Treatment of Double Default Effects within the Basel Regulatory Framework and a Theoretical and an Experimental Investigation of Higher-Order Risk Preferences

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Introduction

PART I of this dissertation is concerned with the treatment of double default effects in credit portfolios within the Basel regulatory framework. In two different contexts, we propose mathematical models to deal with this issue. The first model constitutes the first approach to this problem, i.e., there is no benchmark model. For the second model, we claim that it is superior to the one currently applied. Further, the approach can be used to tackle double default effects in a large class of credit risk models. This requires some explanation.

A bank subject to the Basel regulation must hold capital that can absorb possible losses. The minimum capital requirement, the so-called regulatory capital, should be risk sensitive, i.e., increasing in the default risk of the portfolio. This motivates the application of sophisticated risk management techniques in order to mitigate risk. Likewise, regulatory capital being risk sensitive mitigates the incentive to undertake excessive risk-taking. A major and important change to the First Basel Accord, "Basel I," is that, under the New Basel Accord, "Basel II," minimum capital requirements are much closer related to the actual risk of the credit portfolio. Unlike under Basel I, computation of regulatory capital under Basel II is based on a mathematical portfolio credit risk model, which has the individual default probability of each loan as a crucial input parameter. This parameter may be inferred from a rating agency, or may be estimated by the regulated entity itself by means of an internal model which has to be approved by supervisors.

The actual default probability of an exposure is smaller if the exposure is hedged in some way. This, in fact, happens quite frequently. For example, granting loans and transferring the risk afterwards is a typical practice for a bank. Such a credit risk transfer can be facilitated by use of numerous financial instruments. These include ordinary guarantees, collateral securitization, and credit derivatives such as credit default swaps, to name a few. Consider the simple case where the bank purchases insurance for a loan. Then, the insurance protection seller, i.e., the guarantor, will pay for the lost exposure if the obligor defaults. From the perspective of the regulated bank, the exposure is only lost if both

the obligor and the guarantor default, thus “double default.” The so-called *double default treatment* then specifies to what extent regulatory capital is reduced due to obtaining credit protection.
Basel II is structured into three parts, called pillars. Pillar 1 specifies minimum capital requirements for credit risk, market risk, and operational risk. In this dissertation, we focus on credit risk. When calculating the credit risk component of regulatory capital under Pillar I, *name concentration risk,* among other risks, is neglected. The computation of regulatory capital under Pillar 1 is based on the assumption that all idiosyncratic risk in the credit portfolio, i.e., risk specific to individual borrowers (“names”), has been diversified away. More specifically, it is assumed that portfolios are infinitely fine-grained in the sense that the largest individual exposures account for an infinitely small share of total portfolio exposure. However, the impact of undiversified idiosyncratic risk on regulatory capital can be assessed via a methodology known as *granularity adjustment* (GA). The GA is subject to supervisory discretion under Pillar 2 of Basel II, which lays out principles for the supervisory review process. Pillar 3 is concerned with market discipline and specifies, among other things, information release and accounting requirements, and is not particularly relevant to this treatise.
The first chapter of this dissertation is concerned with the treatment of double default effects within the GA. In fact, this work is the first to propose a method to account for double default effects within the GA, i.e., there is no benchmark model. In the second chapter, we discuss the current treatment of double default effects that is employed under Pillar 1 within the so-called Internal Ratings Based (IRB) approach. We criticize the IRB treatment of double default effects and propose a novel method that could be used instead. Although we illustrate the method with reference to the IRB model, it is actually more general in the sense that it offers an approach to tackle double default effects within any structural credit risk model.

Before we go into more detail on the contribution to research and practical implications for banking regulation of Part I of this dissertation, some general remarks on the Basel regulatory framework seem appropriate. The introduction so far had the objective to quickly explain the topic to readers without extensive background in finance. Much more could be said on the Basel regulatory framework. In particular, far more issues are addressed than have been mentioned.
The New Basel Accord of 2003, Basel II, was finalized in Basel Committee on Banking Supervision (2004). The treatment of double default effects within the IRB approach that

is a topic of this dissertation is inspired by Heitfield and Barger (2003) and appeared as an amendment to Basel II in Basel Committee on Banking Supervision (2005). Later on it was incorporated in a revised version of the Basel II document: Basel Committee on Banking Supervision (2006). This is the version we generally refer to as Basel II. For a qualitative introduction to Basel II see, e.g., Chapter 1.3 in McNeil et al. (2005). Lütkebohmert (2009) contains a more quantitative discussion of Basel II with a focus on the IRB model. A very noteworthy, early, and critical assessment of Basel II is given by Daníelsson et al. (2001).
In response to the financial crisis, a new compendium of reform proposals was published by the Basel Committee in Basel Committee on Banking Supervision (2009). The changes to Basel II proposed in that document were finalized a few months ago in Basel Committee on Banking Supervision (2010), a document that carries "Basel III" in its name. In the context of this treatise, it is important to understand that Basel III comprises amendments to Basel II, rather than replacing it. These amendments leave the relevance and topicality of our contributions unaffected. Further, note that Basel II still is the main body of the Basel banking regulation framework, which is why we generally refer to Basel II rather than to Basel III in this treatise. For recent comprehensive summaries of the Basel III reforms written by practitioners the reader may consult Banh et al. (2011) and Brzenk et al. (2011) as well as an article by Joel Clark in the January 2011 Issue of Risk, p. 9. Some interesting opinions of both regulators and bankers on the recent Basel III reforms can be found on pp. 44 - 49 of that issue.

Mathematical tools may help to improve efficient regulation. However, banking regulation does not boil down to a mathematical problem. Many important issues are more of a psychological, economic, legal, or political nature; see, for example, Ackermann (2010) or Cukierman (2011) for recent discussions of the challenges to banking regulation. The former German Secretary of Finance, Peer Steinbrück, makes ten recommendations for new banking regulation rules; see Steinbrück (2010, pp. 226-233). Generally, we feel that the author suggests that international coordination might pose the most serious obstacle. When making use of mathematical tools—as we do in this thesis—the tradeoff between benefit (e.g., rigor, more accurate results) and cost (e.g., increasing complexity and loss of tractability, which impede implementation) should be carefully weighed out. The recent changes of Basel III are fairly simple from a mathematical point of view. Nevertheless, the time schedule conceded to banks for implementation goes beyond 2018. Further, the United States of America still face problems with the implementation of Basel II. According to Daniel Tarullo, a governor at the Federal Reserve Board, banks and supervisors in the US had devoted substantial resources to the advanced IRB approach to Basel II, but continue to encounter significant difficulties in developing and validating models; see the article by Joel Clark in the December 2010 Issue of Risk, p. 20. According to that article, he further explains: "Although, fortunately, Basel III does not present nearly the degree of technical challenge posed by the advanced approach of Basel II, there will still be a good bit of opaqueness in how some of its components are implemented [...]."

The advantages as well as the disadvantages and assumptions of both our proposals in Chapters One and Two will be discussed in detail. We believe that our proposals constitute an improvement to the current version of Basel II and can contribute to a more stable banking system. The recent global financial crisis has drastically demonstrated the crucial importance of the latter to the real economies all around the world. As indicated in the previous paragraph, many issues have to be addressed, and every one of it is worth to be treated thoroughly. In this dissertation, we address the treatment of double default effects. A main focus in the development of our models has been set on making implementation not too burdensome. They are parsimonious in the sense that they impose little additional structure compared to the existing framework. Further, they do not require extensive data and can be embedded in the current version of Basel II. Therefore, implementation should not require too much time and effort. In the spirit of the previous discussion, we believe that this should be seen as a major advantage.

In **Chapter One**, we propose a treatment of double default effects within the GA for Basel II. Up to now, no such model has been proposed. More specifically, we provide an analytic formula for the GA in an extended single-factor CreditRisk$^+$ setting, incorporating double default effects. It relies on an approximation whose accuracy is verified using Monte Carlo techniques. The formula is a generalization of the GA derived in Gordy and Lütkebohmert (2007). The formula is very flexible in the sense that it fits to various hedging instruments and can accommodate partial hedging. We illustrate why this feature is particularly important under Basel II modeling. Further, it is useful to account, e.g., for a tracking error of the hedging instrument. The GA formula also distinguishes between guarantors that are themselves members of the portfolio and those that are not. The investigation of these features might be insightful for double default modeling in general, i.e., not limited to modeling under Basel II. Computation of the GA is fast and simple, and the GA makes use of inputs required for application of the IRB approach anyway.

Therefore, it is very well suited for application under Pillar 2 of Basel II.

In **Chapter Two**, we first discuss the IRB treatment of double default effects as currently applied under Pillar 1 of Basel II. Under that model, additional correlation is induced between obligors and guarantors. We criticize this approach because correlation is a symmetric measure of dependency, while the actual dependence relationship is asymmetric. Further, the induced correlation is the same for all guarantors and obligors, and it remains entirely unclear how the correlation parameter should be chosen. The approach further assumes that every loan is hedged by a different guarantor. Furthermore, every guarantor is assumed to be external to the portfolio. Therefore, regulatory capital is not sensitive to excessive contracting of the same guarantor.
To overcome these deficiencies, in Chapter Two we then propose a new approach to account for double default effects that can be applied in any model of portfolio credit risk and, in particular, within the IRB approach of Basel II. The model endogenously quantifies the impact of the guarantee payment on the guarantor's unconditional default probability. Within a structural model of portfolio credit risk, the guarantor's loss due to the guarantee payment corresponds to a downward jump in its firm value process. Therefore, we call it an *asset drop model*. In spite of its simplicity, the new approach does not show any of the above-mentioned shortcomings. Thus, it better reflects the risk associated with double defaults. Also this model is easily applicable in terms of data requirements and computational time, and thus is very well suited for application under Pillar 1 of Basel II.

Whereas the first part of this dissertation is concerned with *risk measurement*, **PART II** is concerned with *risk preferences*. The reader should not expect the topics of the two parts of this thesis to overlap much. While the research approach in the first part is rather pragmatic and applied, that of the second part could be seen as more fundamental. In Part II of this thesis, we first provide a theoretical analysis of higher-order risk preferences with reference to statistical moments. These mathematical results are particularly insightful for understanding their relationship to skewness preference and kurtosis aversion. In contrast, the last work presented in this thesis is of an empirical nature. We propose an experimental method to test for risk preferences of order three, i.e., prudence, and present results from a laboratory experiment. This method benefits from the theoretical insights presented before. None of our results are based on expected utility theory (EUT), and we claim that they contribute to the better understanding of this relatively young, unexplored, and promising field of economic research. Both works highlight the relevance of higher-order risk preferences for a comprehensive and more refined perception of risk attitudes, i.e., one that clearly distinguishes risk attitudes from risk aversion as commonly defined.

The concept of *risk aversion* plays a key role in analyzing decision making under risk. An established characterization is that an individual preferring a payoff with certainty over a risky payoff with the same mean is said to be risk averse (e.g., Gollier (2001, p. 18)). Alternatively, Rothschild and Stiglitz (1970) state that a risk averse individual dislikes any mean-preserving spread of the wealth distribution. Within EUT, these two characterizations coincide and are equivalent to the utility function being concave. Throughout this treatise, when speaking of EUT, we assume that the von-Neumann-Morgenstern utility function is sufficiently smooth. Risk aversion then corresponds to the second derivative of the utility function being negative.

It might come as a surprise that risk aversion, according to any of these definitions, does not exhaustively capture risk preferences. Indeed, risk aversion is just one piece in the puzzle that drives economic behavior under risk. For the better understanding of the latter, risk aversion must be complemented by *higher-order risk preferences* such as *prudence* (third-degree risk aversion) or *temperance* (fourth-degree risk aversion). Higher-order risk preferences are the topic of the second part of this dissertation.

Although Kimball (1990) coined the term "prudence," its implications have been used in assessing a precautionary demand for saving much earlier by Leland (1968) and Sandmo (1970). In particular, they show within an EUT setting how a risky future income does not guarantee that a consumer increases saving unless the individual exhibits prudence. In the recent years, as will be discussed later, numerous behavioral traits have been shown to be related to prudence under EUT. Within EUT, prudence and temperance can be defined as the third and fourth derivative of the utility function, respectively, being positive and negative.

In a different strand of literature, Menezes et al. (1980) show that the third derivative of the utility function being positive is equivalent to aversion to increases in downside risk. An increase in downside risk is a density transformation that leaves mean and variance of a distribution unchanged, but decreases its third moment. Thus, another definition of prudence is *downside risk aversion*. Similarly, temperance can be defined as *outer risk aversion*, i.e., aversion to density transformations that increase the fourth moment while leaving the first three moments unchanged. These definitions are independent of the EUT paradigm.

More recently, higher-order risk preferences have been defined by Eeckhoudt and Schlesinger (2006) as preferences over rather simple lottery pairs which imply *proper risk apportionment.* For example, given two equally likely future states, a prudent individual prefers to have an unavoidable zero-mean risk in the state where her wealth is higher. Equivalently, she prefers to have the unavoidable items of a sure loss and a zero-mean risk in different future states rather than in the same state. This new understanding of risk preferences does also not rely on EUT. Further, it can be generalized to the multi-attribute case as shown in Eeckhoudt et al. (2007) or Tsetlin and Winkler (2009). Proper risk apportionment is also related to a *preference for combining "good" with "bad;"* see Eeckhoudt et al. (2009). In this dissertation, we exploit this new definition to analyze higher-order risk preferences theoretically (Chapter Three) as well as to test for them by means of a laboratory experiment (Chapter Four). These projects will be given some more detail in the remainder of this introduction.

In **Chapter Three**, we present a moment characterization of higher-order risk preferences. That is, we compute all (natural) statistical moments of the proper risk apportionment lotteries. Our results, which are generalizations of Roger (2011) and Ekern (1980), give a better understanding of how higher-order risk preferences relate to skewness preference and kurtosis aversion. In particular, we show how higher-order risk preferences relate to the strong notions of skewness and kurtosis referring to *all* odd and even moments, respectively. As moments are well understood, our results should be easily accessible to a wide audience in economics and finance.
More specifically, we show that prudence implies skewness preference, and this preference is robust towards variation in kurtosis. We thus speak of the *kurtosis robustness feature* of prudence. Further, we show that all higher-order risk preferences of odd order imply skewness preference—but for different distributions than prudence—and also have a kurtosis robustness feature. Similar results are presented for temperance and higher-order risk preferences of even order that can be related to kurtosis aversion and have a *skewness robustness feature.*
We also show that the skewness of the zero-mean risks that have to be apportioned according to Eeckhoudt and Schlesinger's definition of proper risk apportionment are the source of the lotteries' statistical generality.
While our results are not based on EUT, an implication within that theory is that all commonly used utility functions exhibit skewness preference and kurtosis aversion.

On the empirical side, there is an extensive literature on the measurement of risk aversion in numerous empirical settings as well as in various experiments. Focusing on experiments, almost as large as the number of experimental studies is the diversity in procedures. Two well established methods based on binary lottery choices are the multiple price list (e.g., Schubert et al. (1999), Holt and Laury (2002), Barr and Packard (2002)) and random lottery pairs technique (e.g., Grether and Plott (1979), Hey and Orme (1994)). An alternative approach comprises a selection task from an ordered set of lotteries (e.g., Binswanger (1980), Eckel and Grossman (2008)). Another prominent method is the Becker-DeGroot-Marschak auction where a certainty equivalent is elicited (Becker et al. (1964), Harrison (1986), Loomes (1988)). The trade-off method proposed in Wakker and Deneffe (1996) and Abdellaoui (2000) is a chained procedure that aims to elicit certainty equivalents and probabilities. It is not based on the assumption of a specific preference functional. In Dohmen et al. (forthcoming), subjects decide between safe and risky options in a variant of the so-called switch multiple price list technique. Recent theory-free approaches to measure risk aversion that, respectively, rely on preference for proper risk apportionment and aversion to mean-preserving spreads are Ebert and Wiesen (2010) and Maier and Rüger (2010). See also Harrison and Rutström (2008) for a comprehensive overview of different experimental methods to elicit risk aversion.
In sharp contrast, there are few empirical studies on prudence. Dynan (1993), Carrol (1994), and Carroll and Kimball (2008) trace prudence indirectly via the precautionary savings motive.
Laboratory experiments could be used to investigate higher-order risk preferences as well as the associated theories and behavioral traits in a more controlled environment. Research in this direction has just started. The first attempt was made by Tarazona-Gomez (2004) who finds weak evidence for prudence. It is based on strong assumptions within EUT, in particular, a truncation of the utility function. The only other studies testing for prudence are Deck and Schlesinger (2010) and the one presented in the last chapter of this dissertation.

In **Chapter Four**, we introduce a simple experimental method to test for prudence in the laboratory. To this end, we present a novel graphical representation of compound lotteries which is easily accessible to subjects and test it for robustness by use of a factorial design. Prudence is observed on the aggregate and individual level. Although we did not do so in the experiment, it is straightforward to adapt our method to also test for temperance. Besides studying experimental methodology, our main focus is on contrasting prudence from skewness preference. We find that prudence does not boil down to skewness preference. The skewness of the zero-mean risk has a significant influence on subjects' decisions. With reference to Chapter Three, we further provide some theoretical explanations for this result. The observed presence of prudence highlights its empirical relevance and motivates further research on its experimental measurement in order to close the significant gap to the respective literature on risk aversion.

This thesis has benefited from numerous comments of various people, including journal referees and (associate) editors. It has also benefited from proof-reading of various people. Chapters One and Two have been developed jointly with Eva Lütkebohmert and are based on Ebert and Lütkebohmert (2011) and Ebert and Lütkebohmert (2009), respectively. Chapter Four owes to the collaboration with Daniel Wiesen and is based on Ebert and Wiesen (forthcoming). Chapter Three is single-authored and is based on Ebert (2010). Following advice from Thomson (1999, p. 180), that chapter and this introduction also make use of the first person plural. The second part of this introduction borrowed from Ebert and Wiesen (2010) several times. The note presented in Appendix B on the skewness of binary lotteries and the use of the latter in experiments is single-authored. Earlier versions of the results can be found in Ebert (2010) and Ebert and Wiesen (2009).

Each of the next four chapters is self-contained. However, the suggested order is supposed to ease comprehension.

Part I

On Double Default Modeling in the Basel Framework

Chapter 1

Treatment of Double Default Effects within the Granularity Adjustment for Basel II

1.1 Introduction

In the portfolio risk factor frameworks that underpin both industry models of credit Value-at-Risk (VaR) and the Internal Ratings-Based (IRB) risk weights of Basel II, credit risk in a portfolio arises from two sources: systematic and idiosyncratic. Idiosyncratic risk represents the effects of risks that are particular to individual borrowers. Under the Asymptotic Single Risk Factor (ASRF) framework on which the IRB approach is based, it is assumed that bank portfolios are *perfectly* fine-grained, in the sense that the largest individual exposures account for an infinitely small share of total portfolio exposure. In such a portfolio, idiosyncratic risk is fully diversified away, so economic capital depends only on systematic risk. Real-world portfolios are not, of course, perfectly fine-grained. The asymptotic assumption might be approximately valid for some of the largest bank portfolios, but would clearly be much less satisfactory for the portfolios of smaller or more specialized institutions. When there are material name concentrations, there will be a residual of undiversified idiosyncratic risk in the portfolio. The IRB formula omits the contribution of this residual to the required economic capital.

The impact of undiversified idiosyncratic risk on portfolio VaR can be assessed via a methodology known as *granularity adjustment* (GA). It is derived as a first-order asymptotic approximation of the effect of diversification in large portfolios. The basic concepts and approximate form for the GA were first introduced by Michael Gordy in 2000 for application in Basel II; see Gordy (2003). It was then substantially refined and put on a more rigorous foundation by Wilde (2001) and Martin and Wilde (2003) using theoretical results from Gouriéroux et al. (2000). Recently, Gordy and Lütkebohmert (2007) proposed and evaluated a GA suitable for application under Pillar 2 of Basel II.[1] Note also that recently the GA methodology to quantify the effect of idiosyncratic risk has proved useful in quite different contexts. Gouriéroux and Monfort (2008) derive GAs for optimal portfolios, i.e., they quantify the error in efficiency if one uses an optimal portfolio consisting of finitely many assets only in order to proxy the true, perfectly diversified market portfolio. In Gagliardini and Gouriéroux (2010), the authors define and compute a GA within a derivative pricing model.

However, none of these methods account for guarantees and general hedging instruments within a credit portfolio. This work aims at filling this gap, since the exclusion of hedging instruments represents, of course, a rather severe limitation as it is not at all rare that credit exposures are hedged in some way. For example, granting loans and transferring the risk afterwards is a typical practice for a bank. The relevance of hedging instruments is also acknowledged by the Basel Committee as Basel II (Basel Committee on Banking Supervision (2006)) discusses extensively credit risk mitigation (CRM) techniques. These include, e.g., ordinary guarantees, collateral securitization, and credit derivatives such as credit default swaps. Today, credit derivatives might be the most common guarantee instrument. Their market has grown rapidly over the first decade of the century. According to the International Swaps and Derivatives Association (ISDA) Market Survey[2], the notional of outstanding credit default swaps peaked at US$62 trillion as of December 2007. After the crisis, it still amounted to US$26 trillion as of June 2010; see O'Kane (2008) for a comparison of several studies on the topic.

