only if*

1. *$\tau : \mathrm{Obj}(\mathbf{C}) \longrightarrow \mathrm{Arr}(\mathbf{D})$. Usually the notation $\tau_A := \tau(A)$ is used.*

2. *If $A \in \mathbf{C}$, then $\mathbf{D} \vDash F(A) \xrightarrow{\tau_A} G(A)$*

3. *If $\mathbf{C} \vDash A \xrightarrow{f} B$, then the following diagram commutes*

$$\begin{tikzcd} F(A) \arrow[r, "\tau_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\ F(B) \arrow[r, "\tau_B"] & G(B) \end{tikzcd}$$

If the categories $\mathbf{C}$ and $\mathbf{D}$ are small (in some universe), then taking as objects all the functors $F : \mathbf{C} \longrightarrow \mathbf{D}$, as arrows the natural transformations between these functors and with the composition of natural transformation defined by

$$(\tau \cdot \sigma)_A := \tau_A \cdot \sigma_A,$$

we obtain a category indicated by the symbol $\mathbf{D}^{\mathbf{C}}$. In this category we can thus say when two functors are isomorphic. In particular a functor $F : \mathbf{C} \longrightarrow \mathbf{Set}$ is called *representable* if

$$\exists\, A \in \mathbf{C} : \quad \mathbf{Set}^{\mathbf{C}} \vDash F \simeq \mathbf{C}(A, -),$$

such an isomorphism is called a *representation*.

A.3 Limits and colimits

Definition A.3.1. *Let $\mathbf{C}$, $\mathbf{I}$ be two categories and $F : \mathbf{I} \longrightarrow \mathbf{C}$ a functor, then we say that*

$$\left(V \xrightarrow{\ f_i\ } F(i) \right)_{i \in \mathbf{I}} \quad \text{is a cone with base } F$$

if and only if:

1. $f : \mathrm{Obj}(\mathbf{I}) \longrightarrow \mathrm{Arr}(\mathbf{C})$. *We will use the notation $f_i := f(i)$ for $i \in \mathbf{I}$.*
2. $V \in \mathbf{C}$
3. $\forall i \in \mathbf{I} : \quad \mathbf{C} \vDash V \xrightarrow{\ f_i\ } F(i)$
4. *If $\mathbf{I} \vDash i \xrightarrow{\ h\ } j$, then in the category $\mathbf{C}$ the following diagram commutes*

$$\begin{tikzcd} & V \arrow[dl, "f_i"'] \arrow[dr, "f_j"] & \\ F(i) \arrow[rr, "F(h)"'] & & F(j) \end{tikzcd}$$

A universal cone with base F is called a limit of F:

Definition A.3.2. *In the previous hypothesis, we say that*

$$\left(L \xrightarrow{\ \mu_i\ } F(i) \right)_{i \in \mathbf{I}} \quad \text{is a limit of } F$$

if and only if:

1. *$\left(L \xrightarrow{\mu_i} F(i)\right)_{i\in\mathbf{I}}$ is a cone with base F*

2. *If $\left(V \xrightarrow{f_i} F(i)\right)_{i\in\mathbf{I}}$ is another cone with base F, then there exists one and only one morphism φ such that, in the category $\mathbf{C}$, the following conditions hold*

   (a) $V \xrightarrow{\varphi} L$

   (b) For every $i \in \mathbf{I}$, we have

$$\begin{tikzcd} L \arrow[r, "\mu_i"] & F(i) \\ V \arrow[u, "\varphi"] \arrow[ur, "f_i"'] & \end{tikzcd}$$

The notions of cocone and of colimit are dual with respect to these, so that the analogous definition can be obtained by simply reversing the directions of all the arrows. It is possible to prove that if a limit exists, it is unique up to isomorphisms in $\mathbf{C}$. For these reasons, if the limit exists, we will denote the corresponding object L by

$$\lim_{i\in\mathbf{I}} F(i).$$

Analogously the colimit will be denoted by

$$\operatorname*{colim}_{i\in\mathbf{I}} F(i).$$

A category $\mathbf{C}$ is said to be *complete* if every functor $F : \mathbf{I} \longrightarrow \mathbf{C}$ defined in a small category $\mathbf{I}$ admits a limit; whereas it is said to be *cocomplete* if each one of such functor admits a colimit.

Example.

1. If $\mathbf{I} = \{0, 1\}$, then the limit $\left(P \xrightarrow{p_i} F(i)\right)_{i\in\{0,1\}}$ of F is given by an object $P \in \mathbf{C}$ and two morphisms

$$F(1) \xleftarrow{p_1} P \xrightarrow{p_0} F(0)$$

   which verify the universal property: if $\left(V \xrightarrow{f_i} F(i)\right)_{i\in\{0,1\}}$ is another pair of morphisms of this form, then there exists one and only one arrow in $\mathbf{C}$

$$\varphi : V \longrightarrow P$$

such that

$$\begin{tikzcd}
F(0) & P \arrow[l, "p_0"'] \arrow[r, "p_1"] & F(1) \\
& V \arrow[ul, "f_0"] \arrow[u, "\varphi"'] \arrow[ur, "f_1"'] &
\end{tikzcd} \tag{A.3.1}$$

Therefore, in this special case the notion of limit of F gives the usual notion of product of the objects $F(0)$, $F(1) \in \mathbf{C}$. In the present work, the unique morphism φ that verifies (A.3.1) is denoted by $\langle f_1, f_2 \rangle$. With the notion of cocone and the same index category $\mathbf{I} = \{0, 1\}$ we obtain the usual notion of sum of two objects.

2. If $\mathbf{I}$ is the category generated by the graph

$$\begin{tikzcd} 2 \arrow[r, "a"] & 0 & 1 \arrow[l, "b"'] \end{tikzcd}$$

then the notion of limit corresponds to the notion of pull-back of the diagram

$$\begin{tikzcd}
& F(1) \arrow[d, "F(b)"] \\
F(2) \arrow[r, "F(a)"'] & F(0)
\end{tikzcd}$$

3. If $\mathbf{I}$ is the category generated by the graph

$$\begin{tikzcd} 0 \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & 1 \end{tikzcd}$$

then the notion of limit corresponds to that of *equalizer* of the diagram

$$\begin{tikzcd} F(0) \arrow[r, shift left, "F(a)"] \arrow[r, shift right, "F(b)"'] & F(1) \end{tikzcd}$$

that is an arrow $E \xrightarrow{e} F(0)$ such that $e \cdot F(a) = e \cdot F(b)$ which is universal among all the arrows that verify these relations.

In case of concrete categories the notion of limit can be simplified using the notion of lifting.

Definition A.3.3. *Let* $\mathbf{C}$ *be a concrete category based on* $\mathbf{D}$ *with faithful functor* $U : \mathbf{C} \longrightarrow \mathbf{D}$. *We will use the notation* $U^{-1}(f)$ *every time* f *is in the image set of the functor* U. *Let* $I \in \mathbf{Set}$. *Then, we say that*

$$\left(C \xrightarrow{\quad\gamma_i\quad} C_i\right)_{i\in I} \textit{ is a } \text{lifting} \textit{ of } \left(D \xrightarrow{\quad\delta_i\quad} D_i\right)_{i\in I}$$

if and only if:

1. $U\left(C \xrightarrow{\quad\gamma_i\quad} C_i\right) = D \xrightarrow{\quad\delta_i\quad} D_i \quad \forall i \in I$

2. *If* $\mathbf{D} \models U(A) \xrightarrow{\quad\varphi\quad} U(C)$ *and for every* $i \in I$ *we have*

$$\mathbf{C} \models A \xrightarrow{\quad U^{-1}(\varphi\cdot\delta_i)\quad} C_i$$

   then

$$\mathbf{C} \models A \xrightarrow{\quad U^{-1}(\varphi)\quad} C$$

The theorem which connects the two concepts is the following.

Theorem A.3.4. *Under the previous hypothesis of Definition A.3.3, let us consider a functor* $F : I \longrightarrow \mathbf{C}$ *and let* $\left(D \xrightarrow{\quad\delta_i\quad} U(F(i))\right)_{i\in I}$ *be the limit of* $U \circ F$ *in the category* $\mathbf{D}$. *Finally, let us suppose that*

$$\left(C \xrightarrow{\quad\gamma_i\quad} F(i)\right)_{i\in I} \textit{ is a lifting of } \left(D \xrightarrow{\quad\delta_i\quad} U(F(i))\right)_{i\in I}.$$

Then

$$\left(C \xrightarrow{\quad\gamma_i\quad} F(i)\right)_{i\in I} \textit{ is the limit of } F$$

A.4 The Yoneda embedding

Every object A of a locally small category $\mathbf{C}$ defines a contravariant functor

$$\mathrm{Y}(A) := \mathbf{C}(-, A) : \mathbf{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}\,.$$

This map Y can be extended to the arrow of $\mathbf{C}$. Indeed, every morphism $f : A \longrightarrow B$ in $\mathbf{C}$ induces a natural transformation

$$\mathrm{Y}(f) := \mathbf{C}(-, f) : \mathrm{Y}(A) \longrightarrow \mathrm{Y}(B),$$

so that, at the end we obtain a functor

$$\mathrm{Y} : \mathbf{C} \longrightarrow \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$$

called the *Yoneda embedding.* The name is justified by the following two results. To state the first one of them, we will use the following language to express a bijection

Definition A.4.1. *Let $\mathcal{A}(x)$, $\mathcal{B}(y)$ and $\mathcal{C}(x,y)$ be three property in the free variables x and y. Then with the statement*

To give $x : A(x)$ is equivalent to give $y : \mathcal{B}(y)$ so that $\mathcal{C}(x,y)$ holds

we mean:

1. $\forall x : \mathcal{A}(x) \Rightarrow \exists!\, y : \mathcal{B}(y)$ *and* $\mathcal{C}(x,y)$
2. $\forall y : \mathcal{B}(y) \Rightarrow \exists!\, x : \mathcal{A}(x)$ *and* $\mathcal{C}(x,y)$

In other words, these properties define a bijection and the property $\mathcal{C}(x,y)$ acts as the formula connecting the objects x and the objects y.

Theorem A.4.2. *Let $\mathbf{C}$ be a locally small category, $F : \mathbf{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ a functor and $C \in \mathbf{C}$. Then to give a natural transformation τ:*

$$\tau : Y(C) \longrightarrow F \tag{A.4.1}$$

is equivalent to give an element s:

$$s \in F(C) \tag{A.4.2}$$

so that the following properties hold:

1. $s = \tau_C\,(1_C)$
2. $\tau_A(g) = F(g)(s) \quad \forall A \in \mathbf{C}\, \forall g \in \mathbf{C}(A,C)$.

As a consequence of this theorem we have the following result, which is cited at Chapter 5 of the present work.

Corollary A.4.3. *The Yoneda embedding is a full and faithful functor.*

A.5 Universal arrows and adjoints

Definition A.5.1. *Let $G : \mathbf{D} \longrightarrow \mathbf{C}$ be a functor and $C \in \mathbf{C}$, then we say that*

$$C \xrightarrow{\ \eta\ } G(D) \text{ is a universal arrow}$$

if and only if:

1. $D \in \mathbf{D}$
2. $\mathbf{C} \vDash C \xrightarrow{\ \eta\ } G(D)$
3. *The pair (D,η) is G-*couniversal[1] *among all the pairs which satisfy the previous two conditions, i.e. if $D_1 \in \mathbf{D}$ and $\mathbf{C} \vDash C \xrightarrow{\ \eta_1\ } G(D_1)$, then there exists one and only one $\mathbf{D}$-morphism φ such that*

[1] Let us note explicitly the inconsistency between the property of co-universality (i.e. the unique morphism φ *starts* from the couniversal object C) and the name "universal arrow". This inconsistency in the name, even if it creates a little bit of confusion, is well established in the practice of category theory.

(a) $\mathbf{D} \models D \xrightarrow{\varphi} D_1$

(b) $\mathbf{C} \models$

$$\begin{tikzcd} C \arrow[r, "\eta"] \arrow[dr, "\eta_1"'] & G(D) \arrow[d, "G(\varphi)"] \\ & G(D_1) \end{tikzcd}$$

The notion of *couniversal arrow* is dual with respect to that of universal arrow.

