Selberg and Ruelle zeta functions and the relative analytic torsion on complete odd-dimensional hyperbolic manifolds of finite volume

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Abstract

Let X be a complete hyperbolic manifold of finite volume and of odd dimension d. Then X can be realized as $X = \Gamma\backslash G/K$, where $G = \mathrm{Spin}(d,1)$, $K = \mathrm{Spin}(d)$ and where Γ is a discrete, torsion-free subgroup of G. Throughout this thesis we assume that for every Γ-cuspidal parabolic subgroup P of G with Langlands decomposition $P = M_P A_P N_P$ one has $\Gamma \cap P = \Gamma \cap N_P$. Firstly, we study Selberg zeta functions on X, prove that these functions have a meromorphic continuation to the complex plane and describe their singularities. Secondly, we define the relative or regularized analytic torsion of X associated to the restriction of a certain representation of G to Γ. We investigate the asymptotic behaviour of this torsion with respect to special sequences of representations of G. Finally, if X is 3-dimensional, we establish a relation between the regularized analytic torsion and the behaviour of a twisted Ruelle zeta function at 0. Our work generalizes results of Fried, Bunke and Olbrich, Bröcker and Wotzke to the non-compact case and results of Müller to the non-compact and higher-dimensional situation.

Contents

1 Introduction

1.1 Statement of the main results

This thesis deals with two aspects of geometry and spectral theory on complete, odd dimensional, not necessarily compact hyperbolic manifolds X of finite volume. Firstly, the Selberg zeta functions, which are dynamical zeta functions defined in terms of the length spectrum and the geodesic flow on the unit sphere bundle of X and a representation of a compact group, are studied. Secondly, we treat the analytic torsion of X with respect to certain representations of its fundamental group. If X is non-compact, this torsion will be introduced as a relative or regularized torsion. For 3-dimensional X, we will relate the regularized analytic torsion associated to certain representations of G to the behaviour of a twisted Ruelle zeta function at 0. This zeta function is a dynamical zeta function which is defined similarly to the Selberg zeta functions. Since X is a locally symmetric space, methods from harmonic analysis, in particular the Selberg trace formula turn out to be our main tool.

Throughout this thesis we let X be a complete hyperbolic manifold of dimension $d = 2n+1$ and of finite volume. If we let $G = \mathrm{Spin}(d, 1)$ and $K = \mathrm{Spin}(d)$, then there exists a discrete, torsion free subgroup $\Gamma \subset G$ such that $X = \Gamma\backslash\mathbb{H}^d$, where $\mathbb{H}^d \cong G/K$ is the d-dimensional hyperbolic space. We assume that for every Γ-cuspidal parabolic subgroup P of G with Langlands decomposition $P = M_P A_P N_P$ one has

$$\Gamma \cap P = \Gamma \cap N_P. \qquad (1.1)$$

The assumption (1.1) is satisfied for example if Γ is neat, i.e. if the group generated by the eigenvalues of any $\gamma \in \Gamma$ contains no root of unity different from 1.

1.1.1 Selberg zeta functions

Our first main topic are Selberg zeta functions. To define these functions, we need to introduce some notation. Let $P_0 = MAN$ be the standard parabolic subgroup of G. Let σ be a finite dimensional unitary representation of M. Then, going back to Selberg, one can associate a dynamical zeta function $Z(s, \sigma)$ to the geodesic flow on the unit sphere bundle SX and the representation σ. This function is defined as follows. Identifying Γ with the fundamental group of X, we obtain a one-to-one correspondence between $\mathrm{C}(\Gamma)$, the set of conjugacy classes of Γ, and the set of free homotopy classes of closed curves in X. For $\gamma \in \Gamma$ we will denote its conjugacy class by $[\gamma]$. Moreover, by $f([\gamma])$ we will denote the free homotopy class of closed paths associated to $[\gamma]$. For a conjugacy class $[\gamma]$ let $\ell(\gamma)$ denote the infimum of the lengths of piecewise smooth loops which belong to $f([\gamma])$. Then it turns out that $\ell(\gamma)$ is non-zero if and only if γ is semisimple and that for semisimple γ there exists a unique closed geodesic $c_{[\gamma]} \in f([\gamma])$ of length $\ell(\gamma)$. Thus let $\mathrm{C}(\Gamma)_{\mathrm{s}}$ denote the set of semisimple conjugacy classes in Γ. Let $\mathfrak{a}$ denote the Lie algebra of A and let $H_1 \in \mathfrak{a}$ be of norm one and positive with respect to the choice of N. Then every non-trivial semisimple element γ of Γ is conjugate to an element of the form $m_\gamma \exp(\ell(\gamma)H_1)$, where $m_\gamma \in M$ is unique up to conjugation in M. Let $\bar{N}$ be the nilpotent subgroup opposite to N and let $\bar{\mathfrak{n}}$ be its Lie algebra. Then for $s \in \mathbb{C}$ with $\mathrm{Re}(s)$ sufficiently large, the Selberg zeta function $Z(s,\sigma)$ is defined as

$$Z(s,\sigma) := \prod_{\substack{[\gamma]\in \mathrm{C}(\Gamma)_{\mathrm{s}}-[1]\\ [\gamma] \text{ prime}}} \prod_{k=0}^{\infty} \det\left(\mathrm{Id} - \sigma(m_\gamma) \otimes S^k \operatorname{Ad}(m_\gamma \exp(\ell(\gamma)H_1))|_{\bar{\mathfrak{n}}} e^{-(s+n)\ell(\gamma)}\right). \qquad (1.2)$$

Here, a semisimple conjugacy class $[\gamma]$ is called prime if the corresponding closed geodesic $c_{[\gamma]}$ is prime. Moreover, $S^k \operatorname{Ad}$ denotes the k-th symmetric power of the adjoint representation. The infinite product in (1.2) converges absolutely only for $\mathrm{Re}(s) >> 0$. Our main result about the Selberg zeta function is the following theorem.

Theorem 1.1. *Let X be a complete odd dimensional hyperbolic manifold of finite volume and assume that its fundamental group satisfies* (1.1). *Let σ be a finite dimensional unitary representation of M. Then the Selberg zeta function $Z(s,\sigma)$ has a meromorphic continuation to $\mathbb{C}$. All its possible singularities (zeroes and poles) and their corresponding orders can be described in terms of the following data.*

- *By the discrete spectrum of a graded differential operator $A(\sigma)$ of Laplace type which acts on a locally homogeneous vector bundle $E(\sigma)$ over X.*
- *By the poles of the scattering matrix $\mathbf{C}(\nu_\sigma : \sigma : s)$ associated to σ and a certain representation ν_σ of K.*
- *Additionally, the Selberg zeta function has singularities which depend on X only via p, the number of cusps of X. They are located on the negative integers if the highest weight of σ is integral and on the negative half integers if the highest weight of σ is half-integral.*

Taking into account that the Selberg zeta function is a dynamical zeta function which is defined by geometric data of the underlying manifold X, namely its length spectrum, Theorem 1.1 provides a relation between the geometry of the possibly non-compact hyperbolic manifold X and the spectrum of certain differential operators. More precisely, the singularities in the first two items correspond to spectral parameters of X since poles of the scattering matrix $\mathbf{C}(\nu_\sigma : \sigma : s)$ are related to poles of the resolvent of $A(\sigma)$. The additional singularities of $Z(s,\sigma)$ on the negative real line arise from the contribution of weighted orbital integrals to the geometric side of the trace formula. For a more precise version of Theorem 1.1 we refer the reader to Theorem 7.4 and Theorem 7.8 below.
Let us now give a brief history of previous results related to Theorem 1.1. If X is compact, it was shown in [Fr2] that $Z(s,\sigma)$ has a meromorphic continuation to $\mathbb{C}$. For compact X Bunke and Olbrich described the singularities of $Z(s,\sigma)$ as in the first item of Theorem 1.1, see [BO]. If X is of finite volume only and satisfies assumption (1.1), the meromorphic continuation of $Z(s,\sigma)$ and a description of its singularities were obtained by Gangolli and Wallach for the trivial representation of M, [GaWa]. In [GP] Gon and Park generalized

their methods to the fundamental representations σ_k of M on $\Lambda^k(\mathbb{C}^{2n})$. However it is not clear whether the methods of Gangolli, Wallach, Gon and Park can be applied to general $\sigma \in \hat{M}$ since they use a special type of a Paley-Wiener theorem for differential forms which prescribes the K-types of a test function in a very specific way.
Our approach to prove Theorem 1.1 combines the methods of Bunke and Olbrich from the compact case with the invariant Selberg trace formula as it is stated in [Ho2]. We would like to emphasize that this is only possible because the Fourier transform of the invariant part associated to the weighted orbital integrals which appears on the geometric side of the trace formula has been determined by Hoffmann [Ho1].
Let σ be a finite dimensional unitary representation of M. Then there is another dynamical zeta function $R(s,\sigma)$ associated to σ and the geodesic flow. Namely, for $\mathrm{Re}(s) >> 0$ put

$$R(s,\sigma) := \prod_{\substack{[\gamma]\in \mathrm{C}(\Gamma)_{\mathrm{s}}-[1] \\ [\gamma] \text{ prime}}} \det\left(\mathrm{Id} - \sigma(m_\gamma)e^{-s\ell(\gamma)}\right).$$

The infinite product converges absolutely and locally uniformly for $\mathrm{Re}(s) >> 0$ and the function $R(s,\sigma)$ is called the Ruelle zeta function associated to σ. One can express the Ruelle zeta function in a standard way as a weighted product of Selberg zeta functions with shifted arguments. Thus Theorem 1.1 implies the following corollary.

Corollary 1.2. *For every finite dimensional unitary representation σ of M the Ruelle zeta function $R(s,\sigma)$ admits a meromorphic continuation to $\mathbb{C}$.*

1.1.2 The asymptotic behaviour of the relative analytic torsion

The second topic of this thesis deals with the relative analytic torsion associated to representations of the fundamental group Γ which arise as restrictions of representations of G. Thus let τ be an irreducible finite dimensional representation of G. Restrict τ to Γ and let E_τ be the associated flat vector-bundle over X. By [MaMu] one can equip E_τ with a canonical metric, called admissible metric. Let $\Delta_p(\tau)$ be the corresponding Laplacian on E_τ valued p-forms. If X is not compact, $\Delta_p(\tau)$ has a continuous spectrum and therefore, the heat operator $\exp(-t\Delta_p(\tau))$ is not trace class. So the usual zeta function regularization can not be used to define the analytic torsion in this case. To overcome this problem, we use the relative trace which was introduced by Müller in [Mü1].
The relative trace is defined as follows. There exists a $u_0 > 0$ such that for every $u > u_0$, X has a natural decomposition as $X = X(u) \sqcup_{\partial X(u)} F(u)$. Here $X(u)$ is a compact manifold with boundary and $F(u)$ is a disjoint union of finitely many cusps. Moreover, one has $X(u) \subset X(u')$ for $u' > u$ and X is the union of all $X(u)$. There exists a naturally defined auxiliary differential operator $T_{\nu_p(\tau),\mathrm{u}}$ which acts on the E_τ-valued p-forms, has purely continuous spectrum and vanishes on the smooth sections supported in the interior of $X(u)$. Furthermore, by [Mü1, Theorem 9.1] for every $t > 0$ the operator $e^{-t\Delta_p(\tau)} - e^{-tT_{\nu_p(\tau),\mathrm{u}}}$ is of trace class. Now the relative trace of $e^{-t\Delta_p(\tau)}$ with respect to the parameter u is defined as

$$\mathrm{Tr}_{\mathrm{rel,u}}\left(e^{-t\Delta_p(\tau)}\right) := \mathrm{Tr}\left(e^{-t\Delta_p(\tau)} - e^{-tT_{\nu_p(\tau),\mathrm{u}}}\right).$$

Another way to regularize the trace in the non-compact case is provided by the b-regularized trace. To define the latter trace, let $K_p(t,x,y)$ be the integral kernel of $e^{-t\Delta_p(\tau)}$. Then the integral of $\operatorname{Tr} K_p(t,x,x)$ over $X(Y)$ has an asymptotic expansion in Y as $Y \to \infty$ and following Melrose [Me], the b-regularized trace is defined as the finite part of this expansion. It coincides with $\operatorname{Tr}_{\mathrm{rel},u}\left(e^{-t\Delta_p(\tau)}\right)$ up to a term which depends on the parameter u and which will be computed explicitly.
It turns out that $\operatorname{Tr}_{\mathrm{rel,u}}\left(e^{-t\Delta_p(\tau)}\right)$ equals the spectral side of the Selberg trace formula applied to the heat operator $\exp(-t\Delta_p(\tau))$ up to a term which depends on u and can be computed explicitly. Using the Selberg trace formula, it follows that $\operatorname{Tr}_{\mathrm{rel,u}}\left(e^{-t\Delta_p(\tau)}\right)$ has an asymptotic expansion as $t \to +0$. Let θ be the standard Cartan involution of G. If τ satisfies $\tau \neq \tau_\theta$, then $\operatorname{Tr}_{\mathrm{rel,u}}\left(e^{-t\Delta_p(\tau)}\right)$ is exponentially decreasing as $t \to \infty$. Thus we can define the relative zeta function $\zeta_{p,\mathrm{u}}(z;\tau)$ of $\Delta_p(\tau)$ by

$$\zeta_{p,\mathrm{u}}(z;\tau) := \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} \operatorname{Tr}_{\mathrm{rel,u}}(e^{-t\Delta_p(\tau)})\, dt.$$

The integral converges absolutely and uniformly on compact subsets of the half-plane $\operatorname{Re}(z) > d/2$ and admits a meromorphic continuation to $\mathbb{C}$ which is regular at $z = 0$. In analogy to the compact case, we now define the relative ζ-regularized determinant $\det_{\mathrm{u}}(\Delta_p(\tau)) \in \mathbb{R}^+$ by

$$\det{}_{\mathrm{u}}(\Delta_p(\tau)) := \exp\left(-\frac{d}{dz}\Big|_{z=0} \zeta_{p,\mathrm{u}}(z,\tau)\right).$$

Now the analytic torsion $T_{X,\mathrm{u}}(\tau) \in \mathbb{R}^+$ with respect to E_τ and the admissible metric is defined by

$$T_{X,\mathrm{u}}(\tau) := \prod_{p=0}^{d} \det{}_{\mathrm{u}}(\Delta_p(\tau))^{\frac{p}{2}(-1)^{p+1}}. \tag{1.3}$$

If X is compact, then X has no cusps, the heat operators are trace class and their relative trace is the same as their trace. Thus in the compact case (1.3) is just the Ray-Singer analytic torsion of X.
We will study the asymptotic behaviour of the analytic torsion for special sequences of representations of G. These representations are defined as follows. Fix natural numbers $\tau_1 \geq \tau_2 \geq \cdots \geq \tau_{n+1}$. For $m \in \mathbb{N}$ let $\tau(m)$ be the finite-dimensional irreducible representation of G with highest weight $(\tau_1 + m, \ldots, \tau_{n+1} + m)$ as in (2.7). By Weyl's dimension formula there exists a constant $C > 0$ such that

$$\dim(\tau(m)) = Cm^{\frac{n(n+1)}{2}} + O(m^{\frac{n(n+1)}{2}-1}), \quad m \to \infty. \tag{1.4}$$

Our main result about the asymptotic behaviour of $\log T_{X,\mathrm{u}}(\tau(m))$ as $m \to \infty$ is the following theorem.

Theorem 1.3. *Let $X = \Gamma\backslash\mathbb{H}^{2n+1}$ be a $(2n+1)$-dimensional complete, oriented, hyperbolic manifold of finite volume. Assume that Γ satisfies* (1.1)*. Let*

$$C(n) := (-1)^{n+1}\frac{\pi}{\operatorname{vol}(S^d)},$$

where S^d is the d-dimensional Euclidean unit-sphere. Then one has

$$\log T_{X,\mathrm{u}}(\tau(m)) = -C(n)\operatorname{vol}(X)m\cdot\dim(\tau(m)) + O\left(m^{\frac{n(n+1)}{2}}\log m\right),$$

as $m \longrightarrow \infty$. Here $\operatorname{vol}(X)$ *is the hyperbolic volume of X.*

We will also study the L^2-torsion $T_X^{(2)}(\tau)$. By the homogeneity of X, to define this torsion as in [Lo], it suffices to assume that X has finite volume. We will show that there exists a polynomial $P_\tau(m)$ of degree $n(n+1)/2+1$ such that

$$\log T_X^{(2)}(\tau(m)) = \operatorname{vol}(X)P_\tau(m). \tag{1.5}$$

The polynomial $P_\tau(m)$ only depends on $\tau_1,\dots,\tau_{n+1}$ and not on X. It is obtained from certain Plancherel polynomials and can in principle be computed explicitly. Its leading term will be computed explicitly and we obtain

$$\log T_X^{(2)}(\tau(m)) = -C(n)\operatorname{vol}(X)m\cdot\dim(\tau(m)) + O(m^{\frac{n(n+1)}{2}}), \tag{1.6}$$

where $C(n)$ is as in Theorem 1.3. Employing Theorem 1.3 we obtain the following theorem.

Theorem 1.4. *Let $X = \Gamma\backslash\mathbb{H}^{2n+1}$ be a $(2n+1)$-dimensional complete, oriented, hyperbolic manifold of finite volume. Assume that Γ satisfies* (1.1)*. Then we have*

$$\log T_{X,\mathrm{u}}(\tau(m)) = \log T_X^{(2)}(\tau(m)) + O(m^{\frac{n(n+1)}{2}}\log m)$$

as $m \to \infty$.

For compact hyperbolic 3-manifolds, a variant of Theorem 1.3 was proved in [Mü4], where the log-term in the remainder can be dropped. Our results also imply a generalization of the results of Müller to higher dimensional compact hyperbolic manifolds. More precisely, if X is compact, the remainder term in our estimate can be improved and we will prove the following theorem.

Theorem 1.5. *Let X be a compact hyperbolic manifold of dimension $d = 2n+1$. Then, if $P_\tau(m)$ is the polynomial of degree $n(n+1)/2+1$ as in* (1.5)*, one has*

$$\log T_X(\tau(m)) = \operatorname{vol}(X)P_\tau(m) + O(e^{-cm}),$$

as $m \to \infty$, where $c > 0$. In particular, one has

$$\log T_X(\tau(m)) = \log T_X^{(2)}(\tau(m)) + O(e^{-cm}),$$

as $m \to \infty$.

Since the representations τ are unimodular, it follows from [Mü3] that for compact X the analytic torsion $T_X(\tau(m))$ equals the Reidemeister torsion $\tau_X(\tau(m))$. The latter is an invariant which is constructed in a combinatorial way out of a smooth triangulation of X. Replacing $T_X(\tau(m))$ by $\tau_X(\tau(m))$ in Theorem 1.5, it follows that the volume of a compact odd-dimensional hyperbolic manifold X is determined by the sequence of Reidemeister torsion invariants $\tau_X(\tau(m))$. Again, for compact hyperbolic 3-manifolds this result was proved in [Mü4]. On the other hand, it is not known if there is an extension of the equality of analytic and Reidemeister torsion to the non-compact setting. This is an interesting problem and Theorem 1.3 could be a first step in this direction.

1.1.3 The twisted Ruelle zeta function at 0 and the regularized analytic torsion in the 3-dimensional case

For a hyperbolic 3-manifold X we also investigate the relation between the regularized analytic torsion $T_X(\tau)$ and the behaviour of the twisted Ruelle zeta function R_τ at 0 for certain representations $\tau \in \hat{G}$. This zeta function is defined as

$$R_\tau(s) := \prod_{\substack{[\gamma]\in \mathrm{C}(\Gamma)_{\mathrm{s}}-[1] \\ [\gamma] \text{ prime}}} \det\left(\mathrm{Id} - \tau(\gamma)e^{-s\ell(\gamma)}\right). \tag{1.7}$$

The infinite product in (1.7) converges for $\mathrm{Re}(s)$ sufficiently large. One can express R_τ as a finite product of Ruelle zeta functions $R(s,\sigma)$ with shifted arguments and so by Corollary 1.2 the function $R_\tau(s)$ has a meromorphic continuation to $\mathbb{C}$. In the 3-dimensional case, we can naturally identify $G = \mathrm{Spin}(3,1)$ with $\mathrm{SL}_2(\mathbb{C})$. From now on, for $m \in \frac{1}{2}\mathbb{N}$ we let $\tau(m)$ be the representation of G with highest weight $me_1 + me_2$. Then $\tau(m)$ corresponds to the $2m$-th symmetric power of the standard representation of $\mathrm{SL}_2(\mathbb{C})$. The representation $\tau(m)$ satisfies $\tau(m) \neq \tau(m)_\theta$ and thus the relative analytic torsion $T_{X,\mathrm{u}}(\tau(m))$ is defined as in the previous section. However, for notational convenience we shall now work with the regularized analytic torsion $T_X(\tau(m))$ which is defined in the same way as above and which coincides with the relative analytic torsion $T_{X,\mathrm{u}}(\tau(m))$ up to a term which depends on u and can be computed explicitly. Our main result for the 3-dimensional case is the following theorem.

Theorem 1.6. *For $m \in \mathbb{N}$ there exists an explicit constant $c(\tau(m)) \in \mathbb{R}^+$, which depends on Γ only via p, the number of cusps of X, and which is determined in in* (10.37) *such that*

$$T_X(\tau(m))^4 = c(\tau(m))\frac{\mathbf{C}(m:0)}{\mathbf{C}(m+1:0)}\lim_{s\to 0}\left(R_{\tau(m)}(s)R_{\tau(m)_\theta}(s)\frac{\mathbf{C}(m+1:m-s)}{\mathbf{C}(m:m+1-s)}\Gamma^{-2p}(s-1)\right).$$

Similarly, there exists an explicit constant $c(\tau(m+1/2)) \in \mathbb{R}^+$, which depends on Γ only via p, the number of cusps of X, and which is determined in (10.38) *such that*

$$\begin{aligned} T_X(\tau(m+1/2))^4 =& c(\tau(m+1/2))\frac{\mathbf{C}(m+1/2:0)}{\mathbf{C}(m+3/2:0)}\lim_{s\to 0}\Bigg(R_{\tau(m+1/2)}(s)R_{\tau(m+1/2)_\theta}(s) \\ & \frac{\mathbf{C}(m+3/2:m+1/2-s)}{\mathbf{C}(m+1/2:m+3/2-s)}\Gamma^{-2p}(s-1)\Bigg). \end{aligned}$$

Here the functions $\mathbf{C}(k:s)$ are meromorphic functions of s which are constructed out of the scattering determinant associated to the representation σ of M with highest weight ke_2 and a certain K-type. They are defined in section 10.2.

Let X be a compact odd-dimensional hyperbolic manifold. Then Bröcker and Wotzke proved that for any $\tau \in \hat{G}$ which satisfies $\tau \neq \tau_\theta$ the Ruelle zeta function $R_\tau(s)$ is regular at 0 and that

$$T_X(\tau)^4 = R_\tau(0)R_{\tau_\theta}(0), \tag{1.8}$$

see [Br], [Wo]. Thus Theorem 1.6 is a generalization of their result in dimension 3 to non-compact hyperbolic 3-manifolds. We have stated and proved Theorem 1.6 only for the representations $\tau(m)$. However, one can obtain a similar result for any $\tau \in \hat{G}$ which satisfies $\tau \neq \tau_\theta$. The result of Bröcker and Wotzke is a generalization of Fried's Theorem who first investigated the relation between the Ruelle zeta function and the analytic torsion associated to a unitary representation of the fundamental group, see [Fr1]. For unitary representations of Γ, the relation between the behaviour of the Ruelle zeta function at 0 and the regularized analytic torsion was studied by Park, [Pa]. Theorem 1.6 does not imply that the Ruelle zeta function $R_{\tau(m)}$ is regular at 0.
On closed hyperbolic manifolds, the results of Fried, Bröcker and Wotzke imply a relation between a spectral invariant, the analytic torsion, and a special value of a geometric zeta function. In Theorem 1.6 the spectrum of certain operators appears on both sides of the equation due to the appearance of the C-matrix on the right hand side. However, it turns out that if we consider the quotient of two analytic torsions $T_X(\tau(m_1))$ and $T_X(\tau(m_2))$, we can eliminate the appearance of the spectrum on one side of the equation. Namely, we have the following corollary.

Corollary 1.7. *Let $m \in \mathbb{N}$. Then for $m \geq 3$ one has*

$$\frac{T_X(\tau(m))}{T_X(\tau(2))} = \left(\frac{c(\tau(m))}{c(\tau(2))}\right)^{1/4} \exp\left(-\frac{1}{\pi}\operatorname{vol}(X)(m(m+1)-6)\right)\prod_{k=3}^{m}\left|R(k,\sigma_k)\right|.$$

where the constants $c(\tau(m))$ and $c(\tau(2))$ are as in Theorem 1.6. Similarly, for $m \geq 2$ one has

$$\frac{T_X(\tau(m+1/2))}{T_X(\tau(3/2))} = \left(\frac{c(\tau(m+1/2))}{c(\tau(3/2))}\right)^{1/4} \exp\left(-\frac{1}{\pi}\operatorname{vol}(X)(m(m+2)-3)\right) \cdot \prod_{k=2}^{m}\left|R(k+1/2,\sigma_{k+1/2})\right|.$$

The Ruelle zeta functions $R(s,\sigma_k)$ appearing in the corollary are regular at $s=k$. Corollary 1.7 is a generalization of [Mü4, equation(8.7), equation(8.8)]. One easily sees that $|\log c(\tau(m))|$ is of order $O(m)$ as $m \to \infty$ and thus Corollary 1.7 and the decay of the $\log|R(k,\sigma_k)|$ also imply Theorem 1.3 in the 3-dimensional case with an improved

remainder term.
Let $\overline{X}$ be the Borel-Serre compactification of X. Then in their preprint [MePo], Menal-Ferrer and Porti studied the relation between the Reidemeister torsion $\tau_X(\tau(m))$ of $\overline{X}$ associated to the restriction of the representation $\tau(m)$ to Γ and Ruelle zeta functions. In order to define the Reidemeister torsion, one has to take into account that the cohomology of $\overline{X}$ with respect to the local system defined by $\tau(m)$ is non-trivial, see [MePo]. If we use [MePo, Theorem 5.8] and make the same assumptions as in this theorem, together with Corollary 1.7 we can compare the regularized analytic torsion and the Reidemeister torsion as follows.

Corollary 1.8. *Let $m \in \mathbb{N}$. Then for $m \geq 3$ one has*

$$\frac{T_X(\tau(m))}{T_X(\tau(2))} = \left(\frac{c(\tau(m))}{c(\tau(2))}\right)^{1/4} \frac{|\tau_X(\tau(m))|}{|\tau_X(\tau(2))|},$$

where the constants $c(\tau(m))$ and $c(\tau(2))$ are as in Theorem 1.6.

We remark that the quotient of regularized torsions in the last equation are non-trivial and can even become arbitrarily large by Theorem 1.3.

1.2 Outline of the proof

We shall now sketch the method for the proof of our main results. Let π_Γ be the right regular representation of G on $L^2(\Gamma\backslash G)$. Then there is a decomposition

$$\pi_\Gamma = \pi_{\Gamma,d} \oplus \pi_{\Gamma,c}.$$

Here the representation $\pi_{\Gamma,d}$ is completely reducible. On the other hand, the representation $\pi_{\Gamma,c}$ is isomorphic to a direct integral over all tempered principal series representations of G. For a K-finite Schwarz function ϕ on G, the operator $\pi_{\Gamma,d}(\phi)$ is trace class and the invariant trace formula as it is stated in [Ho2] expresses $\mathrm{Tr}\,(\pi_{\Gamma,d}(\phi))$ as a sum of invariant distributions on G applied to ϕ.
In order to prove the meromorphic continuation of the Selberg zeta function $Z(s,\sigma)$, we first study the symmetrized Selberg zeta function $S(s,\sigma)$, which is given by $Z(s,\sigma)$ if $\sigma = w_0\sigma$ and by $Z(s,\sigma)Z(s,w_0\sigma)$, if $\sigma \neq w_0\sigma$. Here w_0 is a fixed representative of the restricted Weyl group. As in the compact case ([BO]), there is a K-finite function h_t^σ, belonging to all Harish-Chandra Schwarz spaces, such that the logarithmic derivative of $S(s,\sigma)$ is equal to a certain integral transform of $H(h_t^\sigma)$. Here H is a distribution on G which occurs in the invariant Selberg trace formula. It is built from the semisimple conjugacy classes of Γ. Geometrically, the function h_t^σ arises from the graded fibre trace of the kernel of $e^{-t\tilde{A}(\sigma)}$, where $\tilde{A}(\sigma)$ is a Laplace-type operator which acts on a graded vector bundle $\tilde{E}(\sigma)$ over $\tilde{X}$. Now we apply the invariant trace formula to h_t^σ and compute the integral transform of all involved summands explicitly. In this way we can show that the residues of the logarithmic derivative of $S(s,\sigma)$ are integral. Moreover, we can determine

its poles and the corresponding residue explicitly.
If σ is not invariant under the Weyl group, we introduce the antisymmetric Selberg zeta function $S_a(s,\sigma) := Z(s,\sigma)/Z(s,w_0\sigma)$. The bundle $\tilde{E}(\sigma)$ turns out to be a spinor bundle and there is a canonical twisted Dirac operator $\tilde{D}(\sigma)$ on $\tilde{E}(\sigma)$ such that $\tilde{D}(\sigma)^2 = \tilde{A}(\sigma)$. Now the fibre trace of the kernel of $\mathrm{Tr}(\tilde{D}(\sigma)e^{-t\tilde{D}(\sigma)^2})$ is represented by a K-finite Harish-Chandra Schwarz function k_t^σ and the logarithmic derivative of $S_a(s,\sigma)$ equals an integral transform of $H(k_t^\sigma)$, where the distribution H is as above. Using the invariant trace formula again we obtain a meromorphic continuation of $S_a(s,\sigma)$ and a complete description of its singularities. Putting everything together, we can also complete the proof of Theorem 1.1.

Next we describe the proof of our main results concerning torsion asymptotics. In a first step, we establish a determinant formula which relates the symmetric Selberg zeta function $S(s,\sigma)$ to the relative graded determinant $\det_{\mathrm{gr,u}}(A(\sigma)+s^2)$ of the operator $A(\sigma)$ for certain $s \in \mathbb{C}$. The logarithm of the latter determinant is obtained via the Laplace-Mellin transform applied to the spectral side $J_{spec}(h_t^\sigma)$ of the Selberg trace formula up to an additional summand $\mathcal{LM}R_u(s,\sigma)$ which depends on the auxiliary operators on the cusp and can be computed explicitly. By the non-invariant trace formula, the spectral side equals the geometric side $J_{geo}(h_t^\sigma)$. The geometric side is given by a sum of distributions applied to h_t^σ. One has

$$J_{geo}(h_t^\sigma) = I(h_t^\sigma) + H(h_t^\sigma) + T(h_t^\sigma) + \mathcal{I}(h_t^\sigma) + J(h_t^\sigma), \tag{1.9}$$

where $I(h_t^\sigma)$ is the contribution of the identity conjugacy class of Γ and $H(h_t^\sigma)$ is as above. Moreover, $T(h_t^\sigma)$, $\mathcal{I}(h_t^\sigma)$ and $J(h_t^\sigma)$ are tempered distributions applied to h_t^σ which are constructed out of the parabolic conjugacy classes of Γ. To save notation, for a function $f : \mathbb{R}^+ \to \mathbb{C}$ we shall write

$$\mathcal{LM}f(s) := \frac{d}{dz}\bigg|_{z=0}\left(\frac{1}{\Gamma(z)}\int_0^\infty e^{-ts^2}f(t)t^{z-1}dt\right),$$

if the integral exists for $\mathrm{Re}(z) >> 0$ and admits a meromorphic continuation to $z \in \mathbb{C}$ which is regular at 0. We compute the Laplace-Mellin transform of each term on the right hand side of (1.9) separately. Combining these computations with our computations for the symmetric Selberg zeta function, for certain $s \in \mathbb{C}$ we obtain

$$\begin{aligned}\log\det_{\mathrm{gr,u}}\left(A(\sigma)+s^2\right) = &\log S(s,\sigma) - \mathcal{LM}I(s,\sigma) - \mathcal{LM}T(s,\sigma) - \mathcal{LM}\mathcal{I}(s,\sigma)\\ &- \mathcal{LM}J(s,\sigma) - \mathcal{LM}R_u(s,\sigma).\end{aligned} \tag{1.10}$$

Now we come to the analytic torsion. Let $\tau \in \hat{G}$, $\tau \neq \tau_\theta$. Let

$$K_{\mathrm{u}}(t,\tau) := \sum_{p=0}^{2n+1}(-1)^p p\,\mathrm{Tr}_{\mathrm{rel,u}}(e^{-t\Delta_p(\tau)}).$$

We need to compute the finite part of the Mellin transform of $K_{\mathrm{u}}(t,\tau)$ at 0. Let $\widetilde{E}_\tau$ be the homogeneous vector bundle over $\widetilde{X}=G/K$ associated to τ and let $\widetilde{\Delta}_p(\tau)$ be the Laplacian on $\widetilde{E}_\tau$-valued p-forms on $\widetilde{X}$. The heat operator $e^{-t\widetilde{\Delta}_p(\tau)}$ is a convolution operator with kernel $H_t^{\nu_p(\tau)}\colon G\to \operatorname{End}(\Lambda^p\mathfrak{p}^*\otimes V_\tau)$. Let $h_t^{\nu_p(\tau)}(g)=\operatorname{tr} H_t^{\nu_p(\tau)}(g)$, $g\in G$, and put

$$k_t^\tau=\sum_{p=1}^{d}(-1)^p p h_t^{\nu_p(\tau)}. \tag{1.11}$$

Again it follows from the Selberg trace formula that

$$K_{\mathrm{u}}(t,\tau)=I(k_t^\tau)+H(k_t^\tau)+T(k_t^\tau)+\mathcal{I}(k_t^\tau)+J(k_t^\tau)+R_{\mathrm{u}}(t,\tau), \tag{1.12}$$

where $R_{\mathrm{u}}(t,\tau)$ is a term which depends only on the auxiliary operators on the cusps an can be computed explicitly. Using now a theorem of Kostant on Lie algebra cohomology, we obtain explicitly computable $\sigma_{\tau,k}\in\hat{M}$ and $\lambda_{\tau,k}\in\mathbb{R}^+$ such that the kernel k_t^τ can be rewritten as

$$k_t^\tau=\sum_{k=0}^{n}(-1)^{k+1}e^{-t\lambda_{\tau,k}^2}h_t^{\sigma_{\tau,k}}, \tag{1.13}$$

where the functions $h_t^{\sigma_{\tau,k}}$ are as above. Let $\mathcal{M}I(t,\tau)$ be the Mellin transform of $I(k_t^\tau)$ evaluated at 0. Then one has

$$\log T_X^{(2)}(\tau)=\frac{1}{2}\mathcal{M}I(t,\tau).$$

Let $\Delta_\tau(k)=A(\sigma_{\tau,k})+\lambda_{\tau,k}^2$. We will show that $\det_{\mathrm{gr,u}}(\Delta_\tau(k))$ is canonically defined. From (1.13) one can easily deduce the equality

$$T_{X,\mathrm{u}}(\tau)=\prod_{k=0}^{n}\det{}_{\mathrm{gr,u}}(\Delta_\tau(k))^{(-1)^k}.$$

Now we let $\tau=\tau(m)$ and apply (1.10). The term $\mathcal{M}I(t,\tau(m))$ can be identified with the corresponding weighted sum of the $\mathcal{LM}I(\lambda_{\tau(m),k},\sigma_{\tau(m),k})$ and using the Harish-Chandra Plancherel-Theorem we deduce equation (1.5) and equation (1.6). Thus in order to prove our main result we have to show that the weighted sum of all other terms in (1.10) for $\sigma=\sigma_{\tau(m),k}$, $s=\lambda_{\tau(m),k}$ is of lower order. Firstly, the contribution of the Selberg zeta functions decays exponentially. This proves Theorem 1.5, since all other terms vanish in the compact case. To estimate the other terms in the non-compact case, we again use Hoffmann's computation of the Fourier transform of the weighted orbital integral as well as the explicit formulas for the standard Knapp-Stein intertwining operators. This completes the proof of Theorem 1.3 and of Theorem 1.4.

Let us remark that one can also prove our main result about the asymptotic behaviour of the analytic torsion without using the Selberg zeta functions. The proof is then nearly the same, in particular, the treatment of the identity contribution, which is the leading term, of the weighted orbital integrals and of the Knapp-Stein intertwining term is unaltered. For more details, we refer the reader to [MP2]. However, if one wants to obtain a remainder estimate for the cocompact case as in Theorem 1.5 without using the Selberg zeta functions, one has to impose additional arguments. This was done in [MP1].

To prove Theorem 1.6, we combine our previous results with Wotzke's methods for the proof of equation (1.8). For the 3-dimensional case, these methods are described in [Mü4]. Firstly, as in [Mü4], [Wo], a Theorem of Kostant on Lie algebra cohomology gives

$$T_X(\tau(m))^2 = \frac{\det_{\mathrm{gr}}(A(\sigma_m) + (m+1)^2)}{\det_{\mathrm{gr}}(A(\sigma_{m+1}) + m^2)} \tag{1.14}$$

and

$$R_\tau(m) R_{\tau(m)_\theta}(s) = \frac{S(s+m+1, \sigma_m) S(s-m-1, \sigma_m)}{S(s+m, \sigma_{m+1}) S(s-m, \sigma_{m+1})}. \tag{1.15}$$

To relate the behaviour of $R_\tau(m) R_{\tau(m)_\theta}(s)$ at 0 to $T_X(\tau(m))$ we now want to apply our determinant formula from equation (1.10) to the right hand side of equation (1.15) and combine the result with equation (1.14). However, in contrast to the situation on a closed hyperbolic 3-manifold, this is not possible directly since in the non-compact case equation (1.10) only holds for $\mathrm{Re}(s) > 0$, $\mathrm{Re}(s^2) > 0$. Thus we additionally have to apply a functional equation for those Selberg zeta functions on the right hand side of (1.15) which have a negative argument for s in a neighbourhood of 0. Via the functional equation, the C-matrices appear in Theorem 1.6. We will prove a functional equation for the symmetric Selberg zeta function in section 10.2. In principle, our proof of this functional equation and thus also a theorem similar to Theorem 1.6 carry over to higher dimensions. However, the results would be rather complicated.

1.3 Structure of this thesis

This thesis is organized as follows. In section 2 we fix notations and collect some basic facts about representation theory which are used throughout this thesis. In section 3 we introduce Ruelle and Selberg zeta functions and establish their convergence in some half space. Section 4 is devoted to the right regular representation of G on $L^2(\Gamma\backslash G)$. We first review its decomposition into a discrete and a continuous part. Then we describe some basic properties of the C-matrix associated to the Eisenstein series which are needed in our setting. Finally we recall the MaaßSelberg relations which give an explicit formula for the inner product of truncated Eisenstein series. In section 5 we introduce the relative trace of locally invariant differential operators which act on locally homogeneous vector bundles over X. We first introduce these operators and compute the Fourier transform of the associated heat-kernels on the universal covering. Then we study certain ordinary differential operators on the cusp which are induced by the locally invariant differential operators and compute the local trace of their heat kernels. Finally we define the relative trace and show that it equals the spectral side of the Selberg trace formula applied to the heat kernel on the universal covering space of X up to a term which can be computed explicitly. In section 6 we state the Selberg trace formula in its invariant form and the non-invariant trace formula for the relative trace of the locally invariant differential operators. We also study the Fourier transform of the distribution $\mathcal{I}$ which appears in the trace formula. The proof of Theorem 1.1 will be established in section 7. In section 8 we will study the relative determinant of locally invariant differential operators and prove equation (1.10). In order to define the relative determinant, one needs a short time asymptotic expansion of the relative heat trace which will also be established. In section 9 we introduce the relative analytic torsion as well as the L^2-torsion and prove our main results about the asymptotic behaviour of the analytic torsion. In the final section 10 we apply our results to the 3-dimensional case and prove Theorem 1.6 and Corollary 1.7.

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2 Preliminaries

In this section we will establish some notation and recall some basic facts about representations of the involved Lie groups.

2.1

For $d \in \mathbb{N}$, $d = 2n + 1$ we let $G := \mathrm{Spin}(d, 1)$. The group G is defined as the universal covering group of $\mathrm{SO}_0(d, 1)$, where $\mathrm{SO}_0(d, 1)$ is the identity component of $\mathrm{SO}(d, 1)$. Let $K := \mathrm{Spin}(d)$. Then K is a maximal compact subgroup of G. Put $\tilde{X} := G/K$. Let

$$G = NAK$$

be the standard Iwasawa decomposition of G and let M be the centralizer of A in G. Then we have $M = \mathrm{Spin}(d - 1)$. The Lie algebras of G, K, A, M and N will be denoted by $\mathfrak{g}, \mathfrak{k}, \mathfrak{a}, \mathfrak{m}$ and $\mathfrak{n}$, respectively. Define the standard Cartan involution $\theta : \mathfrak{g} \to \mathfrak{g}$ by

$$\theta(Y) = -Y^t, \quad Y \in \mathfrak{g}.$$

The lift of θ to G will be denoted by the same letter θ. Let

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

be the Cartan decomposition of $\mathfrak{g}$ with respect to θ. Let $x_0 = eK \in \tilde{X}$. Then we have a canonical isomorphism

$$T_{x_0}\tilde{X} \cong \mathfrak{p}. \tag{2.1}$$

We define a symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ by

$$\langle Y_1, Y_2 \rangle := \frac{1}{2(d-1)} B(Y_1, Y_2), \quad Y_1, Y_2 \in \mathfrak{g}. \tag{2.2}$$

By (2.1) the restriction of $\langle \cdot, \cdot \rangle$ to $\mathfrak{p}$ defines an inner product on $T_{x_0}\tilde{X}$ and therefore an invariant metric on $\tilde{X}$. This metric has constant curvature -1 and $\tilde{X}$, equipped with this metric, is isometric to the hyperbolic space $\mathbb{H}^d$.

2.2

Denote by $E_{i,j}$ the matrix in $\mathfrak{g}$ whose entry at the i-th row and j-th column is equal to 1 and all of its other entries are equal to 0. Let

$$H_i := \begin{cases} E_{1,2} + E_{2,1}, & i = 1; \\ \sqrt{-1}(E_{2i-1,2i} - E_{2i,2i-1}), & i = 2, \dots n + 1. \end{cases} \tag{2.3}$$

Then

$$\mathfrak{a} = \mathbb{R}H_1$$

and

$$\mathfrak{b} = \mathbb{R}\sqrt{-1}H_2 + \cdots + \mathbb{R}\sqrt{-1}H_{n+1}$$

is the standard Cartan subalgebra of $\mathfrak{m}$. Moreover $\mathfrak{b}$ is also a Cartan subalgebra of $\mathfrak{k}$, and

$$\mathfrak{h} := \mathfrak{a} \oplus \mathfrak{b}$$

is a Cartan-subalgebra of $\mathfrak{g}$. Define $e_i \in \mathfrak{h}_{\mathbb{C}}^*$, $i = 1, \dots, n+1$, by

$$e_i(H_j) = \delta_{i,j},\ 1 \leq i, j \leq n+1.$$

Then the sets of roots of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$, $(\mathfrak{k}_{\mathbb{C}}, \mathfrak{b}_{\mathbb{C}})$ and $(\mathfrak{m}_{\mathbb{C}}, \mathfrak{b}_{\mathbb{C}})$ are given by

$$\begin{aligned}
&\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) = \{\pm e_i \pm e_j,\ 1 \leq i < j \leq n+1\} \\
&\Delta(\mathfrak{k}_{\mathbb{C}}, \mathfrak{b}_{\mathbb{C}}) = \{\pm e_i,\ 2 \leq i < j \leq n+1\} \sqcup \{\pm e_i \pm e_j,\ 2 \leq i < j \leq n+1\} \\
&\Delta(\mathfrak{m}_{\mathbb{C}}, \mathfrak{b}_{\mathbb{C}}) = \{\pm e_i \pm e_j,\ 2 \leq i < j \leq n+1\}
\end{aligned}$$

(see [Kn1, Section IV,2]). We fix positive systems of roots by

$$\begin{aligned}
&\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) := \{e_i + e_j,\ i \neq j\} \sqcup \{e_i - e_j,\ i < j\} \\
&\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{b}_{\mathbb{C}}) := \{e_i \colon 2 \leq i \leq n+1\} \sqcup \{e_i + e_j,\ i \neq j,\ i, j \geq 2\} \sqcup \{e_i - e_j,\ 2 \leq i < j\} \\
&\Delta^+(\mathfrak{m}_{\mathbb{C}}, \mathfrak{b}_{\mathbb{C}}) := \{e_i + e_j,\ i \neq j,\ i, j \geq 2\} \sqcup \{e_i - e_j,\ 2 \leq i < j\}.
\end{aligned}$$

We let $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{a}_{\mathbb{C}})$ be the set of roots of $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ which do not vanish on $\mathfrak{a}_{\mathbb{C}}$. Then

$$\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{a}_{\mathbb{C}}) = \{e_1 \pm e_j \colon j = 2, \dots, n+1\}.$$

For $\alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ there exists a unique $H'_\alpha \in \mathfrak{h}_{\mathbb{C}}$ such that $B(H, H'_\alpha) = \alpha(H)$ for all $H \in \mathfrak{h}_{\mathbb{C}}$. One has $\alpha(H'_\alpha) \neq 0$. We let

$$H_\alpha := \frac{2}{\alpha(H'_\alpha)} H'_\alpha.$$

One easily sees that

$$H_{\pm e_i \pm e_j} = \pm H_i \pm H_j. \tag{2.4}$$

For $j = 1, \dots, n+1$ we let

$$\rho_j := n + 1 - j.$$

Then the half-sum of positive roots are given by

$$\rho_G := \frac{1}{2} \sum_{\alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})} \alpha = \sum_{j=1}^{n+1} \rho_j e_j \tag{2.5}$$

and

$$\rho_K := \frac{1}{2}\sum_{\alpha\in\Delta^+(\mathfrak{k}_\mathbb{C},\mathfrak{b}_\mathbb{C})}\alpha = \sum_{j=2}^{n+1}(\rho_j+1/2)e_j$$

and

$$\rho_M := \frac{1}{2}\sum_{\alpha\in\Delta^+(\mathfrak{m}_\mathbb{C},\mathfrak{b}_\mathbb{C})}\alpha = \sum_{j=2}^{n+1}\rho_j e_j. \tag{2.6}$$

We let W_G be the Weyl-group of $\Delta(\mathfrak{g}_\mathbb{C},\mathfrak{h}_\mathbb{C})$.

2.3

Let $\mathbb{Z}\left[\frac{1}{2}\right]^j$ be the set of all $(k_1,\dots,k_j)\in\mathbb{Q}^j$ such that either all k_i are integers or all k_i are half integers. Then the finite dimensional irreducible representations $\tau\in\hat{G}$ of G are parametrized by their highest weights

$$\begin{aligned}\Lambda(\tau) =&k_1(\tau)e_1+\dots+k_{n+1}(\tau)e_{n+1},\quad (k_1(\tau),\dots k_{n+1}(\tau))\in\mathbb{Z}\left[\frac{1}{2}\right]^{n+1},\\ &k_1(\tau)\geq k_2(\tau)\geq\dots\geq k_n(\tau)\geq\left|k_{n+1}(\tau)\right|.\end{aligned} \tag{2.7}$$

Furthermore the finite dimensional representations $\nu\in\hat{K}$ of K are parametrized by their highest weights

$$\begin{aligned}\Lambda(\nu) =&k_2(\nu)e_2+\dots+k_{n+1}(\nu)e_{n+1},\quad (k_2(\nu),\dots k_{n+1}(\nu))\in\mathbb{Z}\left[\frac{1}{2}\right]^{n},\\ &k_2(\nu)\geq k_2(\nu)\geq\dots\geq k_n(\nu)\geq k_{n+1}(\nu)\geq 0.\end{aligned} \tag{2.8}$$

Finally the finite dimensional irreducible representations $\sigma\in\hat{M}$ of M are parametrized by their highest weights

$$\begin{aligned}\Lambda(\sigma) =&k_2(\sigma)e_2+\dots+k_{n+1}(\sigma)e_{n+1},\quad (k_2(\sigma),\dots,k_{n+1}(\sigma))\in\mathbb{Z}\left[\frac{1}{2}\right]^{n},\\ &k_2(\sigma)\geq k_3(\sigma)\geq\dots\geq k_n(\sigma)\geq\left|k_{n+1}(\sigma)\right|.\end{aligned} \tag{2.9}$$

For $\tau\in\hat{G}$ let $\tau_\theta:=\tau\circ\theta$. Let $\Lambda(\tau)$ denote the highest weight of τ as in (2.7). Then the highest weight $\Lambda(\tau_\theta)$ of τ_θ is given by

$$\Lambda(\tau_\theta)=k_1(\tau)e_1+\dots+k_n(\tau)e_n-k_{n+1}(\tau)e_{n+1}. \tag{2.10}$$

Moreover, by the Weyl dimension formula [Kn1, Theorem 4.48] we have

$$\begin{aligned}\dim(\tau)=&\prod_{\alpha\in\Delta^+(\mathfrak{g}_\mathbb{C},\mathfrak{h}_\mathbb{C})}\frac{\langle\Lambda(\tau)+\rho_G,\alpha\rangle}{\langle\rho_G,\alpha\rangle}\\ =&\prod_{i=1}^{n}\prod_{j=i+1}^{n+1}\frac{(k_i(\tau)+\rho_i)^2-(k_j(\tau)+\rho_j)^2}{\rho_i^2-\rho_j^2}.\end{aligned} \tag{2.11}$$

Similarly, for $\sigma \in \hat{M}$ with highest weight $\Lambda(\sigma) \in \mathfrak{b}_{\mathbb{C}}^*$ as in (2.9) we have

$$\begin{aligned}\dim(\sigma) &= \prod_{\alpha\in\Delta^+(\mathfrak{m}_{\mathbb{C}},\mathfrak{b}_{\mathbb{C}})} \frac{\langle \Lambda(\sigma)+\rho_M, \alpha\rangle}{\langle \rho_M, \alpha\rangle} \\ &= \prod_{i=2}^{n}\prod_{j=i+1}^{n+1} \frac{(k_i(\sigma)+\rho_i)^2-(k_j(\sigma)+\rho_j)^2}{\rho_i^2-\rho_j^2}.\end{aligned} \tag{2.12}$$

Finally, if $\nu \in \hat{K}$ is of highest weight $\Lambda(\nu) \in \mathfrak{b}_{\mathbb{C}}^*$ as in (2.8) one has

$$\begin{aligned}\dim(\nu) &= \prod_{\alpha\in\Delta^+(\mathfrak{k}_{\mathbb{C}},\mathfrak{b}_{\mathbb{C}})} \frac{\langle \Lambda(\nu)+\rho_K, \alpha\rangle}{\langle \rho_K, \alpha\rangle} \\ &= \prod_{i=2}^{n+1}(k_i(\nu)+\rho_i+1/2)\prod_{j=i+1}^{n+1} \frac{(k_i(\nu)+\rho_i+1/2)^2-(k_j(\nu)+\rho_j+1/2)^2}{(\rho_i+1/2)^2-(\rho_j+1/2)^2}.\end{aligned} \tag{2.13}$$

Let M' be the normalizer of A in K and let $W(A) = M'/M$ be the restricted Weyl-group. It has order two and it acts on the finite-dimensional representations of M as follows. Let $w_0 \in W(A)$ be the non-trivial element and let $m_0 \in M'$ be a representative of w_0. Given $\sigma \in \hat{M}$, the representation $w_0\sigma \in \hat{M}$ is defined by

$$w_0\sigma(m) = \sigma(m_0 m m_0^{-1}), \quad m \in M.$$

Let $\Lambda(\sigma) = k_2(\sigma)e_2 + \cdots + k_{n+1}(\sigma)e_{n+1}$ be the highest weight of σ as in (2.9). Then the highest weight $\Lambda(w_0\sigma)$ of $w_0\sigma$ is given by

$$\Lambda(w_0\sigma) = k_2(\sigma)e_2 + \cdots + k_n(\sigma)e_n - k_{n+1}(\sigma)e_{n+1}. \tag{2.14}$$

2.4

Next we describe the restrictions of the representations to subgroups. Firstly, for the groups G and K we have the following proposition.

Proposition 2.1. *Let $\tau \in \hat{G}$ be of highest weight $k_1(\tau)e_1 + \cdots + k_{n+1}(\tau)e_{n+1}$ as in (2.7). Then τ decomposes with multiplicity one into the representations $\nu \in \hat{K}$ with highest weight $k_2(\nu)e_2 + \cdots + k_{n+1}(\nu)e_{n+1}$ as in (2.8) such that $k_{j-1}(\tau) \geq k_j(\nu) \geq |k_j(\tau)|$ for every $j \in \{2, \ldots, n+1\}$ and such that all $k_j(\nu)$ are integers if all $k_j(\tau)$ are integers resp. such that all $k_j(\nu)$ are half-integers if all $k_j(\tau)$ are half integers.*

Proof. [GW][Theorem 8.1.4] □

Secondly, for the groups K and M one has the following proposition.

Proposition 2.2. *Let $\nu \in \hat{K}$ with highest weight $k_2(\nu)e_2 + \cdots + k_{n+1}(\nu)e_{n+1}$ as in (2.8). Then ν decomposes with multiplicity one into representations $\sigma \in \hat{M}$ with highest weight $k_2(\sigma)e_2 + \cdots + k_{n+1}(\sigma)e_{n+1}$ such that $k_j(\nu) \geq |k_j(\sigma)|$ for every $j \in \{2, \ldots, n+1\}$ and such that all $k_j(\sigma)$ are integers if all $k_j(\nu)$ are integers resp. such that all $k_j(\sigma)$ are half integers if all $k_j(\sigma)$ are half integers.*

Proof. [GW][Theorem 8.1.3] □

Let κ be the spin-representation of K over the spinor space Δ^{2n} as in [Fri, page 14]. Then κ is the representation with highest weight

$$\Lambda(\kappa) = \frac{1}{2}e_2 + \cdots + \frac{1}{2}e_{n+1}. \tag{2.15}$$

By Proposition 2.2 there is an M-invariant splitting

$$\Delta^{2n} = \Delta_+^{2n} \oplus \Delta_-^{2n}$$

such that the restriction of κ to M acts on Δ_+^{2n} as κ^+ and on Δ_-^{2n} as κ^-, where κ^+ is the representation with highest weight $\frac{1}{2}e_2 + \cdots + \frac{1}{2}e_{n+1}$ and κ^- is the representation with highest weight $\frac{1}{2}e_2 + \cdots + \frac{1}{2}e_n - \frac{1}{2}e_{n+1}$. Let $R(K)$ and $R(M)$ be the representation rings of K and M. Let $\iota : M \longrightarrow K$ be the inclusion and let $\iota^* : R(K) \longrightarrow R(M)$ be the induced map. If $R(M)^{W(A)}$ is the subring of $W(A)$-invariant elements of $R(M)$, then clearly ι^* maps $R(K)$ into $R(M)^{W(A)}$.