It is reasonable that a financial institution should be able to decrease its capital requirements if it buys protection for its exposures. This is also important from a regulatory point of view, because it gives banks the incentive to hedge their credit risk. Therefore, in 2005, the Basel Committee made an amendment to the 2003 New Basel Accord concerning the treatment of guarantees in the IRB approach; see Basel Committee on Banking Super-

[1] Two other earlier works on the GA are Emmer and Tasche (2005) and Pykhtin and Dev (2002). See Lütkebohmert (2009) for more information on the development of the GA and a discussion of the different methods.

[2] This data is available at http://www.isda.org/statistics/.

vision (2005).[3] In the New Basel Accord of 2003, banks were allowed to adopt a so-called *substitution approach* to hedged exposures. Roughly speaking, under this approach a bank can compute the risk weighted assets for a hedged position as if the credit exposure was a direct exposure to the obligor's guarantor. Therefore, the benefit to the bank in terms of capital requirements from obtaining the protection may be small or even nonexistent. Since the 2005 amendment, for each hedged exposure the bank can choose between the substitution approach and the so-called *double default treatment.* The latter, inspired by Heitfield and Barger (2003), takes into account that default of a hedged exposure only occurs if both the obligor and the guarantor default ("double default"). There are rather strict requirements on the obligor and the guarantor for application of the double default treatment. Moreover, the parameters chosen in calculating the double default probability are quite conservative. We refer to Grundke (2008) for a meta-study on this issue. It has been shown in Heitfield and Barger (2003) that this double default treatment can lead to a significant decrease in capital requirements under the Advanced IRB approach.

Since the IRB treatment of double default effects is also based on the assumption of an infinitely granular portfolio, it seems natural to investigate the impact of guarantees on possible adjustments for undiversified idiosyncratic risk as represented, for example, by the GA. In this work, we address this issue and derive a GA that takes double default effects into account. The GA is derived as a first-order asymptotic approximation for the effect of diversification in large portfolios within an extended version of the CreditRisk$^+$ model that allows for idiosyncratic recovery risk.[4] Note, however, that our methodology could, in principle, be applied to any model of portfolio credit risk that is based on a conditional independence framework. We derive an analytic solution for the GA in a very general setting with several partially hedged positions where the guarantors can also act as obligors in the portfolio themselves. Moreover, we present some results on the performance of our new formula. In particular, we study the impact of guarantees and double default effects on the risk weighted assets of Basel II. Similar to the revised GA of Gordy and Lütkebohmert (2007), our generalization only requires data inputs which are already available when calculating IRB capital charges and reserve requirements. The fact that the GA is analytical allows for a fast computation and avoids the simulation of rare double default events. Thus, it is very well suited for application under Pillar 2 of

[3]Meanwhile the amendment has also been incorporated in a revised version of the 2003 Basel accord, Basel Committee on Banking Supervision (2006). If not noted otherwise, this is the version we refer to as "Basel II."

[4]CreditRisk$^+$ is a widely used industry model developed by Credit Suisse Financial Products (1997).

Basel II.

In Section 1.2, we start by introducing our basic notations and the CreditRisk$^+$ setting that we apply. Moreover, in this section we provide a review of the GA methodology without guarantees. In Section 1.3, we provide some illustrative examples of our main result and discuss the main difficulties that occur when deriving a GA in the presence of guarantees. In particular, we discuss the various scenarios and interactions between obligors and guarantors that can occur in practice. Section 1.4 gives the main result for an arbitrary number of partially hedged positions in the portfolio and discusses multiple hedging of a single obligor. Here we also provide a numerical example of the performance of our novel GA. The accuracy of the analytical approximation provided by our new GA is studied in Section 1.5 by comparison with simulated GA results. In Section 1.6, we conclude and discuss our assumptions and results. Appendix A.1 provides proofs of our results. A comparison study of our model with the treatment of double default effects within the IRB approach, which will be the topic of the next chapter, can be found in Ebert and Lütkebohmert (2011).

1.2 Notations and basic GA methodology

Our model presents an extension of the GA introduced in Gordy and Lütkebohmert (2007), which is based on the single factor CreditRisk$^+$ model allowing for idiosyncratic recovery risk.[5] Note, however, that our general GA can, in principle, be applied to any risk factor model of portfolio credit risk that is based on a conditional independence framework.
Let X denote the systematic risk factor, which we assume to be unidimensional to achieve consistency with the ASRF framework of Basel II. Denote the probability density function of X by $h(X)$. In our specific setting, we assume X to be gamma distributed with mean 1 and variance $1/\xi$ for some $\xi > 0$.[6] We consider a portfolio consisting of N obligors indexed by $n = 1, 2, \ldots, N$. Suppose that exposures of each obligor have been aggregated so that there is a unique position for each obligor in the portfolio. We refer the reader to Gordy and Lütkebohmert (2007) for a discussion of this assumption. Assume that the first $K \geq 0$ positions are hedged by some guarantors who might or might not be part

[5] For the implementation of the impact of guarantees in fully fledged CreditRisk$^+$ we refer the reader to Schmock (2008) who introduces connected groups of obligors.

[6] For the calibration of the parameter ξ we refer the interested reader to Gordy and Lütkebohmert (2007).

of the portfolio themselves. The remaining $N-K$ positions are unhedged.[7] Denote by EAD_n the exposure at default of obligor n and let

$$s_n = \mathrm{EAD}_n \,/ \sum_{i=1}^{N} \mathrm{EAD}_i$$

be its share of total exposure. Applying an actuarial definition of loss as in the CreditRisk$^+$ model, we define the loss rate of obligor n as $U_n = \mathrm{LGD}_n \cdot \mathrm{D}_n$ where D_n is a default indicator equal to 1 if obligor n defaults and 0 otherwise. Here $\mathrm{LGD}_n \in [0,1]$ denotes the loss given default rate of obligor n, which is assumed to be random and independent of D_n with expectation ELGD_n and volatility VLGD_n. The systematic risk factor X generates correlation across obligor defaults by shifting the default probabilities. Conditional on $X = x$, the default probability of obligor n is

$$\mathrm{PD}_n(x) = \mathrm{PD}_n \cdot (1 - w_n + w_n \cdot x) \tag{1.2.1}$$

where PD_n is the unconditional default probability and w_n is a factor loading specifying the extent to which obligor n depends on the systematic factor X.

We denote the loss variable of a portfolio with K hedged positions and $N-K$ unhedged positions by $L_{K,N-K}$.[8] Note that, in the situation without guarantees, we have conditional independence between obligors in the portfolio and thus can express the portfolio loss as

$$L_{0,N} = \sum_{n=1}^{N} s_n U_n. \tag{1.2.2}$$

Denote the q^{th} percentile of the distribution of some random variable X by $\alpha_q(X)$. For ease of notation we will sometimes use $x_q = \alpha_q(X)$ instead. When economic capital is measured as Value-at-Risk at the q^{th} percentile, we wish to estimate $\alpha_q(L_{K,N-K})$. The IRB formula, however, delivers the q^{th} percentile of the conditional expected loss $\alpha_q(\mathbb{E}[L_{K,N-K}|X])$. The difference

$$\alpha_q(L_{K,N-K}) - \alpha_q(\mathbb{E}[L_{K,N-K}|X]) \tag{1.2.3}$$

is the "exact" adjustment for the effect of undiversified idiosyncratic risk in the portfolio. This interpretation is justified in a conditional independence setting by the fact that $\alpha_q(\mathbb{E}[L_{K,N-K}|X])$ converges to $\alpha_q(L_{K,N-K})$ as the portfolio becomes more and more fine-

[7] In the following, quantities with a subindex n refer to the single obligor n, and are defined for arbitrary $n = 1, \ldots, N$.

[8] In general, when we use notations with two lower subindices, the first index gives the number of hedged positions and the second index gives the number of unhedged positions in the considered portfolio. This will be convenient when we derive the GA for portfolios with K hedged positions.

grained; see Gordy (2003, Proposition 5) for assumptions and a proof of this result. Such an exact adjustment cannot be obtained in analytical form, but we can construct a Taylor series approximation in orders of $1/N$. Therefore, we define the conditional expectation and conditional variance of obligor n's loss variable by $\mu_n(x) = \mathbb{E}[U_n|x]$ and $\sigma_n^2 = \mathbb{V}[U_n|x]$ and, on portfolio level, we define the quantities

$$\mu_{K,N-K}(x) \;=\; \mathbb{E}[L_{K,N-K}|x] \tag{1.2.4}$$

$$\sigma^2_{K,N-K}(x) \;=\; \mathbb{V}[L_{K,N-K}|x]. \tag{1.2.5}$$

Based on theoretical results of Gouriéroux et al. (2000), one can show that a first-order approximation in $1/N$ of equation (1.2.3), which defines our GA, can be obtained as

$$GA_{K,N-K} = \frac{-1}{2h(x_q)}\frac{d}{dx}\left(\frac{\sigma^2_{K,N-K}(x)h(x)}{\mu'_{K,N-K}(x)}\right)\Big|_{x=x_q}. \tag{1.2.6}$$

It is derived by applying a second-order Taylor expansion of the portfolio loss around its conditional mean. This result is independent of the question whether there are some hedged positions in the portfolio, since only the quantities $\mu_{K,N-K}(x)$ and $\sigma_{K,N-K}(x)$ are sensitive to this decision. Gordy and Lütkebohmert (2007) reformulate this result within a CreditRisk$^+$ framework and derive a simple analytic formula for the GA in the case without guarantees, which we will briefly review in the remainder of this section.

Assume a portfolio with N unhedged exposures. First, note that due to the conditional independence framework in the case without hedged positions, the quantities in equations (1.2.4) and (1.2.5) can be expressed as

$$\mu_{0,N}(x) \;=\; \mathbb{E}[L_{0,N}|x] = \sum_{n=1}^{N} s_n\mu_n(x) \tag{1.2.7}$$

$$\sigma^2_{0,N}(x) \;=\; \mathbb{V}[L_{0,N}|x] = \sum_{n=1}^{N} s_n^2\sigma_n^2(x). \tag{1.2.8}$$

By analogy to Gordy and Lütkebohmert (2007), we now reparametrize the inputs of the GA formula (1.2.6), i.e., the quantities $\mu_n(x)$ and $\sigma_n^2(x)$ for $n = 1,\dots,N$. Therefore, for every obligor n let $\mathcal{R}_n$ be the expected loss (EL) reserve requirement and let $\mathcal{K}_n$ be the unexpected loss (UL) capital requirement as a share of EAD_n . In the default-mode setting of CreditRisk$^+$, these quantities can be expressed as

$$\begin{aligned}\mathcal{R}_n &= \mathbb{E}[U_n] = \text{ELGD}_n \cdot \text{PD}_n && (1.2.9)\\ \mathcal{K}_n &= \mathbb{E}[U_n|x_q] - \mathbb{E}[U_n] = \text{ELGD}_n \cdot \text{PD}_n \cdot w_n \cdot (x_q - 1). && (1.2.10)\end{aligned}$$

Furthermore, let $\mathcal{K}_{0,N} = \sum_{n=1}^{N} s_n \mathcal{K}_n$ denote the required capital per unit exposure for the portfolio as a whole. Since the conditional default probability in a CreditRisk$^+$ framework equals $\text{PD}_n(x) = \text{PD}_n \cdot (1 - w_n + w_n \cdot x)$, we obtain

$$\begin{aligned}\mu_n(x_q) &= \mathcal{K}_n + \mathcal{R}_n\\ \mu_n'(x_q) &= \mathcal{K}_n/(x_q - 1) \qquad (1.2.11)\\ \mu_n''(x_q) &= 0.\end{aligned}$$

Moreover, by approximating the Bernoulli-distributed default indicators D_n by Poisson distributed random variables as in CreditRisk$^+$, it can be shown that

$$\sigma_n^2(x) = \mathcal{C}_n \mu_n(x) + \mu_n^2(x) \frac{\text{VLGD}_n^2}{\text{ELGD}_n^2} \qquad (1.2.12)$$

and thus

$$\frac{d}{dx}\sigma_n^2(x_q) = \mathcal{C}_n \mu_n'(x_q) + 2\mu_n'(x_q)\mu_n(x_q) \frac{\text{VLGD}_n^2}{\text{ELGD}_n^2} \qquad (1.2.13)$$

with

$$\mathcal{C}_n = \frac{\text{ELGD}_n^2 + \text{VLGD}_n^2}{\text{ELGD}_n}.$$

Noting that

$$\mu'(x_q) = \sum_{n=1}^{N} s_n \mathcal{K}_n/(x_q - 1) = \mathcal{K}_{0,N}/(x_q - 1) \text{ and } \mu_{0,N}''(x_q) = 0$$

in the case without hedging, one can reformulate equation (1.2.6) as

$$GA_{0,N} = \frac{1}{2\mathcal{K}_{0,N}} \left(\delta \sigma_{0,N}^2(x_q) - (x_q - 1) \frac{d}{dx} \sigma_{0,N}^2(x_q) \right) \qquad (1.2.14)$$

where

$$\delta = -(x_q - 1) \frac{h'(x_q)}{h(x_q)}.$$

Inserting the CreditRisk$^+$ representations of the terms $\mu_{0,N}(x_q)$ and $\sigma_{0,N}^2(x_q)$ and their derivatives, Gordy and Lütkebohmert (2007) obtain

$$
\begin{aligned}
GA_{0,N} \;=\; & \frac{1}{2\mathcal{K}_{0,N}} \sum_{n=1}^{N} s_n^2 \left[\left(\delta \mathcal{C}_n (\mathcal{K}_n + \mathcal{R}_n) + \delta (\mathcal{K}_n + \mathcal{R}_n)^2 \cdot \frac{\mathrm{VLGD}_n^2}{\mathrm{ELGD}_n^2} \right) \right. \\
& \left. -\mathcal{K}_n \left(\mathcal{C}_n + 2(\mathcal{K}_n + \mathcal{R}_n) \cdot \frac{\mathrm{VLGD}_n^2}{\mathrm{ELGD}_n^2} \right) \right].
\end{aligned} \tag{1.2.15}
$$

It is the aim of this work to extend this result to the situation with guarantees and to derive a simple closed-form GA that is able to account for double default effects and which is consistent with the ASRF model underlying Basel II.

1.3 Some illustrative examples and discussion of the methodology

In this section, we provide some illustrative examples of the general GA formula given in Theorem 1. We start by discussing in some detail the main problems that occur in the presence of guarantees. To start with, it therefore suffices to study the case $K = 1$, i.e., we consider a portfolio consisting of an exposure to obligor 1, which is partially hedged by a guarantor g_1, and $N-1$ unhedged positions.[9] Note that partial hedging is of particular importance here since, for the GA computation, exposures to a single obligor first have to be aggregated.[10] Thus, if one exposure to an obligor is hedged and there are also some unhedged exposures to this obligor, we have to face the problem of partial hedging in the GA computation. For $0 \leq \lambda \leq 1$, denote by $(1-\lambda)\,\mathrm{EAD}_1$ the unhedged portion, and by $\lambda\,\mathrm{EAD}_1$ the hedged portion of the exposure to obligor 1. All derivations in this work will be given for the case where there is direct exposure to guarantors. That is, guarantors are themselves obligors in the portfolio. In the current case, we thus let $g_1 = 2$ and let s_2 be the exposure share of the guarantor, obligor 2. The situation where there is actually no direct exposure to the guarantor is then simply obtained as the special case where the exposure $s_2 = 0$.

In this situation, the loss rates associated with the unhedged exposure to obligor 1, the direct exposure to the guarantor, and the hedged exposure to obligor 1 can no longer be treated as conditionally independent. The IRB treatment of double default effects,

[9] From now on, we will think of ordinary guarantees as the hedging instruments although our results can be applied to all types of CRM techniques as indicated in the introduction. For example, the "guarantor" could also be the protection seller within a credit default swap contract.

[10] For a detailed discussion of this problem we refer to Gordy and Lütkebohmert (2007).

however, ignores this issue by not specifying the relationships of the guarantors with the credit portfolio. It is implicitly assumed that there are only perfect full hedges, that guarantors are not obligors in the portfolio themselves, and that different obligors are hedged by different guarantors. To treat the possible interactions appropriately, we construct a "composite instrument" with loss rate $\hat{U}_1$ and exposure share $\hat{s}_1 = \lambda s_1$ consisting of the hedged portion $\lambda\,\mathrm{EAD}_1$ of the exposure to obligor 1. Note that the loss rate of the unhedged portion $(1-\lambda)\,\mathrm{EAD}_1$ of the exposure to obligor 1 is given by U_1. In the following, we will use the notation "hat" for a quantity referring to a *hedged* obligor and its guarantor. Thus, in general, such a quantity will depend on characteristics of both the hedged obligor and its guarantor. Note that, when obligor 1 defaults and the guarantor 2 survives, the latter will pay for the hedged exposure such that the exposure to obligor 1 is only lost in cases when both obligor 1 and obligor 2 default. Therefore, let $\hat{U}_1 = U_1 U_2$. We define the EL capital requirement for the composite instrument as

$$\begin{aligned}
\hat{\mathcal{R}}_1 &\equiv \mathbb{E}[\hat{U}_1] = \mathbb{E}[U_1U_2] = \mathbb{E}[\mathbb{E}[U_1U_2|X]] = \mathbb{E}\left[\mathbb{E}[U_1|X]\cdot\mathbb{E}[U_2|X]\right] \\
&= \mathrm{ELGD}_1\,\mathrm{ELGD}_2\cdot\mathbb{E}[\mathrm{PD}_1\cdot(1+w_1\cdot(X-1))\cdot\mathrm{PD}_2\cdot(1+w_2\cdot(X-1))] \\
&= \mathrm{ELGD}_1\,\mathrm{ELGD}_2\,\mathrm{PD}_1\,\mathrm{PD}_2\cdot(1+w_1w_2\cdot\mathbb{V}[X]) \\
&= \mathcal{R}_1\mathcal{R}_2 + \frac{\mathcal{K}_1\mathcal{K}_2}{(x_q-1)^2\xi},
\end{aligned}$$

which follows from the fact that the Bernoulli random variables D_1 and D_2 are independent conditional on the systematic risk factor X, which is gamma distributed with mean 1 and variance $1/\xi$. Moreover, the UL capital contribution for the composite instrument is given by

$$\begin{aligned}
\hat{\mathcal{K}}_1 &\equiv \mathbb{E}[\hat{U}_1|x_q] - \mathbb{E}[\hat{U}_1] = \mathbb{E}[U_1U_2|x_q] - \mathbb{E}[U_1U_2] \\
&= \mathrm{ELGD}_1\,\mathrm{PD}_1\cdot(1+w_1(x_q-1))\cdot\mathrm{ELGD}_2\,\mathrm{PD}_2\cdot(1+w_2(x_q-1)) - \hat{\mathcal{R}}_1 \\
&= \mathcal{K}_1\mathcal{K}_2 + \mathcal{K}_1\mathcal{R}_2 + \mathcal{R}_1\mathcal{K}_2 - \frac{\mathcal{K}_1\mathcal{K}_2}{(x_q-1)^2\xi}.
\end{aligned}$$

The portfolio loss $L_{1,N-1}$ in case of a single partial hedge can no longer be expressed by equation (1.2.2) but is given by

$$\begin{aligned}
L_{1,N-1} &= L_{0,N-1} + \lambda s_1\hat{U}_1 + (1-\lambda)s_1U_1 \\
&= L_{0,N-1} + s_1U_1\left(\lambda U_2 + (1-\lambda)\right).
\end{aligned} \tag{1.3.1}$$

Note that in the definition of $L_{0,N-1}$ the exposure shares are also defined as $\mathrm{EAD}_n \,/\sum_{i=1}^{N}\mathrm{EAD}_i$, i.e., with respect to the portfolio consisting of N positions.

Remark 1 (Recovery Rate Modeling)**.** *All derivations in this work are given for arbitrary choices of the loss given default variables for both the obligor and its guarantor. That is, double recovery effects can be included in the model. However, there are several good reasons why double recovery effects should not be reflected within the computation of regulatory capital. Also, in the Basel II IRB approach, recognition of double recovery effects is not allowed. Generally, the committee argues that "it is difficult to prescribe conditions on obligor, protection provider, and form of protection that could give sufficient certainty on the prospect of double recovery in the event of double default."*[11] *The bank is only allowed to use the* LGD *estimate of the guarantor instead of the estimate for the obligor. In the computation of the GA, we suggest that recovery from the guarantor should not be recognized at all whenever he is involved in partial hedging or when he is an obligor in the portfolio himself.*[12] *In the former case, it is implicitly assumed that the obligor's recovery is shared proportionally between the hedged and unhedged portions, although we could imagine that the unhedged portion has priority. In the latter case, the guarantor's recovery as an obligor should be dependent on the recovery payment for the guarantee. Thus, a conservative treatment, as is usually preferred by the Basel Committee, would be to completely neglect recovery from the guarantor (except for the direct exposure to the guarantor). This is achieved by setting* $\mathrm{LGD}_{g_n} = 100\%$ *for all* n *in the expressions for the GA we derive.*

Remark 2. *We want to point out here that the loss rates in the above definition of the portfolio loss, equation (1.3.1), are no longer conditionally independent, because the loss rates (*U_2 *for the guarantor,* U_1 *for the unhedged exposure to obligor* 1*, and* $\hat{U}_1$ *for the composite instrument) are conditionally dependent. However, it still makes sense to define the GA as the percentile difference in equation (1.2.3), as long as the exposures that are hedged by internal guarantors are sufficiently small as shares of total portfolio exposure. Otherwise, the asymptotic result underlying the computation of portfolio VaR under the ASRF model breaks down; see Gordy (2003, p. 209) for further details. This problem is more severe for the IRB treatment of double default effects because of the additional correlation assumed in that setting; see Section 1.6 for details.*

[11] See Basel Committee on Banking Supervision (2005, paragraph 206) for more details on the committee's reasons and Basel Committee on Banking Supervision (2006, paragraph 285 ff) for details on the prescribed treatment of recovery rates within the IRB approach. Indeed, proper modeling of double recovery would itself be a topic for future research.