Definition A.5.2. *Let* $\mathbf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{D}$ *be a pair of functors with opposite directions, then we write*

$$F \dashv G \textit{ with unit } \eta,$$

and we read it G is right adjoint of F with unit η*, if and only if:*

1. $1_{\mathbf{C}} \xrightarrow{\eta} F \cdot G$*, i.e.* η *is a natural transformation from the identity functor* $1_{\mathbf{C}}$ *to the composition* $F \cdot G = G \circ F$*.*

2. $C \xrightarrow{\eta_C} G(F(C))$ *is a universal arrow.*

In case of locally small categories, the notion of pair of adjoint functors can be reformulated in the following way

Theorem A.5.3. *If* $\mathbf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{D}$ *and* $\mathbf{C}$*,* $\mathbf{D}$ *are locally small, then to give* η*:*

$$F \dashv G \textit{ with unit } \eta,$$

is equivalent to give a natural transformation ϑ*:*

$$\vartheta : \mathbf{D}\left(F(-), -\right) \xrightarrow[\sim]{} \mathbf{C}\left(-, G(-)\right)$$

so that it results

$$\vartheta_{CD}(\psi) = \eta_C \cdot G(\varphi)$$

for every $c \in \mathbf{C}$*,* $D \in \mathbf{D}$ *and* $\psi \in \mathbf{D}(FC, D)$*.*

In the particular case where the categories $\mathbf{C}$ and $\mathbf{D}$ are generated by preordered sets $(C, \leq)$ and $(D, \preceq)$, a pair of adjoints $F \dashv G$ correspond to a pair of order preserving morphisms such that

$$F(c) \preceq d \iff c \leq G(d)$$

(a so called *Galois connection*). In case of concrete categories based on **Set**, the notion of cartesian closedness is fully presented in Chapter 5. In more abstract categories, it is defined in the following way.

Definition A.5.4. *We say that* $(\mathbf{C}, \times, T, \pi, \varepsilon, h)$ *is a cartesian closed category if and only if:*

1. $\mathbf{C}$ *is locally small*
2. *For every objects* A*,* $B \in \mathbf{C}$*, the diagram*

$$A \xleftarrow{\pi^1_{AB}} A \times B \xrightarrow{\pi^2_{AB}} B$$

   is a product

3. T *is a terminal object, i.e. for every* $A \in \mathbf{C}$ *there exists one and only one morphism* t *such that*

$$\mathbf{C} \vDash A \xrightarrow{t} T$$

4. *For every* $A \in \mathbf{C}$

$$(-) \times A \dashv h(A, -) \text{ with counit } \varepsilon^A$$

Appendix B

A comparison with other theories of infinitesimals

It is not easy to clarify in a few pages the relationships between our theory of Fermat reals and other, more developed and well established theories of actual infinitesimals. Nevertheless, in this chapter we want to sketch a first comparison, mostly underlining the conceptual differences instead of the technical ones, hoping in this way to clarify the foundational and philosophical choices we made in the present work.

Our focus will fall on the most studied theories like NSA, SDG and surreal numbers, or on constructions having analogies with our Fermat reals like Weil functors and the Levi-Civita field, but we will not dedicate a section to more algebraic theories whose first aim is not to develop properties of infinitesimals or infinities and related applications, but instead to construct a general framework for the study of fields extending the reals (like formal power series or super-real fields). In the case of surreal numbers and the Levi-Civita field we will also give a short presentation of the topic.

A general distinction criterion to classify a theory of infinitesimals is the possibility to establish a dialogueue between potential infinitesimals and actual infinitesimals. On the one hand of this dialogue there are potential infinitesimals, represented by some kind of functions $i : E \longrightarrow \mathbb{R}$ defined on a directed set $(E, \leq)$, like sequences $i : \mathbb{N} \longrightarrow \mathbb{R}$ or functions defined on a subset E of $\mathbb{R}$, and such that

$$\lim_{(E,\leq)} i = 0. \tag{B.0.1}$$

Classical example are, of course, $i(n) = \frac{1}{n}$ for $n \in \mathbb{N}_{>0}$ and $i(t) = t$ for $t \in \mathbb{R}_{\geq 0}$. On the other hand, there are actual infinitesimals as elements $d \in R$ of a suitable ring R. The dialogue can be realized, if any, in several ways, using e.g. the standard part and the limit (B.0.1), or through some connection between the order relation defined on R and the order of the directed set $(E, \leq)$, or through the ring operations of R and pointwise operations on the set of potential infinitesimals. From our point of view, it is very natural to see this dialogue as an advantage, if the theory permits this possibility. First of all, it is a dialogue between two different, but from several aspects equivalent, instruments to formalize natural phenomena and mathematical problems, and hence it seems natural to expect a close relation between them. Secondly, this dialogue can remarkably increase our intuition on actual infinitesimals and can suggest further generalizations. For example, in the context of Fermat reals, it seems very natural to try a generalization taking some stochastic processes $(x_t)_{t\in\mathbb{R}_{\geq 0}}$ instead of our little-oh polynomials, creating in this way "stochastic infinitesimals".

Theories with a, more or less strong, dialogue between potential infinitesimals and actual infinitesimals are: NSA, the theory of surreal numbers and our theory of Fermat reals.

This dialogue, and hence the consequent generalizations or intuitions, are more difficult in formal algebraic approaches to infinitesimals. Very roughly, these approaches can be summarized following the spirit of J. Conway's citation on pag. 103: if one needs some kind of infinitesimal d, add this new symbol to $\mathbb{R}$ and impose to it the properties you need. In this class of theories we can inscribe all the other theories: SDG, Weil functors, differential geometry over a base ring, and Levi-Civita field. They can be thought of as theories generated by two different elementary ideas: the ring of dual numbers $\mathbb{R}[\varepsilon]/\left\langle\varepsilon^2=0\right\rangle$ (firstly generalized by the strongly stimulating and influential article Weil [1953]) and the fields of formal power series. The distinction between these two classes of theories, those that try a dialogue with potential infinitesimals and those approaching formally the problem, is essentially philosophical and at the end choosing one of them rather than the other one is more of a personal opinion than a rational choice. First of all, the distinction is not always so crisp, and (non constructive) NSA represents a case where the above mentioned dialogue cannot always be performed. Moreover, it is also surely important to note that formal theories of infinitesimals are able to reach a great formal power and flexibility, and sometimes through them a sort of a posteriori intuition about actual infinitesimals can be gained.

B.1 Nonstandard Analysis

A basic request in the construction of NSA is to extend the real field by a larger field ${}^*\mathbb{R} \supseteq \mathbb{R}$. As a consequence of this request, in NSA every non zero infinitesimal is invertible and so we cannot have non trivial nilpotent elements (in a field $h^2 = 0$ always implies $h = 0$). On the contrary, in the theory of Fermat reals we aim at obtaining a ring extending the reals, and, as a result of our choices, we cannot have non-nilpotent infinitesimals, in particular they cannot be invertible. In the present work, our first aim was to obtain a meaningful theory from the point of view of the intuitive interpretation, to the disadvantage of some formal property, only partially inherited from the real field. Vice versa every construction in NSA has, as one of its primary aims, to obtain the inheritance of *all the properties* of the reals through the transfer principle. This way of thinking conducts NSA towards the necessity to extend every function $f : \mathbb{R} \longrightarrow \mathbb{R}$, e.g. $f = \sin$, from $\mathbb{R}$ to ${}^*\mathbb{R}$, and to the property that any sequence of standard reals $(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$, even the more strange, e.g. $(\sin(n))_{n\in\mathbb{N}}$, represents one and only one hyperreal.

Of course, in the present work we followed a completely different way: to define the ring of Fermat reals ${}^\bullet\mathbb{R}$ we restrict our construction to the use of little-oh polynomials $x \in \mathbb{R}_o[t]$ only, and therefore we can extend only smooth functions from $\mathbb{R}$ to ${}^\bullet\mathbb{R}$. Obviously, our purpose is to develop infinitesimal instruments for smooth differential geometry only, and we have not the aim of developing an alternative foundation for all mathematics, like NSA does. In exchange, not every property is transferred to ${}^\bullet\mathbb{R}$, e.g. our, as presently developed, is not a meaningful framework where to talk of a $\mathcal{C}^1$ but not smooth function $f : {}^\bullet\mathbb{R} \longrightarrow {}^\bullet\mathbb{R}$.

In NSA, this attention to formally inherit every property of the reals implies that on the one hand we have the greatest formal strength, but on the other hand we need a higher formal control and sometimes we lose the intuitive point of view. We can argue for the truth of this assertion from two points of view: the first one is connected with the necessity to use a form of the axiom of choice to construct the non principal ultrafilter needed to define ${}^*\mathbb{R}$. In the second one, we will study more formally the classical motivation used to introduce ${}^*\mathbb{R}$: two sequences of reals are equivalent if they agree almost everywhere on a "large" set.

It is rather interesting to recall here that the work of Schmieden and Laugwitz [1958] predates by a few year the construction of ${}^*\mathbb{R}$ by A. Robinson. In Schmieden and Laugwitz [1958] using the the filter of co-finite sets and not an ultrafilter, a ring extending the real field $\mathbb{R}$ and containing infinitesimals and infinities is constructed. This work has been of great inspiration for subsequent works in constructive non-standard analysis like Palmgren [1995, 1997, 1998], where a field extending the reals is developed constructively, with a related transfer theorem, but without a standard part map. Because of their construtive nature, in these works, no use of the axiom of choice is made.

To study the relationships between the axiom of choice and the hyperreals, we start from Connes et al. [2000], where the author argued that in NSA it is impossible to give an example of nonstandard infinitesimal, even "to name" it. More precisely, A. Connes asserts that to any infinitesimal $e \in {}^*\mathbb{R}_{\neq 0}$ it is possible to associate, in a canonical way, a non Lebesgue-measurable subset of $(0, 1)$. The following result of Solovay [1970]

Theorem B.1.1. *There exists a model of the Zermelo-Fraenkel theory of sets without axiom of choice (**ZF**), in which every subset of* $\mathbb{R}$ *is Lebesgue measurable.*

would show us the impossibility, in the point of view of A. Connes, to give an example of infinitesimal in NSA. These affirmations, not proved in Connes et al. [2000], can be formalized using the following results:

Theorem B.1.2. *Let* $e \in {}^*\mathbb{R}_{\neq 0}$ *be an infinitesimal, and set*

$$\mathcal{U}_e := \left\{ X \subseteq \mathbb{N} \mid \left[\frac{1}{e}\right] \in {}^*X \right\},$$

where $[x]$ *is the integer part of the hyperreal* x*. Then* $\mathcal{U}_e$ *is an ultrafilter on* $\mathbb{N}$ *containing the filter of all co-finite sets.*

Proof: Directly from the definitions of ultrafilter and from the properties of the operator ${}^*(-)$.

■

The second result we need is due to Sierpiński [1938] and does not need the axiom of choice to be proved:

Theorem B.1.3. *Let* $f : \mathcal{P}(\mathbb{N}) \longrightarrow \{0,1\}$ *be a finitely additive measure defined on every subset of* $\mathbb{N}$*. For each* $x \in (0,1)$*, let*

$$x = \frac{1}{2^{n_1}} + \frac{1}{2^{n_2}} + \frac{1}{2^{n_3}} + \dots$$

be the binary representation of x*, and set*

$$\varphi(x) := f(\{n_1, n_2, n_3, \dots\}).$$

Then the function $\varphi : (0,1) \longrightarrow \{0,1\}$ *is not Lebesgue measurable and hence* $\varphi^{-1}(\{1\})$ *is a non Lebesgue-measurable subset of* $(0,1)$*.*

■

Using these results the sentence of A. Connes is now more clear: to any $e \in {}^*\mathbb{R}_{\neq 0}$ infinitesimal we can associate the ultrafilter $\mathcal{U}_e$ on $\mathbb{N}$; to this ultrafilter we can associate the finitely additive measure $f_e(S) := 1$ if $S \in \mathcal{U}_e$ and $f_e(S) := 0$ if $S \notin \mathcal{U}_e$; to this measure we can finally associate the non Lebesgue-measurable subset of $(0,1)$ given by $S_e := \varphi_e^{-1}(\{1\})$, where φ_e is defined as in Theorem B.1.3. The association $e \mapsto S_e$ is canonical in the sense that it does not depend on the axiom of choice. But the result of Solovay, i.e. Theorem B.1.1, proves that it is impossible to construct a non Lebesgue-measurable set without using some form of the axiom of choice, so the association $e \mapsto S_e$ shows the impossibility to define ${}^*\mathbb{R}$ without some form of this axiom[1]. This is the technical result. Whether this can be interpreted as "it is impossible to give an example of infinitesimal in NSA" or not, it depends on how one means the words "to give an example". It seems indeed, undeniable that if one accepts the axiom of choice and $\mathcal{U}$ is an ultrafilter on $\mathbb{N}$ containing the filter of co-finite sets, then the hyperreal

$$e := \left[\left(\frac{1}{n}\right)_{n\in\mathbb{N}}\right]_{\mathcal{U}} \in {}^*\mathbb{R}$$

is an example of infinitesimal.