Proposition 2.3. *The map ι is an isomorphism from $R(K)$ onto $R(M)^{W(A)}$. Explicitly, let $\sigma \in \hat{M}$ be of highest weight $\Lambda(\sigma)$ as in (2.9) and assume that $k_{n+1}(\sigma) > 0$. Let $\nu(\sigma) \in \hat{K}$ be the representation of highest weight*

$$\Lambda\left(\nu(\sigma)\right) := \sum_{j=2}^{n+1} \left(k_j(\sigma) - 1/2\right) e_j. \tag{2.16}$$

Then one has

$$\sigma - w_0\sigma = \left(\kappa^+ - \kappa^-\right) \otimes \iota^*\nu(\sigma).$$

Moreover, $\nu(\sigma) \otimes \kappa$ splits as $\nu(\sigma) \otimes \kappa = \nu^+(\sigma) \oplus \nu^-(\sigma)$ such that

$$\sigma + w_0\sigma = \iota^*\nu^+(\sigma) - \iota^*\nu^-(\sigma).$$

Here one has

$$\nu^{\pm}(\sigma) = \sum_{\substack{\mu\in\{0,1\}^n,\, c(\mu)=\pm 1 \\ \Lambda(\sigma)-\mu \text{ as in } (2.8)}} (-1)^{c(\mu)} \nu\left(\Lambda(\sigma) - \mu\right), \tag{2.17}$$

where $c(\mu) := \#\{1 \in \mu\}$ and where $\nu\left(\Lambda(\sigma) - \mu\right)$ denotes the representation of K with highest weight $\Lambda(\sigma) - \mu$.

Proof. This is proved by Bunke and Olbrich, [BO] , Proposition 1.1. □

2.5

Measures are normalized as follows. Let $\rho := ne_1$. Every $a \in A$ can be written as $a = \exp \log a$, where $\log a \in \mathfrak{a}$ is unique. For $t \in \mathbb{R}$, we let $a(t) := \exp(tH_1)$. If $g \in G$, we define $n(g) \in N$, $H(g) \in \mathbb{R}$ and $\kappa(g) \in K$ by

$$g = n(g)a(H(g))\kappa(g).$$

Normalize the Haar-measure on K such that K has volume 1. We let

$$\langle X, Y \rangle_\theta := -\frac{1}{2(d-1)} B(X, \theta(Y)). \tag{2.18}$$

We fix an isometric identification of $\mathbb{R}^{2n}$ with $\mathfrak{n}$ with respect to the inner product $\langle \cdot, \cdot \rangle_\theta$. We give $\mathfrak{n}$ the measure induced from the Lebesgue measure under this identification. Moreover, we identify $\mathfrak{n}$ and N by the exponential map and we will denote by dn the Haar measure on N induced from the measure on $\mathfrak{n}$ under this identification. We normalize the Haar measure on G by setting

$$\int_G f(g)dg = \int_N \int_{\mathbb{R}} \int_K e^{-2nt} f(na(t)k) dk dt dn. \tag{2.19}$$

Let $\Gamma \subset G$ be a discrete subgroup. We equip $\Gamma\backslash G$ and $\tilde{X}$ with the induced quotient measures, where Γ carry the counting measure. Let $\pi : G \to \Gamma\backslash G$ be the projection. For $f \in C_c(G)$ define $\bar{f} : \Gamma\backslash G \to \mathbb{C}$ by

$$\bar{f}(\pi(g)) := \sum_{\gamma \in \Gamma} f(\gamma g).$$

Then the quotient measure dx on $\Gamma\backslash G$ is uniquely characterized by the property that for $f \in C_c(G)$ one has

$$\int_G f(g)dg = \int_{\Gamma\backslash G} \bar{f}(x)dx$$

for $f \in C_C(G)$. A corresponding equality holds for the quotient measure on $\widetilde{X}$. We note the following lemma.

Lemma 2.4. *Let $\Gamma' \subset \Gamma$ be a subgroup. Let $f : \Gamma'\backslash G \to \mathbb{R}^+$ be measurable. Let $\pi : \Gamma'\backslash G \to \Gamma\backslash G$ be the projection. Then one has*

$$\int_{\Gamma'\backslash G} f(y)dy = \int_{\Gamma\backslash G} \sum_{\gamma \in \Gamma'\backslash\Gamma} f(\gamma' \pi^{-1} x)dx.$$

Proof. By [Ra, Lemma 1.1] the map $f \mapsto \bar{f}$ is surjective. This easily implies the proposition. $\square$

2.6

We parametrize the principal series as follows. Given $\sigma \in \hat{M}$ with $(\sigma, V_\sigma) \in \sigma$, let $\mathcal{H}^\sigma$ denote the space of measurable functions $f\colon K \to V_\sigma$ satisfying

$$f(mk) = \sigma(m)f(k), \quad \forall k \in K, \forall m \in M, \quad \text{and} \quad \int_K \| f(k) \|^2 \; dk = \| f \|^2 < \infty.$$

Then for $\lambda \in \mathbb{C}$ and $f \in H^\sigma$ let

$$\pi_{\sigma,\lambda}(g)f(k) := e^{(i\lambda+n)H(kg)} f(\kappa(kg)).$$

Recall that the representations $\pi_{\sigma,\lambda}$ are unitary iff $\lambda \in \mathbb{R}$. Moreover, for $\lambda \in \mathbb{R} - \{0\}$ and $\sigma \in \hat{M}$ the representations $\pi_{\sigma,\lambda}$ are irreducible and $\pi_{\sigma,\lambda}$ and $\pi_{\sigma',\lambda'}$, $\lambda, \lambda' \in \mathbb{C}$ are equivalent iff either $\sigma = \sigma'$, $\lambda = \lambda'$ or $\sigma' = w_0\sigma$, $\lambda' = -\lambda$. The restriction of $\pi_{\sigma,\lambda}$ to K coincides with the induced representation $\mathrm{Ind}_M^K(\sigma)$. Hence by Frobenius reciprocity [Kn1, p.208] for every $\nu \in \hat{K}$ one has

$$[\pi_{\sigma,\lambda} : \nu] = [\nu : \sigma]\,. \tag{2.20}$$

2.7

We establish some facts about infinitesimal characters. Let $U(\mathfrak{g}_\mathbb{C})$ denote the universal enveloping algebra of $\mathfrak{g}_\mathbb{C}$ and let $Z(U(\mathfrak{g}_\mathbb{C}))$ be its center. Let $\Omega \in Z(U(\mathfrak{g}_\mathbb{C}))$ be the Casimir element with respect to the Killing form normalized as in (2.2). Let $I(\mathfrak{h}_\mathbb{C})$ be the Weyl-group invariant elements of the symmetric algebra $S(\mathfrak{h}_\mathbb{C})$ of $\mathfrak{h}_\mathbb{C}$. Let

$$\gamma : Z(U(\mathfrak{g}_\mathbb{C})) \longrightarrow I(\mathfrak{h}_\mathbb{C}) \tag{2.21}$$

be the Harish-Chandra isomorphism [Kn1, Section VIII,5]. Every $\Lambda \in \mathfrak{h}_\mathbb{C}^*$ defines a homomorphism

$$\chi_\Lambda : Z(U(\mathfrak{g}_\mathbb{C})) \longrightarrow \mathbb{C}$$

by

$$\chi_\Lambda(Z) := \Lambda(\gamma(Z)).$$

Let τ be an irreducible finite-dimensional representation of G with highest weight $\Lambda(\tau)$. Its infinitesimal character will also be denoted by τ, i.e. every $Z \in Z(U(\mathfrak{g}_\mathbb{C}))$ acts by $\tau(Z) \cdot \mathrm{Id}$. It follows from the definition of γ that

$$\tau(Z) = \chi_{\Lambda(\tau)+\rho_G}(Z) = \chi_{w(\Lambda(\tau)+\rho_G)}(Z); \quad Z \in Z(U(\mathfrak{g}_\mathbb{C})),\ w \in W. \tag{2.22}$$

Moreover, a standard computation gives

$$\gamma(\Omega) = \sum_{j=1}^{n+1} H_j^2 - \sum_{j=1}^{n+1} \rho_j^2, \tag{2.23}$$

where the H_j are defined by (2.3). Thus, if the highest weight $\Lambda(\tau)$ of τ is as in (2.7) one obtains

$$\tau(\Omega) = \sum_{j=1}^{n+1} (k_j(\tau) + \rho_j)^2 - \sum_{j=1}^{n+1} \rho_j^2 \tag{2.24}$$

Let Ω_M be the Casimir operator of $\mathfrak{m}$ with respect to the restriction of the normalized killing form on $\mathfrak{g}$ to $\mathfrak{m}$. Then Ω_M lies in the center of $U(m_{\mathbb{C}})$. If $\sigma \in \hat{M}$, its infinitesimal character will be denoted by σ too. Then if $\Lambda(\sigma)$ is as in (2.9), the same argument as above gives

$$\sigma(\Omega_M) = \sum_{j=2}^{n+1} (k_j(\sigma) + \rho_j)^2 - \sum_{j=2}^{n+1} \rho_j^2. \tag{2.25}$$

Finally let Ω_K be the Casimir operator of $\mathfrak{k}$ with respect to the restriction of the normalized killing form on $\mathfrak{g}$ to $\mathfrak{k}$. Then Ω_K belongs to $Z(U(\mathfrak{k}_{\mathbb{C}}))$, the center of the universal enveloping algebra of $\mathfrak{k}_{\mathbb{C}}$. If $\nu \in \hat{K}$, we will denote the infinitesimal character of ν by ν too. If the highest weight $\Lambda(\nu)$ of ν is as in (2.8), an argument analogous to the one above gives

$$\nu(\Omega_K) = \sum_{j=2}^{n+1} \left(k_j(\nu) + \rho_j + \frac{1}{2}\right)^2 - \sum_{j=2}^{n+1} \left(\rho_j + \frac{1}{2}\right)^2. \tag{2.26}$$

2.8

Now we come to the infinitesimal character of $\pi_{\sigma,\lambda}$.

Proposition 2.5. *Let $\sigma \in \hat{M}$ with highest weight $\Lambda(\sigma) \in \mathfrak{b}_{\mathbb{C}}^*$. Then the infinitesimal character of $\pi_{\sigma,\lambda}$ equals $\chi_{\Lambda(\sigma)+\rho_M+i\lambda e_1}$.*

Proof. [Kn1], Proposition 8.22. □

Corollary 2.6. *For $\sigma \in \hat{M}$ with highest weight $\Lambda(\sigma)$ given by (2.9), let*

$$c(\sigma) := \sum_{j=2}^{n+1} (k_j(\sigma) + \rho_j)^2 - \sum_{j=1}^{n+1} \rho_j^2. \tag{2.27}$$

Then for the Casimir element $\Omega \in Z(\mathfrak{g}_{\mathbb{C}})$ one has

$$\pi_{\sigma,\lambda}(\Omega) = -\lambda^2 + c(\sigma).$$

Proof. This follows form equation (2.23) and proposition 2.5. □

2.9

Let $\tau \in \hat{G}$ and let $\Lambda(\tau) = \tau_1 e_1 + \cdots + \tau_{n+1} e_{n+1}$ be its highest weight. For $w \in W$ let $l(w)$ denote its length with respect to the simple roots which define the positive roots above. Let

$$W^1 := \{w \in W_G \colon w^{-1}\alpha > 0 \,\forall \alpha \in \Delta(\mathfrak{m}_{\mathbb{C}}, \mathfrak{b}_{\mathbb{C}})\}$$

Let V_τ be the representation space of τ. For $k = 0, \dots, 2n$ let $H^k(\mathfrak{n}, V_\tau)$ be the cohomology of $\mathfrak{n}$ with coefficients in V_τ. Then $H^k(\mathfrak{n}, V_\tau)$ is an MA module. In our case, the theorem of Kostant states:

Proposition 2.7. *In the sense of MA-modules one has*

$$H^k(\mathfrak{n}; V_\tau) \cong \sum_{\substack{w \in W^1 \\ l(w)=k}} V_{\tau(w)},$$

where $V_{\tau(w)}$ is the MA module of highest weight $w(\Lambda(\tau) + \rho_G) - \rho_G$.

Proof. See [BW, Theorem III.3]. □

Corollary 2.8. *As MA-modules we have*

$$\bigoplus_{k=0}^{2n} (-1)^k \Lambda^k \mathfrak{n}^* \otimes V_\tau = \bigoplus_{w \in W^1} (-1)^{l(w)} V_{\tau(w)}.$$

Proof. This follows from proposition 2.7 and the Poincare principle [Ko, (7.2.3)]. □

For $w \in W^1$ let $\sigma_{\tau,w}$ be the representation of M with highest weight

$$\Lambda(\sigma_{\tau,w}) := w(\Lambda(\tau) + \rho_G)|_{\mathfrak{b}_{\mathbb{C}}} - \rho_M \tag{2.28}$$

and let $\lambda_{\tau,w} \in \mathbb{C}$ such that

$$w(\Lambda(\tau) + \rho_G)|_{\mathfrak{a}_{\mathbb{C}}} = \lambda_{\tau,w} e_1. \tag{2.29}$$

For $k = 0, \dots n$ let

$$\lambda_{\tau,k} = \tau_{k+1} + n - k \tag{2.30}$$

and let $\sigma_{\tau,k}$ be the representation of G with highest weight

$$\Lambda(\sigma_{\tau,k}) := (\tau_1 + 1)e_2 + \cdots + (\tau_k + 1)e_{k+1} + \tau_{k+2} e_{k+2} + \cdots + \tau_{n+1} e_{n+1}. \tag{2.31}$$

Then by the computations in [BW, Chapter VI.3] one has

$$\begin{aligned} \{(\lambda_{\tau,w}, \sigma_{\tau,w}, l(w)) \colon w \in W^1\} = {} & \{(\lambda_{\tau,k}, \sigma_{\tau,k}, k) \colon k = 0, \dots, n\} \\ & \sqcup \{(-\lambda_{\tau,k}, w_0 \sigma_{\tau,k}, 2n - k) \colon k = 0, \dots, n\}. \end{aligned} \tag{2.32}$$

Remark 2.9. Corollary 2.8 was first proved by U. Bröcker[Br] by an elementary but tedious computation without using the theorem of Kostant. Also the convenient notation $\sigma_{\tau,k}$ and $\lambda_{\tau,k}$ is due to him.

We will also need the following proposition.

Proposition 2.10. *For every $w \in W^1$ one has*

$$\tau(\Omega) = \lambda_{\tau,w}^2 + c(\sigma_{\tau,w}).$$

Proof. Using (2.22) and (2.23) one gets

$$\tau(\Omega) = \chi_{\Lambda(\tau)+\rho_G}(\Omega) = \chi_{w(\Lambda(\tau)+\rho_G)}(\Omega) = \chi_{\Lambda(\sigma_{\tau,w})+\rho_M+\lambda_\tau(w)e_1}(\Omega) = \lambda_{\tau,w}^2 + c(\sigma_{\tau,w}).$$

□

2.10

For $\sigma \in \hat{M}$ and $\lambda \in \mathbb{R}$ let $\mu_\sigma(\lambda)$ be the Plancherel measure associated to $\pi_{\sigma,\lambda}$. Then, since $\mathrm{rk}(G) > \mathrm{rk}(K)$, $\mu_\sigma(\lambda)$ is a polynomial in λ of degree $2n$. Let $\langle \cdot, \cdot \rangle$ be the bilinear form defined by (2.2). Let $\Lambda(\sigma) \in \mathfrak{b}_{\mathbb{C}}^*$ be the highest weight of σ as in (2.9). Then by Theorem 13.2 in [Kn1] there exists a constant $c(n) \in \mathbb{R}$, $c(n) \neq 0$ such that one has

$$\mu_\sigma(\lambda) = -c(n) \prod_{\alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})} \frac{\langle i\lambda e_1 + \Lambda(\sigma) + \rho_M, \alpha \rangle}{\langle \rho_G, \alpha \rangle}.$$

By the computations in [Ol] and our normalization one has

$$c(n) = (-1)^{n+1} \frac{1}{2\,\mathrm{vol}(S^d)}, \tag{2.33}$$

where S^d is the d-dimensional Euclidean unit sphere. For $z \in \mathbb{C}$ let

$$P_\sigma(z) = -c(n) \prod_{\alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})} \frac{\langle z e_1 + \Lambda(\sigma) + \rho_M, \alpha \rangle}{\langle \rho_G, \alpha \rangle}. \tag{2.34}$$

One easily sees that

$$P_\sigma(z) = P_{w_0\sigma}(z) = P_\sigma(-z). \tag{2.35}$$

2.11

We let Γ be a discrete, torsion free subgroup of G with $\mathrm{vol}(\Gamma\backslash G) < \infty$. Let $X := \Gamma\backslash G/K$. We equip X with the quotient measure induced from G and with the Riemannian metric induced from $\widetilde{X}$. Let $\mathfrak{P}$ be a fixed set of representatives of Γ-inequivalent cuspidal parabolic subgroups of G. Then $\mathfrak{P}$ is finite. Let $p := \#\mathfrak{P}$. Let

$$P_0 := MAN.$$

Without loss of generality we will assume that $P_0 \in \mathfrak{P}$. For every $P \in \mathfrak{P}$, there exists a $k_P \in K$ such that $P = N_P A_P M_P$ with $N_P = k_P N k_P^{-1}$, $A_P = k_P A k_P^{-1}$, $M_P = k_P M k_P^{-1}$. We let $k_{P_0} = 1$. We will assume from now on that for each $P \in \mathfrak{P}$ one has

$$\Gamma \cap P = \Gamma \cap N_P. \tag{2.36}$$

Since N_P is abelian, $\Gamma \cap N_P \backslash N \cong T^{2n}$, the flat 2n-torus. For $P \in \mathfrak{P}$ let $a_P(t) := k_P a(t) k_P^{-1}$. If $g \in G$, we define $n_P(g) \in N_P$, $H_P(g) \in \mathbb{R}$ and $\kappa_P(g) \in K$ by

$$g = n_P(g) a_P(H_P(g)) \kappa_P(g).$$

Now for each $P \in \mathfrak{P}$ define an identification ι_P of $(0,\infty)$ with A_P by $\iota_P(t) := a_P(\log(t))$. For $Y > 0$, let $A_P^0[Y] := \iota_P(Y,\infty)$ and $A_P[Y] := \iota_P[Y,\infty)$. Then there exists a $Y_0 > 0$ and for every $Y \geq Y_0$ a compact connected subset $C(Y)$ of G, $C(Y') \supseteq C(Y)$ for $Y' \geq Y$ such that in the sense of a disjoint union one has

$$G = \Gamma \cdot C(Y) \sqcup \bigsqcup_{P \in \mathfrak{P}} \Gamma \cdot N_P A_P^0[Y] K \tag{2.37}$$

and such that

$$\gamma \cdot N_P A_P^0[Y] K \cap N_P A_P^0[Y] K \neq \emptyset \Leftrightarrow \gamma \in \Gamma \cap N_P. \tag{2.38}$$

The measures on N_P and A_P will be the measures induced from N and A via the conjugation with k_P. Let f be integrable over $\Gamma\backslash G$. Then identifying f with a measurable function on G it follows from (2.19), (2.37) and (2.38) that for every $Y \geq Y_0$ one has

$$\int_{\Gamma\backslash G} f(x)dx = \int_{C(Y)} f(g)dg + \sum_{P \in \mathfrak{P}} \int_{\Gamma \cap N_P \backslash N_P} \int_{\log Y}^{\infty} \int_K e^{-2nt} f(n_P a_P(t) k) dn_P dt dk \tag{2.39}$$

Moreover one clearly has

$$\int_{\Gamma\backslash G} f(x)dx = \lim_{Y\to\infty} \int_{C(Y)} f(x)dx. \tag{2.40}$$

If for $Y \geq Y_0$ one lets

$$F_{P,Y} := A_P[Y] \times \Gamma \cap N_P \backslash N_P \cong [Y,\infty) \times \Gamma \cap N_P \backslash N_P \tag{2.41}$$

it follows from (2.37) and (2.38) that there exists a compact manifold $X(Y)$ with smooth boundary such that X has a decomposition as

$$X = X(Y) \cup \bigsqcup_{P \in \mathfrak{P}} F_{P,Y} \tag{2.42}$$

with $X(Y) \cap F_{P,Y} = \partial X(Y) = \partial F_{P,Y}$ and $F_{P,Y} \cap F_{P',Y} = \emptyset$ if $P \neq P'$. Let g_{N_P} be the metric on $\Gamma_{N_P} \backslash N_P$ induced by $-\frac{1}{4n} B_\theta$. Then the metric on $F_{P,Y}$ is given by

$$\frac{1}{y^2} dy^2 + \frac{1}{y^2} g_{N_P}. \tag{2.43}$$

Finally, since N_P is abelian, one can canonically identify $\Gamma_{N_P} \backslash N_P$ with T^{2n} and under this identification g_{N_P} induces the standard metric on T^{2n}.
If $P \in \mathfrak{P}$, $P' \in \mathfrak{P}$ we will say that $\sigma_P \in \hat{M}_P$ and $\sigma_{P'} \in \hat{M}_{P'}$ are associated if for all $m_{P'} \in M_{P'}$ one has $\sigma_{P'}(m_{P'}) = \sigma_P(k_P k_{P'}^{-1} m_{P'} k_{P'} k_P^{-1})$. For $\sigma_P \in \hat{M}_P$ we will denote by $\boldsymbol{\sigma}$ the set of all $\sigma_{P'} \in \hat{M}_{P'}$ associated to σ_P, where P' runs through $\mathfrak{P}$.

3 Ruelle and Selberg zeta functions

In this section we give a preliminary examination of the Ruelle and Selberg zeta functions. Let $\mathrm{C}(\Gamma)$ denote the set of conjugacy classes of Γ. Then there is a canonical one-to one correspondence between $\mathrm{C}(\Gamma)$ and the set of free homotopy classes of closed paths in X. For $\gamma \in \Gamma$ we will denote its conjugacy class by $[\gamma]$. Moreover, by $f([\gamma])$ we will denote the free homotopy class of closed paths associated to $[\gamma]$. Now for $[\gamma] \in \mathrm{C}(\Gamma)$ we let

$$\ell(\gamma) := \inf_{x \in \widetilde{X}} d(x, \gamma x), \tag{3.1}$$

where $\ell(\gamma)$ is well defined since the metric on $\widetilde{X}$ is Γ-invariant. Then $\ell(\gamma)$ is the infimum over all lengths of the piecewise smooth curves belonging to $f([\gamma])$.
We let $\mathrm{C}(\Gamma)_{\mathrm{s}}$ be the set of conjugacy classes $[\gamma]$ such that γ is semisimple. Moreover we let $\mathrm{C}(\Gamma)_{\mathrm{par}}$ be the set of conjugacy classes $[\gamma]$ such that γ is Γ-conjugate to an element of $\Gamma \cap N_P$, $P \in \mathfrak{P}$. Then by [War1, Lemma 5.3] and (2.36) we have

$$\mathrm{C}(\Gamma) = \mathrm{C}(\Gamma)_{\mathrm{s}} \cup \mathrm{C}(\Gamma)_{\mathrm{par}}, \quad \mathrm{C}(\Gamma)_{\mathrm{s}} \cap \mathrm{C}(\Gamma)_{\mathrm{par}} = [1] \,.$$

Finally, the semisimple elements of Γ will be denoted by Γ_{s}. Now let $[\gamma] \in \mathrm{C}(\Gamma)_{\mathrm{s}}$. By the G-invariance of the metric d we have

$$\ell(\gamma) = \inf_{xK \in \widetilde{X}} d(1, x^{-1}\gamma x). \tag{3.2}$$

By Proposition 1.4.3.4 in [War2] the set $\{g\gamma g^{-1} \colon g \in G\}$ is closed in G. Thus, since Γ is torsion-free, for the non-trivial semisimple conjugacy classes $[\gamma]$ we have $\ell(\gamma) > 0$ and the infimum in (3.1) is attained. We shall now recall its computation. We start with the following lemma.

Lemma 3.1. *Let $\gamma \in \Gamma_{\mathrm{s}} - \{1\}$. Then there exists a $g \in G$, a $m_\gamma \in M$ and a $t(\gamma) > 0$ such that $g^{-1}\gamma g = m_\gamma \exp t_\gamma H_1$. Here t_γ is unique and m_γ is determined up to conjugacy in M.*

Proof. This follows from [Wal, Lemma 6.6]. The proof given there does not use that Γ is cocompact. $\square$

Now we can compute $\ell(\gamma)$ explicitly for every $[\gamma] \in \mathrm{C}(\Gamma)_{\mathrm{s}}$.

Proposition 3.2. *Let $[\gamma] \in \mathrm{C}(\Gamma)_{\mathrm{s}}$ be non-trivial. Then we have $\ell(\gamma) = t_\gamma$, where t_γ is as in Lemma 3.1. Moreover there exists a unique closed geodesic $c_{[\gamma]} \in f([\gamma])$ of length $\ell(\gamma)$. Explicitly, if $g \in G$ is such that $\gamma = g \exp t_\gamma H_1 m_\gamma g^{-1}$ and if $\pi : G \to X$ is the projection, the curve $c_{[\gamma]} : [0, \ell(\gamma)] \to X$ is given as $c_{[\gamma]}(t) := g \exp tH_1$.*

Proof. By Lemma 3.1 there exists a $g \in G$ such that $g \exp(t_\gamma H_1) m_\gamma g^{-1} = \gamma$. Let $\pi : G \to \widetilde{X}$ be the projection. Define a curve $\widetilde{c}_{[\gamma]} : \mathbb{R} \to \widetilde{X}$ by $\widetilde{c}_{[\gamma]}(t) := \pi(g \exp tH_1)$. Then $\widetilde{c}_{[\gamma]}$ is a geodesic and γ translates $\widetilde{c}_{[\gamma]}$ as $\gamma \cdot \widetilde{c}_{[\gamma]}(t) = \widetilde{c}_{[\gamma]}(t_\gamma + t)$. Thus, since $\widetilde{X}$ is of negative curvature, we can apply [BON, Proposition 4.2] and conclude that the elements which minimize (3.1) are exactly the elements $\widetilde{c}_{[\gamma]}(t)$, $t \in \mathbb{R}$. It follows that $\ell(\gamma) = t_\gamma$ and that $c_{[\gamma]}$ is unique and is of the required form. $\square$

We will also need the following definition.

Definition 3.3. A non-trivial semisimple conjugacy class $[\gamma] \in \mathrm{C}(\Gamma)_{\mathrm{s}} - [1]$ is called prime if $\ell(\gamma)$ is the smallest period of the geodesic $c_{[\gamma]}$, i.e. if $c_{[\gamma]}(t+s) = c_{[\gamma]}(t)$ for all t implies that $s = k \cdot \ell(\gamma)$ with $k \in \mathbb{Z}$. A semisimple non-trivial element $\gamma \in \Gamma$ is called prime if $[\gamma]$ is prime.

If $\phi : [0, t_0] \to X$ is a closed curve and $n \in \mathbb{N}$, we will denote by ϕ^n the closed curve from $[0, nt_0]$ to X which is given as $\phi^n(t) := \phi(s(t))$, where $s(t) \in [0, t_0]$ is such that $t = s(t) + kt_0$, $k \in \mathbb{N}_0$. One has the following proposition.

Proposition 3.4. *Let $\gamma \in \Gamma - \{1\}$ be semisimple. Let $Z(\gamma)$ be the centralizer of γ in Γ. Then $Z(\gamma)$ is infinite cyclic. Moreover there exists a semisimple prime element $\gamma_0 \in \Gamma$ such that $Z(\gamma)$ is generated by γ_0 and such that $\gamma = \gamma_0^{n_\Gamma(\gamma)}$ with $n_\Gamma(\gamma) \in \mathbb{N}$. One has $\ell(\gamma) = n_\Gamma(\gamma)\ell(\gamma_0)$ and $c_{[\gamma]} = (c_{[\gamma_0]})^{n_\Gamma(\gamma)}$.*

Proof. By Lemma 3.1 and Proposition 3.2 we have $\gamma = g \exp(\ell(\gamma) H_1) m_\gamma g^{-1}$ for an element $g \in G$. As in the proof of Lemma 4.1 in [Ga] it now follows easily that there exists a $\gamma_0 = g \exp(\ell(\gamma_0) H_1) m_\gamma g^{-1} \in \Gamma$ which generates $Z(\gamma)$ and such that $\gamma = \gamma_0^{n_\Gamma(\gamma)}$ for a $n_\Gamma(\gamma) \in \mathbb{N}$. If one uses Proposition 3.2, the proof is completed. $\square$

Apparently the element γ_0 from Proposition 3.4 is unique and thus the element $n_\Gamma(\gamma)$ is well-defined. Now for the parabolic elements we have the following easy proposition.

Proposition 3.5. *Let $[\gamma] \in \mathrm{C}(\Gamma)_{\mathrm{par}}$. Then $\ell(\gamma) = 0$.*

Proof. By (2.36) we can assume that there is a $P \in \mathfrak{P}$ such that $\gamma \in \Gamma \cap N_P$. Write $\gamma = \exp Y_\gamma$, where $Y_\gamma \in \mathfrak{n}_P$. Then for every $t \in \mathbb{R}$ the curve $c(s) := \exp(sY_\gamma) a_P(t) K$, $s \in [0, 1]$ connects $a_P(t)K$ and $\gamma a_P(t) K$ and is of length $e^{-t} \|Y_\gamma\|$. This proves the proposition. $\square$

Next we turn to a preliminary estimation of the growth of the numbers $\ell(\gamma)$, where $[\gamma] \in \mathrm{C}(\Gamma)_{\mathrm{s}}$. This is crucial in order to ensure the convergence of the infinite products which define the Ruelle and Selberg zeta functions in some half space. We keep the notations of section 2.11. Let $\pi : G \to G/K$ be the projection. Using (2.37) one can show that there exist an $u_0 > 0$, a compact connected set $\mathcal{F}(u_0) \subset \widetilde{X}$ and for every $P \in \mathfrak{P}$ a compact connected subset $\omega_P \subset N_P$ such that the set

$$\mathcal{F} := \mathcal{F}(u_0) \sqcup \bigsqcup_{P \in \mathfrak{P}} \pi(\omega_P A_P[\exp(u_0)])$$

satisfies

$$\widetilde{X} = \Gamma \cdot \mathcal{F}; \quad \mathrm{vol}(\mathcal{F}) = \mathrm{vol}(\mathring{\mathcal{F}}) < \infty; \quad \gamma\mathring{\mathcal{F}} \cap \mathring{\mathcal{F}} = \emptyset,\ \gamma \in \Gamma - 1. \tag{3.3}$$

Here $\mathring{\mathcal{F}}$ denotes the interior of $\mathcal{F}$. For $u \geq u_0$ we put

$$\mathcal{F}(u) := \mathcal{F} - \bigsqcup_{P \in \mathfrak{P}} \pi(\omega_P A_P[\exp u]). \tag{3.4}$$

Then $\mathcal{F}(u)$ is a compact connected set too. For $x \in \widetilde{X}$ and $r > 0$ we let $B_r(x)$ be the metric ball around x of radius r.

Lemma 3.6. *Let $x_0 \in \mathcal{F}(u_0)$. Then there exists a constant C such that for all $u \geq u_0$ one has*

$$\mathcal{F}(u) \subset B_{u+C}(x_0). \tag{3.5}$$

Moreover, for all $x, y \in \widetilde{X}$ and all $P \in \mathfrak{P}$, $x = n_P(x)a_P(x)K$, $y = n_P(y)a_P(y)K$ one has

$$d(x,y) \geq d(a_P(x)K, a_P(y)K) = |a_P(x) - a_P(y)|\,. \tag{3.6}$$

Proof. Since $\mathcal{F}(u_0)$ is compact, there exists a constant C_1 such that $\mathcal{F}(u_0) \subset B_{C_1}(x_0)$. On the other hand let $P \in \mathfrak{P}$ and let $y \in \pi(\omega_P A_P[u_0])$. Write $x_0 = n_P(x_0)a_P(x_0)K$, $y = n_P(y)a_P(y)K$. Then one has

$$\begin{aligned} d(x_0, y) &\leq d(n_P(x_0)a_P(x_0)K, n_P(x_0)a_P(y)K) + d(n_P(x_0)a_P(y)K, n_P(y)a_P(y)K) \\ &= d(a_P(x_0)K, a_P(y)K) + d(1K, a_P(y)^{-1}n_P(x_0)^{-1}n_P(y)a_P(y)K) \\ &\leq d(1K, a_P(x_0)K) + d(1K, a_P(y)) + d(1K, a_P(y)^{-1}n_P(x_0)^{-1}n_P(y)a_P(y)K). \end{aligned}$$

Since $H_P(y) > 0$ by assumption and since $n_P(y) \in \omega_P$, there exists a compact subset C of $\widetilde{X}$ such $a_P(y)^{-1}n_P(x_0)^{-1}n_P(y)a_P(y)K \in C$ for every $y \in \pi\left(\omega_P A_P[u_0]\right)$. Thus there exists a constant C_1' such that for every $u \geq u_0$ and every $y \in \pi(\omega_P A_P[u_0]) - \pi(\omega_P A_P[u])$ one has

$$d(x_0, y) \leq C_1' + u.$$

This proves (3.5). For every $P \in \mathfrak{P}$ one can canonically identify $\widetilde{X}$ with $N_P A_P$. Let g_{N_P} denote the metric on N_P as above and let g_{A_P} denote the restriction of the metric on $\widetilde{X}$ to A_P. Then the metric g on $N_P A_P$ under this identification becomes $g(an) = e^{H_P(a)} g_{N_P}(n) \times g_{A_P}(a)$. Thus the length of every piecewise smooth curve joining x and y is greater or equal than the length of its projection to A_P which is a curve joining $a_P(x)$ and $a_P(y)$. This gives (3.6). $\square$

Lemma 3.7. *Let $x \in \mathcal{F}$. Assume that there exists a $\gamma \in \Gamma_{\mathrm{s}}$, $\gamma \neq 1$, such that $d(x, \gamma x) \leq R$. Then one has $x \in \mathcal{F}(u_0 + R + 1)$.*

Proof. Let $x \in \mathcal{F}$ and assume that there exists a $P \in \mathfrak{P}$ such that $x \in \pi(\omega_P A_P[u_0+R+1])$. Then if one writes $\gamma x = n_P(\gamma x) a_P(\gamma x) K$, equation (2.38) and the fact that $\gamma \notin \Gamma \cap N_P$ give $a_P(\gamma x) \leq u_0$. Thus by (3.6) one has $d(x, \gamma x) > R$. $\square$

Now we can prove the following proposition about the growth of the length spectrum.

Proposition 3.8. *There exists a constant C such that for every $R > 0$ one has*

$$\#\{[\gamma] \in \mathrm{C}(\Gamma)_{\mathrm{s}} \colon \ell(\gamma) \leq R\} \leq C e^{8nR}.$$

Proof. By (3.3) for every $[\gamma_0] \in \mathrm{C}(\Gamma)_{\mathrm{s}}$ we have

$$\ell(\gamma_0) = \inf\{d(x, \gamma x) \colon x \in \mathcal{F} \colon \gamma \in [\gamma_0]\}.$$

Thus together with Lemma 3.7 we get

$$\#\{[\gamma] \in \mathrm{C}(\Gamma)_{\mathrm{s}} - [1] : \ell(\gamma) \leq R\} = \#\{\gamma \in \Gamma_{\mathrm{s}} - 1 \colon \exists x \in \mathcal{F}(u_0 + R + 1) \colon d(x, \gamma x) \leq R\}.$$

We fix an $x_0 \in \mathcal{F}(u_0)$. We let C be as in (3.5) and we let $C_1 := C + u_0 + 1$. Let $\gamma \in \Gamma_{\mathrm{s}} - 1$ and $x \in \mathcal{F}(u_0 + R + 1)$ such that $d(x, \gamma x) \leq R$. Let $y \in \gamma \cdot \mathcal{F}(u_0 + R + 1)$. Then using (3.5) and the Γ-invariance of d we obtain

$$d(y, x_0) \leq d(y, \gamma x_0) + d(\gamma x_0, \gamma x) + d(\gamma x, x) + d(x, x_0) \leq 4(R + C_1).$$

In other words, we have

$$\bigcup_{\substack{\{\gamma \in \Gamma_{\mathrm{s}} - 1 \colon \\ \exists x \in \mathcal{F}(u_0 + R + 1) \colon d(x, \gamma x) \leq R\}}} \gamma \cdot \mathcal{F}(u_0 + R + 1) \subseteq B_{4(R + C_1)}(x_0). \tag{3.7}$$

By (3.3) the union on the left hand side of (3.7) is disjoint up to a set of measure zero. Thus if we let $C_2 := 1/\operatorname{vol}(\mathcal{F}(u_0 + 1))$ it follows that

$$\#\{\gamma \in \Gamma_{\mathrm{s}} - 1 \colon \exists x \in \mathcal{F}(u_0 + R + 1) \colon d(x, \gamma x) \leq R\} \leq C_2 \operatorname{vol} B_{4(R + C_1)}(x_0).$$

It follows from (2.19) that there exists a constant C_3 such that

$$\operatorname{vol}\left(B_{4(R + C_1)}(x_0)\right) \leq C_3 e^{8nR}.$$

This proves the proposition. $\square$

Remark 3.9. The estimate in Proposition 3.8 can be improved considerably. Namely, Gangolli and Warner proved that

$$\#\{[\gamma] \in \mathrm{C}(\Gamma)_{\mathrm{s}} \colon \ell(\gamma) \leq R\} \sim \frac{e^{2nR}}{2nR}, \tag{3.8}$$

as $R \to \infty$, see [GaWa, Proposition 5.4]. However, to prove this asymptotic, Gangolli and Warner already use the Selberg zeta function associated to the trivial representation of M. In [GaWa] the convergence of the infinite product defining the Selberg zeta function in some half-space is established by inserting suitable functions into the Selberg trace formula. Since we want to keep our work independent from [GaWa] and since we want to obtain a more direct definition and convergence of Ruelle and Selberg zeta functions, we proved the weaker estimate in Proposition 3.8 in an elementary way.

Now we can define Ruelle and Selberg zeta functions. With the results just obtained, the geometric meaning and the basic properties of these functions can be established in the same way as it was done in [BO, p. 96-99] and we shall follow Bunke and Olbrich here. Let SX be the unit sphere bundle of X. Then since K acts transitively on the unit-sphere of $\mathfrak{p}$ by the adjoint representation, there is a natural isomorphism $\Gamma\backslash G/M \cong SX$. Namely, if $\pi : G \to X$ is the projection one sends $\Gamma gM \in \Gamma\backslash G/M$ to the tangential vector $\frac{d}{dt}|_{t=0}\pi(g\exp tH_1)$ at $\pi(g)$. This map is well-defined since M and A commute. Let ϕ be the geodesic flow on SX. Then ϕ is given on $\Gamma\backslash G/M$ as

$$\phi(t, \Gamma gM) := \phi_t(\Gamma g) := \Gamma g \exp -tH_1 M, \quad t \in \mathbb{R},\ g \in G.$$

Now let σ be a unitary finite dimensional representation of M on V_σ and let $V(\sigma) := \Gamma\backslash(G \times_\sigma V_\sigma)$. Then $V(\sigma)$ is a vector bundle over $S(X) \cong \Gamma\backslash G/M$. Moreover the geodesic flow ϕ lifts to a flow ϕ_σ on $V(\sigma)$ by

$$\phi_\sigma(t, [\Gamma g, v]) := [\Gamma g \exp(-tH_1), v]\,.$$

Again this map is well-defined since M and A commute. Now consider a free homotopy class of closed paths in X and write the homotopy class as $f([\gamma])$, $[\gamma] \in \mathrm{C}(\Gamma)$. Let $\ell(\gamma)$ be as in (3.2). Then by Proposition 3.2 and Proposition 3.5, $\ell(\gamma)$ is not zero if and only if $[\gamma] \in \mathrm{C}(\Gamma)_s - [1]$. Thus let $[\gamma] \in \mathrm{C}(\Gamma)_s - [1]$. Then by Proposition 3.2 there exists a $g \in G$ such that we can write $\gamma = g \exp(\ell(\gamma)H_1) m_\gamma g^{-1}$. If $\pi : G \to X$ is the projection, the restriction of the geodesic

$$c_{[\gamma]} : \mathbb{R} \to X,\ c_{[\gamma]} = \pi(g \exp(tH_1)) \tag{3.9}$$

to $[0, \ell(\gamma)]$ belongs to $f([\gamma])$ and is of length $\ell(\gamma)$. Via the flow ϕ_σ, this closed geodesic defines an endomorphism $\mu_\sigma(c_{[\gamma]})$ on the fibre of $V(\sigma)$ over ΓgM. One easily sees that

$$\mu_\sigma(c_{[\gamma]})\,([\Gamma g, v]) = [\Gamma g, \sigma(m_\gamma)v]\,.$$

In particular, if we regard $\mu_\sigma(c_{[\gamma]})$ as an endomorphism on V_σ this endomorphism is independent of g, the starting point of the closed geodesic. Now we can define the Ruelle zeta function.

Definition 3.10. Let $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 8n$. Let σ be a finite dimensional representation of M. Then the Ruelle zeta function associated to σ is defined as

$$R(s,\sigma) := \prod_{\substack{[\gamma]\in \mathrm{C}(\Gamma)_{\mathrm{s}}-[1] \\ [\gamma] \text{ prime}}} \det\left(\mathrm{Id} - \mu_\sigma(c_{[\gamma]})e^{-s\ell(\gamma)}\right) = \prod_{\substack{[\gamma]\in \mathrm{C}(\Gamma)_{\mathrm{s}}-[1] \\ [\gamma] \text{ prime}}} \det\left(\mathrm{Id} - \sigma(m_\gamma)e^{-s\ell(\gamma)}\right).$$

First we have to assure convergence of the infinite product defining $R(s,\sigma)$.

Proposition 3.11. *The infinite product in Definition 3.10 converges absolutely and locally uniformly for* $\mathrm{Re}(s) > 8n$.

Proof. One has

$$\begin{aligned} \log R(s,\sigma) &= \sum_{\substack{[\gamma]\in \mathrm{C}(\Gamma)_{\mathrm{s}}-[1] \\ [\gamma] \text{ prime}}} \mathrm{Tr}\log\left(\mathrm{Id} - \sigma(m_\gamma)e^{-s\ell(\gamma)}\right) \\ &= -\sum_{\substack{[\gamma]\in \mathrm{C}(\Gamma)_{\mathrm{s}}-[1] \\ [\gamma] \text{ prime}}} \sum_{k=1}^{\infty} \frac{\mathrm{Tr}\left(\sigma(m_\gamma)^k\right)e^{-ks\ell(\gamma)}}{k} \\ &= -\sum_{[\gamma]\in \mathrm{C}(\Gamma)_{\mathrm{s}}-[1]} \frac{\mathrm{Tr}\left(\sigma(m_\gamma)\right)e^{-s\ell(\gamma)}}{n_\Gamma(\gamma)}. \end{aligned} \tag{3.10}$$

For every $[\gamma] \in \mathrm{C}(\Gamma)_{\mathrm{s}}$ one can estimate $|\mathrm{Tr}\left(\sigma(m_\gamma)\right)| \le \dim(\sigma)$. Thus by Proposition 3.8 the last series converges absolutely and locally uniformly for $\mathrm{Re}(s) > 8n$. ☐

Next we introduce Selberg zeta functions. Let ϕ be the geodesic flow as above. Then if TSX is the tangent-bundle of SX one defines a map $d\phi$ by

$$d\phi : \mathbb{R}\times TSX \to TSX; \quad d\phi(t, Y_x) := d\phi_t(x)(Y_x), \quad t\in\mathbb{R},\ x\in SX,\ Y\in T_xSX.$$

The isomorphism $SX \cong \Gamma\backslash G/M$ induces an isomorphism

$$TSX \cong \Gamma\backslash G \times_{\mathrm{Ad}} (\bar{\mathfrak{n}}\oplus\mathfrak{a}\oplus\mathfrak{n}).$$

With respect to this isomorphism the map $d\phi$ is given by

$$d\phi(t,[\Gamma g, Y]) = [\Gamma g\exp -tH_1, \mathrm{Ad}(\exp tH_1)Y], \quad Y\in\bar{\mathfrak{n}}\oplus\mathfrak{a}\oplus\mathfrak{n},\ g\in G.$$

Now the spaces $\bar{\mathfrak{n}}$, $\mathfrak{a}$ and $\mathfrak{n}$ are invariant under $\mathrm{Ad}(A)$ and $\mathrm{Ad}(\exp tH_1)$ acts on these spaces by $e^{-t}\cdot\mathrm{Id}$ respectively Id respectively $e^t\cdot\mathrm{Id}$. Thus ϕ has the Anosov property, i.e. there exists a $d\phi$-invariant splitting

$$TSX = T^{\mathrm{s}}SX \oplus T^0SX \oplus T^{\mathrm{u}}SX, \tag{3.11}$$

such that $d\phi|_{T^0SX} = \mathrm{Id}$ and such that for every $Y_{\mathrm{s}} \in T^{\mathrm{s}}SX$, $Y_{\mathrm{u}} \in T^{\mathrm{u}}SX$, one has $\|d\phi_t(Y_{\mathrm{s}})\| \sim ce^{-t}$, $\|d\phi_t(Y_{\mathrm{u}})\| \sim c'e^t$ as $t\to 0$ with $c = \|Y_{\mathrm{s}}\|$, $c' = \|Y_{\mathrm{u}}\|$. Now let $[\gamma]\in \mathrm{C}_{\mathrm{s}}(\Gamma)$

and let $c_{[\gamma]}$ be the associated closed geodesic of length $\ell(\gamma)$. Then $d\phi$ induces an endomorphism $P(c_{[\gamma]})$ on $T_{c_{[\gamma]}(0)}SX$ which according to the splitting in (3.11) decomposes as

$$P(c_{[\gamma]}) = P^{\mathrm{s}}(c_{[\gamma]}) \oplus \mathrm{Id} \oplus P^{\mathrm{u}}(c_{[\gamma]}).$$

If we identify $T^{\mathrm{s}}_{c_{[\gamma]}(0)}SX \cong \bar{\mathfrak{n}}$ respectively $T^{\mathrm{u}}_{c_{[\gamma]}(0)}SX \cong \mathfrak{n}$, it follows that $P^{\mathrm{s}}(c_{[\gamma]})$ and $P^{\mathrm{u}}(c_{[\gamma]})$ act as $\mathrm{Ad}\left(m_\gamma \exp\left(\ell(\gamma)H_1\right)\right)$ on these spaces. In particular they can be defined independent of the starting point of $c_{[\gamma]}$. Now we can define the Selberg zeta function.

Definition 3.12. Let $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 8n$. Let σ be a finite dimensional representation of M. Then the Selberg zeta function associated to σ is defined as

$$\begin{aligned} Z(s,\sigma) &:= \prod_{\substack{[\gamma]\in \mathrm{C}(\Gamma)_{\mathrm{s}}-[1] \\ [\gamma] \text{ prime}}} \prod_{k=0}^{\infty} \det\left(\mathrm{Id} - \mu_\sigma(c_{[\gamma]}) \otimes S^k P^{\mathrm{s}}(c_{[\gamma]}) e^{-(s+n)\ell(\gamma)}\right) \\ &= \prod_{\substack{[\gamma]\in \mathrm{C}(\Gamma)_{\mathrm{s}}-[1] \\ [\gamma] \text{ prime}}} \prod_{k=0}^{\infty} \det\left(\mathrm{Id} - \sigma(m_\gamma) \otimes S^k \,\mathrm{Ad}(m_\gamma \exp(\ell(\gamma)H_1))|_{\bar{\mathfrak{n}}} e^{-(s+n)\ell(\gamma)}\right) \end{aligned}$$

Here S^k denotes the k-th symmetric power of an endomorphism.

Again we have to ensure the convergence of the infinite product involved.

Proposition 3.13. *The infinite product in Definition 3.12 converges absolutely and locally uniformly for* $\mathrm{Re}(s) > 8n$.

Proof. Let us set $a_\gamma := \exp\left(\ell(\gamma)H_1\right)$. Then we have

$$\begin{aligned} \log Z(s,\sigma) &= \sum_{\substack{[\gamma]\in \mathrm{C}(\Gamma)_{\mathrm{s}}-[1] \\ [\gamma] \text{ prime}}} \sum_{k=0}^{\infty} \mathrm{Tr} \log\left(\mathrm{Id} - \sigma(m_\gamma) \otimes S^k(\mathrm{Ad}(m_\gamma a_\gamma)_{\bar{\mathfrak{n}}}) e^{-(s+n)\ell(\gamma)}\right) \\ &= - \sum_{\substack{[\gamma]\in \mathrm{C}(\Gamma)_{\mathrm{s}}-[1] \\ [\gamma] \text{ prime}}} \sum_{k=0}^{\infty} \sum_{l=1}^{\infty} \frac{\mathrm{Tr}\left(\left(\sigma(m_\gamma) \otimes S^k(\mathrm{Ad}(m_\gamma a_\gamma)|_{\bar{\mathfrak{n}}}) e^{-(s+n)\ell(\gamma)}\right)^l\right)}{l} \\ &= - \sum_{[\gamma]\in \mathrm{C}(\Gamma)_{\mathrm{s}}-[1]} \sum_{k=0}^{\infty} \frac{\mathrm{Tr}\left(\sigma(m_\gamma) \otimes S^k(\mathrm{Ad}(m_\gamma a_\gamma)|_{\bar{\mathfrak{n}}}) e^{-(s+n)\ell(\gamma)}\right)}{n_\Gamma(\gamma)} \\ &= - \sum_{[\gamma]\in \mathrm{C}(\Gamma)_{\mathrm{s}}-[1]} \frac{\mathrm{Tr}\left(\sigma(m_\gamma)\right) e^{-(s+n)\ell(\gamma)}}{n_\Gamma(\gamma) \det(\mathrm{Id} - \mathrm{Ad}(m_\gamma a_\gamma)|_{\bar{\mathfrak{n}}})}. \end{aligned} \tag{3.12}$$

It is easy to see that

$$\det(\mathrm{Id} - \mathrm{Ad}(m_\gamma a_\gamma)|_{\bar{\mathfrak{n}}}) \geq \left(1 - e^{-l(\gamma)}\right)^n.$$

Moreover, if we let

$$c := \min\{\ell(\gamma), [\gamma] \in \mathrm{C}(\Gamma)_{\mathrm{s}} - [1]\} \tag{3.13}$$

it follows from Proposition 3.8 that $c > 0$. Hence there exists a constant C such that for all $[\gamma] \in \mathrm{C}(\Gamma)_{\mathrm{s}} - [1]$ one has

$$\frac{1}{\det(\mathrm{Id} - \mathrm{Ad}(m_\gamma a_\gamma)|_{\bar{\mathfrak{n}}})} < C. \tag{3.14}$$

Again for every $[\gamma] \in \mathrm{C}(\Gamma)_{\mathrm{s}}$ one has $|\mathrm{Tr}\,(\sigma(m_\gamma))| \leq \dim(\sigma)$. The Proposition follows from Proposition 3.8. □

The proof of the preceding proposition also gives information about the asymptotic behaviour of $|\log Z(s,\sigma)|$ as $m \to \infty$.

Proposition 3.14. *There exists a constant C such that for every $\sigma \in \hat{M}$ and every $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 16n$ one has*

$$|\log Z(s,\sigma)| \leq C \dim(\sigma) e^{-\mathrm{Re}(s)/2}.$$

Proof. By our assumption on s, by Proposition 3.13, by (3.12) and by (3.14) we can estimate

$$|\log Z(s,\sigma)| \leq C \dim(\sigma) e^{-\mathrm{Re}(s)/2} \sum_{[\gamma] \in \mathrm{C}(\Gamma)_{\mathrm{s}} - [1]} e^{-(\mathrm{Re}(s)/2+n)\ell(\gamma)},$$

where the last series converges by Proposition 3.8. □

Finally, as in [BO, Proposition 3.4] the Ruelle zeta function can be expressed as a weighted product of Selberg zeta functions with shifted arguments. For $q = 0, \ldots, 2n$ let σ_q be the representation $\Lambda^q \mathrm{Ad}$ of M on $\Lambda^q \bar{\mathfrak{n}}$.