[12] We thank an anonymous referee for this observation and the following justification.

To obtain the GA we must compute the conditional expectation $\mu_{1,N-1}(x)$ and the conditional variance $\sigma^2_{1,N-1}(x)$ referring to the above definition of loss, equation (1.3.1), and also derivatives of these expressions. Since, in the current case, no other obligor in the portfolio is hedged by guarantor 2, all of the $N-2$ ordinary obligors are independent of obligor 2 and the composite instrument conditional on the systematic risk factor X. Thus, our approach will be to express $L_{1,N-1}$ as a deviation from $L_{0,N-2}$, $\mu_{1,N-1}(x)$ as a deviation from $\mu_{0,N-2}(x)$, and so on. We then show that these quantities can also be expressed as deviations from $L_{0,N-1}, \mu_{0,N-1}(x)$ and so on. In this way, the GA computation can partially be traced back to the one in Gordy and Lütkebohmert (2007) that was sketched in Section 1.2. This is the main idea for the proof of our first result, which is summarized in the following Proposition. The proof is given in Appendix A.1.

Proposition 1 (The GA formula in the case of a single partial hedge)**.** *The GA for the case where a portion λ of the exposure to obligor 1 is hedged by obligor 2 is given by*

$$\begin{aligned}\widetilde{GA}_{1,N-1} \quad &= \quad \frac{\mathcal{K}_{0,N-1}}{\mathcal{K}_{1,N-1}(\lambda)}\overline{GA}_{0,N} + \frac{s_1\lambda\mathcal{K}_1\mathcal{K}_2}{(\mathcal{K}_{1,N-1}(\lambda))^2}\sigma^2_{0,N-1}(x_q) \\ &\quad + \frac{\left(s_1^2\hat{\mathcal{C}}_1(\lambda) + 2s_1s_2\lambda\mathcal{C}_2\right)}{2\mathcal{K}_{1,N-1}(\lambda)}\Big(\delta(\hat{\mathcal{K}}_1 + \hat{\mathcal{R}}_1) - (\mathcal{K}_1(\mathcal{K}_2+\mathcal{R}_2) + \mathcal{K}_2(\mathcal{K}_1+\mathcal{R}_1))\Big)\end{aligned} \tag{1.3.2}$$

where

$$\mathcal{K}_{1,N-1}(\lambda) := \mathcal{K}_{0,N-1} + s_1\left(\lambda\left(\mathcal{K}_1(\mathcal{K}_2+\mathcal{R}_2) + \mathcal{K}_2(\mathcal{K}_1+\mathcal{R}_1)\right) + (1-\lambda)\mathcal{K}_1\right)$$

and

$$\begin{aligned}\overline{GA}_{0,N} \quad := \quad GA_{0,N-1} + \frac{s_1^2(1-\lambda)^2}{2\mathcal{K}_{0,N-1}}&\left[\delta\left(\mathcal{C}_1(\mathcal{K}_1+\mathcal{R}_1) + (\mathcal{K}_1+\mathcal{R}_1)^2\frac{\mathrm{VLGD}_1^2}{\mathrm{ELGD}_1^2}\right)\right. \\ &\left. - \left(2\mathcal{K}_1(\mathcal{K}_1+\mathcal{R}_1)\frac{\mathrm{VLGD}_1^2}{\mathrm{ELGD}_1^2} + \mathcal{C}_1\mathcal{K}_1\right)\right].\end{aligned} \tag{1.3.3}$$

Here $GA_{0,N-1}$ is the GA formula for the portfolio with $N-1$ ordinary obligors, equation (1.2.15). Furthermore, we used the notation

$$\hat{\mathcal{C}}_1(\lambda) := \lambda^2\mathcal{C}_1\mathcal{C}_2 + 2\lambda(1-\lambda)\mathcal{C}_1. \tag{1.3.4}$$

The notation $\widetilde{GA}_{K,N-K}$ indicates that we simplified the expression for the GA by neglecting terms that are of order $\mathcal{O}(\frac{1}{N^2}\cdot \mathrm{PD}^3 \cdot \mathrm{ELGD}^3)$ or even higher. These terms would contribute little to the GA.[13]

[13]For more details on this argument see the proof and Remark 3.

The second term in equation (1.3.3) is the standard GA contribution of the non-hedged part $(1-\lambda)s_1$ of the exposure to obligor 1 (compare with equation (1.2.15)). Thus, in the first term of equation (1.3.2) we have summarized the contribution to the GA belonging to the unhedged part of the portfolio, i.e., to exposures $\mathrm{EAD}_2, \ldots, \mathrm{EAD}_N$, and to the unhedged portion $(1-\lambda)\,\mathrm{EAD}_1$ of the exposure to obligor 1. The third term of equation (1.3.2) depends only on the hedged obligor and its guarantor. It represents the contribution to the GA that is purely due to the hedged exposure to obligor 1. Note that this term also contains a part that vanishes when there is no direct exposure to the guarantor, i.e., when $s_2 = 0$, which leads to a reduction of the GA. The second term depends on all obligors in the portfolio. Hence, there is no additive decomposition of $\widetilde{GA}_{1,N-1}$ into the portfolio components belonging to the $N-1$ ordinary obligors and the hedged position and its guarantor. Note that, for $\lambda = 1$, we have $\hat{\mathcal{C}}_1(\lambda) = \mathcal{C}_1\mathcal{C}_2$ and $\overline{GA}_{0,N} = GA_{0,N-1}$.

Remark 3. *Studying equation (1.3.2) in more detail, we will see that double default effects are second-order effects $\mathcal{O}(1/N^2)$ in the GA. Therefore, we assume a homogeneous portfolio where each exposure share equals $s_n = 1/N$ and PDs and ELGDs are constant for all obligors. Assume that the exposure to obligor 1 is fully hedged by obligor 2, i.e., $\lambda = 1$. Recall that, by the definition of $\mathcal{K}_{1,N-1}(\lambda)$, for such a portfolio we have*

$$\begin{aligned}\mathcal{K}_{1,N-1}(\lambda) &= \sum_{n=2}^{N} s_n\mathcal{K}_n + s_1(\mathcal{K}_1(\mathcal{K}_2+\mathcal{R}_2) + \mathcal{K}_2(\mathcal{K}_1+\mathcal{R}_1)) \\ &= \frac{N-1}{N}\mathcal{K}_1 + \frac{1}{N}(2\mathcal{K}_1^2 + 2\mathcal{K}_1\mathcal{R}_1).\end{aligned}$$

Thus, for large N the terms

$$\mathcal{K}_n/\mathcal{K}_{1,N-1}(\lambda) = N/(N-1+2(\mathcal{K}_1+\mathcal{R}_1))$$

are approximately equal to 1. Similarly, one can show that $\mathcal{K}_{0,N-1}/\mathcal{K}_{1,N-1}(\lambda)$ is also approximately equal to 1. Moreover, one can easily show that, for a homogeneous portfolio, $GA_{0,N-1}$ is proportional to $1/N$. Thus, the first term in equation (1.3.2) is of order $1/N$. Furthermore, for a homogeneous portfolio the quantity

$$\sigma_{0,N-1}^2(x_q) = \sum_{n=2}^{N} s_n^2\sigma_n^2(x_q) = \frac{1}{N^2}\sum_{n=2}^{N}\sigma_n^2(x_q)$$

is proportional to $(N-1)/N^2$. Hence, for large N, the second term in equation (1.3.2) is approximately proportional to $1/N^2$. Similarly, we obtain that the third term is proportional to $1/N^2$. Hence, the main contribution to the portfolio GA comes from the unhedged part of the portfolio, while double default effects still contribute second-order to the GA. Therefore, *in terms of the GA, a bank will be rewarded significantly with lower capital requirements when buying credit protection.*

We now extend the previous model by allowing for several hedged positions in the portfolio. For the analysis it is sufficient to consider only two hedged positions as this illustrates all possible interactions between obligors and guarantors, and the extension to more than two hedged positions will be straightforward. Let us first generalize the notations from the previous situation to the case with several guarantees. Therefore, consider a portfolio where the exposures to the first K obligors are partially hedged by some guarantors $g_1, \ldots, g_K \in \{K+1, \ldots, N\}$.[14] Denote the hedged fraction of the loan to obligor $n \in \{1, \ldots, K\}$ by $\lambda_n \in [0,1]$ and define the vector $\lambda = (\lambda_1, \ldots, \lambda_K) \in [0,1]^K$. We define composite instruments for all hedged obligors by $\hat{U}_n = U_n \cdot U_{g_n}$ for $n = 1, \ldots, K$. The portfolio loss is then given by

$$L_{K,N-K} \quad = \quad L_{0,N-K} + \sum_{n=1}^{K} s_n \left(\lambda_n \hat{U}_n + (1-\lambda_n) U_n \right). \tag{1.3.5}$$

Moreover, we generalize the definition for the EL and UL capital of the composite instruments for arbitrary n as follows

$$\begin{aligned} \hat{\mathcal{R}}_n &= \mathcal{R}_n \mathcal{R}_{g_n} + \frac{\mathcal{K}_n \mathcal{K}_{g_n}}{(x_q - 1)^2 \xi} \\ \hat{\mathcal{K}}_n &= \mathcal{K}_n \mathcal{K}_{g_n} + \mathcal{K}_n \mathcal{R}_{g_n} + \mathcal{R}_n \mathcal{K}_{g_n} - \frac{\mathcal{K}_n \mathcal{K}_{g_n}}{(x_q - 1)^2 \xi}. \end{aligned}$$

Furthermore, we also extend the definition of $\hat{\mathcal{C}}_1(\lambda)$ to

$$\hat{\mathcal{C}}_n(\lambda_n) = \lambda_n^2 \mathcal{C}_n \mathcal{C}_{g_n} + 2\lambda_n (1-\lambda_n) \mathcal{C}_n, \tag{1.3.6}$$

and we generalize the notation $\mathcal{K}_{1,N-1}(\lambda)$ to the case of K partially hedged positions by setting

$$\mathcal{K}_{K,N-K}(\lambda) = \mathcal{K}_{0,N-K} + \sum_{k=1}^{K} s_k \left[\lambda_k (\mathcal{K}_k (\mathcal{K}_{g_k} + \mathcal{R}_{g_k}) + \mathcal{K}_{g_k} (\mathcal{K}_k + \mathcal{R}_k)) + (1-\lambda_k) \mathcal{K}_k \right].$$

Finally, we naturally extend the definition of $\overline{GA}_{0,N}$ to the case with K partially hedged loans.

In the case of two guarantees, we have to distinguish between two different scenarios.

[14] We will discuss the case $g_n \in \{1, \ldots, K\}$ in Remark 4.

First, it is possible that two different guarantors hedge two different obligors. Therefore, we consider a portfolio with two partially hedged obligors (1 and 2) and $N-2$ ordinary obligors $(3,\dots,N)$ where $g_1 \neq g_2$. The portfolio loss is then given by

$$L_{2,N-2} \quad = \quad L_{0,N-2} + s_1\left(\lambda_1 \hat{U}_1 + (1-\lambda_1)U_1\right) + s_2\left(\lambda_2 \hat{U}_2 + (1-\lambda_2)U_2\right). \qquad (1.3.7)$$

Note that, in the above equation, terms referring to the hedged obligor 1 are conditionally independent from those referring to the hedged obligor 2. This is why we can compute the conditional mean and conditional variance of the corresponding composite instruments for the hedged exposure to obligor 1 and obligor 2 separately by applying the same methods as in the case of a single hedged position; see Appendix A.1 for details.

Another possible scenario with two guarantees is that one guarantor hedges two different obligors. Similarly to the previous case, we consider a portfolio with two hedged obligors (1 and 2) and $N-2$ ordinary obligors $(3,4,\dots,N)$. However, the obligors now have the same guarantor $g_1 = g_2$. For ease of notation let $g_1 = g_2 = 3$. Then, the portfolio loss is given by

$$\begin{aligned} L_{2,N-2} \quad = \quad & L_{0,N-3} + \Big(s_3U_3 + s_1\lambda_1U_1U_3 + s_2\lambda_2U_2U_3\Big) \\ & + \Big(s_1(1-\lambda_1)U_1 + s_2(1-\lambda_2)U_2\Big). \end{aligned} \qquad (1.3.8)$$

Neglecting third- and higher-order terms in EL and UL capital contributions, one can show that the expressions for $\mu_{2,N-2}(x_q)$ and $\sigma^2_{2,N-2}(x_q)$ and their derivatives do not depend on whether both obligors have different guarantors or the same guarantor. Consequently, the formula for the GA also has to be the same as in the case with different guarantors. It is summarized in the following proposition. For the proof we refer the reader to Appendix A.1. It can be shown that the GA in the case of the same guarantor is larger, but only in third-order terms, which are neglected in our simplified version.

Proposition 2 (The GA formula in the case of two partial hedges)**.** *The GA in the case where a portion* λ_1 *of the exposure to obligor* 1 *is hedged by guarantor* g_1 *and a portion* λ_2

of the exposure to obligor 2 is hedged by guarantor g_2 is given by

$$\begin{aligned}\widetilde{GA}_{2,N-2} \quad &= \quad \frac{\mathcal{K}_{0,N-2}}{\mathcal{K}_{2,N-2}(\lambda)}\overline{GA}_{0,N} + \frac{s_1\lambda_1\mathcal{K}_1\mathcal{K}_{g_1} + s_2\lambda_2\mathcal{K}_2\mathcal{K}_{g_2}}{(\mathcal{K}_{2,N-2}(\lambda))^2}\sigma^2_{0,N-2}(x_q) \\ &+\frac{\left(s_1^2\hat{\mathcal{C}}_1(\lambda) + s_1 s_{g_1}\lambda_1\mathcal{C}_{g_1}(\lambda)\right)}{2\mathcal{K}_{2,N-2}(\lambda)}\left(\delta(\hat{\mathcal{K}}_1 + \hat{\mathcal{R}}_1) - (\mathcal{K}_1(\mathcal{K}_{g_1} + \mathcal{R}_{g_1}) + \mathcal{K}_{g_1}(\mathcal{K}_1 + \mathcal{R}_1))\right) \\ &+\frac{\left(s_2^2\hat{\mathcal{C}}_2(\lambda) + s_2 s_{g_2}\lambda_2\mathcal{C}_{g_2}(\lambda)\right)}{2\mathcal{K}_{2,N-2}(\lambda)}\left(\delta(\hat{\mathcal{K}}_2 + \hat{\mathcal{R}}_2) - (\mathcal{K}_2(\mathcal{K}_{g_2} + \mathcal{R}_{g_2}) + \mathcal{K}_{g_2}(\mathcal{K}_2 + \mathcal{R}_2))\right)\end{aligned} \tag{1.3.9}$$

where we again neglected terms that are of order $\mathcal{O}(\frac{1}{N^2}\cdot \mathrm{PD}^3\cdot \mathrm{ELGD}^3)$ or higher.

1.4 GA for a portfolio with several guarantees

In this section, we provide a general formula for the GA of a portfolio with several guarantees. Here we not only extend the previous result from 2 to K hedged obligors in the portfolio,[15] but we further allow for different parts of the exposure to the same obligor to be hedged by several distinct guarantors.[16] This generalization is necessary for several applications. Suppose, for example, there are three loans to obligor 1 (indexed by $1, 2$, and 3) in the portfolio. Loans 1 and 2 are guaranteed by two different guarantors $g_{1,1}$ and $g_{1,2}$, respectively, whereas loan 3 is unhedged.[17] For the computation of the GA all three loans first have to be aggregated into a single loan. Let $\lambda_{1,1}$ and $\lambda_{1,2}$ denote the fractions of the first and second loan to obligor 1, respectively, on the aggregated position. The fraction $1-\lambda_{1,1}-\lambda_{1,2}$ of the aggregated position is the unhedged part. In this section, we will derive the contribution of such a partially hedged obligor 1 to the GA.

More generally, suppose we have a portfolio with N obligors of which the first $K \leq N$ are hedged and the entries of the tuple $\lambda_n = (\lambda_{n,1}, \ldots, \lambda_{n,j_n})$ are the portions of the exposure EAD_n to obligor n ($n = 1, \ldots, K$), which are hedged by guarantors $g_{n,1}, \ldots, g_{n,j_n}$, respectively. Denote by Λ the collection of all tuples $\lambda_1, \ldots, \lambda_K$. In this situation, the portfolio loss can be written as

$$L_{K,N-K} = L_{0,N-K} + \sum_{n=1}^{K}\sum_{i=1}^{j_n} s_n\lambda_{n,i}U_nU_{g_{n,i}} + s_n\left(1-\sum_{i=1}^{j_n}\lambda_{n,i}\right)U_n.$$

[15]From the computations in the previous section, this is essentially straightforward.

[16]For the case when the same exposure is hedged by more than one guarantor see Remark 4.

[17]For simplicity, think of full hedges, although the argument works as well for partial hedges.

To write down the final version of the GA, we generalize the notations of Section 1.3. First we naturally generalize the notation $\mathcal{K}_{K,N-K}(\lambda)$ to the case of multiple hedges per obligor, which we then denote by $\mathcal{K}_{K,N-K}(\Lambda)$, and we generalize the notation $\hat{\mathcal{C}}_n(\lambda_n)$ in the following way

$$\hat{\mathcal{C}}_n(\lambda_{n,i}) = \lambda_{n,i}^2 \mathcal{C}_n \mathcal{C}_{g_{n,i}} + 2\lambda_{n,i}\left(1 - \sum_{i=1}^{j_n} \lambda_{n,i}\right)\mathcal{C}_n.$$

Similarly, the notation $\overline{GA}_{0,N}$ is adapted by replacing the terms $1-\lambda_n$ by the quantities $1 - \sum_{i=1}^{j_n} \lambda_{n,i}$.

We can now formulate our main result: a single analytic formula for the GA that applies to any of the aforementioned hedging situations.[18]

Theorem 1 (The general GA formula)**.** *Consider a portfolio with an arbitrary number of hedged positions where every hedging instrument may be any type of credit risk mitigation technique. Exposures to the same obligor may be hedged by different guarantors and for every exposure only parts may be hedged. Guarantors may or may not be obligors in the portfolio themselves and they may hedge exposures of more than one obligor. The total exposure share of the positions that are hedged by guarantors who are part of the portfolio themselves, however, has to be sufficiently small such that the asymptotic result underlying the ASRF model still holds. With the notations above, the GA of such a portfolio can be computed by means of the following analytic formula*

$$\begin{aligned}\widetilde{GA}_{K,N-K} &= \frac{\mathcal{K}_{0,N-K}}{\mathcal{K}_{K,N-K}(\Lambda)}\overline{GA}_{0,N} + \frac{\sigma^2_{0,N-K}(x_q)}{(\mathcal{K}_{K,N-K}(\Lambda))^2}\sum_{n=1}^{K}\sum_{i=1}^{j_n} s_n\lambda_{n,i}\mathcal{K}_n\mathcal{K}_{g_{n,i}} \\ &\quad + \frac{1}{2\mathcal{K}_{K,N-K}(\Lambda)}\sum_{n=1}^{K}\sum_{i=1}^{j_n}\left(s_n^2\hat{\mathcal{C}}_{n,i}(\lambda_{n,i}) + 2s_n s_{g_{n,i}}\lambda_{n,i}\mathcal{C}_{g_{n,i}}\right) \qquad (1.4.1) \\ &\quad \cdot \left(\delta(\hat{\mathcal{K}}_{n,i} + \hat{\mathcal{R}}_{n,i}) - (\mathcal{K}_n(\mathcal{K}_{g_{n,i}} + \mathcal{R}_{g_{n,i}}) + \mathcal{K}_{g_{n,i}}(\mathcal{K}_n + \mathcal{R}_n))\right).\end{aligned}$$

The notation $\widetilde{GA}_{K,N-K}$ indicates that we simplified the expression for the GA by neglecting terms that are of order $\mathcal{O}(\frac{1}{N^2} \cdot \mathrm{PD}^3 \cdot \mathrm{ELGD}^3)$ or even higher. These terms would contribute little to the GA.

Remark 4. *Note that, from the previous derivations it is obvious that a loan that is hedged by several guarantors will contribute only third-order terms to the GA. The same is true when a guarantor itself is hedged. In these cases, we suggest a substitution approach as applied by Basel Committee on Banking Supervision (2006). That is, whenever there*

[18] By treating the case of multiple hedging of the same exposure as proposed in Remark 4, the formula indeed applies to all possible hedging combinations.

are multiple guarantors to a single loan, the risk manager can choose one guarantor whose characteristics (i.e., PD, ELGD, EL, and UL capital contributions) enter the GA formula.

Before we investigate the accuracy of our analytical GA by comparison with simulation results in Section 1.5 and discuss our main result in Section 1.6, we provide a numerical example in order to study the impact of hedging on the GA.