The last example seems a typical solution to several problems of NSA related to the existence of ultrafilters, and can be synthesized in the sentence "the ultrapower construction is intuitively clear once the ultrafilter is fixed". For example, an ultrafilter $\mathcal{U}$ on $\mathbb{N}$ containing the filter of co-finite sets is frequently presented as a possible notion of "large sets of natural numbers" and the basic equivalence relation

$$(x_n)_n \sim (y_n)_n \qquad :\Longleftrightarrow \qquad \{n \in \mathbb{N} \,|\, x_n = y_n\} \in \mathcal{U}$$

is hence interpreted as "the two sequences of real numbers are almost everywhere equal, i.e. they agree on a large set (with respect to the notion of large sets given by $\mathcal{U}$)". We want to show now that this intuition is not always correct, despite of the "natural" choice of the ultrafilter $\mathcal{U}$.

To compare two elements of an ultrafilter, i.e. two infinite subsets of $\mathbb{N}$ we will use the notion of *natural density* (also called *asymptotic density*, see e.g. Tenenbaum [1995]):

Definition B.1.4. *If $A \subseteq \mathbb{N}$ and $n \in \mathbb{N}$, we will set $A_{\le n} := \{a \in A \,|\, a \le n\}$. Now let A, B be subsets of $\mathbb{N}$, we will say that* there exists the (natural) density of A with respect to B *iff there exists the limit*

$$\rho(A, B) := \lim_{n\to+\infty} \frac{\mathrm{card}(A_{\le n})}{\mathrm{card}(B_{\le n})} \in \mathbb{R} \cup \{+\infty\}.$$

The set of pairs (A, B) for which the density $\rho(A, B)$ is defined will be denoted by $\mathcal{D}$.

For example if $P := \{2n \,|\, n \in \mathbb{N}\}$ is the set of even numbers, then $\rho(P, \mathbb{N}) = \frac{1}{2}$, that is the set of even number is dense $\frac{1}{2}$ with respect to the set of all the natural numbers.

The notion of natural density has the following properties:

[1] Let us note explicitly, that Theorem B.1.1 refers to the full version of the axiom of choice. Indeed, it is well known, see e.g. Albeverio et al. [1988] and references therein, that the existence of an ultrafilter on $\mathbb{N}$ is less stronger than the full axiom of choice. Roughly speaking, we have just proved that if we are able to construct the hyperreal field ${}^*\mathbb{R}$, then some form of the axiom of choice must holds, not necessarily the full one.

Theorem B.1.5. *Let A and B be subsets of $\mathbb{N}$, then we have:*

1. $\rho(A,B) = \frac{\mathrm{card}(A)}{\mathrm{card}(B)}$ *if A and B are both finite.*
2. $\rho(A,B) = 0$ *if A is finite and B is infinite; vice versa* $\rho(A,B) = +\infty$.
3. $\rho(A,B) \le 1$ *if* $A \subseteq B$ *and* $(A,B) \in \mathcal{D}$.
4. $\rho(-,B)$ *is finitely additive.*
5. $\rho(m+A, m+B) = \rho(A,B)$ *if* $(A,B) \in \mathcal{D}$, *i.e. the natural density is translation invariant.*
6. $\rho\left(\{h\cdot n \,|\, n \in \mathbb{N}\}, \mathbb{N}\right) = \frac{1}{h}$ *if* $h \in \mathbb{N}_{\ne 0}$.
7. *If* (A,B), $(C,D) \in \mathcal{D}$, *then the following implications are true:*

   (a) $A \cap C = \emptyset \quad \Longrightarrow \quad (A \cup C, B) \in \mathcal{D}$

   (b) $A \subseteq B \quad \Longrightarrow \quad (B \setminus A, B) \in \mathcal{D}$

   (c) $A \cup C = B \quad \Longrightarrow \quad (A \cap C, B) \in \mathcal{D}$

Proof: see Tenenbaum [1995] and references therein.

■

Our first aim is to generalize the conclusion *6*. of this theorem and secondly to prove that given an infinite element $P \in \mathcal{U}$ of a fixed ultrafilter, we can always find in the ultrafilter a subset $S \subseteq P$ having density $\frac{1}{2}$ with respect to P. This means, intuitively, that an ultrafilter is closed not only with respect to supersets, but also with respect to suitable subsets.

Lemma B.1.6. *Let $b : \mathbb{N} \longrightarrow \mathbb{N}$ be a strictly increasing sequence of natural numbers, and set for simplicity of notations*

$$B := \{b_n \,|\, n \in \mathbb{N}\}.$$

Then we have

$$\rho\left(\{b_{h\cdot n} \,|\, n \in \mathbb{N}\}, B\right) = \frac{1}{h} \quad \forall h \in \mathbb{N}_{\ne 0}.$$

Proof: Let $\mathrm{int}(r)$ be the integer part of the real $r \in \mathbb{R}$, i.e. the greatest integer number greater or equal to r, and for simplicity of notations set $B_h := \{b_{h\cdot n} \,|\, n \in \mathbb{N}\}$. We first want to prove that

$$\mathrm{card}(B_h)_{\le n} = \mathrm{int}\left(\frac{\mathrm{card}(B_{\le n}) - 1}{h}\right) + 1.$$

Indeed, since b is strictly increasing, we have

$$\mathrm{card}(B_{\le n}) = \max\{k \,|\, b_k \le n\} + 1 =: K + 1 \tag{B.1.1}$$

$$\operatorname{card}(B_h)_{\leq n} = \max\{k \mid b_{h\cdot k} \leq n\} + 1 =: H + 1, \tag{B.1.2}$$

so that we want to prove that $H + 1 = \operatorname{int}\left(\frac{K}{h}\right) + 1$, i.e. that $H = \operatorname{int}\left(\frac{K}{h}\right)$. In fact from (B.1.2) we have $b_{h\cdot H} \leq n$ and hence $h \cdot H \leq K$ from (B.1.1), i.e. $H \leq \frac{K}{h}$ and $H \leq \operatorname{int}\left(\frac{K}{h}\right)$. To prove the opposite, let us consider a generic integer $m \leq \frac{K}{h}$ and let us prove that $m \leq H$. In fact, since b is increasing we have $b_{h\cdot m} \leq b_K$ and $b_K \leq n$ from (B.1.1). Hence $b_{h\cdot m} \leq n$ and from (B.1.2) we obtain the conclusion $m \leq H$.

Now we can evaluate the limit (in the sense that this limit exists if and only if any one of the limits in this series of equalities exists)

$$\lim_{n\to+\infty} \frac{\operatorname{card}(B_h)_{\leq n}}{\operatorname{card}(B_{\leq n})} = \lim_{n\to+\infty} \frac{1}{\operatorname{card}(B_{\leq n})} \cdot \left\{ \operatorname{int}\left(\frac{\operatorname{card}(B_{\leq n}) - 1}{h} \right) + 1 \right\}.$$

Let, for simplicity, $\beta_n := \operatorname{card}(B_{\leq n})$ and note that $\beta_n \to +\infty$ because b is strictly increasing. Finally, let $\operatorname{frac}(r) := r - \operatorname{int}(r) \in [0, 1)$ be the fractional part of the generic real $r \in \mathbb{R}$. With these notations, our limit becomes

$$\begin{aligned}
&\lim_{n\to+\infty} \frac{1}{\beta_n} \cdot \left\{ \frac{\beta_n - 1}{h} - \operatorname{frac}\left(\frac{\beta_n - 1}{h} \right) + 1 \right\} = \\
&\qquad = \lim_{n\to+\infty} \left\{ \frac{1}{h} \cdot \left(1 - \frac{1}{\beta_n} \right) - \frac{1}{\beta_n} \cdot \operatorname{frac}\left(\frac{\beta_n - 1}{h} \right) + \frac{1}{\beta_n} \right\} = \frac{1}{h}
\end{aligned}$$

since $\beta_n \to +\infty$ and the fractional part is limited.

■

Now we can prove that if P is an infinite element of a given ultrafilter $\mathcal{U}$, then in $\mathcal{U}$ we can also find a subset of P with one half of the elements of P.

Lemma B.1.7. *Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$, $P \in \mathcal{U}$ an infinite element of the ultrafilter and finally $n \in \mathbb{N}_{\neq 0}$. Then we can always find an $S \in \mathcal{U}$ such that*

1. *$S \subseteq P$*
2. *Either $\rho(S, P) = \frac{1}{n}$ or $\rho(S, P) = 1 - \frac{1}{n}$.*

Therefore we have

$$\forall P \in \mathcal{U} : \ P \text{ infinite} \implies \exists\, S \in \mathcal{U} : \ S \subseteq P \text{ and } \rho(S, P) = \frac{1}{2}.$$

Proof: Since P is infinite, setting

$$\begin{aligned}
p_0 &:= \min(P) \\
p_{n+1} &:= \min\left(P \setminus \{p_0, \ldots, p_n\}\right)
\end{aligned}$$

we obtain a strictly increasing sequence of natural numbers. Setting $S' := \{p_{n\cdot k} \mid k \in \mathbb{N}\}$ from Lemma B.1.6 we have $\rho(S', P) = \frac{1}{n}$. Therefore, if $S' \in \mathcal{U}$, we have the conclusion for $S := S'$. Otherwise, we have $\mathbb{N} \setminus S' \in \mathcal{U}$, so that setting $S := (\mathbb{N} \setminus S') \cap P = P \setminus S'$ we obtain $S \in \mathcal{U}$ and

$$\rho(S, P) = \rho(P \setminus S', P) = 1 - \rho(S', P) = 1 - \frac{1}{n}.$$

The second part of the conclusion follows setting $n = 2$.

■

Now we only have to apply recursively this lemma to obtain that in any ultrafilter we can always find elements with arbitrary small density:

Theorem B.1.8. *Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$ and $P \in \mathcal{U}$ with P infinite, then we can find a sequence $(P_n)_n$ of elements of $\mathcal{U}$ such that*

1. $P_0 = P$
2. *For every $n \in \mathbb{N}$*
$$P_{n+1} \subseteq P_n$$
$$\rho(P_{n+1}, P_n) = \frac{1}{2}$$
$$\rho(P_n, P) = \frac{1}{2^n}.$$

Therefore in any ultrafilter we can always find elements of arbitrary small density, i.e.

$$\forall \varepsilon > 0 \; \exists S \in \mathcal{U} : \; \rho(S, \mathbb{N}) < \varepsilon.$$

Proof: Set $P_0 := P$ and apply recursively the previous Lemma B.1.7 (note that following the proof of this lemma, we can affirm that we are not applying here the axiom of countable choice) we obtain

$$P_n \in \mathcal{U} \quad , \quad P_{n+1} \subseteq P_n \quad , \quad \rho(P_{n+1}, P_n) = \frac{1}{2}.$$

But

$$\frac{\mathrm{card}(P_n)_{\le k}}{\mathrm{card}(P_0)_{\le k}} = \frac{\mathrm{card}(P_n)_{\le k}}{\mathrm{card}(P_{n-1})_{\le k}} \cdot \frac{\mathrm{card}(P_{n-1})_{\le k}}{\mathrm{card}(P_{n-2})_{\le k}} \cdot \ldots \cdot \frac{\mathrm{card}(P_1)_{\le k}}{\mathrm{card}(P_0)_{\le k}}.$$

Therefore, for $k \to +\infty$ we obtain $\rho(P_n, P) = \frac{1}{2^n}$. The final sentence of the statement follows from the previous one if $P := \mathbb{N}$ and if we take n such that $2^{-n} < \varepsilon$.