Proposition 3.15. *Let σ be a finite dimensional representation of M. Then one has*

$$R(s,\sigma) = \prod_{q=0}^{2n} Z(s - n + q, \sigma \otimes \sigma_q)^{(-1)^q}.$$

Proof. For every $[\gamma] \in \mathrm{C}(\Gamma)_{\mathrm{s}} - [1]$ one has

$$\mathrm{Tr}\,\Lambda^q \mathrm{Ad}(m_\gamma a_\gamma)|_{\bar{\mathfrak{n}}} = e^{-q\ell(\gamma)}\,\mathrm{Tr}\,\sigma_q(m_\gamma).$$

Thus one has

$$\det(\mathrm{Id} - \mathrm{Ad}(m_\gamma a_\gamma)|_{\bar{\mathfrak{n}}}) = \sum_{q=0}^{2n} (-1)^q\,\mathrm{Tr}\,\Lambda^q \mathrm{Ad}(m_\gamma a_\gamma)|_{\bar{\mathfrak{n}}} = \sum_{q=0}^{2n} (-1)^q e^{-q\ell(\gamma)}\,\mathrm{Tr}\,\sigma_q(m_\gamma).$$

Using (3.10) and (3.12) the proposition follows. □

4 The right regular representation of G on $L^2(\Gamma\backslash G)$

4.1 The decomposition of the right regular representation

Let π_Γ be the right-regular representation of G on $L^2(\Gamma\backslash G)$. In this section we describe some basic properties of π_Γ. The main references are [La], [HC1] [War1].
There exists an orthogonal decomposition

$$L^2(\Gamma\backslash G) = L^2_d(\Gamma\backslash G) \oplus L^2_c(\Gamma\backslash G) \tag{4.1}$$

of $L^2(\Gamma\backslash G)$ into closed π_Γ-invariant subspaces. The restriction of π_Γ to $L^2_d(\Gamma\backslash G)$ decomposes into the orthogonal direct sum of irreducible unitary representations of G and the multiplicity of each irreducible unitary representation of G in this decomposition is finite. On the other hand, by the theory of Eisenstein series, the restriction of π_Γ to $L^2_c(\Gamma\backslash G)$ is isomorphic to the direct integral over all unitary principal-series representations of G. The definition and some basic properties of the Eisenstein series will now be described briefly. Let $P \in \mathfrak{P}$. Let $\nu \in \hat{K}$ and $\sigma_P \in \hat{M}_P$ such that $[\nu : \sigma_P] \neq 0$. Then we let $\mathcal{E}_P(\sigma_P, \nu)$ be the set of all continuous functions Φ on G which are left-invariant under $N_P A_P$ such that for all $x \in G$ the function $m \mapsto \Phi_P(mx)$ belongs to $L^2(M, \sigma_P)$, the σ_P-isotypical component of the right regular representation of M_P and such that for all $x \in G$ the function $k \mapsto \Phi_P(xk)$ belongs to the ν-isotypical component of the right regular representation of K. By (4.3) below, $\mathcal{E}_P(\sigma_P, \nu)$ is finite dimensional. We define an inner product on $\mathcal{E}_P(\sigma_P, \nu)$ as follows. Any element of $\mathcal{E}_P(\sigma_P, \nu)$ can be identified canonically with a function on K. For $\Phi, \Psi \in \mathcal{E}_P(\sigma_P, \nu)$ we now set

$$\langle \Phi, \Psi \rangle := \operatorname{vol}(\Gamma \cap N_P \backslash N_P) \int_K \Phi(k)\bar{\Psi}(k)dk. \tag{4.2}$$

Now we define the Hilbert space $\mathcal{E}_P(\sigma_P, \nu)$ as

$$\mathcal{E}_P(\sigma_P) := \bigoplus_{\substack{\nu \in \hat{K} \\ [\nu:\sigma_P]\neq 0}} \mathcal{E}_P(\sigma_P, \nu).$$

For $\lambda \in \mathbb{C}$ let $\pi_{\Gamma,\sigma_P,\lambda}$ be the representation of G on $\mathcal{E}_P(\sigma_P)$ defined by

$$\pi_{\Gamma,\sigma_P,\lambda}(g)\Phi(n_P a_P k) := e^{(\lambda+n)H_P(kg)}\Phi(kg), \quad n_P \in N_P,\ a_P \in A_P, k \in K, \quad \Phi \in \mathcal{E}_P(\sigma_P, \nu).$$

For $\sigma_P \in \hat{M}_P$, $\nu \in \hat{K}$ put

$$\boldsymbol{\mathcal{E}}(\boldsymbol{\sigma}, \nu) := \bigoplus_{\sigma_{P'} \in \boldsymbol{\sigma}} \mathcal{E}_{P'}(\sigma_{P'}, \nu); \quad \boldsymbol{\mathcal{E}}(\boldsymbol{\sigma}) := \bigoplus_{\sigma_{P'} \in \boldsymbol{\sigma}} \mathcal{E}_{P'}(\sigma_{P'}).$$

Moreover we define a representation $\boldsymbol{\pi}_{\Gamma,\boldsymbol{\sigma},\lambda}$ of G on $\boldsymbol{\mathcal{E}}(\boldsymbol{\sigma})$ by

$$\boldsymbol{\pi}_{\Gamma,\boldsymbol{\sigma},\lambda} := \bigoplus_{\sigma_P \in \boldsymbol{\sigma}} \pi_{\Gamma,\sigma_P,\lambda}.$$

The pair $(\boldsymbol{\mathcal{E}}(\boldsymbol{\sigma}), \boldsymbol{\pi}_{\Gamma,\boldsymbol{\sigma},\lambda})$ can be related to the the principal series as follows. For $\sigma \in \hat{M}$ let

$$\mathrm{Hom}_M(V_\sigma, L^2(M))$$

denote the set of intertwining operators between σ and the right regular representation. The dimension of this space equals the degree of σ. For every $\nu \in \hat{K}$ let $(\mathcal{H}^\sigma)^\nu$ be the ν-isotypical component of the representations $\pi_{\sigma,\lambda}$. Then for every $\sigma_P \in \boldsymbol{\sigma}$ and $\nu \in \hat{K}$, $[\nu : \sigma] \neq 0$ we define

$$I_{P,\sigma}(\nu) : \mathrm{Hom}_M(V_\sigma, L^2(M)) \otimes (\mathcal{H}^\sigma)^\nu \to \mathcal{E}_P(\sigma_P, \nu)$$

by

$$I_{P,\sigma}(\nu)(u \otimes \Phi)(g) := u \circ \Phi(mk_p^{-1}\kappa_P(g))(m^{-1}) \text{ for almost all } m \in M. \tag{4.3}$$

Then it is easy to see that $I_{P,\sigma}(\nu)$ is an isometry. Thus we can define an isometry

$$I_{P,\sigma} : \mathrm{Hom}_M(V_\sigma, L^2(M)) \otimes \mathcal{H}^\sigma \to \mathcal{E}_P(\sigma_P)$$

as the direct sum of the $I_{P,\sigma}(\nu)$. Now let

$$\boldsymbol{\mathcal{L}}(\sigma) := \bigoplus_{P \in \mathfrak{P}} \mathrm{Hom}_M(V_\sigma, L^2(M))$$

and let

$$\mathbf{I}_\sigma : \boldsymbol{\mathcal{L}}(\sigma) \otimes \mathcal{H}^\sigma \longrightarrow \boldsymbol{\mathcal{E}}(\boldsymbol{\sigma}) \tag{4.4}$$

be the isomorphism which is the direct sum of the $I_{P,\sigma}$. Then $\mathbf{I}_\sigma$ is an isomorphism and an intertwining operator between the representations $1 \otimes \pi_{\sigma,\lambda}$ and $\boldsymbol{\pi}_{\Gamma,\sigma,i\lambda}$, where 1 stands for the trivial representation of G on $\boldsymbol{\mathcal{L}}(\sigma)$.
For $\Phi_P \in \mathcal{E}_P(\sigma_P, \nu)$ and $\lambda \in \mathbb{C}$ let

$$\Phi_{P,\lambda}(g) := e^{(\lambda+n)(H_P(x))}\Phi_P(g). \tag{4.5}$$

Then for $x \in \Gamma\backslash G$, $x = \Gamma g$ one defines

$$E(\Phi_P : \lambda : x) := \sum_{\gamma \in \Gamma \cap N_P \backslash \Gamma} \Phi_{P,\lambda}(\gamma g). \tag{4.6}$$

On $\Gamma\backslash G \times \{\lambda \in \mathbb{C} \colon \mathrm{Re}(\lambda) > n\}$ the series (4.6) is absolutely and locally uniformly convergent. As a function of λ, it has a meromorphic continuation to $\mathbb{C}$ with only finitely many poles in the strip $0 < \mathrm{Re}(\lambda) \leq n$ which are located on $(0, n]$. Moreover, it has no poles on the line $\mathrm{Re}(\lambda) = 0$.
For $\Phi = (\Phi_P)_{P \in \mathfrak{P}} \in \boldsymbol{\mathcal{E}}(\boldsymbol{\sigma}, \nu)$ and $x \in \Gamma\backslash G$ let

$$\mathbf{E}(\Phi : \lambda : x) = \sum_{P \in \mathfrak{P}} E(\Phi_P : \lambda : x).$$

Let $P, P' \in \mathfrak{P}$ and let $\sigma_P \in \hat{M}_P$, $\sigma_{P'} \in \hat{M}_{P'}$ be associated. For $\Phi_P \in \mathcal{E}_P(\sigma_P, \nu)$ and $g \in G$ let

$$E_{P'}(\Phi_P : g : \lambda) := \frac{1}{\operatorname{vol}(\Gamma \cap N_{P'} \backslash N_{P'})} \int_{\Gamma \cap N_{P'} \backslash N_{P'}} E(\Phi_P : ng : \lambda) dn$$

be the constant term of $E(\Phi_P : - : \lambda)$ along P'. Then there exists a meromorphic function

$$C_{P|P'}(\nu : \sigma_P : \lambda) : \mathcal{E}(\sigma_P, \nu) \longrightarrow \mathcal{E}(w_{P'}\sigma_{P'}, \nu),$$

where $w_{P'}$ is the non-trivial element of $W(A_{P'})$, such that for $P \neq P'$ one has

$$E_{P'}(\Phi_P : g : \lambda) = (C_{P|P'}(\nu : \sigma_P : \lambda)\Phi_P)_{-\lambda}(g) \tag{4.7}$$

and such that

$$E_P(\Phi_P : g : \lambda) = \Phi_{P,\lambda}(g) + (C_{P|P}(\nu : \sigma_P : \lambda)\Phi_P)_{-\lambda}(g). \tag{4.8}$$

For $\Phi = (\Phi_P)_{P \in \mathfrak{P}} \in \boldsymbol{\mathcal{E}}(\boldsymbol{\sigma}, \nu)$ and $P' \in \mathfrak{P}$ one defines

$$E_{P'}(\Phi : g : \lambda) := \sum_{P \in \mathfrak{P}} E_{P'}(\Phi_P : g : \lambda). \tag{4.9}$$

Next we define

$$C_{P|P'}(\sigma_P : \lambda) := \bigoplus_{\nu} C_{P|P'}(\nu : \sigma_P : \lambda).$$

Moreover we let $\mathbf{C}(\nu : \sigma : \lambda)$ resp. $\mathbf{C}(\sigma : \lambda)$ be the maps built from the $C_{P'|P''}(\sigma_{P'}, \nu, \lambda)$ resp. the $C_{P'|P''}(\sigma_{P'}, \lambda)$, where $\sigma_{P'} \in \boldsymbol{\sigma}$. Then one has

$$\mathbf{C}(w_0\sigma : \lambda)\mathbf{C}(\sigma : -\lambda) = \mathrm{Id}; \quad \mathbf{C}(\sigma : \lambda)^* = \mathbf{C}(w_0\sigma : \bar{\lambda}), \tag{4.10}$$

For $\Phi \in \boldsymbol{\mathcal{E}}(\boldsymbol{\sigma}, \nu)$ the Eisenstein series satisfies the functional equation

$$\mathbf{E}(\Phi : x : \lambda) = \mathbf{E}(\mathbf{C}(\nu : \sigma : \lambda)\Phi : x : -\lambda). \tag{4.11}$$

The representations $\boldsymbol{\pi}_{\Gamma,\boldsymbol{\sigma},\lambda}$ and $\boldsymbol{\pi}_{\Gamma,\boldsymbol{w\sigma},-\lambda}$ are equivalent. Moreover, it follows easily from (4.11) that if $f \in C_c^\infty(G)$ is of left and right K-type ν one has

$$\mathbf{C}(\nu : \sigma : \lambda)\boldsymbol{\pi}_{\Gamma,\boldsymbol{\sigma},\lambda}(f) = \boldsymbol{\pi}_{\Gamma,\boldsymbol{w\sigma},-\lambda}(f)\mathbf{C}(\nu : \sigma : \lambda). \tag{4.12}$$

Hence $\mathbf{C}(\sigma : \lambda)$ is an intertwining operator between $\boldsymbol{\pi}_{\Gamma,\boldsymbol{\sigma},\lambda}$ and $\boldsymbol{\pi}_{\Gamma,\boldsymbol{w\sigma},-\lambda}$. Using the theory of Eisenstein series, one introduces $L^2_c(\Gamma\backslash G)$ as a closed subspace of $L^2(\Gamma\backslash G)$ which is π_Γ isomorphic to the direct sum

$$\bigoplus_{\sigma \in \hat{M}} \int_0^\infty \oplus \boldsymbol{\pi}_{\Gamma,\boldsymbol{\sigma},i\lambda} d\lambda.$$

Here $\int_{\mathbb{R}} \oplus \boldsymbol{\pi}_{\Gamma,\boldsymbol{\sigma},i\lambda} d\lambda$ stands for the direct integral over the representations $\boldsymbol{\pi}_{\Gamma,\boldsymbol{\sigma},i\lambda}$.
At the end of this section we remark that the space $L^2_d(\Gamma\backslash G)$ admits a further decomposition

$$L^2_d(\Gamma\backslash G) = L^2_{\text{cusp}}(\Gamma\backslash G) \oplus L^2_{\text{res}}(\Gamma\backslash G).$$

The space $L^2_{\text{cusp}}(\Gamma\backslash G)$ is the space spanned by the cusp forms, i.e. the square integrable functions f, which for all $P \in \mathfrak{P}$ satisfy

$$f^0_P(x) := \int_{\Gamma\cap N_P\backslash N_P} f(nx)dn = 0 \quad \text{for almost all } x \in G.$$

One does not know much about $L^2_{\text{cusp}}(\Gamma\backslash G)$ and its size in general. On the other hand, let $\Phi \in \boldsymbol{\mathcal{E}}(\boldsymbol{\sigma}, \nu)$. Let $s_0 \in (0, n]$ be a pole of $E(\Phi : - : -)$. Then the function $x \mapsto \text{Res}|_{s=s_0} E(\Phi : s : x)$ is square integrable on $\Gamma\backslash G$ and $L^2_{\text{res}}(\Gamma\backslash G)$ is spanned by all these residues of Eisenstein series.

4.2 Some properties of the C-matrix

Let $\sigma \in \hat{M}$. According to [Ho2], the map $\mathbf{I}_\sigma$ from (4.4) can be used to relate the intertwining operators $\mathbf{C}(\sigma : s)$ to the Knapp-Stein intertwining operators of the principal series representations associated to P_0. One obtains a result similar to the adelic case, where the C-matrix factorizes into a product of the Knapp-Stein intertwining operator and the intertwining operators at the finite places.
We first briefly recall the definition of the Knapp-Stein operators. Let $\bar{N} := \Theta(N)$ and let $\bar{P}_0 := \bar{N}AK$ be the parabolic subgroup opposite to P_0. Let $\sigma \in \hat{M}$. For $\Phi \in \mathcal{H}^\sigma$ we define a function Φ_λ on G by

$$\Phi_\lambda(nak) := e^{(i\lambda+n)H(a)}\Phi(k).$$

This definition differs from (4.5) by the factor i. For $\text{Im}(\lambda) < 0$ and $\Phi \in (\mathcal{H}^\sigma)^K$ the integral

$$J_{\bar{P}_0|P_0}(\sigma,\lambda)(\Phi)(k) := \int_{\bar{N}} \Phi_\lambda(\bar{n}k)d\bar{n} = \int_N \Phi_\lambda(w_0 n w_0^{-1} k)dn \tag{4.13}$$

is convergent and $J_{\bar{P}_0|P_0}(\sigma,\lambda)$ extends to an intertwining operator $J_{\bar{P}_0|P_0}(\sigma,\lambda) : \mathcal{H}^\sigma \longrightarrow \mathcal{H}^\sigma$ between $\pi_{\sigma,\lambda}$ and $\pi_{\sigma,\lambda,\bar{P}_0}$, where $\pi_{\sigma,\lambda,\bar{P}_0}$ denotes the principal series representation associated to σ, λ and $\bar{P}_0$. Moreover, by [KS], as an operator-valued function $J_{\bar{P}_0|P_0}(\sigma,\lambda)$ has a meromorphic continuation to $\mathbb{C}$. If $\sigma \neq w_0\sigma$, $J_{\bar{P}_0|P_0}(\sigma,\lambda)$ has no poles on $i\mathbb{R}$ and is invertible there. If $\sigma = w_0\sigma$, $J_{\bar{P}_0|P_0}(\sigma,\lambda)$ is regular and invertible on $\mathbb{R} - \{0\}$. Let $m_0 \in M'$ be a representative for w_0 as in section 2.3. Then one defines an operator $A(w_0) : \mathcal{H}^\sigma \to \mathcal{H}^{w_0\sigma}$ by $A(w_0)\Phi(k) := \Phi(m_0 k)$. Then $A(w_0)$ intertwines $\pi_{\sigma,\lambda,\bar{P}_0}$ and $\pi_{w_0\sigma,-\lambda}$. Thus the operator

$$J_{P_0}(\sigma,\lambda) := A(w_0)J_{\bar{P}_0|P_0}(\sigma,\lambda) \tag{4.14}$$

is, wherever it is defined, an intertwining operator between $\pi_{\sigma,\lambda}$ and $\pi_{w_0\sigma,-\lambda}$. If $\nu \in \hat{K}$ with $[\nu : \sigma] \neq 0$ we denote by $J_{P_0}(\nu,\sigma,\lambda)$ the restriction of $J_{P_0}(\sigma,\lambda)$ to a map from $(\mathcal{H}^\sigma)^\nu$ to $(\mathcal{H}^{w_0\sigma})^\nu$, the ν-isotypical components of $\mathcal{H}^\sigma$ resp. $\mathcal{H}^{w_0\sigma}$. Now the C-matrix is related to the Knapp-Stein operator as follows.

Proposition 4.1. *There exist a meromorphic* $\operatorname{Hom}(\boldsymbol{\mathcal{L}}(\sigma), \boldsymbol{\mathcal{L}}(w_0\sigma))$*-valued function* $\mathbf{T}(\sigma,\lambda)$ *which is regular on* $i\mathbb{R}-\{0\}$ *such that in the sense of meromorphic functions one has*

$$\mathbf{C}(\nu:\sigma:s)(\mathbf{I}_\sigma(u\otimes\Phi)) = \mathbf{I}_{w_0\sigma}\left(\mathbf{T}(\sigma,s)(u)\otimes J_{P_0}(\nu,\sigma,-is)(\Phi)\right)$$

for all $u\in\boldsymbol{\mathcal{L}}(\sigma)$ *and all* $\Phi\in(\mathcal{H}^\sigma)^\nu$.

Proof. The proposition is proved in [Ho2, Theorem 7.1] for the more general setting of a rank-one lattice and a Hecke operator. □

Proposition 4.1 gives the following corollary.

Corollary 4.2. *Let* f *be a* K*-finite Schwarz-function. Then in the sense of meromorphic functions one has*

$$\begin{aligned}
&\operatorname{Tr}\left(\boldsymbol{\pi}_{\Gamma,\boldsymbol{\sigma},s}(f)\mathbf{C}(\sigma:s)^{-1}\frac{d}{ds}\mathbf{C}(\sigma:s)\right)\\
&=\operatorname{Tr}\left(\mathbf{T}(\sigma,s)^{-1}\frac{d}{ds}\mathbf{T}(\sigma,s)\right)\Theta_{\sigma,-is}(f)\\
&-i\dim(\sigma)p\operatorname{Tr}\left(\pi_{\sigma,-is}(f)J_{\bar{P}_0|P_0}(\sigma,-is)^{-1}\frac{d}{dz}J_{\bar{P}_0|P_0}(\sigma,-is)\right).
\end{aligned}$$

Proof. First we remark that since f is K-finite all traces are taken in a finite dimensional vector space. It follows from (4.14), (4.16) and (4.17) that $J_{P_0}(\nu,\sigma,\lambda)$ is invertible as a meromorphic function of λ. Thus the same holds for the function $\mathbf{T}(\sigma,\lambda)$. By Proposition 4.1 and since $\boldsymbol{I}_\sigma$ is an intertwining operator between $1\otimes\pi_{\sigma,\lambda}$ and $\boldsymbol{\pi}_{\Gamma,\boldsymbol{\sigma},i\lambda}$, we have

$$\begin{aligned}
&\operatorname{Tr}\left(\boldsymbol{\pi}_{\Gamma,\boldsymbol{\sigma},s}(f)\mathbf{C}(\sigma:s)^{-1}\frac{d}{ds}\mathbf{C}(\sigma:s)\right)\\
&=\operatorname{Tr}\left(1\otimes\pi_{\sigma,-is}(f)\left(\mathbf{T}(\sigma,s)\otimes J_{P_0}(\sigma,-is)\right)^{-1}\frac{d}{ds}\left(\mathbf{T}(\sigma,s)\otimes J_{P_0}(\sigma,-is)\right)\right).
\end{aligned}$$

One has $\dim\left(\boldsymbol{\mathcal{L}}(\sigma)\right)=p\dim(\sigma)$ and by (4.14) one has

$$J_{P_0}(\sigma,-is)^{-1}\frac{d}{ds}J_{P_0}(\sigma,-is)=J_{\bar{P}_0|P_0}(\sigma,-is)^{-1}\frac{d}{ds}J_{\bar{P}_0|P_0}(\sigma,-is). \tag{4.15}$$

This proves the corollary. □

Let $\nu\in\hat{K}$ be a K-type of $\pi_{\sigma,\lambda}$. Since $[\nu:\sigma]=1$, it follows from Frobenius reciprocity and Schur's Lemma that

$$J_{\bar{P}_0|P_0}(\sigma,\lambda)|_{(\mathcal{H}^\sigma)^\nu}=c_\nu(\sigma:\lambda)\cdot\operatorname{Id}, \tag{4.16}$$

where $c_\nu(\sigma:\lambda)\in\mathbb{C}$. The function $\lambda\mapsto c_\nu(\sigma:\lambda)$ can be computed explicitly. If $k_2(\sigma)e_2+\cdots+k_{n+1}(\sigma)e_{n+1}$ is the highest weight of σ and $k_2(\nu)e_2+\cdots+k_{n+1}(\nu)e_{n+1}$ is the highest weight of ν, by Theorem 8.2 in [EKM] one has, taking the different parametrization into account:

$$c_\nu(\sigma : \lambda) = \alpha(n)\frac{\prod_{j=2}^{n+1}\Gamma(i\lambda - k_j(\sigma) - \rho_j)\prod_{j=2}^{n+1}\Gamma(i\lambda + k_j(\sigma) + \rho_j)}{\prod_{j=2}^{n+1}\Gamma(i\lambda - k_j(\nu) - \rho_j)\prod_{j=2}^{n+1}\Gamma(i\lambda + k_j(\nu) + \rho_j + 1)}, \tag{4.17}$$

where $\alpha(n)$ is a constant depending only on n. Thus, if all $k_j(\sigma)$ are integral one gets

$$c_\nu(\sigma : z)^{-1}\frac{d}{dz}c_\nu(\sigma : z) = \sum_{j=2}^{n+1}\sum_{\substack{|k_j(\sigma)|<l \\ \leq k_j(\nu)}}\frac{i}{iz - l - \rho_j} - \sum_{j=2}^{n+1}\sum_{|k_j(\sigma)|\leq l\leq k_j(\nu)}\frac{i}{iz + l + \rho_j}. \tag{4.18}$$

If all $k_j(\sigma)$ are half integral, a similar formula holds. For notational convenience, if $\nu \in \hat{K}$ and $\sigma \in \hat{M}$ with $[\nu : \sigma] = 0$ we let $c_\nu(\sigma : z) := 0$. Since $\dim(\boldsymbol{\mathcal{L}}(\sigma)) = p\dim(\sigma)$ it follows with Proposition 4.1, (4.15) and (4.16) that for every $\nu \in \hat{K}$ with $[\nu : \sigma] \neq 0$ one has

$$\begin{aligned}\operatorname{Tr}\left(\mathbf{T}(\sigma, i\lambda)^{-1}\frac{d}{ds}\mathbf{T}(\sigma, i\lambda)\right) =& \frac{1}{\dim(\nu)}\operatorname{Tr}\left(\mathbf{C}(\nu : \boldsymbol{\sigma} : i\lambda)^{-1}\frac{d}{ds}\mathbf{C}(\nu : \sigma : i\lambda)\right) \\ &+ ip\dim(\sigma)c(\nu : \sigma : \lambda)^{-1}\frac{d}{d\lambda}c(\nu : \sigma : \lambda).\end{aligned} \tag{4.19}$$

Now for $\sigma \in \hat{M}$ with highest weight $k_2(\sigma)e_2 + \cdots + k_{n+1}(\sigma)e_{n+1}$ as in (2.9) we let $\nu_\sigma \in \hat{K}$ be the representation of K with highest weight $k_2(\sigma)e_2 + \cdots + |k_{n+1}(\sigma)|\, e_{n+1}$. Then by Proposition 2.2 we have $[\nu_\sigma : \sigma] = 1$ and thus using (4.18) and (4.19) we get

$$\begin{aligned}\operatorname{Tr}\left(\mathbf{T}(\sigma, i\lambda)^{-1}\frac{d}{ds}\mathbf{T}(\sigma, i\lambda)\right) =& \frac{1}{\dim(\nu)}\operatorname{Tr}\left(\mathbf{C}(\nu_\sigma : \boldsymbol{\sigma} : i\lambda)^{-1}\frac{d}{ds}\mathbf{C}(\nu_\sigma : \sigma : i\lambda)\right) \\ &+ \sum_{j=2}^{n+1}\frac{p\dim(\sigma)}{i\lambda + |k_j(\sigma)| + \rho_j}.\end{aligned} \tag{4.20}$$

Finally we recall the factorization of the C-matrix into an infinite product involving its zeroes and poles. Let $\sigma \in \hat{M}$ and $\nu \in \hat{K}$ with $[\nu : \sigma] \neq 0$. The restrictions of the representations $\pi_{\sigma,\lambda}$ and $\pi_{w_0\sigma,-\lambda}$ to K are independent of the parameter λ and are unitarily equivalent via the map $A(w_0)^{-1} : \mathcal{H}^{w_0\sigma} \to \mathcal{H}^{\sigma}$. If we tensor $A(w_0)^{-1}$ with an isometry $I'(\sigma) : \boldsymbol{\mathcal{L}}(w_0\sigma) \to \boldsymbol{\mathcal{L}}(\sigma)$ and use the isomorphism $\mathbf{I}_\sigma$ we obtain an isometry $I(\sigma) : \boldsymbol{\mathcal{E}}(w_0\sigma) \to \boldsymbol{\mathcal{E}}(\sigma)$ which maps $\boldsymbol{\mathcal{E}}(w_0\sigma, \nu)$ to $\boldsymbol{\mathcal{E}}(\sigma, \nu)$ for every $\nu \in \hat{K}$. Moreover by (4.14) and Proposition 4.1 for all $u \in \boldsymbol{\mathcal{L}}(\sigma)$ and all $\Phi \in (\mathcal{H}^\sigma)^\nu$ we have

$$I(\sigma) \circ \mathbf{C}(\nu : \sigma : s) \circ \mathbf{I}_\sigma = \mathbf{I}_\sigma \circ \left((I'(\sigma) \circ \mathbf{T}(\sigma, s)) \otimes J_{\bar{P}_0|P_0}(\nu, \sigma, -is)\right).$$

Thus using (4.16) it follows that the multiplicity of each pole of $\det(I(\sigma) \circ \mathbf{C}(\nu : \sigma : s))$ is divisible by $\dim(\nu)$. Let $\{\beta\}$ and $\{\eta\}$ denote the set of poles of $\det(I(\sigma) \circ \mathbf{C}(\nu : \sigma : s))$ on

$(0,n]$ respectively $\{s \in \mathbb{C}\colon \operatorname{Re}(s) < 0\}$, counted with multiplicity divided by $\dim(\nu)$. Then the set $\{\beta\}$ is finite and by [Mü2, Lemma 6.6] the series

$$\sum_{\eta} \frac{\operatorname{Re}(\eta)}{|\eta|^2} \tag{4.21}$$

converges absolutely. Moreover by [Mü2, Theorem 6.9] one has

$$\begin{aligned} &\frac{1}{\dim(\nu)} \operatorname{Tr}\left(\mathbf{C}(\nu : \sigma : s)^{-1} \frac{d}{ds} \mathbf{C}(\nu : \sigma : s)\right) \\ &= \log q(\sigma) + \sum_{\{\beta\}} \left(\frac{1}{s+\beta} - \frac{1}{s-\beta}\right) + \sum_{\eta} \left(\frac{1}{s+\overline{\eta}} - \frac{1}{s-\eta}\right), \end{aligned} \tag{4.22}$$

where $q(\sigma) \in \mathbb{R}^+$ and where the sum converges absolutely.

4.3 The Maaß-Selberg relations

The Maaß-Selberg relations compute the inner product of two truncated Eisenstein series. They play an important role for the study of the relative trace which will be defined below. Thus for the sake of completeness we include a proof here. Our proof adapts the proof of Theorem 12.10 in [Bo] to our situation. Let us first introduce the truncation operator. Let $P \in \mathfrak{P}$. Let Y_0 be as in section 2.11 and let $Y \geq Y_0$. For $P \in \mathfrak{P}$ let $\chi_{P,Y}$ be the characteristic function of $N_P A_P^0[Y]\, K \subset G$. Now let $\sigma \in \hat{M}$, $\nu \in \hat{K}$, $[\sigma : \nu] \neq 0$. Let $\Phi \in \boldsymbol{\mathcal{E}}(\boldsymbol{\sigma} : \nu)$. Then for $Y \geq Y_0$ and $x \in \Gamma\backslash G$, $x = \Gamma g$ we let

$$E^Y(\Phi : s : x) := E(\Phi : s : x) - \sum_{P \in \mathfrak{P}} \sum_{\gamma \in \Gamma \cap N_P \backslash \Gamma} \chi_{P,Y}(\gamma g) E_P(\Phi : s : \gamma g),$$

where $E_P(\Phi : x : s)$ is as in (4.9). By (2.38) at most one summand in this sum is not zero. By [HC1] the function $E^Y(\Phi : x : s)$ lies in $L^2(\Gamma\backslash G)$. One easily sees that

$$\int_{\Gamma\backslash G} E^Y(\Phi : s : x)\overline{E^Y}(\Psi : s' : x)dx = \int_{\Gamma\backslash G} E^Y(\Phi : s : x)\overline{E}(\Psi : s' : x)dx, \tag{4.23}$$

where $\Phi, \Psi \in \boldsymbol{\mathcal{E}}(\boldsymbol{\sigma} : \nu)$. If $\sigma = w_0\sigma$, we let

$$\overline{\boldsymbol{\mathcal{E}}}(\boldsymbol{\sigma}, \nu) := \boldsymbol{\mathcal{E}}(\boldsymbol{\sigma}, \nu); \quad \overline{\boldsymbol{\mathcal{E}}}(\boldsymbol{\sigma}) := \boldsymbol{\mathcal{E}}(\boldsymbol{\sigma}).$$

If $\sigma \neq w_0\sigma$ we let

$$\overline{\boldsymbol{\mathcal{E}}}(\boldsymbol{\sigma}, \nu) := \boldsymbol{\mathcal{E}}(\boldsymbol{\sigma}, \nu) \oplus \boldsymbol{\mathcal{E}}(\boldsymbol{w\sigma}, \nu); \quad \overline{\boldsymbol{\mathcal{E}}}(\boldsymbol{\sigma}) := \boldsymbol{\mathcal{E}}(\boldsymbol{\sigma}) \oplus \boldsymbol{\mathcal{E}}(\boldsymbol{w\sigma}),$$

where the direct sum is a direct sum of Hilbert spaces. Let $P \in \mathfrak{P}$, $\sigma_P \in \hat{M}_P$. Assume that $\sigma_P \neq w_P\sigma_P$. Then we remark that the inner product on $\mathcal{E}(\sigma_P) \oplus \mathcal{E}(w_P\sigma_P) \subset \overline{\boldsymbol{\mathcal{E}}}(\boldsymbol{\sigma})$ is still given as in (4.2). We first prove the following lemma.

Lemma 4.3. *Let $\sigma \in \hat{M}$, $\nu \in \hat{K}$, $[\nu : \sigma] \neq 0$. Let $\Phi, \Psi \in \boldsymbol{\mathcal{E}}(\boldsymbol{\sigma} : \nu)$. Let $s, s' \in \mathbb{C}$ such that $(s + \overline{s'})(s - \overline{s'}) \neq 0$. Let $\langle \cdot, \cdot \rangle$ denote the inner product on $\overline{\boldsymbol{\mathcal{E}}}(\boldsymbol{\sigma}, \nu)$. Then in the sense of meromorphic functions one has*

$$\begin{aligned}
&\int_{\Gamma\backslash G} E^Y(x : \Phi : s)\overline{E^Y}(x : \Psi : s')dx \\
=&\frac{Y^{s+\overline{s'}}}{s+\overline{s'}} \langle \Phi, \Psi \rangle + \frac{Y^{s-\overline{s'}}}{s-\overline{s'}} \langle \Phi, \mathbf{C}(\sigma : s')\Psi \rangle + \frac{Y^{\overline{s'}-s}}{\overline{s'}-s} \langle \mathbf{C}(\sigma : s)\Phi, \Psi \rangle \\
-&\sum_{P \in \mathfrak{P}} \frac{Y^{-s-\overline{s'}}}{s+\overline{s'}} \langle \mathbf{C}(\sigma : s)\Phi, \mathbf{C}(\sigma : s')\Psi \rangle .
\end{aligned}$$

Proof. To save notation, if $P, P' \in \mathfrak{P}$ we write $C_{P|P'}(s)$ for $C_{P|P'}(\nu : \sigma : s)$. Moreover we let $\delta_{P|P'} = 1$ if $P = P'$ and $\delta_{P|P'} = 0$ if $P \neq P'$. We assume that $\mathrm{Re}(s) > \mathrm{Re}(s') > n$. This suffices by meromorphic continuation. Moreover, by bilinearity we can assume that there exist $P_1, P_2 \in \mathfrak{P}$ such that $\Phi \in \mathcal{E}_{P_1}(\sigma_{P_1}, \nu)$, $\Psi \in \mathcal{E}_{P_2}(\sigma_{P_2}, \nu)$. By (2.19), for every $P \in \mathfrak{P}$ we have

$$\int_{\Gamma\cap N_P\backslash G} f(x)dx := \int_{\Gamma\cap N_P\backslash N_P} \int_{\mathbb{R}} \int_K e^{-2nt} f(n_P a_P(t)k) d\bar{n}_P dt dk. \tag{4.24}$$

First assume that $P \neq P_1$. Let $v_P := \mathrm{vol}\,(\Gamma \cap N_P \backslash N_P)$. Then

$$\begin{aligned}
&\int_{\Gamma\backslash G} \sum_{\gamma \in \Gamma\cap N_P\backslash\Gamma} \chi_{P,Y}(\gamma x) E_P(\Phi : s : \gamma x)\overline{E}(\Psi : s' : x)dx \\
=&\int_{\Gamma\cap N_P\backslash G} \chi_{P,Y}(x) E_P(\Phi : s : x)\overline{E}(\Psi : s' : x)dx \\
=&\int_{\Gamma\cap N_P\backslash N_P} \int_{\mathbb{R}} \int_K e^{-2nt} \chi_{P,Y}(a_P(t)k) E_P(\Phi : s : a_P(t)k)\overline{E}(\Psi : s' : n_P a_P(t)k) dk dt dn_P \\
=&v_P \int_{\mathbb{R}} \int_K e^{-2nt} \chi_{P,Y}(a_P(t)k) E_P(\Phi : s : a_P(t)k)\overline{E_P}(\Psi : s' : a_P(t)k) dk dt \\
=&\int_{\log Y}^{\infty} e^{-2nt} e^{(-s+n)t} e^{(-\overline{s'}+n)t} dt \left\langle C_{P_1|P}(s)\Phi, C_{P_2|P}(s')\Psi \right\rangle \\
&+ \delta_{P|P_2} \int_{\log Y}^{\infty} e^{-2nt} e^{(-s+n)t} e^{(\overline{s'}+n)t} dt \left\langle C_{P_1|P}(s)\Phi, \Psi \right\rangle \\
=&\frac{Y^{-s-\overline{s'}}}{s+\overline{s'}} \left\langle C_{P_1|P}(s)\Phi, C_{P_2|P}(s')\Psi \right\rangle + \delta_{P|P_2} \frac{1}{s-\overline{s'}} Y^{-s+\overline{s'}} \left\langle C_{P_1|P}(s)\Phi, \Psi \right\rangle .
\end{aligned}$$

Here the first equality follows from Lemma 2.4, the second equality follows from (4.24) and the fourth equality follows from (4.7) and (4.8). Using these arguments again, together with (4.6) we get

$$\int_{\Gamma\backslash G}\big(E(\Phi:s:x)-\sum_{\gamma\in\Gamma\cap N_{P_1}\backslash\Gamma}\chi_{P_1,Y}(\gamma x)E_{P_1}(\Phi:s:\gamma x)\big)\overline{E}(\Psi:s':x)dx$$

$$=\int_{\Gamma\backslash G}\sum_{\gamma\in\Gamma\cap N_{P_1}\backslash\Gamma}((1-\chi_{P_1,Y}(\gamma x))\Phi_{P_1,s}(\gamma x)\overline{E}(\Psi:s':x)dx$$

$$-\int_{\Gamma\backslash G}\sum_{\gamma\in\Gamma\cap N_{P_1}\backslash\Gamma}\chi_{P_1,Y}(\gamma x)(C_{P_1|P_1}(s)\Phi)_{P_1,-s}(\gamma x)\overline{E}(\Psi:s':x)dx$$

$$=\delta_{P_1|P_2}\int_{-\infty}^{\log Y}e^{-2nt}e^{(s+n)t}e^{(\overline{s'}+n)t}dt\left\langle\Phi,\Psi\right\rangle+\int_{-\infty}^{\log Y}e^{-2nt}e^{(s+n)t}e^{(-\overline{s'}+n)t}dt\left\langle\Phi,C_{P_2|P_1}(s')\Psi\right\rangle$$

$$-\delta_{P_1|P_2}\int_{\log Y}^{\infty}e^{-2nt}e^{(-s+n)t}e^{(\overline{s'}+n)t}dt\left\langle C_{P_1|P_1}\Phi,\Psi\right\rangle$$

$$-\int_{\log Y}^{\infty}e^{-2nt}e^{(-s+n)t}e^{(-\overline{s'}+n)t}dt\left\langle C_{P_1|P_1}(s)\Phi,C_{P_2|P_1}(s')\Psi\right\rangle$$

$$=\delta_{P_1|P_2}\frac{Y^{s+\overline{s'}}}{s+\overline{s'}}\left\langle\Phi,\Psi\right\rangle+\frac{Y^{s-\overline{s'}}}{s-\overline{s'}}\left\langle\Phi,C_{P_2|P_1}(s')\Psi\right\rangle+\delta_{P_1|P_2}\frac{Y^{\overline{s'}-s}}{\overline{s'}-s}\left\langle C_{P_1|P_1}(s)\Phi,\Psi\right\rangle$$

$$-\frac{Y^{-s-\overline{s'}}}{s+\overline{s'}}\left\langle C_{P_1|P_1}(s)\Phi,C_{P_2|P_1}(s')\Psi\right\rangle.$$

By (4.23) one has

$$\int_{\Gamma\backslash G}E^Y(\Phi:s:x)\overline{E^Y}(\Psi:s':x)dx$$

$$=\int_{\Gamma\backslash G}\big(E(\Phi:s:x)-\sum_{\gamma\in\Gamma\cap N_{P_1}\backslash\Gamma}\chi_{P_1,Y}(\gamma x)E_{P_1}(\Phi:s:\gamma x)\big)\overline{E}(\Psi:s':x)dx$$

$$-\sum_{\substack{P\in\mathfrak{P}\\P\neq P_1}}\int_{\Gamma\backslash G}\sum_{\gamma\in\Gamma\cap N_P\backslash\Gamma}\chi_{P,Y}(\gamma x)E_P(\Phi:s:\gamma x)\overline{E}(\Psi:s':x)dx.$$

This proves the proposition. □

Passing to the limit in Lemma 4.3, we obtain the Maaß-Selberg relations.

Corollary 4.4. *Let $\lambda\in\mathbb{R}-\{0\}$. Then under the same assumptions as in the preceding lemma one has*

$$\int_{\Gamma\backslash G}E^Y(\Phi:i\lambda:x)\overline{E^Y}(\Psi:i\lambda:x)dx=2\left\langle\Phi,\Psi\right\rangle\log Y+\frac{Y^{2i\lambda}}{2i\lambda}\left\langle\Phi,\mathbf{C}(\sigma:i\lambda)\Psi\right\rangle$$

$$-\frac{Y^{-2i\lambda}}{2i\lambda}\left\langle\mathbf{C}(\sigma:i\lambda)\Phi,\Psi\right\rangle-\left\langle\mathbf{C}(\sigma:-i\lambda)\frac{d}{dz}\mathbf{C}(\sigma:i\lambda)\Phi,\Psi\right\rangle.$$

Proof. Let $\varepsilon > 0$. Then by Lemma 4.3 we have

$$\begin{aligned}&\int_{\Gamma\backslash G} E^Y(\Phi : i\lambda+\varepsilon : x)\overline{E^Y}(\Psi : i\lambda+\varepsilon : x)dx\\ =&\frac{Y^{2\varepsilon}}{2\varepsilon}\langle\Phi,\Psi\rangle - \frac{Y^{-2\varepsilon}}{2\varepsilon}\langle \mathbf{C}(\sigma : i\lambda+\varepsilon)\Phi, \mathbf{C}(\sigma : i\lambda+\varepsilon)\Psi\rangle\\ +&\frac{Y^{2i\lambda}}{2i\lambda}\langle\Phi, \mathbf{C}(\sigma : i\lambda+\varepsilon)\Psi\rangle - \frac{Y^{-2i\lambda}}{2i\lambda}\langle \mathbf{C}(\sigma : i\lambda+\varepsilon)\Phi, \Psi\rangle\end{aligned}$$

By (4.10) we have

$$\begin{aligned}&\frac{Y^{2\varepsilon}}{2\varepsilon}\langle\Phi,\Psi\rangle - \frac{Y^{-2\varepsilon}}{2\varepsilon}\langle \mathbf{C}(\sigma : i\lambda+\varepsilon)\Phi, \mathbf{C}(\sigma : i\lambda+\varepsilon)\Psi\rangle\\ =&\frac{Y^{2\varepsilon}-Y^{-2\varepsilon}}{2\varepsilon}\langle\Phi,\Psi\rangle - \frac{Y^{-2\varepsilon}}{2\varepsilon}\langle \mathbf{C}(\sigma : -i\lambda+\epsilon)\left(\mathbf{C}(\sigma : i\lambda+\epsilon) - \mathbf{C}(\sigma : i\lambda-\epsilon)\right)\Phi, \Psi\rangle .\end{aligned}$$

Letting $\varepsilon \to 0$, the lemma follows. □

Corollary 4.5. *Let $\{e_1,\ldots,e_N\}$ be an orthonormal base of $\boldsymbol{\mathcal{E}}(\boldsymbol{\sigma} : \nu)$. Denote the corresponding matrix entries by $\mathbf{C}_{j,k}(\sigma : s)$. Let $a_0 > 0$. Then the $\mathbf{C}_{j,k}(\sigma : s)$ are bounded on the set $\{s \in \mathbb{C}\colon a_0 \geq |\mathrm{Re}(s)| \geq 0,\ |\mathrm{Im}(s)| > 1\}$.*

Proof. By Lemma 4.3 with $s = s' = i\lambda+\mu$, $\lambda\mu \neq 0$ one has

$$\begin{aligned}&\sum_k |\mathbf{C}_{kj}(\sigma : i\lambda+\mu)|^2 + 2\mu Y^{2\mu}\int_{\Gamma\backslash G} E^Y(e_j : i\lambda+\mu : x)\overline{E^Y}(e_j : i\lambda+\mu : x)dx\\ &\leq Y^{4\mu} + \frac{2\mu Y^{2\mu}}{|\lambda|}|\mathbf{C}_{j,j}(\sigma : i\lambda+\mu)| .\end{aligned}$$

This implies the corollary. □

5 The relative trace of Bochner Laplace operators on locally homogeneous vector bundles

5.1 Bochner Laplace operators

Regard G as a principal K-fibre bundle over $\tilde{X}$. By the invariance of $\mathfrak{p}$ under $\mathrm{Ad}(K)$, the assignment

$$T_g^{\mathrm{hor}} := \left\{ \left.\frac{d}{dt}\right|_{t=0} g\exp tX \colon X \in \mathfrak{p}\right\}$$

defines a horizontal distribution on G. This connection is called the canonical connection. Let ν be a finite-dimensional unitary representation of K on $(V_\nu, \langle\cdot,\cdot\rangle_\nu)$. Let

$$\tilde{E}_\nu := G \times_\nu V_\nu$$

be the associated homogeneous vector bundle over $\tilde{X}$. Then $\langle\cdot,\cdot\rangle_\nu$ induces a G-invariant metric $\tilde{B}_\nu$ on $\tilde{E}_\nu$. Let $\widetilde{\nabla}^\nu$ be the connection on $\tilde{E}_\nu$ induced by the canonical connection. Then $\widetilde{\nabla}^\nu$ is G-invariant. Let

$$E_\nu := \Gamma\backslash(G \times_\nu V_\nu)$$

be the associated locally homogeneous bundle over X. Since $\tilde{B}_\nu$ and $\widetilde{\nabla}^\nu$ are G-invariant, they push down to a metric B_ν and a connection ∇^ν on E_ν. Let

$$C^\infty(G,\nu) := \{f : G \to V_\nu \colon f \in C^\infty,\ f(gk) = \nu(k^{-1})f(g),\ \forall g \in G, \forall k \in K\}. \tag{5.1}$$

Let

$$C^\infty(\Gamma\backslash G,\nu) := \{f \in C^\infty(G,\nu) \colon f(\gamma g) = f(g)\, \forall g \in G, \forall \gamma \in \Gamma\}\,. \tag{5.2}$$

Let $C^\infty(X,E_\nu)$ denote the space of smooth sections of E_ν. Then there is a canonical isomorphism

$$A : C^\infty(X,E_\nu) \cong C^\infty(\Gamma\backslash G,\nu).$$

There is also a corresponding isometry for the space $L^2(X,E_\nu)$ of L^2-sections of E_ν. For every $X \in \mathfrak{g}$, $g \in G$ and every $f \in C^\infty(X,E_\nu)$ one has

$$A(\nabla^\nu_{L(g)_*X} f)(g) = \frac{d}{dt}|_{t=0} Af(g \exp tX).$$

Let $\widetilde{\Delta}_\nu = \widetilde{\nabla^\nu}^*\widetilde{\nabla}^\nu$ be the Bochner-Laplace operator of $\widetilde{E}_\nu$. Since $\widetilde{X}$ is complete, $\widetilde{\Delta}_\nu$ with domain the smooth compactly supported sections is essentially self-adjoint [Ch]. Its self-adjoint extension will be denoted by $\widetilde{\Delta}_\nu$ too. By [Mi1, Proposition 1.1] on $C^\infty(G,\nu)$ one has

$$\widetilde{\Delta}_\nu = -\Omega + \nu(\Omega_K), \tag{5.3}$$

where Ω_K is as in section 2.8. Let $\tilde{A}_\nu$ be the differential operator on $\widetilde{E}_\nu$ which acts as $-\Omega$ on $C_c^\infty(G,\nu)$. Then it follows from (5.3) that $\tilde{A}_\nu$ is bounded from below and essentially selfadjoint. Its selfadjoint extension will be denoted by $\tilde{A}_\nu$ too. Let $e^{-t\tilde{A}_\nu}$ be the corresponding heat semigroup on $L^2(G,\nu)$, where $L^2(G,\nu)$ is defined analogously to (5.1). Then the same arguments as in [CY, section1] imply that there exists a function

$$K_t^\nu \in C^\infty(G \times G, \mathrm{End}(V_\nu)), \tag{5.4}$$

which is symmetric in the G-variables and for which $g' \mapsto K_t^\nu(g,g')$ belongs to $L^2(G,\mathrm{End}(V_\nu))$ for each $g \in G$ such that

$$K_t^\nu(gk, g'k') = \nu(k^{-1})K_t^\nu(g,g')\nu(k'),\ \forall g,g' \in G,\ \forall k,k' \in K$$

and such that

$$e^{-t\tilde{A}_\nu}\phi(g)=\int_G K_t^\nu(g,g')\phi(g')dg',\ \phi\in L^2(G,\nu).$$

Since Ω is G-invariant, K_t^ν is invariant under the diagonal action of G. Hence there exists a function

$$H_t^\nu:G\longrightarrow \operatorname{End}(V_\nu);\quad H_t^\nu(k^{-1}gk')=\nu(k)^{-1}\circ H_t^\nu(g)\circ\nu(k'),\ \forall k,k'\in K,\forall g\in G \tag{5.5}$$

such that

$$K_t^\nu(g,g')=H_t^\nu(g^{-1}g'),\quad \forall g,g'\in G. \tag{5.6}$$

Thus one has

$$(e^{-t\tilde{A}_\nu}\phi)(g)=\int_G H_t^\nu(g^{-1}g')\phi(g')dg',\quad \phi\in L^2(G,\nu),\quad g\in G. \tag{5.7}$$

By the arguments of [BM, Proposition 2.4], H_t^ν belongs to all Harish-Chandra Schwarz spaces $(\mathcal{C}^q(G)\otimes\operatorname{End}(V_\nu))$, $q>0$.
Now we pass to the quotient $X=\Gamma\backslash\widetilde{X}$. Let $\Delta_\nu=\nabla^{\nu*}\nabla^\nu$ be the closure of the Bochner-Laplace operator with domain the smooth compactly supported sections of E_ν. Then Δ_ν is selfadjoint and by (5.3) it induces the operator $-\Omega+\nu(\Omega_K)$ on $C^\infty(\Gamma\backslash G,\nu)$. Thus if we let A_ν be the operator $-\Omega$ on $C_c^\infty(\Gamma\backslash G,\nu)$, then A_ν is bounded from below and essentially selfadjoint. The closure of A_ν will be denoted by A_ν too. Let e^{-tA_ν} be the heat-semigroup of A_ν on $L^2(\Gamma\backslash G,\nu)$. Let

$$H^\nu(t;x,x'):=\sum_{\gamma\in\Gamma}H_t^\nu(g^{-1}\gamma g'), \tag{5.8}$$

where $x,x'\in\Gamma\backslash G$, $x=\Gamma g$, $x'=\Gamma g'$. By [War1, Chapter 4] this series converges absolutely and locally uniformly. It follows from (5.7) that

$$(e^{-tA_\nu}\phi)(x)=\int_{\Gamma\backslash G}H^\nu(t;x,x')\phi(x')dx',\quad \phi\in L^2(\Gamma\backslash G,\nu),\quad x\in\Gamma\backslash G.$$

Now we let

$$h_t^\nu(g):=\operatorname{tr}H_t^\nu(g), \tag{5.9}$$

where tr denotes the trace in $\operatorname{End}V_\nu$, and define an operator $\pi_\Gamma(h_t^\nu)$ acting on $L^2(\Gamma\backslash G)$ by

$$\pi_\Gamma(h_t^\nu)f(x):=\int_G h_t^\nu(g)f(xg)dg.$$

Then $\pi_\Gamma(h_t^\nu)$ is an integral-operator on $L^2(\Gamma\backslash G)$, whose kernel is given by

$$h^\nu(t;x,x'):=\operatorname{tr}H^\nu(t;x,x'). \tag{5.10}$$

We shall now compute the Fourier transform of h_t^ν. Let π be a unitary admissible representation of G on a Hilbert space $\mathcal{H}_\pi$. Let $\check{\nu}$ be the contragredient representation of ν and let $P_{\check{\nu}}(\pi)$ be the projection of $\mathcal{H}_\pi$ onto $\mathcal{H}_\pi^{\check{\nu}}$, the $\check{\nu}$-isotypical component of $\mathcal{H}_\pi$. By assumption $\mathcal{H}_\pi^{\check{\nu}}$ is finite dimensional. By [On, § 4, Proposition 4 and § 7, Proposition 3], we have $\check{\nu} \cong \nu$. Thus together with (5.5) one obtains

$$\pi(h_t^\nu) = P_{\check{\nu}}(\pi)\pi(h_t^\nu)P_{\check{\nu}}(\pi) = P_\nu(\pi)\pi(h_t^\nu)P_\nu(\pi). \tag{5.11}$$

The restriction of $\pi(h_t^\nu)$ to $\mathcal{H}_\pi^\nu$ will be denoted by $\pi(h_t^\nu)$ too. Define a bounded operator on $\mathcal{H}_\pi \otimes V_\nu$ by

$$\tilde{\pi}(H_t^\nu(g)) := \int_G \pi(g) \otimes H_t^\nu(g) dg. \tag{5.12}$$

Then relative to the splitting

$$\mathcal{H}_\pi \otimes V_\nu = (\mathcal{H}_\pi \otimes V_\nu)^K \oplus \Big((\mathcal{H}_\pi \otimes V_\nu)^K\Big)^\perp,$$

$\tilde{\pi}(H_t^\nu)$ has the form

$$\begin{pmatrix} \pi(H_t^\nu) & 0 \\ 0 & 0 \end{pmatrix},$$

where $\pi(H_t^\nu)$ acts on $(\mathcal{H}_\pi \otimes V_\nu)^K$. It follows as in [BM, Corollary 2.2] that

$$\pi(H_t^\nu) = e^{t\pi(\Omega)}\,\mathrm{Id}, \tag{5.13}$$

where Id is the identity on $(\mathcal{H}_\pi \otimes V_\nu)^K$. Now let $A : \mathcal{H}_\pi \to \mathcal{H}_\pi$ be a bounded operator which is an intertwining operator for $\pi|_K$. Then $A \otimes \mathrm{Id}$ acts on $(\mathcal{H}_\pi \otimes V_\nu)^K$. Denotes this operator on $(\mathcal{H}_\pi \otimes V_\nu)^K$ by $\tilde{A}$. Then by the same argument as in [BM, Lemma 5.1] one has

$$\mathrm{Tr}\,(A \circ \pi(h_t^\nu)) = \mathrm{Tr}\,\Big(\tilde{A} \circ \tilde{\pi}(H_t^\nu)\Big).$$

Together with (5.13) we obtain

$$\mathrm{Tr}\,(A \circ \pi(h_t^\nu)) = e^{t\pi(\Omega)} \cdot \mathrm{Tr}\,\tilde{A}. \tag{5.14}$$

Using this equation we obtain the following proposition.

Proposition 5.1. *For $\sigma \in \hat{M}$ and $\lambda \in \mathbb{R}$ let $\Theta_{\sigma,\lambda}$ be the global character of $\pi_{\sigma,\lambda}$. Then one has*

$$\Theta_{\sigma,\lambda}(h_t^\nu) = e^{t(c(\sigma)-\lambda^2)}; \quad \mathrm{Tr}\,(\boldsymbol{\pi}_{\Gamma,\sigma,i\lambda}(h_t^\nu)) = p\dim(\sigma)e^{t(c(\sigma)-\lambda^2)}$$

for $[\nu : \sigma] \neq 0$ and $\Theta_{\sigma,\lambda}(h_t^\nu) = \mathrm{Tr}\,(\boldsymbol{\pi}_{\Gamma,\sigma,i\lambda}(h_t^\nu)) = 0$ otherwise. Here $c(\sigma)$ is as in (2.27).

Proof. Let $\pi \in \hat{G}$ and let Θ_π be its global character. Taking $A = \mathrm{Id}$ in (5.14), one obtains

$$\Theta_\pi(h_t^\nu) = e^{t\pi(\Omega)} \cdot \dim(\mathcal{H}_\pi \otimes V_\nu)^K = e^{t\pi(\Omega)} \cdot [\pi : \check{\nu}] = e^{t\pi(\Omega)} \cdot [\pi : \nu]\,,$$

where we used $\nu \cong \check{\nu}$. By Proposition 2.2 for all $\nu \in \hat{K}$ and all $\sigma \in \hat{M}$ we have $[\nu : \sigma] \leq 1$. Thus the proposition follows from (2.20), Corollary 2.6 and the intertwining property of the map $\mathbf{I}_\sigma$. ∎

5.2 The Bochner Laplace operator on the cusp

Let $P \in \mathfrak{P}$, let Y_0 be as in section 2.11 and let $u \in \mathbb{R}^+$ with $u \geq Y_0$. Then the decomposition (2.37) gives a natural inclusion $I_{P,u}$ of $L^2\left([u,\infty), y^{-d}dy; V_\nu\right)$ into the space $L^2(\Gamma\backslash G, \nu)$. Namely, let $\phi \in L^2\left([u,\infty), y^{-d}dy; V_\nu\right)$ and let $\chi_{P,u}$ be as in section 4.3. Then define a function $I_{P,u}\phi$ in $L^2(\Gamma\backslash G, \nu)$ by

$$I_{P,u}\phi(x) = \frac{1}{\sqrt{\mathrm{vol}(\Gamma \cap N_P\backslash N_P)}} \sum_{\gamma \in \Gamma\cap N_P\backslash\Gamma} \chi_{P,u}(\gamma g)\nu(\kappa_P(\gamma g))^{-1}\phi(e^{H_P(\gamma g)}),$$

where $x \in \Gamma\backslash G$, $x = \Gamma g$. By (2.38) and (2.39) the assignment $\phi \mapsto I_{P,u}\phi$ embeds the space $L^2\left([u,\infty), y^{-d}dy; V_\nu\right)$ isometrically into $L^2(\Gamma\backslash G, \nu)$. For $f \in L^2(\Gamma\backslash G, \nu)$ and $y \in \Gamma\backslash G$, $y = \Gamma g$ let

$$f_{P,u}(y) := \frac{1}{\mathrm{vol}(\Gamma \cap N_P\backslash N_P)} \sum_{\gamma \in \Gamma\cap N_P\backslash\Gamma} \chi_{P,u}(\gamma g) \int_{\Gamma\cap N_P\backslash N_P} f(n_P\gamma g)dn_P. \tag{5.15}$$

Then $f \mapsto f_{P,u}$ is the orthogonal projection of $L^2(\Gamma\backslash G, \nu)$ onto $I_{P,u}\left(L^2\left([u,\infty), y^{-d}; V_\nu\right)\right)$. Since Ω is G-invariant, one has $\Omega f_{P,u}(y) = (\Omega f)_{P,u}(y)$ for every $f \in C^\infty(\Gamma\backslash G, \nu)$. Thus $-\Omega$ induces in a canonical way a differential operator T_ν on $C^\infty([u,\infty); V_\nu)$. The operator T_ν can be computed explicitly. Let Ω_M be as in section 2.8 and define an endomorphism $L(\nu)$ of V_ν by $L(\nu) := -\nu|_M(\Omega_M)$, where $\nu|_M$ denotes the restriction of ν to M.