Example 1. *Consider an artificial portfolio, P, which is the most concentrated portfolio that is admissible under the EU large exposure rules.*[19] *For this purpose, we divide a total exposure of €6000 into one loan of size €45, forty-five loans of size €47, and thirty-two loans of size €120.*[20] *We assume a constant* PD *of 1% and a constant* ELGD *of 45%. Now assume that all thirty-two loans of size €120 are completely hedged by different guarantors who are not part of the portfolio themselves. For these guarantors we assume a constant* PD *of 0.1% and a constant* ELGD *of 45%. Moreover, we fix the effective maturity for all obligors and guarantors to* $M = 1$ *year.*
Our generalized GA formula (1.4.1) leads to an add-on for undiversified idiosyncratic risk of $\widetilde{GA}_{32,46} = 0.86\%$ *of total exposure, i.e., €51.60.*[21] *To study the impact of hedging on economic capital we computed the IRB capital for portfolio P using the IRB treatment of double defaults.*[22] *Then, the regulatory capital for portfolio P with thirty-two guarantees equals 4.98% or €298.80. Hence, our novel GA formula leads to an add-on on regulatory capital of 17.27%. We now compare this result with the analogous computations when guarantees are neglected. The GA formula for the portfolio P without hedged exposures yields a GA of* $\mathrm{GA} = 1.66\%$ *of total exposure, i.e., €99.60. The corresponding regulatory capital for the portfolio without guarantees is 6.21% of total exposure. Thus, the add-on on regulatory capital due to the GA for the portfolio without hedged exposures is 26.73%. Hence, accounting for guarantees within the computation of the GA can significantly reduce the capital requirement for undiversified idiosyncratic risk. In our example of portfolio P, the reduction is by approximately 35%. Table 1.1 summarizes the results of our example.*

Remark 5. *Note that, for a homogeneous portfolio where all exposures have the same size and PDs and ELGDs are also identical for all obligors, hedging can also have the*

[19] According to the EU rules, banks are not allowed to have an exposure that requires 25% or more of regulatory capital. Moreover, the sum of all large exposures, i.e., exposures that require at least 10% of regulatory capital, must not account for more than 8 times the regulatory capital. For more details see Directive 93/6/EEC of March 15, 1993 on the capital adequacy of investment firms and credit institutions.

[20] For a detailed derivation of this portfolio from the EU large exposure rules we refer to Düllmann and Masschelein (2007).

[21] In our numerical results, we always fix the variance parameter of the systematic risk factor as $\xi = 0.125$. Moreover, we computed the variance of LGD as $\mathrm{VLGD}_n^2 = \frac{1}{4} ELGD_n \cdot (1 - ELGD_n)$.

[22] See Chapter 2 for more details on this approach.

Table 1.1: Impact of guarantees on GA and IRB capital requirements

Portfolio P	GA (in %)	IRB capital (in%)	add-on for GA (in %)
without guarantees	1.66	6.21	26.73
with guarantees	0.86	4.98	17.27

The GA is computed using equation (1.4.1) both for the portfolio without hedged exposures and the portfolio with thirty-two guarantees. The regulatory capital is computed using the IRB treatment of double default effects for the portfolio with guarantees and the standard IRB formula for the portfolio without guarantees. The add-on for the GA on regulatory capital is defined as the quotient of the GA and IRB capital.

opposite effect and increase the GA. This is due to the fact that hedging can shift the exposure distribution of the portfolio to a more concentrated distribution. For such a homogeneous portfolio, for example, the exposure distribution is uniform and the portfolio can be considered as almost perfectly diversified for large N. When we assume now that some of the exposures in the portfolio are guaranteed by some other obligors in the portfolio, the portfolio becomes more concentrated and thus the GA increases.

1.5 Numerical validation of the analytical GA formula

In this section, we study the performance of our new GA formula. We therefore compare our analytical GA, equation (1.4.1), for a portfolio with hedged exposures to simulation results based on VaR computations within the CreditRisk$^+$ model. That is, we compute

$$\alpha_q(L) - \alpha_q(\mathbb{E}[L|X]) \tag{1.5.1}$$

numerically by Monte Carlo simulation, where the portfolio loss variable L is modeled as

$$L = \sum_{n=1}^{N} s_n \,\mathrm{LGD}_n \,\mathrm{D}_n \,\mathrm{D}_{g_n},$$

such that double recovery effects are neglected and where $\mathrm{D}_{g_n} = 1$ if the exposure to obligor n is unhedged. In CreditRisk$^+$, the default indicator variables D_n are Bernoulli-distributed with parameter given by the stochastic default probability $\mathrm{PD}_n(X)$, which depends on the systematic risk factor X that is gamma distributed with mean 1 and variance $1/\xi$. Moreover, the conditional PDs can be expressed as

$$\mathrm{PD}_n(X) = \mathrm{PD}_n \cdot (1 - w_n + w_n X)\,.$$

Thus, we need to specify the factor loadings w_n in the CreditRisk$^+$ model. In our GA formula, although it is formulated within a generalized CreditRisk$^+$ setting, we did not need to specify these factor loadings as we parametrized our final formula with respect to the IRB model. The latter is derived within a single-factor mark-to-market Vasicek model that is closest in spirit to KMV Portfolio Manager, the actuarial counterpart of which is a two-state CreditMetrics approach. Thus, to compare our GA results with the simulation results, we use a factor transformation relating the gamma distributed factor X in CreditRisk$^+$ to the standard normally distributed risk factor Y in CreditMetrics. In the latter model, the systematic risk factor is weighted with the asset correlation ρ_n for obligor n, i.e., the conditional default probability is

$$\mathrm{PD}_n(Y) = \mathbb{P}(\{\sqrt{\rho_n} \cdot Y + \sqrt{1-\rho_n} \cdot \epsilon_n \leq C_n\}) \tag{1.5.2}$$

where ϵ_n denotes the standard normally distributed idiosyncratic shock and C_n denotes the default threshold of obligor n, which is given by $\Phi^{-1}(\mathrm{PD}_n)$.

Following the methods used in Gordy (2000), we compare the two models by their factor distributions and conditional default probabilities, and not in terms of matching the first moments of the loss distributions. Hence, the problem to be solved is to find a parametrization such that the conditional default probabilities in CreditRisk$^+$ and CreditMetrics agree. As shown in Gordy (2000), the variance $V_n^{\text{CreditMetrics}}$ of the default probabilities $\mathrm{PD}_n(y)$ conditional on the systematic risk factor $Y = y$ is given by

$$V_n^{\text{CreditMetrics}} = \mathbb{V}[\mathrm{PD}_n(y)] = \Phi_2(C_n, C_n, \rho_n) - \mathrm{PD}_n^2 \tag{1.5.3}$$

where $\Phi_2(\cdot, \cdot, \rho)$ is the bivariate standard normal cumulative distribution function with correlation ρ. In CreditRisk$^+$, the variance of the default probability is given by

$$V_n^{\text{CreditRisk}^+} = \mathbb{V}[\mathrm{PD}_n \cdot (1 - w_n + w_n x)] = (\mathrm{PD}_n \cdot w_n \cdot \sqrt{1/\xi})^2. \tag{1.5.4}$$

Matching the variances of the default probabilities of both models leads to

$$\Phi_2(C_n, C_n, \rho_n) - \mathrm{PD}_n^2 = (\mathrm{PD}_n \cdot w_n \cdot \sqrt{1/\xi})^2 \tag{1.5.5}$$

and thus

$$w_n = \sqrt{\xi \cdot \frac{\Phi_2(C_n, C_n, \rho_n) - \mathrm{PD}_n^2}{\mathrm{PD}_n^2}}. \tag{1.5.6}$$

We used this relation to specify the factor loadings in the computation of the simulated GA.

For the comparison analysis between the analytical and the simulated GA, we constructed a sequence of stylized portfolios consisting of 1000 loans.[23] The first 100 loans are fully hedged by external guarantors. The remaining 900 loans are unhedged. All obligors in the portfolio have a PD of 2%, whereas the guarantors have a PD of 1%. The ELGD for unhedged exposures as well as for guarantors is assumed to be 45%. For hedged exposures we follow the IRB treatment of double default effects and set ELGD = 100%. Effective maturity is set to 1 year. We start with a homogeneous portfolio where all exposures are of size 1. We then successively increase name concentration risk in the portfolio by increasing the size of the hedged exposures from 1 to 100 in steps of 10. For these portfolios we computed the GA by simulating equation (1.5.1) within the above described CreditRisk$^+$ setting. In particular, we compute the factor loadings using relation (1.5.6) and insert for ρ_n the asset correlation specified in the IRB approach, which we also use in the analytical GA computation. The results are summarized in Table 1.2. The simulation study shows that the analytical GA performs very well for different degrees of exposure concentration. The GA becomes more accurate as the portfolio becomes more concentrated. For very homogenous portfolios (where the GA is rather small in absolute terms anyway), the analytical approximation might slightly overstate the granularity adjustment.

Remark 6 (Simulation of the GA in practice)**.** *The simulation of the very rare double default events and the study of their impact on VaR can be extremely time-consuming, even for the relatively high default probabilities chosen in the example. To compute the unconditional VaR of the portfolio loss rate, one has to infer default or survival of any obligor (and guarantor) from the realization of a Bernoulli random variable. This has to be done for every realization of the systematic risk factor. It is because of the simple structure of the considered example portfolio (in particular because there are only two different exposure sizes) that this process can be simplified. In the example, for every realization of the systematic risk factor, it suffices to draw two binomial samples instead (for the hedged and unhedged parts, respectively). For a realistic portfolio, however, simulation might not be practicable. This problem is critical even without double default and is a primary motivation for the analytical approximation used in the IRB approach. Likewise, and in particular with double default, it motivates the analytical approximation of the GA.*

[23] The choice of the portfolios is motivated in Remark 6.

Table 1.2: Analytical and simulated GA

Size of hedged exposures	Analytical GA (in %)	Simulated GA (in %)
1	0.11	0.08 (0.02)
10	0.10	0.07 (0.01)
20	0.14	0.10 (0.01)
30	0.17	0.15 (0.02)
40	0.20	0.18 (0.02)
50	0.22	0.19 (0.01))
60	0.24	0.22 (0.02)
70	0.25	0.23 (0.02)
80	0.26	0.23(0.02)
90	0.27	0.25 (0.02)
100	0.28	0.27 (0.02)

Table 1.2 shows analytical and simulated GA results for eleven portfolios of 1000 loans each, where the first 100 loans are hedged by external guarantors and their exposure sizes increase from 1 to 100. The remaining 900 loans are unhedged and have a constant exposure of size 1. Borrower PDs are 2% while guarantors have a PD of 1%. ELGDs are set to 45% for unhedged positions and 100% for hedged positions. The analytical GA in column 2 is computed using equation (1.4.1). The simulated GA is computed using equation (1.5.1). The means and standard errors in parentheses that are reported in column 3 are obtained from ten identical runs with $m = 2000000$ simulation steps each. Effective maturity is 1 year and the variance parameter ξ in the gamma distribution is set to 0.125.

1.6 Discussion and conclusion to Chapter 1

In this chapter, we derived a granularity adjustment (GA) that accounts for credit risk mitigation techniques in a very general setting. The derivation of our main result, Theorem 1, is rather complex because it considers all possible interactions between obligors and guarantors that can occur in practice. However, it relies on a simple model of double default that allows for an analytical solution. Therefore, simulations of the very rare double default events can be avoided. Moreover, the GA is parsimonious with respect to data requirements as its inputs are needed for the computation of Pillar 1 regulatory capital under the IRB approach anyway. This is a very important quality since the data inputs can pose the most serious obstacle for practical application. Thus, our general GA formula is very well suited for application under Pillar 2 of Basel II.

Let us now discuss the underlying assumptions of our main result, formula (1.4.1), in more detail. Here, we will focus only on the assumptions related to the treatment of double default effects in the GA. For a discussion of the general assumptions of the GA methodology we refer the reader to Gordy and Lütkebohmert (2007) and Lütkebohmert (2009). The latter also contains a comparison with related approaches.

Our model of double default effects is based on the assumption that the loss rate of the exposure to an obligor that is hedged by a guarantor is given by the product of the individual loss rates, which are assumed to be independent conditional on the systematic risk factor. Thus, we implicitly assume that the obligor's default (triggering the guarantee payment) is not an excessive burden to the guarantor. The same problem arises in the IRB treatment of double default effects. To mitigate it, conditions on obligors and guarantors can be imposed in order to qualify for their hedging relationship to be accounted for; see Basel Committee on Banking Supervision (2005) and Grundke (2008) for a discussion of the conditions).

The IRB treatment of double default effects further assumes some additional correlation since the obligor and its guarantor are correlated not only through the systematic risk factor but also through an additional factor. It should be noted, however, that correlation cannot capture the asymmetry in their relationship, i.e., the guarantor should suffer much more from the default of the obligor than vice versa. Therefore, we argue that assuming extra high correlation, as is implied by the dependence on an additional factor in the IRB approach, is problematic, in particular, when there is direct exposure to the guarantor. Given the default of the guarantor, this would imply a higher probability of default for the obligor, which does not seem to be empirically justified. A better approach, in our opinion, would be to increase the guarantor's unconditional default probability appropriately as this also captures the above-mentioned asymmetry. Such an *asset drop model* will be developed the next chapter. Within a simple structural model of default, Grundke (2008) shows that the additional correlation of 0.7 fixed in the IRB treatment of double default effects approximately corresponds to an increase of 100% in the guarantors unconditional probability of default.

We further note that, under the ASRF model that underpins Basel II, one must be careful when introducing additional correlation between obligors in the portfolio. The exposure shares of obligors that are correlated through more than the common risk factor must be sufficiently small. This is because, otherwise, the asymptotic result underlying the computation of portfolio VaR under the ASRF model breaks down; see Gordy (2003, p. 209) for further details. This might be the case if, for example, several loans in the portfolio are guaranteed by a large insurance company and, in particular, if there is direct exposure to that guarantor. This problem is not addressed in Basel Committee on Banking Supervision (2005).

As our GA formula is parametrized to achieve consistency with the IRB approach, one could also argue for computing the GA with double default effects in a *two-step approach,* where, in a first step, we compute the GA *without* considering double default effects (and obtain the result of Gordy and Lütkebohmert (2007): that is, equation (1.2.15)). In a second step, we could then compute the UL capital requirement $\mathcal{K}_n^{DD}$ for a hedged obligor n as in the IRB treatment of double default effects, and insert this parameter instead of $\mathcal{K}_n$ in the GA formula. This two-step procedure, however, essentially ignores any interaction of the guarantor with the rest of the portfolio. That is, it even ignores the common dependence induced by the systematic risk factor. Hence, roughly speaking, under a two-step approach the computation of EL and UL for a given portfolio and the computation of the GA are solved separately (rather than jointly) and then are put together naively. This, of course, implies a fairly easy derivation, however, with the shortcoming of missing any mathematical justification.

In contrast to this procedure, the *bottom-up approach* we used to derive the GA given by formula (1.4.1) incorporates double default effects right in the beginning. More precisely, our treatment of double default effects enters the model setup (the portfolio loss distribution) rather than just the model's "solution": the final GA formula. Thus, it avoids the inconsistencies and disadvantages involved with a two-step procedure. The drawback is that this fully rigorous derivation is much more complex. In the current case, however, we saw that the derivation is tractable and even leads to a rather simple (in terms of parameters) analytical solution that can be implemented easily. This solution correctly incorporates all the different interactions between obligors and guarantors that can occur. In the case of our Example 1, the two-step method would lead to a GA of 1.44% of total exposure, i.e., €86.40. Thus, the capital reducing effect of the guarantees would be much lower in this approach than in our rigorous model-based approach.

Chapter 2

Improved Double Default Modeling for the Basel Framework - An Endogenous Asset Drop Model without Additional Correlation

2.1 Introduction

In 2005, the Basel Committee made an amendment (Basel Committee on Banking Supervision (2005)) to the original New Basel Accord of 2003 (Basel Committee on Banking Supervision (2003, 2004)) that deals with the treatment of hedged exposures in credit portfolios.[1] In the original New Basel Accord of 2003, within the Internal Ratings Based (IRB) approach, banks are allowed to adopt a so-called *substitution approach* to hedged exposures. Roughly speaking, under this approach a bank can compute the risk weighted assets for a hedged position as if the credit exposure was a direct exposure to the obligor's guarantor. Therefore, the bank may have only a small or even no benefit in terms of capital requirements from obtaining the protection. Since the 2005 amendment, for each

[1] Meanwhile the amendment also has been incorporated in a revised version of the 2003 New Basel Accord, Basel Committee on Banking Supervision (2006). If not noted otherwise, this is the version we refer to as "Basel II."

hedged exposure the bank can choose between the substitution approach and the so-called *double default treatment*. The latter, inspired by Heitfield and Barger (2003), takes into account that default of a hedged exposure only occurs if both the obligor and the guarantor default ("double default"), and thus seems to be more sophisticated and realistic than the substitution approach.

The recent global financial crisis drastically demonstrated the importance of how to treat hedged exposures in credit portfolios. However, the literature on the treatment of double default effects within the computation of economic capital is scarce. This is particularly true for the literature on the computation of regulatory capital under Basel II. Given that the former model sets a benchmark for the quantification of minimum capital requirements for hedged exposures of banks in the European Union, this seems to be unjustified.

There is no doubt that hedging exposures is rather a natural act than a rare exception. For example, granting loans and transferring the risk afterwards is a typical practice for a bank. This can be implemented through numerous instruments (referred to as credit risk mitigation (CRM) techniques in Basel II) such as ordinary guarantees, collateral securitization, and credit derivatives. The latter comprise, for example, credit default swaps and bundled credit packages such as credit loan obligations. This is also why CRM techniques were discussed extensively in Basel II in the first place and why the Basel Committee chose to improve on the earlier version by introducing the treatment of double default effects in 2005. After all, through the regulatory treatment of double default effects, the Basel Committee sets incentives for banks to obtain credit protection. In the aftermath of the global financial crisis, the Basel Committee is again largely concerned with making improvements to the treatment of counterparty risk in Basel II in general; see Basel Committee on Banking Supervision (2009, 2010). However, in these documents and the related consultative documents, which can be summarized under the term Basel III, the Basel Committee has not addressed the treatment of double default effects. More generally, no structural modifications have been made concerning the computation of risk weighted assets within the IRB approach of Basel II. In this chapter, we motivate and propose a new methodology to treat double default effects in structural credit risk models. In particular, we are concerned with the computation of regulatory capital in the IRB approach of Basel II.[2]

[2] As the IRB approach has been part of the Basel II reforms, we speak of our proposed treatment as being a modification to the IRB approach of Basel II. Of course, our treatment likewise applies to Basel III as the model underlying the IRB approach has not changed. See also the Introduction to this thesis for a clarification.

To motivate our new method, we first review the IRB treatment of double default effects. While this approach constitutes an important first step in modeling double default under Basel II, we will show that it also has severe shortcomings. Most importantly, we argue that imposing additional correlation between obligors and guarantors is unsuitable to capture their essentially asymmetric relationship appropriately. We also show that this approach, in general, violates some of the assumptions of the Asymptotic Single Risk Factor (ASRF) model (see Gordy (2003)), which represents the mathematical basis of the IRB approach. Furthermore, it is implicitly assumed within the IRB treatment of double defaults that guarantors are external. That is, it is assumed that there is no direct exposure to guarantors. It is also assumed that every loan in the portfolio is hedged by a different guarantor. This leads to underestimation of the associated concentration risk.
The major contribution of this work is a new method to account for double default effects in the computation of economic capital. It can be used within all structural models of credit risk and, in particular, in the IRB approach of Basel II. The model does not exhibit any of the deficiencies we point out for the IRB treatment of double defaults. Instead of modeling the relationship between an obligor and its guarantor through dependency on an additional stochastic risk factor, we adjust the guarantor's default probability appropriately if the hedged obligor defaults. The model is endogenous as it actually quantifies the increase of the guarantor's default probability instead of exogenously imposing a numerical value as it is done in the IRB treatment of double default effects for the additional correlation parameters. The idea behind the model is to quantify the size of the downward jump of the guarantor's firm value process in case of the obligor's default which triggers the guarantee payment. We therefore call this approach an *asset drop* model. Practical application of the model is straightforward since it does not require extensive data. Moreover, due to its simple analytic representation, economic capital can be computed almost instantaneously.

Structural models with (downward) jumps have been considered previously in the literature, e.g., in the jump diffusion model of Zhou (2001b). Bivariate versions of the latter were introduced in Zhou (2001a) and Hull and White (2001). These approaches have also been used to model default dependencies in the counterparty risk literature, in particular for evaluating the credit value adjustment (CVA) for credit default swaps (CDSs); see, e.g., Lipton and Sepp (2009), Brigo and Chourdakis (2009), and references therein. In these models, jumps occur randomly rather than being triggered by a specific event as in our model. That is, we provide an explanation for the jump time as well as for the jump size. Moreover, in contrast to our approach, the above-mentioned literature models dependencies symmetrically by correlating the asset processes. Most importantly, none of the papers deals with the computation of regulatory capital.

Parts of the CVA literature (e.g., Pykhtin and Zhu (2007), Gregory (2009), and Pykhtin (2010)) explicitly focus on the estimation of exposure at default (EAD), i.e., on estimating the loss in market value when the contract terminates. Similarly, Taplin et al. (2007) and Valvonis (2008) investigate the credit conversion factor used to account for possible (retail) overdrafts. This literature can be understood as being complementary to our work, also in order to consider the price, value, and market risk of a guarantee. Similarly, one could calculate the refinancing costs that occur if a guarantor defaults and the guarantee should be reestablished. If a collateral serves as a guarantee, the jump size could be taken as its expected exposure at default.