■

In the precise sense given by this theorem, we can hence affirm that in any ultrafilter on $\mathbb{N}$ we can always find also "arbitrary small" sets. For example, if we set $\varepsilon := 10^{-100}$, we can find $S \in \mathcal{U}$ with density $\rho(S, \mathbb{N}) < 10^{-100}$. The characteristic function of S

$$x_n := \begin{cases} 1 & \text{if } n \in S \\ 0 & \text{if } n \notin S \end{cases} \qquad \forall n \in \mathbb{N}$$

generates, modulo $\mathcal{U}$, an hyperreal $y := [(x_n)_n]_{\mathcal{U}} \in {}^*\mathbb{R}$ equal to 1 but (with respect to the density $\rho(-,\mathbb{N})$) almost always equal to 0. Finally, the set S of indexes $n \in \mathbb{N}$ where $x_n = 1$ has a density strongly lower with respect to the set $\mathbb{N} \setminus S$ of indexes where $x_n = 0$, in fact

$$\begin{aligned}\rho(S, \mathbb{N} \setminus S) &= \lim_{n \to +\infty} \frac{\operatorname{card}(S_{\le n})}{n+1} \cdot \frac{n+1}{\operatorname{card}(\mathbb{N} \setminus S)_{\le n}} = \\ &= \frac{\rho(S, \mathbb{N})}{\rho(\mathbb{N} \setminus S, \mathbb{N})} \le \frac{10^{-100}}{1 - 10^{-100}}.\end{aligned}$$

See also Chapter 1, where we already compared NSA with the basic aims of the present work on Fermat reals.

B.2 Synthetic differential geometry

We have already mentioned, several times, the relationships between Fermat reals and SDG, and we have already presented very briefly the main ideas for the construction of a model in SDG (see Section 5.5). For these reasons, here we essentially summarize and underline the differences between the two theories.

There are many analogies between SDG and Fermat reals, so that sometimes the proofs of several theorems remain almost unchanged. But the differences are so important that, in spite of the similarities, these theories can be said to describe "different kind of infinitesimals".
We have already noted (see Section 4.1) that one of the most important differences is that for the Fermat reals we have $h \cdot k = 0$ if $h^2 = k^2 = 0$, whereas this is not the case for SDG, where first order infinitesimals $h, k \in \Delta := \{d \,|\, d^2 = 0\}$ with $h \cdot k$ not necessarily equal zero, sometimes play an important role. Note that, as shown in the proof of Schwarz theorem using infinitesimals (see Section 3.6), to bypass this difference, sometimes completely new ideas are required (to compare our proof with that of SDG, see e.g. Kock [1981], Lavendhomme [1996]). Because of these diversities, in our derivation formula we are forced to state $\exists!\, m \in \mathbb{R}_2$ and not $\exists!\, m \in {}^\bullet\mathbb{R}$ (see 12.1). This is essentially the only important difference between this formula and the Kock-Lawvere axiom. Indeed to differentiate a generic smooth map $f : {}^\bullet\mathbb{R} \longrightarrow {}^\bullet\mathbb{R}$ we need the Fermat method (see Section 11.2) i.e. the notion of "smooth incremental ratio".

Another point of view regarding the relationships between Fermat reals and SDG concerns models of SDG. As we hint in Section 5.5, these models are topos of not simple construction, so that we are almost compelled to work with the internal language of the topos itself, that is in intuitionistic logic. If on the one hand this implies that "all our spaces and functions are smooth", and so we do not have to prove this, e.g. after every definition, on the other hand it requires a more strong formal control of the Mathematics one is doing.

Everyone can be in agreement or not with the assertion whether it is difficult or easy to learn to work in intuitionistic logic and after to translate the results using topos based models. Anyway we think undeniable that the formal beauty achieved by SDG can hardly be reached using a theory based on classical logic. It suffices to say, as a simple example, that to prove the infinitesimal linearity of M^N (starting from M, N generic inf-linear spaces), it suffices to fix $n \in N$, to note that $t_i(-, n)$ are tangent vectors at $f(n)$, to consider their parallelogram $p(-, n)$, and automatically, thanks to the use of intuitionistic logic, p is smooth without any need to use directly the sheaf property to prove it. See our Theorem 13.1.5 to compare this proof with the proof of the analogous statement in our context.

On the other hand, if we need a partition of unity, we are forced to assume a suitable axiom for the existence of bump functions (whose definition, in the models, necessarily uses the law of the excluded middle).

Indeed, we think that, as we hint in Chapter 10, the best properties of the theory of Fermat reals can be obtained using an "intuitionistic interpretation". We can say that also this theory "proves" that the best logic to deal with nilpotent infinitesimals in differential geometry is the intuitionistic one and not the classical one. All the efforts done in the present work can be framed into an attempt to obtain a sufficiently simple model of nilpotent infinitesimals, having a strong intuitive interpretation but, at the same time, without forcing the reader to switch to intuitionistic logic. Indeed, we think that the best result in the theory of Fermat reals would be to prove that the category of smooth spaces $\mathcal{C}^\infty$ and that of Fermat spaces ${}^\bullet\mathcal{C}^\infty$ are really topoi: in this way the reader working in this theory would have the possibility to use the internal language of these topoi, in intuitionistic logic, and at the same time a sufficiently simple model to work directly in classical logic or to interpret the results obtained using the internal language. We plan to achieve some steps in this direction in future works.

Moreover, from the intuitive, classical, point of view, SDG sometimes presents counter-intuitive properties. For example, it is a little strange to think that we do not have "examples" of infinitesimals in SDG (it is only possible to prove that $\neg\neg\exists d \in \Delta$), so that, e.g., we cannot construct a physical theory containing a fixed infinitesimal parameter; moreover any $d \in \Delta$ is at the same time negative $d \leq 0$ and positive $d \geq 0$; finally the definition of the Lie brackets using $h \cdot k$ for $h, k \in \Delta$, i.e.

$$[X, Y]_{h \cdot k} = Y_{-k} \circ X_{-h} \circ Y_k \circ X_h,$$

is very far to the usual definitions given on manifolds.

B.3 Weil functors

Weil functors (in the following WF; see Kolár et al. [1993] and Kriegl and Michor [1996]) represent a way to introduce some kind of useful infinitesimal method without the need to possess a non-trivial background in mathematical logic. The construction of WF does not achieve the construction of a whole "infinitesimal universe", like in the theory of Fermat reals or in NSA and SDG, but it defines functors $T_A : \mathbf{Man} \longrightarrow \mathbf{Man}$, related to certain geometrical constructions of interest, starting from a *Weil algebra*. A Weil algebra is a real commutative algebra with unit of the form $A = \mathbb{R} \cdot 1 \oplus N$, where N is a finite dimensional ideal of nilpotent elements. The flexibility of its input A gives a corresponding flexibility to the construction of these functors. But, generally speaking, if one changes the geometrical problem, one has also to change the algebra A and so the corresponding functor T_A. E.g., if $A = \mathbb{R}[x]/\langle x^2 \rangle$, then T_A is the ordinary tangent bundle functor, whereas if $B = R[x,y]/\langle x^2, y^2 \rangle$, then $T_B = T_A \circ T_A$ is the second tangent bundle. The definition of a WF starting from a generic Weil algebra A is very long, and we refer the reader e.g. to Kriegl and Michor 1997, 1996, Kolár et al. 1993. Note that, in the previous example, x, $y \in B$ verify $x^2 = y^2 = 0$ but $x \cdot y \neq 0$. This provides us the first difference between WF and Fermat reals. In fact ${}^\bullet\mathbb{R} = \mathbb{R} \cdot 1 \oplus D_\infty$ and $\dim_\mathbb{R} D_\infty = \infty$, so that using the infinitesimals of ${}^\bullet\mathbb{R}$ we can generate a large family of Weil algebras, e.g. any $A = \mathbb{R} \cdot 1 \oplus N \subset \mathbb{R} \cdot 1 \oplus D_k$ (which represents k-th order infinitesimal Taylor's formulas) where N is an $\mathbb{R}$-finite dimensional ideal of infinitesimals taken in D_k. On the other hand, not every algebra can be generated in this way, e.g. the previous $B = R[x,y]/\langle x^2, y^2 \rangle$. But using exponential objects of $\boldsymbol{\mathcal{C}}^\infty$ and ${}^\bullet\boldsymbol{\mathcal{C}}^\infty$ we can give a simple infinitesimal representation of a large class of WF. For $\alpha_1, \ldots, \alpha_c \in \mathbb{N}^n$, $c \geq n$, let

$$D_k^\alpha := \{h \in D_{k_1} \times \ldots \times D_{k_n} \mid h^{\alpha_i} = 0 \quad \forall i = 1, \ldots, c\}.$$

E.g. if $k_1 = (3,0)$, $k_2 = (0,2)$ and $\alpha = (1,1)$, then $D_k^\alpha = \{(h,k) \in D_3 \times D_2 \mid h \cdot k = 0\}$. To any infinitesimal object D_k^α there is associated a corresponding Taylor's formula: let $f = {}^\bullet g|_{D_k^\alpha}$, with $g \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$, then

$$f(h) = \sum_{r \in \iota(\alpha)} \frac{h^r}{r!} \cdot m_r \quad \forall h \in D_k^\alpha. \tag{B.3.1}$$

Here $\iota(\alpha) := \{r \in \mathbb{N}^n \mid \exists h \in D_k^\alpha : h^r \neq 0, |r| \leq k\}$ is the set of multi-indexes $r \in \mathbb{N}^n$ corresponding to a non zero power h^r, and $k := \max(k_1, \ldots, k_n)$. The coefficients $m_r = \frac{\partial^r g}{\partial x^r}(0) \in \mathbb{R}$ are uniquely determined by the formula (B.3.1). We can therefore proceed generalizing the definition 13.1.1 of standard tangent functor.

Definition B.3.1. *If $M \in$ **Man** is a manifold, we call $M^{D_k^\alpha}$ the $\boldsymbol{\mathcal{C}}^\infty$ object* *with support set*

$$\left|M^{D^\alpha_k}\right| := \{{}^\bullet f|_{D^\alpha_k} \,:\, f \in \mathcal{C}^\infty(\mathbb{R}^n, M)\},$$

and with generalized elements of type U (open in $\mathbb{R}^u$) defined by:

$$d \in_U M^{D^\alpha_k} \quad :\Longleftrightarrow \quad d : U \longrightarrow \left|M^{D^\alpha_k}\right| \quad \text{and} \quad d \cdot i \in_{\bar{U}} {}^\bullet M^{D^\alpha_k},$$

where $i : \left|M^{D^\alpha_k}\right| \hookrightarrow {}^\bullet M^{D^\alpha_k}$ is the inclusion.

Let us note explicitly that writing $M^{D^\alpha_k}$ we are doing an abuse of notation because this is not an exponential object. We can extend this definition to the arrows of **Man** by setting $f^{D^\alpha_k}(t) := t \cdot f \in N^{D^\alpha_k}$, where $t \in M^{D^\alpha_k}$ and $f \in \mathbf{Man}(M, N)$. With these definitions we obtain a product preserving functor $(-)^{D^\alpha_k} : \mathbf{Man} \longrightarrow \mathbf{Man}$. Finally we have a natural transformation $e_0 : (-)^{D^\alpha_k} \longrightarrow 1_{\mathbf{Man}}$ defined by evaluation at $0 \in \mathbb{R}^n$: $e_0(M)(t) := t(0)$. The functor $(-)^{D^\alpha_k}$ and the natural transformation e_0 verify the "locality condition" of Theorem 1.36.1 in Kolár et al. [1993]: if U is open in M and $i : U \hookrightarrow M$ is the inclusion, then $U^{D^\alpha_k} = e_0(M)^{-1}(U)$ and $i^{D^\alpha_k}$ is the inclusion of $U^{D^\alpha_k}$ in $M^{D^\alpha_k}$. We can thus apply the above cited theorem to obtain that $(-)^{D^\alpha_k}$ is a Weil functor, whose algebra is

$$\mathrm{Al}\left((-)^{D^\alpha_k}\right) = \mathbb{R}^{D^\alpha_k}.$$

Not every Weil functor has this simple infinitesimal representation. E.g., the second tangent bundle $(-)^D \circ (-)^D$ is not of type $(-)^{D^\alpha_k}$; indeed it is easy to prove that the only possible candidate could be $D^\alpha_k = D \times D$, but $(\mathbb{R}^D)^D$ is a four dimensional manifold, whereas $\mathbb{R}^{D\times D}$ has dimension three. We do not have this kind of problems with the functor $(-)^{D^\alpha_k} = {}^\bullet\mathcal{C}^\infty(D^\alpha_k, -) : {}^\bullet\mathcal{C}^\infty \longrightarrow {}^\bullet\mathcal{C}^\infty$ which generalizes the previous one as well as $\mathrm{T}M = {}^\bullet M^D$ generalizes the standard tangent functor. In fact because of cartesian closedness we have

$$\left(X^{D^\alpha_k}\right)^{D^\beta_h} \simeq X^{D^\alpha_k \times D^\beta_h}$$

and $D^\alpha_k \times D^\beta_h$ is again of type D^α_k.