Lemma 5.2. *One has* $T_\nu = -y^2\frac{d^2}{dy^2} + (d-2)y\frac{d}{dy} + L(\nu)$.

Proof. For $\alpha \in \Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ let $\mathfrak{g}_\mathbb{C}^\alpha$ be the corresponding root space. Then one can choose X_α in $\mathfrak{g}_\mathbb{C}^\alpha$, $X_{-\alpha} \in \mathfrak{g}_\mathbb{C}^{-\alpha}$ such that $B(X_\alpha, X_{-\alpha}) = 1$, $[X_\alpha, X_{-\alpha}] = H_\alpha$, where H_α is as in section 2.2. By (2.4) one has

$$\sum_{\alpha\in\Delta^+(\mathfrak{g}_\mathbb{C},\mathfrak{a}_\mathbb{C})} H_\alpha = 2nH_1.$$

Thus by the definition of Ω and Ω_M one has

$$\begin{aligned}\Omega =& \sum_{i=1}^{n+1} H_i^2 + \sum_{\alpha\in\Delta^+(\mathfrak{g}_\mathbb{C},\mathfrak{h}_\mathbb{C})} (X_\alpha X_{-\alpha} + X_{-\alpha}X_\alpha)\\ =& H_1^2 + \sum_{\alpha\in\Delta^+(\mathfrak{g}_\mathbb{C},\mathfrak{a}_\mathbb{C})} (H_\alpha + 2X_{-\alpha}X_\alpha) + \sum_{i=2}^{n+1} H_i^2 + \sum_{\alpha\in\Delta^+(\mathfrak{m}_\mathbb{C},\mathfrak{b}_\mathbb{C})} (X_\alpha X_{-\alpha} + X_{-\alpha}X_\alpha)\\ =& H_1^2 + 2nH_1 + \Omega_M \qquad \mod U(\mathfrak{g}_\mathbb{C})\mathfrak{n}_\mathbb{C}.\end{aligned}$$

Now the element H_1 induces the differential operator $-y\frac{d}{dy}$ on $(0,\infty)$ under ι_P. Since Ω_M is invariant under the anti-involution of $U(\mathfrak{m}_\mathbb{C})$ induced by $Y \mapsto -Y$, $Y \in \mathfrak{m}_\mathbb{C}$, the proposition follows. □

Consider the differential operator

$$T_\nu := -y^2\frac{d^2}{dy^2} + (d-2)y\frac{d}{dy} + L(\nu)$$

acting on $C_c^\infty([u,\infty);V_\nu) \subset L^2\left([u,\infty), y^{-d}dy; V_\nu\right)$. Its selfadjoint extension with repsect to the Neumannn boundary condition at u will be denoted by T_ν too. First consider the operator $L(\nu)$.

Lemma 5.3. *For $\sigma \in \hat{M}$ let P_σ denote the projection from V_ν to the σ-isotypical subspace of $\nu|_M$. Then one has*

$$L(\nu) = \sum_{\substack{\sigma\in\hat{M}\\ [\nu:\sigma]\neq 0}} -\left(c(\sigma) + \frac{(d-1)^2}{4}\right)P_\sigma,$$

where $c(\sigma)$ is as in (2.27)*.*

Proof. By (2.25) and (2.27) one has

$$c(\sigma) = \sigma(\Omega_M) - \frac{(d-1)^2}{4}.$$

This implies the lemma. □

Now consider the differential operator

$$T_0 := -y^2\frac{d^2}{dy^2} + (d-2)y\frac{d}{dy} \tag{5.16}$$

acting on $C_c^\infty([u,\infty)) \subset L^2(([b,\infty);y^{-d})$. The self-adjoint extension of T_0 with respect to the Neumann boundary condition at u will be denoted by T_0 too. The spectral reslution of T_0 is described in [Mü1, Chapter IV]. For $s \in \mathbb{C} - (d-1)$ let

$$c(s) := \frac{s}{s-d+1}u^{2s-d+1} \tag{5.17}$$

Then $c(s)$ satisfies

$$c(s)c(d-1-s) = 1. \tag{5.18}$$

Let

$$\eta(y,s) := y^s + c(s)y^{d-1-s}.$$

Then $\eta(y,s)$ is a generalized eigenfunction of T_0 satisfying the Neumann boundary condition. For $\phi \in C_c^\infty([u,\infty])$ let

$$J\phi(\lambda) := \int_u^\infty \phi(y)\eta\left(y, \frac{d-1}{2} + i\lambda\right)y^{-d}dy. \tag{5.19}$$

Then J can be extended to an isometry from $L^2\left([u,\infty),y^{-d}dy\right)$ onto $L^2(\mathbb{R}^+;\frac{d\lambda}{2\pi})$. The adjoint operator of J is given by

$$J^*\psi(y)=\frac{1}{2\pi}\int_0^\infty\psi(\lambda)\overline{\eta}\left(y,\frac{d-1}{2}+i\lambda\right)d\lambda. \tag{5.20}$$

For ϕ in the domain of T_0 one has

$$JT_0\phi(\lambda)=\left(\frac{(d-1)^2}{4}+\lambda^2\right)J\phi(\lambda). \tag{5.21}$$

We start with the following lemma.

Lemma 5.4. *One has*

$$\begin{aligned}&\frac{1}{2\pi}\int_0^\infty e^{-t\lambda^2}\int_u^Y\eta\left(y,\frac{d-1}{2}+i\lambda\right)\overline{\eta}\left(y,\frac{d-1}{2}+i\lambda\right)y^{-d}dyd\lambda\\ =&\frac{\log Y-\log u}{\sqrt{4\pi t}}-\frac{1}{2\pi}\int_0^\infty e^{-t\lambda^2}\frac{d-1}{\frac{(d-1)^2}{4}+\lambda^2}d\lambda-\frac{1}{4}+o(1),\end{aligned}$$

as $Y\to\infty$.

Proof. The proof is analogous to the classical proof of the Maaß-Selberg relations. Assume first that $(s-\overline{s'})(s+\overline{s'}-(d-1))\neq 0$. Let F_P^Y be as in (2.41) and let $\Delta=-\operatorname{div}\operatorname{grad}$ be the Laplacian on F_P^Y with respect to the metric $y^{-2}dy^2+y^{-2}g_P$. Then one has $\Delta f=T_0 f$ for $f\in C^\infty([u,\infty))$. We have $\overline{s'}(d-1-\overline{s'})-s(d-1-s)=(s-\overline{s'})(s+\overline{s'}-(d-1))$. Thus applying Greens's formula and the boundary condition of $\eta(y,s)$ and $\overline{\eta}(y,s')$ at $y=u$ one gets

$$\begin{aligned}&\int_u^Y\eta(y,s)\overline{\eta}(y,s')\frac{dy}{y^d}\\ =&\frac{1}{(s-\overline{s'})(s+\overline{s'}-(d-1))}\int_u^Y\left(\eta(y,s)\overline{\Delta\eta}(y,s')-\Delta\eta(y,s)\overline{\eta}(y,s')\right)\frac{dy}{y^d}\\ =&\frac{1}{(s-\overline{s'})(s+\overline{s'}-(d-1))}\frac{1}{Y^{d-2}}\left(\overline{\eta}(Y,s')\frac{d}{dy}\Big|_{y=Y}\eta(y,s)-\eta(Y,s)\frac{d}{dy}\Big|_{y=Y}\overline{\eta}(y,s')\right)\\ =&\frac{Y^{s+\overline{s'}-(d-1)}}{s+\overline{s'}-(d-1)}+\frac{c(s)\overline{c}(s')Y^{d-1-s-\overline{s'}}}{d-1-s-\overline{s'}}+\frac{\overline{c}(s')Y^{s-\overline{s'}}}{s-\overline{s'}}+\frac{c(s)Y^{\overline{s'}-s}}{\overline{s'}-s}.\end{aligned}$$

Now one proceeds as in the proof of corollary 4.4. Thus if one lets s and s' tend to $\frac{d-1}{2}+i\lambda$ in the last equation and uses (5.18) one gets

$$\begin{aligned}&\int_b^Y\eta\left(y,\frac{d-1}{2}+i\lambda\right)\overline{\eta}\left(y,\frac{d-1}{2}+i\lambda\right)\frac{dy}{y^d}\\ =&2\log Y-\frac{c'\left(\frac{d-1}{2}+i\lambda\right)}{c(\frac{d-1}{2}+i\lambda)}+\frac{c(\frac{d-1}{2}-i\lambda)Y^{2i\lambda}}{2i\lambda}+\frac{c(\frac{d-1}{2}+i\lambda)Y^{-2i\lambda}}{-2i\lambda}.\end{aligned}$$

By (5.17) one has

$$\frac{c'\left(\frac{d-1}{2}+i\lambda\right)}{c(\frac{d-1}{2}+i\lambda)} = \frac{d-1}{\frac{(d-1)^2}{4}+\lambda^2} + 2\log u.$$

Finally one has

$$\int_0^\infty e^{-t\lambda^2}\left(\frac{c(\frac{d-1}{2}-i\lambda)Y^{2i\lambda}}{2i\lambda} + \frac{c(\frac{d-1}{2}+i\lambda)Y^{-2i\lambda}}{-2i\lambda}\right)d\lambda$$
$$= \int_0^\infty e^{-t\lambda^2} c(\frac{d-1}{2}-i\lambda)\frac{Y^{2i\lambda}-Y^{-2i\lambda}}{2i\lambda}d\lambda + \int_0^\infty e^{-t\lambda^2}Y^{-2i\lambda}\frac{c(\frac{d-1}{2}-i\lambda)-c(\frac{d-1}{2}+i\lambda)}{2i\lambda}d\lambda.$$

The second term tends to zero for $Y\to\infty$ by the Riemann Lebesgue Lemma. Using that

$$\phi(0) = \lim_{\mu\to\infty}\frac{2}{\pi}\int_0^\infty \phi(t)\frac{\sin(\mu t)}{t}dt \tag{5.22}$$

for every $\phi\in\mathcal{S}(\mathbb{R})$ and using the explicit form of $c(s)$ the lemma is proved. □

Now we can study the heat semigroup e^{-tT_ν} of T_ν on $L^2([u,\infty), y^{-d}dy)$.

Corollary 5.5. *Let $\Phi_\nu(t,y,y')$ be the integral kernel of e^{-tT_ν} and let $\phi_t^\nu(y) := \operatorname{tr}\Phi_\nu(t,y,y)$. Then one has*

$$\int_u^Y \phi_t^\nu(y)y^{-d}dy = \sum_{\substack{\sigma\in\hat{M}\\ [\nu:\sigma]\neq 0}} e^{tc(\sigma)}\dim(\sigma)\left(\frac{\log Y-\log u}{\sqrt{4\pi t}} - \frac{1}{4} - \frac{1}{2\pi}\int_0^\infty e^{-t\lambda^2}\frac{(d-1)}{\frac{(d-1)^2}{4}+\lambda^2}d\lambda\right)$$
$$+ o(1),$$

as $Y\to\infty$.

Proof. By Lemma 5.3 and by (5.19), (5.20), (5.21) one has

$$\Phi_\nu(t,y,y') = \frac{1}{2\pi}\sum_{\substack{\sigma\in\hat{M}\\ [\nu:\sigma]\neq 0}} e^{tc(\sigma)}\int_0^\infty e^{-t\lambda^2}\overline{\eta}\left(y,\frac{d-1}{2}+i\lambda\right)\eta\left(y',\frac{d-1}{2}+i\lambda\right)d\lambda\cdot P_\sigma.$$

The corollary follows from Lemma 5.4. □

5.3 The relative trace

The decomposition (4.1) induces a decomposition of $L^2(\Gamma\backslash G,\nu)\cong\left(L^2(\Gamma\backslash G,\nu)\otimes V_\nu\right)^K$ as

$$L^2(\Gamma\backslash G,\nu) = L^2_d(\Gamma\backslash G,\nu)\oplus L^2_c(\Gamma\backslash G,\nu).$$

This decomposition is invariant under A_ν in the sense of unbounded operators. Let A_ν^d denote the restriction of A_ν to $L^2_d(\Gamma\backslash G,\nu)$ in the sense of unbounded operators. Since $\pi_{\Gamma,d}$ decomposes discretely into a direct sum of irreducible unitary representations of G, there exists a subset $J\subset\mathbb{N}$, a sequence λ_j, $j\in J$ of real numbers and an orthonormal base ϕ_j, $j\in J$ of $L^2_d(\Gamma\backslash G,\nu)$ such that $A_\nu\phi_j=\lambda_j\phi_j$ for every $j\in J$. The set J may be finite. For $\lambda\in[0,\infty)$ let

$$N(\lambda):=\#\{j\in J\colon \lambda_j\leq\lambda\}.$$

By Theorem I.1 in [Do1] and Theorem 9.1 in [Do2], there exists a constant C such that

$$N(\lambda)\leq C(1+\lambda)^{\frac{d}{2}}. \tag{5.23}$$

Hence the sum

$$\sum_j e^{-t\lambda_j}$$

converges, the operator $e^{-tA_\nu^d}$ is of trace class and one has

$$\operatorname{Tr}\left(e^{-tA_\nu^d}\right)=\sum_j e^{-t\lambda_j}. \tag{5.24}$$

Let h_t^ν be as in (5.9). Then relative to the decompositiiton in (4.1), $\pi_\Gamma(h_t^\nu)$ splits as

$$\pi_\Gamma(h_t^\nu)=\pi_{\Gamma,d}(h_t^\nu)\oplus\pi_{\Gamma,c}(h_t^\nu). \tag{5.25}$$

According to this decomposition, the function $h^\nu(t;x,x')$ from (5.10) decomposes as

$$h^\nu(t;x,x')=h_d^\nu(t;x,x')+h_c^\nu(t;x,x'), \tag{5.26}$$

where $h_d^\nu(t;x,x')$ resp. $h_c^\nu(t;x,x')$ are smooth and denote the kernels of $\pi_{\Gamma,d}(h_t^\nu)$ resp. $\pi_{\Gamma,c}(h_t^\nu)$ on $L^2_d(\Gamma\backslash G)$ resp. on $L^2_c(\Gamma\backslash G)$. Moreover, the function $h_d^\nu(t;x,x)$ is absolutely integrable, the operator $\pi_{\Gamma,d}(h_t^\nu)$ is of trace class and one has

$$\operatorname{Tr}(e^{-tA_\nu^d})=\operatorname{Tr}(\pi_{\Gamma,d}(h_t^\nu))=\int_{\Gamma\backslash G}h_d^\nu(t;x,x)dx. \tag{5.27}$$

The kernel $h_c^\nu(t;x,x')$ can be computed using the explicit realization of $\pi_{\Gamma,c}$. Namely, for each $\sigma\in\hat{M}$ with $[\nu:\sigma]\neq0$ let $\{e(\sigma)_1,\dots,e(\sigma)_n\}$ be an orthonormal base of $\boldsymbol{\mathcal{E}}(\nu:\boldsymbol{\sigma})$. Then by [War1, Theorem 4.7], equation (5.11) and Frobenius reciprocity one has

$$\begin{aligned}&h_c^\nu(t;x,x')\\=&\frac{1}{4\pi}\sum_{\substack{\sigma\in\hat{M}\\ [\nu:\sigma]\neq0}}\sum_{k,l}\int_{\mathbb{R}}\left\langle\boldsymbol{\pi}_{\Gamma,\boldsymbol{\sigma},i\lambda}(h_t^\nu)e(\sigma)_l,e(\sigma)_k\right\rangle\boldsymbol{E}(e(\sigma)_k:i\lambda:x)\overline{\boldsymbol{E}}(e(\sigma)_l:i\lambda:x')d\lambda.\end{aligned} \tag{5.28}$$

Next we want to compare the heat semigroup of A_ν with the heat semigroup of an auxiliary operator $T_{\nu,u}$. This operator is defined as follows. Let Y_0 be as in section 2.11 and let $u \geq Y_0$. For every $P \in \mathfrak{P}$ let $T^P_{\nu,u}$ be the operator T_ν defined in section 5.2 regarded as an operator on $\mathcal{D}(T^P_{\nu,u}) \subset L^2(\Gamma\backslash G, \nu)$ via the embedding $I_{P,u}$ from section 5.2. Then we put

$$T_{\nu,u} := \bigoplus_{P\in\mathfrak{P}} T^P_{\nu,u}.$$

By [Mü1, Theorem 9.1] for every $t > 0$ the difference $e^{-tA_\nu} - e^{-tT_{\nu,u}}$ is of trace class. Thus we can define the relative trace $\mathrm{Tr}_{\mathrm{rel,u}}(e^{-tA_\nu})$ with respect to the parameter u by

$$\mathrm{Tr}_{\mathrm{rel,u}}(e^{-tA_\nu}) := \mathrm{Tr}\left(e^{-tA_\nu} - e^{-tT_{\nu,u}}\right).$$

We now compute the relative trace using Corollary 4.4, Corollary 5.5 and equation (5.28). First we prove the following lemma.

Lemma 5.6. *Let $C(Y)$ be as in equation* (2.37)*. Then one has*

$$\int_{C(Y)} h^\nu_c(t;x,x)dx = \sum_{\substack{\sigma\in\hat{M};\sigma=w_0\sigma\\ [\nu:\sigma]\neq 0}} \frac{\mathrm{Tr}\left(\boldsymbol{\pi}_{\Gamma,\sigma,0}(h^\nu_t)\mathbf{C}(\nu:\sigma:0)\right)}{4} + \sum_{\substack{\sigma\in\hat{M}\\ [\nu:\sigma]\neq 0}} \left(\frac{pe^{tc(\sigma)}\log Y\dim(\sigma)}{\sqrt{4\pi t}}\right.$$
$$\left. - \frac{1}{4\pi}\int_{\mathbb{R}} \mathrm{Tr}\left(\boldsymbol{\pi}_{\Gamma,\sigma,i\lambda}(h^\nu_t)\mathbf{C}(\nu:\sigma:-i\lambda)\frac{d}{dz}\mathbf{C}(\nu:\sigma:i\lambda)\right)d\lambda\right) + o(1),$$

as $Y \to \infty$.

Proof. The argument on page 82 in [War1] can be extended to $h^\nu_t \in \mathcal{C}(G)$ and thus the integral

$$\int_{\mathbb{R}}\int_{\Gamma\backslash G}\left|\sum_{k,l}\langle\boldsymbol{\pi}_{\Gamma,\boldsymbol{\sigma},i\lambda}(h^\nu_t)e(\sigma)_l, e(\sigma)_k\rangle\,\mathbf{E}^Y(e(\sigma)_k:i\lambda:x)\overline{\mathbf{E}^Y}(e(\sigma)_l:i\lambda:x)\right|dxd\lambda \qquad (5.29)$$

is finite. Hence, if one defines

$$h^\nu_c(t;x;Y) := \frac{1}{4\pi}\sum_{k,l}\int_{\mathbb{R}}\langle\boldsymbol{\pi}_{\Gamma,\boldsymbol{\sigma},i\lambda}(h^\nu_t)e(\sigma)_l, e(\sigma)_k\rangle\,\mathbf{E}^Y(e(\sigma)_k:i\lambda:x)\overline{\mathbf{E}^Y}(e(\sigma)_l:i\lambda:x)d\lambda,$$

then $h^\nu_c(t;x;Y)$ is absolutely integrable over $\Gamma\backslash G$. Moreover, by the definition of the trunctation operator one has $h^\nu_c(t;x;Y) = h^\nu_c(t;x,x)$ on $C(Y)$. Thus one has

$$\int_{C(Y)} h^\nu_c(t,x,x)dx = \int_{\Gamma\backslash G} h^\nu_c(t;x;Y)dx + o(1),$$

as $Y \to \infty$. Now by the finiteness of the integral in (5.29), integrating $h_c^\nu(t;x;Y)$ over $\Gamma\backslash G$ one can interchange the order of integration. Applying Corollary 4.4 and (4.10) one gets

$$\int_{C(Y)} h_c^\nu(t,x,x)dx = \sum_{\substack{\sigma\in\hat{M}\\ [\nu:\sigma]\neq 0}} \left(\frac{\log Y}{2\pi} \int_{\mathbb{R}} \operatorname{Tr} \boldsymbol{\pi}_{\Gamma,\sigma,i\lambda}(h_t^\nu) d\lambda \right.$$
$$-\frac{1}{4\pi}\int_{\mathbb{R}} \operatorname{Tr}\left(\boldsymbol{\pi}_{\Gamma,\sigma,i\lambda}(h_t^\nu)\mathbf{C}(\sigma:-i\lambda)\frac{d}{dz}\mathbf{C}(\sigma:i\lambda)\right) d\lambda$$
$$-\frac{1}{4\pi}\sum_{k,l}\int_{\mathbb{R}} Y^{-2i\lambda}\left\langle \boldsymbol{\pi}_{\Gamma,\sigma,i\lambda}(h_t^\nu)e_l, e_k\right\rangle \frac{\left\langle \mathbf{C}(\sigma:i\lambda)e(\sigma)_k, e(\sigma)_l\right\rangle - \left\langle \mathbf{C}(\sigma:-i\lambda)e(\sigma)_k, e(\sigma)_l\right\rangle}{2i\lambda} d\lambda$$
$$\left. +\frac{1}{4\pi}\sum_{k,l}\int_{\mathbb{R}} \frac{Y^{2i\lambda}-Y^{-2i\lambda}}{2i\lambda}\left\langle \boldsymbol{\pi}_{\Gamma,\sigma,i\lambda}(h_t^\nu)e(\sigma)_l, e(\sigma)_k\right\rangle \left\langle \mathbf{C}(\sigma:-i\lambda)e(\sigma)_k, e(\sigma)_l\right\rangle d\lambda \right) + o(1),$$

as $Y \to \infty$. By [War2, Theorem E2], for every σ, k and l the function

$$\lambda \mapsto \left\langle \boldsymbol{\pi}_{\Gamma,\sigma,i\lambda}(h_t^\nu)e(\sigma)_l, e(\sigma)_k\right\rangle$$

belongs to $\mathcal{S}(\mathbb{R})$. Thus using Corollary 4.5 and the Riemann-Lebesgue Lemma it follows that the integral in the third line of the last equation tends to zero as Y tends to ∞. Moreover it follows together with (5.22) that the integral in the fourth line tends to

$$\operatorname{Tr}\left(\boldsymbol{\pi}_{\Gamma,\sigma,0}(h_t^\nu)\mathbf{C}(\nu:\sigma:0)\right)$$

as $Y \to \infty$. Using Proposition 5.1, the lemma follows. □

Now we obtain an explicit formula for the relative trace.

Proposition 5.7. *The relative trace is given as*

$$\operatorname{Tr}_{\mathrm{rel,u}}(e^{-tA_\nu}) = \operatorname{Tr}\left(\pi_{\Gamma,d}(h_t^\nu)\right) + \sum_{\substack{\sigma\in\hat{M};\sigma=w_0\sigma\\ [\nu:\sigma]\neq 0}} \frac{\operatorname{Tr}\left(\boldsymbol{\pi}_{\Gamma,\sigma,0}(h_t^\nu)\mathbf{C}(\nu:\sigma:0)\right)}{4}$$
$$-\sum_{\substack{\sigma\in\hat{M}\\ [\nu:\sigma]\neq 0}} \frac{1}{4\pi}\int_{\mathbb{R}} \operatorname{Tr}\left(\boldsymbol{\pi}_{\Gamma,\sigma,i\lambda}(h_t^\nu)\mathbf{C}(\nu:\sigma:-i\lambda)\frac{d}{dz}\mathbf{C}(\nu:\sigma:i\lambda)\right) d\lambda$$
$$+\sum_{\substack{\sigma\in\hat{M}\\ [\nu:\sigma]\neq 0}} e^{tc(\sigma)} p\dim(\sigma)\left(\frac{\log u}{\sqrt{4\pi t}} + \frac{1}{4} + \frac{1}{2\pi}\int_0^\infty e^{-t\lambda^2}\frac{(d-1)}{\frac{(d-1)^2}{4}+\lambda^2}d\lambda \right).$$

Proof. Let $\Phi_{\nu,u}(t;x,y)$ be the integral kernel of $e^{-tT_{\nu,u}}$ and let $\phi_{\nu,u}(t;x) := \operatorname{Tr}\Phi_{\nu,u}(t;x,x)$. By [Mü1, Theorem 9.1] one has

$$\operatorname{Tr}_{\mathrm{rel,u}}(e^{-tA_\nu}) = \int_{\Gamma\backslash G} \left(h^\nu(t;x,x) - \phi_{\nu,u}(t;x)\right) dx,$$

where the right hand side is absolutely integrable. Thus using (5.26), (5.27) and (2.40) we obtain

$$\begin{aligned}\operatorname{Tr}_{\mathrm{rel,u}}(e^{-tA_\nu}) &= \operatorname{Tr}\left(\pi_{\Gamma,d}(h_t^\nu)\right) + \int_{\Gamma\backslash G} \left(h_c^\nu(t;x,x) - \phi_{\nu,u}(t;x)\right) dx \\ &= \operatorname{Tr}\left(\pi_{\Gamma,d}(h_t^\nu)\right) + \lim_{Y\to\infty} \int_{C(Y)} \left(h_c^\nu(t;x,x) - \phi_{\nu,u}(t;x)\right) dx.\end{aligned}$$

Using Corollary 5.5 and Lemma 5.6 the proposition follows. □

Remark 5.8. For $\sigma \in \hat{M}$, $[\nu : \sigma] \neq 0$ let $\widetilde{\mathbf{C}}(\nu : \sigma : z) := (\mathbf{C}(\nu : \sigma : z) \otimes \operatorname{Id})|_{(\boldsymbol{\mathcal{E}}(\boldsymbol{\sigma})\otimes V_\nu)^K}$. Using the intertwining properties of $\mathbf{I}_\sigma$ and $\mathbf{C}(\sigma : \lambda)$, equation (5.14) and corollary 2.6 one gets

$$\operatorname{Tr}\left(\boldsymbol{\pi}_{\Gamma,\sigma,0}(h_t^\nu)\mathbf{C}(\nu : \sigma : 0)\right) = e^{tc(\sigma)} \operatorname{Tr} \widetilde{\mathbf{C}}(\nu : \sigma : 0). \tag{5.30}$$

Moreover, the restriction of the representations $\boldsymbol{\pi}_{\Gamma,\sigma,i\lambda}$ to K is independent of the parameter λ. Thus the operator

$$\mathbf{C}(\nu : \sigma : -i\lambda)\frac{d}{dz}\mathbf{C}(\nu : \sigma : i\lambda)$$

is again an intertwining operator for the restriction of $\boldsymbol{\pi}_{\Gamma,\sigma,i\lambda}$ to K and one also has

$$\begin{aligned}&\operatorname{Tr}\left(\boldsymbol{\pi}_{\Gamma,\sigma,i\lambda}(h_t^\nu)\mathbf{C}(\nu : \sigma : -i\lambda)\frac{d}{dz}\mathbf{C}(\nu : \sigma : i\lambda)\right) \\ =&e^{-t\left(\lambda^2 - c(\sigma)\right)} \operatorname{Tr}\left(\widetilde{\mathbf{C}}(\nu : \sigma : -i\lambda)\frac{d}{dz}\widetilde{\mathbf{C}}(\nu : \sigma : i\lambda)\right).\end{aligned}$$

Remark 5.9. By equation (5.27) and Lemma 5.6, the integral of $h^\nu(t;x,x)$ over $C(Y)$ has an asymptotic expansion in Y as $Y \to \infty$. Then one can take the constant term in this expansion as a definition of the regularized trace $\operatorname{Tr}_{\mathrm{reg}}(e^{-tA_\nu})$ of e^{-tA_ν}. Thus one has

$$\begin{aligned}\operatorname{Tr}_{\mathrm{reg}}(e^{-tA_\nu}) = \operatorname{Tr}\left(\pi_{\Gamma,d}(h_t^\nu)\right) &+ \sum_{\substack{\sigma\in\hat{M};\sigma=w_0\sigma \\ [\nu:\sigma]\neq 0}} \frac{\operatorname{Tr}\left(\boldsymbol{\pi}_{\Gamma,\sigma,0}(h_t^\nu)\mathbf{C}(\nu : \sigma : 0)\right)}{4} \\ &- \sum_{\substack{\sigma\in\hat{M} \\ [\nu:\sigma]\neq 0}} \frac{1}{4\pi}\int_{\mathbb{R}} \operatorname{Tr}\left(\boldsymbol{\pi}_{\Gamma,\sigma,i\lambda}(h_t^\nu)\mathbf{C}(\nu : \sigma : -i\lambda)\frac{d}{dz}\mathbf{C}(\nu : \sigma : i\lambda)\right) d\lambda.\end{aligned}$$

This regularization of the trace is similar to the b-trace of Melrose [Me] and was also used by Park [Pa] for certain $\nu \in \hat{K}$. Moreover the difference between the relative and the regularized trace of e^{-tA_ν} is given by the last line in the equation of Proposition 5.7.

6 The trace formula

6.1 Statement of the trace formula

Let α be a K-finite Schwarz function. Define an operator $\pi_\Gamma(\alpha)$ on $L^2(\Gamma\backslash G)$ by

$$\pi_\Gamma(\alpha)f(x) := \int_G \alpha(g)f(gx)dg.$$

Then relative to the decompostition (4.1) one has a splitting

$$\pi_\Gamma(\alpha) = \pi_{\Gamma,d}(\alpha) \oplus \pi_{\Gamma,c}(\alpha).$$

It easily follows from (5.23) that the operator $\pi_{\Gamma,d}(\alpha)$ is of trace class. In this section we recall the Selberg trace formula for $\operatorname{Tr}(\pi_{\Gamma,d}(\alpha))$. In this way we also obtain a formula for the relative trace $\operatorname{Tr}_{\mathrm{rel,u}}(e^{-tA_\nu})$ defined in the previous section.
First we introduce the distributions involved. Let

$$I(\alpha) := \operatorname{vol}(\Gamma\backslash G)\alpha(1).$$

By [HC2, Theorem 3], the Plancherel theorem can be applied to α. Four groups of real rank one which do not possess a compact Cartan subgroup it is stated in [Kn1, Theorem 13.2]. Thus if $P_\sigma(z)$ is as in section 2.10 one obtains

$$I(\alpha) = \operatorname{vol}(X)\sum_{\sigma\in\hat{M}}\int_{\mathbb{R}} P_\sigma(i\lambda)\Theta_{\sigma,\lambda}(\alpha)d\lambda, \tag{6.1}$$

where the sum is finite since α is K-finite.
Next let $\mathrm{C}(\Gamma)_{\mathrm{s}}$ be the set of semisimple conjugacy classes of Γ as in section 3. Then define

$$H(\alpha) := \int_{\Gamma\backslash G}\sum_{\gamma\in\Gamma_{\mathrm{s}}-1}\alpha(x^{-1}\gamma x)dx.$$

By [War1, Lemma 8.1] the integral converges absolutely. Its Fourier transform can be computed as follows. For $[\gamma]\in \mathrm{C}(\Gamma)_{\mathrm{s}}-[1]$ let $m_\gamma\in M$ and $\ell(\gamma)\in\mathbb{R}^+$ be as in Lemma 3.1 and Proposition 3.2. Let $a_\gamma := \exp\ell(\gamma)H_1$. Moreover let γ_0 be as in Proposition 3.4. Then one puts

$$L(\gamma,\sigma) := \frac{\overline{\operatorname{Tr}(\sigma)(m_\gamma)}}{\det\left(\operatorname{Id}-\operatorname{Ad}(m_\gamma a_\gamma)|_{\bar{\mathfrak{n}}}\right)}e^{-n\ell(\gamma)}. \tag{6.2}$$

Then proceeding as in [Wal, Chapter 6] and using [Ga, equation 4.6] one obtains

$$H(\alpha) = \sum_{\sigma\in\hat{M}}\sum_{[\gamma]\in\mathrm{C}(\Gamma)_{\mathrm{s}}-[1]}\frac{l(\gamma_0)}{2\pi}L(\gamma,\sigma)\int_{-\infty}^{\infty}\Theta_{\sigma,\lambda}(\alpha)e^{-i\ell(\gamma)\lambda}d\lambda, \tag{6.3}$$

where the sum is finite since α is K-finite.
Next let $P \in \mathfrak{P}$. Let $\{e_1, \dots, e_{2n}\}$ be an orthonormal base of $\mathfrak{n}_P$, let $L = \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_{2n}$ and let $A : \mathfrak{n}_P \to \mathfrak{n}_P$ such that $\exp A(L) = \Gamma \cap N_P$. Let $T := A^* A$. Then the Epstein-zeta function $\zeta_T(s)$ is defined by

$$\zeta_T(s) := \sum_{z \in L} (z^\tau T z)^{-s}. \tag{6.4}$$

By [Ter, Chapter 1.4, Theorem 1] this series converges absolutely fo $\mathrm{Re}(s) > n$ and ζ_T has a meromorphic continuation to $\mathbb{C}$ with a simple pole at $s = n$. Let R_T be the residue of ζ_T at $s = n$. Then by [Ter, Chapter 1.4, Theorem 1] one has

$$R_T = \frac{\pi^n}{\Gamma(n)\sqrt{\det T}} = \frac{\mathrm{vol}\,(S^{2n-1})}{2\,\mathrm{vol}\,(\Gamma \cap N_P \backslash N_P)}. \tag{6.5}$$

Now for every $\eta \in \Gamma \cap N_P - \{1\}$ let $X_\eta := \log \eta$. Let $\|\cdot\|$ be the norm induced on $\mathfrak{n}_P$ by the restriction of $-\frac{1}{4n}B(\cdot, \theta\cdot)$ to $\mathfrak{n}_P$. Then for $\mathrm{Re}(s) > 0$ the Epstein zeta function ζ_P is defined by

$$\zeta_P(s) := \sum_{\eta \in \Gamma \cap N_P - \{1\}} \|X_\eta\|^{-2n(1+s)}. \tag{6.6}$$

It follows that ζ_P has a meromorphic continuation to $\mathbb{C}$ with a simple pole at 0. Let $C_P(\Gamma)$ be the constant term of ζ_P at $s = 0$ and let $R_P(\Gamma)$ be the residue of ζ_P at $s = 0$. Then by (6.5) one has

$$C'_P(\Gamma) := \frac{R_P(\Gamma) 2n\,\mathrm{vol}(\Gamma \cap N_P \backslash N_P)}{\mathrm{vol}(S^{2n-1})} = 1. \tag{6.7}$$

Now let

$$\begin{aligned}
T_P(\alpha) &:= \int_K \int_{N_P} \alpha(k n_P k^{-1}) dk dn_P = \int_K \int_N \alpha(knk^{-1}) dn; \\
T(\alpha) &:= \sum_{P \in \mathfrak{P}} C_P(\Gamma) \frac{\mathrm{vol}(\Gamma \cap N_P \backslash N_P)}{\mathrm{vol}(S^{2n-1})} T_P(\alpha); \\
T'_P(\alpha) &:= \int_K \int_{N_P} \alpha(k n_P k^{-1}) \log \|\log n_P\| dn_P dk.
\end{aligned}$$

Then T and $T_{P'}$ are tempered distributions. Let

$$C(\Gamma) := \sum_{P \in \mathfrak{P}} C_P(\Gamma) \frac{\mathrm{vol}(\Gamma \cap N_P \backslash N_P)}{\mathrm{vol}(S^{2n-1})}. \tag{6.8}$$

The distributions T is invariant. Applying the Fourier inversion formula and the Peter-Weyl-Theorem to equation 10.21 in [Kn1], one obtains the Fourier transform of T as:

$$T(\alpha) = \sum_{\sigma \in \hat{M}} \frac{\dim(\sigma)}{2\pi} C(\Gamma) \int_{\mathbb{R}} \Theta_{\sigma,\lambda}(\alpha) d\lambda, \tag{6.9}$$

see [Wal, Lemma 6.3]. The distributions T'_P are not invariant. However, they can be made invariant using the standard Knapp-Stein intertwining operators as follows. Let $\epsilon > 0$ be such that 0 is the only possible pole of the operators $J_{\bar{P}_0|P_0}(\sigma, z)$, $J_{\bar{P}_0|P_0}(\sigma, z)^{-1}$, $\boldsymbol{T}(\sigma, z)$, $\boldsymbol{T}(\sigma, z)^{-1}$ on $\{z \in \mathbb{C}\colon\ |z| < 2\epsilon\}$ for all $\sigma \in \hat{M}$ which satisfy $[\nu : \sigma] \neq 0$, ν a K-type of α. Let H_ϵ be the half-circle from $-\epsilon$ to ϵ in the lower half-plane, oriented counter-clockwise. Let D_ϵ be the path which is the union of $(-\infty, -\epsilon]$, H_ϵ and $[\epsilon, \infty)$. Let

$$J_\sigma(\alpha) := \frac{p \dim \sigma}{4\pi i} \int_{D_\epsilon} \operatorname{Tr}\left(J_{\bar{P}_0|P_0}(\sigma, z)^{-1} \frac{d}{dz} J_{\bar{P}_0|P_0}(\sigma, z) \pi_{\sigma,z}(\alpha) \right) dz.$$

The change of contour is only neccessary if $J_{\bar{P}_0|P_0}(\sigma, s)$ has a pole at 0, i.e. if $\sigma = w_0\sigma$. Let

$$J(\alpha) := -\sum_{\sigma \in \hat{M}} J_\sigma(\alpha). \tag{6.10}$$

Then by equation (4.16), equation (5.5) and Proposition 5.1 one has

$$J(h_t^\nu) = -\frac{p}{4\pi i} \sum_{\sigma \in \hat{M}} [\nu : \sigma] \dim(\sigma) \int_{D_\epsilon} e^{-t(z^2 - c(\sigma))} c_\nu(\sigma : z)^{-1} \frac{d}{dz} c_\nu(\sigma : z) dz. \tag{6.11}$$

Now we define a distribution $\mathcal{I}$ by

$$\mathcal{I}(\alpha) := \sum_{P \in \mathfrak{P}} T'_P(\alpha) - J(\alpha). \tag{6.12}$$

Then $\mathcal{I}$ is an invariant distribution. This can be seen as follows. Using the formula for $J_M(m, \alpha)$ on p. 92 of [Ho1], we get $J_{M_P}(1, \alpha) = T'_P(\alpha)$. Next using the formula for the invariant distribution $I_P(m, \alpha)$ on p. 93 of [Ho1] and formula (8) of [Ho1], it follows that

$$I_P(1, \alpha) = T'_P(\alpha) + \sum_{\sigma \in \hat{M}_0} \frac{\dim(\sigma)}{4\pi i} \int_{D_\epsilon} \operatorname{Tr}\left(J_{\bar{P}_0|P_0}(\sigma, z)^{-1} \frac{d}{dz} J_{\bar{P}_0|P_0}(\sigma, z) \pi_{\sigma,z}(\alpha) \right) dz.$$

Summing over $P \in \mathfrak{P}$, we get

$$\sum_{P \in \mathfrak{P}} I_P(1, \alpha) = \mathcal{I}(\alpha) - J(\alpha),$$

which proves our claim.
For $\sigma \in \hat{M}$ let

$$\mathcal{S}_\sigma(\alpha) := \frac{1}{4\pi} \int_{D_\epsilon} \operatorname{Tr}\left(\boldsymbol{T}(\sigma, iz)^{-1} \frac{d}{ds} \boldsymbol{T}(\sigma, iz) \right) \Theta_{\sigma,z}(\alpha) dz,$$

where $\boldsymbol{T}(\sigma, iz)$ is as in Proposition 4.1 and let

$$\mathcal{S}(\alpha) := \sum_{\sigma \in \hat{M}} \mathcal{S}_\sigma(\alpha), \tag{6.13}$$

where the sum is finite since α is K-finite. By Corollary 4.2 we then have

$$\sum_{\sigma\in\hat{M}}\frac{1}{4\pi}\int_{\mathbb{R}}\operatorname{Tr}\left(\boldsymbol{\pi}_{\Gamma,\sigma,i\lambda}(\alpha)\mathbf{C}(\sigma:-i\lambda)\frac{d}{dz}\mathbf{C}(\sigma:i\lambda)\right)d\lambda=\mathcal{S}(\alpha)-J(\alpha). \tag{6.14}$$

For $\sigma\in\hat{M}$ let

$$R_\sigma(\alpha):=\begin{cases}-\frac{1}{4}\operatorname{Tr}\left(\mathbf{C}(\sigma:0)\boldsymbol{\pi}_{\Gamma,\boldsymbol{\sigma},0}(\alpha)\right), & \sigma=\ w_0\sigma\\ 0, & \sigma\neq\ w_0\sigma.\end{cases}$$

Then $R_\sigma(\alpha)$ can be transformed further as follows. We use the notations of section 4.1 and section 4.2. Let $\sigma\in\hat{M}$ with $\sigma=w_0\sigma$. Let $J_{\bar{P}_0|P_0}(\sigma,\lambda)$ be as in (4.13). Then $J_{\bar{P}_0|P_0}(\sigma,\lambda)$ might have a pole at $\lambda=0$. However, if the meromorphic function $r_{\bar{P}_0|P_0}(\sigma:\lambda)$ is as in [Ho2, page 113-114], the operator

$$R_{P_0}(\sigma:\lambda)=A(w_0)r_{\bar{P}_0|P_0}(\sigma:\lambda)^{-1}J_{\bar{P}_0|P_0}(\sigma:\lambda)$$

is defined and invertible for $\lambda\in\mathbb{R}$, and one has

$$R_{P_0}(\sigma:0)^*=R_{P_0}(\sigma:0);\quad R_{P_0}(\sigma:0)^{-1}=R_{P_0}(\sigma:0). \tag{6.15}$$

By [KS, Proposition 49, Proposition 53] the representation $\pi_{\sigma,0}$ is irreducible. Moreover, $R_{P_0}(\sigma:0)$ satisfies $R_{P_0}(\sigma:0)\circ\pi_{\sigma,0}=\pi_{\sigma,0}\circ R_{P_0}(\sigma:0)$. Thus by [Kn1, Corollary 8.13], $R_{P_0}(\sigma:0)$ is a scalar operator. Togehter with (6.15) it follows that $(R_{P_0}(\sigma:0))^2=\pm\operatorname{Id}$. Now define a meromorphic $\operatorname{Hom}(\boldsymbol{\mathcal{L}}(\sigma),\boldsymbol{\mathcal{L}}(\sigma))$-valued function $\mathbf{S}(\sigma:s)$ by

$$\mathbf{S}(\sigma:s):=r_{\bar{P}_0|P_0}(\sigma:s)\mathbf{T}(\sigma:s),$$

where $\mathbf{T}(\sigma:s)$ is as in Proposition 4.1. Then, since $\mathbf{C}(\sigma:0)$ is defined and invertible, it follows from Proposition 4.1 that $\mathbf{S}(\sigma:s)$ is defined at $s=0$ and that

$$\mathbf{I}_\sigma^{-1}\mathbf{C}(\sigma:0)\mathbf{I}_\sigma=\mathbf{S}(\sigma:0)\otimes R_{P_0}(\sigma:0),$$

where $\mathbf{I}_\sigma$ is as in (4.4). Using (4.10) and (6.15) it follows that

$$\mathbf{S}(\sigma:0)^*=\mathbf{S}(\sigma:0);\quad \mathbf{S}(\sigma:0)^{-1}=\mathbf{S}(\sigma:0).$$

Hence $\mathbf{S}(\sigma:0)$ is diagonalizable with eigenvalues ±1. Putting everything together, it follows that there exist natural numbers $c_1(\sigma),c_2(\sigma)$ with $c_1(\sigma)+c_2(\sigma)=p\dim(\sigma)$ such that one has

$$R_\sigma(\alpha)=\frac{c_1(\sigma)-c_2(\sigma)}{4}\Theta_{\sigma,0}(\alpha) \tag{6.16}$$

for every K-finite Schwarz function α. Now one defines

$$R(\alpha):=\sum_{\sigma\in\hat{M}}R_\sigma(\alpha). \tag{6.17}$$

This sum is finite since α is K-finite.
Finally, we let

$$R_{\mathrm{u}}(t,\nu) := \sum_{\substack{\sigma\in\hat{M}\\ [\nu:\sigma]\neq 0}} e^{tc(\sigma)} p \dim(\sigma)\left(\frac{\log u}{\sqrt{4\pi t}} + \frac{1}{4} + \frac{1}{2\pi}\int_0^\infty e^{-t\lambda^2}\frac{(d-1)}{\frac{(d-1)^2}{4}+\lambda^2}d\lambda\right). \qquad (6.18)$$

Then by Proposition 5.7, $R_{\mathrm{u}}(t,\nu)$ enters in the formula for $\mathrm{Tr}_{\mathrm{rel}}(e^{-A_\nu})$. However, $R^{\nu}_{t,u}$ only depends on the parameter u and on the number p of cusps of X. Now we can state the trace formula and the relative trace formula.

Theorem 6.1. *Let α be a K-finite Schwarz function. Then one has*

$$\mathrm{Tr}\left(\pi_{\Gamma,d}(\alpha)\right) = I(\alpha) + H(\alpha) + T(\alpha) + \mathcal{I}(\alpha) + R(\alpha) + \mathcal{S}(\alpha). \qquad (6.19)$$

Let $\nu \in \hat{K}$ and let A_ν be the differential operator from section 5.1. Then one has

$$\mathrm{Tr}_{\mathrm{rel,u}}(e^{-tA_\nu}) = I(h^\nu_t) + H(h^\nu_t) + T(h^\nu_t) + \mathcal{I}(h^\nu_t) + J(h^\nu_t) + R_{\mathrm{u}}(t,\nu). \qquad (6.20)$$

Proof. Equation (6.19) is a special case of the invariant trace formula stated in [Ho2, Theorem 6.4]. It follows if one combines [War1, Theorem 8.4], the Theorem on page 299 in [OW], (6.7) and (6.14). Here one has to take into account that our normalizations are different from those of [OW]. Equation (6.20) follows from Proposition 5.7, equation (6.14) and equation (6.19). □

6.2 The Fourier transform of the distribution $\mathcal{I}$

The Fourier transform of the distribution $\mathcal{I}$ was computed in [Ho1]. We will state his result and draw some consequences of it. For $\sigma \in \hat{M}$ with highest weight $k_2(\sigma)e_2 + \cdots + k_{n+1}(\sigma)e_{n+1}$ and $\lambda \in \mathbb{R}$ define $\lambda_\sigma \in (\mathfrak{h})^*_{\mathbb{C}}$ by

$$\lambda_\sigma := i\lambda e_1 + \sum_{j=2}^{n+1}(k_j(\sigma)+\rho_j)e_j.$$

Let $S(\mathfrak{b}_{\mathbb{C}})$ be the symmetric algebra of $\mathfrak{b}_{\mathbb{C}}$. Define $\Pi \in S(\mathfrak{b}_{\mathbb{C}})$ by

$$\Pi := \prod_{\alpha\in\Delta^+(\mathfrak{m}_{\mathbb{C}},\mathfrak{b}_{\mathbb{C}})} H_\alpha. \qquad (6.21)$$

Let $\xi \in \mathfrak{b}^*_{\mathbb{C}}$, $\xi = \xi_2 e_2 + \cdots + \xi_{n+1}e_{n+1}$. Then it follows from (2.4) that

$$\Pi(\xi) = \prod_{2\leq i<j\leq n+1}(\xi_i - \xi_j)(\xi_i + \xi_j). \qquad (6.22)$$

If τ is any permutation of $\{2,\dots,n+1\}$ and

$$\xi_\tau := \xi_2 e_{\tau(2)} + \dots + \xi_{n+1} e_{\tau(n+1)}$$

it follows from (6.22) that

$$\Pi(\xi_\tau) = \pm\Pi(\xi). \tag{6.23}$$

The restriction of the Killing form to $\mathfrak{h}_\mathbb{C}$ defines a non-degenerate symmetric bilinear form. We will identify $\mathfrak{h}_\mathbb{C}^*$ with $\mathfrak{h}_\mathbb{C}$ via this form and denote the induced symmetric bilinear form on $\mathfrak{h}_\mathbb{C}^*$ by $\langle\cdot,\cdot\rangle$. Then for $\alpha \in \Delta^+(\mathfrak{g}_\mathbb{C},\mathfrak{h}_\mathbb{C})$ we denote by $s_\alpha : \mathfrak{h}_\mathbb{C}^* \to \mathfrak{h}_\mathbb{C}^*$ the reflection $s_\alpha(x) = x - 2\frac{\langle x,\alpha\rangle}{\langle\alpha,\alpha\rangle}\alpha$. Now the Fourier transform of $\mathcal{I}$ is computed as follows.

Theorem 6.2. *For every K-finite $\alpha \in \mathcal{C}^2(G)$ one has*

$$\mathcal{I}(\alpha) = \frac{p}{4\pi}\sum_{\sigma\in\hat{M}}\int_\mathbb{R}\Omega(\check{\sigma},-\lambda)\Theta_{\sigma,\lambda}(\alpha)d\lambda,$$

where

$$\Omega(\sigma,\lambda) := -2\dim(\sigma)\gamma - \frac{1}{2}\sum_{\alpha\in\Delta^+(\mathfrak{g}_\mathbb{C},\mathfrak{a}_\mathbb{C})}\frac{\Pi(s_\alpha\lambda_\sigma)}{\Pi(\rho_M)}\left(\psi(1+\lambda_\sigma(H_\alpha)) + \psi(1-\lambda_\sigma(H_\alpha))\right).$$

Here ψ denotes the digamma function and γ denotes the Euler-Mascheroni constant. Moreover $\check{\sigma}$ denotes the contragredient representation of σ and Π is as in (6.21).

Proof. This follows from [Ho1, Theorem 5], [Ho1, Theorem 6], [Ho1, Corollary on page 96]. □

For later purposes, we shall now transform the functions $\Omega(\lambda,\sigma)$ a little bit further. We start with the following elementary lemma.

Lemma 6.3. *One has*

$$\sum_{\alpha\in\Delta^+(\mathfrak{g}_\mathbb{C},\mathfrak{a}_\mathbb{C})}\frac{\Pi(s_\alpha\lambda_\sigma)}{\Pi(\rho_M)} = 2\dim\sigma.$$

Proof. This is proved in [Ho1, page 95] but can also be seen as follows. Write $\Lambda(\sigma)+\rho_M = \xi_2 e_2 + \dots + \xi_{n+1}e_{n+1}$. Then if $\alpha = e_1 \pm e_j$, one has

$$s_\alpha(\lambda_\sigma) = \mp\xi_j e_1 + \xi_2 e_2 + \dots + \xi_{j-1}e_{j-1} \mp i\lambda e_j + \xi_{j+1}e_{j+1} + \dots + \xi_{n+1}e_{n+1}. \tag{6.24}$$

Using (2.14) and (6.22) it follows that

$$\Pi(s_{e_1+e_j}(\lambda_\sigma)) = \Pi(s_{e_1-e_j}(\lambda_\sigma));\quad \Pi(s_{e_1+e_j}(\lambda_\sigma)) = \Pi(s_{e_1+e_j}(\lambda_{w_0\sigma})). \tag{6.25}$$

Thus by (2.12) and (6.22) for $\alpha = e_1 \pm e_j$ one gets

$$\begin{aligned}\frac{\Pi(s_\alpha(\lambda_\sigma))}{\Pi(\rho_M)} &= \frac{(-1)^j}{\Pi(\rho_M)} \prod_{\substack{2\le k<l\le n+1\\ k,l\neq j}} (\xi_k^2 - \xi_l^2) \prod_{\substack{p=2\\ p\neq j}}^{n+1} \left(-\lambda^2 - \xi_p^2\right) \\ &= \frac{1}{\Pi(\rho_M)} \prod_{2\le k<l\le n+1} (\xi_k^2 - \xi_l^2) \prod_{\substack{p=2\\ p\neq j}}^{n+1} \frac{-\lambda^2 - \xi_p^2}{\xi_j^2 - \xi_p^2} \\ &= \dim(\sigma) \prod_{\substack{p=2\\ p\neq j}}^{n+1} \frac{-\lambda^2 - \xi_p^2}{\xi_j^2 - \xi_p^2}. \end{aligned} \tag{6.26}$$

Now the expression

$$\sum_{j=2}^{n+1} \prod_{\substack{p=2\\ p\neq j}}^{n+1} \frac{s - \xi_p^2}{\xi_j^2 - \xi_p^2}$$

is a polynomial in s of degree $n-1$ which is equal to 1 at the n different points $\xi_2^2, \dots, \xi_{n+1}^2$. Thus one has

$$\sum_{j=2}^{n+1} \prod_{\substack{p=2\\ p\neq j}}^{n+1} \frac{-\lambda^2 - \xi_p^2}{\xi_j^2 - \xi_p^2} = 1$$

for every λ. This proves the lemma. □

For $j = 2, \dots, n+1$ and $\lambda \in \mathbb{C}$ let

$$P_j(\sigma, \lambda) := \frac{\Pi(s_{e_1+e_j}\lambda_\sigma)}{\Pi(\rho_M)}. \tag{6.27}$$

Then if σ is of highest weight $k_2(\sigma)e_2 + \dots + k_{n+1}(\sigma)e_{n+1}$ as in (2.9) it follows from (6.26) that

$$P_j(\sigma, \lambda) = \dim(\sigma) \prod_{\substack{p=2\\ p\neq j}}^{n+1} \frac{-\lambda^2 - (k_p(\sigma) + \rho_p)^2}{(k_j(\sigma) + \rho_j)^2 - (k_p(\sigma) - \rho_p)^2}. \tag{6.28}$$

In particular $P_j(\sigma, \lambda)$ is an even polynomial in λ of degree $2n-2$.