Closer to our model is the contagion model of Leung and Kwok (2005). There, upward jumps in the default intensity of an entity occur whenever another entity defaults. This allows for an asymmetric dependency structure between obligor and guarantor which has to be specified exogenously.

While the mentioned literature focuses on the proper pricing of guarantees like CDSs by evaluating the CVA, our work deals with the impact of guarantees on regulatory capital. That is, once the guarantee has been obtained (irrespectively of its price, CVA, or current market value), by how much should credit risk sensitive regulatory capital be reduced? Although the IRB treatment of double default effects is largely applied in practice, this question has not been answered so far. To the best of our knowledge, the only paper that directly addresses the IRB model of double default is Grundke (2008). However, it is not concerned with the IRB model and the latter's assumptions, but rather with the appropriate parameter choices within the model of Heitfield and Barger (2003).

The remainder of the chapter is structured as follows. In Section 2.2, we provide a review of the IRB treatment of double default effects, and we reveal several severe shortcomings of the approach. Section 2.3 contains our new asset drop model to account for double default effects which can be used in structural models of credit risk and, in particular, in the IRB approach of Basel II. We also implement our method within some examples and compare the results to the current IRB treatment of double default effects. A discussion and concluding remarks are given in Section 2.4.

2.2 Review and discussion of the IRB treatment of double defaults

Within the IRB approach of Pillar 1 in Basel II, banks may choose between the simple *substitution approach* outlined in the Introduction and a double default approach where risk weighted assets for exposures subject to double default are calculated as follows.[3] Assume the exposure to obligor n is hedged by guarantor g_n. Within the double default treatment in the IRB approach, one first computes the unexpected loss (UL) capital requirement $\mathcal{K}_n$ for the hedged obligor n in the same way as the one for an unhedged exposure[4] with LGD_n replaced by the loss given default LGD_{g_n} of the guarantor. In the computation of the maturity adjustment, the default probability is chosen as the minimum of the obligor's default probability PD_n and the guarantor's default probability PD_{g_n}. Then, the UL capital requirement $\mathcal{K}_n^{DD}$ for the hedged exposure is calculated by multiplying $\mathcal{K}_n$ by an adjustment factor depending on the PD of the guarantor, namely

$$\mathcal{K}_n^{DD} = \mathcal{K}_n \cdot (0.15 + 160 \cdot \mathrm{PD}_{g_n}). \tag{2.2.1}$$

Finally, the risk weighted asset amount for the hedged exposure is computed in the same way as for unhedged exposures. Note that the multiplier $(0.15 + 160 \cdot \mathrm{PD}_{g_n})$ is derived as a linear approximation to the UL capital requirement for hedged exposures. For the computation of the latter, i.e., to derive the exact conditional expected loss function for a hedged exposure, the ASRF framework, which also presents the basis for the computation of the risk weighted assets in the IRB approach, is used in an extended version. Specifically, it is assumed that the asset returns r_n and r_{g_n} of an obligor and its guarantor, respectively, are no longer conditionally independent given the systematic risk factor X. They also depend on an additional risk factor Z_{n,g_n} which only affects the obligor and its guarantor. More precisely,

$$r_n = \sqrt{\rho_n} X + \sqrt{1-\rho_n}\left(\sqrt{\psi_{n,g_n}} Z_{n,g_n} + \sqrt{1-\psi_{n,g_n}}\epsilon_n\right) \tag{2.2.2}$$

where ρ_n is the asset correlation of obligor n, ψ_{n,g_n} is a factor specifying the sensitivity of obligor n to the factor Z_{n,g_n}, and ϵ_n is the idiosyncratic risk factor of obligor n. By implicitly assuming that all hedges are perfect full hedges, guarantors are themselves not

[3] See Basel Committee on Banking Supervision (2006), paragraph 284.

[4] The latter is defined in paragraphs 272 and 273 of Basel Committee on Banking Supervision (2006).

obligors in the portfolio, and different obligors are hedged by different guarantors, the joint default probability of the obligor and its guarantor can be computed explicitly as[5]

$$\begin{aligned} & \mathbb{P}\left(\{\text{default of obligor n}\} \cap \{\text{default of guarantor}\, g_n\}\right) \\ = \quad & \Phi_2\left(\Phi^{-1}(\mathrm{PD}_n), \Phi^{-1}(\mathrm{PD}_{g_n}); \rho_{n,g_n}\right) \end{aligned} \tag{2.2.3}$$

where ρ_{n,g_n} is the correlation between obligor n and its guarantor g_n and $\Phi_2(\cdot, \cdot; \rho)$ denotes the cumulative distribution function of the bivariate standard normal distribution with correlation ρ. Therefore, the conditional expected loss function for a hedged exposure is given by

$$\begin{aligned} & \mathbb{E}\left[\mathbb{1}_{\{r_n \le c_n\}} \mathbb{1}_{\{r_{g_n} \le c_{g_n}\}} \,\mathrm{LGD}_n \,\mathrm{LGD}_{g_n} \,|X\right] = \mathrm{LGD}_n \,\mathrm{LGD}_{g_n} \cdot \\ & \Phi_2\left(\frac{\Phi^{-1}(\mathrm{PD}_n) - \sqrt{\rho_n}X}{\sqrt{1-\rho_n}}, \frac{\Phi^{-1}(\mathrm{PD}_{g_n}) - \sqrt{\rho_{g_n}}X}{\sqrt{1-\rho_{g_n}}}; \frac{\rho_{n,g_n} - \sqrt{\rho_n \rho_{g_n}}}{\sqrt{(1-\rho_n)(1-\rho_{g_n})}}\right) \end{aligned} \tag{2.2.4}$$

for default thresholds c_n and c_{g_n} for obligor n and its guarantor g_n, respectively. One obtains the IRB risk weight function for a hedged exposure with effective maturity of one year by inserting $\Phi^{-1}(0.001)$ for X, subtracting the expected loss

$$\Phi_2(\Phi^{-1}(\mathrm{PD}_n), \Phi^{-1}(\mathrm{PD}_{g_n}); \rho_{n,g_n}) \cdot \mathrm{LGD}_n \,\mathrm{LGD}_{g_n}, \tag{2.2.5}$$

and multiplying with 12.5 and 1.06. Since the expected loss should in general be rather small, in Basel Committee on Banking Supervision (2005) this term is set equal to zero. Moreover, it is assumed that there are no double recovery effects and thus $\mathrm{LGD}_n = 1$. Within the IRB treatment of double default effects, however, the linear approximation (2.2.1) of the exact conditional expected loss function (2.2.4) is used which holds for the parameter values specified before.[6]

Let us now discuss the assumptions underlying this approach in more detail. First let us investigate how well correlation in general suits to model the dependency between a guarantor and an obligor. Positive correlation implies that default of the obligor makes the default of the guarantor more likely. This seems very reasonable as the guarantor suffers from the guarantee payment, and if it is large, it might even drag him into default. Vice

[5] For more details on the derivation see, e.g., Grundke (2008, pp. 40-41).

[6] Grundke (2008) explains this approximation in greater detail and illustrates its accuracy. For a comprehensive and more detailed overview of the double default treatment we refer to his paper and the original paper by Heitfield and Barger (2003).

versa, however, it seems neither theoretically nor empirically justified that the default of the guarantor implies a similar pain to the hedged obligor.[7] Note that the obligor, in general, will not even know whether the bank that granted the loan obtained credit protection at all. And if so, the obligor will not know the name of the guarantor. Essentially, for the hedged obligor, the pain from the default of the guarantor should not be heavier than the pain from the default of any other firm in the economy. It will influence the default probability of the obligor only through shifts in the state of the systematic risk factor. As correlation necessarily introduces a symmetric dependency between two random variables, it can never capture the asymmetric relationship that holds between a guarantor and an obligor.

Before we continue, let us consider a case where modeling the dependency between a guarantor and an obligor symmetrically could be justified. Suppose, first, there is no direct exposure to guarantors and, second, every guarantor hedges exactly one position in the portfolio. In this case, one is interested in the double default, but not specifically in the default of the guarantor. The unconditional dependence of the guarantor with the rest of the portfolio is ignored, but this can be compensated perfectly by choosing the additional correlation sufficiently high. Essentially, in that case the obligor and its guarantor (that interacts with the obligor and nobody else) constitute a conditionally independent unit in the portfolio. Then, correlation can be used reasonably to model the default dependency between the obligor and its guarantor. The default event of obligor 1 can be simply replaced with the less likely double default event.

The IRB treatment of double default effects simply makes no distinction, whether or not a guarantor is itself an obligor in the portfolio or if it guarantees for several obligors. The implicit approach undertaken in the IRB model for *any* hedging constellation is the one just explained.

If one of the two assumptions above is violated, then application of the IRB treatment of double default effects is no more rigorous. When applying the IRB treatment of double defaults, the interactions of each guarantor with the rest of the portfolio are ignored. To be more precise, if the guarantor itself is in the portfolio, it would be treated as any other obligor in the portfolio, i.e., conditionally independent from the obligor it guarantees for. Its expected loss is computed as if it was not involved in a hedging relationship, i.e., with an unchanged default probability and a correlation parameter as used for obligors rather

[7]For a discussion of wrong-way risk and the market risk of guarantees see Remark 8 at the end of this section.

than for guarantors. If a guarantor hedges several positions, this problem becomes even more severe. Moreover, this implies that excessive contracting of the same guarantor is not reflected in the computation of economic capital.

Further, note that the IRB treatment of double default effects is generally unsuited to deal with the above situations because of the additional correlation assumption. If the guarantor is itself in the portfolio, its default will significantly increase the default probability of the obligor, what is an unappreciated consequence as mentioned before. If, on the other hand, the guarantor hedges more than one obligor, say 3 hedges 1 and 2, then the default of 1 increases the guarantor's default probability which itself increases the default probability of 2. That is, 1 and 2 are no more conditionally independent because they share the same "contagious" guarantor. In general, this seems to be very unreasonable as there need not be any business relationship between 1 and 2, or there even might be a negative relationship between them such that default of 1 should actually decrease the default probability of obligor 2.[8] Thus, we conclude that the IRB treatment of double default effects can only be used reasonably if every obligor in the portfolio has a different guarantor and if there is no direct exposure to any of these guarantors.

Remark 7 (Consistency with the ASRF model)**.** *From a theoretical or mathematical point of view, the introduction of additional correlation within the IRB approach leads to some problems as a main assumption underlying this framework is violated. Suppose that a guarantor hedges several obligors or that a guarantor is internal in the sense that there is also direct exposure to the guarantor. In this case, additional correlation violates the conditional independence assumption on which the ASRF model is based. Conditional independence between the obligor loss variables, however, is required because the ASRF model relies on a law of large numbers. Let us mention here, however, that the violation of the conditional independence assumption underlying the ASRF model will essentially occur in any approach that correctly accounts for interactions resulting from double default effects. In this situation, the asymptotic result used in the approximation of value-at-risk* $\alpha_q(L)$ *of the portfolio loss by the expected portfolio loss* $\mathbb{E}[L|\alpha_q(X)]$ *conditional on the quantile* $\alpha_q(X)$ *of the systematic risk factor only holds when the hedged exposure shares and the direct exposure shares to guarantors are sufficiently small.*

Finally, let us also mention another deficiency of the IRB treatment of double default

[8] Similarly to the argument before, also note that 1 and 2 will not know wether there is a guarantor. And if so, they will not know who it is.

effects which is highly relevant for practical applications. It concerns the parameter choice of the conditional correlation parameters. While not questioning the assumption of imposing additional correlation between an obligor and its guarantor in general, in a recent and long overdue empirical study, Grundke (2008) investigates the numerical values of the correlation parameters $\rho_{g_n} = 0.7$ and $\rho_{n,g_n} = 0.5$ set by the Basel Committee. To this end, he reviews empirical studies on default correlation and further initiates new simulation studies, which yield rather different results. While the empirical studies he considers imply that the parameters are chosen overly conservative, the simulation experiments "show that the assumed values are not unrealistic for capturing the intended effects."[9] He also notes that the appropriateness of the parameter choice actually depends, for example, on the size of the guarantor and the amount guaranteed. Within the IRB treatment of double default effects, correlation parameters are independent of these quantities. Implicitly this means, for instance, that a small bank and a large insurance company would suffer equally from any guarantee payment.

Remark 8 (Wrong-way risk)**.** *It might be argued that not the obligor, but the bank whose regulatory capital we aim to compute will be affected by the guarantor's default. This phenomenon, sometimes referred to as wrong-way risk, might be due to a loss in market value of the defaulted hedging product. For example, if the bank decides to obtain a new guarantee, this loss in market value had to be realized immediately as replacement costs. It should be clear, however, that this effect will not justify a symmetric dependency structure.[10] Moreover, we propose not to dilute this effect with the Pillar 1 capital requirements. Also in the current treatment of double default effects within the IRB approach, the price or market value of guarantees is not reflected, and this seems well justified. Given the existence of a guarantee, the bank should benefit from smaller capital requirements (depending on the quality of the guarantee). If there is no guarantee (or of it has defaulted), it should not. Price, market value, or possible replacement costs of the guarantee should be reflected on the market risk side. The CVA literature mentioned in the introduction offers appropriate tools for its risk assessment.*

[9] See Grundke (2008, p. 58).

[10] Within the model we will propose it is straightforward to incorporate such a reverse feedback effect while still having some asymmetry. This can be achieved, e.g., by introducing an additional drop in the asset value of the obligor by the market value of the hedging product.

2.3 The asset drop technique as an alternative approach

In this section, we will present an alternative method to account for double default effects in credit portfolios that does not rely on additional correlation between obligor and guarantor. It does capture their asymmetric relationship, i.e., that the guarantor should suffer much more from the obligor's default (triggering the guarantee payment) than vice versa. Further, our method distinguishes the case where there is direct exposure to the guarantor from the case where the guarantor is external to the portfolio. Furthermore, we properly treat the situation where a guarantor hedges several obligors.

Instead of modeling the relationship between guarantor and obligor through dependency on an additional stochastic risk factor, we adjust the guarantor's default probability appropriately if the obligor defaults. Our model is endogenous as it actually quantifies the increase in the guarantor's default probability instead of exogenously imposing numerical values as it is done in case of the additional correlation parameters ρ_{n,g_n} in the IRB treatment of double default effects. The increase in the guarantor's default probability in our new approach depends on the size of the guarantee payment as well as on the size of the guarantor measured in terms of its asset value. The method is very well suited for practical applications as it does not pose extensive data requirements. Moreover, due to the simple analytical representation of economic capital when incorporated in the IRB model, it can be computed almost instantaneously.

2.3.1 Methodology

Within a structural model of default, the guarantee payment that occurs to the guarantor corresponds to a downward jump in its firm value process or, equivalently, in the firm's asset return. This causes the unconditional default probability to increase by a growth factor $(1 + \lambda_{n,g_n})$. This qualitative observation can be found in Grundke (2008, p. 53).[11]

To illustrate the idea of the approach, let us first consider the simple case where obligor 1 is hedged by a guarantor, g_1, which is external to the portfolio. That is, the guarantor is itself not an obligor in the portfolio. We want to quantify the impact of obligor 1's default on the guarantor's unconditional default probability. In the current situation, the default of the

[11] In order to assess the conservativeness of the parameter choices for the additional correlation in the treatment of double default effects in the IRB approach, Grundke shows that the additional correlation approximately translates into an increase of 100% in the guarantor's unconditional PD . In principle, one could use Grundke's calculation to (numerically) obtain individual additional correlation parameters from our estimate of λ_{n,g_n}.

guarantor is only of interest if obligor 1 defaults as well. If solely the guarantor defaults, there is no loss as there is no direct exposure to the guarantor. Thus, our objective is to compute the guarantor's (increased) default probability when the hedged obligor already has defaulted such that the guarantee payment has been triggered. The loss due to the guarantee payment may cause the guarantor's default or may make it more likely. For simplicity and for consistency with the IRB approach, we illustrate the method within an extension of the model of Merton (1974). However, in principle, our new approach can also be applied in more sophisticated structural credit risk models which are, e.g., driven by Lévy processes.

In the IRB approach, we consider a two-period model with a 1-year horizon where time t is today and T refers to one year in the future. Our input parameters are the initial firm value $V_{g_1}(t)$ of the guarantor g_1, i.e., the firm's value at time t, which is taken, e.g., from the balance sheet, or inferred from the current stock price, as well as an estimate of its volatility σ_{g_1}. We further need the (non-portfolio specific) default probability PD_{g_1} that could be obtained from a rating agency or from an internal model and the risk-free interest rate r. In Merton's model, it is assumed that the asset value process of guarantor g_1 follows a geometric Brownian motion of the form

$$V_{g_1}(T) = V_{g_1}(t) \cdot e^{(\mu_{g_1} - \frac{1}{2}\sigma_{g_1}^2)(T-t) + \sigma_{g_1} W_{T-t}} \tag{2.3.1}$$

where W_{T-t} is a standard Brownian motion and B_{g_1} is the guarantor's debt value. Under the risk-neutral measure, one then obtains the unconditional default probability of guarantor g_1 as

$$\mathrm{PD}_{g_1} = \mathbb{P}(V_{g_1}(T) < B_{g_1}) = 1 - \Phi\left(\frac{\ln(V_{g_1}(t)/B_{g_1}) + (r - \frac{1}{2}\sigma_{g_1}^2)(T-t)}{\sigma_{g_1}\sqrt{T-t}}\right). \tag{2.3.2}$$

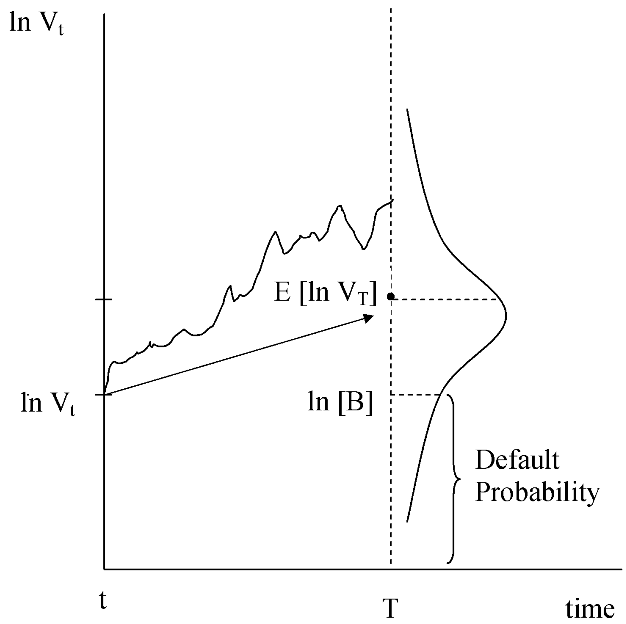
From this, one can compute the default threshold B_{g_1} of guarantor g_1 implied by Merton's model as

$$B_{g_1} = V_{g_1}(t) \cdot \exp\left(-\Phi^{-1}(1 - \mathrm{PD}_{g_1}) \cdot \sigma_{g_1}\sqrt{T_t} + \left(r - \frac{1}{2}\sigma_{g_1}^2\right)(t-t)\right). \tag{2.3.3}$$

Figure 2.1 illustrates the mechanism of the Merton model.[12]
Our asset drop model represents an extension of Merton's model. If obligor 1 defaults, this

[12]The illustrations in Figures 2.1 and 2.2 have been kindly provided by John O'Keefe who discussed an earlier version of this work at the Australasian Banking and Finance Conference 2009.

Figure 2.1: Probability of default in the Merton model



The asset value process V_t follows a geometric Brownian motion such that the log asset-returns are normally distributed with mean $\mathbb{E}[\ln V_T]$ at maturity T. If the asset value at maturity falls below the value of the firm's liabilities B, the firm will default.

corresponds to a drop in the asset value V_{g_1} of the guarantor by the nominal $\hat{E}_{1,g_1}$ that g_1 guarantees for obligor 1.[13] Hence, we model the asset value process of the guarantor g_1 as

$$V_{g_1}(T) = V_{g_1}(t) \cdot e^{(\mu_{g_1} - \frac{1}{2}\sigma_{g_1}^2)(T-t) + \sigma_{g_1} W_{T-t}} - \hat{E}_{1,g_1} \cdot \mathbb{1}_{\{V_1(T) \le B_1\}}. \tag{2.3.4}$$

Thus, our model represents a jump-diffusion model in the sense that the jump time is determined by the stopping time $\mathbb{1}_{\{V_1(T) \le B_1\}}$, i.e., by the default time of obligor 1 triggering the guarantee payment. Moreover, the jump size is deterministic and given by the nominal $\hat{E}_{1,g_1}$ that g_1 guarantees for obligor 1. We refer to this type of model as a Bernoulli mixture model.[14] The guarantor defaults with the increased probability PD'_{g_1} when the guarantee

[13] At this point, it can be seen that the model is, in principle, capable to capture also other dependencies such as business-to-business relationships. For example, if it is known that the guarantor has a direct claim of E_{1,g_1} to obligor 1, it might be reasonable to continue the computation with the higher asset drop $\hat{E}_{1,g_1} + E_{1,g_1}$. To appropriately treat risky collateral, $\hat{E}_{1,g_1}$ could be taken as expected exposure at default.