Summarizing, we can affirm that WF permit to consider nilpotent infinitesimals which are more algebraic and hence more general than those occurring in the Fermat reals. The typical example is the WF T_B for $B = R[x, y]/\langle x^2, y^2\rangle$, corresponding to the second tangent bundle. On the other hand, WF do not permit to consider an extension of the real field with the addition of new infinitesimal points (like in our framework, where we have the extension from $\mathbb{R}$ to ${}^\bullet\mathbb{R}$), and hence they do not permit to consider properties like order between infinitesimals, an extension functor analogous of the Fermat functor and the related properties, like the transfer theorem, tangent vectors as infinitesimal curves, infinitesimal parallelograms to add tangent vectors, infinitesimal fluxes, and so on. This implies that with the WF we do not have "a framework with the possibility to extend standard spaces adding infinitesimals", but we are forced to consider a new WF for every geometrical construction we are considering. Finally, the general definition of WF works on the category of smooth manifolds modelled on convenient vector spaces, because it needs the existence of charts (see Kriegl and Michor [1997, 1996], Kolár et al. [1993]), and we already mentioned (see Section 5.3) that this category is not cartesian closed. Therefore, WF cannot be defined for spaces like N^M, where M is a non-compact manifold. On the contrary, we have seen (see Chapter 13) that some results of infinitesimal differential geometry can be obtained also for spaces of the form ${}^\bullet N^{\bullet M}$, where M is a generic manifold.

Finally, a recent approach similar in essence to Weil functors is *differential geometry over a general base ring*, see Bertram [2008] and references therein. The basic idea is to develop, as far as possible, all the topics of differential geometry not dealing with integration theory, in the framework of manifolds modelled over a generic topological module V over a topological ring $\mathbb{K}$. This of course, includes ordinary finite dimensional real or complex manifolds, but also infinite dimensional manifolds modelled on Banach spaces and even on the hyper-vector spaces ${}^*\mathbb{R}^n$. One of the basic results is that in this way the tangent functor TM becomes a manifold over the scalar extension $V \oplus \varepsilon V$, i.e. over the module of all the expressions of the form $u + \varepsilon v$ over the ring $\mathbb{K}[\varepsilon]$ of dual numbers over $\mathbb{K}$, i.e. $\mathbb{K}[\varepsilon] := \mathbb{K} \oplus \varepsilon\mathbb{K} := \mathbb{K}[x]/\left(x^2\right)$. The process can be iterated obtaining that the double tangent bundle T^2M is a manifold over $V \oplus \varepsilon_1 V \oplus \varepsilon_2 V \oplus \varepsilon_1\varepsilon_2 V$, which is a module over $\mathbb{K}[\varepsilon_1, \varepsilon_2] := \mathbb{K}[x, y]/\left(x^2, y^2\right)$. Analogous results are available for T^kM and for the jet bundle J^kM. The theory is appealing for its generality and for the possibility to obtain in a simple way a context with formal infinitesimals. This construction does not deal with cartesian closedness and hence generic spaces like $\mathbf{Man}(M, N)$ cannot be considered.

B.4 Surreal numbers

Surreal numbers has been introduced by J.H. Conway and presented in Knuth [1974] and in Conway [1976][2]. One of the most surprising features of surreal numbers is that starting from a simple set of rules it is possible to construct a rich algebraic structure containing the real numbers as well as infinite and infinitesimals, but also all the ordinal numbers, the hyperreals of NSA, the Levi-Civita field and the field of rational functions. Indeed, in a precise sense we will see later, the ordered field **No** of surreal numbers is the largest possible ordered field or, in other words, the above mentioned simple rules for the construction of surreal numbers, represent the most general

[2]Really, the same field of numbers has been predate by Cuesta Dutari [1954] (in Spanish) and Harzheim [1964] (in German).

way to obtain a notion of number culminating in an ordered field.

There are two basic ideas to introduce surreal numbers: the first is to have the possibility to construct numbers in a transfinite-recursive way using a notion analogous to that of Dedekind cut (called *Conway cut*). If we have a totally ordered set $(N,<)$, a Conway cut is simply a pair (L,R) of subsets L, $R\subseteq N$ such that

$$\forall l\in L\,\forall r\in R:\quad l<r, \tag{B.4.1}$$

in this case we will simply write $L<R$. This is exactly the notion of Dedekind cut without the condition that the subsets L, R have to be contiguous (i.e. without the condition that $\forall\varepsilon>0\,\exists\, l\in L\,\exists\, r\in R:\ |l-r|<\varepsilon$). Exactly because we do not have this further condition, we need another condition for a pair (L,R) to identify a unique "number". Indeed, the second idea, intuitively stated, is that every Conway cut identifies uniquely *the simplest number* x between L and R:

$$\forall l\in L\,\forall r\in R:\quad l<x<r. \tag{B.4.2}$$

We can intuitively represent a Conway cut and the associated simplest number in the following way

$$\overline{\qquad L \qquad}\quad x \quad\overline{\qquad R \qquad}$$

A little more formally, the class **No** of surreal numbers is introduced by Conway using a suitable set of rules. We can think at these rules as axioms defining a suitable structure $(\mathbf{No},\le,\{-\,|\,-\})$. In the following, as usual, $x<y$ means $x\le y$ and $x\ne y$.

Construction If L, $R\subseteq\mathbf{No}$ and $L<R$, then $\{L\,|\,R\}\in\mathbf{No}$, that is starting from a Conway cut (L,R) we can construct a surreal with $\{L\,|\,R\}\in\mathbf{No}$.

Surjectivity If $x\in\mathbf{No}$, then there exist L, $R\subseteq\mathbf{No}$ such that $L<R$ and $x=\{L\,|\,R\}$, that is all surreal numbers can be constructed starting from a Conway cut.

Inequality If $x=\{L_x\,|\,R_x\}$ and $y=\{L_y\,|\,R_y\}$ are well defined[3], then $x\le y$ if and only if $L_x<\{y\}$ and $\{x\}<R_y$, i.e. $l_x<y$ and $x<r_y$ for every $l_x\in L_x$ and every $r_y\in R_y$. This rule can be represented in the following way

$$\overline{\qquad L_x \qquad}\quad x \quad\overline{\qquad R_x \qquad}\ \cdots\ \overline{\qquad L_y \qquad}\quad y \quad\overline{\qquad R_y \qquad}$$

[3]That is $L_x<R_x$ and $L_y<R_y$. Let us note that using a notation like $x=\{L_x\,|\,R_x\}$ we do not mean that a number $x\in\mathbf{No}$ uniquely determines the subsets L_x and R_x.

Equality If x, $y \in \mathbf{No}$, then $x = y$ if and only if $x \leq y$ and $y \leq x$, that is equality between surreal numbers can be defined starting from the order relation $\leq$.

Starting from these simple rules/axioms we can already construct several meaningful examples of numbers in $\mathbf{No}$. From the definition (B.4.1) of $L < R$ we see that always $L < \emptyset$ and $\emptyset < R$ for every L, $R \subseteq \mathbf{No}$. So we have $\emptyset < \emptyset$ and from the Construction rule $\{\emptyset \,|\, \emptyset\} \in \mathbf{No}$. Therefore, $\mathbf{No}$ is not empty and we can iterate the process. For simplicity, we will write $\{|\} := \{\emptyset \,|\, \emptyset\}$, $\{L \,|\} := \{L \,|\, \emptyset\}$, $\{|\, R\} := \{\emptyset \,|\, R\}$, $\{x_1, \ldots, x_n \,|\, R\} := \{\{x_1, \ldots, x_n\} \,|\, R\}$ and $\{L \,|\, x_1, \ldots, x_n\} := \{L \,|\, \{x_1, \ldots, x_n\}\}$. Hence we have, e.g.,

$$\begin{aligned} &\{|\} \in \mathbf{No} \\ &x \in \mathbf{No} \quad \Longrightarrow \quad \{|x\}, \ \{x|\} \in \mathbf{No}. \end{aligned}$$

But the understanding of the class $\mathbf{No}$ has a great improvement if we introduce the above mentioned interpretation of simplicity. Conway's idea is that a number $x \in \mathbf{No}$ is simpler than $y \in \mathbf{No}$ if x is defined before y in the previous iterative process (using Conway's terminology: x was born before y). So, $\{|\}$ is the simplest number[4] and this justify the definition $\{|\} =: 0$. On the next step, we have e.g. $\{|0\}$ and $\{0|\}$ which have the same degree of simplicity (because they are both defined in the second step of the iterative process). To interpret these numbers we have hence to use the idea of simplicity expressed in (B.4.2): $\{0|\}$ is the simplest number greater than 0 and $\{|0\}$ is the simplest number less than 0. This justify the definition $1 := \{0|\}$ and $-1 := \{|0\}$. Up to isomorphisms we can hence affirm that $\mathbb{N}$, $\mathbb{Z} \subseteq \mathbf{No}$. Another meaningful example based on this interpretation is the number $\{0 \,|\, 1\}$ which has to be thought as the simplest number between 0 and 1, i.e. $\frac{1}{2} := \{0 \,|\, 1\}$. From the Inequality rule we can prove that $\{|\, x\} \leq 0 \leq \{x \,|\}$ for every $x \in \mathbf{No}$, and that $1 \not\leq 0$, hence $0 < 1$ follows from the Equality rule. Analogously one can prove that $\ldots < -3 < -2 < -1 < 0 < 1 < 2 < 3 < \ldots$ Moreover, we can also easily see that e.g. $\{-1 \,|\, 1\} = 0$ so we note that different subsets L, $R \subseteq \mathbf{No}$ can define the same number $x = \{L \,|, R\}$.
But now we also have that $\mathbb{N} \subseteq \mathbf{No}$ and hence we can form the number $\mathbb{N} + 1 := \{\mathbb{N} \,|\}$ and this catch a glimpse of the possibility to extend all this using transfinite induction.

Instead of further proceeding with Conway's approach to $\mathbf{No}$ we want to sketch his point of view to the foundational questions arising from his construction. These ideas are precisely stated in the Appendix to Part Zero of [Conway, 1976]. The mainstream's approach to a topic like $\mathbf{No}$, where

[4] Of course, at this stage of developement and using this not-strictly formal point of view, our use of the notion of "simplicity" is only informal and it is natural to ask for a more formal definition, considering, moreover, its uniqueness. This will be done in the next section.

one must use e.g. transfinite induction, is to fix a formal theory of sets (like Zermelo-Fraenkel **ZF** theory or von Neumann-Bernay-Gödel **NBG** theory) and to formalize every construction inside that theory. Sometimes, this formalization can conduct to a theory far from the original intuition, because different formalizations are possible of a given informal construction. Let us note explicitly that here J. Conway's term of judgment is the searching for a good dialectic between informal constructions and their formal counterpart, which has been a leading thread of all the present work. Conway's foundational point of view can be summarized citing, as in [Conway, 1976]:

> It seems to us, however, that mathematics has now reached the stage where formalization within some particular axiomatic theory is irrelevant, even for foundational studies. It should be possible to specify conditions on a mathematical theory which would suffice for embeddability within **ZF** (supplemented by additional axioms of infinity if necessary), but which do not otherwise restrict the possible constructions in that theory.

The Cuesta Dutari definition of No

From our point of view, one has the best result if there exists a formalization respecting the above mentioned good formal-informal dialectic, and hence a great effort has to be dedicated to the searching for this best formalization, if any, before assuming a point of view so general and radical like J. Conway's one. This type of formalization for surreal numbers is possible and is given by [Cuesta Dutari, 1954]. In this appendix we only sketch the first steps of this development; for a complete treatment, see [Alling, 1987]. The set theory we will consider is **NBG**.