Proposition 6.4. *Let $\sigma \in \hat{M}$ be of highest weight $k_2(\sigma)e_2 + \dots + k_{n+1}(\sigma)e_{n+1}$. Then one has*

$$\Omega(\lambda, \sigma) = \Omega(\lambda, w_0\sigma) = \Omega(\lambda, \check{\sigma}); \quad \Omega(\lambda, \sigma) = \Omega(-\lambda, \sigma). \tag{6.29}$$

Moreover one can write

$$\Omega(\lambda,\sigma)=\Omega_1(\lambda,\sigma)+\Omega_2(\lambda,\sigma),$$

where $\Omega_1(\lambda,\sigma)$ *and* $\Omega_2(\lambda,\sigma)$ *are defined as follows. If all* $k_j(\sigma)$ *are non-negative integers, let* $m_0 := m_0(\sigma) := k_{n+1}(\sigma)$. *Then one puts*

$$\Omega_1(\lambda,\sigma) := -\dim(\sigma)\left(2\gamma+\psi(1+i\lambda)+\psi(1-i\lambda)+\sum_{1\le l<m_0}\frac{2l}{l^2+\lambda^2}\right)$$

and

$$\Omega_2(\lambda,\sigma) := -\sum_{j=2}^{n+1}P_j(\sigma,\lambda)\left(\sum_{\substack{m_0\le l\\ <k_j(\sigma)+\rho_j}}\frac{2l}{l^2+\lambda^2}+\frac{(k_j(\sigma)+\rho_j)}{(k_j(\sigma)+\rho_j)^2+\lambda^2}\right).$$

If all $k_j(\sigma)$ *are positive half integers let* $m_0 := m_0(\sigma) := k_{n+1}(\sigma)-1/2$. *Then one puts*

$$\Omega_1(\lambda,\sigma) := -\dim(\sigma)\left(2\gamma+\psi\left(\frac{1}{2}+i\lambda\right)+\psi\left(\frac{1}{2}-i\lambda\right)+\sum_{0\le l<m_0}\frac{2\,(l+1/2)}{(l+1/2)^2+\lambda^2}\right)$$

and

$$\Omega_2(\lambda,\sigma) := -\sum_{j=2}^{n+1}P_j(\sigma,\lambda)\left(\sum_{\substack{m_0\le l\\ <k_j(\sigma)+\rho_j-1/2}}\frac{2\,(l+1/2)}{(l+1/2)^2+\lambda^2}+\frac{(k_j(\sigma)+\rho_j)}{(k_j(\sigma)+\rho_j)^2+\lambda^2}\right).$$

Finally, if $k_{n+1}(\sigma)<0$ *one puts* $\Omega_1(\sigma,\lambda)=\Omega_1(w_0\sigma,\lambda)$, $\Omega_2(\sigma,\lambda)=\Omega_2(w_0\sigma,\lambda)$.

Proof. Let $j\in\{2,\dots,n+1\}$. We have

$$\lambda_\sigma(H_{e_1\pm e_j}) = i\lambda \pm (k_j(\sigma)+\rho_j)\,. \tag{6.30}$$

Now recall that $\rho_{n+1}=0$ and that the highest weight of $w_0\sigma$ is given by $k_2(\sigma)e_2+\dots+k_n(\sigma)e_n-k_{n+1}(\sigma)e_{n+1}$. Moreover recall that for $M=\mathrm{Spin}(n)$ one has $\check{\sigma}\cong\sigma$ if n is odd and $\check{\sigma}\cong w_0\sigma$ if n is even. Thus (6.25) and (6.30) imply (6.29). We can assume that $k_{n+1}(\sigma)\ge 0$. Assume first that all $k_j(\sigma)$ are integers. Then using $\psi(z+1)=\frac{1}{z}+\psi(z)$, (6.25) and (6.30) we obtain

$$\begin{aligned}
&\frac{\Pi(s_{e_1+e_j}\lambda_\sigma)}{\Pi(\rho_M)}\left(\psi(1+\lambda_\sigma(H_{e_1+e_j}))+\psi(1-\lambda_\sigma(H_{e_1+e_j}))\right)\\
&+\frac{\Pi(s_{e_1-e_j}\lambda_\sigma)}{\Pi(\rho_M)}\left(\psi(1+\lambda_\sigma(H_{e_1-e_j}))+\psi(1-\lambda_\sigma(H_{e_1-e_j}))\right)\\
&=2\frac{\Pi(s_{e_1+e_j}\lambda_\sigma)}{\Pi(\rho_M)}\Bigg(\psi(1+i\lambda)+\psi(1-i\lambda)+\sum_{1\le l<m_0}\frac{2l}{l^2+\lambda^2}\\
&\quad+\sum_{m_0\le l<k_j(\sigma)+\rho_j}\frac{2l}{l^2+\lambda^2}+\frac{(k_j(\sigma)+\rho_j)}{(k_j(\sigma)+\rho_j)^2+\lambda^2}\Bigg).
\end{aligned}$$

Using Lemma 6.3 and (6.25) the proposition follows. If all $k_j(\sigma)$ are half integers, one proceeds similarly. $\square$

The following considerations will be needed in order to show that the residues of the logarithmic derivative of the symmetrized Selberg zeta function are integral.

Lemma 6.5. *Let $\sigma \in \hat{M}$ with highest weight $k_2(\sigma)e_2 + \cdots + k_{n+1}(\sigma)e_{n+1}$ as in (2.9). Assume that all $k_j(\sigma)$ are integers with $k_{n+1}(\sigma) \geq 0$. For every $j \in \{2, \ldots, n+1\}$ and every $l \in \mathbb{N}$ with $k_{n+1}(\sigma) \leq l \leq k_j(\sigma) + \rho_j$ let*

$$\xi_{j,l}(\sigma) := \sum_{2 \leq i < j} (k_i(\sigma) + \rho_i)e_i + le_j + \sum_{j < i \leq n+1} (k_i(\sigma) + \rho_i)e_i.$$

Then $\frac{\Pi(\xi_{j,l}(\sigma))}{\Pi(\rho_M)}$ is an integer.
Assume that all $k_j(\sigma)$ are half-integers with $k_{n+1}(\sigma) > 0$. For every $j \in \{2, \ldots, n+1\}$ and for every $l \in \mathbb{N}$ with $k_{n+1}(\sigma) - 1/2 \leq l \leq k_j(\sigma) + \rho_j - 1/2$ let

$$\xi_{j,l}(\sigma) := \sum_{2 \leq i < j} (k_i(\sigma) + \rho_i)e_i + (l + 1/2)e_j + \sum_{j < i \leq n+1} (k_i(\sigma) + \rho_i)e_i.$$

Then $\frac{\Pi(\xi_{j,l}(\sigma))}{\Pi(\rho_M)}$ is an integer.

Proof. Suppose first that $k_{n+1}(\sigma) = l$. If $j < n+1$, one has $\Pi(\xi_{j,l}) = 0$ by (6.22). If $j = n+1$, one has

$$\frac{\Pi\left(\xi_{j,l}(\sigma)\right)}{\Pi(\rho_M)} = \dim(\sigma)$$

by the Weyl dimension formula (2.12).
Now suppose that $l > k_{n+1}(\sigma)$. Then there exists a minimal $\nu \in \{2, \ldots, n\}$ such that $l - \rho_\nu \geq k_{\nu+1}(\sigma)$. We have $\nu \geq j$. If $\nu > 2$ we have $l - \rho_{\nu-1} < k_\nu(\sigma)$. Moreover, if for $\nu > 2$ we have $l - \rho_{\nu-1} = k_\nu(\sigma) - 1$, we have $l = k_\nu(\sigma) + \rho_\nu$ and so in this case we have

$$\frac{\Pi\left(\xi_{j,l}(\sigma)\right)}{\Pi(\rho_M)} = \dim(\sigma)$$

for $\nu = j$ and $\Pi\left(\xi_{j,l}(\sigma)\right) = 0$ for $\nu > j$ by (2.12) and (6.22). Thus it remains to consider the case that for $\nu > 2$ one has $k_\nu(\sigma) - 1 \geq l - \rho_{\nu-1} + 1 = l - \rho_\nu$. Then we define $\Lambda' \in \mathfrak{b}_{\mathbb{C}}^*$ by

$$\Lambda' := \sum_{2 \leq i < j} k_i(\sigma)e_i + \sum_{j < i \leq \nu} \left(k_i(\sigma) - 1\right)e_{i-1} + (l - \rho_\nu)e_\nu + \sum_{\nu < i \leq n+1} k_i(\sigma)e_i.$$

It follows from (2.9) that Λ' is the highest weight $\Lambda(\sigma')$ of a representation σ' of M. By the Weyl dimension formula (2.12) and by (6.23) one has

$$\dim(\sigma') = \frac{\Pi(\Lambda(\sigma') + \rho_M)}{\Pi(\rho)} = \pm \frac{\Pi\left(\xi_{j,l}(\sigma)\right)}{\Pi(\rho_M)}.$$

If all $k_i(\sigma)$ are half integers, one proceeds in the same way. $\square$

The preceding lemma gives the following corollary.

Corollary 6.6. *Let $\sigma \in \hat{M}$ with highest weight $k_2(\sigma)e_2 + \cdots + k_{n+1}(\sigma)e_{n+1}$ as in* (2.9)*. Assume that all $k_j(\sigma)$ are integers. For every $2 \leq j \leq n+1$ and every l with $|k_{n+1}(\sigma)| \leq l \leq |k_j(\sigma)| + \rho_j$ let*

$$c_{j,l}(\sigma) := P_j(\sigma, il)$$

Then $c_{j,l}(\sigma)$ is an integer.
Assume that all $k_j(\sigma)$ are non-zero half integers. For every $2 \leq j \leq n+1$ and every l with $|k_{n+1}(\sigma)| - 1/2 \leq l \leq |k_j(\sigma)| + \rho_j - 1/2$ let

$$c_{j,l}(\sigma) := P_j\left(\sigma, i(l+1/2)\right).$$

Then $c_{j,l}(\sigma)$ is an integer.

Proof. By (6.26) we can assumme that $k_{n+1}(\sigma) \geq 0$. The corollary follows from equation (6.24) and Lemma 6.5. □

All in all, we have obtained the following proposition.

Proposition 6.7. *Let $\sigma \in \hat{M}$ with highest weight $k_2(\sigma)e_2 + \cdots + k_{n+1}(\sigma)e_{n+1}$. Assume that all $k_j(\sigma)$ are integers. Let $m_0 := |k_{n+1}(\sigma)|$. Then there exists an even polynomial $Q(\sigma, \lambda)$ of degree $\leq 2n-4$ and for every $j = 2, \ldots, n+1$, every l with $m_0 \leq l < |k_j(\sigma)| + \rho_j - 1$ there exist integers $c_{j,l}(\sigma)$ such that*

$$\begin{aligned}\Omega(\sigma,\lambda) = &-\dim(\sigma)\left(2\gamma + \psi(1+i\lambda) + \psi(1-i\lambda) + \sum_{1\leq l<m_0} \frac{2l}{\lambda^2+l^2}\right)\\ &-\sum_{j=2}^{n+1} \sum_{\substack{m_0\leq l< \\ |k_j(\sigma)|+\rho_j}} c_{j,l}(\sigma)\left(\frac{1}{l+i\lambda} + \frac{1}{l-i\lambda}\right)\\ &-\sum_{j=2}^{n+1} \frac{\dim(\sigma)}{2}\left(\frac{1}{|k_j(\sigma)|+\rho_j+i\lambda} + \frac{1}{|k_j(\sigma)|+\rho_j-i\lambda}\right) - Q(\sigma,\lambda).\end{aligned}$$

Assume that all $k_j(\sigma)$ are non-zero half integers. Let $m_0 := |k_{n+1}(\sigma)| - 1/2$. Then there exists an even polynomial $Q(\sigma, \lambda)$ of degree $\leq 2n-4$ and for every $j = 2, \ldots, n+1$, every l with $m_0 \leq l < |k_j(\sigma)| + \rho_j - 1/2$ there exist integers $c_{j,l}(\sigma)$ such that

$$\begin{aligned}\Omega(\sigma,\lambda) = &-\dim(\sigma)\left(2\gamma + \psi\left(\frac{1}{2}+i\lambda\right) + \psi\left(\frac{1}{2}-i\lambda\right) + \sum_{0\leq l<m_0} \frac{2(l+1/2)}{(l+1/2)^2+\lambda^2}\right)\\ &-\sum_{j=2}^{n+1} \sum_{\substack{m_0\leq l< \\ |k_j(\sigma)|+\rho_j-1/2}} c_{j,l}(\sigma)\left(\frac{1}{l+1/2+i\lambda} + \frac{1}{l+1/2-i\lambda}\right)\\ &-\sum_{j=2}^{n+1} \frac{\dim(\sigma)}{2}\left(\frac{1}{|k_j(\sigma)|+\rho_j+i\lambda} + \frac{1}{|k_j(\sigma)|+\rho_j-i\lambda}\right) - Q(\sigma,\lambda).\end{aligned}$$

For every $\sigma \in \hat{M}$ one has $Q(\sigma, \lambda) = Q(w_0\sigma, \lambda)$.

Proof. Assume that all $k_j(\sigma)$ are integers. For every $j = 2, \dots, n+1$, $P_j(\sigma, \lambda)$ is an even polynomial. Thus for every l with $m_0 \leq l \leq |k_j(\sigma)| + \rho_j$ we can write

$$\frac{P_j(\sigma,\lambda)2l}{l^2+\lambda^2} = Q_{j,l}(\sigma,\lambda) + c_{j,l}(\sigma)\,\frac{2l}{l^2+\lambda^2},$$

where

$$Q_{j,l}(\sigma,\lambda) := \frac{P_j(\sigma,\lambda) - P_j(\sigma,il)}{l+i\lambda} + \frac{P_j(\sigma,\lambda) - P_j(\sigma,il)}{l-i\lambda} \tag{6.31}$$

is an even polynomial and where

$$c_{j,l}(\sigma) := P_j(\sigma, il) \tag{6.32}$$

is integral by corollary 6.6. Using (2.14) and (6.28) it follows that

$$Q_{j,l}(\sigma,\lambda) = Q_{j,l}(w_0\sigma,\lambda).$$

Using (6.28) it follows that

$$P_j\left(\sigma, i(|k_j(\sigma)| + \rho_j)\right) = \dim(\sigma).$$

Thus if one lets

$$Q(\sigma,\lambda) := \sum_{j=2}^{n+1} \sum_{\substack{m_0 \leq l \\ < |k_j(\sigma)|+\rho_j}} Q_{j,l}(\sigma,\lambda) + \frac{1}{2} \sum_{\substack{l=|k_j(\sigma)|+\rho_j \\ 2\leq j\leq n+1}} Q_{j,l}(\sigma,\lambda), \tag{6.33}$$

the proposition follows from Proposition 6.4. If all $k_j(\sigma)$ are half-integers one proceeds in the same way. □

7 The Selberg zeta function

7.1 The symmetric Selberg zeta function

Let $\sigma \in \hat{M}$. For $\mathrm{Re}(s) > 8n$ we define the symmetric Selberg zeta function by

$$S(s,\sigma) := \begin{cases} Z(s,\sigma)Z(s,w_0\sigma), & \sigma \neq w_0\sigma; \\ Z(s,\sigma), & \sigma = w_0\sigma, \end{cases}$$

where $Z(s,\sigma)$ is the Selberg zeta function defined in Definition 3.12. In this section we want to prove that $S(s,\sigma)$ has a meromorphic continuation to $\mathbb{C}$.

By Proposition 2.17 there exist unique integers $m_\nu(\sigma) \in \{-1, 0, 1\}$ which are zero except for finitely many $\nu \in \hat{K}$ such that for $\sigma = w_0\sigma$ one has

$$\sigma = \sum_{\nu \in \hat{K}} m_\nu(\sigma)\iota^*\nu \tag{7.1}$$

and such that for $\sigma \neq w_0\sigma$ one has

$$\sigma + w_0\sigma = \sum_{\nu \in \hat{K}} m_\nu(\sigma)\iota^*\nu. \tag{7.2}$$

Let $E(\sigma)$ be the bundle

$$E(\sigma) := \bigoplus_{\substack{\nu \\ m_\nu(\sigma) \neq 0}} E_\nu. \tag{7.3}$$

Then $E(\sigma)$ has a grading

$$E(\sigma) = E^+(\sigma) \oplus E^-(\sigma) \tag{7.4}$$

defined by the sign of $m_\nu(\sigma)$. For every $\nu \in \hat{K}$ let A_ν be the operator on $C^\infty(X, E_\nu)$ induced by $-\Omega$. Let $A(\sigma)$ be the operator acting on $C^\infty(X, E(\sigma))$ defined by

$$A(\sigma) := \bigoplus_{\substack{\nu \\ m_\nu(\sigma) \neq 0}} A_\nu + c(\sigma),$$

where $c(\sigma)$ is as in (2.27). Let

$$\tilde{E}(\sigma) := \bigoplus_{\substack{\nu \\ m_\nu(\sigma) \neq 0}} \tilde{E}_\nu, \tag{7.5}$$

where $\tilde{E}_\nu$ is as in section 5.1. Then $\tilde{E}(\sigma)$ has again a grading defined by the sign of $m_\nu(\sigma)$. Let $\tilde{A}(\sigma)$ be the lift of $A(\sigma)$ to $\tilde{E}(\sigma)$. Let

$$h_t^\sigma(g) := e^{-tc(\sigma)} \sum_{\substack{\nu \\ m_\nu(\sigma) \neq 0}} m_\nu(\sigma) h_t^\nu(g), \tag{7.6}$$

where h_t^ν is as in (5.9). Then by (7.2) and Proposition 5.1, for a principal series representation $\pi_{\sigma',\lambda}$, $\sigma' \in \hat{M}$, $\lambda \in \mathbb{R}$ we have

$$\Theta_{\sigma',\lambda}(h_t^\sigma) = e^{-t\lambda^2} \quad \text{for } \sigma' \in \{\sigma, w_0\sigma\}; \qquad \Theta_{\sigma',\lambda}(h_t^\sigma) = 0, \quad \text{otherwise.} \tag{7.7}$$

The following generalized resolvent formula is due to Bunke and Olbrich.

Proposition 7.1. *Let $s_1, \dots, s_N$ be complex numbers with $s_i^2 \neq s_{i'}^2$ for $i \neq i'$. Then for every $z \in \mathbb{C} - \{-s_1^2, \dots, -s_N^2\}$ one has*

$$\sum_{i=1}^{N} \frac{1}{s_i^2 + z} \prod_{\substack{i'=1 \\ i' \neq i}}^{N} \frac{1}{s_{i'}^2 - s_i^2} = \prod_{i=1}^{N} \frac{1}{s_i^2 + z}.$$

Proof. This is proved by Bunke and Olbrich, [BO, Lemma 3.5]. □

We will also need the following lemma.

Lemma 7.2. *Let $s_1, \dots, s_N$ be complex numbers with $s_i^2 \neq s_{i'}^2$ for $i \neq i'$. Then one has*

$$\sum_{i=1}^{N} e^{-ts_i^2} \prod_{\substack{i'=1 \\ i' \neq i}}^{N} \frac{1}{s_{i'}^2 - s_i^2} = O(t^{N-1}),$$

as $t \to 0$.

Proof. One has

$$\sum_{i=1}^{N} e^{-ts_i^2} \prod_{\substack{i'=1 \\ i' \neq i}}^{N} \frac{1}{s_{i'}^2 - s_i^2} = \left(\sum_{k=0}^{N-2} \frac{(-t)^k}{k!} \sum_{i=1}^{N} s_i^{2k} \prod_{\substack{i'=1 \\ i' \neq i}}^{N} \frac{1}{s_{i'}^2 - s_i^2} + f(t) \right),$$

where $f(t)$ depends on $s_1, \dots, s_N$ and satisfies

$$f(t) = O(t^{N-1}),\ t \to 0.$$

By [BO, Lemma 3.6], correcting some misprints, for every $0 \leq k \leq N-2$ one has

$$\sum_{i=1}^{N} s_i^{2k} \prod_{\substack{i'=1 \\ i' \neq i}}^{N} \frac{1}{s_{i'}^2 - s_i^2} = 0.$$

This proves the proposition. □

By (5.3) and the definition of $A(\sigma)$ there exists a $\lambda_0(\sigma) \in \mathbb{R}$ such that $A(\sigma) \geq \lambda_0(\sigma)$. Let $\lambda_0 < \lambda_1 < \dots$ be the eigenvalues of $A(\sigma)$, which might be either a finite or an infinte sequence. For each λ_k let $\mathcal{E}(\lambda_k)$ be the eigenspace of $A(\sigma)$ with eigenvalue λ_k. Let

$$m_{\mathrm{s}}(\lambda_k, \sigma) = \dim_{\mathrm{gr}} \mathcal{E}(\lambda_{\mathrm{k}}). \tag{7.8}$$

For $\lambda > 0$ let

$$N(\lambda) := \sum_{\lambda_k \leq \lambda} |m_{\mathrm{s}}(\lambda_k, \sigma)|$$

be the counting function of the eigenvalues of $A(\sigma)$. Then by Theorem I.1 in [Do1] and Theorem 9.1 in [Do2], there exists a constant C such that

$$|N(\lambda)| \leq C(1+\lambda)^{\frac{d}{2}}. \tag{7.9}$$

Now let $N \in \mathbb{N}$ with $N > d/2$ and choose distinct points $s_1, \ldots, s_N$, such that $\operatorname{Re}(s_i) > 8n$ and such that $\operatorname{Re}(s_i^2) > \max\{0, -\lambda_0(\sigma)\}$ for all i. Then by (7.9), the sum

$$\sum_k m_{\mathrm{s}}(\lambda_k,\sigma)\prod_{i=1}^{N}\frac{1}{\lambda_k+s_i^2}$$

converges absolutely. Let

$$A(\sigma)_d := \bigoplus_{\substack{\nu\in\hat{K}\\ m_\nu(\sigma)\neq 0}} A_\nu^d + c(\sigma),$$

where A_ν^d is as in section 5.1. Then by Propostition 7.1 we have

$$\sum_k\prod_{i=1}^{N} m_{\mathrm{s}}(\lambda_k,\sigma)\frac{1}{\lambda_k+s_i^2} = \int_0^\infty \sum_{i=1}^{N}\operatorname{Tr}_{\mathrm{s}} e^{-t\left(A(\sigma)_d+s_i^2\right)}\prod_{\substack{i'=1\\ i'\neq i}}^{N}\frac{1}{s_{i'}^2-s_i^2}dt, \tag{7.10}$$

where the super-trace is taken with respect to the grading defined in (7.4). We compute the right hand side of (7.10) using the invariant trace formula stated in equation (6.19) of Theorem 6.1. To save notation we let

$$\epsilon(\sigma) = 1, \text{ if } \sigma = w_0\sigma; \quad \epsilon(\sigma) = 2, \text{ if } \sigma \neq w_0\sigma. \tag{7.11}$$

We begin with the identity contribution. By (2.35), (6.1) and (7.7) one has

$$I(h_t^\sigma) = \epsilon(\sigma)\operatorname{vol}(X)\int_{\mathbb{R}} e^{-t\lambda^2}P_\sigma(\sqrt{-1}\lambda)d\lambda.$$

We now assure that the identity contribution can be integrated individually.

Lemma 7.3. *The integral*

$$\int_0^\infty\int_{\mathbb{R}}\sum_{i=1}^{N}P_\sigma(\sqrt{-1}\lambda)e^{-t(\lambda^2+s_i^2)}\prod_{\substack{i'=1\\ i'\neq i}}^{N}\frac{1}{s_{i'}^2-s_i^2}d\lambda dt \tag{7.12}$$

converges absolutely.

Proof. There exists an $\epsilon > 0$ such that for $t \to \infty$ one has

$$\int_{\mathbb{R}} \left| \sum_{i=1}^{N} P_\sigma(\sqrt{-1}\lambda) e^{-t(\lambda^2+s_i^2)} \prod_{\substack{i'=1\\ i'\neq i}}^{N} \frac{1}{s_{i'}^2 - s_i^2} \right| d\lambda = O(e^{-t\epsilon}).$$

On the other hand, by a change of variables one has

$$\int_{\mathbb{R}} \left| P_\sigma(\sqrt{-1}\lambda) e^{-t\lambda^2} \right| d\lambda = O(t^{-d/2}),$$

as $t \to +0$. Thus togther with Lemma 7.2 it follows that there exists an $\epsilon > 0$ such that

$$\int_{\mathbb{R}} \left| \sum_{i=1}^{N} P_\sigma(\sqrt{-1}\lambda) e^{-t(\lambda^2+s_i^2)} \prod_{\substack{i'=1\\ i'\neq i}}^{N} \frac{1}{s_{i'}^2 - s_i^2} \right| d\lambda = O(t^{-1+\epsilon}),$$

as $t \to +0$. This proves the Lemma. □

Now by Lemma 7.3 we can interchange the order of integration in (7.12) and thus using Proposition 7.1 one gets

$$\int_0^\infty \int_{\mathbb{R}} \sum_{i=1}^{N} P_\sigma(\sqrt{-1}\lambda) e^{-t(\lambda^2+s_i^2)} \prod_{\substack{i'=1\\ i'\neq i}}^{N} \frac{1}{s_{i'}^2 - s_i^2} d\lambda dt = \int_{\mathbb{R}} P_\sigma(\sqrt{-1}\lambda) \prod_{i=1}^{N} \frac{1}{\lambda^2 + s_i^2} d\lambda.$$

Thus, since $P_\sigma(z)$ is an even polynomial, one computes

$$\int_0^\infty \sum_{i=1}^{N} I(h_t^\sigma) e^{-ts_i^2} \prod_{\substack{i'=1\\ i'\neq i}}^{N} \frac{1}{s_{i'}^2 - s_i^2} dt = \epsilon(\sigma)\pi \mathrm{vol}(X) \sum_{i=1}^{N} P_\sigma(s_i) \frac{1}{s_i} \prod_{\substack{i'=1\\ i'\neq i}}^{N} \frac{1}{s_{i'}^2 - s_i^2}. \tag{7.13}$$

Next we come to the hyperbolic contribution. For $\gamma \in \mathrm{C}(\Gamma)_{\mathrm{s}}$ let

$$L_{\mathrm{sym}}(\gamma,\sigma) := L(\gamma,\sigma) \text{ if } \sigma = w_0\sigma; \quad L_{\mathrm{sym}}(\gamma,\sigma) = L(\gamma,\sigma) + L(\gamma, w_0\sigma) \text{ if } \sigma \neq w_0\sigma. \tag{7.14}$$

Here $L(\gamma,\sigma)$ is as in (6.2). Then using (6.3) we obtain

$$\int_0^\infty \sum_{i=1}^{N} H(h_t^\sigma) e^{-ts_i^2} \prod_{\substack{i'=1\\ i'\neq i}}^{N} \frac{1}{s_{i'}^2 - s_i^2} dt = \sum_{i=1}^{N} \sum_{[\gamma]\in \mathrm{C}(\Gamma)_{\mathrm{s}} - [1]} \ell(\gamma_0) L_{\mathrm{sym}}(\gamma,\sigma) e^{-s_i l(\gamma)} \frac{1}{2s_i} \prod_{\substack{i'=1\\ i'\neq i}}^{N} \frac{1}{s_{i'}^2 - s_i^2}.$$

Since $\mathrm{Re}(s_i) > 8n$ for every i, the sum on the right hand side converges absolutely by (3.14) and Proposition 3.8 and thus the integral exists. By [GW, section 3.2.5] if n is even we have $\check{\sigma} = \sigma$ and if n is odd we have $\check{\sigma} = w_0\sigma$ for every $\sigma \in \hat{M}$. Thus, using (3.12) and the definition of $L(\gamma, \sigma)$ in (6.2), we obtain

$$\int_0^\infty \sum_{i=1}^N H(h_t^\sigma) e^{-ts_i^2} \prod_{\substack{i'=1 \\ i' \neq i}}^N \frac{1}{s_{i'}^2 - s_i^2} dt = \sum_{i=1}^N \frac{d}{ds}\Big|_{s=s_i} \log S(s,\sigma) \frac{1}{2s_i} \prod_{\substack{i'=1 \\ i' \neq i}}^N \frac{1}{s_{i'}^2 - s_i^2}. \tag{7.15}$$

We now compute the contribution of the distribution $\mathcal{I}$. It will be shown in section 8.1 that $\mathcal{I}(h_t^\sigma) = O(t^{-\frac{d-1}{2}})$, as $t \to +0$. Thus, since $\mathcal{I}(h_t^\sigma)$ is bounded for $t \to \infty$, it follows together with Lemma 7.2 that the integral

$$\int_0^\infty \sum_{i=1}^N \mathcal{I}(h_t^\sigma) e^{-ts_i^2} \prod_{\substack{i'=1 \\ i' \neq i}}^N \frac{1}{s_{i'}^2 - s_i^2} dt$$

converges absolutely. We keep the notations of Proposition 6.7. If all $k_j(\sigma)$ are integers let

$$\begin{aligned}\mathcal{I}_1(\sigma; s_1, \dots, s_N) =& -\epsilon(\sigma)\frac{\dim(\sigma)p}{2} \sum_{i=1}^N \cdot \left(\psi(1+s_i) + \sum_{1 \leq l < m_0} \frac{1}{l+s_i}\right) \frac{1}{s_i} \prod_{\substack{i'=1 \\ i' \neq i}}^N \frac{1}{s_{i'}^2 - s_i^2} \\ & - \frac{p\epsilon(\sigma)}{2} \sum_{j=2}^{n+1} \sum_{\substack{m_0 \leq l \\ < |k_j(\sigma)| + \rho_j}} c_{j,l}(\sigma) \sum_{i=1}^N \frac{1}{l+s_i} \frac{1}{s_i} \prod_{\substack{i'=1 \\ i' \neq i}}^N \frac{1}{s_{i'}^2 - s_i^2}\end{aligned}$$

and if all $k_j(\sigma)$ are half-integers we let

$$\begin{aligned}\mathcal{I}_1(\sigma; s_1, \dots, s_N) =& -\epsilon(\sigma)\frac{\dim(\sigma)p}{2} \sum_{i=1}^N \left(\psi\left(\frac{1}{2}+s_i\right) + \sum_{0 \leq l < m_0} \frac{1}{l+1/2+s_i}\right) \frac{1}{s_i} \prod_{\substack{i'=1 \\ i' \neq i}}^N \frac{1}{s_{i'}^2 - s_i^2} \\ & - \frac{p\epsilon(\sigma)}{2} \sum_{j=2}^{n+1} \sum_{\substack{m_0 \leq l \\ < |k_j(\sigma)| + \rho_j - 1/2}} c_{j,l}(\sigma) \sum_{i=1}^N \frac{1}{l+1/2+s_i} \frac{1}{s_i} \prod_{\substack{i'=1 \\ i' \neq i}}^N \frac{1}{s_{i'}^2 - s_i^2}.\end{aligned}$$

For $\sigma \neq w_0\sigma$ we let

$$\mathcal{I}_2(\sigma; s_1, \dots, s_N) = -\epsilon(\sigma)\frac{\dim(\sigma)p}{4} \sum_{j=2}^{n+1} \sum_{i=1}^N \frac{1}{|k_j(\sigma)| + \rho_j + s_i} \frac{1}{s_i} \prod_{\substack{i'=1 \\ i' \neq i}}^N \frac{1}{s_{i'}^2 - s_i^2}$$

and for $\sigma = w_0\sigma$ we let

$$\mathcal{I}_2(\sigma; s_1, \dots, s_N) = -\epsilon(\sigma)\frac{\dim(\sigma)p}{4} \sum_{j=2}^{n} \sum_{i=1}^N \frac{1}{|k_j(\sigma)| + \rho_j + s_i} \frac{1}{s_i} \prod_{\substack{i'=1 \\ i' \neq i}}^N \frac{1}{s_{i'}^2 - s_i^2}.$$

Finally, we let

$$\begin{aligned}
&\mathcal{I}_3(\sigma; s_1, \dots, s_N) \\
:= &-\epsilon(\sigma)\frac{p}{4}\sum_{i=1}^{N} Q(\sigma, is_i)\frac{1}{s_i}\prod_{\substack{i'=1\\ i'\neq i}}^{N}\frac{1}{s_{i'}^2 - s_i^2} \;-\epsilon(\sigma)\frac{\dim(\sigma)\gamma p}{2}\sum_{i=1}^{N}\frac{1}{s_i}\prod_{\substack{i'=1\\ i'\neq i}}^{N}\frac{1}{s_{i'}^2 - s_i^2}.
\end{aligned}$$

Then using Theorem 6.2, Proposition 6.7, equation (7.7) and Proposition 7.1 we compute:

$$\begin{aligned}
&\int_0^\infty \sum_{i=1}^{N}\mathcal{I}(h_t^\sigma)e^{-ts_i^2}\prod_{\substack{i'=1\\ i'\neq i}}^{N}\frac{1}{s_{i'}^2 - s_i^2}dt \\
=&\mathcal{I}_1(\sigma; s_1, \dots, s_N) + \mathcal{I}_2(\sigma; s_1, \dots, s_N) + \mathcal{I}_3(\sigma; s_1, \dots, s_N).
\end{aligned} \tag{7.16}$$

Here the order of integration in the occuring integrals can be interchanged using Lemma 7.2 and the results of section 8.1 again. Next we treat the contribution of the distribution $\mathcal{S}$ defined in (6.13). For $\sigma \in \hat{M}$ let $\nu_\sigma \in \hat{K}$ be as in section 4.2. Moreover let $\{\beta(\sigma)\}$, $\{\beta(w_0\sigma)\}$, and $\{\eta(\sigma)\}$, $\{\eta(w_0\sigma)\}$ denote the poles of $\det\left(I(\sigma)\circ \mathbf{C}(\nu_\sigma : \sigma : s)\right)$ resp. of $\det\left(I(w\sigma)\circ \mathbf{C}(\nu_\sigma : w_0\sigma : s)\right)$ on $(0, n]$ and on $\{s \in \mathbb{C}\colon \operatorname{Re}(s) < 0\}$, counted with multiplicity divided by $\dim(\nu_\sigma)$ as in section 4.2. By the absolute convergence of the series in (4.21), by (4.20), (4.22) and (7.7) the integral

$$\int_0^\infty \sum_{i=1}^{N}\mathcal{S}(h_t^\sigma)e^{-ts_i^2}\prod_{\substack{i'=1\\ i'\neq i}}^{N}\frac{1}{s_{i'}^2 - s_i^2}dt$$

converges absolutely and one has

$$\begin{aligned}
&\int_0^\infty \sum_{i=1}^N \mathcal{S}(h_t^\sigma)e^{-ts_i^2}\prod_{\substack{i'=1\\ i'\neq i}}^N \frac{1}{s_{i'}^2-s_i^2}dt\\
=&\frac{\epsilon(\sigma)}{8}\sum_{\{\eta(\sigma)\}}\sum_{i=1}^N\left(\frac{1}{\eta(\sigma)-s_i}+\frac{1}{\bar{\eta}(\sigma)-s_i}\right)\frac{1}{s_i}\prod_{\substack{i'=1\\ i\neq i'}}^N\frac{1}{s_{i'}^2-s_i^2}\\
&+\frac{\epsilon(\sigma)}{4}\sum_{\{\beta(\sigma)\}}\sum_{i=1}^N\frac{1}{s_i+\beta(\sigma)}\frac{1}{s_i}\prod_{\substack{i'=1\\ i'\neq i}}^N\frac{1}{s_{i'}^2-s_i^2}\\
&+\frac{\epsilon(\sigma)}{8}\sum_{\{\eta(w_0\sigma)\}}\sum_{i=1}^N\left(\frac{1}{\eta(w_0\sigma)-s_i}+\frac{1}{\bar{\eta}(w_0\sigma)-s_i}\right)\frac{1}{s_i}\prod_{\substack{i'=1\\ i'\neq i}}^N\frac{1}{s_{i'}^2-s_i^2}\\
&+\frac{\epsilon(\sigma)}{4}\sum_{\{\beta(w_0\sigma)\}}\sum_{i=1}^N\frac{1}{s_i+\beta(w_0\sigma)}\frac{1}{s_i}\prod_{\substack{i'=1\\ i'\neq i}}^N\frac{1}{s_{i'}^2-s_i^2}\\
&+\frac{\epsilon(\sigma)}{8}\sum_{i=1}^N\left(\log q(\sigma)+\log q(w_0\sigma)\right)\frac{1}{s_i}\prod_{\substack{i'=1\\ i'\neq i}}^N\frac{1}{s_{i'}^2-s_i^2}\\
&+\frac{\epsilon(\sigma)}{4}\sum_{j=2}^{n+1}\sum_{i=1}^N\frac{p\dim(\sigma)}{|k_j(\sigma)|+\rho_j+s_i}\frac{1}{s_i}\prod_{\substack{i'=1\\ i'\neq i}}^N\frac{1}{s_{i'}^2-s_i^2}.
\end{aligned}$$

The residual contribution is nonzero only if $\sigma = w_0\sigma$ and using (6.16) it follows that in this case we have

$$\int_0^\infty \sum_{i=1}^N R(h_t^\sigma)e^{-ts_i^2}\prod_{\substack{i'=1\\ i'\neq i}}^N \frac{1}{s_{i'}^2-s_i^2}dt = \frac{c_1(\sigma)-c_2(\sigma)}{4}\sum_{i=1}^N\frac{1}{s_i^2}\cdot\prod_{i'=1}^N\frac{1}{s_{i'}^2-s_i^2},$$

where the existence of the integral is obvious. Finally, we treat the contribution of the distribution T. Using (6.9) and (7.7) we obtain

$$\int_0^\infty \sum_{i=1}^N T(h_t^\sigma)e^{-ts_i^2}\prod_{\substack{i'=1\\ i'\neq i}}^N \frac{1}{s_{i'}^2-s_i^2}dt = \frac{\epsilon(\sigma)\dim(\sigma)}{2}C(\Gamma)\sum_{i=1}^N\frac{1}{s_i}\prod_{\substack{i'=1\\ i'\neq i}}^N\frac{1}{s_{i'}^2-s_i^2}, \tag{7.17}$$

where the existence of the integral is obvious again. Put

$$c_\Gamma(\sigma) := \epsilon(\sigma)\left(\dim(\sigma)C(\Gamma)-\dim(\sigma)\gamma p\right).$$

If all $k_j(\sigma)$ are integers, we let

$$\Xi(s,\sigma) := \exp\left(2\pi\mathrm{vol}(X)\epsilon(\sigma)\int_0^s P_\sigma(r)dr - \epsilon(\sigma)\frac{p}{2}\int_0^s Q(\sigma,ir)dr + sc_\Gamma(\sigma)\right) \cdot (\Gamma(1+s))^{-p\epsilon(\sigma)\dim(\sigma)} \cdot S(s,\sigma).$$

If all $k_j(\sigma)$ are half integers, we let

$$\Xi(s,\sigma) := \exp\left(2\pi\mathrm{vol}(X)\epsilon(\sigma)\int_0^s P_\sigma(r)dr - \epsilon(\sigma)\frac{p}{2}\int_0^s Q(\sigma,ir)dr + sc_\Gamma(\sigma)\right) \cdot \left(\Gamma\left(\frac{1}{2}+s\right)\right)^{-p\epsilon(\sigma)\dim(\sigma)} \cdot S(s,\sigma).$$

Let λ_k be an eigenvalue of $A(\sigma)$ such that $\lambda_k \neq 0$. Then let

$$t_k^{\pm} = \pm i\sqrt{\lambda_k}. \tag{7.18}$$

Here for $\lambda_k < 0$ we choose the square root $\sqrt{\lambda_k}$ which has positive imaginary part. We now come to the main result of this section. We say that a meromorphic function $f : \mathbb{C} \to \mathbb{C}$ has a singularity of order $k \in \mathbb{Z}$ at $s_0 \in \mathbb{C}$ if the limit

$$\lim_{s\to s_0}(s-s_0)^{-k}f(s)$$

exists and is not zero.

Theorem 7.4. *Let $\sigma \in \hat{M}$. Then the symmetric Selberg zeta function has a meromorphic continuation to $\mathbb{C}$. Its singularities associated to spectral parameters are located as follows.*

1. *At the points $t_k^{\pm}$ of order $m_{\mathrm{s}}(\lambda_k,\sigma)$, where $m_{\mathrm{s}}(\lambda_k,\sigma)$ is as in (7.8) and the $t_k^{\pm}$ are as in (7.18).*

2. *At the point $s = 0$ of order $2m_{\mathrm{s}}(0,\sigma)$ if $\sigma \neq w_0\sigma$ and of order $2m_{\mathrm{s}}(0,\sigma) - c_1(\sigma)$ if $\sigma = w_0\sigma$. Here $c_1(\sigma)$ is as in (6.16).*

3. *At the points $-\beta(\sigma)$ of order $-\epsilon(\sigma)m(\beta(\sigma))$. Here $\beta(\sigma)$ are the poles of $\det(I(\sigma)\circ\mathbf{C}(\nu_\sigma : \sigma : s))$ on $(0,n]$ and $m(\beta(\sigma))$ is the corresponding multiplicity divided by $\dim(\nu_\sigma)$.*

4. *At the points $\eta(\sigma)$ of order $\epsilon(\sigma)m(\eta(\sigma))/2$. Here $\eta(\sigma)$ are the poles of $\det(I(\sigma)\circ\mathbf{C}(\nu_\sigma : \sigma : s))$ with negative real part and $m(\eta(\sigma))$ is the corresponding multiplicity divided by $\dim(\nu_\sigma)$.*

5. *At the points $\eta(w_0\sigma)$ of order $\epsilon(\sigma)m(\eta(w_0\sigma))/2$. Here $\eta(w_0\sigma)$ are the poles of $\det(I(w_0\sigma)\circ\mathbf{C}(\nu_\sigma : w_0\sigma : s))$ with negative real part and $m(\eta(w_0\sigma))$ is the corresponding multiplicity divided by $\dim(\nu_\sigma)$.*

Here $\epsilon(\sigma)$ is as in (7.11). If $\sigma = w_0\sigma$, the orders of the singularities at $\eta(\sigma)$ in item 4 and item 5 have to be added so that they are indeed integral.
Moreover, the symmetric Selberg zeta function has singularities which depend on Γ only via p, the number of cusps of Γ. If all $k_j(\sigma)$ are integers, they are located at the negative integers and if all $k_j(\sigma)$ are half integers, they are located at the negative half integers.
Explicitly, if all $k_j(\sigma)$ are integers, they are given as follows.

1. *At the points $-l$, $l \in \mathbb{N}$, $l \geq m_0$ of order $-p\epsilon(\sigma)\dim(\sigma)$.*
2. *At the points $-l$, $l \in \mathbb{N}$, $m_0 \leq l < |k_j(\sigma)| + \rho_j$, $j = 2, \ldots, n+1$ of order $p\epsilon(\sigma)c_{j,l}(\sigma)$.*

Here m_0 and the $c_{j,l}(\sigma)$ are as in Proposition 6.7. If all $k_j(\sigma)$ are half integers, they are given in the same way if $\mathbb{N}$ is replaced by $\frac{1}{2}\mathbb{N} - \mathbb{N}$.
The above enumeration exhausts all possible singularities of non-zero order of $S(s,\sigma)$. Here we use the convention that the orders of overlapping singularities in the enumeration add up.

Proof. Assume first that all $k_j(\sigma)$ are integers. We let $s_1 =: s$ and fix $s_2, \ldots, s_N$. Let

$$K(\sigma, s) = 0, \text{ if } \sigma \neq w_0\sigma;\ K(\sigma, s) = -\frac{c_1(\sigma)}{s}, \text{ if } \sigma = w_0\sigma.$$

By (4.10), η is a pole of $\det\left(I(\sigma) \circ \mathbf{C}(\nu_\sigma : \sigma : s)\right)$ if and only if $\bar{\eta}$ is a pole of $\det\left(I(w_0\sigma) \circ \mathbf{C}(\nu_\sigma : w_0\sigma : s)\right)$ and their orders are equal. Thus if we multiply both sides of (7.10) by

$$2s \prod_{i'=2}^{N} \left(s_{i'}^2 - s^2\right)$$

and use that for $\sigma = w_0\sigma$ we have

$$k_{n+1}(\sigma) = 0; \quad c_1(\sigma) + c_2(\sigma) = p\dim(\sigma),$$

we obtain using Propositon 7.1 and the above computations:

$$\begin{aligned}\frac{\Xi'(s,\sigma)}{\Xi(s,\sigma)} =& 2s \sum_k m_{\mathrm{s}}(\lambda_k, \sigma) \left(\frac{1}{s^2 + \lambda_k} + C(\lambda_k; s, s_2, \ldots, s_N) \right) \\ & + \sum_{1 \leq l < m_0} p\epsilon(\sigma)\dim(\sigma) \frac{1}{l+s} + \sum_{j=2}^{n+1} \sum_{\substack{m_0 \leq l < \\ |k_j(\sigma)| + \rho_j}} p\epsilon(\sigma) c_{j,l}(\sigma) \cdot \frac{1}{l+s} + K(\sigma, s) \\ & + \sum_{\eta(\sigma)} \left(\frac{\epsilon(\sigma)}{2(s - \eta(\sigma))} + \frac{\epsilon(\sigma)}{2(s - \bar{\eta}(\sigma))} \right) - \sum_{\beta(\sigma)} \frac{\epsilon(\sigma)}{s + \beta(\sigma)} - \frac{\epsilon(\sigma)\log q(\sigma)}{4} \\ & - \frac{\epsilon(\sigma)\log q(w_0\sigma)}{4} + sC(\sigma; s, s_2, \ldots, s_N). \end{aligned} \tag{7.19}$$

Here $C(\lambda_k; s, s_2, \ldots, s_N)$ and $C(\sigma; s, s_2, \ldots, s_N)$ are polynomials in s which are of degree at most $2(N-2)$ and which depend on the parameters. All infinite series involved converge absolutely and locally uniformly by the above arguments. Hence the logarithmic derivative of Ξ has a meromorphic continuation to $\mathbb{C}$. For $\lambda_k \neq 0$, one has

$$\frac{2s}{s^2+\lambda_k} = \frac{1}{s-t_k^+} + \frac{1}{s-t_k^-}.$$

The $c_{j,l}(\sigma)$ are integral by Corollary 6.6. Finally, for $\sigma = w_0\sigma$, $c_1(\sigma)$ is integral by construction. Hence the residues of the logarithmic derivatives of Ξ are integral. If all $k_j(\sigma)$ are half integers, we proceed in the same way. This proves the theorem. □

7.2 The twisted Dirac Operator on $\tilde{X}$

In this section we introduce certain twisted Dirac operators on $\tilde{X}$ and compute the Fourier transform of the corresponding heat-kernel. We keep the notations of section 2.4 Let $\mathrm{Cl}(\mathfrak{p})$ be the Clifford-algebra of $\mathfrak{p}$ with respect to the normalized Killing form, which defines a scalar product on $\mathfrak{p}$. We regard K as a multiplicative subgroup of $\mathrm{Cl}(\mathfrak{p})$ in such a way that the action of K on $\mathfrak{p} \subset \mathrm{Cl}(\mathfrak{p})$ by conjugation coincides with the action Ad of K on $\mathfrak{p}$, i.e. we fix a homogeneous spin-structure. Let $\mathrm{Cl}(\tilde{X}) := G \times_{\mathrm{Ad}} \mathrm{Cl}(\mathfrak{p})$ be the Clifford bundle of $\tilde{X}$, where Ad denotes the extension of Ad to a representation of K on $\mathrm{Cl}(\mathfrak{p})$. Let Δ^{2n} be the Spinor space as in [Fri, page 14] and let $\tilde{S} = G \times_\kappa \Delta^{2n}$ be the spinor bundle of $\tilde{X}$. Let

$$\mathrm{c} : \mathrm{Cl}(\mathfrak{p}) \otimes \Delta^{2n} \longrightarrow \Delta^{2n},\ X \otimes v \mapsto \mathrm{c}(X)v \tag{7.20}$$

denote the Clifford multiplication. Let H_1 be as in (2.3). Since M centralizes $\mathfrak{a}$, $\mathrm{c}(H_1)$ maps the spaces $\Delta^{2n}_{\pm}$ into themselves. Since $\kappa^{\pm}$ are irreducible representations of M, $\mathrm{c}(H_1)$ acts diagonally on $\Delta^{2n}_{\pm}$. It follows from the explicit construction of the Clifford multiplication [Fri, page 13-14] that $\mathrm{c}(H_1)$ can not act diagonally on Δ^{2n}. Thus, since $\mathrm{c}(H_1)^2 = -\mathrm{Id}$, there is an $\epsilon \in \{\pm 1\}$ such that $\epsilon\,\mathrm{c}(H_1)$ acts on the spaces $\Delta^{2n}_{\pm}$ with eigenvalues $\mp i$.
Let $\nu \in \hat{K}$ be an irreducible unitary representation of K on V_ν and let $\tilde{E}_\nu$ be the homogeneous vector-bundle as in section 5.1. Let $\tilde{D}_\nu$ be the twisted Dirac operator on $\tilde{E}_\nu \otimes \tilde{S}$ multiplied by ϵ. Let $C^\infty(G, \kappa \otimes \nu)$ be as in (5.1). We identify the smooth sections of $\tilde{E}_\nu \otimes \tilde{S}$ with $C^\infty(G, \nu \otimes \kappa)$ as in section 5.1. Then, if $X_1, \ldots, X_{2n+1}$ is an orthonormal base of $\mathfrak{p}$, on $C^\infty(G, \nu \otimes \kappa) \cong \left(C^\infty(G) \otimes V_\nu \otimes \Delta\right)^K$ one has

$$\tilde{D}_\nu f(g) = \epsilon \sum_{i=1}^{2n+1} R(X_i) \otimes \mathrm{Id} \otimes \mathrm{c}(X_i).$$

By [Ch], $\tilde{D}_\nu$ and $\tilde{D}_\nu^2$ with domain $C_c^\infty(G, \nu \otimes \kappa) \subset L^2(G, \nu \otimes \kappa)$ are essentially selfadjoint. Their closures will be denoted by $\tilde{D}_\nu$ and $\tilde{D}_\nu^2$ too. We next recall the Parthasarathy formula for $\tilde{D}_\nu^2$. In [AS], it was assumed that $\mathrm{rk}(G) = \mathrm{rk}(K)$. However, one can proceed exactly as in [AS, page 53-54] to obtain [AS, equation (A 9)], i.e.

$$\tilde{D}_\nu^2 = -R(\Omega) + (\nu(\Omega_K) - \kappa(\Omega_K))\,\mathrm{Id}\,. \tag{7.21}$$

Now we want to study the operators $\tilde{D}_\nu e^{-t\tilde{D}_\nu^2}$. First consider the heat semigroup $e^{-t\tilde{D}_\nu^2}$. Let $c(\nu) := \nu(\Omega_K) - \kappa(\Omega_K)$. Then it follows from (5.7) and (7.21) that on $L^2(G, \nu \otimes \kappa)$ one has

$$e^{-t\tilde{D}_\nu^2}\phi(g) = e^{-tc(\nu)} \int_G H_t^{\nu\otimes\kappa}(g^{-1}g')\phi(g')dg',$$

where $H_t^{\nu\otimes\kappa}$ is as in (5.5). Since $H_t^{\nu\otimes\kappa}$ belongs to all Harish-Chandra Schwarz spaces $(\mathcal{C}^q(G) \otimes \mathrm{End}\,(V_\nu \otimes \Delta^{2n}))^K$, $q > 0$ and since these spaces are stable under the differential action of $U(\mathfrak{g}_\mathbb{C})$, it follows that $\tilde{D}_\nu e^{-t\tilde{D}_\nu^2}$ acts on $L^2(G, \nu \otimes \kappa)$ as an integral operator with smooth kernel

$$K_t^\nu \in \left(\mathcal{C}^q(G) \otimes \mathrm{End}\left(V_\nu \otimes \Delta^{2n}\right)\right)^K,$$

where K_t^ν is given as

$$K_t^\nu(g) := \epsilon e^{-tc(\nu)} \sum_{i=1}^{2n+1} \mathrm{Id} \otimes \mathrm{c}(X_i) \circ \frac{d}{dt}\bigg|_{t=0} H_t^{\nu\otimes\kappa}(\exp -tX_i g).$$

Define a K-finite Schwarz function k_t^ν by

$$k_t^\nu := \mathrm{Tr}\, K_t^\nu, \tag{7.22}$$

where Tr denotes the trace in $\mathrm{End}\,(V_\nu \otimes \Delta^{2n})$.
The Fourier transform of k_t^ν was computed by Moscovici and Stanton in [MS]. We shall recall some details of their argument. Let π be an admissible unitary representation of G on $\mathcal{H}_\pi$. Let $\mathcal{H}_\pi^\infty$ be the set of smooth vectors of $\mathcal{H}_\pi$. Define a bounded operator on $\mathcal{H}_\pi \otimes V_\nu \otimes \Delta^{2n}$ by

$$\tilde{\pi}(K_t^\nu) := \int_G \pi(g) \otimes K_t^\nu(g)dg. \tag{7.23}$$

Then arguing as in [BM, Lemma 5.1] one obtains

$$\mathrm{Tr}\,\pi(k_t^\nu) = \mathrm{Tr}\left(\tilde{\pi}(K_t^\nu)\right).$$

To compute $\mathrm{Tr}\,(\tilde{\pi}(K_t^\nu))$, let X_i be an orthonormal base of $\mathfrak{p}$. Define and operator

$$\tilde{D}_\nu(\pi) : \left(\mathcal{H}_\pi^\infty \otimes V_\nu \otimes \Delta^{2n}\right)^K \to \left(\mathcal{H}_\pi^\infty \otimes V_\nu \otimes \Delta^{2n}\right)^K$$

by

$$\tilde{D}_\nu(\pi) := \epsilon \sum_{i=1}^{2n+1} \pi(X_i) \otimes \mathrm{Id} \otimes \mathrm{c}(X_i).$$

It is easliy verified that $\tilde{D}_\nu(\pi)$ is independent of the choice of the orthonormal base. Moreover, if $k \in K$, $v \in \left(\mathcal{H}_\pi^\infty \otimes V_\nu \otimes \Delta^{2n}\right)^K$, one has

$$(\pi \otimes \nu \otimes \kappa)(k)\tilde{D}_\nu(\pi)v = \epsilon \sum_{i=1}^{2n+1} \pi(\mathrm{Ad}(k)X_i) \otimes \mathrm{Id} \otimes c(\mathrm{Ad}(k)X_i) = \tilde{D}_\nu(\pi)v,$$

since the $\mathrm{Ad}(k)X_i$ also form an orthonormal base of $\mathfrak{p}$. Hence $\tilde{D}_\nu(\pi)$ does indeed map $\left(\mathcal{H}_\pi^\infty \otimes V_\nu \otimes \Delta^{2n}\right)^K$ to itself. Now a change of variables in (7.23) gives

$$\tilde{\pi}(K_t^\nu) = e^{-tc(\nu)}\tilde{D}_\nu(\pi) \circ \tilde{\pi}(H_t^{\nu\otimes\kappa}),$$

where $\tilde{\pi}(H_t^{\nu\otimes\kappa})$ is as in (5.12). As in section 5.1 , relative to the splitting

$$\mathcal{H}_\pi \otimes \Delta^{2n} \otimes V_\nu = (\mathcal{H}_\pi \otimes \Delta^{2n} \otimes V_\nu)^K \oplus \left((\mathcal{H}_\pi \otimes \Delta^{2n} \otimes V_\nu)^K\right)^\perp,$$

$\tilde{\pi}(H_t^{\nu\otimes\kappa})$ has the form

$$\begin{pmatrix} \pi(H_t^{\nu\otimes\kappa}) & 0 \\ 0 & 0 \end{pmatrix},$$

where $\pi(H_t^{\nu\otimes\kappa})$ acts on $\left(\mathcal{H}_\pi \otimes V_\nu \otimes \Delta^{2n}\right)^K$ by $e^{t\pi(\Omega)}\,\mathrm{Id}$. One has $\left(\mathcal{H}_\pi \otimes V_\nu \otimes \Delta^{2n}\right)^K = \left(\mathcal{H}_\pi^\infty \otimes V_\nu \otimes \Delta^{2n}\right)^K$. Hence one gets

$$\mathrm{Tr}\,(\pi(k_t^\nu)) = e^{t(\pi(\Omega)-c(\nu))}\,\mathrm{Tr}\,\tilde{D}_\nu(\pi). \tag{7.24}$$

It remains to compute $\mathrm{Tr}\,\tilde{D}_\nu(\pi)$. For a principal series representation $\pi = \pi_{\sigma,\lambda}$, this was carried out in [MS, Chapter 3].