[14] Note that a classical jump diffusion model as, e.g., in Zhou (2001b) is not suitable to model double default effects for the following reason. In that model, jumps are driven by a Poisson process with intensity λ, and the jump amplitude is stochastic as well. The main idea of our double default model is that we explicitly model the time when the asset value drops, i.e., jumps downward. Therefore, we consider the default time of the obligor that is hedged. This then leads to a Bernoulli-mixture model as stated above. Moreover, in our setting, the jump amplitude is deterministic as the amount that is guaranteed should be known in advance.

payment has been triggered, i.e., under the risk-neutral measure the increased default probability of g_1 is given by

$$\begin{aligned}
\mathrm{PD}'_{g_1} &= \mathbb{P}\left(V_{g_1}(T) \leq B_{g_1} | V_1(T) \leq B_1\right) \\
&= \mathbb{P}\left(V_{g_1}(t) \cdot e^{(r-\frac{1}{2}\sigma_{g_1}^2)(T-t)+\sigma_{g_1} W_{T-t}} - \hat{E}_{1,g_1}\right) \\
&= 1 - \Phi\left(\frac{\ln\left(\frac{V_{g_1}(t)}{B_{g_1}+\hat{E}_{1,g_1}}\right) + \left(r - \frac{1}{2}\sigma_{g_1}^2\right)(T-t)}{\sigma_{g_1}\sqrt{T-t}}\right).
\end{aligned} \tag{2.3.5}$$

Similarly, the guarantor defaults with the probability PD_{g_1} if obligor 1 survives, i.e.,

$$\begin{aligned}
\mathrm{PD}_{g_1} &= \mathbb{P}\left(V_{g_1}(T) \leq B_{g_1} | V_1(T) \geq B_1\right) \\
&= \mathbb{P}\left(V_{g_1}(t) \cdot e^{(r-\frac{1}{2}\sigma_{g_1}^2)(T-t)+\sigma_{g_1} W_{T-t}}\right) \\
&= 1 - \Phi\left(\frac{\ln\left(V_{g_1}(t)/B_{g_1}\right) + \left(r - \frac{1}{2}\sigma_{g_1}^2\right)(T-t)}{\sigma_{g_1}\sqrt{T-t}}\right).
\end{aligned} \tag{2.3.6}$$

Figure 2.2 illustrates the functioning of our new asset drop approach. In particular, it shows how the guarantor's PD increases when the guarantee payment has been triggered. Note that B_{g_1} is the default threshold of guarantor g_1 in case the hedged obligor 1 has not defaulted. Thus, B_{g_1} can be computed from the guarantor's observed rating according to the classical Merton model by equation (2.3.3). Thus, we can compute the increased PD'_{g_1} of the guarantor due to the obligor's default using equations (2.3.3) and (2.3.5). This then provides an analytic formula for the unconditional default growth rate λ_{1,g_1}, i.e., the relative increase of the guarantor's default probability due to the hedged obligor's default. It is defined as

$$\lambda_{1,g_1} = \frac{\mathrm{PD}'_{g_1} - \mathrm{PD}_{g_1}}{\mathrm{PD}_{g_1}} \tag{2.3.7}$$

such that

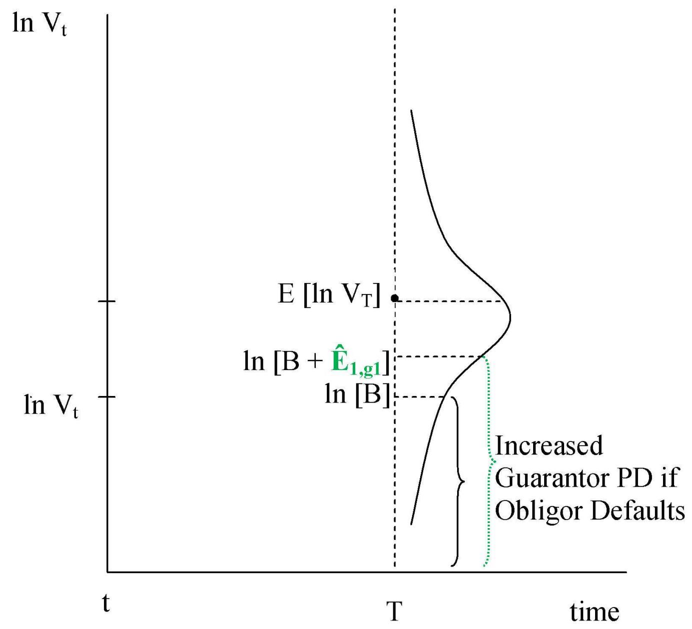
$$\mathrm{PD}'_{g_1} = \mathrm{PD}_{g_1} \cdot (1 + \lambda_{1,g_1}). \tag{2.3.8}$$

We now illustrate how this approach can be incorporated in the IRB model for the computation of economic capital. The probability distribution of the loss variable L_1 of obligor 1 is in our setting given by

$$\mathbb{P}(L_1 = l) = \begin{cases} \mathrm{PD}'_{g_1}\,\mathrm{PD}_1 & \text{for } l = s_1\,\mathrm{LGD}_{g_1} \\ (1 - \mathrm{PD}'_{g_1})\,\mathrm{PD}_1 + (1 - \mathrm{PD}_1) & \text{for } l = 0. \end{cases} \tag{2.3.9}$$

In order to respect double recovery effects, LGD_{g_1} could be multiplied by LGD_1. How-

Figure 2.2: Probability of default in the asset drop model



The asset value process V_t follows a Bernoulli-mixture model of the form (2.3.4) such that the asset value of the guarantor drops by the guarantee's nominal $\hat{E}_{1,g_1}$ in case the hedged obligor defaults. Otherwise, the asset value of the guarantor is log-normally distributed with mean $\mathbb{E}[\ln V_T]$ at maturity T. If the hedged obligor has defaulted and if the asset value of the guarantor at maturity falls below the value of the firm's liabilities B plus the guarantee's nominal $\hat{E}_{1,g_1}$, the guarantor will default as well. Hence, the default of the hedged obligor leads to an increase in the guarantor's default probability.

ever, for several reasons, double recovery is not reflected in the current Basel II framework. Therefore, also in the following we always set $\mathrm{LGD}_1 = 1$ such that only recovery of the guarantor is accounted for. Then, the expected loss for obligor 1 is $\mathbb{E}[L_1] = s_1 \,\mathrm{LGD}_{g_1}\,\mathrm{PD}_1\,\mathrm{PD}'_{g_1}$, and the expected loss conditional on a realization x_q of the systematic risk factor X is

$$\mathbb{E}[L_1|x_q] = s_1\,\mathrm{LGD}_{g_1}\,\mathrm{PD}_1(x_q)\,\mathrm{PD}'_{g_1}(x_q)$$

where the conditional PDs are computed as in the IRB approach by

$$\mathrm{PD}_1(X) = \Phi\left(\frac{\Phi^{-1}(\mathrm{PD}_1) - \sqrt{\rho_1}X}{\sqrt{1-\rho_1}}\right)$$

and analogously for PD'_{g_1}. Hence, the unexpected loss capital requirement $\mathcal{K}_1$ for the hedged exposure s_1 is[15]

$$\mathcal{K}_1 = \mathrm{LGD}_{g_1}(\mathrm{PD}_1(x_q)\,\mathrm{PD}'_{g_1}(x_q) - \mathrm{PD}_1\,\mathrm{PD}'_{g_1}).$$

Hence, to compute the IRB capital charges for the hedged exposure to obligor 1, one simply inserts the double default probability $\mathrm{PD}'_{g_1}\,\mathrm{PD}_1$ instead of PD_1 in the formula for the IRB risk weight functions.

Remark 9 (Convexity of effective guarantor PD)**.** *By taking derivatives in equations (2.3.3) and (2.3.5), it can be shown that* PD'_g *is convex in the guarantee nominal. This convexity sets an incentive for banks to contract several distinct guarantors for various loans. Suppose, for example, there are two identical loans and two guarantors with exactly the same characteristics. Then, the overall increase in default probability is smaller if each guarantor is contracted for one of the loans compared to when one guarantor is chosen to guarantee both loans. Thus, also the bank's economic capital will be smaller if it diversifies its guarantor risk. In particular, as will be shown explicitly in Example 2, excessive contracting of the same guarantor will significantly increase economic capital. This definitely is an appreciated consequence from a regulatory point of view. However, the effect is not reflected in the current treatment of double default effects within the IRB approach. Under that approach, economic capital does not depend on whether a hundred loans are hedged by one single guarantor, or whether every loan is hedged by one out of a hundred different guarantors.*

Example 2 (Computation of effective PD with the asset drop technique)**.** *Consider two medium-sized banks,* g_1 *and* g_2*, which according to their balance sheets have total asset values of* $V_{g_1}(t) = 50$ *and* $V_{g_2}(t) = 10$ *billion Euros, respectively. Both firm value volatilities are estimated to be* $\sigma^2_{g_1} = \sigma^2_{g_2} = 30\%$*. Assume both to have the same rating which translates into an unconditional default probability of* $\mathrm{PD}_{g_1} = \mathrm{PD}_{g_2} = 0.5\%$*. The market's risk-free interest rate is* $r = 0.02\%$*. Assume a 1-year time horizon. Using formula (2.3.3), we can compute the implicit default threshold for the larger bank in the Merton model and obtain* $B_{g_1} = 22.517.068$ *billion Euros. Likewise, for the smaller bank, we obtain* $B_{g_2} = 4.502.414$ *billion Euros. Figure 2.3 shows the effective default probabilities* PD'_{g_1} *and* PD'_{g_2} *of the two banks as a function of the expected guarantee payment* $\hat{E}_{1,g_1} \equiv \hat{E}_{1,g_2}$*. These have been computed with the asset drop technique according to equation (2.3.5). When the expected guarantee payment is, e.g., 400 million Euros, then the effective default probability of the smaller bank would be* $\mathrm{PD}'_{g_2} = 1.09\%$*, which corresponds to an increase by a factor* $(1 + \lambda_{1,g_2}) = 2.19$*, i.e.,* $\lambda_{1,g_2} = 1.19$*. This means that a financial institution which has no*

[15]The Basel II economic capital for the hedged exposure 1 is obtained by multiplying $\mathcal{K}_1$ with the scaling factor 1.06 and the maturity adjustment MA_1 where we insert $\mathrm{PD}_1\,\mathrm{PD}'_{g_1}$ instead of PD_1.

Figure 2.3: Effective PD computed with the asset drop technique

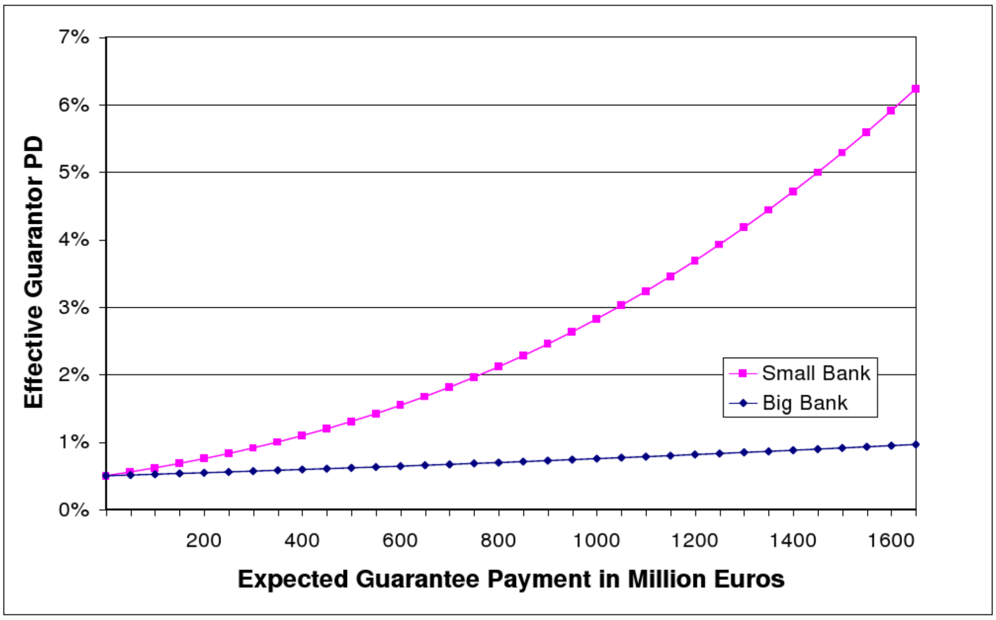


Figure 2.3 shows the effective guarantor default probability $\mathrm{PD}'_{g_n} = \mathrm{PD}_{g_n}(1+\lambda_{n,g_n})$ for two banks as a function of the expected guarantee payment $\hat{E}_{n,g_n}$. For a large bank (diamond line) the graph is moderately increasing, and a guarantee payment of 1600 million Euros would roughly double its initial default probability. For a smaller bank (square line) the initial default probability already doubles when it has to make a payment of 275 million Euros. From this graph we also see the convexity of the relationship. This implies higher capital requirements if the same guarantor is used for several transactions.

direct exposure to g_2 *and which buys protection from the latter for its 400 million exposure to obligor* 1 *will use this increased default probability when computing its economic capital due to obligor* 1. *This is intuitive as* g_2*'s default is only of interest when obligor* 1 *already has defaulted. For the larger bank the guarantee payment corresponds to a less significant loss. Its effective PD would only increase by a factor* $(1+\lambda_{1,g_1}) = 1.18$ *to* $\mathrm{PD}'_{g_1} = 0.59\%$. *Note also that the relationship is convex as already mentioned in Remark 9. Also note from equations (2.3.3) and (2.3.5) that the increase in* PD *is scale invariant with respect to the firm size and the loan nominal. Thus, for example, a true global player with 100 times the firm size of the large bank considered here could guarantee 100 times as much as the large bank while suffering from the same increase in* PD .

2.3.2 Generalizations

Let us now consider the more complicated case where there is direct exposure to the guarantor. Denote the exposure share of obligor 1 by s_1 and assume that it is fully hedged by guarantor g_1. Denote the direct exposure share to the guarantor by s_{g_1}. In this case, we also have to focus on the default of the guarantor itself, i.e., a loss also occurs if the guarantor defaults and the hedged obligor survives. In this situation, in a sense, there are two appropriate default probabilities of the guarantor. If obligor 1 already has defaulted, the default probability of the guarantor is given by PD'_{g_1}. Otherwise, it is given by PD_{g_1}. To compute the contribution to economic capital of the hedged obligor and its guarantor within in the IRB approach, we have to compute the conditional expected loss of both. As we do not want to reflect double recovery effects (similarly to the treatment in Basel II), we set $\mathrm{LGD}_1 = 1$ for a hedged exposure. The probability distribution of the joint loss variable L_{1,g_1} of obligor 1 and its guarantor g_1 is then

$$\mathbb{P}(L_{1,g_1} = l) = \begin{cases} \mathrm{PD}'_{g_1}\,\mathrm{PD}_1 & \text{for } l = s_1\,\mathrm{LGD}_{g_1} \\ & \qquad +s_{g_1}\,\mathrm{LGD}_{g_1} \\ \mathrm{PD}_{g_1}(1-\mathrm{PD}_1) & \text{for } l = s_{g_1}\,\mathrm{LGD}_{g_1} \\ (1-\mathrm{PD}'_{g_1})\,\mathrm{PD}_1 + (1-\mathrm{PD}_{g_1})(1-\mathrm{PD}_1) & \text{for } l = 0. \end{cases} \tag{2.3.10}$$

Note that the increased unconditional default probability PD'_{g_1} occurs together with PD_1, i.e., with the probability that obligor 1 defaults as in these situations the guarantee payment is triggered. The first case corresponds to the situation where both the obligor and the guarantor default, i.e., to the double default case. In the second case, only the guarantor defaults such that only the direct exposure to g_1 is lost. The third case comprises the hedging case, i.e., the obligor defaults and the guarantor succeeds in delivering the guarantee payment (although its default probability has increased) and the case where both the guarantor and the obligor survive. Thus, no loss occurs in this case. The expected loss can be computed as

$$\begin{aligned} \mathbb{E}[L_{1,g_1}] &= \mathrm{PD}'_{g_1}\,\mathrm{PD}_1(s_{g_1}\,\mathrm{LGD}_{g_1} + s_1\,\mathrm{LGD}_{g_1}) \\ &\quad + \mathrm{PD}_{g_1}(1-\mathrm{PD}_1)s_{g_1}\,\mathrm{LGD}_{g_1} \\ &= s_{g_1}\,\mathrm{LGD}_{g_1}\Big(\mathrm{PD}_{g_1} + \mathrm{PD}_1\cdot(\mathrm{PD}'_{g_1} - \mathrm{PD}_{g_1})\Big) \\ &\quad + s_1\,\mathrm{LGD}_{g_1}\,\mathrm{PD}'_{g_1}\,\mathrm{PD}_1\,. \end{aligned}$$

This can be reformulated as

$$\mathbb{E}[L_{1,g_1}] = s_{g_1}\,\mathrm{LGD}_{g_1}\,\mathrm{PD}_{g_1}(1+\lambda_{1,g_1}\,\mathrm{PD}_1) + s_1\,\mathrm{LGD}_{g_1}\,\mathrm{PD}'_{g_1}\,\mathrm{PD}_1\,. \tag{2.3.11}$$

The probability that the exposure s_{g_1} in the first term is lost equals the expected default probability of the guarantor. The probability that the hedged exposure s_1 in the second term is lost, on the other hand, equals the default probability of the guarantor conditional on obligor 1's default. The second term in equation (2.3.11) is the expected loss due to obligor 1 that only occurs in the situation of double default. This term is the same as in the case where the guarantor is external. The first term in equation (2.3.11) is the expected loss due to obligor 2 whose default probability increases if it has to exercise the guarantee payment. Therefore, the expected loss due to an obligor increases if it is involved in a hedging activity because its expected PD increases. This fact is ignored in the treatment of double default effects in the IRB approach since guarantors are implicitly treated as external.[16]

The derivation of economic capital for the hedged exposure and its guarantor is obtained as follows. The conditional expected loss can be obtained as in the model underlying the IRB treatment of double default effects when there is no additional correlation. Denote by r_n and r_{g_n} the log asset return of obligor n and its guarantor g_n, respectively. Let the conditional default probabilities be defined as in the IRB model by

$$\mathrm{PD}_n(X) = \Phi\left(\frac{\Phi^{-1}(\mathrm{PD}_n) - \sqrt{\rho_n}X}{\sqrt{1-\rho_n}}\right) \tag{2.3.12}$$

for $n = 1$ or g_1 and analogously for $\mathrm{PD}'_{g_1}(X)$. Then, in our setting we have

$$\begin{aligned}
\mathbb{E}[L_{1,g_1}|X] &= s_1\,\mathrm{LGD}_{g_1}\,\mathbb{E}[\mathbb{1}_{\{r_1<c_1\}}\mathbb{1}_{\{r_{g_1}<c'_{g_1}\}}|X] \\
&\quad + s_{g_1}\,\mathrm{LGD}_{g_1}\,\mathbb{E}[\mathbb{1}_{\{r_{g_1}<c_{g_1}\}}\mathbb{1}_{\{r_1\geq c_1\}} + \mathbb{1}_{\{r_{g_1}<c'_{g_1}\}}\mathbb{1}_{\{r_1<c_1\}}|X] \\
&= s_1\,\mathrm{LGD}_{g_1}\,\mathrm{PD}_1(X)\,\mathrm{PD}'_{g_1}(X) \\
&\quad + s_{g_1}\,\mathrm{LGD}_{g_1}\Big(\mathrm{PD}_{g_1}(X)(1-\mathrm{PD}_1(X)) + \mathrm{PD}'_{g_1}(X)\,\mathrm{PD}_1(X)\Big)
\end{aligned} \tag{2.3.13}$$

where we again neglected double recovery effects. Note that the loss variables for s_1 and s_{g_1} in the above equation are stochastically dependent conditional on X. Thus, approximating the value-at-risk $\alpha_q(L)$ by the conditional expected portfolio loss as it is done in the IRB approach only makes sense within a double default treatment when the hedged exposure shares and the direct exposure shares to guarantors are sufficiently small; see Remark 7 for more details.

Partial hedging and the case where a guarantor hedges multiple obligors in a portfolio can

[16] Note, again, that under the IRB approach it would not be reasonable to take into account direct exposure to a guarantor as the additional correlation would induce an unrealistic dependency between obligor and guarantor.

be approached with the same technique just presented, and the results are straightforward. A detailed treatment of these situations under Pillar 2 of Basel II has been presented in the previous chapter.