Definition B.4.1. *Let* $(T, \leq)$ *a totally ordered set, then we say that* (L, R) is a Cuesta Dutari cut in T *iff*

1. $L,\ M \subseteq T$
2. $L < R$
3. $L \cup R = T$.

Moreover, we will denote by $\mathrm{CD}(T)$ *the class of all Cuesta Dutari cuts in* T*.*

Essentially a Cuesta Dutari cuts is a Conway cut with the additional condition $L \cup R = T$. Since $(\emptyset, T)$ and $(T, \emptyset)$ are always Cuesta Dutari cuts, we have that $\mathrm{CD}(T)$ is never empty.

If we think than each Cuesta Dutari cut identifies a new number, the union $T \cup \mathrm{CD}(T)$ can be thought of as a *completion* of the totally ordered set $(T, \leq)$ if we can extend the order relation $\leq$ to this union. This is done in the following

Definition B.4.2. *Let $(T, \leq)$ be a totally ordered set, then on the* Cuesta Dutari completion $\chi(T) := T \cup \mathrm{CD}(T)$ of T *we define the order relation:*

1. *If $x, y \in T$ then we will say that x is less than or equal to y iff $x \leq y$ in T. Because of this first case, the order relation on $\chi(T)$ will be denoted again by the symbol $\leq$.*
2. *If $x \in T$ and $y = (L, R) \in \mathrm{CD}(T)$, then:*
   (a) $x \in L \quad \Longrightarrow \quad x < y$
   (b) $x \in R \quad \Longrightarrow \quad y < x$
3. *If $x = (L_x, R_x)$, $y = (L_y, R_y) \in \chi(T)$, then $x < y$ iff $L_x \subset L_y$.*

It is indeed possible to prove (see Alling [1987]) that $(\chi(T), \leq)$ is a totally ordered set. For example if we take t, $\tau \in T$ with $t < \tau$, we can consider the cut $c = ((-\infty, t], [\tau, +\infty))$ and we have $t < c < \tau$. If $x = (L, R) \in \mathrm{CD}(T)$, then $L < \{x\} < R$ and, as a further example, $(\emptyset, T)$ is the least element of $\chi(T)$, whereas $(T, \emptyset)$ is the greatest element.

So, how can we form 0 using Cuesta Dutari cuts? We do not have to think at the Cuesta Dutari completion as a final completion starting from a single given ordered set $(T, \leq)$ but, instead, as a tool for a transfinite-recursive construction:

Definition B.4.3. *Let* **On** *be the class of all ordinals, we define by transfinite recursion the family $(T_\alpha)_{\alpha \in \mathbf{On}}$ of ordered sets given by:*

1. *T_0 is the empty set ordered with the empty relation,*
2. *For every $\beta \in \mathbf{On}$:*
   (a) *If $\alpha + 1 = \beta$, then $T_\beta := \chi(T_\alpha)$*
   (b) *If β is a non-zero limit ordinal, then $T_\beta := \bigcup_{\alpha<\beta} T_\alpha$.*

Finally we set $\mathbf{No} := \bigcup_{\alpha \in \mathbf{On}} T_\alpha$.

So, e.g., $0 := (\emptyset, \emptyset) \in T_1 = \chi(T_0) \subset \mathbf{No}$. The ordinal index α in the previous transfinite recursive definition gives the notion of simplicity of a number $x \in \mathbf{No}$, that is its *birthday* using Conway's terminology.

Definition B.4.4. *If $x \in \mathbf{No}$, we define the* birth-order *function by*

$$b(x) := \min_{\leq} \{\alpha \in \mathbf{No} \,|\, x \in T_\alpha\},$$

where $\leq$ is the order relation defined on **No**.

So we have, e.g., $b(0)=0$, $b(1)=b(-1)=1$, $b(2)=b\left(\frac{1}{2}\right)=b\left(-\frac{1}{2}\right)=2$.

At this point, the most important result is Conway's simplicity theorem: it states that every Conway cut (L,R) determines uniquely the simplest surreal number filling the gap between the subsets L and R:

Theorem B.4.5. *Let L, $R \subseteq \mathbf{No}$ with $L<R$, then there exist one and only one $x \in \mathbf{No}$ such that:*

1. $L<\{x\}<R$
2. *If $y \in \mathbf{No} \setminus \{x\}$ verifies $L<\{y\}<R$, then $b(x)<b(y)$.*

This unique x will be denoted by $\{L \,|\, R\}$.

For a proof see Alling [1987].

Ring operations on No

On the class of surreal numbers we can define all the field operations, the simplest one being subtraction:

$$x=\{L \,|\, R\} \quad \Longrightarrow \quad -x:=\{-R \,|\, -L\}.$$

The definition of sum and product requires some motivations. Let $x=\{L_x \,|\, R_x\}$ and $y=\{L_y \,|\, R_y\}$ be surreal numbers, so that

$$l_x<x<r_x \tag{B.4.3}$$
$$l_y<y<r_y \tag{B.4.4}$$

for every $l_x \in L_x$, $r_x \in R_x$, $l_y \in L_y$ and $r_y \in R_y$. Since we want $\mathbf{No}$ to be an ordered group with respect to addition, we must have:

$$\begin{gathered}
l_x+y<x+y \\
x+l_y<x+y \\
x+y<r_x+y \\
x+y<x+r_y .
\end{gathered}$$

Hence Conway defines $x+y$ as the simplest number verifying these inequalities, i.e. using transfinite recursion we can define

$$x+y:=\{(L_x+y)\cup(x+L_y) \,|\, (R_x+y)\cup(x+R_y)\},$$

where, e.g., $L_x+y:=\{l_x+y \,:\, l_x \in L_x\}$. Analogously we can proceed to justify the definition of product. From (B.4.3) and (B.4.4), in the hypothesis that $\mathbf{No}$ be an ordered group under multiplication, we must have that $x-l_x$,

$y - l_y$, $r_x - x$ and $r_y - y$ are all greater than zero. Taking all the products of these terms involving an x and a y we have

$$\begin{aligned}
0 &< (x - l_x) \cdot (y - l_y) = xy - l_x y - x l_y + l_x l_y \\
0 &< (r_x - x) \cdot (r_y - y) = xy - r_x y - x r_y + r_x r_y \\
0 &< (x - l_x) \cdot (r_y - y) = -xy + l_x y + x r_y - l_x r_y \\
0 &< (r_x - x) \cdot (y - l_y) = -xy + r_x y + x l_y + r_x l_y.
\end{aligned}$$

As a consequence, from these we get inequalities bounding xy:

$$\begin{gathered}
l_x y + x l_y - l_x l_y < xy < l_x y + x r_y - l_x r_y \\
r_x y + x r_y - r_x r_y < xy < r_x y + x l_y + r_x l_y.
\end{gathered}$$

We can hence define (once again by transfinite recursion):

$$\begin{aligned}
&L_{x \cdot y} := (L_x y + x L_y - L_x L_y) \cup (R_x y + x R_y - R_x R_y) \\
&R_{x \cdot y} := (L_x y + x R_y - L_x R_y) \cup (R_x y + x L_y + R_x L_y) \\
&x \cdot y := \{L_{x \cdot y} \,|\, R_{x \cdot y}\},
\end{aligned}$$

where e.g. $L_x y := \{l_x \cdot y \,:\, l_x \in L_x\}$ and $L_x L_y := \{l_x \cdot l_y \,|\, l_x \in L_x\,,\, l_y \in L_y\}$.Using these definitions we can prove that **No** verifies the axioms of an ordered field.

Examples of surreal numbers

As we already sketched, up to isomorphism we have $n = \{0, 1, 2, \ldots, n - 1 \,|\} \in \mathbb{N} \subseteq \mathbf{No}$, $-n = \{| -n + 1, -n + 2, \ldots, -2, -1, 0\} \in \mathbb{Z} \subseteq \mathbf{No}$, but also $\omega := \{\mathbb{N} \,|\}$. It results $n < \omega$ for every $n \in \mathbb{N}$ and hence **No** is a non-Archimedean field. Moreover, because we have an ordered field containing the integers, we also have $\mathbb{Q} \subseteq \mathbf{No}$, i.e. all the rationals can be seen as surreal numbers. Finally, using Dedekind cuts we can also identify $\mathbb{R}$ with a subfield of **No**.

But using transfinite induction we can also define

$$\begin{aligned}
&\varphi(0) := 0 \\
&\varphi(\beta) := \{\{\varphi(\alpha) \,:\, \alpha < \beta\} \,|\} \in \mathbf{No} \quad \forall \beta \in \mathbf{On},
\end{aligned}$$

and we can prove that φ is an order-preserving map from the class of all ordinals **On** into **No**, with birthday function verifying $b(\varphi(\beta)) = \beta$. This also proves that **No** is a proper class and not a set because φ, being order preserving, is injective. For this reason usually one says that **No** is a Field, with the capital initial letter to underline that it verifies the axioms of a field, but its support set is a proper class.

We have already seen that in **No** we have infinities like $\omega = \{\mathbb{N}\,|\}$, but we can also easily construct infinitesimals like

$$\varepsilon := \left\{ 0 \,\middle|\, \left\{ \frac{1}{n} : n \in \mathbb{N}_{>0} \right\} \right\},$$

which is strictly positive but smaller than any strictly positive real.

All these examples can conduct us toward the conjecture that the class **No** is some kind of "universal" field containing every possible extension of the real field. Indeed we have the following theorem (see Conway [1976] for the proof; see also Ehrlich [1988] for a more general and systematic treatment)

Theorem B.4.6. *The field* **No** *verifies the following properties*

1. **No** *is an ordered Field*
2. *If*

$$A \text{ is an ordered subfield of } \mathbf{No}$$
$$A \text{ is an ordered subfield of } B,$$

with $|A|$ *and* $|B|$ *sets and not proper classes, then there exist*

$$B' \text{ ordered subfield of } \mathbf{No}$$
$$f : B \xrightarrow[\sim]{} B' \text{ isormophism of ordered fields}$$

such that $f|_A = 1_A$.

Moreover, if F *verifies these properties 1. and 2 (like* **No** *does). then* $F \simeq$ **No** *as an ordered field.*

We can represent the situation in the following way: if we have (as diagram of morphisms between ordered fields)

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, hook] & \mathbf{No} \\ B & \end{tikzcd}$$

then we can complete it with the commutative diagram

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, hook] & \mathbf{No} \\ B \arrow[r, "f", "\sim"'] & B' \arrow[u, hook] \end{tikzcd}$$

From this point of view the field of surreal numbers is remarkably inclusive[5]. For example applying the previous theorem with $A = \mathbb{R}$ and $B = {}^*\mathbb{R}$, we obtain that **No** contains, up to isomorphism, the hyperreals of NSA.

Comparison with Fermat reals

The first comparison between surreals and Fermat reals comes from the previous Theorem (B.4.6) which cannot be applied to the ring ${}^\bullet\mathbb{R}$. More trivially, the existence of non-zero nilpotent infinitesimals is not compatible with field axioms.

Moreover, the construction of **No** is deeply based on order properties and produces a single numeric field and not a category of extended spaces, including manifolds, like our ${}^\bullet\mathcal{C}^\infty$.

The field **No** has many remarkable properties, it is a real closed field, there is the possibility to define exponential and logarithm and even a notion of Riemann integral (see Fornasiero [2004]). On the other hand, like any other non Archimedean ordered field, **No** is totally disconnected, therefore we have examples of functions differentiable on an interval with everywhere zero derivative which are not constant, we do not have the uniqueness of the primitive of a continuous function and we do not have uniqueness in the simplest initial value problem: $y'(x) = 0$, $y(0) = 0$. This cannot be directly compared with our results regarding the development of the calculus on the Fermat reals (see Chapters 11 and 12) because our results are applicable to smooth functions only and not to a lower degree of differentiability.

From a methodological point of view, as we have already sketched above, Conway's construction seems to be based on the search of a theory with strong intuitive meaning, essentially due to Conway's simplicity theorem (B.4.5). Formalization like Cuesta Dutari [1954] and Alling [1987] permit to obtain a good dialectic between formal theory and intuitive interpretation, which is also the leading design of the present work.