Proposition 7.5. *Let $\sigma \in \hat{M}$, $\lambda \in \mathbb{R}$. Then one has*

$$\mathrm{Tr}\,\tilde{D}_\nu(\pi_{\sigma,\lambda}) = \lambda\left(\dim\left(V_\sigma \otimes V_\nu \otimes \Delta_+^{2n}\right)^M - \dim\left(V_\sigma \otimes V_\nu \otimes \Delta_-^{2n}\right)^M\right).$$

Proof. This follows from [MS, Proposition 3.6], where an i has to be added by the arguments of Chapter 3 in [MS]. □

Now let $\sigma \in \hat{M}$ and assume that $\sigma \neq w_0\sigma$, $k_{n+1}(\sigma) > 0$. Let $\nu(\sigma) \in \hat{K}$ be of highest weight (2.16). Let $\tilde{E}(\sigma)$ be the vector bundle over $\tilde{X}$ as in (7.5). Then by Proposition 2.3 one has $\tilde{E}(\sigma) = \tilde{E}_{\nu(\sigma)} \otimes \tilde{S}$. Let $\tilde{D}(\sigma) := \tilde{D}_{\nu(\sigma)}$. By (2.15), (2.16) and (2.26) on gets

$$\begin{aligned} c(\nu(\sigma)) =& \nu(\sigma)(\Omega_K) - \kappa(\Omega_K) = \sum_{j=2}^{n+1}(k_j(\sigma)+\rho_j)^2 - \sum_{j=2}^{n+1}(\rho_j+1)^2 \\ =& \sum_{j=2}^{n+1}(k_j(\sigma)+\rho_j)^2 - \sum_{j=1}^{n+1}(\rho_j)^2 = c(\sigma), \end{aligned} \tag{7.25}$$

where $c(\sigma)$ is as in (2.27). Thus by (7.21) one has

$$\tilde{A}(\sigma) = \tilde{D}(\sigma)^2, \tag{7.26}$$

where $\tilde{A}(\sigma)$ is as in section 7.1. Moreover, let

$$k_t^{\sigma} := k_t^{\nu(\sigma)}, \tag{7.27}$$

where $k_t^{\nu(\sigma)}$ is as in (7.22). Then one obtains the folllowing corollary.

Corollary 7.6. *Let $\sigma \in \hat{M}$, $k_{n+1}(\sigma) > 0$. Then for $\lambda \in \mathbb{R}$ one has*

$$\Theta_{\sigma,\lambda}(k_t^{\sigma}) = (-1)^n \lambda e^{-t\lambda^2}; \quad \Theta_{w_0\sigma,\lambda}(k_t^{\sigma}) = (-1)^{n+1} \lambda e^{-t\lambda^2}.$$

Moreover, if $\sigma' \in \hat{M}$, $\sigma' \notin \{\sigma, w_0\sigma\}$, for every $\lambda \in \mathbb{R}$ one has

$$\Theta_{\sigma',\lambda}(k_t^{\sigma}) = 0.$$

Proof. Let $\check{\sigma}$ be the contragredient representation of σ. By [GW, section 3.2.5] if n is even one has $\check{\sigma} = \sigma$ and if n is odd one has $\check{\sigma} = w_0\sigma$. Hence by Propostition 2.3, for $\sigma' \in \hat{M}$ one has

$$\dim \left(V_{\sigma'} \otimes V_{\nu(\sigma)} \otimes \Delta_+^{2n} \right)^M - \dim \left(V_{\sigma'} \otimes V_{\nu(\sigma)} \otimes \Delta_-^{2n} \right)^M = (-1)^n \left[\sigma - w_0\sigma : \sigma' \right].$$

The corollary follows from Corollary 2.6, equation (7.24), Proposition 7.5 and equation (7.25). □

7.3 The antisymmetric Selberg zeta function

Let $\sigma \in \hat{M}$ with $k_{n+1}(\sigma) > 0$. Let $E(\sigma) = \Gamma \backslash \tilde{E}(\sigma)$ be the locally homogeneous vector bundle over X as in section 7.1. Let $\tilde{D}(\sigma)$ be as in the previous section. Then $\tilde{D}(\sigma)$ pushes down to an operator $D(\sigma)$ on the smooth sections of $E(\sigma)$. By (7.26) one has

$$D(\sigma)^2 = A(\sigma), \tag{7.28}$$

where $A(\sigma)$ is the operator from section 7.1. The operator $D(\sigma)$ has a well defined restriction to $L^2_d(\Gamma \backslash G, \nu(\sigma) \otimes \kappa)$ in the sense of unbounded operators. This restriction will be denoted by $D(\sigma)_d$. The space $L^2_d(\Gamma \backslash G, \nu(\sigma) \otimes \kappa)$ is an orthogonal direct sum of finite dimensional eigenspaces of $D(\sigma)_d^2$. On these spaces $D(\sigma)_d$ acts as a symmetric operator, and thus $L^2_d(\Gamma \backslash G, \nu(\sigma) \otimes \kappa)$ is also the orthogonal direct sum of eigenspaces of $D(\sigma)_d$. Let $\{\mu\}_k$ be the sequence of eigenvalues of $D(\sigma)_d$ with $\mu_k = \mu_{k'}$ iff $k = k'$. This sequence might be finite or infinite. Let $\mathcal{E}(\mu_k)$ be the eigenspace of $D(\sigma)$ corresponding to the eigenvalue μ_k and let $d(\mu_k, \sigma) := \dim(\mathcal{E}(\mu_k))$. For $\lambda > 0$ let

$$N(\lambda) := \sum_{|\mu_k| \leq \lambda} d(\mu_k, \sigma).$$

Then by [Do1, Theorem I.1] and [Do2, Theorem 9.1] there exists a constant $C > 0$ such that

$$N(\lambda) \leq C(1+\lambda)^{2n+1}. \tag{7.29}$$

It follows that the sum

$$\sum_k d(\mu_k, \sigma)\,|\mu_k|\,e^{-t\mu_k^2}$$

is convergent. Hence the operator $D(\sigma)_d\, e^{-tD(\sigma)_d^2}$ is of trace class. Moreover, one has

$$\operatorname{Tr}\left(D(\sigma)_d\, e^{-tD(\sigma)_d^2}\right) = \operatorname{Tr}\left(\pi_{\Gamma,d}(k_t^\sigma)\right), \tag{7.30}$$

where k_t^σ is as in (7.27). Let $N \in \mathbb{N}$, $N > n+1$. We choose distinct points $s_1, \dots, s_N$ such that $\operatorname{Re}(s_i) > 8n$, $\operatorname{Re}(s_i^2) > 0$ for all i. Then by (7.29), the sum

$$\sum_k d(\mu_k, \sigma)\mu_k \prod_{i=1}^N \frac{1}{\mu_k^2 + s_i^2}$$

converges absolutely. By Proposition 7.1 we have

$$2i \sum_k d(\mu_k, \sigma)\mu_k \prod_{i=1}^N \frac{1}{\mu_k^2 + s_i^2} = 2i \int_0^\infty \sum_{i=1}^N e^{-ts_i^2} \operatorname{Tr}\left(D(\sigma)_d\, e^{-tD(\sigma)_d^2}\right) \prod_{\substack{i'=1 \\ i' \neq i}}^N \frac{1}{s_{i'}^2 - s_i^2} dt. \tag{7.31}$$

We compute the right hand side of (7.31) using (7.30) and the invariant trace formula in equation (6.19) of Theorem 6.1. By (2.35) we have $P_\sigma(\lambda) = P_{w_0\sigma}(\lambda)$. By Proposition 6.4 we have $\Omega(\sigma, \lambda) = \Omega(w_0\sigma, \lambda)$. Thus, applying (6.1), (6.9), Theorem 6.2 and Corollary 7.6 we obtain

$$I(k_t^\sigma) = T(k_t^\sigma) = \mathcal{I}(k_t^\sigma) = 0. \tag{7.32}$$

Using equation (6.3) and Corollary 7.6, we get

$$\begin{aligned} &2i \int_0^\infty \sum_{i=1}^N e^{-ts_i^2} H(k_t^\sigma) \prod_{\substack{i'=1 \\ i' \neq i}}^N \frac{1}{s_{i'}^2 - s_i^2} dt \\ &= (-1)^n \sum_{i=1}^N \sum_{[\gamma] \in \mathrm{C}(\Gamma)_{\mathrm{s}} - [1]} l(\gamma_0)(L(\gamma, \sigma) - L(\gamma, w_0\sigma))e^{-l(\gamma)s_i} \prod_{\substack{i'=1 \\ i' \neq i}}^N \frac{1}{s_{i'}^2 - s_i^2}. \end{aligned}$$

Here, since $\operatorname{Re}(s) > 8n$, the sum on the right hand side converges absolutely by Proposition 3.8 and equation (3.14) and thus the integral converges absolutely. For $\operatorname{Re}(s) > 8n$ we define the anti-symmetric Selberg-zeta function by

$$S_a(s, \sigma) := \frac{Z(s, \sigma)}{Z(s, w_0\sigma)}.$$

Again, by [GW, section 3.2.5] if n is even we have $\overline{\mathrm{Tr}(\sigma)} = \mathrm{Tr}(\sigma)$ and if n is odd we have $\overline{\mathrm{Tr}(\sigma)} = \mathrm{Tr}(w_0\sigma)$ for every $\sigma \in \hat{M}$. Thus using (3.12) and the definition of $L(\gamma,\sigma)$ in (6.2) it follows that

$$2i\int_0^\infty \sum_{i=1}^N e^{-ts_i^2} H(k_t^\sigma) \prod_{\substack{i'=1\\ i'\neq i}}^N \frac{1}{s_{i'}^2 - s_i^2} dt = \sum_{i=1}^N \frac{d}{ds}\Big|_{s=s_i} \log S_a(s,\sigma) \prod_{\substack{i'=1\\ i'\neq i}}^N \frac{1}{s_{i'}^2 - s_i^2}. \tag{7.33}$$

Using equation (6.16) and Corollary 7.6 one obtains

$$R(k_t^\sigma) = 0. \tag{7.34}$$

Now we come to the contribution of the distribution $\mathcal{S}$. Let $\{\eta(\sigma)\}$ resp. $\{\eta(w_0\sigma)\}$ be the poles of $\det(I(\sigma)\circ \mathbf{C}(\nu(\sigma):\sigma:s))$ resp. $\det(I(w_0\sigma)\circ \mathbf{C}(\nu(\sigma):w_0\sigma:s))$ on $\{s \in \mathbb{C}\colon \mathrm{Re}(s) < 0\}$ counted with multiplicity divided by $\dim(\nu(\sigma))$. By (4.10), the poles of $\det(I(\sigma)\circ \mathbf{C}(\nu(\sigma):\sigma:s))$ and $\det(I(w_0\sigma)\circ \mathbf{C}(\nu(\sigma):w_0\sigma:s))$ on $(0,n]$ coincide. Thus using (4.22), (4.20) and Corollary 7.6, one obtains

$$\begin{aligned}
&2i\int_0^\infty \sum_{i=1}^N e^{-ts_i^2} S(k_t^\sigma) \prod_{\substack{i'=1\\ i\neq i'}}^N \frac{1}{s_{i'}^2 - s_i^2} dt \\
=&\frac{(-1)^n}{2}\sum_{\eta(\sigma)}\sum_{i=1}^N \left(\frac{1}{s_i - \bar{\eta}(\sigma)} - \frac{1}{s_i - \eta(\sigma)}\right)\prod_{\substack{i'=1\\ i'\neq i}}^N \frac{1}{s_{i'}^2 - s_i^2} \\
&+ \frac{(-1)^{n+1}}{2}\sum_{\eta(w_0\sigma)}\sum_{i=1}^N \left(\frac{1}{s_i - \bar{\eta}(w_0\sigma)} - \frac{1}{s_i - \eta(w_0\sigma)}\right)\prod_{\substack{i'=1\\ i'\neq i}}^N \frac{1}{s_{i'}^2 - s_i^2}.
\end{aligned} \tag{7.35}$$

Let $\sigma \in \hat{M}$ with $k_{n+1}(\sigma) < 0$. Then we define the twisted Dirac operator $D(\sigma)$ as $D(\sigma) := -D(w_0\sigma)$, where $D(w_0\sigma)$ is as in section 7.2. We can now prove our main result about the antisymmetric Selberg zeta function.

Theorem 7.7. *Let $\sigma \in \hat{M}$. Then the antisymmetric Selberg zeta function $S_a(s,\sigma)$ has a meromorphic continuation to $\mathbb{C}$ and its singularities are located as follows.*

1. *At the points $\pm i\mu_k$ of order $d(\pm\mu_k,\sigma) - d(\mp\mu_k,\sigma)$, where μ_k is a non-zero eigenvalue of the twisted Dirac operator $D(\sigma)$ and $d(\pm\mu_k,\sigma)$ is the dimension of the eigenspace corresponding to $\pm\mu_k$.*

2. *At the points $\eta(\sigma)$ of order $(-1)^n m(\eta(\sigma))$. Here $\eta(\sigma)$ are the poles of* $\det(I(\sigma)\circ \mathbf{C}(\nu_\sigma:\sigma:s))$ *with negative real part and $m(\eta(\sigma))$ is the corresponding multiplicity divided by* $\dim(\nu_\sigma)$.

3. *At the points $\eta(w_0\sigma)$ of order $(-1)^{n+1} m(\eta(w_0\sigma))$. Here $\eta(w_0\sigma)$ are the poles of* $\det(I(w_0\sigma)\circ \mathbf{C}(\nu_\sigma:w_0\sigma:s))$ *with negative real part and $m(\eta(w_0\sigma))$ is the corresponding multiplicity divided by* $\dim(\nu_\sigma)$.

The above enumeration exhausts all possible singularities of non-zero order of $S_a(s,\sigma)$ if at overlapping singularities the orders are added up.

Proof. Clearly we can assume that $k_{n+1}(\sigma) > 0$. We let $s_1 =: s$, $\mathrm{Re}(s) > 8n$, $\mathrm{Re}(s^2) > 0$ and fix distinct $s_2, \dots, s_N$ with $\mathrm{Re}(s_i) > 8n$, $\mathrm{Re}(s_i^2) > 0$. Then we multiply both sides of (7.31) by

$$\prod_{i'=2}^{N} \left(s_{i'}^2 - s^2\right)$$

and apply Proposition 7.1 to the left hand side and equation (6.19) of Theorem 6.1 to the right hand side. Using (7.32), (7.33), (7.34) and (7.35) we obtain

$$\begin{aligned}\frac{S_a'(s,\sigma)}{S_a(s,\sigma)} =& \sum_k \left(\frac{d(\mu_k,\sigma)}{s-i\mu_k} - \frac{d(\mu_k,\sigma)}{s+i\mu_k} + C(\mu_k; s, s_2, \dots, s_N)\right)\\ &+ \frac{(-1)^n}{2} \sum_{\eta(\sigma)} \left(\frac{1}{\bar{\eta}(\sigma) - s} + \frac{1}{s-\eta(\sigma)} + C(\eta(\sigma); s, s_2, \dots, s_N)\right)\\ &+ \frac{(-1)^{n+1}}{2} \sum_{\eta(w_0\sigma)} \left(\frac{1}{\bar{\eta}(w_0\sigma) - s} + \frac{1}{s-\eta(w_0\sigma)} + C(\eta(w_0\sigma); s, s_2, \dots, s_N)\right).\end{aligned}$$

Here $C(\mu_k; s, s_2, \dots, s_N)$, $C(\eta(\sigma); s, s_2, \dots, s_N)$, $C(\eta(w_0\sigma); s, s_2, \dots, s_N)$ are polynomials in s of degree at most $2(N-2)$. All involved series converge absolutely and locally uniformly by the above arguments. Now we use that by (4.10), $\eta(\sigma)$ is a pole of $\det\left(I(\sigma) \circ \mathbf{C}(\nu_\sigma : \sigma : s)\right)$ if and only if $\bar{\eta}(\sigma)$ is a pole of $\det\left(I(w_0\sigma) \circ \mathbf{C}(\nu_\sigma : w_0\sigma : s)\right)$ and that the orders of the poles are equal. Thus we obtain

$$\begin{aligned}\frac{S_a'(s,\sigma)}{S_a(s,\sigma)} =& \sum_k \left(\frac{d(\mu_k,\sigma)}{s-i\mu_k} - \frac{d(\mu_k,\sigma)}{s+i\mu_k} + C(\mu_k; s, s_2, \dots, s_N)\right)\\ &+ (-1)^n \sum_{\eta(\sigma)} \left(\frac{1}{s-\eta(\sigma)} + C'(\eta(\sigma); s, s_2, \dots, s_N)\right)\\ &+ (-1)^{n+1} \sum_{\eta(w_0\sigma)} \left(\frac{1}{s-\eta(w_0\sigma)} + C'(\eta(w_0\sigma); s, s_2, \dots, s_N)\right). \qquad (7.36)\end{aligned}$$

This proves the theorem. □

We can now complete our proof of the meromorphic continuation of the Selberg zeta function $Z(s,\sigma)$ and describe its singularities.

Theorem 7.8. *Let $\sigma \in \hat{M}$. Then the Selberg zeta function $Z(s,\sigma)$ has a meromorphic continuation to $\mathbb{C}$. If $\sigma = w_0\sigma$, the singularities of the Selberg zeta function $Z(s,\sigma)$ are as in Theorem 7.4.*
Assume that $\sigma \neq w_0\sigma$. Then the singularities of the Selberg zeta function $Z(s,\sigma)$ associated to spectral parameters are located as follows.

1. *At the points* $\pm i\mu_k$ *of order* $\frac{1}{2}\left(m_{\mathrm{s}}(\mu_k^2,\sigma)+d(\pm\mu_k,\sigma)-d(\mp\mu_k,\sigma)\right)$*, where* μ_k *is a non-zero eigenvalue of the twisted Dirac operator* $D(\sigma)$*,* $d(\mu_k,\sigma)$ *is the dimension of the corresponding eigenspace and* $m_{\mathrm{s}}(\mu_k^2,\sigma)$ *is as in* (7.8)*.*

2. *At the point* $s=0$ *of order* $m_{\mathrm{s}}(0,\sigma)$*.*

3. *At the points* $-\beta(\sigma)$ *of order* $-m(\beta(\sigma))$*. Here* $\beta(\sigma)$ *are the poles of* $\det\left(I(\sigma)\circ\mathbf{C}(\nu_\sigma:\sigma:s)\right)$ *on* $(0,n]$ *and* $m(\beta(\sigma))$ *is the corresponding multiplicity divided by* $\dim(\nu_\sigma)$*.*

4. *If* n *is even at the points* $\eta(\sigma)$ *of order* $m(\eta(\sigma))$*. Here* $\eta(\sigma)$ *are the poles of* $\det\left(I(\sigma)\circ\mathbf{C}(\nu_\sigma:\sigma:s)\right)$ *with negative real part and* $m(\eta(\sigma))$ *is the corresponding multiplicity divided by* $\dim(\nu_\sigma)$*.*

5. *If* n *is odd at the points* $\eta(w_0\sigma)$ *of order* $m(\eta(w_0\sigma))$*. Here* $\eta(w_0\sigma)$ *are the poles of* $\det\left(I(w_0\sigma)\circ\mathbf{C}(\nu_\sigma:w_0\sigma:s)\right)$ *with negative real part and* $m(\eta(w_0\sigma))$ *is the corresponding multiplicity divided by* $\dim(\nu_\sigma)$*.*

Moreover, the Selberg zeta function $Z(s,\sigma)$ *has singularities which depend on* Γ *only via* p*, the number of cusps of* Γ*. If all* $k_j(\sigma)$ *are integers, they are located at the negative integers and if all* $k_j(\sigma)$ *are half integers, they are located at the negative half integers.*
Explicitly, if all $k_j(\sigma)$ *are integers, they are given as follows.*

1. *At the points* $-l$*,* $l\in\mathbb{N}$*,* $l\geq m_0$ *of order* $-p\dim(\sigma)$*.*

2. *At the points* $-l$*,* $l\in\mathbb{N}$*,* $m_0\leq l<|k_j(\sigma)|+\rho_j$*,* $j=2,\ldots,n+1$ *of order* $pc_{j,l}(\sigma)$*.*

Here m_0 *and the* $c_{j,l}(\sigma)$ *are as in Proposition 6.4. If all* $k_j(\sigma)$ *are half integers, they can be described in the same way if* $\mathbb{N}$ *is replaced by* $\frac{1}{2}\mathbb{N}-\mathbb{N}$*.*
The above enumeration exhausts all possible singularities of non-zero order of $Z(s,\sigma)$ *if at overlapping singularities the orders are added up.*

Proof. If $\sigma=w_0\sigma$, the Theorem follows from Theorem 7.4. Thus we can assume that $\sigma\neq w_0\sigma$. We have

$$\frac{Z'(s,\sigma)}{Z(s,\sigma)}=\frac{1}{2}\left(\frac{S'(s,\sigma)}{S(s,\sigma)}+\frac{S_a'(s,\sigma)}{S_a(s,\sigma)}\right).$$

If we combine Theorem 7.4 and Theorem 7.7, it follows that the logarithmic derivative of $Z(s,\sigma)$ has a meromorphic continuation to $\mathbb{C}$. It remains to show that its residues are integral. By (7.28), the eigenvalues λ_k of $A(\sigma)$ are given by the μ_k^2. For every k let $\mathcal{E}(\pm\mu_k)$ be the eigenspaces of $D(\sigma)$ corresponding to the eigenvalue $\pm\mu_k$ and let $\mathcal{E}(\mu_k^2)$ be the eigenspace of $A(\sigma)$ corresponding to the eigenvalue μ_k^2. Then one has $\mathcal{E}(\mu_k^2)=\mathcal{E}(\mu_k)\oplus\mathcal{E}(-\mu_k)$. Recall that $m_{\mathrm{s}}(\mu_k^2,\sigma)=\dim_{\mathrm{gr}}\mathcal{E}(\mu_k^2)$. Thus $m_{\mathrm{s}}(\mu_k^2,\sigma)+d(\mu_k,\sigma)-d(-\mu_k,\sigma)$ is even. Since $\sigma\neq w_0\sigma$ by assumption, we have $\epsilon(\sigma)=2$ and thus the theorem follows. □

We conclude this section with the following corollary.

Corollary 7.9. *For every $\sigma \in \hat{M}$ the Ruelle zeta function $R(s,\sigma)$ has a meromorphic contination to $\mathbb{C}$.*

Proof. This follows immediately from Proposition 3.15 and Theorem 7.8. □

8 The relative determinant and the Selberg zeta function

8.1 The asymptotic expansion of the relative trace and the definition of the relative determinant

Let ν be a finite-dimensional unitary representation of K and let E_ν be the associated locally homogeneous bundle over X defined in section 5.1. Let A_ν be the differential operator which acts on E_ν and is induced by $-\Omega$. In this section we want to introduce the relative determinant of $A_\nu + s$ for certain $s \in \mathbb{C}$. As on compact manifolds, in order to do so we need an asymptotic expansion of the relative trace of e^{-tA_ν} as $t \to +0$. To begin with, we prove some elementary lemmas.

Lemma 8.1. *Let $c \in \mathbb{R}$ and let $\phi_1(t) := \int_{\mathbb{R}} e^{-t\lambda^2} \frac{1}{\lambda^2+c^2} d\lambda$. Then there exist $a_j \in \mathbb{C}$ such that*

$$\phi_1(t) \sim \sum_{j=0}^{\infty} a_j t^{\frac{j}{2}}$$

as $t \to +0$.

Proof. We have

$$\phi_1(t) = e^{tc^2} \int_{\mathbb{R}} \frac{e^{-t\left(\lambda^2+c^2\right)}}{\lambda^2+c^2} d\lambda.$$

One has

$$\frac{d}{dt} \int_{\mathbb{R}} \frac{e^{-t\left(\lambda^2+c^2\right)}}{\lambda^2+c^2} d\lambda = -\frac{\sqrt{\pi}}{\sqrt{t}}.$$

Thus one has

$$\int_{\mathbb{R}} \frac{e^{-t\left(\lambda^2+c^2\right)}}{\lambda^2+c^2} d\lambda = C - 2\sqrt{\pi t}.$$

Writing e^{tc^2} as a power series, the proposition follows. □

Lemma 8.2. *Let $\phi_2(t) := \int_{\mathbb{R}} e^{-t\lambda^2}\psi(1+i\lambda)d\lambda$. Then there exist complex coefficients a_j, b_j, c_j such that*

$$\phi_2(t) \sim \sum_{j=0}^{\infty} a_j t^{j-1/2} + \sum_{j=0}^{\infty} b_j t^{j-1/2} \log t + \sum_{j=0}^{\infty} c_j t^j,$$

as $t \to +0$.

Proof. The asymptotic behaviour of the Laplace transform at 0 of functions which admit suitable asymptotic expansions at infinitiy has been treated in [HL].
Recall that

$$\psi(z+1) = \log z + \frac{1}{2z} - \sum_{k=1}^{N} \frac{B_{2k}}{2k} \cdot \frac{1}{z^{2k}} + R_N(z),\ N \in \mathbb{N}. \tag{8.1}$$

Here B_i are the Bernoulli-numbers and

$$R_N(z) = O(z^{-2N-2}),\ z \to \infty$$

uniformly on sectors $-\pi + \delta < \arg(z) < \pi - \delta$. Consider

$$\phi_2^+(t) := \int_0^\infty e^{-t\lambda^2}\psi(1+i\lambda)d\lambda.$$

Let χ be the characteristic function of $[1, \infty)$. Define a function

$$g(\lambda) := \psi(1+i\lambda) - \log i\lambda - \frac{\chi(\lambda)}{2i\lambda}$$

and define a function

$$h(\lambda) := \frac{g(\sqrt{\lambda})}{2\sqrt{\lambda}}.$$

Then by (8.1) there is an asymptotic expansion

$$h(\lambda) \sim \sum_{k=1}^{\infty} a_k \lambda^{-k-1/2}, \quad \lambda \to \infty. \tag{8.2}$$

First define

$$\psi_2^+(t) := \int_0^\infty e^{-t\lambda^2} g(\lambda)d\lambda = \int_0^\infty e^{-t\lambda} h(\lambda)d\lambda.$$

Then by (8.2) and [HL, Corollary 5.2] one obtains

$$\psi_2^+(t) \sim \sum_{k=0}^{\infty} a'_k t^{k+1/2} + \sum_{k=0}^{\infty} c'_k t^k$$

for complex a'_k, c'_k. Next we have

$$\int_0^\infty e^{-t\lambda^2} \log \lambda \, d\lambda = t^{-1/2} \int_0^\infty e^{-\lambda^2} \log \lambda d\lambda - \frac{\sqrt{\pi}}{4} t^{-1/2} \log t.$$

Finally we have

$$\begin{aligned}\int_1^\infty e^{-t\lambda^2} \lambda^{-1} d\lambda &= \int_{\sqrt{t}}^1 e^{-\lambda^2} \lambda^{-1} d\lambda + \int_1^\infty e^{-\lambda^2} \lambda^{-1} d\lambda \\ &= \int_{\sqrt{t}}^1 \sum_{k=0}^\infty (-1)^k \frac{\lambda^{2k-1}}{k!} d\lambda + C \\ &= -\log \sqrt{t} + \sum_{k=1}^\infty (-1)^{k+1} \frac{t^k}{k! 2k} + C'.\end{aligned}$$

Putting everything together, we obtain the desired asymptotic expansion for ϕ_2^+. For the integral over $(-\infty, 0]$ we proceed similarly.
Alternatively, one can also proceed as in [Koy, page 156-157, page 165-166]. The methods of [HL] and [Koy] are closely related. □

Lemma 8.3. *Let $P(z) := \sum_{j=0}^N a_j z^{2j}$ be an even polynomial. Then there exist $a'_j \in \mathbb{C}$ such that*

$$\int_{\mathbb{R}} e^{-t\lambda^2} P(\lambda) d\lambda = \sum_{j=0}^N a'_j t^{-j-\frac{1}{2}}.$$

Proof. This follows from a change of variables. □

Proposition 8.4. *There exist coefficients a_j, b_j, c_j such that one has*

$$\operatorname{Tr}_{\mathrm{rel,u}}(e^{-tA_\nu}) \sim \sum_{j=0}^\infty a_j t^{j-\frac{d}{2}} + \sum_{j=0}^\infty b_j t^{j-\frac{1}{2}} \log t + \sum_{j=0}^\infty c_j t^j$$

as $t \to +0$.

Proof. Assume that all $k_j(\nu)$ are integers. We write $\operatorname{Tr}_{\mathrm{rel,u}}(e^{-tA_\nu})$ as in equation (6.20) of Theorem 6.1 and derive an asymptotic expansion of each summand separately. We can always ignore additional factors of the form e^{-tc}, $c > 0$ by expanding this term in a power series. The term $I(h_t^\nu)$ has the desired asymptotic expansion by Proposition 5.1, equation (6.1) and Lemma 8.3 . Second, by Proposition 3.8, (3.14), (6.3) and Proposition 5.1 we have $H(h_t^\nu) = O(e^{-\frac{c}{t}})$ for a constant $c > 0$. By Proposition 5.1 and equation (6.9) the term $T(h_t^\nu)$ has an asymptotic expansion starting with $t^{-\frac{1}{2}}$. For every $\sigma \in \hat{M}$ with $[\nu : \sigma] \neq 0$ we write $\Omega(\lambda, \sigma)$ as in Proposition 6.7. Then by Theorem 6.2, Propostion 5.1 Proposition 6.7, Lemma 8.1, Lemma 8.2 and Lemma 8.3 it follows that the term $\mathcal{I}(h_t^\nu)$ has the claimed asymptotic expansion in t. The term $J(h_t^\nu)$ has the claimed asymptotic expanion by Proposition 5.1, equation (4.18), equation (6.11) and Lemma 8.1. Finally, the term $R_u(t, \nu)$ has the claimed asymptotic expansion by Lemma 8.1. If all $k_j(\nu)$ are half-integers, one can proceed in the same way. □

Now we introduce the relative zeta function associated to A_ν. By (5.3), there exists a smallest eigenvalue $\lambda_\nu \in \mathbb{R}$ of A_ν^d. Now we define $b(\nu) \in \mathbb{R}$ by

$$b(\nu) := \max\left\{\{c(\sigma)\colon \sigma \in \hat{M}\colon\ [\nu : \sigma] \neq 0\} \sqcup \{-\lambda_\nu\}\right\}, \tag{8.3}$$

where the constants $c(\sigma)$ are as in (2.27).

Proposition 8.5. *Let $s \in \mathbb{C}$ with $\mathrm{Re}(s) > b(\nu)$. Then for $\mathrm{Re}(z) > \frac{d}{2}$ the integral*

$$\xi_{\nu,\mathrm{u}}(s,z) := \int_0^\infty t^{z-1}\,\mathrm{Tr}_{\mathrm{rel,u}}(e^{-t(A_\nu+s)})dt \tag{8.4}$$

converges and $\xi_{\nu,\mathrm{u}}$ is holomorphic on $\{(s,z) \in \mathbb{C}\times\mathbb{C}\colon\ \mathrm{Re}(s) > b(\nu)\colon\ \mathrm{Re}(z) > \frac{d}{2}\}$. Moreover, $\xi_{\nu,\mathrm{u}}(s,z)$ has a continuation to a holomorphic function on $\{(s,z) \in \mathbb{C}\times\mathbb{C}\colon\ \mathrm{Re}(s) > b(\nu)\colon z \neq -j, z \neq d/2 - j, j \in \mathbb{N}_0\}$. For every $s \in \mathbb{C}$ with $\mathrm{Re}(s) > b(\nu)$ the function $z \mapsto \xi_{\nu,\mathrm{u}}(s,z)$ has an at most simple pole at $z = 0$ with

$$\mathrm{Res}\big|_{z=0}\xi_{\nu,\mathrm{u}}(s,z) = c_0, \tag{8.5}$$

where c_0 is the constant from Proposition 8.4.

Proof. By (5.24), Proposition 5.7 and Remark 5.8, there exists a constant C such that one can estimate

$$\left|\mathrm{Tr}_{\mathrm{rel,u}}\, e^{-t(A_\nu+s)}\right| \leq Ce^{-t(\mathrm{Re}(s)-b(\nu))}. \tag{8.6}$$

Thus the integral

$$\int_1^\infty t^{z-1}\,\mathrm{Tr}_{\mathrm{rel,u}}\, e^{-t(A_\nu+s)}$$

converges absolutely for all $\{(z,s) \in \mathbb{C}\times\mathbb{C}\colon\ \mathrm{Re}(s) > b(\nu)\}$ and is holomorphic there. Moreover, by Proposition 8.4 one has $\mathrm{Tr}_{\mathrm{rel,u}}\, e^{-t(A_\nu+s)} = O(t^{-d/2})$ for $t \to +0$, locally uniformly in s and thus the integral in (8.4) converges on the prescribed domain and is holomorphic in z and s. Secondly, expanding e^{-ts} in a power series, Proposition 8.4 gives an asymptotic expansion

$$\mathrm{Tr}_{\mathrm{rel,u}}\, e^{-t(A_\nu+s)} \sim \sum_{j=0}^\infty a_j(s)t^{j-\frac{d}{2}} + \sum_{j=0}^\infty b_j(s)t^{j-\frac{1}{2}}\log t + \sum_{j=0}^\infty c_j(s)t^j$$

as $t \to +0$ which holds locally uniformly in s. Here the coefficients $a_j(s)$, $b_j(s)$ and $c_j(s)$ depend holomorphically on s and one has $c_0(s) = c_0$ for every s since d is odd. Thus the meromorphic continuation and equation (8.5) follow from standard methods which are described for example in [Gi]. □

Now we can define the relative determinant proceeding as on a compact manifold. By Proposition 8.5, for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > b(\nu)$ the function $\xi_{\nu,\mathrm{u}}(s,z)/\Gamma(z)$ is regular at $z = 0$.

Definition 8.6. Let $s \in \mathbb{C}$ with $\mathrm{Re}(s) > b(\nu)$. Then we define the relative determinant of $A_\nu + s$ with respect to the parameter u by

$$\det_{\mathrm{u}}(A_\nu + s) := \exp\left(-\frac{\partial}{\partial z}\Big|_{z=0}\frac{\xi_{\nu,\mathrm{u}}(s,z)}{\Gamma(z)}\right).$$

Remark 8.7. Let us now put the constant $b(\nu)$ in a more natural context. Let A_ν^c denote the restriction of A_ν to $L^2_c(\Gamma\backslash G, \nu)$ in the sense of unbounded operators. Using the theory of Eisenstein series and Corollary 2.6, one can establish in a standard way a unitary equivalence between A_ν^c and the direct sum of the operators of multiplication by $-c(\sigma)+\lambda^2$ acting on $L^2\left([0,\infty), \boldsymbol{\mathcal{E}}(\boldsymbol{\sigma},\nu); d\lambda\right)$. Here the direct sum is over all $\sigma \in \hat{M}$ such that $[\nu : \sigma] \neq 0$, $\boldsymbol{\mathcal{E}}(\boldsymbol{\sigma},\nu)$ is as in section 4.1 and $d\lambda$ is the Lebesgue measure. Thus $-b(\nu)$ is just the infimum of the spectrum of A_ν. This explains the estimate in (8.6). However, since we did not explicitly establish the spectral representation of A_ν^c, we chose the more artificial definition in (8.3) for $b(\nu)$ here.

8.2 The relative graded determinant of the auxiliary operators

In this section we study the relative graded determinant of the operators $A(\sigma_0) + s$ from section 7.1. Let $\sigma_0 \in \hat{M}$ and assume that all $k_j(\sigma_0)$ are integral. For $\nu \in \hat{K}$ let $m_\nu(\sigma_0)$ be as in (7.1) resp. (7.2). Then we define $b(\sigma_0) \in \mathbb{R}$ by

$$b(\sigma_0) := \max\{b(\nu) - c(\sigma_0) \colon \nu \in \hat{K} \colon m_\nu(\sigma_0) \neq 0\}, \tag{8.7}$$

where $c(\sigma_0)$ is as in (2.27) and where $b(\nu)$ is as in (8.3). Let $s \in \mathbb{C}$ with $\mathrm{Re}(s) > b(\sigma_0)$. It follows from Proposition 8.5 that for every $\nu \in \hat{K}$ with $m_\nu(\sigma_0) \neq 0$ the relative determinant $\det_{\mathrm{u}}\left(A_\nu + c(\sigma_0) + s\right) \in \mathbb{R}^+$ is defined. Thus we can define the relative graded determinant $\det_{\mathrm{gr,u}}(A(\sigma_0) + s) \in \mathbb{R}^+$ of $A(\sigma_0) + s$ with respect to the parameter u by

$$\det_{\mathrm{gr,u}}(A(\sigma_0) + s) := \prod_{\substack{\nu\in\hat{K} \\ m_\nu(\sigma_0)\neq 0}} \left(\det_{\mathrm{u}}(A_\nu + c(\sigma_0) + s)\right)^{m_\nu(\sigma_0)}.$$

We now examine the function $s \mapsto \det_{\mathrm{u}}(A(\sigma_0) + s^2)$, $\mathrm{Re}(s) > 0$, $\mathrm{Re}(s^2) > b(\sigma_0)$. More precisely, we use the relative trace formula from Theorem 6.1 and study the Mellin transform of each summand on the right hand side of (6.20) separately. We first establish some auxiliary results for the computation of the involved Mellin transforms.

Lemma 8.8. *Let* $s \in \mathbb{C}$ *with* $\mathrm{Re}(s) > 0$, $\mathrm{Re}(s^2) > 0$, *let* $z \in \mathbb{C}$, $\mathrm{Re}(z) > 0$ *and let* $j \in [0,\infty)$. *Define*

$$\xi(s,z) := \frac{1}{\pi}\int_0^\infty t^{z-1}e^{-ts^2}\int_{D_\epsilon}\frac{e^{-t\zeta^2}}{i\zeta + j}d\zeta dt,$$

where D_ϵ is a contour as in section 6.1. Then for every $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$, $\mathrm{Re}(s^2) > 0$ the function $\xi(s,z)$ has a meromorophic continuation to $z \in \mathbb{C}$ with a simple pole at $z = 0$. Moreover, one has

$$\frac{\partial}{\partial z}\bigg|_{z=0} \frac{\xi(s,z)}{\Gamma(z)} = -2\log(s+j).$$

Proof. If $j = 0$, one has

$$\xi(s,z) = s^{-2z}\Gamma(z)$$

and the proposition follows. Assume that $j > 0$. Then one has

$$\xi(s,z) = \frac{j}{\pi}\int_0^\infty t^{z-1}e^{-ts^2}\int_{\mathbb{R}} \frac{e^{-t\lambda^2}}{\lambda^2+j^2}d\lambda dt.$$

Thus the existence of the integral and the meromorphic continuation follow from Lemma 8.1 and standard methods. Let

$$\phi(s,z) := \frac{j}{\pi}\int_0^\infty t^{z-1}\int_{\mathbb{R}} \frac{e^{-t\left(\lambda^2+s^2\right)}}{\lambda^2+j^2}d\lambda dt - \frac{j}{\pi}\int_0^\infty t^{z-1}\int_{\mathbb{R}} \frac{e^{-t\left(\lambda^2+j^2\right)}}{\lambda^2+j^2}d\lambda dt.$$

Then, since $e^{-ts^2} - e^{-tj^2} = O(t)$, $t \to 0$, it follows from Lemma 8.1 that the integral converges for $\mathrm{Re}(z) > -1$ and is holomorphic in $s \in \mathbb{C}$, $\mathrm{Re}(s) > 0$, $\mathrm{Re}(s^2) > 0$. One has

$$\frac{\partial}{\partial s}\phi(s,0) = -\frac{2js}{\pi}\int_0^\infty\int_{\mathbb{R}} \frac{e^{-t\left(\lambda^2+s^2\right)}}{\lambda^2+j^2}d\lambda dt = \frac{-2}{s+j}.$$

Since $\phi(j,0) = 0$, one has

$$\phi(s,0) = -2\log(s+j) + 2\log 2j. \tag{8.8}$$

On the other hand for $\mathrm{Re}(z) > 0$ one has

$$\xi(j,z) = \frac{j}{\pi z}\int_0^\infty \left(\frac{d}{dt}t^z\right)\int_{\mathbb{R}} \frac{e^{-t\left(\lambda^2+j^2\right)}}{\lambda^2+j^2}d\lambda dt = \frac{j^{-2z}}{\sqrt{\pi}\, z}\Gamma\left(z+\frac{1}{2}\right).$$

Hence for $z \to 0$ one has

$$\xi(j,z) = \frac{1}{z} - 2\log j + \frac{\Gamma'(\frac{1}{2})}{\sqrt{\pi}} + O(z) = \frac{1}{z} - 2\log j + \psi\left(\frac{1}{2}\right) + O(z).$$

Together with (8.8) this gives for $z \to 0$:

$$\begin{aligned}\xi(s,z) =& \frac{1}{z} - 2\log j + \psi\left(\frac{1}{2}\right) - 2\log(s+j) + 2\log 2j + O(z)\\ =& \frac{1}{z} - 2\log(s+j) - \gamma + O(z),\end{aligned}$$

where we used $\psi(\frac{1}{2}) = -2\log 2 - \gamma$. Since for $z \to 0$ one has

$$\frac{1}{\Gamma(z)} = z + \gamma z^2 + O(z^3), \tag{8.9}$$

the proposition follows. □

Lemma 8.9. *Let $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 0$, $\operatorname{Re}(s^2) > 0$, let $z \in \mathbb{C}$, $\operatorname{Re}(z) > 1/2$. Define*

$$\tilde{\xi}(s,z) := \frac{1}{\pi}\int_0^\infty t^{z-1}e^{-ts^2}\int_{\mathbb{R}} e^{-t\lambda^2}\psi\left(1+i\lambda\right)d\lambda dt.$$

Then for every $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 0$, $\operatorname{Re}(s^2) > 0$ the function $\tilde{\xi}(s,z)$ has a meromorophic continuation to $z \in \mathbb{C}$ with an at most simple poles in $z = 0$. Moreover there exists a constant $C'(\psi)$ such that

$$\frac{\partial}{\partial z}\bigg|_{z=0}\frac{\xi(s,z)}{\Gamma(z)} = -2\log\Gamma\left(1+s\right) + C'(\psi).$$

Proof. The convergence of the integral and the statement about the meromorphic continuation follow from Lemma 8.2 and standard methods. Fix $s_0 \in (0,\infty)$. Then, since $e^{-ts^2} - e^{-ts_0^2} = O(t)$ as $t \to +0$, it follows from Lemma 8.2 that the integral

$$\tilde{\phi}(s,z) := \int_0^\infty t^{z-1}\int_{\mathbb{R}}\left(e^{-t\left(\lambda^2+s^2\right)} - e^{-t\left(\lambda^2+s_0^2\right)}\right)\psi\left(1+i\lambda\right)d\lambda dt$$

converges for $\operatorname{Re}(z) > -\frac{1}{2}$ and is holomorphic in $s \in \mathbb{C}$, $\operatorname{Re}(s) > 0$, $\operatorname{Re}(s^2) > 0$. One has

$$\frac{\partial}{\partial s}\tilde{\phi}(s,0) = -2s\int_{\mathbb{R}}\frac{\psi\left(1+i\lambda\right)}{\lambda^2+s^2}d\lambda = -2\pi\psi(1+s).$$

This proves the lemma. □

Proposition 8.10. *Let P be an even polynomial of degree $2N$. Let $s \in \mathbb{C}$, $\operatorname{Re}(s) > 0$, $\operatorname{Re}(s^2) > 0$. For $\operatorname{Re}(z) > N + \frac{1}{2}$ the integral*

$$E(s,z) := \int_0^\infty t^{z-1}e^{-ts^2}\int_{\mathbb{R}} e^{-t\lambda^2}P(i\lambda)d\lambda dt$$

converges absolutely. Moreover, for every $s \in \mathbb{C}$, $\operatorname{Re}(s) > 0$, $\operatorname{Re}(s^2) > 0$ the function $z \mapsto E(z,s)$ has a meromorphic continuation to $\mathbb{C}$. Moreover $E(s,z)$ is regular at $z = 0$ and one has

$$E(s,0) = -2\pi\int_0^s P(\lambda)d\lambda.$$

Proof. If $s \in \mathbb{R}^+$, the lemma is proved in [Fr1], Lemma 2 and Lemma 3. The general case follows from analytic continuation. □

Now fix $\sigma_0 \in \hat{M}$. For notational convenice we assume that all $k_j(\sigma_0)$, defined as in (2.9) are integers. We treat the contribution of each summand on the right hand side of (6.20) to the relative graded determinant of $A(\sigma_0) + s^2$ separately. In the sequel, we shall write $\mathcal{LM}$ to indicate that the Laplace-Mellin transform of a function is taken, allthoug we take the Laplace-transform in s^2 rather than in s.
First, the idenditiy contriubtion is easily treated.

Proposition 8.11. *Let $s \in \mathbb{C}$, $\operatorname{Re}(s) > 0$, $\operatorname{Re}(s^2) > 0$. For $\operatorname{Re}(z) > d/2$, the integral*

$$\mathcal{LM}I(s,z,\sigma_0) := \int_0^\infty t^{z-1} e^{-ts^2} I(h_t^{\sigma_0}) dt$$

converges absolutely. Moreover, $\mathcal{LM}I(s,z,\sigma_0)$ has a meromorphic continuation to $z \in \mathbb{C}$ and is regular at $z = 0$. Let

$$\mathcal{LM}I(s,\sigma_0) := \mathcal{LM}I(s,z,\sigma_0)\big|_{z=0}.$$

Then one has

$$\mathcal{LM}I(s,\sigma_0) = -2\pi \operatorname{vol}(X)\epsilon(\sigma_0) \int_0^s P_{\sigma_0}(r) dr.$$

Proof. This follows from equation (2.35), equation (6.1), equation (7.7) and Proposition 8.10. □

We now prove an estimate for the hyperbolic contribution to the relative graded determinant of $A(\sigma_0)$.

Proposition 8.12. *Let $s \in \mathbb{R}$, $s > n$. For every $z \in \mathbb{C}$ the integral*

$$\mathcal{LM}H(s,z,\sigma_0) := \int_0^\infty t^{z-1} e^{-ts^2} H(h_t^{\sigma_0}) dt$$

converges absolutely and $\mathcal{LM}H(s,z,\sigma_0)$ is an entire function of z. Let

$$\mathcal{LM}H(s,\sigma_0) := \mathcal{LM}H(s,z,\sigma_0)\big|_{z=0}.$$

Then there exists a constant C depending on σ_0 such that for every s with $s > \sqrt{2}n$ one has

$$|\mathcal{LM}H(s,\sigma_0)| \leq Cs^{-2}. \tag{8.10}$$

Proof. By (6.3) and (7.7) we have

$$\mathcal{LM}H(s,z,\sigma_0) = \int_0^\infty t^{z-1} e^{-ts^2} \sum_{[\gamma] \in \mathrm{C}(\Gamma)_{\mathrm{s}} - [1]} \ell(\gamma_0) L_{\mathrm{sym}}(\gamma;\sigma_0) \frac{e^{-\ell(\gamma)^2/4t}}{(4\pi t)^{\frac{1}{2}}} dt. \tag{8.11}$$

Let

$$f(t) := \sum_{[\gamma]\in \mathrm{C}(\Gamma)_{\mathrm{s}}-[1]} \ell(\gamma_0) L_{\mathrm{sym}}(\gamma;\sigma_0)\frac{e^{-\ell(\gamma)^2/4t}}{(4\pi t)^{\frac{1}{2}}}.$$

We have

$$|f(t)| \leq 2\dim(\sigma_0) \sum_{[\gamma]\in \mathrm{C}(\Gamma)_{\mathrm{s}}-[1]} \ell(\gamma_0) L(\gamma;1)\frac{e^{-\ell(\gamma)^2/4t}}{(4\pi t)^{\frac{1}{2}}},$$

where 1 stands for the trivial representation of M. Now let Δ_0 be the Laplace operator acting on $C^\infty(X)$. Then by (5.3), Δ_0 is induced by $-\Omega$. Let Δ_0^d be as in section 5.1. By (2.27), Proposition 5.1, equation (6.19) in Theorem 6.1, Theorem 6.2 and the formulas stated in section 6.1, we have

$$\begin{aligned} e^{-tn^2} \sum_{[\gamma]\in \mathrm{C}(\Gamma)_{\mathrm{s}}-[1]} \ell(\gamma_0) L(\gamma;1)\frac{e^{-\ell(\gamma)^2/4t}}{(4\pi t)^{\frac{1}{2}}} &= \operatorname{Tr}\left(e^{-t\Delta_0^d}\right) + e^{-tn^2}\frac{c_2(1)-c_1(1)}{4} \\ -\int_{D_\epsilon} e^{-t(z^2+n^2)} \Bigg(\mathrm{vol}(X)P_1(iz)+\Omega(1,z)&+\frac{1}{2\pi}C(\Gamma)+\frac{1}{4\pi}\operatorname{Tr}\left(\boldsymbol{T}(1,iz)^{-1}\frac{d}{ds}\boldsymbol{T}(1,iz)\right)\Bigg)\,dz. \end{aligned}$$

Here D_ϵ is the contour from section 6.1. The right hand side of this equation is bounded for $t \geq 1$. Thus there exists a constant C_0 such that

$$|f(t)| \leq C_0 e^{tn^2}, \quad t \geq 1. \tag{8.12}$$

For $s > n$ and $z \in \mathbb{C}$ put

$$G_0(s,z;\sigma_0) := \int_1^\infty t^{z-1} e^{-ts^2} f(t)\, dt.$$

Then it follows from (8.12) that $G_0(s,z;\sigma_0)$ is an entire function of z and that for $s > \sqrt{2}n$ we can estimate

$$|G_0(s,0;\sigma_0)| \leq \int_1^\infty t^{-1}e^{-ts^2}|f(t)|\, dt \leq C_1 e^{-\frac{s^2}{4}}, \quad s > \sqrt{2}n. \tag{8.13}$$

Next we consider the integral from 0 to 1. Using (3.14) it follows that there exists a constant C_2 such that for every $[\gamma] \in \mathrm{C}(\Gamma)_{\mathrm{s}} - [1]$ one has

$$\ell(\gamma_0)|L_{\mathrm{sym}}(\gamma;\sigma_0)| \leq \frac{2\dim(\sigma_0)\ell(\gamma)e^{-n\ell(\gamma)}}{\det(\mathrm{Id}-\mathrm{Ad}(m_\gamma a_\gamma)|_{\bar{\mathfrak{n}}})} \leq C_2. \tag{8.14}$$

Let c be as in (3.13). Then $c > 0$ by Proposition 3.8. Using (8.14) and Proposition 3.8, it follows that there exists $C_3 > 0$ such that

$$|f(t)| \leq C_3 e^{-\sqrt{c}/t}, \quad 0 < t \leq 1. \tag{8.15}$$

For $s > 0$ and $z \in \mathbb{C}$ put

$$G_1(s,z;\sigma_0) = \int_0^1 t^{z-1} e^{-ts^2} f(t)\, dt.$$

By (8.15), $G_1(s,z;\sigma_0)$ is an entire function of z and its value at zero can be estimated as follows

$$|G_1(s,0;\sigma_0)| \leq \int_0^1 t^{-1} e^{-ts^2} |f(t)|\, dt \leq C_4 \int_0^1 e^{-ts^2} e^{-\frac{\sqrt{c}}{2t}}\, dt. \tag{8.16}$$

To estimate the integral on the right hand side, we integrate by parts which gives

$$\int_0^1 e^{-ts^2} e^{-\frac{\sqrt{c}}{2t}}\, dt = -\frac{1}{s^2} e^{-s^2} e^{-\frac{\sqrt{c}}{2}} + \frac{\sqrt{c}}{2s^2} \int_0^1 t^{-2} e^{-ts^2} e^{-\frac{\sqrt{c}}{2t}}\, dt. \tag{8.17}$$

Thus there exists a constant C_5 such that

$$|G_1(s,0;\sigma_0)| \leq C_5 s^{-2}, \quad s > 0. \tag{8.18}$$

Together with (8.13) the Proposition follows. □

Next we treat the contribution of the distribution $\mathcal{I}$.

Proposition 8.13. *Let $s \in \mathbb{C}$, $\mathrm{Re}(s) > 0$, $\mathrm{Re}(s^2) > 0$. For $z \in \mathbb{C}$, $\mathrm{Re}(z) > d/2$ the integral*

$$\mathcal{LMI}(s,z,\sigma_0) := \int_0^\infty t^{z-1} e^{-ts^2} \mathcal{I}(h_t^{\sigma_0}) dt$$

converges absolutely. Moreover, the function $z \mapsto \mathcal{LMI}(s,z,\sigma_0)$ has a meromorphic continuation to $z \in \mathbb{C}$ with an at most simple pole at $z = 0$. Let $C(\psi) := -C'(\psi)/2$, where $C'(\psi)$ is as in Lemma 8.9. Let

$$\mathcal{LMI}(s,\sigma_0) := \frac{\partial}{\partial z}\Big|_{z=0} \frac{\mathcal{LMI}(s,z,\sigma_0)}{\Gamma(z)}.$$

Then one has

$$\begin{aligned}\mathcal{LMI}(s,\sigma_0) =& p\epsilon(\sigma_0)\Big(\dim(\sigma_0) \sum_{1 \leq l < m_0} \log(s+l) + \sum_{j=2}^{n+1} \sum_{m_0 \leq l < |k_j(\sigma_0)|+\rho_j} c_{j,l}(\sigma_0) \log(s+l)\Big) \\ &+ p\epsilon(\sigma_0)\dim(\sigma_0)\Big(\log\Gamma(1+s) + C(\psi) + s\gamma + \frac{1}{2} \sum_{\substack{j=2 \\ |k_j(\sigma)|+\rho_j>0}}^{n+1} \log(s + |k_j(\sigma_0)| + \rho_j)\Big) \\ &+ \frac{p\epsilon(\sigma_0)}{2} \int_0^s Q(\sigma_0, ir) dr.\end{aligned}$$

Here the notations are as in Proposition 6.7.

Proof. This follows from Theorem 6.2, Proposition 6.7, equation (7.7), Lemma 8.8, Lemma 8.9 and Lemma 8.10. □

For the distribution T we have the following proposition.

Proposition 8.14. *Let* $s \in \mathbb{C}$, $\mathrm{Re}(s^2) > 0$, $\mathrm{Re}(s) > 0$. *For* $\mathrm{Re}(z) > 1/2$ *the integral*

$$\mathcal{LM}T(s, z, \sigma_0) := \int_0^\infty t^{z-1} e^{-ts^2} T(h_t^{\sigma_0}) dt$$

conveges absolutely. Moreover, the function $z \mapsto \mathcal{LM}T(s, z, \sigma_0)$ *has a meromorphic continuation to* $\mathbb{C}$ *which is regular at* 0. *Let*

$$\mathcal{LM}T(s, \sigma_0) := \mathcal{LM}T(s, z, \sigma_0)\big|_{z=0}.$$

Then one has

$$\mathcal{LM}T(s, \sigma_0) = -C(\Gamma)\epsilon(\sigma_0)\dim(\sigma_0)s,$$

where $C(\Gamma)$ *is as in* (6.8).