Example 3 (Comparison of EC computed with the IRB treatment of double default effects and with the asset drop technique)**.** *Consider a portfolio with* $N = 110$ *obligors. The first* $n = 1, \ldots, 10$ *loans in the portfolio are hedged by guarantors* $101, \ldots, 110$ *who also act as obligors in the portfolio. Assume the exposures to equal* $\mathrm{EAD}_n = 1$ *for all* $n = 1, \ldots, 110$. *The PDs are assumed to be 1% for* $n = 1, \ldots, 100$, *and* 0.1% *for the guarantors* $n = 101, \ldots, 110$. *As in the IRB approach, let LGDs be 45% for all unhedged obligors* $n = 11, \ldots, 110$. *Hedged exposures are assigned an LGD of 100% in order to neglect double recovery effects, i.e.,* $\mathrm{LGD}_n = 100\%$ *for* $n = 1, \ldots, 10$. *We assume an effective maturity of* $M = 1$ *year for all obligors and guarantors in the portfolio. Value-at-risk is computed at the 99.9% percentile level. The IRB treatment of double default effects yields an economic capital of* 5.40% *of total exposure.*[17] *This is lower than the value obtained when neglecting double default effects entirely, which equals* 5.79%. *Denoting by* x_q *the* q^{th} *percentile of the systematic risk factor* X, *we calculated the IRB capital with the asset drop technique as*

$$\begin{aligned}
&\sum_{n=1}^{10} s_n \,\mathrm{LGD}_{g_n} \left[\mathrm{PD}_n(x_q)\widetilde{\mathrm{PD}}'_{g_n}(x_q)) - \mathrm{PD}_n \,\mathrm{PD}_{g_n}(1+\lambda_{n,g_n})\right] \\
&+ \sum_{n=11}^{100} s_n \,\mathrm{LGD}_n \left(\mathrm{PD}_n(x_q) - \mathrm{PD}_n\right) \\
&+ \sum_{n=101}^{110} s_{g_n} \,\mathrm{LGD}_{g_n} \Big[\mathrm{PD}_{g_n}(x_q)\cdot(1-\mathrm{PD}_n(x_q)) + \mathrm{PD}'_{g_n}(x_q)\,\mathrm{PD}_n(x_q) \\
&\qquad\qquad - \mathrm{PD}_{g_n}(1+\mathrm{PD}_n \cdot\lambda_{n,g_n})\Big].
\end{aligned} \tag{2.3.14}$$

In the above equation, $\widetilde{\mathrm{PD}}'_{g_n}(x_q)$ *denotes the conditional increased default probability for the guarantor computed via equation (2.3.12) with PD equal to* $\mathrm{PD}_{g_n}(1+\lambda_{n,g_n})$ *and asset correlation parameter* ρ *set to* 0.7. *The latter value is the increased correlation parameter chosen in the IRB treatment for exposures subject to double default. Although the choice of this parameter might be questionable, it is used here for reasons of better comparability of our model with the IRB treatment of double default effects. Figure 2.4 shows the influence of the parameter* λ *through the increased default probability* $\mathrm{PD}'_{g_n} = \mathrm{PD}_{g_n}(1+\lambda)$ *of the guarantor on the IRB capital computed within the asset drop approach. Here we chose a constant level of* λ *for all hedged obligors in the portfolio. With increasing* λ *the IRB capital also increases. This is very intuitive because higher values of* λ *mean that the*

[17] This computation is based on the approximation in equation (2.2.1) as this is the one applied in practice.

Figure 2.4: Influence of increased guarantor PD on EC

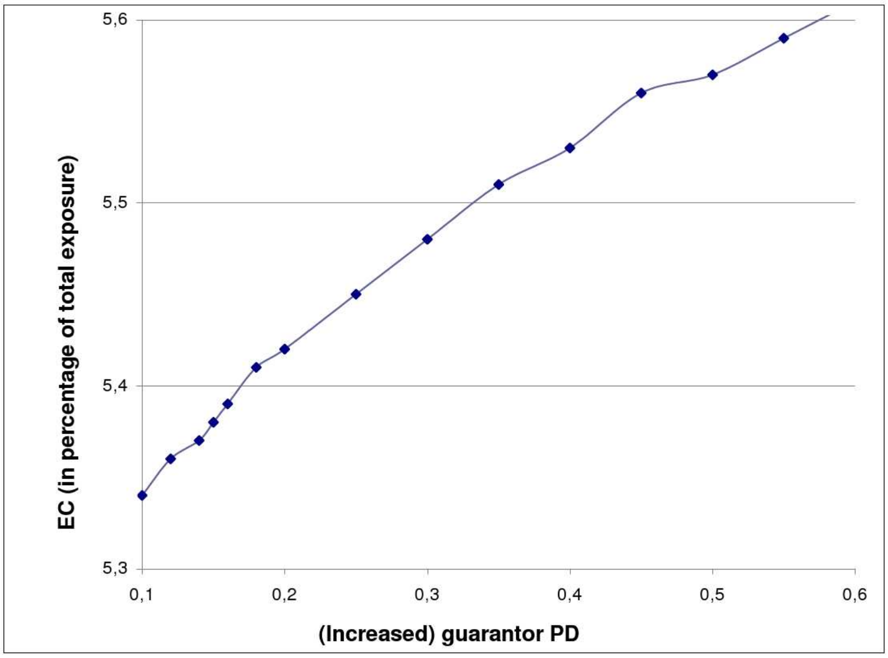


Figure 2.4 shows the influence of the parameter λ through the increased guarantor default probability $\mathrm{PD}'_{g_n} = \mathrm{PD}_{g_n}(1+\lambda_{n,g_n})$ on regulatory capital computed within the asset drop model. λ increases from 0.0 to 5.0 leading to an increase in EC from 5.34% to 5.61% of total portfolio exposure. For $\lambda = 0.7$ ($\mathrm{PD}'_{g_n} = 0.17\%$), the asset drop model leads to the same $EC = 5.40\%$ as the IRB treatment of double defaults.

expected default probabilities of the guarantors increase. This obviously results in higher capital requirements. For $\lambda = 0.7$ ($\mathrm{PD}'_{g_n} = 0.17\%$), our new asset drop method leads to the same economic capital as the one computed within the IRB treatment of double defaults, i.e., $EC = 5.40\%$ of total portfolio exposure.

2.4 Conclusion to Chapter 2

In this chapter, we pointed out several severe problems of the treatment of double default effects applied under Pillar 1 in the Basel framework's IRB approach. Our main criticism is that this treatment relies on the assumption of additional correlation between obligors and guarantors. Thus, it fails to model their asymmetric dependence structure appropriately, i.e., that the guarantor should suffer much more from the obligor's default triggering the guarantee payment than vice versa. The particular choice for the additional correlation parameter is the same for all obligors and guarantors, and it remains entirely unclear how specific guarantor and obligor characteristics could be reflected in this parameter.

Further, all guarantors are treated as distinct for different obligors, and are assumed to be external to the portfolio. Thus, if there is direct exposure to guarantors or if several obligors have the same guarantor, then additional dependencies and concentrations in the credit portfolio are ignored. Hence, excessive contracting of the same guarantor is also not reflected in the computation of economic capital.

To overcome these deficiencies, we proposed a new approach to account for double default effects that can be applied in any model of portfolio credit risk and, in particular, under the IRB approach. It is easily applicable in terms of data requirements and computational time. Specifically, compared to the model of Heitfield and Barger (2003) underlying the IRB treatment of double defaults, we require in addition the total values of the firms' assets, which can be directly inferred from the balance sheets; this should not be too much of a burden for any bank. Moreover, it should be obvious that these quantities should be reflected in any good model of double default.

In spite of its simplicity, our new approach does not show any of the above-mentioned shortcomings, and thereby better reflects the risk associated with double defaults. The model endogenously quantifies the impact of the guarantee payment on the guarantor's unconditional default probability. Within a structural model of portfolio credit risk, the guarantor's loss due to the guarantee payment corresponds to a downward jump in its firm value process. The jump size is determined endogenously by the underlying assumed credit risk model. This new *asset drop technique* could also be used to model other dependencies within a conditional independence framework such as, for example, default contagion effects through business-to-business dependencies.

Part II

On Higher-Order Risk Preferences

Chapter 3

Moment Characterization of Higher-Order Risk Preferences

3.1 Introduction

It is well known that risk aversion only partially describes individuals' risk preferences. Numerous behavioral traits stem from higher-order risk preferences such as prudence or temperance. The most prominent one is that prudence is necessary and sufficient for a precautionary savings motive. That means, the awareness of uncertainty in future payoffs such as income raises an individual's optimal saving today. Although the term "prudence" was coined by Kimball (1990), its relationship to saving behavior was noted earlier by Leland (1968) and Sandmo (1970). Since then, a large body of literature on the behavioral implications of higher-order risk preferences has emerged. An overview with a focus on prudence will be given in the next chapter.

These predictions are derived from models based on expected utility theory (EUT). Under EUT, assuming differentiability of a utility function u (as we do throughout this treatise), risk aversion, prudence, and temperance are equivalent to $u'' < 0$, $u''' > 0$ and $u^{(4)} < 0$, respectively. More generally, Ekern (1980) defines a decision maker as being nth-degree risk averse if and only if $sgn(u^{(n)}) = (-1)^{n+1}$. Prudence, for example, is also widely assumed because it is necessary (but not sufficient) for decreasing absolute risk aversion. In this spirit, nth-degree risk aversion for some order n often serves as a necessary condition for numerous stronger preference specifications such as proper risk aversion (Pratt and Zeckhauser (1987)) and standard risk aversion (Kimball (1993)). It is important to note that "all the commonly used utility functions" exhibit nth-degree risk aversion for all n;

see Brockett and Golden (1987). Thus, it is interesting to study this property that has been labeled *mixed risk aversion* by Caballé and Pomansky (1996).

We study nth-degree risk aversion and mixed risk aversion by using a novel approach based on the proper risk apportionment model of Eeckhoudt and Schlesinger (2006). They give another definition of nth-degree risk aversion as a preference over lotteries and show equivalence to Ekern's definition. The lottery preferences can be interpreted as the desire to disaggregate unavoidable risks and losses, i.e., to apportion them properly across different states of nature. These lotteries allow for studying risk attitudes outside EUT. Furthermore, one can exploit the simplicity of defining risk preferences via proper risk apportionment for both theoretical and empirical purposes.[1] Also, the remarkable equivalence between the lottery preferences and nth-degree risk aversion motivates the study of their statistical properties.

In this chapter, we compute *all moments* of the proper risk apportionment lotteries of *all orders*. Thus, we actually present a characterization of the lotteries and, implicitly, of higher-order risk preferences. This is because the sequence of moments uniquely determines the distribution of a bounded random variable.[2] The characterization provides insights into the statistical structure of the proper risk apportionment model and why a preference over relatively simple lotteries can imply nth-degree risk aversion. Most interestingly, it provides a better understanding of the relationships between higher-order risk preferences, skewness preference, and kurtosis aversion. Since the notions of skewness and kurtosis refer to moments, these results should be accessible to a wide audience. Here it should be noted that preference implications based on a *finite* number of moments are generally flawed; see Brockett and Kahane (1992) and Brockett and Garven (1998). We, however, relate higher-order risk preferences to the strong notions of skewness and kurtosis referring to *all* odd and even moments, respectively. While none of our results is based on EUT, within EUT our results imply that all of the commonly used utility functions exhibit both skewness preference and kurtosis aversion. This is good news for economic

[1]For example, the lotteries are used by Gollier (2010) to investigate ecological discounting, by Maier and Rüger (2009) to investigate reference-dependent risk preferences of higher orders, and by Jindapon (2010) to define probability premia of higher order. The recent application of the lotteries in economic experiments will be discussed in the next chapter. Generally, we will save up some motivating examples for prudence as well as more details for that chapter.

[2]This is known as the solution to the Hausdorff moment problem in the probability literature; see Hausdorff (1921). The assumption of boundedness is unproblematic from an economic point of view as there is not an infinite amount of money. Thus, the assumption is standard in the literature on decision making under risk. A stronger assumption often made is that distributions are defined on a compactum, which implies boundedness.

modeling as it is consistent with the stylized fact that investors are skewness seeking and kurtosis averse.[3]

Our results build upon the recent work of Roger (2011), who made an important contribution in achieving the characterization. He computed all moments of the proper risk apportionment lotteries for the special case where the risks that have to be apportioned are *symmetric*. However, we will show that the *asymmetry* (or skewness) of these risks is just the origin of the proper risk apportionment model's statistical generality. We also generalize the early results of Ekern (1980), who considered differences in moments up to order n for nth-degree risk aversion. Further, Ekern's results, unlike ours, are limited to random variables with a compact support. Ekern (1980), in turn, is a generalization of Menezes et al. (1980) who showed that prudence is equivalent to downside risk aversion. A downside risk increase is a mean-variance preserving density transformation shifting variation from the right to the left of the distribution, thereby decreasing its third moment. This is in analogy to the mean-preserving spread of Rothschild and Stiglitz (1970) disliked by a risk averse individual. For more on prudence and skewness see also Chiu (2005, 2010). Menezes and Wang (2005) illustrated that an individual dislikes increases in outer risk which leave the first three moments of a distribution unchanged and increase the fourth moment, if and only if she is temperate. Edginess (5th-degree risk aversion) has been considered by Lajeri-Chaherli (2004) and will also be related to skewness preference in this chapter.

More specifically, the results of this chapter are presented as follows. In Section 3.2, we review the proper risk apportionment model of Eeckhoudt and Schlesinger, and discuss skewness and kurtosis and how they relate to *all* odd and even moments, respectively. In particular, we illustrate how skewness and kurtosis manifest in discrete (lottery) distributions.

In Section 3.3, we explicitly compute all moments of the prudence and temperance lotteries. We show that distributions preferred by a prudent decision maker must have larger skewness *as defined by larger odd moments of any order*, but they may or may not have larger kurtosis *as defined by larger even moments of any order*. We refer to this as the

[3]There is a substantial amount of evidence for skewness preference; see, e.g., Boyer et al. (2010) and the many references therein. There are less fully fledged research papers on kurtosis aversion. Some evidence for kurtosis aversion is presented in Dittmar (2002), and results in Guiso et al. (1996) are consistent with temperate behavior; see Deck and Schlesinger (2010) for a brief discussion. In the experiment of Deck and Schlesinger, intemperance was observed, while in Ebert and Wiesen (2010) we observed temperance.

kurtosis robustness feature of prudence. We show that whether the prudent lottery choice has the smaller or larger kurtosis solely depends on the skewness of the risk that has to be apportioned. This helps to explain an experimental result presented in the next chapter, i.e., that significantly more prudent decisions are made when a left-skewed risk has to be apportioned. In this situation, someone who is skewness seeking might choose imprudently because he dislikes the larger kurtosis associated with the prudent lottery choice. Likewise, though not as clear-cut, we show that temperance implies a preference for distributions with smaller kurtosis as defined by smaller even moments and which is robust towards variation in the odd moments. This is referred to as the *skewness robustness feature* of temperance.

In Section 3.4, we generalize these results and investigate all moments of the proper risk apportionment lotteries of all orders. We show that all higher-order risk preferences of odd and even order (not only prudence and temperance), respectively, are related to skewness preference and kurtosis aversion in a complementary way. That is, there are distributions that only differ in their skewness and preference between them is determined by prudence. However, there are also distributions that only differ in their skewness, but preference between them is not determined by prudence, but, for example, by edginess. This should raise more interest in these concepts which are generally regarded as rather abstract. In both Sections 3.3 and 3.4, we will discuss how our results relate to those of Roger (2011) and which of his results are specific to the symmetry of the zero-mean risks.

In Section 3.5, we conclude and discuss implications of our results for EUT. All proofs are given in Appendix A.2.

3.2 Proper risk apportionment, skewness, kurtosis, and moments

We first define the lotteries of Eeckhoudt and Schlesinger (2006) and explain the importance of proper risk apportionment. Let X be Bernoulli-distributed with parameter p. Throughout this chapter, let $p = 0.5$. Let $k > 0$ such that the amount $-k$ can be interpreted as a sure reduction in wealth. For all $n \in \mathbb{N}$, let ϵ_n be a zero-mean risk (i.e., $\mathbb{E}[\epsilon_n] = 0$) with finite moments. The lotteries for monotonicity and risk aversion, respectively, are given by $A_1 = -k,\ B_1 = 0$ and $A_2 = \epsilon_1,\ B_2 = 0$. For the first two so-called higher-order risk preferences, prudence and temperance, the lotteries are

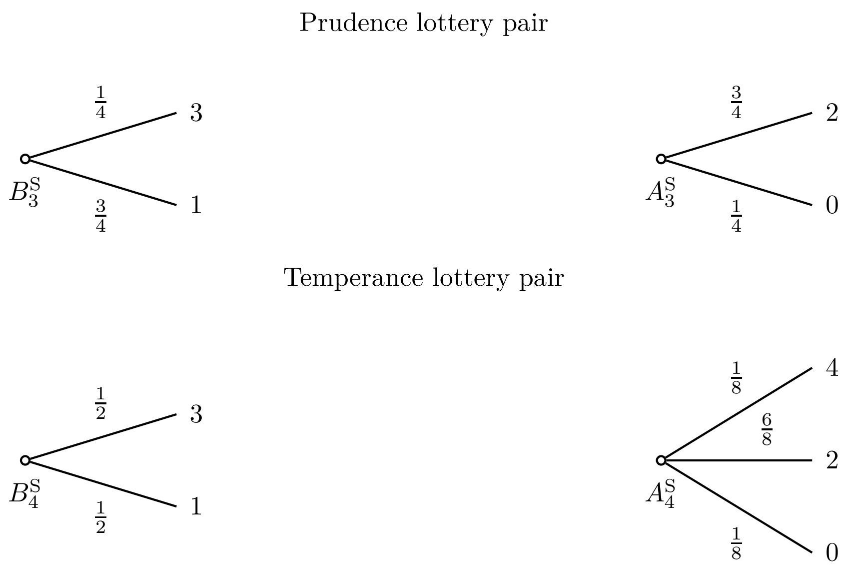
$$\begin{aligned} A_3 &= X \cdot 0 + (1-X)\,(\epsilon_1 - k) = XB_1 + (1-X)(A_1 + \epsilon_1) \\ B_3 &= X(-k) + (1-X)\epsilon_1 = XA_1 + (1-X)(B_1 + \epsilon_1) \end{aligned}$$

and

$$\begin{aligned} A_4 &= X \cdot 0 + (1-X)(\epsilon_1 + \epsilon_2) = XB_2 + (1-X)(A_2 + \epsilon_2) \\ B_4 &= X\epsilon_1 + (1-X)\epsilon_2 = XA_2 + (1-X)(B_2 + \epsilon_2). \end{aligned}$$

Examples of these lotteries are shown in Figure 3.1 where outcomes have been aggregated. Therefore, the lotteries appear as multinomial rather than compound. For higher orders,

Figure 3.1: Examples of a prudence and a temperance lottery pair with symmetric (S) zero-mean risks



The prudence lotteries depicted in this figure are constructed with initial wealth $x = 2$, fixed loss $-k = -1$, and the zero-mean risk ϵ yields 1 or -1 with equal probability. For the temperance lotteries, initial wealth is $x = 2$ and the zero-mean risks ϵ_1 and ϵ_2 both yield 1 or -1 with equal probability. Outcomes have been aggregated.

proper risk apportionment of order n is defined iteratively by continuing the previously illustrated nesting process, i.e.,

$$\begin{aligned} A_n &= XB_{n-2} + (1-X)\left(A_{n-2} + \epsilon_{\lfloor n/2 \rfloor}\right) \\ B_n &= XA_{n-2} + (1-X)\left(B_{n-2} + \epsilon_{\lfloor n/2 \rfloor}\right) \end{aligned}$$

where $\llcorner n/2 \lrcorner$ is the largest integer smaller than or equal to $n/2$. An agent exhibits *proper risk apportionment of order n* if she prefers B_n over A_n for all wealth levels x, for all sure losses $-k$ and, in particular, for all zero-mean risks ϵ. A prudent decision maker, for example, will prefer to disaggregate the sure loss $-k$ and the zero-mean risk ϵ. That is, she prefers to have the two unavoidable items in different rather than in the same of two equally likely states of nature. In other words, she disaggregates the two "harms" of a sure loss and a zero-mean risk.[4] A financial economist might speak of a preference for diversification. An equivalent interpretation is that the additional risk is preferred when wealth is higher. These numerous interpretations already illustrate the implicit generality of the preference. Moreover, preference between the proper risk apportionment lotteries has strong implications within EUT.

Theorem 2 (Eeckhoudt and Schlesinger, 2006)**.** *Within EUT with differentiable utility function u, proper risk apportionment of order n is equivalent to the condition* $sgn(u^{(n)}) = (-1)^{n+1}$.

Thus, the lottery preference of B_1 over A_1 is equivalent to an increasing utility function within the differentiable EUT. This is very intuitive because preference of B_1 over A_1 for all $-k$ simply corresponds to preferring more to less (no matter how much more). Likewise, the lottery preference of B_2 over A_2 is equivalent to a concave utility function within the differentiable EUT, i.e., to risk aversion. Also this is intuitive as preference for the expected value of a prospect over the prospect itself is a well-established and theory-free definition of risk aversion; see, e.g., Wakker (2010, p. 52). While none of the results in this chapter are based on EUT, the above theorem tells us how to interpret them under the assumption of EUT.

Next we review the qualitative definitions of skewness and kurtosis, respectively. For the purpose of this chapter, it will be particularly insightful to discuss how skewness and kurtosis are reflected in discrete (lottery) distributions. This will be done with reference to Figure 3.1.

Generally, a distribution is *right-skewed* if it has a longer right tail and that tail has less probability mass than the left tail. This is true for lottery B_3^{S} in Figure 3.1 because the low outcome 1 has a small distance to the mean of 1.5, whereas the high outcome 3 has a large distance to the mean. In general, any binary lottery is right-skewed if and only if the

[4]This interpretation from Eeckhoudt and Schlesinger (2006) requires the decision maker to be risk averse such that a zero-mean risk indeed constitutes a harm.

higher outcome occurs with the smaller probability.[5] Thus, lottery A_3^{S} in Figure 3.1 (which also has mean 1.5) is left-skewed. The particular lottery pair $(A_3^{\mathrm{S}}, B_3^{\mathrm{S}})$ has been introduced in Mao (1970) and motivated the definition of downside risk aversion in Menezes et al. (1980). A downside risk averse decision maker will prefer B_3^{S} over A_3^{S}. She rather opts for the smaller outcome 1 most of the time such that she is safe with respect to the worst outcome 0 that can occur when taking A_3^{S} instead. Choice B_3^{S} also implies a small chance of winning the high prize (outcome 3).