B.5 Levi-Civita field

The Levi-Civita field (from now on: LCF) originally appeared in Levi-Civita [1893] and Levi-Civita [1898], but it was subsequently rediscovered by Ostrowski [1935], Neder [1941–1943], Berz [1992] and Berz [1994] (to whom, together with K. Shamseddine, we can attribute the modern development of the topic). For an account of Levi-Civita's work see also Laugwitz [1975]. For a detailed work in this topic and the proofs of the theorems we will state in this section, see e.g. Shamseddine [1999]. Because of the several analogies

[5]From this point of view the name "surreal numbers" is less meaningful than the original Conway's "numbers" without any adjective.

between our Fermat reals and the LCF, we will introduce this topic with a certain level of detail.

To motivate the introduction of the LCF we start from the idea to add to the real field $\mathbb{R}$ a single new invertible infinitesimal number d to which we want to apply all the ring operations but also arbitrary roots. Hence, we would like to be able to form numbers like $3+d+2d^2$, but also like d^{-1}, $d^{1/2}$, $d^{-1/2}$, $2-d^3+4d^{3/5}-\frac{1}{2}d^{-2/3}$. This can be easily obtained using purely algebraic methods, e.g. considering formal power series of the form

$$x=\sum_{q\in\mathbb{Q}}x_q\cdot d^q. \tag{B.5.1}$$

There is no problem in defining the sum pointwise

$$x+y:=\sum_{q\in\mathbb{Q}}x_q d^q+\sum_{q\in\mathbb{Q}}y_q d^q:=\sum_{q\in\mathbb{Q}}\left(x_q+y_q\right)d^q,$$

but we can recognize a first limitation considering the product, which is defined, as usual for formal power series, as

$$x\cdot y:=\sum_{q\in\mathbb{Q}}\left(\sum_{r+s=q}x_r\cdot y_s\right)\cdot d^q. \tag{B.5.2}$$

Indeed, the sum $\sum_{r+s=q}x_r\cdot y_s$ can have an infinite number of addends, depending on how many non-zero coefficients x_r and y_s we have in the factors x and y, and hence the previous definition of product $x\cdot y$ can be meaningless for generic formal power series x, y. Because we want that the definition (B.5.2) works for every pair x, y, we must introduce a limitation on the number of coefficients in our formal power series. In other words, we must limit the number of non zero coefficients in the formal series. For example we can have

$$\bar{x}=d^{-3}+d^{-2}+d^{-1}+1+d+d^2+d^3+d^4+d^5+\ldots,$$

or

$$\bar{y}=\ldots+d^{-5}+d^{-4}+d^{-3}+d^{-2}+d^{-1}+1+d+d^2.$$

More generally, the equation $r+s=q$ may have infinitely many solutions if there is an accumulation point for the indexes $s\in\mathbb{Q}$ such that $y_s\neq 0$. In the LCF the choice fall on power series with a finite number of exponents "on the left", i.e. such that

$$\forall q\in\mathbb{Q}:\ \operatorname{card}\left\{r\in\mathbb{Q}\,|\,x_r\neq 0\,,\ r\leq q\right\}\ \text{is finite.} \tag{B.5.3}$$

From our point of view, which is not near to the formal point of view expressed in the construction of the LCF - remember that at present we do not have a notion of convergence for our series - it seems hard to motivate this choice instead of the limitation "on the right". Moreover, let us note explicitly that if we do not want to introduce limitations on the exponents we consider in our power series, then we are forced to say that the product $x \cdot y$ is not always defined, but only for those pairs x, y such that the sum $\sum_{r+s=q} x_r \cdot y_s$ converges with respect to some notion of convergence[6]. This may seem strange from an algebraic point of view, but it can be considered more common in the calculus, where, e.g. in the standard Schwartz's theory of distribution, the product of two distributions is not always defined and can be considered in some cases only (see e.g. Colombeau [1992]), or where the set of convergent or divergent real sequences is not closed with respect to pointwise product.

Leaving the intuitive motivations to arrive to a more formal mathematics, we can introduce our formal power series thinking of the corresponding definition for polynomials: identifying a polynomial $a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$ with the n-tuple of its coefficients $(a_0, a_1, a_2, \ldots, a_n)$ is equivalent to say that a polynomial is a function of the form $a : \{0, 1, 2, \ldots, n\} \longrightarrow \mathbb{R}$. Analogously we can define

Definition B.5.1. *The support set $\mathcal{R}$ of the LCF is the set of all the functions $x : \mathbb{Q} \longrightarrow \mathbb{R}$ with* left-finite support*, i.e. such that*

$$\forall q \in \mathbb{Q} : \ \operatorname{card} \{ r \in \mathbb{Q} \mid x_r \neq 0 \ , \ r \leq q \} \ \textit{ is finite.}$$

So, the value[7] $x(q) =: x[q]$ has to be thought as the coefficient of the addend $x[q] \cdot d^q$. The ring operations are defined for $q \in \mathbb{Q}$ as

$$\begin{aligned} (x + y)[q] &:= x[q] + y[q] \\ (x \cdot y)[q] &:= \sum_{\substack{r, s \in \mathbb{Q} \\ r+s=q}} x[r] \cdot y[s], \end{aligned}$$

and we can verify that $(\mathcal{R}, +, \cdot)$ becomes a field (see e.g. Shamseddine [1999]). In a formal power series like (B.5.1) the leading term $x_m \cdot d^m$ with the lower value $m \in \mathbb{Q}$ of the exponent determines the behavior of the number from several points of view, e.g. with respect to order. For this reason we introduce the following notations:

Definition B.5.2. *Let x, $y \in \mathcal{R}$, then*

1. $\operatorname{supp}(x) := \{ q \in \mathbb{Q} \mid x[q] \neq 0 \}$

[6] This notion may be trivial, e.g. if we consider only those pairs for which for every $q \in \mathbb{Q}$, there is only a finite number of exponents r, s such that $r + s = q$.

[7] The notation with square brackets $x[q]$ permits to avoid confusion when one consider functions defined on the LCF $\mathcal{R}$.

2. $\lambda(x) := \min(\mathrm{supp}(x))$ *for* $x \neq 0$ *and* $\lambda(0) := +\infty$. *The term* $\lambda(x)$ *is called* order of magnitude.

3. $x \sim y \quad :\Longleftrightarrow \quad \lambda(x) = \lambda(y)$. *This relation is called* agreement of order of magnitude.

4. $x \approx y \quad :\Longleftrightarrow \quad \lambda(x) = \lambda(y) \quad and \quad x[\lambda(x)] = y[\lambda(y)]$

5. $x =_r y \quad :\Longleftrightarrow \quad \forall q \in \mathbb{Q}_{\leq r}: \ x[q] = y[q]$

If $r > 0$, the relation $=_r$ is the analogous of our equality up to k-th order infinitesimals (see Chapter 3): if $x =_r y$, then $x - y$ is given by sum of infinitesimals $a_q d^q$ of order $d > r$. E.g. we can expect to have $1+d+d^2-2d^3 =_2 1+d+d^2+4d^5$ or $d^{n+1} =_n 0$. So, it appears sufficiently clear that, even if we do not have a ring with nilpotent elements, the equivalence relation $=_r$ can supply a possible alternative language.

In the LCF we can prove the existence of roots:

Theorem B.5.3. *Let* $x \in \mathcal{R}$ *be non zero and* $n \in \mathbb{N}_{>0}$. *If* n *is even and the leading term* $x[\lambda(x)] > 0$, *then* x *has two* n*-th roots in* $\mathcal{R}$. *If* n *is even and* $x[\lambda(x)] < 0$, *then* x *has no* n*-th roots in* $\mathcal{R}$. *If* n *is odd, then* x *has a unique* n*-th root in* $\mathcal{R}$.

Exactly like in the study of polynomials, the particular number $d[q] := 1$ for $q = 1$ and $d[q] := 0$ otherwise, works as the independent variable in our formal power series, and the equality (B.5.1) can now be proved for every $x \in \mathcal{R}$ because $(d^r)[q] = 1$ if $q = r$ and $(d^r)[q] = 0$ otherwise. Let us note explicitly that if $r = \frac{p}{q} \in \mathbb{Q}$ with $p, q \in \mathbb{Z}$, $q > 0$, then $d^r = \sqrt[q]{d^p}$, so we need the previous Theorem B.5.3.

Obviously, the embedding of the reals is given by $r \in \mathbb{R} \mapsto r[-] \in \mathcal{R}$, where $r[0] = r$ and $r[q] = 0$ otherwise, but it is now also clear that formal Laurent series (and hence also D. Tall's superreal numbers, see e.g. Tall [1980]), i.e. numbers of the form

$$x = \sum_{k=-N}^{+\infty} x_k \cdot d^k$$

are embedded in the LCF.

Essential for the development of the LCF as an ordered field but also for the different notions of continuity and differentiability of functions $f : \mathcal{R} \longrightarrow \mathcal{R}$ is the order relation. As hinted above, we can define the order relation by comparison of the leading terms

Definition B.5.4. *If* $x, y \in \mathcal{R}$, *we define*

$$\begin{aligned} x > 0 \quad &:\Longleftrightarrow \quad x[\lambda(x)] > 0 \\ x > y \quad &:\Longleftrightarrow \quad x - y > 0 \end{aligned}$$

With this relation the LCF becomes a totally ordered field extending the real field.

In the framework of Fermat reals, the natural topology is the final one with respect to which any figure is continuous (see Section 6.2). As we mentioned ibidem, for Fermat reals, and more generally for Fermat spaces, the topology is a byproduct of the diffeological structure. Using this structure, we have a natural way to define smooth maps between $\mathcal{C}^\infty$-spaces and hence to extend these maps to the corresponding Fermat spaces, without any particular focusing on the topology. In the LCF, it is not so clear what functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ can be extended to the whole $\mathcal{R}$ and hence the approach is different and mimics the classical approach of calculus. The next step is hence to use the order relation to define a corresponding order topology.

Definition B.5.5. *Because the order relation on $\mathcal{R}$ is total, we can define the absolute value in the usual way*

$$|x| := \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Moreover, we say that a subset $U \subseteq \mathcal{R}$ is open in the order topology *iff*

$$\forall u \in U \; \exists \delta \in \mathcal{R}_{>0} : \quad \{x \in \mathcal{R} \, : \, |x - u| < \delta\} \subseteq U.$$

For example the sequence $(d^n)_{n \in \mathbb{N}}$ converges to the zero sequence in the order topology. Using the same idea, that is the formal analogy with the reals $\mathbb{R}$, we can define continuity, differentiability and convergence of sequences

Definition B.5.6. *Let $D \subseteq \mathcal{R}$ and $f : D \longrightarrow \mathcal{R}$, then we say that f* is topologically continuous at $x_0 \in D$ *iff*

$$\forall \varepsilon \in \mathcal{R}_{>0} \; \exists \delta \in \mathcal{R}_{>0} : \; \forall x \in D : \; |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon.$$

Definition B.5.7. *Let $D \subseteq \mathcal{R}$ and $f : D \longrightarrow \mathcal{R}$, then we say that f* is topologically differentiable at $x_0 \in D$ *iff there exists a number $l \in \mathcal{R}$ such that*

$$\forall \varepsilon \in \mathcal{R}_{>0} \; \exists \delta \in \mathcal{R}_{>0} \, \forall x \in D : \; 0 < |x - x_0| < \delta \Rightarrow \left| \frac{f(x) - f(x_0)}{x - x_0} - l \right| < \varepsilon.$$

Definition B.5.8. *Let $s : \mathbb{N} \longrightarrow \mathcal{R}$, then we say that s* converges strongly to $s \in \mathcal{R}$ iff

$$\forall \varepsilon \in \mathcal{R}_{>0} \; \exists N \in \mathbb{N} : \; \forall n \in \mathbb{N} : \; n \geq N \Rightarrow |s_n - s| < \varepsilon.$$

It is interesting that now, using this notion of convergence, we can associate to our formal power series a notion of convergence:

Theorem B.5.9. *Let $x \in \mathcal{R}$ and define recursively*

$$q_0 : = \lambda(x)$$

$$q_{k+1} := \begin{cases} \min\left(\mathrm{supp}(x) \setminus \{q_0, q_1, \dots, q_k\}\right) & \text{if } \mathrm{supp}(x) \supset \{q_0, q_1, \dots, q_k\} \\ 0 & \text{otherwise} \end{cases}$$

then the sequence $n \in \mathbb{N} \mapsto \sum_{k=0}^{n} x[q_k] \cdot d^{q_k} \in \mathcal{R}$ converges strongly to x, so that we can write

$$x = \sum_{k=0}^{+\infty} x[q_k] \cdot d^{q_k}.$$

Note that this theorem does not realize the above mentioned dialogue between potential infinitesimals and actual infinitesimals because, trivially, in its statement there is no mention of any such potential infinitesimal, i.e. of a function of the form $i : E \longrightarrow \mathbb{R}$ convergent to zero, instead in the statement convergence is understood in the LCF sense.