Proof. By (6.9) and (7.7) one has

$$\mathcal{LM}T(s, z, \sigma_0) = \epsilon(\sigma_0)\dim(\sigma_0)\frac{C(\Gamma)}{2\sqrt{\pi}} s^{-2z+1}\Gamma\left(z - \frac{1}{2}\right).$$

This gives the proposition. □

Now we come to the contribution of the non-invariant distribution J. By (7.6) and (6.11) we have

$$\begin{aligned} J(h_t^{\sigma_0}) =& e^{-tc(\sigma_0)} \sum_{\nu \in \hat{K}} m_\nu(\sigma_0) J(h_t^\nu) \\ =& -\frac{pe^{-tc(\sigma_0)}}{4\pi i} \sum_{\sigma \in \hat{M}} \dim(\sigma) \sum_{\nu \in \hat{K}} m_\nu(\sigma_0) [\nu : \sigma] \\ & \times \int_{D_\epsilon} c_\nu(\sigma : z)^{-1} \frac{d}{dz} c_\nu(\sigma : z) e^{-t(z^2 - c(\sigma))} dz \end{aligned} \tag{8.19}$$

To study the right hand side, we will need the following lemma.

Lemma 8.15. *Let* $\sigma_0 \in \hat{M}$. *For* $\sigma \in \hat{M}$ *let*

$$f(z, \sigma) := \sum_{\nu \in \hat{K}} m_\nu(\sigma_0) [\nu : \sigma] c_\nu(\sigma : z)^{-1} \frac{d}{dz} c_\nu(\sigma : z). \tag{8.20}$$

Let $m_0 := |k_{n+1}(\sigma_0)|$. *Then one has*

$$f(z, \sigma) = \sum_{\nu \in \hat{K}} m_\nu(\sigma_0) [\nu : \sigma] \sum_{j=2}^{n+1} \left(\sum_{\substack{m_0 \le l \le k_j(\nu) \\ l > |k_j(\sigma)|}} \frac{i}{iz - l - \rho_j} + \sum_{\substack{m_0 \le l \le k_j(\nu) \\ l \ge |k_j(\sigma)|}} \frac{i}{-iz - l - \rho_j} \right).$$

Proof. If $k_{n+1}(\sigma_0) = 0$, the Lemma follows directly from (4.18). Thus we may assume that $k_{n+1}(\sigma_0) \neq 0$. Then by Proposition 2.3 for every $\nu \in \hat{K}$ with $m_\nu(\sigma_0) \neq 0$ and every $j = 2, \dots, n+1$ one has

$$m_0 - 1 \leq k_j(\nu).$$

Thus using (4.18) one can write

$$\begin{aligned} f(z,\sigma) = &\sum_{\nu\in\hat{K}} m_\nu(\sigma_0)\,[\nu:\sigma] \sum_{j=2}^{n+1} \left(\sum_{\substack{m_0 \leq l \leq k_j(\nu) \\ l > |k_j(\sigma)|}} \frac{i}{iz - l - \rho_j} + \sum_{\substack{m_0 \leq l \leq k_j(\nu) \\ l \geq |k_j(\sigma)|}} \frac{i}{-iz - l - \rho_j} \right) \\ &+ \sum_{\nu\in\hat{K}} m_\nu(\sigma_0)\,[\nu:\sigma] \sum_{j=2}^{n+1} \left(\sum_{\substack{l=0 \\ l > |k_j(\sigma)|}}^{m_0-1} \frac{i}{iz - l - \rho_j} + \sum_{\substack{l=0 \\ l \geq |k_j(\sigma)|}}^{m_0-1} \frac{i}{-iz - l - \rho_j} \right). \end{aligned} \tag{8.21}$$

Now if $\sigma = \sigma_0$ or $\sigma = w_0\sigma_0$ the sum in the second row of (8.21) is zero. On the other hand, assume that $\sigma \neq \sigma_0$, $\sigma \neq w_0\sigma_0$. Then, since $R(M)$ is the free abelian group generated by the representations $\sigma \in \hat{M}$, one has

$$\sum_{\nu\in\hat{K}} m_\nu(\sigma_0)\,[\nu:\sigma] = 0.$$

Thus in this case the sum in the second row of (8.21) is zero too. This proves the proposition. □

We also have the following lemma.

Lemma 8.16. *Let $\sigma_0 \in \hat{M}$. Then for every $\sigma \in \hat{M}$ such that $[\nu:\sigma]\,m_\nu(\sigma_0) \neq 0$ one has $c(\sigma_0) - c(\sigma) \geq 0$.*

Proof. This follows from Propostion 2.2, equation (2.17) and equation (2.27). □

Now we can compute the contribution of the distribution J.

Proposition 8.17. *Let $s \in \mathbb{C}$, $\operatorname{Re}(s) > 0$, $\operatorname{Re}(s^2) > 0$. For $\operatorname{Re}(z) > 0$ the integral*

$$\mathcal{LM}J(s,z,\sigma_0) := \int_0^\infty t^{z-1} e^{-ts^2} J(h_t^{\sigma_0})\,dt$$

converges absolutely. Moreover, the function $z \mapsto \mathcal{LM}J(s,z,\sigma_0)$ has a meromorphic continuation to $\mathbb{C}$ with only simple poles. Let

$$\mathcal{LM}J(s,\sigma_0) := \frac{\partial}{\partial z}\bigg|_{z=0} \frac{\mathcal{LM}J(s,z,\sigma_0)}{\Gamma(z)}.$$

Then if m_0 is as in Lemma 8.15 one has

$$\begin{aligned}&\mathcal{LM}J(s,\sigma_0)\\&=-p\sum_{\sigma\in\hat{M}}\sum_{\nu\in\hat{K}}m_\nu(\sigma_0)\,[\nu:\sigma]\dim(\sigma)\sum_{j=2}^{n+1}\sum_{\substack{m_0\leq l\leq k_j(\nu)\\ l>|k_j(\sigma)|}}\log\left(\sqrt{s^2+c(\sigma_0)-c(\sigma)}+l+\rho_j\right)\\&\quad-\frac{p}{2}\sum_{\sigma\in\hat{M}}\sum_{\nu\in\hat{K}}m_\nu(\sigma_0)\,[\nu:\sigma]\dim(\sigma)\sum_{\substack{j=2\\|k_j(\sigma)|\geq m_0}}^{n+1}\log\left(\sqrt{s^2+c(\sigma_0)-c(\sigma)}+|k_j(\sigma)|+\rho_j\right).\end{aligned}$$

Proof. By Lemma 8.16, for every $\sigma\in\hat{M}$ such that $[\nu:\sigma]\,m_\nu(\sigma_0)\neq 0$ we have $c(\sigma_0)-c(\sigma)\geq 0$. Thus the proposition follows from Lemma 8.8, equation (8.19) and Lemma 8.15. □

Finally we treat the contribution of the auxiliary operators on the cusp to the relative graded determinant. Therefore we let

$$R_{\mathrm{u}}(t,\sigma_0):=\sum_{\nu\in\hat{K}}m_\nu(\sigma_0)e^{-tc(\sigma_0)}R_{\mathrm{u}}(t,\nu),\tag{8.22}$$

where $R_{\mathrm{u}}(t,\nu)$ is as in (6.18). Then we have the following lemma.

Lemma 8.18. *One has*

$$R_{\mathrm{u}}(t,\sigma_0)=\epsilon(\sigma_0)p\dim(\sigma_0)\left(\frac{\log u}{\sqrt{4\pi t}}+\frac{1}{4}+\frac{1}{2\pi}\int_0^\infty e^{-t\lambda^2}\frac{d-1}{\frac{(d-1)^2}{4}+\lambda^2}\right).$$

Proof. By (6.18) one has

$$\begin{aligned}R_{\mathrm{u}}(t,\sigma_0)=&\sum_{\sigma\in\hat{M}}\sum_{\nu\in\hat{K}}m_\nu(\sigma_0)\,[\nu:\sigma]\,e^{-t(c(\sigma_0)-c(\sigma))}p\dim(\sigma)\Bigg(\frac{\log u}{\sqrt{4\pi t}}+\frac{1}{4}\\&+\frac{1}{2\pi}\int_0^\infty e^{-t\lambda^2}\frac{d-1}{\frac{(d-1)^2}{4}+\lambda^2}d\lambda\Bigg).\end{aligned}$$

As in the proof of Lemma 8.15, if $\sigma\neq\sigma_0$, $\sigma\neq w_0\sigma_0$, we have

$$\sum_{\nu\in\hat{K}}m_\nu(\sigma_0)\,[\nu:\sigma]=0.$$

On the other hand, the same argument implies that for $\sigma=\sigma_0$, $\sigma=w_0\sigma_0$ we have

$$\sum_{\nu\in\hat{K}}m_\nu(\sigma_0)\,[\nu:\sigma]=1.$$

This proves the lemma. □

Proposition 8.19. *Let $s \in \mathbb{C}$, $\operatorname{Re}(s) > 0$, $\operatorname{Re}(s^2) > 0$. For $z \in \mathbb{C}$, $\operatorname{Re}(z) > 0$ the integral*

$$\mathcal{LM}R_u(s, z, \sigma_0) := \int_0^\infty t^{z-1} e^{-ts^2} R_{\mathrm{u}}(t, \sigma_0) dt$$

converges absolutely. Moreover, the function $z \mapsto \mathcal{LM}R_u(s, z, \sigma_0)$ has a meromorphic continuation to $\mathbb{C}$ with only simple poles. Let

$$\mathcal{LM}R_u(s, \sigma_0) := \frac{\partial}{\partial z}\Big|_{z=0} \frac{\mathcal{LM}R_u(s, z, \sigma_0)}{\Gamma(z)}.$$

Then one has

$$\mathcal{LM}R_u(s, \sigma_0) = -p \dim(\sigma_0)\epsilon(\sigma_0) \left(s \log u + \log\big(s + (d-1)/2\big) + \frac{\log s}{2} \right).$$

Proof. This follows from Lemma 8.8 and Lemma 8.18. □

We summarize our computations in the following proposition.

Proposition 8.20. *Let $s \in \mathbb{R}$, $s > \max\{b(\sigma_0), n\}$. Then one has*

$$\det_{\mathrm{gr,u}}(A(\sigma_0) + s^2) = \exp\bigg(- \mathcal{LM}I(s, \sigma_0) - \mathcal{LM}H(s, \sigma_0) - \mathcal{LM}T(s, \sigma_0) - \mathcal{LM}\mathcal{I}(s, \sigma_0) - \mathcal{LM}J(s, \sigma_0) - \mathcal{LM}R_u(s, \sigma_0)\bigg),$$

where the summands on the right hand side of this equation are defined as in the previous propositions.

Proof. This follows immediately from equation (6.20) and from the previous propositions. □

8.3 The determinant formula for the symmetric Selberg zeta function

In this section we want to relate the symmetric Selberg Zeta function $S(s, \sigma_0)$, $\sigma_0 \in \hat{M}$, to the relative graded determinant $\det_{\mathrm{gr,u}}(A(\sigma_0) + s^2)$, where $\det_{\mathrm{gr,u}}(A(\sigma_0) + s^2)$ is as in section 8.2. To do so, we will generalize the methods of Bunke and Olbrich ([BO, Chapter 3.3.3.]) to the non compact setting and combine them with our previous computations. Let us assume that all $k_j(\sigma_0)$ are integral.
Let $b(\sigma_0)$ be as in (8.7). For $s \in \mathbb{C}$, $\operatorname{Re}(s) > b(\sigma_0)$ and $z \in \mathbb{C}$ let

$$\xi_{A(\sigma_0),\mathrm{u}}(s, z) := \sum_{\substack{\nu \in \hat{K} \\ m_\nu(\sigma_0) \neq 0}} m_\nu(\sigma_0) \xi_{\nu,\mathrm{u}}(s + c(\sigma_0), z),$$

where $\xi_{\nu,\mathrm{u}}$ is as in (8.4). Then, by definition, for $s \in \mathbb{C}$ with $\operatorname{Re}(s^2) > b(\sigma_0)$ one has

$$\log\big(\det_{\mathrm{gr,u}}(A(\sigma_0) + s^2)\big) = -\frac{\partial}{\partial z}\Big|_{z=0} \frac{\xi_{A(\sigma_0),\mathrm{u}}(s^2, z)}{\Gamma(z)}.$$

Lemma 8.21. *Let $N \in \mathbb{N}$. Let $s_1, \ldots, s_N$ be N distinct complex numbers. Then one has*

$$\sum_{i=1}^{N} e^{-ts_i^2} \operatorname{Tr}^{\mathrm{s}}_{\mathrm{rel,u}} e^{-tA(\sigma_0)} \prod_{\substack{i'=1 \\ i' \neq i}}^{N} \frac{1}{s_{i'}^2 - s_i^2} = O(t^{N-1-d/2})$$

as $t \to +0$.

Proof. By Proposition 8.4 one has

$$\operatorname{Tr}^{\mathrm{s}}_{\mathrm{rel,u}} \left(e^{-tA(\sigma_0)}\right) = O(t^{-d/2}),\ t \to +0.$$

Thus together with Lemma 7.2, the Lemma follows. □

Lemma 8.22. *Let $N \in \mathbb{N}$ with $N > d/2$. Let $s_1, \ldots, s_N$ be N distinct complex numbers with $\operatorname{Re}(s_i^2) > b(\sigma_0)$ for all i. Then one has*

$$\int_0^\infty \sum_{i=1}^{N} e^{-ts_i^2} \operatorname{Tr}^{\mathrm{s}}_{\mathrm{rel,u}} e^{-tA(\sigma_0)} \prod_{\substack{i'=1 \\ i' \neq i}}^{N} \frac{1}{s_{i'}^2 - s_i^2} dt = \sum_{i=1}^{N} \frac{d}{ds}\Big|_{s=s_i^2} \log \det\nolimits_{\mathrm{gr,u}} \left(A(\sigma_0) + s\right) \prod_{\substack{i'=1 \\ i' \neq i}}^{N} \frac{1}{s_{i'}^2 - s_i^2}.$$

Proof. By the definition of $b(\sigma_0)$ and by (8.6) there exists a $c > 0$ such that

$$\sum_{i=1}^{N} e^{-ts_i^2} \operatorname{Tr}^{\mathrm{s}}_{\mathrm{rel,u}} e^{-tA(\sigma_0)} \prod_{\substack{i'=1 \\ i' \neq i}}^{N} \frac{1}{s_{i'}^2 - s_i^2} = O(e^{-ct}),$$

as $t \to \infty$. Together with Lemma 8.21 it follows that for $\operatorname{Re}(z) > -1/2$ the integral

$$\int_0^\infty t^z \sum_{i=1}^{N} e^{-ts_i^2} \operatorname{Tr}^{\mathrm{s}}_{\mathrm{rel,u}} e^{-tA(\sigma_0)} \prod_{\substack{i'=1 \\ i' \neq i}}^{N} \frac{1}{s_{i'}^2 - s_i^2} dt$$

converges absoluetly. It follows from the definition of $\xi_{A(\sigma_0),\mathrm{u}}(s,z)$ and from meromorphic continuation that

$$\xi_{A(\sigma_0),\mathrm{u}}(s, z+1) = -\frac{\partial}{\partial s} \xi_{A(\sigma_0),\mathrm{u}}(s, z). \tag{8.23}$$

Finally, by (8.5) there exists a constant $\alpha(\sigma_0)$ such that for all s with $\operatorname{Re}(s) > b(\sigma_0)$ one has

$$\operatorname{Res}\Big|_{z=0} \xi_{A(\sigma_0),\mathrm{u}}(s, z) = \alpha(\sigma_0)$$

and so by (8.9) for all such s one has

$$\log \det\nolimits_{\mathrm{gr,u}}(A(\sigma_0) + s) = -\lim_{z \to 0} \left(\xi_{A(\sigma_0),\mathrm{u}}(s, z) - \frac{\alpha(\sigma_0)}{z} + \gamma \alpha(\sigma_0) \right). \tag{8.24}$$

Thus we can write

$$\begin{aligned}
&\int_0^\infty \sum_{i=1}^N \operatorname{Tr}^{\mathrm{s}}_{\mathrm{rel,u}} e^{-t(A(\sigma_0)+s_i^2)} \prod_{\substack{i'=1\\ i'\neq i}}^N \frac{1}{s_{i'}^2-s_i^2}dt\\
=&\lim_{z\to 0}\int_0^\infty t^z \sum_{i=1}^N \operatorname{Tr}^{\mathrm{s}}_{\mathrm{rel,u}} e^{-t(A(\sigma_0)+s_i^2)} \prod_{\substack{i'=1\\ i'\neq i}}^N \frac{1}{s_{i'}^2-s_i^2}dt\\
=&\lim_{z\to 0}\sum_{i=1}^N \xi_{A(\sigma_0),\mathrm{u}}(s_i^2,z+1) \prod_{\substack{i'=1\\ i'\neq i}}^N \frac{1}{s_{i'}^2-s_i^2}\\
=&-\lim_{z\to 0}\sum_{i=1}^N \frac{\partial}{\partial s}\Big|_{s=s_i^2} \xi_{A(\sigma_0),\mathrm{u}}(s,z) \prod_{\substack{i'=1\\ i'\neq i}}^N \frac{1}{s_{i'}^2-s_i^2} &&(+)\\
=&-\lim_{z\to 0}\sum_{i=1}^N \frac{\partial}{\partial s}\Big|_{s=s_i^2} \left(\xi_{A(\sigma_0),\mathrm{u}}(s,z)-\frac{\alpha(\sigma_0)}{z}+\gamma\alpha(\sigma_0)\right) \prod_{\substack{i'=1\\ i'\neq i}}^N \frac{1}{s_{i'}^2-s_i^2} &&(++)\\
=&\sum_{i=1}^N \frac{d}{ds}\Big|_{s=s_i^2} \det\nolimits_{\mathrm{gr}}\left(A(\sigma_0)+s\right) \prod_{\substack{i'=1\\ i'\neq i}}^N \frac{1}{s_{i'}^2-s_i^2}. &&(+++)
\end{aligned}$$

Here $(+)$ follows from(8.23). In $(++)$ limit and differentiation can be interchanged since the way in which one continues $\xi_{A(\sigma_0),\mathrm{u}}(s,z)$ meromorphically easily implies that there exists a neighbourhood U of 0 such that the function

$$(s,z)\mapsto \xi_{A(\sigma_0),\mathrm{u}}(s,z)-\frac{\alpha(\sigma_0)}{z}+\gamma\alpha(\sigma_0)$$

is holomorphic on $\{(s,z)\in\mathbb{C}^2\colon\ \mathrm{Re}(s)>b(\sigma_0)\colon z\in U\}$. Thus $(+++)$ follows from (8.24). This proves the lemma. □

Now we can prove a determinant formula for the Selberg zeta function.

Proposition 8.23. *Let* $s\in\mathbb{C}$, $\mathrm{Re}(s)>0$, $\mathrm{Re}(s^2)>\max\{0,b(\sigma_0)\}$. *Then we have*

$$\begin{aligned}
S(s,\sigma_0)=&\det\nolimits_{\mathrm{gr,u}}\left(A(\sigma_0)+s^2\right)\exp\bigl(\mathcal{LMI}(s,\sigma_0)+\mathcal{LMT}(s,\sigma_0)+\mathcal{LM}\mathcal{I}(s,\sigma_0)\\
&+\mathcal{LMJ}(s,\sigma_0)+\mathcal{LMR}_u(s,\sigma_0)\bigr),
\end{aligned}$$

where the notations are as in section 8.2.

Proof. Let $N\in\mathbb{N}$ with $N>d/2$. Let $s_1,\dots,s_N$ be N distinct complex numbers with

$\mathrm{Re}(s_i^2) > b(\sigma_0)$. Then by (6.20) we have

$$\int_0^\infty \sum_{i=1}^N e^{-ts_i^2} \operatorname{Tr}^{\mathrm{s}}_{\mathrm{rel,u}} e^{-tA(\sigma_0)} \prod_{\substack{i'=1 \\ i'\neq i}}^N \frac{1}{s_{i'}^2 - s_i^2} dt \tag{8.25}$$

$$= \int_0^\infty \sum_{i=1}^N e^{-ts_i^2} \left(I(h_t^{\sigma_0}) + H(h_t^{\sigma_0}) + \mathcal{I}(h_t^{\sigma_0}) + T(h_t^{\sigma_0}) + J(h_t^{\sigma_0}) + R_{\mathrm{u}}(t,\sigma_0) \right) \prod_{\substack{i'=1 \\ i'\neq i}}^N \frac{1}{s_{i'}^2 - s_i^2} dt,$$

where the integral exists by the arguments in the proof of Lemma 8.22. Using equation (8.19), Lemma 8.15 and Lemma 8.16, for every $s \in \mathbb{C}$ with $\mathrm{Re}(s^2) > 0$ we get

$$\int_0^\infty e^{-ts^2} J(h_t^{\sigma_0}) dt = \tag{8.26}$$

$$p \sum_{\sigma\in\hat{M}} \sum_{\nu\in\hat{K}} m_\nu(\sigma_0) \sum_{j=2}^{n+1} \sum_{\substack{m_0 \leq l \leq k_j(\nu) \\ l > |k_j(\sigma)|}} \frac{[\nu:\sigma] \dim(\sigma)}{2\sqrt{s^2 + c(\sigma_0) - c(\sigma)} \left(\sqrt{s^2 + c(\sigma_0) - c(\sigma)} + l + \rho_j \right)} +$$

$$\frac{p}{2} \sum_{\sigma\in\hat{M}} \sum_{\nu\in\hat{K}} m_\nu(\sigma_0) \sum_{\substack{j=2 \\ |k_j(\sigma)| \geq m_0}}^{n+1} \frac{[\nu:\sigma] \dim(\sigma)}{2\sqrt{s^2 + c(\sigma_0) - c(\sigma)} \left(\sqrt{s^2 + c(\sigma_0) - c(\sigma)} + |k_j(\sigma_0)| + \rho_j \right)}.$$

Using Lemma 8.18, for every $s \in \mathbb{C}$ with $\mathrm{Re}(s^2) > 0$ we obtain

$$\int_0^\infty e^{-ts^2} R_{\mathrm{u}}(t,\sigma_0) dt = p\epsilon(\sigma_0) \dim(\sigma_0) \left(\frac{\log u}{2s} + \frac{1}{4s^2} + \frac{1}{2s(s + \frac{d-1}{2})} \right). \tag{8.27}$$

We fix distinct complex numbers $s_2, \ldots, s_N$ with $\mathrm{Re}(s_i) > 8n$ and $\mathrm{Re}(s_i^2) > \max\{b(\sigma_0), 8n\}$. Then we let $s_1 =: s$ vary under the restriction that $s \in \mathbb{R}$, $s > \max\{b(\sigma_0), 8n\}$ and $s \notin \{s_2, \ldots, s_N\}$. If we multiply both sides of (8.25) by

$$2s \prod_{i'=2}^N (s_{i'}^2 - s^2)$$

and use Lemma 8.22, equation (7.13), equation (7.15), equation (7.16), equation (7.17), equation (8.26) and equation (8.27) it follows that

$$
\begin{aligned}
&\frac{d}{ds}\log S(s,\sigma_0)\\
=&\frac{d}{ds}\log\det{}_{\mathrm{gr,u}}\left(A(\sigma)+s^2\right)-2\pi\epsilon(\sigma_0)\operatorname{vol}(X)\,P_{\sigma_0}(s)\\
&-p\sum_{\sigma\in\hat{M}}\sum_{\nu\in\hat{K}}m_\nu(\sigma_0)\,[\nu:\sigma]\dim(\sigma)\sum_{j=2}^{n+1}\sum_{\substack{m_0\leq l\leq k_j(\nu)\\ l>|k_j(\sigma)|}}\frac{d}{ds}\log\left(\sqrt{s^2+c(\sigma_0)-c(\sigma)}+l+\rho_j\right)\\
&-\frac{p}{2}\sum_{\sigma\in\hat{M}}\sum_{\nu\in\hat{K}}m_\nu(\sigma_0)\,[\nu:\sigma]\dim(\sigma)\sum_{\substack{j=2\\ |k_j(\sigma)|\geq m_0}}^{n+1}\frac{d}{ds}\log\left(\sqrt{s^2+c(\sigma_0)-c(\sigma)}+|k_j(\sigma)|+\rho_j\right)\\
&+p\epsilon(\sigma_0)\sum_{j=2}^{n+1}\sum_{\substack{m_0\leq l<\\ |k_j(\sigma_0)|+\rho_j}}\frac{c_{j,l}(\sigma_0)}{l+s}+\sum_{\substack{j=2\\ |k_j(\sigma)|+\rho_j>0}}^{n+1}\frac{p\dim(\sigma_0)\epsilon(\sigma_0)}{2(|k_j(\sigma_0)|+\rho_j+s)}\\
&+p\epsilon(\sigma_0)\dim(\sigma_0)\left(\psi(1+s)+\sum_{1\leq l<m_0}\frac{1}{l+s}\right)+\epsilon(\sigma_0)\frac{p}{2}Q(\sigma_0,is)+\epsilon(\sigma_0)\dim(\sigma_0)\gamma p\\
&-\epsilon(\sigma_0)\dim(\sigma_0)C(\Gamma)-p\epsilon(\sigma_0)\dim(\sigma_0)\left(\log u+\frac{1}{2s}+\frac{1}{s+\frac{d-1}{2}}\right)+p(s),
\end{aligned}
$$

where $p(s)$ is a polynomial in s depending on $s_2, \dots, s_N$. Thus we get

$$\log S(s,\sigma_0) = \log \det_{\mathrm{gr,u}} \left(A(\sigma_0) + s^2\right) - 2\pi\epsilon(\sigma_0)\operatorname{vol}(X) \int_0^s P_{\sigma_0}(r)dr$$

$$- p \sum_{\sigma\in\hat{M}} \sum_{\nu\in\hat{K}} m_\nu(\sigma_0)\,[\nu:\sigma]\dim(\sigma) \sum_{j=2}^{n+1} \sum_{\substack{m_0\le l\le k_j(\nu)\\ l>|k_j(\sigma)|}} \log\left(\sqrt{s^2+c(\sigma_0)-c(\sigma)} + l + \rho_j\right)$$

$$- \frac{p}{2} \sum_{\sigma\in\hat{M}} \sum_{\nu\in\hat{K}} m_\nu(\sigma_0)\,[\nu:\sigma]\dim(\sigma) \sum_{\substack{j=2\\ |k_j(\sigma)|\ge m_0}}^{n+1} \log\left(\sqrt{s^2+c(\sigma_0)-c(\sigma)} + |k_j(\sigma)| + \rho_j\right)$$

$$+ p\epsilon(\sigma_0) \sum_{j=2}^{n+1} \sum_{\substack{m_0\le l<\\ |k_j(\sigma_0)|+\rho_j}} c_{j,l}(\sigma_0)\log(l+s)$$

$$+ \sum_{\substack{j=2\\ |k_j(\sigma_0)|+\rho_j>0}}^{n+1} \frac{p\epsilon(\sigma_0)\dim(\sigma_0)}{2}\log\left(|k_j(\sigma_0)|+\rho_j+s\right) + \frac{p\epsilon(\sigma_0)}{2}\int_0^s Q(\sigma_0, ir)dr$$

$$+ p\epsilon(\sigma_0)\dim(\sigma_0)\left(\log\Gamma(1+s) + \sum_{1\le l<m_0}\log(l+s) + s\gamma\right) - s\epsilon(\sigma_0)\dim(\sigma_0)C(\Gamma)$$

$$- p\epsilon(\sigma_0)\dim(\sigma_0)\left(s\log u + \frac{\log s}{2} + \log\left(s+\frac{d-1}{2}\right)\right) + P(s),$$

where $P(s)$ is again a polynomial in s depending on $s_2, \dots, s_N$. In the last equation we express $\log\det_{\mathrm{gr,u}}(A(\sigma_0)+s^2)$ using Proposition 8.20. Then by Proposition 8.11, Proposition 8.13, Proposition 8.14 Proposition 8.17 and Proposition 8.19 we obtain

$$\log S(s,\sigma_0) = -\mathcal{LMH}(s,\sigma_0) + P(s) - p\epsilon(\sigma_0)\dim(\sigma_0)C(\psi),$$

where $\mathcal{LMH}(s;\sigma_0)$ is as in Proposition 8.12 and $C(\psi)$ is as in Proposition 8.13. Now we let $s\to\infty$. Then, since $S(s,\sigma_0)$ tends to zero by Lemma 3.14 and since $\mathcal{LMH}(s;\sigma_0)$ tends to zero by Proposition 8.12, it follows that the polynomial $P(s)$ is constant, $P(s) = p\epsilon(\sigma_0)\dim(\sigma_0)C(\psi)$. Thus we have

$$\log S(s,\sigma_0) = -\mathcal{LMH}(s,\sigma_0).$$

If we now apply Proposition 8.20 again, the proposition is proved for $s\in\mathbb{R}$ with $s > \max\{b(\sigma_0), 8n\}$. This set is contained in the connected set $\{s\in\mathbb{C}\colon \operatorname{Re}(s)>0\colon \operatorname{Re}(s^2) > \max\{b(\sigma_0), 0\}\}$ and both sides of the equation in the proposition are meromorphic on this set by the results obtained previously. This proves the proposition. □

Remark 8.24. If all $k_j(\sigma_0)$ are half-integral, one can proceed in a similar way and obtain a similiar formula as in Proposition 8.23.

Let us finally note the following proposition which defines the region in which Proposition 8.23 can be applied more precisely.

Proposition 8.25. *Let $\sigma_0 \in \hat{M}$ and assume that σ_0 satisfies $\sigma_0 \neq w_0\sigma_0$. Then one has $b(\sigma_0) = 0$.*

Proof. Let $A(\sigma_0)$ be as in section 7.1 and let $\lambda(\sigma_0) \in \mathbb{R}$ be the smallest eigenavlue of $A(\sigma_0)^d$. Then one has

$$b(\sigma_0) = \max\{c(\sigma) - c(\sigma_0) \colon m_\nu(\sigma_0)\,[\nu : \sigma] \neq 0\} \sqcup \{-\lambda(\sigma_0)\}.$$

By Lemma 8.16 we have $c(\sigma_0) - c(\sigma) \geq 0$ for every $\sigma \in \hat{M}$ with $m_\nu(\sigma_0)\,[\nu : \sigma] \neq 0$, $\nu \in \hat{K}$. Moreover the twisted Dirac operator $D(\sigma_0)$ can be defined as in section 7.2 and by (7.26) we have $A(\sigma_0) = D(\sigma_0)^2$. Thus we have $A(\sigma_0)^d \geq 0$. This proves the Lemma. □

9 The relative analytic torsion

9.1 Definition and basic properties

Let τ be an irreducible finite dimensional representation of G on V_τ. Let E'_τ be the flat vector bundle associated to the restriction of τ to Γ. Then E'_τ is canonically isomorphic to the locally homogeneous vector bundle E_τ associated to $\tau|_K$. By [MaMu], there exists an inner product $\langle\cdot,\cdot\rangle$ on V_τ such that

1. $\langle\tau(Y)u, v\rangle = -\langle u, \tau(Y)v\rangle$ for all $Y \in \mathfrak{k}$, $u, v \in V_\tau$
2. $\langle\tau(Y)u, v\rangle = \langle u, \tau(Y)v\rangle$ for all $Y \in \mathfrak{p}$, $u, v \in V_\tau$.

Such an inner product is called admissible. It is unique up to scaling. Fix an admissible inner product. Since $\tau|_K$ is unitary with respect to this inner product, it induces a metric on E_τ which will be called admissible too. Let $\Lambda^p(E_\tau)$ be the bundle of E_τ valued p-forms on X. Let

$$\nu_p(\tau) := \Lambda^p \operatorname{Ad}^* \otimes \tau :\ K \to \operatorname{GL}(\Lambda^p\mathfrak{p}^* \otimes V_\tau). \tag{9.1}$$

Then one has canonically

$$\Lambda^p(E_\tau) \cong \Gamma\backslash(G \times_{\nu_p(\tau)} \Lambda^p\mathfrak{p}^* \otimes V_\tau). \tag{9.2}$$

If $\Lambda^p(X, E_\tau)$ are the smooth E_τ-valued p-forms on X, the isomorphism (9.2) induces an isomorphism

$$\Lambda^p(X, E_\tau) \cong C^\infty(\Gamma\backslash G, \nu_p(\tau)), \tag{9.3}$$

where the latter space is as in (5.2). A corresponding isomorphism also holds for the L^2-spaces. Let $\Delta_p(\tau)$ be the Hodge-Laplacian on $\Lambda^p(X, E_\tau)$ with respect to the admissible inner product. By [MaMu, equation (6.9)], on $C^\infty(\Gamma\backslash G, \nu_p(\tau))$ one has

$$\Delta_p(\tau) = -\Omega + \tau(\Omega)\,\mathrm{Id}\,. \tag{9.4}$$

In order to define the analytic torsion $T_{X,\mathrm{u}}(\tau)$ for certain $\tau \in \hat{G}$, we need to show that for every $p = 0, \dots, d$ the determinant $\det_{\mathrm{u}}(\Delta_p(\tau))$ can be defined as in section 8.1. Thus we have to study the numbers $b(\nu_p(\tau)) - \tau(\Omega)$, where $b(\nu_p(\tau))$ is as in (8.3). We begin with the following lemma.

Lemma 9.1. *Let τ be an irreducible finite dimensional representation of G with highest weight $\tau_1 e_1 + \cdots + \tau_{n+1} e_{n+1}$ as in* (2.7)*. Then*

$$\tau(\Omega) - c(\sigma) \geq \tau_{n+1}^2$$

for all $\sigma \in \hat{M}$ with $[\nu_p(\tau)\colon \sigma] \neq 0$. Moreover assume that $\sigma \in \hat{M}$ is such that $[\nu_p(\tau)\colon \sigma] \neq 0$ and such that $\sigma = w_0\sigma$. Then one has

$$\tau(\Omega) - c(\sigma) \geq (\tau_n + 1)^2 + \tau_{n+1}^2 \geq 1 + \tau_{n+1}^2.$$

Proof. For $p = 0, ..., d$ let

$$\nu_p := \Lambda^p \operatorname{Ad}^*\colon K \to \mathrm{GL}(\Lambda^p \mathfrak{p}^*).$$

Recall that $\nu_p(\tau) = \nu_p \otimes \tau|_K$. Let $\nu \in \hat{K}$ with $[\nu_p(\tau) : \nu] \neq 0$. Then by [Kn2, Proposition 9.72] there exists a $\nu' \in \hat{K}$ with $[\tau : \nu'] \neq 0$ of highest weight $\Lambda(\nu') \in \mathfrak{b}_{\mathbb{C}}^*$ and a $\mu \in \mathfrak{b}_{\mathbb{C}}^*$ which is a weight of ν_p such that the highest weight $\Lambda(\nu)$ of ν is given by $\mu + \Lambda(\nu')$. Now let $\nu' \in \hat{K}$ be such that $[\tau : \nu'] \neq 0$. Let $\Lambda(\nu')$ be the highest weight of ν' as in (2.8). Then by Proposition 2.1 we have

$$\tau_{j-1} \geq k_j(\nu'), \quad j = 2, \dots, n+1.$$

Moreover, every weight $\mu \in \mathfrak{b}_{\mathbb{C}}^*$ of ν_p is given as

$$\mu = \pm e_{j_1} \pm \cdots \pm e_{j_p}, \quad 2 \leq j_1 < \cdots < j_p \leq n+1.$$

Thus, if $\nu \in \hat{K}$ is such that $[\nu_p(\tau) : \nu] \neq 0$, the highest weight $\Lambda(\nu)$ of ν, given as in (2.8), satisfies

$$\tau_{j-1} + 1 \geq k_j(\nu), \quad j \in \{2, \dots, n+1\}.$$

Let $\sigma \in \hat{M}$ be such that $[\nu_p(\tau) : \sigma] \neq 0$. Using Proposition 2.2 one obtains

$$\tau_{j-1} + 1 \geq |k_j(\sigma)|$$

for every $j \in \{2, \dots, n+1\}$, where the $k_j(\sigma)$ are as in (2.9). Furthermore note that by (2.5) we have $\rho_{j-1} = \rho_j + 1$. Using (2.24), (2.27) and $\rho_{n+1} = 0$, we get

$$c(\sigma) = \sum_{j=2}^{n+1} (k_j(\sigma) + \rho_j)^2 - \sum_{j=1}^{n+1} \rho_j^2 \leq \sum_{j=2}^{n+1} (\tau_{j-1} + \rho_{j-1})^2 - \sum_{j=1}^{n+1} \rho_j^2 = \tau(\Omega) - \tau_{n+1}^2.$$

Now assume that σ additionally satisfies $\sigma = w_0\sigma$. This is equivalent to $k_{n+1}(\sigma) = 0$ by (2.14). Thus since $\rho_{n+1} = 0$, $\rho_n = 1$ we get

$$c(\sigma) = \sum_{j=2}^{n} (k_j(\sigma) + \rho_j)^2 - \sum_{j=1}^{n+1} \rho_j^2 \leq \sum_{j=2}^{n} (\tau_{j-1} + \rho_{j-1})^2 - \sum_{j=1}^{n+1} \rho_j^2 = \tau(\Omega) - (\tau_n + 1)^2 - \tau_{n+1}^2.$$

Finally by (2.7) we have $\tau_n \geq 0$. This proves the lemma. □

Now we study the point spectrum of the operators $\Delta_p(\tau)$.

Lemma 9.2. *Let $\tau \in \hat{G}$ and assume that $\tau \neq \tau_\theta$. For $p \in \{0, \dots, d\}$ let λ_0 be an eigenvalue of $\Delta_p(\tau)$. Then one has $\lambda_0 \geq 1/4$.*

Proof. If $\tau \neq \tau_\theta$ one has $|\tau_{n+1}| \geq 1/2$. Let $\hat{G}_{\mathrm{un}}$ be the unitary dual of G. Recall that if λ_0 is an eigenvalue of $\Delta_p(\tau)$, there exists a $\pi \in \hat{G}_{\mathrm{un}}$ with $[\pi : \check{\nu}_p(\tau)] = [\pi : \nu_p(\tau)] \neq 0$ such that

$$\lambda_0 = -\pi(\Omega) + \tau(\Omega).$$

Here we use that every $\nu \in \hat{K}$ is self-contragredient. Since $\mathrm{rk}(G) > \mathrm{rk}(K)$, it follows from [Kn1, Theorem 8.54] and [Tr, Corollary 6.2] that $\hat{G}_{\mathrm{un}}$ consist of the unitary principal-series representations $\pi_{\sigma,\lambda}$, $\sigma \in \hat{M}$, $\lambda \in \mathbb{R}$ and the complementary series representations $\pi^c_{\sigma,\lambda}$, $\sigma \in \hat{M}$, $\lambda \in \mathbb{R}$. First consider a unitary principal series representation $\pi_{\sigma,\lambda}$. Then by Frobenius reciprocity [Kn1, page 208], $[\pi_{\sigma,\lambda} : \nu_p(\tau)]$ is non zero iff $[\nu_p(\tau) : \sigma]$ is non zero. Thus together with Corollary 2.6 and Lemma 9.1, for every $\lambda \in \mathbb{R}$ one has

$$-\pi_{\sigma,\lambda}(\Omega) + \tau(\Omega) = -c(\sigma) + \lambda^2 + \tau(\Omega) \geq 1/4.$$

Next consider a complementary series representation $\pi^c_{\sigma,\lambda}$. Again it follows from Frobenius reciprocity that $[\pi_{\sigma,\lambda} : \nu_p(\tau)]$ is non zero iff $[\nu_p(\tau) : \sigma]$ is non zero. Moreover by [KS, Proposition 49, Propostion 53], if $\pi^c_{\sigma,\lambda}$ belongs to the complementary series one has $\sigma = w_0\sigma$ and $0 < \lambda < 1$. Recall that by Corollary 2.6 one has

$$\pi^c_{\sigma,\lambda}(\Omega) = c(\sigma) + \lambda^2.$$

Thus together with Lemma 9.1 one gets

$$-\pi^c_{\sigma,\lambda}(\Omega) + \tau(\Omega) = -\lambda^2 - c(\sigma) + \tau(\Omega) \geq -\lambda^2 + 1 + \tau_{n+1}^2 \geq 1/4.$$

This proves the lemma. □

Now we obtain the following corollary.

Corollary 9.3. *Let $\tau \in \hat{G}$ and assume that $\tau \neq \tau_\theta$. Then one has $b(\nu_p(\tau)) - \tau(\Omega) \leq -1/4$ for every $p = 0, \dots, d$, where $b(\nu_p(\tau))$ is as in* (8.3).

Proof. The condition $\tau \neq \tau_\theta$ is equivalent to $\tau_{n+1}^2 \geq 1/4$. Thus the corollary follows immediately from (9.4), Lemma 9.1 and Lemma 9.2. □

Now let $\tau \in \hat{G}$ and assume that $\tau \neq \tau_\theta$. Then by equation (9.4) and Corollary 9.3 the relative determinants $\det_{\mathrm{u}}(\Delta_p(\tau)) \in \mathbb{R}^+$ of the operators $\Delta_p(\tau)$, $p = 0, \dots, d$ are defined as in section 8.1. In analogy to the compact case we now define the relative analytic torsion $T_{X,\mathrm{u}}(\tau) \in \mathbb{R}^+$ associated to the the bundle E_τ and the admissible metric by

$$T_{X,\mathrm{u}}(\tau) := \prod_{p=0}^{d} \det_{\mathrm{u}}(\Delta_p(\tau))^{\frac{p(-1)^{p+1}}{2}}.$$

Let

$$K_{\mathrm{u}}(t,\tau) := \sum_{p=0}^{d} (-1)^p p \operatorname{Tr}_{\mathrm{rel,u}}(e^{-t\Delta_p(\tau)}).$$

Then by the definition of the analytic torsion one has

$$\log T_{X,\mathrm{u}}(\tau) = \frac{1}{2} \frac{d}{dz}\bigg|_{z=0} \left(\frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} K_{\mathrm{u}}(t,\tau) dt \right), \tag{9.5}$$

where by Proposition 8.5 the right hand side is defined near $z = 0$ by analytic continuation of the Mellin transform. For every $p = 0, \dots, d$ let $\nu_p(\tau)$ be the representation defined by (9.1) and let $h_t^{\nu_p(\tau)}$ be as in (5.9). Put

$$k_t^\tau := e^{-t\tau(\Omega)} \sum_{p=0}^{d} (-1)^p p h_t^{\nu_p(\tau)}. \tag{9.6}$$

Moreover, if $R_{\mathrm{u}}(t, \nu_p(\tau))$ is as in (6.18), put

$$R_{\mathrm{u}}(t,\tau) := e^{-t\tau(\Omega)} \sum_{p=0}^{d} (-1)^p p R_{\mathrm{u}}(t, \nu_p(\tau)).$$

Then by equation (6.20) in Theorem 6.1 we have

$$K_{\mathrm{u}}(t,\tau) = I(k_t^\tau) + H(k_t^\tau) + T(k_t^\tau) + \mathcal{I}(k_t^\tau) + J(k_t^\tau) + R_{\mathrm{u}}(t,\tau). \tag{9.7}$$

Next we want to express $T_{X,\mathrm{u}}(\tau)$ using the relative graded determinants of certain auxiliary differential operators. These operators are defined as follows. For $k = 0, \dots, n$ let $\lambda_{\tau,k} \in \mathbb{R}$ be as in (2.30) and let $\sigma_{\tau,k}$ be the representation of M with highest weight (2.31). Let

$E(\sigma_{\tau(m),k})$ be the graded vector bundle over X as in section 7.1. We define a differential operator $\Delta_\tau(k)$ acting on the graded bundle $E(\sigma_{\tau,k})$ by

$$\Delta_\tau(k) := A(\sigma_{\tau,k}) + \lambda_{\tau,k}^2, \tag{9.8}$$

where $A(\sigma_{\tau,k})$ is as in section 7.1. We now want to assure that the relative graded determinant $\det_{\mathrm{gr,u}}(\Delta_\tau(k))$ is defined for $\tau \neq \tau_\theta$.

Lemma 9.4. *Let $\tau \in \hat{G}$ and assume that $\tau \neq \tau_\theta$. For $k = 0, \dots, n$ let $b(\sigma_{\tau,k})$ be as in (8.7). Then $b(\sigma_{\tau,k}) = 0$.*

Proof. Since $\tau \neq \tau_\theta$ we have $\sigma_{\tau(m),k} \neq w_0\sigma_{\tau(m),k}$ by (2.10), (2.14) and (2.31). Thus the lemma follows from Proposition 8.25. □

Now let $\tau \in \hat{G}$, $\tau \neq \tau_\theta$. Then by (2.10) and (2.30) one has $\lambda_{\tau,k}^2 \geq 1/4$. Thus by Lemma 9.4, the relative graded determinant $\det_{\mathrm{gr,u}}(\Delta_\tau(k)) = \det_{\mathrm{gr,u}}(A(\sigma_{\tau,k}) + \lambda_{\tau,k}^2) \in \mathbb{R}^+$ is defined for every $k = 0, \dots, n$ as in section 8.2. Furthermore, the following proposition holds.

Proposition 9.5. *Let $\tau \in \hat{G}$, $\tau \neq \tau_\theta$. For $k = 0, \dots, n$ let $h_t^{\sigma_{\tau,k}}$ be as in (7.6). Then one has*

$$k_t^\tau = \sum_{k=0}^{n} (-1)^{k+1} e^{-t\lambda_{\tau,k}^2} h_t^{\sigma_{\tau,k}}. \tag{9.9}$$

Proof. It is easy to see that as M-modules $\mathfrak{p}$ and $\mathfrak{a} \oplus \mathfrak{n}$ are equivalent. Thus in the sense of M-modules one has

$$\sum_{p=0}^{d} (-1)^p p\, \Lambda^p \mathfrak{p}^* = \sum_{p=0}^{d} (-1)^p p \left(\Lambda^p \mathfrak{n}^* + \Lambda^{p-1} \mathfrak{n}^* \right) = \sum_{p=0}^{d-1} (-1)^{p+1} \Lambda^p \mathfrak{n}^*. \tag{9.10}$$

Let $i^* \colon R(K) \to R(M)^{W(A)}$ be the restriction map. Then it follows from (9.10), Corollary 2.8 and (2.32) that we have

$$\sum_{p=0}^{d} (-1)^p p\, i^*(\nu_p(\tau)) = \sum_{k=0}^{n} (-1)^{k+1} (\sigma_{\tau,k} + w_0\sigma_{\tau,k}). \tag{9.11}$$

Since $\tau \neq \tau_\theta$ we have $\sigma_{\tau,k} \neq w_0\sigma_{\tau,k}$ for all k by (2.10), (2.14) and (2.31). Thus as in (7.2) we can write

$$\sigma_{\tau,k} + w_0\sigma_{\tau,k} = \sum_{\nu \in \hat{K}} m_\nu(\sigma_{\tau,k}) i^*(\nu).$$

Moreover, the restriction map i^* is injective. Therefore in $R(K)$ the following equality holds:

$$\sum_{p=0}^{d} (-1)^p p \nu_p(\tau) = \sum_{k=0}^{n} (-1)^{k+1} \sum_{\nu \in \hat{K}} m_\nu(\sigma_{\tau,k}) \nu.$$

Since $R(K)$ is a free abelian group generated by the representations $\nu \in \hat{K}$, it follows that for every $\nu \in \hat{K}$ one has

$$\sum_{p=0}^{d}(-1)^p p\,[\nu_p(\tau):\nu] = \sum_{k=0}^{n}(-1)^{k+1} m_\nu(\sigma_{\tau,k}). \tag{9.12}$$

Moreover let us remark that if ν, ν_1, ν_2 are finite dimensional unitary representations of K with $\nu = \nu_1 \oplus \nu_2$ one has

$$h_t^\nu = h_t^{\nu_1} + h_t^{\nu_2}. \tag{9.13}$$

Thus one computes

$$\begin{aligned}
k_t^\tau = \sum_{p=0}^{d}(-1)^p p e^{-t\tau(\Omega)} h_t^{\nu_p(\tau)} &= \sum_{p=0}^{d}(-1)^p p \sum_{\nu\in\hat{K}} [\nu_p(\tau):\nu]\, e^{-t\tau(\Omega)} h_t^\nu \\
&= \sum_{\nu\in\hat{K}} \sum_{p=0}^{d}(-1)^p p\,[\nu_p(\tau):\nu]\, e^{-t\tau(\Omega)} h_t^\nu \\
&= \sum_{\nu\in\hat{K}} \sum_{k=0}^{n}(-1)^{k+1} m_\nu(\sigma_{\tau,k}) e^{-t(\tau(\Omega))} h_t^\nu \qquad (+) \\
&= \sum_{k=0}^{n}(-1)^{k+1} \sum_{\nu\in\hat{K}} m_\nu(\sigma_{\tau,k}) e^{-t(\tau(\Omega))} h_t^\nu \\
&= \sum_{k=0}^{n}(-1)^{k+1} \sum_{\nu\in\hat{K}} m_\nu(\sigma_{\tau,k}) e^{-t\left(\lambda_{\tau,k}^2 + c(\sigma_{\tau,k})\right)} h_t^\nu \qquad (++) \\
&= \sum_{k=0}^{n}(-1)^{k+1} e^{-t\lambda_{\tau,k}^2} h_t^{\sigma_{\tau,k}}. \qquad (+++)
\end{aligned}$$

Here the second equality in the first line follows from (9.13), (+) is (9.12), (++) follows from Proposition 2.10 and $(+++)$ follows from (7.6).

$\square$

Corollary 9.6. *Let $\tau \in \hat{G}$, $\tau \neq \tau_\theta$. Then one has*

$$T_{X,\mathrm{u}}(\tau)^2 = \prod_{k=0}^{n} \det_{\mathrm{gr,u}}\left(\Delta_\tau(k)\right)^{(-1)^k}.$$

Proof. If ν, ν_1, ν_2 are finite dimensional unitary representations of K with $\nu = \nu_1 \oplus \nu_2$ one has $R_{\mathrm{u}}(t,\nu) = R_{\mathrm{u}}(t,\nu_1) + R_{\mathrm{u}}(t,\nu_2)$. Thus using (9.12) and Proposition 2.10 one gets

$$R_{\mathrm{u}}(t,\tau) = e^{-t\tau(\Omega)} \sum_{k=0}^{n}(-1)^{k+1} \sum_{\nu\in\hat{K}} m_\nu(\sigma_{\tau,k}) R_{\mathrm{u}}(t,\nu) = \sum_{k=0}^{n}(-1)^{k+1} e^{-t\lambda_{\tau,k}^2} R_{\mathrm{u}}(t,\sigma_{\tau,k}),$$

where $R_{\mathrm{u}}(t,\sigma_{\tau,k})$ is as in (8.22). Applying Proposition 8.20, equation (9.5), equation (9.7), and Proposition 9.5, the corollary follows. $\square$

9.2 L^2-torsion

In this section we briefly discuss the L^2-torsion $T_X^{(2)}(\tau)$. Although X is not compact, the L^2-torsion can be defined as in the compact case [Lo]. Let $\tau \in \hat{G}$ be of highest weight $\tau_1 e_1 + \cdots + \tau_{n+1} e_{n+1}$ and assume that $\tau_{n+1} \neq 0$. Let $\widetilde{\Delta}_p(\tau)$ be the Laplace operator on $\widetilde{E}_\tau$-valued p-forms on $\widetilde{X}$. Let $\nu_p(\tau)$ be as in (9.1) and let $C^\infty(G, \nu_p(\tau))$ be as in (5.1). By (9.4), on $C^\infty(G, \nu_p(\tau)$ the kernel of $e^{-t\widetilde{\Delta}_p(\tau)}$ is given by $e^{-t\tau(\Omega)} H_t^{\nu_p(\tau)}$ where $H_t^{\nu_p(\tau)}$ is as in (5.5). Then the Γ-trace of $e^{-t\widetilde{\Delta}_p(\tau)}$ (see [Lo] for its definition) is given by

$$\operatorname{Tr}_\Gamma\left(e^{-t\widetilde{\Delta}_p(\tau)}\right) = \operatorname{vol}(X) e^{-t\tau(\Omega)} h_t^{\nu_p(\tau)}(1), \tag{9.14}$$

where $h_t^{\nu_p(\tau)}(1)$ is as in (5.9). Using the Plancherel theorem and Proposition 5.1, we get

$$\operatorname{Tr}_\Gamma\left(e^{-t\widetilde{\Delta}_p(\tau)}\right) = \operatorname{vol}(X) \sum_{\substack{\sigma \in \hat{M} \\ [\nu_p(\tau):\sigma] \neq 0}} e^{-t(\tau(\Omega) - c(\sigma))} \int_{\mathbb{R}} e^{-t\lambda^2} P_\sigma(i\lambda)\, d\lambda. \tag{9.15}$$

Here $P_\sigma(z)$ is as in (2.34). Since $P_\sigma(z)$ is an even polynomial of degree $d-1$ it follows from Lemma 8.3 that we have an asymptotic expansion

$$\operatorname{Tr}_\Gamma\left(e^{-t\widetilde{\Delta}_p(\tau)}\right) \sim \sum_{j=0}^\infty a_j t^{j-d/2}, \quad t \to 0.$$

Since we are assuming that the highest weight of τ satisfies $\tau_{n+1} \neq 0$, it follows from Lemma 9.1 and (9.15) that there exists $c > 0$ such that

$$\operatorname{Tr}_\Gamma\left(e^{-t\widetilde{\Delta}_p(\tau)}\right) = O\left(e^{-ct}\right),$$

as $t \to \infty$. Therefore the Mellin transform

$$\int_0^\infty \operatorname{Tr}_\Gamma\left(e^{-t\widetilde{\Delta}_p(\tau)}\right) t^{z-1}\, dt$$

converges absolutely and uniformly on compact subsets of $\operatorname{Re}(z) > d/2$ and admits a meromorphic extension to $\mathbb{C}$. Thus we can define the L^2-torsion $T_X^{(2)}(\tau) \in \mathbb{R}^+$ by

$$\log T_X^{(2)}(\tau) = \frac{1}{2} \frac{d}{dz}\bigg|_{z=0} \left(\frac{1}{\Gamma(z)} \sum_{p=1}^d (-1)^p p \int_{\mathbb{R}} \operatorname{Tr}_\Gamma\left(e^{-t\widetilde{\Delta}_p(\tau)}\right)\, t^{z-1}\, dt \right). \tag{9.16}$$

Now recall that the contribution of the identity $I(k_t^\tau)$ to the right hand side of (9.7) is given by

$$I(t,\tau) := \operatorname{vol}(X) k_t^\tau(1).$$

Let

$$\mathcal{M}I(z,\tau) := \int_0^\infty I(t,\tau) t^{z-1}\, dt$$

be the Mellin transform. Using (9.6) and the considerations above, it follows that the integral converges for $\mathrm{Re}(z) > d/2$ and has a meromorphic extension to $\mathbb{C}$ which is regular at $z = 0$. Let $\mathcal{M}I(\tau)$ be its value at $z = 0$. Then by (9.6), (9.14), and (9.16) we have

$$\log T_X^{(2)}(\tau) = \frac{1}{2}\mathcal{M}I(\tau). \tag{9.17}$$

For $k = 0, \dots, n$ let the $\lambda_{\tau,k}$ be as in (2.30) and let $\sigma_{\tau,k} \in \hat{M}$ be of highest weight $\Lambda(\sigma_{\tau,k})$ as in (2.31). Then, since $\tau_{n+1} \neq 0$, one has $\lambda_{\tau,k} \neq 0$ for every $k = 0, \dots, n$. Thus for every k, $\mathcal{LM}I(s; \sigma_{\tau,k})$ is defined at $s = |\lambda_{\tau,k}|$, where $\mathcal{LM}I(s; \sigma_{\tau,k})$ is as in Proposition 8.11. Moreover, by equation (9.17) and Proposition 9.5 we have

$$\log T_X^{(2)}(\tau) = \frac{1}{2}\mathcal{M}I(\tau) = \frac{1}{2}\sum_{k=0}^{n}(-1)^{k+1}\mathcal{LM}I(s; \sigma_{\tau,k})|_{s=|\lambda_{\tau,k}|}. \tag{9.18}$$

9.3 The asymptotic behaviour of the relative analytic torsion

We fix natural numbers $\tau_1, \dots, \tau_{n+1}$ with $\tau_1 \geq \tau_2 \geq \dots \geq \tau_{n+1}$ and for $m \in \mathbb{N}$ we let $\tau(m)$ be the representation of G with highest weight $(m + \tau_1)e_1 + \dots + (m + \tau_{n+1})e_{n+1}$. Then the relative analytic torsion $T_{X,\mathrm{u}}(\tau(m))$ is defined as in section 9.1. We want to study the asymptotic behaviour of $\log T_{X,\mathrm{u}}(\tau(m))$ as $m \to \infty$.
To begin with, let $\lambda_{\tau(m),k} \in \mathbb{R}$ and $\sigma_{\tau(m),k} \in \hat{M}$ with highest weight $\Lambda(\sigma_{\tau(m),k})$ be as in (2.30) resp. (2.31). One has

$$\lambda_{\tau(m),k} = m + \tau_{k+1} + n - k. \tag{9.19}$$

Thus, since $\tau_k > 0$ for every k by assumption, it follows that

$$\lambda_{\tau(m),k} > 0, \quad k = 0, \dots, n. \tag{9.20}$$

Moreover, one has

$$\begin{aligned}\Lambda(\sigma_{\tau(m),k}) =&(m + \tau_1 + 1)e_2 + \dots + (m + \tau_k + 1)e_{k+1} \\ &+ (m + \tau_{k+2})e_{k+2} + \dots + (m + \tau_{n+1})e_{n+1}.\end{aligned} \tag{9.21}$$

Using the notations of section 8.2, Proposition 8.23, Lemma 9.4 and Corollary 9.6 give

$$\begin{aligned}&\log T_{X,\mathrm{u}}(\tau(m)) \\ =&\frac{1}{2}\sum_{k=0}^{n}(-1)^{k+1}\Big(-\log S(s, \sigma_{\tau(m),k}) + \mathcal{LM}I(s, \sigma_{\tau(m),k}) + \mathcal{LM}T(s, \sigma_{\tau(m),k}) \\ &+ \mathcal{LM}\mathcal{I}(s, \sigma_{\tau(m),k}) + \mathcal{LM}J(s, \sigma_{\tau(m),k}) + \mathcal{LM}R_{\mathrm{u}}(s, \sigma_{\tau(m),k})\Big)\Big|_{s=\lambda_{\tau(m),k}}.\end{aligned} \tag{9.22}$$

We will treat each summand on the right hand side of (9.22) separately.

We first treat the identity contribution. We want to apply Proposition 8.10 to the right hand side of (9.18). Thus we have to examine the Plancherel polynomial further.

Lemma 9.7. *The Plancherel polynomial $P_{\sigma_{\tau(m),k}}(t)$ is given by*

$$P_{\sigma_{\tau(m),k}}(t) = -c(n)(-1)^k \dim(\tau(m)) \prod_{\substack{j=0\\ j\neq k}}^{n} \frac{t^2 - \lambda^2_{\tau(m),j}}{\lambda^2_{\tau(m),k} - \lambda^2_{\tau(m),j}}.$$

where $c(n)$ is the constant occurring in the description of the Plancherel polynomial in equation (2.34).

Proof. This is proved in Bröcker's thesis [Br, p. 60]. For convenience, we recall the proof. To save notation, put

$$\lambda_i := \lambda_{\tau(m),i}, \quad i = 0, \dots, n.$$

By (2.31) we have

$$\begin{aligned}\Lambda(\sigma_{\tau(m),k}) + \rho_M &= \sum_{i=2}^{k+1} (\tau_{i-1} + m + 2 + n - i)e_i + \sum_{i=k+2}^{n+1} (\tau_i + m + n + 1 - i)e_i \\ &= \sum_{i=2}^{k+1} \lambda_{i-2} e_i + \sum_{i=k+2}^{n+1} \lambda_{i-1} e_i.\end{aligned}$$

Hence by (2.34) and (2.11) we have

$$\begin{aligned}P_{\sigma_{\tau(m),k}}(t) =& -c(n) \prod_{j=1}^{n} \prod_{q=j+1}^{n+1} \frac{\left\langle te_1 + \sum_{i=2}^{k+1} \lambda_{i-2}e_i + \sum_{i=k+2}^{n+1} \lambda_{i-1}e_i, e_j + e_q \right\rangle}{\left\langle \sum_{l=1}^{n+1} \rho_l e_l, e_j + e_q \right\rangle} \\ & \cdot \prod_{j=1}^{n} \prod_{q=j+1}^{n+1} \frac{\left\langle te_1 + \sum_{i=2}^{k+1} \lambda_{i-2}e_i + \sum_{i=k+2}^{n+1} \lambda_{i-1}e_i, e_j - e_q \right\rangle}{\left\langle \sum_{l=1}^{n+1} \rho_l e_l, e_j - e_q \right\rangle} \\ =& -c(n) \prod_{\substack{0\leq i\leq n\\ i\neq k}} \left(t^2 - \lambda_i^2\right) \prod_{\substack{0\leq j<i\leq n\\ i,j\neq k}} \left(\lambda_j^2 - \lambda_i^2\right) \prod_{1\leq j<i\leq n+1} (\rho_j^2 - \rho_i^2)^{-1} \qquad (9.23) \\ =& -c(n)(-1)^k \prod_{0\leq j<i\leq n} \frac{\lambda_j^2 - \lambda_i^2}{\rho_{j+1}^2 - \rho_{i+1}^2} \prod_{\substack{j=0\\ j\neq k}}^{n} \frac{t^2 - \lambda_j^2}{\lambda_k^2 - \lambda_j^2} \\ =& -c(n)(-1)^k \dim(\tau(m)) \prod_{\substack{j=0\\ j\neq k}}^{n} \frac{t^2 - \lambda_j^2}{\lambda_k^2 - \lambda_j^2}.\end{aligned}$$

□

Lemma 9.8. *For every sequence $s_0, \dots, s_n$, $s_i \neq s_j$ for $i \neq j$, one has*

$$\sum_{k=0}^{n} \prod_{\substack{j=0\\ j\neq k}}^{n} \frac{t - s_j}{s_k - s_j} = 1.$$

Proof. The expression is a polynomial in t of order n and is equal to 1 at the $n+1$ points $s_0, \dots, s_n$. ☐

Corollary 9.9. *One has*

$$\sum_{k=0}^{n}(-1)^k P_{\sigma_{\tau(m),k}}(t) = -c(n)\dim(\tau(m)).$$

Proof. This follows from Lemma 9.7 and Lemma 9.8. ☐

Proposition 9.10. *Let $c(n)$ be the constant from* (2.34)*. Then one has*

$$\sum_{k=0}^{n}(-1)^k \int_0^{\lambda_{\tau(m),k}} P_{\sigma_{\tau(m),k}}(t)dt = -c(n)m\dim(\tau(m)) + O(m^{\frac{n(n+1)}{2}}),$$

as $m \to \infty$.

Proof. By (2.30) we have $\lambda_{\tau(m),0} > \lambda_{\tau(m),1} > \dots > \lambda_{\tau(m),n}$. By Lemma 9.7 and Corollary 9.9, we have

$$\begin{aligned}
\sum_{k=0}^{n}(-1)^k \int_0^{\lambda_{\tau(m),k}} P_{\sigma_{\tau(m),k}}(t)dt &= \int_0^{\lambda_{\tau(m),n}} \sum_{k=0}^{n}(-1)^k P_{\sigma_{\tau(m),k}}(t)dt \\
&\quad - c(n)\dim(\tau(m)) \sum_{k=0}^{n-1} \int_{\lambda_{\tau(m),n}}^{\lambda_{\tau(m),k}} \prod_{\substack{j=0\\ j\neq k}}^{n} \frac{t^2-\lambda^2_{\tau(m),j}}{\lambda^2_{\tau(m),k}-\lambda^2_{\tau(m),j}}dt \\
&= -c(n)\lambda_{\tau(m),n}\dim\tau(m) \\
&\quad - c(n)\dim\tau(m) \sum_{k=0}^{n-1} \int_{\lambda_{\tau(m),n}}^{\lambda_{\tau(m),k}} \prod_{\substack{j=0\\ j\neq k}}^{n} \frac{t^2-\lambda^2_{\tau(m),j}}{\lambda^2_{\tau(m),k}-\lambda^2_{\tau(m),j}}\,dt.
\end{aligned} \tag{9.24}$$

Now recall that by (2.30) we have

$$\lambda_{\tau(m),k} = \tau_{k+1} + m + n - k. \tag{9.25}$$

It follows easily from (2.11) that there exists a constant $C > 0$ such that

$$\dim\tau(m) = Cm^{\frac{n(n+1)}{2}} + O(m^{\frac{n(n+1)}{2}-1}), \quad m \longrightarrow \infty. \tag{9.26}$$

Together with (9.26) it follows that the first term on the right hand side of (9.24) equals $-c(n)m\dim\tau(m) + O(m^{\frac{n(n+1)}{2}})$. Furthermore we have

$$\int_{\lambda_{\tau(m),n}}^{\lambda_{\tau(m),k}} \prod_{\substack{j=0\\ j\neq k}}^{n} \frac{t^2-\lambda^2_{\tau(m),j}}{\lambda^2_{\tau(m),k}-\lambda^2_{\tau(m),j}}\,dt = \int_{\tau_{n+1}}^{\tau_{k+1}+n-k} \prod_{\substack{j=0\\ j\neq k}}^{n} \frac{(t+m)^2-\lambda^2_{\tau(m),j}}{\lambda^2_{\tau(m),k}-\lambda^2_{\tau(m),j}}\,dt. \tag{9.27}$$

Using (9.25), a direct computation shows that the integrand on right hand side is bounded as $m \to \infty$. Hence we get

$$\sum_{k=0}^{n-1} \int_{\lambda_{\tau(m),n}}^{\lambda_{\tau(m),k}} \prod_{\substack{j=0 \\ j\neq k}}^{n} \frac{t^2 - \lambda_{\tau(m),j}^2}{\lambda_{\tau(m),k}^2 - \lambda_{\tau(m),j}^2} = O(1), \text{ as } m \to \infty. \tag{9.28}$$

Using (9.26), we can estimate the second term on the right hand side of (9.24) by $O(m^{\frac{n(n+1)}{2}})$. This concludes the proof. ∎

We can summarize the asymptotic behaviour of the identity contribution respectively the L^2-torsion in the following proposition.

Proposition 9.11. *There exists a polynomial $P_\tau(m)$, whose coefficients depend only on τ, such that*

$$\log T_X^{(2)}(\tau(m)) = \frac{1}{2}\mathcal{M}I(\tau(m)) = \operatorname{vol}(X)P_\tau(m).$$

Moreover, if $c(n) \neq 0$ is the constant from (2.34), *one has*

$$P_\tau(m) = -2\pi c(n)m \dim(\tau(m)) + O(m^{\frac{n(n+1)}{2}}),$$

as $m \to \infty$.

Proof. By (9.21) we have $\epsilon(\sigma_{\tau(m),k}) = 2$ for every k. Thus by Proposition 8.11 and equation (9.18) one has

$$\log T_X^{(2)}(\tau(m)) = \frac{1}{2}\mathcal{M}I(\tau(m)) = 2\pi \operatorname{vol}(X) \sum_{k=0}^{n} (-1)^k \int_0^{\lambda_{\tau(m),k}} P_{\sigma_{\tau(m),k}}(t)\, dt.$$

By (9.19) and by the explicit form of the Plancherel polynomials given in the first equation of (9.23) it follows that

$$P_\tau(m) := 2\pi \sum_{k=0}^{n} (-1)^k \int_0^{\lambda_{\tau(m),k}} P_{\sigma_{\tau(m),k}}(t)\, dt$$

is a polynomial in m whose coefficients depend only on τ. The proposition follows from Proposition 9.10. ∎

Applying (2.33) it follows that equations (1.5) and (1.6) are proved and in order to prove Theorem 1.3, Theorem 1.4 and Theorem 1.5, we have to study the asymptotic behaviour of the summands in (9.22) which are different from the identity contribution. Concerning the contribution of the Selberg zeta functions, we have the following proposition.

Proposition 9.12. *Let $k \in \{0, \dots, n\}$. Then there exists a constant C such that*

$$\left|\log S(s, \sigma_{\tau(m),k})\big|_{s=\lambda_{\tau(m),k}}\right| \leq Ce^{-m/2}$$

for all $m \geq 16n$.

Proof. First by (9.19) we have $\lambda_{\tau(m),k} > 16n$ if $m > 16n$ and thus $S(s, \sigma_{\tau(m),k})$ is regular at $s = \lambda_{\tau(m),k}$. Moreover by (2.12) and (9.21) there exists a constant $C > 0$ such that

$$\dim(\sigma_{\tau(m),k}) \leq Cm^{\frac{n(n-1)}{2}} \tag{9.29}$$

for every m. If we use Proposition 3.14 and (9.19), the proposition follows. □

Now if X is compact, the distributions T, $\mathcal{I}$, J and the contribution of the auxiliary operator on the cusp vanish by definition. Thus it follows from equation (9.22) that for compact X we have

$$\log T_X(\tau(m)) = \frac{1}{2}\mathcal{M}I(\tau(m)) + \frac{1}{2}\sum_{k=0}^{n}(-1)^k \log S(s, \sigma_{\tau(m),k})\big|_{s=\lambda_{\tau(m),k}}.$$

Employing Propostion 9.11 and Proposition 9.12, Theorem 1.5 is proved.

We continue with the investigation of the asymptotic behaviour of the summands occuring on the right hand side of (9.22). The contribution of the distribution T can be estimated as follows.

Proposition 9.13. *For $k = 0, \dots, n$ let $\mathcal{LMT}(\sigma_{\tau(m),k}, s)$ be as in Proposition 8.14. Let*

$$\mathcal{M}T(\tau(m)) := \sum_{k=0}^{n}(-1)^{k+1}\mathcal{LM}T(s, \sigma_{\tau(m),k})|_{s=\lambda_{\tau(m),k}}.$$

Then there exists a constant C such that fore every m one has

$$|\mathcal{M}T(\tau(m))| \leq Cm^{\frac{n(n+1)}{2}}.$$

Proof. This follows immediately from Proposition 8.14, (9.19) and (9.29). □

Next we treat the contribution of the distribution $\mathcal{I}$ to the relative analytic torsion. Therefore we let

$$\mathcal{M}\mathcal{I}(\tau(m)) := \sum_{k=0}^{n}(-1)^{k+1}\mathcal{LM}\mathcal{I}(s, \sigma_{\tau(m),k})|_{s=\lambda_{\tau(m),k}},$$

where $\mathcal{LM}\mathcal{I}(s, \sigma_{\tau(m),k})$ is as in Proposition 8.13. By (9.20) each $\mathcal{LM}\mathcal{I}(s, \sigma_{\tau(m),k})$ is defined at $s = \lambda_{\tau(m),k}$. In order to estimate $\mathcal{M}\mathcal{I}(\tau(m))$ as $m \to \infty$, we start with the following three lemmas.

Lemma 9.14. *There exists a constant C such that for every m one has*

$$\sum_{k=0}^{n}(-1)^{k+1}\dim(\sigma_{\tau(m),k})\left(\log\Gamma(k_{n+1}(\sigma_{\tau(m),k})+\lambda_{\tau(m),k})+\gamma\lambda_{\tau(m),k}+C(\psi)\right)\leq Cm^{\frac{n(n+1)}{2}},$$

where $C(\psi)$ is as in Proposition 8.13.

Proof. By (2.32) and Corollary 2.8 one has

$$2\sum_{k=0}^{n}(-1)^{k+1}\dim\sigma_{\tau(m),k}=-\dim(\tau)\sum_{p=0}^{2n}(-1)^{p}\dim\Lambda^{p}\mathfrak{n}^{*}=0.$$

Thus the sum in the Lemma equals

$$\sum_{k=0}^{n}(-1)^{k+1}\dim(\sigma_{\tau(m),k})\left(\log\frac{\Gamma(k_{n+1}(\sigma_{\tau(m),k})+\lambda_{\tau(m),k})}{\Gamma(2m)}+\gamma\lambda_{\tau(m),k}\right).$$

It follows from (9.19) and (9.21) that there exists a constant C which is independent of m such that

$$\log\frac{\Gamma(k_{n+1}(\sigma_{\tau(m),k})+\lambda_{\tau(m),k})}{\Gamma(2m)}\leq C\log m.$$

Using (9.19) and (9.29) the proposition is proved. ∎

Lemma 9.15. *Let $k\in\{0,\dots,n\}$ and let $j\in\{2,\dots,n+1\}$. Then there exists a constant C such that for every m one has*

$$\left|P_{j}(\sigma_{\tau(m),k},\lambda)\right|\leq Cm^{\frac{(n-1)(n-2)}{2}}\sum_{i=0}^{2(n-1)}(1+|\lambda|)^{i}m^{2(n-1)-i}\tag{9.30}$$

and such that

$$\left|\frac{d}{d\lambda}P_{j}(\sigma_{\tau(m),k},\lambda)\right|\leq Cm^{\frac{(n-1)(n-2)}{2}}\sum_{i=0}^{2(n-1)-1}(1+|\lambda|)^{i}m^{2(n-1)-i}\tag{9.31}$$

for all $\lambda\in\mathbb{C}$. Here the polynomial $P_j(\sigma_{\tau(m),k},z)$ is as in (6.27).

Proof. This follows easily from (6.28), (9.21) and (9.29). ∎

Lemma 9.16. *Let $k\in\{0,\dots,n\}$ and let $j\in\{2,\dots,n+1\}$. As in (6.31), for $l\in\mathbb{N}$ with $k_{n+1}(\sigma_{\tau(m),k})\leq l\leq k_j(\sigma_{\tau(m),k})+\rho_j$ define an even polynomial $Q_{j,l}(s,\sigma_{\tau(m),k})$ by*

$$Q_{j,l}(\sigma_{\tau(m),k},i\lambda)=\frac{P_{j}(\sigma_{\tau(m),k},i\lambda)-P_{j}(\sigma_{\tau(m),k},il)}{l-\lambda}+\frac{P_{j}(\sigma_{\tau(m),k},i\lambda)-P_{j}(\sigma_{\tau(m),k},il)}{l+\lambda},$$

where the even polynomial $P_j(\sigma_{\tau(m),k},s)$ is defined as in (6.27). Then there exists a constant C such that for every m one has

$$\left|\int_{0}^{\lambda_{\tau(m),k}}Q_{j,l}(\sigma_{\tau(m),k},i\lambda)d\lambda\right|\leq Cm^{\frac{n(n+1)}{2}}.$$

Proof. Using the fact that $P_j(z,\sigma)$ is an even polynomial, equation (9.19) and equation (9.31) give the estimate

$$\left|\int_0^{\lambda_{\tau(m),k}} Q_{j,l}(\sigma_{\tau(m),k}, i\lambda)d\lambda\right| \leq 2\lambda_{\tau(m),k} \max_{|\xi|\leq l+\lambda_{\tau(m),k}} \left|\frac{d}{d\lambda}\Big|_{\lambda=\xi} P_j(\sigma, i\lambda)\right| \leq Cm^{\frac{n(n+1)}{2}}.$$

□

Now we can estimate $\mathcal{MI}(\tau(m))$ as $m \to \infty$.

Proposition 9.17. *There exists a constant C such that for every m one has*

$$\mathcal{MI}(\tau(m)) \leq Cm^{\frac{n(n+1)}{2}}.$$

Proof. Let $k \in \{0,\dots,n\}$. By (9.21) we have $\epsilon(\sigma_{\tau(m),k}) = 2$. Moreover, by Proposition 8.13, (6.31), (6.32) and (6.33) one obtains

$$\begin{aligned}
&\mathcal{LMI}(\lambda_{\tau(m),k}, \sigma_{\tau(m),k}) = 2p\dim(\sigma_{\tau(m),k})\left(\log\Gamma(k_{n+1}(\sigma_{\tau(m),k}) + \lambda_{\tau(m),k}) + \gamma\lambda_{\tau(m),k} + C(\psi)\right)\\
&+ p\sum_{j=2}^{n+1} \sum_{\substack{k_{n+1}(\sigma_{\tau(m),k})\leq l\\ <k_j(\sigma_{\tau(m),k})+\rho_j}} \left(2P_j(\sigma_{\tau(m),k}, il)\log\left(l+\lambda_{\tau(m),k}\right) + \int_0^{\lambda_{\tau(m),k}} Q_{j,l}(\sigma_{\tau(m),k}, i\lambda)d\lambda\right)\\
&+ p\sum_{\substack{j=2\\ l=k_j(\sigma_{\tau(m),k})+\rho_j}}^{n+1} \left(\dim(\sigma_{\tau(m),k})\log\left(l+\lambda_{\tau(m),k}\right) + \frac{1}{2}\int_0^{\lambda_{\tau(m),k}} Q_{j,l}(\sigma_{\tau(m),k}, i\lambda)d\lambda\right).
\end{aligned}$$

By (9.21) for every $j \in \{2,\dots,n+1\}$ one has $m \leq k_j(\sigma_{\tau(m),k}) + \rho_j \leq m + \tau_1 + n$ and by (9.19) we have $\lambda_{\tau(m),k} \leq m + \tau_1 + n$. Thus if we apply Lemma 9.14, Lemma 9.15, Lemma 9.16 and the estimate (9.29), the proposition is proved. □

Now we treat the asymptotics of the contribution of the non-invariant distribution J to $\log T_X(\tau(m))$. Define

$$\mathcal{M}J(\tau(m)) := \sum_{k=0}^{n}(-1)^{k+1}\mathcal{LM}J(s, \sigma_{\tau(m),k})|_{s=\lambda_{\tau(m),k}},$$

where $\mathcal{LM}J(s;\sigma_{\tau(m),k})$ is as in Proposition 8.17. By (9.20), each $\mathcal{LM}J(s;\sigma_{\tau(m),k})$ is defined at $s = \lambda_{\tau(m),k}$. Moreover, the asymptotic behaviour of $\mathcal{M}J(\tau(m))$ can be estimated using the following proposition.

Proposition 9.18. *Let $k \in \{0,\dots,n\}$. Then there exists a constant C such that for every $m > 1$ one has*

$$\left|\mathcal{LM}J(s, \sigma_{\tau(m),k})|_{s=\lambda_{\tau(m),k}}\right| \leq Cm^{\frac{n(n+1)}{2}}\log m.$$

Proof. By (9.21) we have $m \leq k_{n+1}(\sigma_{\tau(m),k})$. Moreover, by (2.17) and (9.21), if $\nu \in \hat{K}$ is such that $m_\nu(\sigma_{\tau(m),k}) \neq 0$, one has

$$k_j(\nu) \leq m + \tau_1 + 1$$

for every $j \in \{2, \dots, n+1\}$. Thus applying Proposition 8.17 we can estimate

$$\begin{aligned}\left|\mathcal{LMJ}(\lambda_{\tau(m),k}, \sigma_{\tau(m),k})\right| \leq p \sum_{\nu \in \hat{K}} \left|m_\nu(\sigma_{\tau(m),k})\right| \sum_{\sigma \in \hat{M}} [\nu : \sigma] \dim(\sigma) \\ \cdot \sum_{j=2}^{n+1} \sum_{l=m}^{m+\tau_1+1} \left|\log\left(\sqrt{\lambda_{\tau(m),k}^2 + c(\sigma_{\tau(m),k}) - c(\sigma)} + l + \rho_j\right)\right|.\end{aligned}$$

Let $\nu \in \hat{K}$ such that $m_\nu(\sigma_{\tau(m),k}) \neq 0$. Let $\sigma \in \hat{M}$ such that $[\nu : \sigma] \neq 0$. Then by Lemma 8.16 and (9.19) we have

$$m \leq \sqrt{\lambda_{\tau(m),k}^2 + c(\sigma_{\tau(m),k}) - c(\sigma)} \leq \sqrt{\lambda_{\tau(m),k}^2 + c(\sigma_{\tau(m),k})}.$$

Hence by (2.27), (9.19) and (9.21) there exists a constant C_1 which is independent of ν and σ such that for every m we can estimate

$$m \leq \sqrt{\lambda_{\tau(m),k}^2 + c(\sigma_{\tau(m),k}) - c(\sigma)} \leq C_1 m$$

Thus there exists a constant C_2 which is independent of ν and σ such that for every $m > 1$ we can estimate

$$\sum_{j=2}^{n+1} \sum_{l=m}^{m+\tau_1+1} \left|\log\left(\sqrt{\lambda_{\tau(m),k}^2 + c(\sigma_{\tau(m),k}) - c(\sigma)} + l + \rho_j\right)\right| \leq C_2 \log m.$$

All in all, it follows that there exists a constant C_3 such that for every $m > 1$ we can estimate

$$\begin{aligned}\left|\mathcal{LMJ}(\lambda_{\tau(m),k}, \sigma_{\tau(m),k})\right| \leq & C_3 \log m \sum_{\nu \in \hat{K}} \left|m_\nu(\sigma_{\tau(m),k})\right| \sum_{\sigma \in \hat{M}} [\nu : \sigma] \dim(\sigma) \\ = & C_3 \log m \sum_{\nu \in \hat{K}} \left|m_\nu(\sigma_{\tau(m),k})\right| \dim(\nu).\end{aligned}$$

Now by (2.17) the number of $\nu \in \hat{K}$ with $m_\nu(\sigma_{\tau(m),k}) \neq 0$ is bounded by 2^n and one has $\left|m_\nu(\sigma_{\tau(m),k})\right| \leq 1$ for every $\nu \in \hat{K}$. Moreover, by (2.17), (9.21) and (2.13) there exists a constant C_4 which is independent of m such that for each $\nu \in \hat{K}$ with $m_\nu(\sigma_{\tau(m),k}) \neq 0$ one has

$$\dim(\nu) \leq C_4 m^{\frac{n(n+1)}{2}}.$$

This proves the proposition. ∎

To complete our investigation of the asymptotic behaviour of $\log T_{X,\mathrm{u}}(\tau(m))$ as $m \to \infty$, we have to deal with the contribution of the auxiliary differential operators on the cusp which depends on the parameter u. Thus let

$$\mathcal{M}R_{\mathrm{u}}(\tau(m)) := \sum_{k=0}^{n}(-1)^{k+1}\mathcal{L}\mathcal{M}R_{\mathrm{u}}(s,\sigma_{\tau(m),k})\Big|_{s=\lambda_{\tau(m),k}},$$

where $\mathcal{L}\mathcal{M}R_{\mathrm{u}}(s,\sigma_{\tau(m),k})$ is as in Proposition 8.19. By (9.20), $\mathcal{L}\mathcal{M}R_{\mathrm{u}}(s,\sigma_{\tau(m),k})$ is defined at $s = \lambda_{\tau(m),k}$. Moreover, if we combine Proposition 8.19, equation (9.19) and equation (9.29) , it follows that there exists a constant $C > 0$ such that for every m one has

$$\mathcal{M}R_{\mathrm{u}}(\tau(m)) \leq Cm^{\frac{n(n+1)}{2}}. \tag{9.32}$$

Now Theorem 1.3 and Theorem 1.4 follow if one combines equation (9.22), equation (9.18), Proposition 9.11, Proposition 9.12 Proposition 9.13, Proposition 9.17, Proposition 9.18 and equation (9.32)

10 The Ruelle zeta function and the relative analytic torsion in the 3-dimensional case

10.1 Preliminaries

In this section we treat the case of a hyperbolic 3-manifold in more detail. Our goal is to relate the relative analytic torsion $T_X(\tau)$ to the behaviour of a certain twisted Ruelle zeta function R_τ at 0 for certain representations $\tau \in \hat{G}$ which satisfy $\tau \neq \tau_\theta$. This is an extension of results obtained by Bröcker and Wotzke to the non-compact case.
Let us first introduce some notation. From now on, we will let $G = \mathrm{Spin}(3,1)$. Then there is a canonical isomorphism $G \cong \mathrm{SL}_2(\mathbb{C})$. This isomorphism induces an isomorphism $K \cong \mathrm{SU}(2)$, $M \cong \mathrm{SO}(2)$. For $l \in \frac{1}{2}\mathbb{N}$ we let ν_l be the representation of K with highest weight le_2. Similarly, for $k \in \frac{1}{2}\mathbb{Z}$ we let σ_k be the representation of M with highest weight ke_2. Then σ_k is one-dimensional. Our parametrization is different from the parametrization of [Mü4] but consistent with the notation used before. Proposition 2.3 has the following explicit form.

Lemma 10.1. *For $\sigma \in \hat{M}$, $\sigma \neq w_0\sigma$, and $\nu \in \hat{K}$ let $m_\nu(\sigma)$ be as in* (7.2)*. Then if $\sigma = \sigma_k$, with $k \in \frac{1}{2}\mathbb{N}$, $k > 1/2$, we have $m_\nu(\sigma) = 1$ for $\nu = \nu_k$, $m_\nu(\sigma) = -1$ for $\nu = \nu_{k-1}$ and $m_\nu(\sigma) = 0$ otherwise.*

Proof. This follows from Proposition 2.3 but is also proved directly in [Mü4, equation (4.2)]. □

Some of the results obtained previously can be made more explicit in the 3-dimensional case, so let us restate them. Firstly, Theorem 6.2 and Proposition 6.7 simplify as follows.

Proposition 10.2. *Let $\Omega(\sigma_k, \lambda)$ be as in Theorem 6.2. Then for $k \in \mathbb{N}_0$ we have*

$$\begin{aligned}\Omega(\sigma_k, \lambda) = &- 2\gamma - \psi(1 + i\lambda) - \psi(1 - i\lambda) \\ &- \sum_{1 \leq l < k} \frac{2l}{\lambda^2 + l^2} - \frac{k}{\lambda^2 + k^2}.\end{aligned}$$

and

$$\begin{aligned}\Omega(\sigma_{k+1/2}, \lambda) = &- 2\gamma - \psi\left(\frac{1}{2} + i\lambda\right) - \psi\left(\frac{1}{2} - i\lambda\right) \\ &- \sum_{0 \leq l < k} \frac{2(l + 1/2)}{\lambda^2 + (l + 1/2)^2} - \frac{k + 1/2}{\lambda^2 + (k + 1/2)^2}.\end{aligned}$$

Proof. This follows easily from Theorem 6.2 and Proposition 6.7. □

Secondly, the contribution of the Knapp-Stein intertwining operators to the relative determinant of the auxiliary operators can be computed explicitly as follows.

Proposition 10.3. *For $\sigma \in \hat{M}$ let $\mathcal{LMJ}(s, \sigma)$ be as in Proposition 8.17. Then for $k \in \mathbb{N}$ one has*

$$\mathcal{LMJ}(s, \sigma_k) = -2p \sum_{j=1}^{k-1} \log\left(\sqrt{s^2 + k^2 - j^2} + k\right) - p \log\left(\sqrt{s^2 + k^2} + k\right) - p \log(s + k)$$

and

$$\begin{aligned}\mathcal{LMJ}(s, \sigma_{k+1/2}) = &- 2p \sum_{j=0}^{k-1} \log\left(\sqrt{s^2 + (k + 1/2)^2 - (j + 1/2)^2} + k + 1/2\right) \\ &- p \log(s + k + 1/2).\end{aligned}$$

Proof. The first equation follows from equation (2.27), Proposition 8.17 and Lemma 10.1. The second equation follows from equation (2.27), equation (4.17), Lemma 8.8, and Lemma 10.1. □

We can now formulate the determinant formula for the symmetric Selberg zeta function. For convenience, from now on we shall work with regularized determinants and regularized torsions defined according to Remark 5.9 rather than with the corresponding relative objects. Then it follows easily from the construction that all the results obtained before continue to hold for the regularized objects if one drops the terms which depend on the parameter u.

For our purposes it suffices to study the case where σ is not invariant under the restricted Weyl group, i.e. σ is non-trivial.

Proposition 10.4. *Let $s \in \mathbb{C}$, $\mathrm{Re}(s) > 0$, $\mathrm{Re}(s^2) > 0$. Let $k \in \frac{1}{2}\mathbb{N}$. Let p be the number of cusps of X and let $c_\Gamma := 2\,(C(\Gamma) - \gamma p)$, where $C(\Gamma)$ is as in (6.8). Then there exists a constant $c \neq 0$ depending only on the parity of $2k$ such that*

$$\begin{aligned} S(s, \sigma_k) = & e^{cp} \det\nolimits_{\mathrm{gr}}\big(A(\sigma) + s^2\big) \exp\left(-4\pi \operatorname{vol}(X) \int_0^s P_{\sigma_k}(r)dr\right)\Gamma^{2p}(s+k) \\ & \cdot (s+k)^p \exp\left(\mathcal{LMJ}(s,\sigma_k) - sc_\Gamma\right). \end{aligned}$$

Proof. By Proposition 8.25 we have $b(\sigma_k) = 0$. Thus for $k \in \mathbb{N}$, the proposition follows from Proposition 8.23 and Proposition 10.2. For $k \in \frac{1}{2}\mathbb{N} - \mathbb{N}$, we can replace the function $\psi(1+i\lambda)$ in Lemma 8.9 by $\psi(1/2+i\lambda)$ and obtain an analogous statement. Then together with Proposition 10.2 and the arguments of section 8.2 and section 8.3, the determinant formula follows also for $k \in \frac{1}{2}\mathbb{N} - \mathbb{N}$. □

Let $\tau \in \hat{G}$ be a finite dimensional irreducible representation of G. Then we define the twisted Ruelle zeta function $R_\tau(s)$ by

$$R_\tau(s) := \prod_{\substack{[\gamma] \in \mathrm{C}(\Gamma)_{\mathrm{s}} - [1] \\ [\gamma] \text{ prime}}} \det\left(\mathrm{Id} - \tau(\gamma) e^{-s\ell(\gamma)}\right). \tag{10.33}$$

Arguing as in [Mü4, section 3] and employing Proposition 3.8, it follows that the infinite product in (10.33) converges for $\mathrm{Re}(s)$ sufficiently large. As in [Mü4, equation (3.14)], the function R_τ can be expressed as a product of Ruelle zeta functions $R(s,\sigma)$ with shifted arguments. Thus by Corollary 7.9 the function $R_\tau(s)$ has a meromorphic continuation to $\mathbb{C}$. Moreover, using Kostants theorem, for $\tau \neq \tau_\theta$ one can express the symmetrized Ruelle zeta function in terms of symmetrized Selberg zeta functions as

$$R_\tau(s) R_{\tau_\theta}(s) = \prod_{w \in W^1} S(s - \lambda_{\tau,w}, \sigma_{\tau,w})^{(-1)^{l(w)}}, \tag{10.34}$$

where the notations are as in section 2.9. This identity is, up to a sign, one of the key results obtained by Wotzke in his thesis, see [Wo, Satz 6.1]. For the 3-dimensional case, equation (10.34) is proved in [Mü4, Proposition 3.5]. The proof doesn't use the cocompactness of Γ. From now on, for $m \in \frac{1}{2}\mathbb{N}$ we let $\tau(m)$ denote the representation of G with highest weight $me_1 + me_2$. Then, if we identify $G \cong \mathrm{SL}_2(\mathbb{C})$, $\tau(m)$ is the $2m$-th symmetric power of the standard representation of $\mathrm{SL}_2(\mathbb{C})$. Thus, $\tau(m)$ corresponds to the representation $\tau_{2m,0} = \tau_{2m}$ of [Mü4]. It follows from (10.34), (2.32), (9.19) and (9.21) that

$$R_{\tau(m)}(s) R_{\tau(m)_\theta}(s) = \frac{S(s+m+1, \sigma_m) S(s-m-1, \sigma_m)}{S(s+m, \sigma_{m+1}) S(s-m, \sigma_{m+1})}. \tag{10.35}$$

Moreover, together with Corollary 9.6 it follows that

$$T_X(\tau(m))^2 = \frac{\det_{\mathrm{gr}}(A(\sigma_m) + (m+1)^2)}{\det_{\mathrm{gr}}(A(\sigma_{m+1}) + m^2)}. \tag{10.36}$$

10.2 The functional equation of the symmetric Ruelle and Selberg zeta functions

Let $\sigma \in \hat{M}$. Using (7.19), we want to prove a functional equation for the symmetrized Selberg zeta function $S(s,\sigma)$.
Let us first symmetrize the scattering matrices. For $\sigma \in \hat{M}$, $\sigma \neq w_0\sigma$ and $\nu \in \hat{K}$ we let $\overline{\boldsymbol{\mathcal{E}}}(\boldsymbol{\sigma},\nu) := \boldsymbol{\mathcal{E}}(\boldsymbol{\sigma},\nu) \oplus \boldsymbol{\mathcal{E}}(\boldsymbol{w_0\sigma},\nu)$ and for $s \in \mathbb{C}$ we let

$$\overline{\mathbf{C}}(\sigma:\nu:s): \ \ \overline{\boldsymbol{\mathcal{E}}}(\boldsymbol{\sigma},\nu) \to \overline{\boldsymbol{\mathcal{E}}}(\boldsymbol{\sigma},\nu)$$

be equal to

$$\overline{\mathbf{C}}(\sigma:\nu:s) := \mathbf{C}(\sigma:\nu:s) \oplus \mathbf{C}(w_0\sigma:\nu:s).$$

For $\sigma \in \hat{M}$ we let ν_σ be as in section 4.2. Then, if $\sigma = \sigma_k$ we have $\nu_\sigma = \nu_{|k|}$. By the arguments of section 4.2, $\left(\det \overline{\mathbf{C}}(\sigma:\nu:s)\right)^{\frac{1}{\dim(\nu)}}$ is canonically defined. To save notation, for $k \in \frac{1}{2}\mathbb{N}$ we shall write

$$\left(\det \overline{\mathbf{C}}(\sigma_k:\nu_k:s)\right)^{\frac{1}{\dim(\nu)}} =: \mathbf{C}(k:s).$$

Then $\mathbf{C}(k:s)$ is a meromorphic function of s which has no zeroes and poles for $s \in i\mathbb{R}$. By (4.10) it satisfies $\mathbf{C}(k:s)\mathbf{C}(k:-s) = 1$.
We have the following functional equation for the symmetrized Selberg zeta function.

Proposition 10.5. *Let $k \in \frac{1}{2}\mathbb{N}$ and let c_Γ be as in Proposition 10.4. Then the symmetric Selberg zeta function $S(s,\sigma_k)$ satisfies the functional equation*

$$S(-s,\sigma_k) = S(s,\sigma_k)\exp\left(8\pi \operatorname{vol}(X)\int_0^s P_{\sigma_k}(r)dr + 2c_\Gamma s\right)\frac{(\Gamma(-s+k))^{2p}}{(\Gamma(s+k))^{2p}}\mathbf{C}(k:s)\mathbf{C}(k:0)^{-1}.$$

Proof. Let

$$\Xi(s,\sigma_k) := \exp\left(4\pi\operatorname{vol}(X)\int_0^s P_{\sigma_k}(r)dr + sc_\Gamma\right)\left(\Gamma\left(s+k\right)\right)^{-2p}\cdot S(s,\sigma_k).$$

It follows from (7.19), Proposition 10.2, (4.10) and (4.22) that

$$\frac{\Xi'(s,\sigma_k)}{\Xi(s,\sigma_k)} + \frac{\Xi'(-s,\sigma_k)}{\Xi(-s,\sigma_k)} = -\frac{d}{ds}\log \mathbf{C}(k:s).$$

Hence the logarithmic derivative of

$$\frac{\Xi(s,\sigma)}{\Xi(-s,\sigma)}\mathbf{C}(k:s)$$

is zero and so this function is constant. Now the order of the singularity of the function $\Xi(s,\sigma_k)$ at 0 is the same as the order of the singularity of $S(s,\sigma)$ at 0. This order is even by Theorem 7.4. Since $P_{\sigma_k}(r)$ is an even polynomial, the proposition follows. □

Let $\sigma \in \hat{M}$, $\sigma \neq w_0\sigma$. Then we define the symmetrized Ruelle zeta function $R_{\mathrm{sym}}(s,\sigma)$ by

$$R_{\mathrm{sym}}(s,\sigma) = R(s,\sigma)R(s,w_0\sigma).$$

It satisfies the following functional equation.

Proposition 10.6. *Let $k \in \frac{1}{2}\mathbb{N}$. Then the symmetrized Ruelle zeta function $R_{\mathrm{sym}}(s,\sigma_k)$ satisfies the functional equation*

$$\begin{aligned} R_{\mathrm{sym}}(-s,\sigma_k) =& R_{\mathrm{sym}}(s,\sigma_k)\exp\left(-\frac{8}{\pi}\operatorname{vol}(X)s\right)\frac{\mathbf{C}(k:s-1)\mathbf{C}(k:s+1)}{\mathbf{C}(k+1:s)\mathbf{C}(k-1:s)} \\ &\cdot \frac{\mathbf{C}(k+1:0)\mathbf{C}(k-1:0)}{\mathbf{C}(k:0)^2}. \end{aligned}$$

Proof. By Proposition 3.15 for every $k \in \frac{1}{2}\mathbb{Z}$ we have

$$R_{\mathrm{sym}}(s,\sigma_k) = \frac{S(s+1,\sigma_k)S(s-1,\sigma_k)}{S(s,\sigma_{k+1})S(s,\sigma_{k-1})}.$$

Moreover, using (2.33) and (2.34) we obtain

$$\int_0^{s+1} P_{\sigma_k}(r)dr + \int_0^{s-1} P_{\sigma_k}(r)dr - \int_0^{s} P_{\sigma_{k+1}}(r)dr - \int_0^{s} P_{\sigma_{k-1}}(r)dr = -\frac{s}{\pi^2}.$$

Thus the proposition follows from Proposition 10.5. □

To prove Corollary 1.7, we will also need the following proposition.

Proposition 10.7. *Let $m \in \mathbb{N}$, $m \geq 3$. Then*

$$\begin{aligned} R_{\tau(m)}(s)R_{\tau(m)_\theta}(s) =& R_{\tau(2)}(s)R_{\tau(2)_\theta}(s)\frac{\mathbf{C}(m:m+1-s)}{\mathbf{C}(m+1:m-s)}\frac{\mathbf{C}(3:2-s)}{\mathbf{C}(2:3-s)}\frac{\mathbf{C}(m+1:0)}{\mathbf{C}(m:0)}\frac{\mathbf{C}(2:0)}{\mathbf{C}(3:0)} \\ &\cdot \prod_{k=3}^{m} R_{\mathrm{sym}}(k-s,\sigma_k)R_{\mathrm{sym}}(k+s,\sigma_k)\exp\left(-\frac{8}{\pi}\operatorname{vol}(X)(k-s)\right). \end{aligned}$$

Similarly, for $m \in \mathbb{N}$ $m \geq 2$ one has

$$\begin{aligned} R_{\tau(m+1/2)}(s)R_{\tau(m+1/2)_\theta}(s) =& R_{\tau(3/2)}(s)R_{\tau(3/2)_\theta}(s)\cdot\frac{\mathbf{C}(m+1/2:m+3/2-s)}{\mathbf{C}(m+3/2:m+1/2-s)}\frac{\mathbf{C}(5/2:3/2-s)}{\mathbf{C}(3/2:5/2-s)} \\ &\cdot \frac{\mathbf{C}(m+3/2:0)}{\mathbf{C}(m+1/2:0)}\frac{\mathbf{C}(3/2:0)}{\mathbf{C}(5/2:0)} \\ &\cdot \prod_{k=2}^{m} R_{\mathrm{sym}}(k+1/2+s,\sigma_{k+1/2})R_{\mathrm{sym}}(k+1/2-s,\sigma_{k+1/2}) \\ &\cdot \prod_{k=2}^{m} \exp\left(-\frac{8}{\pi}\operatorname{vol}(X)(k+1/2-s)\right). \end{aligned}$$

Proof. Let $m \in \mathbb{N}$, $m \geq 3$. Applying [Mü4, equation (3.14)], we can symmetrize [Mü4, equation (8.2)] and obtain

$$R_{\tau(m)}(s)R_{\tau(m)_\theta}(s) = R_{\tau(2)}(s)R_{\tau(2)_\theta}(s)\prod_{k=3}^{m} R_{\mathrm{sym}}(s+k,\sigma_k)R_{\mathrm{sym}}(s-k,\sigma_k).$$

Similarly, for $m \in \mathbb{N}$, $m \geq 2$, symmetrizing [Mü4, equation (8.3)] we have

$$\begin{aligned} &R_{\tau(m+1/2)}(s)R_{\tau(m+1/2)_\theta}(s) \\ =&R_{\tau(3/2)}(s)R_{\tau(3/2)_\theta}(s)\prod_{k=2}^{m} R_{\mathrm{sym}}(s+k+1/2,\sigma_{k+1/2})R_{\mathrm{sym}}(s-k-1/2,\sigma_{k+1/2}). \end{aligned}$$

Applying Proposition 10.6, the proposition follows.

□

10.3 Proof of the main results in the 3-dimensional case

In order to relate the behaviour of $R_\tau(m)R_{\tau(m)_\theta}$ at 0 to the relative analytic torsion $T_X(\tau(m))$, we want to apply the determinant formula for the symmetric Selberg zeta function from Proposition 10.4 to the right hand side of (10.35) and combine it with equation (10.36).

However, in contrast to a compact hyperbolic manifold, this is not possible directly since the determinant formula for the Selberg zeta function is valid only for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$, $\mathrm{Re}(s^2) > 0$. Thus we first have to apply the functional equation for the symmetric Selberg zeta function from Proposition 10.5. We obtain the following proposition.

Proposition 10.8. *One has*

$$\begin{aligned} R_{\tau(m)}(s)R_{\tau(m)_\theta}(s) =&e^{2c_\Gamma}\frac{S(-s+m+1,\sigma_m)S(s+m+1,\sigma_m)\mathbf{C}(m+1:0)}{S(-s+m,\sigma_{m+1})S(s+m,\sigma_{m+1})\mathbf{C}(m:0)} \\ &\cdot\frac{\mathbf{C}(m:m+1-s)\Gamma^{2p}(s-1)\exp\left(8\pi\,\mathrm{vol}(X)\int_0^{-s+m+1}P_{\sigma_m}(r)dr\right)}{\mathbf{C}(m+1:m-s)\Gamma^{2p}(s+1)\exp\left(8\pi\,\mathrm{vol}(X)\int_0^{-s+m}P_{\sigma_{m+1}}(r)dr\right)}. \end{aligned}$$

Proof. By (10.35) one has

$$R_{\tau(m)}(s)R_{\tau(m)_\theta}(s) = \frac{S(s-m-1,\sigma_m)S(s+m+1,\sigma_m)}{S(s-m,\sigma_{m+1})S(s+m,\sigma_{m+1})}$$

Now by proposition 10.5 we have

$$\begin{aligned} \frac{S(s-m-1,\sigma_m)}{S(s-m,\sigma_{m+1})} =&e^{2c_\Gamma}\frac{S(-s+m+1,\sigma_m)\exp\left(8\pi\,\mathrm{vol}(X)\int_0^{-s+m+1}P_{\sigma_m}(r)dr\right)}{S(-s+m,\sigma_{m+1})\exp\left(8\pi\,\mathrm{vol}(X)\int_0^{-s+m}P_{\sigma_{m+1}}(r)dr\right)} \\ &\cdot\frac{\mathbf{C}(m+1:0)\mathbf{C}(m:m+1-s)\Gamma^{2p}(s-1)}{\mathbf{C}(m:0)\mathbf{C}(m+1:m-s)\Gamma^{2p}(s+1)}. \end{aligned}$$

This proves the proposition. □

Now we can prove Theorem 1.6 and state it more precisely.

Theorem 10.9. *For $m \in \mathbb{N}$ we define a constant $c(\tau(m))$ by*

$$c(\tau(m)) := \left(\frac{\prod_{j=1}^{m-1} \sqrt{(m+1)^2 + m^2 - j^2} + m}{\prod_{j=1}^{m} \sqrt{(m+1)^2 + m^2 - j^2} + m + 1} \right)^{4p} \left(\frac{\sqrt{(m+1)^2 + m^2} + m}{\sqrt{(m+1)^2 + m^2} + m + 1} \right)^{2p}. \tag{10.37}$$

Similarly, for $m \in \mathbb{N}$ we define a constant $c(\tau(m+1/2))$ by

$$c(\tau(m+1/2)) := \left(\frac{\prod_{j=0}^{m-1} \sqrt{(m+3/2)^2 + (m+1/2)^2 - (j+1/2)^2} + m + 1/2}{\prod_{j=0}^{m} \sqrt{(m+3/2)^2 + (m+1/2)^2 - (j+1/2)^2} + m + 3/2} \right)^{4p}. \tag{10.38}$$

Then one has

$$T_X(\tau(m))^4 = c(\tau(m)) \frac{\mathbf{C}(m:0)}{\mathbf{C}(m+1:0)} \lim_{s \to 0} \left(R_{\tau(m)}(s) R_{\tau(m)_\theta}(s) \frac{\mathbf{C}(m+1:m-s)}{\mathbf{C}(m:m+1-s)} \Gamma^{-2p}(s-1) \right)$$

and

$$\begin{aligned} T_X(\tau(m+1/2))^4 = & c(\tau(m+1/2)) \frac{\mathbf{C}(m+1/2:0)}{\mathbf{C}(m+3/2:0)} \cdot \lim_{s \to 0} \Bigg(R_{\tau(m+1/2)}(s) R_{\tau(m+1/2)_\theta}(s) \\ & \frac{\mathbf{C}(m+3/2:m+1/2-s)}{\mathbf{C}(m+1/2:m+3/2-s)} \Gamma^{-2p}(s-1) \Bigg). \end{aligned}$$

Proof. Let $m \in \mathbb{N}$. To save notation, let us first introduce two auxiliary functions. Let

$$\begin{aligned} P_{\tau(m)}(s) := \exp\Bigg(& -4\pi \operatorname{vol}(X) \int_0^{s+m+1} P_{\sigma_m}(r) dr + 4\pi \operatorname{vol}(X) \int_0^{-s+m+1} P_{\sigma_m}(r) dr \\ & - 4\pi \operatorname{vol}(X) \int_0^{-s+m} P_{\sigma_{m+1}}(r) dr + 4\pi \operatorname{vol}(X) \int_0^{s+m} P_{\sigma_{m+1}}(r) dr \Bigg). \end{aligned}$$

Moreover, let

$$\begin{aligned} J_{\tau(m)}(s) := \exp\Bigg(& \mathcal{LMJ}(-s+m, \sigma_{m+1}) + \mathcal{LMJ}(s+m, \sigma_{m+1}) \\ & - \mathcal{LMJ}(-s+m+1, \sigma_m) - \mathcal{LMJ}(s+m+1, \sigma_m) \Bigg). \end{aligned}$$

Then by Proposition 10.8, Proposition 10.4 and equation (10.36) one has

$$\begin{aligned}
&\frac{\mathbf{C}(m:0)}{\mathbf{C}(m+1:0)}\lim_{s\to 0}\left(R_{\tau(m)}(s)R_{\tau(m)_\theta}(s)\frac{\mathbf{C}(m+1:m-s)\Gamma^{2p}(s+1)}{\mathbf{C}(m:m+1-s)\Gamma^{2p}(s-1)}J_{\tau(m)}(s)\right)\\
=&\lim_{s\to 0}\left(e^{2c_\Gamma}\frac{S(s+m+1,\sigma_m)S(-s+m+1,\sigma_m)}{S(s+m,\sigma_{m+1})S(-s+m,\sigma_{m+1})}J_{\tau(m)}(s)\right)\\
&\cdot\lim_{s\to 0}\exp\left(8\pi\operatorname{vol}(X)\int_0^{-s+m+1}P_{\sigma_m}(r)dr-8\pi\operatorname{vol}(X)\int_0^{-s+m}P_{\sigma_{m+1}}(r)dr\right)\\
=&\lim_{s\to 0}\frac{\det_{\mathrm{gr}}\left(A(\sigma_m)+(s+m+1)^2\right)\det_{\mathrm{gr}}\left(A(\sigma_m)+(-s+m+1)^2\right)}{\det_{\mathrm{gr}}\left(A(\sigma_{m+1})+(s+m)^2\right)\det_{\mathrm{gr}}\left(A(\sigma_{m+1})+(-s+m)^2\right)}P_{\tau(m)}(s)\\
=&\frac{\det^2_{\mathrm{gr}}\left(A(\sigma_m)+(m+1)^2\right)}{\det^2_{\mathrm{gr}}\left(A(\sigma_{m+1})+m^2\right)}\\
=&T_X(\tau(m))^4.
\end{aligned}$$

Here the determinant-functions are holomorphic for s in a neighbourhood of 0 by Proposition 8.25. The function $P_{\tau(m)}(s)$ is an entire function of s and one has $P_{\tau(m)}(0)=1$. Furthermore, by Proposition 10.3 the function $J_{\tau(m)}(s)$ is entire for s in a neighbourhood of zero and one has $J_{\tau(m)}(0)=c(\tau(m))$. For $\tau(m+1/2)$ we argue in the same way. This proves the theorem. □

Let us finally turn to the proof of Corollary 1.7. Let $m\in\mathbb{N}$, $m\geq 3$. By (3.8) and (3.10) the infinite product defining the Ruelle zeta functions $R(s,\sigma_k)$ converges absolutely for $s\in\mathbb{C}$ with $\operatorname{Re}(s)>2$. By Theorem 10.9 and Proposition 10.7 we have

$$\begin{aligned}
T_X(\tau(m))^4=&c(\tau(m))\frac{\mathbf{C}(m:0)}{\mathbf{C}(m+1:0)}\lim_{s\to 0}\left(R_{\tau(m)}(s)R_{\tau(m)_\theta}(s)\frac{\mathbf{C}(m+1:m-s)}{\mathbf{C}(m:m+1-s)}\Gamma^{-2p}(s-1)\right)\\
=&c(\tau(m))\frac{\mathbf{C}(2:0)}{\mathbf{C}(3:0)}\lim_{s\to 0}\left(R_{\tau(2)}(s)R_{\tau(2)_\theta}(s)\frac{\mathbf{C}(3:2-s)}{\mathbf{C}(2:3-s)}\Gamma^{-2p}(s-1)\right)\\
&\prod_{k=3}^{m}\exp\left(-\frac{8}{\pi}\operatorname{vol}(X)k\right)R_{\mathrm{sym}}(k,\sigma_k)^2\\
=&\frac{c(\tau(m))}{c(\tau(2)}T_X(\tau(2))^4\exp\left(-\frac{4}{\pi}\operatorname{vol}(X)(m(m+1)-6)\right)\prod_{k=3}^{m}R_{\mathrm{sym}}(k,\sigma_k)^2.
\end{aligned}$$

Now one has $\overline{\sigma}=w_0\sigma$ and so by the definition of the Ruelle zeta function and by meromorphic continuation one gets

$$\overline{R(\bar{s},w_0\sigma)}=R(s,\sigma).$$

Thus one has

$$R_{\mathrm{sym}}(k,\sigma_k)=\left|R(k,\sigma_k)\right|^2.$$

This proves the first equation in Corollary 1.7. The second equation is obtained in the same way.

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