We can now give some motivations for the choice of the domain $\mathbb{Q}$ for the elements of the LCF $\mathcal{R}$: why is there, in the definition of $\mathcal{R}$, the field $\mathbb{Q}$ instead of $\mathbb{R}$ like in our Fermat reals[8]? The answer can be anticipated saying that $\mathbb{Q}$ is the simplest domain to obtain some of the desired properties. To render this statement more precise we need the notion of skeleton group.

Definition B.5.10. *Let $(F, +, \cdot, <)$ be a totally ordered field and define in it the absolute value as usual. Let a, $b \in F_{\neq 0}$, then we say*

$$a \ll b \qquad :\iff \qquad \forall n \in \mathbb{N} : \; n \cdot a < b$$

and we will read it a is infinitely smaller than b. *Moreover, we will say*

$$a \sim b \qquad :\iff \qquad \neg(|a| \ll |b|) \quad \text{and} \quad \neg(|b| \ll |a|).$$

The relation $\sim$ is an equivalence relation, and we will denote by

$$S_F := \{[a]_\sim \mid a \in F_{\neq 0}\}$$

the set of all its equivalence classes. Moreover, it is possible to prove that the following definitions are correct:

$$[a]_\sim \cdot [b]_\sim : = [a \cdot b]_\sim$$
$$[a]_\sim^{-1} : = [a^{-1}]_\sim$$

$$[a]_\sim < [b]_\sim \qquad :\iff \qquad \forall n \in \mathbb{N} : \; n|a| < |b|.$$

[8] Recall e.g. that we can consider infinitesimal $\mathrm{d}t_a$ for every real number $a \geq 1$.

It is possible to prove that $(S_F, \cdot, <)$ is a totally ordered group, called the *skeleton group of* F. This notion is naturally tied with the notion of non-Archimedean field. Indeed, the skeleton group of the real field is trivial $S_{\mathbb{R}} = \{[1]_\sim\}$, but it is not so for non-Archimedean fields, as stated in the following

Theorem B.5.11. *Let F be a totally ordered non-Archimedean field, then*

$$\mathbb{Z} \subseteq S_F.$$

Moreover, if F admits roots of positive elements, then

$$\mathbb{Q} \subseteq S_F.$$

This motivates why we take $\mathbb{Q}$ as domain of our functions $x \in \mathcal{R}$, that is as exponents of d in our formal power series: it is the smallest set of exponents that permits to have a non-Archimedean field and roots of positive elements.

But the idea to follow formal analogies to define continuity and differentiability (see Definitions B.5.6 and B.5.7) presents several problems (not characteristic of the LCF, but common to every non-Archimedean totally ordered field): e.g. the function $f : [0,1] \longrightarrow \mathcal{R}$ defined by

$$f(x) := \begin{cases} 0 & \text{if } x \text{ is infinitely small} \\ 1 & \text{if } x \text{ is finite} \end{cases}$$

is topologically continuous and topologically differentiable, but it does not assume the value $d \in [0,1]$ even though $f(0) < d < f(1)$, hence it does not verifies the intermediate value theorem. Moreover, $f'(x) = 0$ for all $x \in [0,1]$ but f is not constant on $[0,1]$. Therefore, also the simplest initial value problem $y' = 0$, $y(0) = 0$ does not have a unique solution. This is due to the fact that infinitesimals are totally disconnected from finite numbers because $d \ll r$ for every $r \in \mathbb{R}_{>0}$, and this is a general problem of non-Archimedean fields. On the other hand, as we have seen in Chapter 13, in the context of Fermat reals, we do not encounter this type of problems.

The solution adopted in the framework of the LCF is to change the notion of topological continuity introducing a Lipschitz condition:

Definition B.5.12. *Let $a < b$ be given in $\mathcal{R}$, let $I \in \{(a,b), [a,b], (a,b], [a,b]\}$ be an interval of $\mathcal{R}$ and $f : I \longrightarrow \mathcal{R}$, then we say that f* is continuous in I *iff*

$$\exists\, M \in \mathcal{R} :\ \forall x \in I :\ |f(x) - f(y)| \le M \cdot |x - y|\,.$$

This is very interesting for our comparison because a Lipschitz condition is essential both for our Fermat reals (see e.g. the Definitions 2.8.1 and 2.1.1, but also Giordano [2004] where the Lipschitz condition is even more needed) and also for Frölicher and Kriegl [1988], Kriegl and Michor [1997]. The corresponding definition of differentiability recalls our approach with smooth incremental ratios (see Section 11.2):

Definition B.5.13. *Under the same hypothesis as in the previous definition we say that f* is differentiable on *I iff there exists a continuous function*

$$f'[-,-] : I \times I \longrightarrow \mathcal{R}$$

such that

$$\forall x, y \in I : \ f(y) = f(x) + f'[x,y] \cdot (y - x).$$

As usual we will set $f'(x) := f'[x,x]$.

This definition is also very interesting, because, even if the approach of the LCF is a formal approach to infinitesimals, some of the chosen solutions are very similar to those adopted in non formal approaches to infinitesimals. For example, note the requirement that $f'[-,-]$ should be continuous, and hence it should verify a Lipschitz condition, in analogy to the Lipschitz differentiability introduced in Frölicher and Kriegl [1988].

A natural problem dealing with LCF is what class of functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ can be extended to a meaningful subclass of $\mathcal{R}$ with the possibility to generalize to them some properties, like the intermediate value theorem, an inverse function theorem, the maximum theorem, the mean value theorem, Rolle's theorem, the existence of primitive functions, or the constancy principle. Because of the left-finiteness of our formal power series $x \in \mathcal{R}$, the most natural class of functions are those locally expandable in power series (the term *analytic* is used for real functions only in the context of the LCF). The solution adopted in Berz [1994] and Shamseddine [1999] (see also Shamseddine and Berz [2007] and references therein for a more recent article) is to define a notion of convergence of power series with coefficients in $\mathcal{R}$, to prove for them the above mentioned theorems, and hence to show that standard power series in $\mathbb{R}$ are included as a particular case of this notion of convergence in $\mathcal{R}$. It is also interesting to note that this concept of convergence is not the one derived from the formal analogies with the real case (see e.g. the Definition B.5.6) but it is rather derived from a family of seminorms. For more details on this development, see the above mentioned references.

The left-finiteness of the Levi-Civita numbers permits to represent them on a computer. Indeed, for every $r \in \mathbb{Q}$ the amount of information we have to store in the power series (B.5.1), up to the terms $x_q d^q$ with $q \leq r$, is finite and we can represent all these numbers with the usual precision available in a computer. Therefore, using the equivalence relation $=_r$ (see Definition B.5.2) we can implement a calculus of Levi-Civita numbers on a computer. Obviously, this is possible for rare cases only in other theories of infinitesimals[9] and it founds useful applications in automatic differentiation (see e.g. Berz [1992]). Here the problem is to find a computer algorithm to calculate the derivatives, up to a given order, of computationally complicated functions like (see Shamseddine [1999])

$$f(x)=\frac{\sin(x^3+2x+1)+\frac{3+\cos(\sin(\ln|1+x|))}{\exp\left(\tanh\left(\sinh\left(\cosh\left(\frac{\sin(\cos(\tan(\exp(x))))}{\cos(\sin(\exp(\tan(x+2))))}\right)\right)\right)\right)}}{2+\sin(\sinh(\cos(\tan^{-1}(\ln(\exp(x)+x^2+3)))))},$$

obtained by composition of elementary functions like sin, cos, exp, ..., the Heaviside function

$$H(x):=\begin{cases}1 & \text{if } x\ge 0\\ 0 & \text{otherwise}\end{cases}$$

and of the field operations. These are called *computer functions*, and can be extended to a suitable subset of $\mathcal{R}$ using their expansion in power series. The property that permits to compute these derivatives is expressed in the following theorem and it presents strong analogies with the calculus in our Fermat reals:

Theorem B.5.14. *Let f be a computer function continuous at $x_0\in\mathbb{R}$ and extendable to $x_0\pm d$. Then f is n-times differentiable at x_0 if and only if there exist real numbers $\alpha_1,\dots,\alpha_n$ such that*

$$f(x_0+d)=_n\sum_{j=0}^n\frac{\alpha_j}{j!}\cdot d^j$$

$$f(x_0-d)=_n\sum_{j=0}^n(-1)^j\frac{\alpha_j}{j!}\cdot d^j.$$

Moreover, in this case we have $f^{(j)}(x_0)=\alpha_j$ for $j=0,\dots,n$.

In Shamseddine [1999] one can find several examples of computation of derivatives using these formulas, and of non smooth functions whose regularity is proved using this theorem. A software, called COSY INFINITY, has also been created, which is suitable for the computation of derivatives of functions using the LCF (see Berz et al. [1996], Shamseddine [1999] and references therein also for the comparison with other methods of computation of derivatives).

Comparison with Fermat reals

We have tried to introduce the LCF with a certain detail, due to the many analogies that one can see between the LCF theory and our own theory,

[9] We only mentioned here that the surreal numbers has been implemented in the computer based proof assistant Coq, see Mamane [2006].

even if the two approaches are very different from the philosophical point of view. We can underline several points where the LCF can be considered a better framework with respect to Fermat reals, and several others where we can state the opposite:

- ☺ The LCF is a field and not a ring. This is surely reassuring for some readers, even if we have mentioned in connection with some deep problems that the theory of non-Archimedean fields find in the development of the calculus. On the other hand, the availability of infinities can be very useful to express e.g. the Riemann integral as an infinite sum or to define Dirac delta functions as ordinary functions, like in NSA.

- ☺ In the LCF we have the existence of roots for every positive numbers, in particular also for infinitesimals. Of course, this is incompatible with a ring containing nilpotent elements and indeed, using Fermat reals, we are able to define roots only for invertible numbers. We hinted in Section 14 to the possibility to define the square root of an infinitesimal Fermat real $h \in D_\infty$ as the simplest $k \in D_{2\cdot\omega(h)}$ such that $k^2 = h$, but this notion, even if useful, does *not* verify the usual equality $\sqrt{k^2} = |k|$.

- ☺ In the present work, we chose to develop a theory of smooth functions only, so as to obtain the simplest results useful for smooth differential geometry. There is the possibility to extend some of our results to functions which only belong to $\mathcal{C}^n$, keeping present some of the ideas used in Giordano [2004]. An example in this direction is given by Theorem 14.6.1. But at present, the theory of Fermat reals and Fermat spaces is not developed in this direction. The possibility to define continuity and differentiability in the theory of the LCF is hence interesting. Because the theory of LCF is not a theory of smooth functions only, we have the possibility to prove a useful theorem like B.5.14, even if that theorem is applicable only to computer functions expandable to $x_0 \pm d$, i.e. to a class smaller than the one considered in the previous analogous Theorem 14.6.1.

On the other hand we have:

- ☹ Until now, the theory of LCF permits to extend the real field only, and not the general case of smooth manifolds, like in the case of Fermat reals.

- ☹ The calculus with nilpotent infinitesimals seems easier, for smooth functions, with respect to the use of the equivalence relation $=_r$. As we mentioned above, because on the right the power series of $\mathcal{R}$ are not necessarily finite, the functions that naturally extends from $\mathbb{R}$ to the LCF are the analytical ones. So, we have the methodological contradiction that the LCF permits to develop a meaningful notion of continuity and differentiability, but at the same time, because of the form of the formal power series considered in the LCF, the best results are for functions locally expandable in power series and not for a lower degree of regularity. At the same time each Fermat number need only a finite number of reals to be stored in a computer, without any need of the equivalence relation $=_r$, and hence $^\bullet\mathbb{R}$ can also be implemented in a computer.

☹ Because of the formal approach to infinitesimals, the intuitive meaning of $\mathcal{R}$ as connected to potential infinitesimals of $\mathbb{R}$ is missing. For example, at the best of our knowledge, there is no idea about how it would be possible to extend a given function $f : \mathbb{R} \longrightarrow \mathbb{R}$ to infinitely large numbers in $\mathcal{R}$.

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