Now we consider the lotteries B_4^{S} and A_4^{S} in Figure 3.1 to discuss kurtosis. Generally, large kurtosis of a distribution implies peakedness and fat tails. Peakedness means that there is a high probability (a "peak" in the frequency distribution) of outcomes close to the mean. Fat tails mean that there is a chance of extreme outcomes (compared to the mean) to occur, i.e., such outcomes have a heavy probability mass.[6] This is true for lottery A_4 which has a probability peak of 6/8 at its mean, which is 2. Lottery B_4^{S}, in contrast, has no probability mass at its mean (which is also 2), and its outcomes are also less extreme compared to those of lottery B_4^{S}. Thus, lottery A_4^{S} has a larger kurtosis than lottery B_4^{S}.

Now we discuss statistical moments and how they relate to skewness and kurtosis. We denote the pth (non-standardized) central moment of a random variable Z by

$$\mathbb{M}_p(Z) = \mathbb{E}[(Z - \mathbb{E}[Z])^p].$$

When speaking of moments, we always refer to (non-standardized) central moments. It is important to note that in this chapter skewness and kurtosis do not refer to the third and fourth moment, respectively. If not noted otherwise, they refer to the qualitative features discussed above. One reason is that the third and fourth moment, respectively, might fail to indicate that a distribution is more skewed or leptokurtic than another one.[7] On the other hand, all higher odd and even moments share reasonable properties of a skewness and kurtosis measure, respectively; see van Zwet (1964). In general, the link between any *finite* number of moments and preference is flawed. For example, for any utility function u with $u' > 0$ and $u'' < 0$, there exist random variables X and Y such that X has the higher mean and the lower variance, but u prefers Y to X; see Brockett and Kahane (1992) and

[5]A formal proof of this result and more explanations to skewness in binary lotteries are presented in Appendix B. In particular, the skewness of lottery A_3^{S} is discussed with reference to its probability mass function, which is plotted in Figure B.1.

[6]Using the normal distribution as a benchmark for absolute values, however, can be misleading; see Kaplansky (1945).

[7]We give such an example for the third moment in Figure 3.3.

Brockett and Garven (1998) for explicit examples. Therefore, a more reliable requirement for a distribution to be more skewed is that *all* odd moments are at least as large as the corresponding moments of the distribution in comparison. Likewise, for a distribution to be more leptokurtic, *all* its even moments are required to be larger. The results in our discussion of higher-order risk preferences, skewness preference and kurtosis aversion can be based on these strong notions of skewness and kurtosis.[8]

3.3 Moment characterizations of prudence and temperance

In this section, we present the statistical characterizations of prudence and temperance in terms of moments. The following Propositions 3 to 6 generalize Propositions 1 to 4 in Roger (2011) to arbitrary zero-mean risks. Propositions 3 to 6 are also generalizations of results in Ekern (1980) in that they consider all moments rather than only moments $1, 2, \ldots, n$ where n is the considered degree of risk aversion. Further, Ekern's results are limited to random variables with a compact support, whereas our results hold for random variables with arbitrary support.
We start with Proposition 3 which presents a statistical characterization of prudence in terms of moments.

Proposition 3 (All moments of the prudence lotteries)**.** *For $p \in \mathbb{N}$, we have*

$$\text{(1)} \qquad \mathbb{M}_p(A_3) = \begin{cases} \left(\frac{k}{2}\right)^p + \frac{1}{2}\sum_{j=2}^{p} \binom{p}{j} \mathbb{E}\left[\epsilon_1^j\right] \left(-\frac{k}{2}\right)^{p-j} & , \ p \text{ even} \\ \frac{1}{2}\sum_{j=2}^{p} \binom{p}{j} \mathbb{E}\left[\epsilon_1^j\right] \left(-\frac{k}{2}\right)^{p-j} & , \ p \text{ odd} \end{cases}$$

$$\text{(2)} \qquad \mathbb{M}_p(B_3) = \begin{cases} \left(\frac{k}{2}\right)^p + \frac{1}{2}\sum_{j=2}^{p} \binom{p}{j} \mathbb{E}\left[\epsilon_1^j\right] \left(\frac{k}{2}\right)^{p-j} & , \ p \text{ even} \\ \frac{1}{2}\sum_{j=2}^{p} \binom{p}{j} \mathbb{E}\left[\epsilon_1^j\right] \left(\frac{k}{2}\right)^{p-j} & , \ p \text{ odd} \end{cases}$$

$$\text{(3)} \qquad \mathbb{M}_p(B_3) - \mathbb{M}_p(A_3) = \begin{cases} \frac{1}{2}\sum_{\substack{j=2 \\ j \text{ odd}}}^{p} \binom{p}{j} \mathbb{E}\left[\epsilon_1^j\right] \left(\frac{k}{2}\right)^{p-j} & , \ p \text{ even} \\ \frac{1}{2}\sum_{\substack{j=2 \\ j \text{ even}}}^{p} \binom{p}{j} \mathbb{E}\left[\epsilon_1^j\right] \left(\frac{k}{2}\right)^{p-j} & , \ p \text{ odd.} \end{cases}$$

Further, the difference $\mathbb{M}_p(B_3) - \mathbb{M}_p(A_3)$ is strictly positive for all p odd. For all p even, it can be positive, negative, or zero.

From Menezes et al. (1980), we already knew that the prudence lotteries have equal mean

[8]For more on moments and other measures of skewness see, e.g., MacGillivray (1986).

and variance and that B_3 has a larger third moment. These results are recovered from part (3) in Proposition 3 by considering $p = 1, 2, 3$. Firstly, let us discuss the implication from part (3) stating that *all* odd moments for the prudent lottery choice B_3 are strictly larger than those of the corresponding imprudent lottery A_3. This shows that the prudent lottery choice B_3 is indeed more skewed to the right (not only in an approximate third-order sense), for all possible zero-mean risks. Secondly, part (3) implies that the even moments may not be identical as proven for symmetric zero-mean risks ϵ_1 in Roger (2011). Roger's result is obtained as a special case from part (3), as symmetry of a random variable implies all its odd moments to be zero. Proposition 3 shows that in that case, and only in that case, lotteries A_3 and B_3 have equal kurtosis. This can also be seen qualitatively from our sample lottery pair in Figure 3.1. Both lotteries A_3^{S} and B_3^{S} have a 3/4-probability peak at an outcome close to the mean (distance of 0.5) which are 2 and 1, respectively. The "extreme" outcomes of lotteries A_3^{S} and B_3^{S} are 0 and 3, respectively. Both have a distance of 1.5 from the mean and occur with equal probability. Thus, indeed, lotteries A_3^{S} and B_3^{S} are equally peaked and heavy-tailed.

In the general case, the even moments of the prudent choice can be larger or smaller than those of the imprudent choice. They are larger (smaller) if and only if the zero-mean risks to be apportioned are right-skewed (left-skewed).[9] An example is given in Figure 3.2. Both B_3^{L} and B_3^{R} are, respectively, more skewed to the right than A_3^{L} and A_3^{R}. However, whereas B_3^{R} has a larger kurtosis than A_3^{R}, B_3^{L} has a smaller kurtosis than A_3^{L}. Qualitatively, lottery B_3^{R} has a 7/8 probability peak at 1 which is close to the mean of 1.5. It also has a very extreme outcome 5. Lottery A_3^{R}, in contrast, has only a 4/8 probability peak at the outcome 2 which is close to the mean. Both remaining outcomes, 0 and 4, are less extreme than 5 as their distance to the mean of 1.5 is smaller. This is in accordance with the result on moments proven in Proposition 3. Analogous arguments apply to lottery pair $(A_3^{\mathrm{L}}, B_3^{\mathrm{L}})$ where the zero-mean risk is left-skewed and thus A_3^{L} has the larger kurtosis.

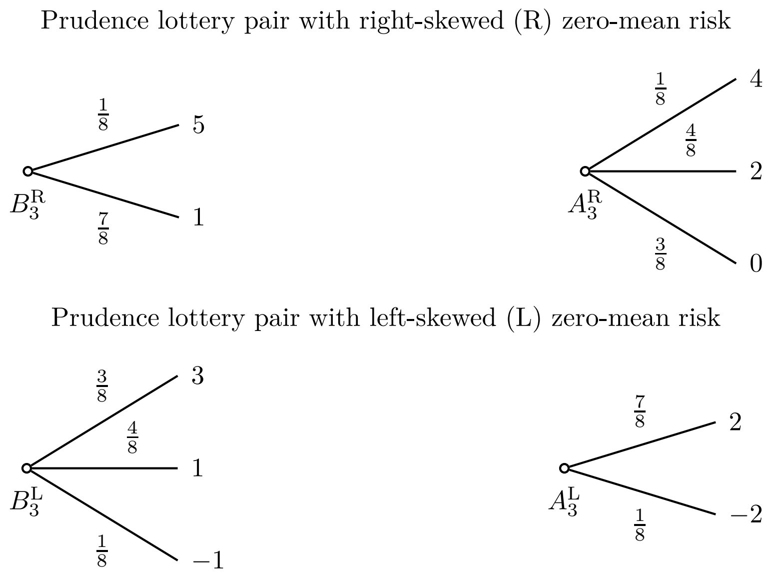
Therefore, prudence must be understood as a preference for large skewness (i.e., large odd moments of all orders) that is robust towards variation in kurtosis (i.e., differences in even moments of all orders). We refer to this as the *kurtosis robustness feature of prudence.* That is, prudence not only determines preference between distributions that purely differ

[9] Throughout Part II of this thesis, when referring to "skewness," the reader should carefully pay attention to wether we refer to the proper risk apportionment lotteries A_n and B_n themselves or to the zero-mean risks (the ϵ's) that have to be apportioned and that are part of the proper risk apportionment lotteries. With kurtosis we will always refer to the proper risk apportionment lotteries, because the kurtosis of the zero-mean risks turns out to be not particularly interesting.

in their skewness. Prudence implies preference for distributions with larger skewness, independently of whether they have the larger or smaller kurtosis.

Thus, the restriction to symmetric zero-mean risks in the proper risk apportionment model of Eeckhoudt and Schlesinger (2006) is rather severe from a statistical point of view. It reduces prudence to "pure" *skewness seeking* (distributions with larger odd moments are preferred) and neglects the kurtosis robustness feature. Empirical support for the kurtosis robustness feature will be presented in the next chapter where we will conclude that also empirically there is more to prudence than skewness seeking. A prudent decision is made more frequently when the zero-mean risk is left-skewed, i.e., the even moments are larger for the imprudent choice. Next we present a characterization of temperance in terms of moments.

Figure 3.2: Examples of prudence lottery pairs with skewed zero-mean risks



In this figure, the prudence lotteries with the right-skewed zero-mean risk are constructed with initial wealth $x = 2$, loss $-k = -1$, and the zero-mean risk ϵ_1 yields 3 with probability $1/4$ and -1 with probability $3/4$. For the prudence lotteries with the left-skewed zero-mean risk, initial wealth is $x = 2$, the loss is $-k = -1$, and the zero-mean risk ϵ_1 yields -3 with probability $1/4$ and 1 with probability $3/4$. Outcomes have been aggregated.

Proposition 4 (All moments of the temperance lotteries)**.** *For $p \in \mathbb{N}$, we have*

$$\text{(1)} \qquad \mathbb{M}_p(A_4) = \frac{1}{2}\sum_{j=0}^{p}\binom{p}{j}\mathbb{E}\left[\epsilon_1^j\right]\mathbb{E}\left[\epsilon_1^{p-j}\right]$$

$$\text{(2)} \qquad \mathbb{M}_p(B_4) = \frac{1}{2}\left(\mathbb{E}\left[\epsilon_2^p\right] + \mathbb{E}\left[\epsilon_1^p\right]\right)$$

$$\text{(3)} \qquad \mathbb{M}_p(B_4) - \mathbb{M}_p(A_4) = -\frac{1}{2}\left(\sum_{j=2}^{p-1}\binom{p}{j}\mathbb{E}\left[\epsilon_1^j\right]\mathbb{E}\left[\epsilon_2^{p-j}\right]\right).$$

Further, for $p > 4$ odd the difference $\mathbb{M}_p(B_3) - \mathbb{M}_p(A_3)$ can be positive, negative, or zero.

Roger (2011) further shows that in the case of symmetric zero-mean risks $\mathbb{M}_p(A_n) = \mathbb{M}_p(B_n) = 0 \;\forall\; p$ odd. For illustrative purposes, consider the case of $p = 5$ and $n = 4$. Using part (3) of Lemma 1, we have

$$\begin{aligned}
\mathbb{M}_5(B_4) - \mathbb{M}_5(A_4) &= -\frac{1}{2}\left(\sum_{j=2}^{5-1}\binom{5}{j}\mathbb{E}\left[\epsilon_1^j\right]\mathbb{E}\left[\epsilon_2^{5-j}\right]\right)\\
&= -\frac{1}{2}\left(\binom{5}{2}\mathbb{E}\left[\epsilon_1^2\right]\mathbb{E}\left[\epsilon_2^3\right] + \binom{5}{3}\mathbb{E}\left[\epsilon_1^3\right]\mathbb{E}\left[\epsilon_2^2\right] + 0\right),
\end{aligned}$$

which can be positive, negative, or zero, depending on the third moments of the zero-mean risks. The proof in the appendix essentially generalizes this example to all odd moments. We interpret the last statement of Proposition 4 as the *skewness robustness feature* of temperance. Roger also shows that $\mathbb{M}_p(B_4) - \mathbb{M}_p(A_4) < 0$ holds for all $p > n$ even. This we cannot prove in the general case. To see the reason why, in part (3) of Lemma 1 set $p = 6$ and $n = 4$, i.e.,

$$\begin{aligned}
\mathbb{M}_6(B_4) - \mathbb{M}_6(A_4) &= \frac{1}{2}\left(\sum_{j=2}^{5}\binom{6}{j}\mathbb{E}\left[\epsilon_1^j\right]\mathbb{E}\left[\epsilon_2^{6-j}\right]\right)\\
&= -\frac{1}{2}\left(0 + \binom{6}{2}\mathbb{E}\left[\epsilon_1^2\right]\mathbb{E}\left[\epsilon_2^4\right] + \binom{6}{3}\mathbb{E}\left[\epsilon_1^3\right]\mathbb{E}\left[\epsilon_2^3\right] + \binom{6}{4}\mathbb{E}\left[\epsilon_1^4\right]\mathbb{E}\left[\epsilon_2^2\right] + 0\right).
\end{aligned}$$

This expression might become positive if the middle term is negative. This could happen if and only if the two zero-mean risks are adversely skewed. However, we could conjecture that for all random variables ϵ_1 and ϵ_2 this is not possible. Using part (3) of Proposition 4, the conjecture can be validated or dismissed for any risks specifically considered. Evidently, it is true if both zero-mean risks are symmetric or skewed in the same direction. For prudence we obtained the clear statement that proper risk apportionment implies preference for large odd moments of all orders that is robust towards variation in the even moments. Analogously, we find some evidence that temperance is a preference for small even moments (kurtosis aversion) that is robust towards variation in the odd moments (skewness robustness).

3.4 Higher-order generalizations

In this section, we generalize the results from the previous section to risk apportionment of orders higher than 4. Lemma 1 presents recursive formulae that can be used to compute any moment of a proper risk apportionment lottery of any order and thus completes our moment characterization of higher-order risk preferences.

Lemma 1. *For $n \geq 3$ (even or odd), we have the following recursive formulae*

$$(1) \qquad \mathbb{M}_p(A_n) = \frac{1}{2}\left(\mathbb{M}_p(B_{n-2}) + \mathbb{M}_p(A_{n-2}) + \sum_{j=2}^{p}\binom{p}{j}\mathbb{E}\left[\epsilon^j_{\llcorner n/2 \lrcorner}\right]\mathbb{M}_{p-j}(A_{n-2})\right)$$

$$(2) \qquad \mathbb{M}_p(B_n) = \frac{1}{2}\left(\mathbb{M}_p(A_{n-2}) + \mathbb{M}_p(B_{n-2}) + \sum_{j=2}^{p}\binom{p}{j}\mathbb{E}\left[\epsilon^j_{\llcorner n/2 \lrcorner}\right]\mathbb{M}_{p-j}(B_{n-2})\right)$$

$$(3) \qquad \mathbb{M}_p\left(B_n\right) - \mathbb{M}_p\left(A_n\right) = \frac{1}{2}\left(\sum_{j=2}^{p}\binom{p}{j}\mathbb{E}\left[\epsilon^j_{\llcorner n/2 \lrcorner}\right]\left(\mathbb{M}_{p-j}(B_{n-2}) - \mathbb{M}_{p-j}(A_{n-2})\right)\right).$$

We now investigate how our Proposition 4 (which applies to temperance lotteries with arbitrary zero-mean risks) and Roger's Proposition 3 (which applies to all proper risk apportionment lotteries of even order, but with symmetric zero-mean risks) generalize to higher even orders.

Proposition 5. *Let $n \geq 4$.*

(1) $\mathbb{M}_p\left(A_n\right) - \mathbb{M}_p\left(B_n\right) = 0,$ *for* $1 \leq p < n$

(2) $\mathbb{M}_p\left(A_n\right) > \mathbb{M}_p\left(B_n\right),$ *for* $p = n.$

Further, for $p > n$ odd the difference $\mathbb{M}_p\left(B_3\right) - \mathbb{M}_p\left(A_3\right)$ can be positive, negative, or zero.

Parts (1) and (2) generalize results in Ekern (1980), whose proofs relied on an iterated integral technique, to random variables with arbitrary support. According to the last statement, all higher-order risk preferences of even order have a skewness robustness feature, i.e., the preferred lottery may or may not have larger odd moments of any order. As Roger shows for symmetric zero-mean risks, we also conjecture (as we did in the case of temperance) that in the general case $\mathbb{M}_p\left(A_n\right) > \mathbb{M}_p\left(B_n\right)$ for $p \geq n$ even is true. For any lotteries specifically considered this can be checked by using the equation in part (3) of Lemma 1. The next proposition generalizes Roger's Proposition 4.

Proposition 6. $n \geq 3$ *odd.*

(1) $\mathbb{M}_p\left(A_n\right) = \mathbb{M}_p\left(B_n\right)$ *for* $p < n$

(2) $\mathbb{M}_p\left(B_n\right) - \mathbb{M}_p\left(A_n\right) > 0$ *for* $p = n$.

Further, for $p > n$ *even the difference* $\mathbb{M}_p\left(B_3\right) - \mathbb{M}_p\left(A_3\right)$ *can be positive, negative, or zero.*

The last statement shows that all higher-order risk preferences of odd order have a kurtosis robustness feature. Parts (1) and (2) generalize results of Ekern (1980) to random variables with arbitrary support. Under the symmetry assumption, for $n \geq 3$ odd Roger (2011) further obtained

$(1a)$	$\mathbb{M}_p(A_n) = \mathbb{M}_p(B_n) = 0$	$\forall\ p < n$ odd,
$(2')$	$\mathbb{M}_p(A_n) = -\mathbb{M}_p(B_n) < 0$	$\forall\ p\ \geq n$ odd,
(3)	$\mathbb{M}_p(A_n) = \mathbb{M}_p(B_n)$	$\forall\ p > n$ even.

While (1a) trivially holds for prudence, in general only the first equality is true. The following is a counterexample for the second inequality. For $n = 5$ and $p = 3$, the recursive formula derived in part (3) of Lemma 1 gives

$$\mathbb{M}_3(A_5) = \frac{1}{2}\left(\mathbb{M}_3(B_3) + \mathbb{M}_3(A_3) + \sum_{j=2}^{3}\binom{3}{j}\mathbb{E}[\epsilon_2^j]\mathbb{M}_{3-j}(A_3)\right).$$

From Proposition 3, $\mathbb{M}_3(B_3) = \frac{1}{2}\left(\binom{3}{2}\mathbb{E}[\epsilon_1^2]\frac{k}{2} + \mathbb{E}[\epsilon_1^3]\right)$, $\mathbb{M}_3(A_3) = \frac{1}{2}\left(\binom{3}{2}\mathbb{E}[\epsilon_1^2]\left(-\frac{k}{2}\right) + \mathbb{E}[\epsilon_1^3]\right)$, and $\mathbb{M}_1(A_3) = 0$ such that

$$\mathbb{M}_3(A_5) = 2\mathbb{E}[\epsilon_1^3]$$

which can be negative, positive, or zero, depending on the asymmetry of the zero-mean risks.

A counterexample for (3) is given by the fourth moment of the prudence lotteries, i.e., $n = 3$ and $p = 4$, as discussed subsequent to Proposition 3.

The equality in (2') is not true, and a counterexample is given by the third moment of the prudence lotteries; see parts (1) and (2) of Proposition 3. Also, in the general case of arbitrary zero-mean risks, we cannot prove the inequality $\mathbb{M}_p(B_n) - \mathbb{M}_p(A_n) > 0$ for p odd, which is redundant from (2'). To see the reason why, take $n = 5$ and $p = 7$ in part (3) of Lemma 1 and impute the expressions for the moments of the prudence lotteries stated in Proposition 3. We get

$$\begin{aligned}\mathbb{M}_7(B_5) - \mathbb{M}_7(A_5) \quad = \quad & \frac{1}{2}\Bigg\{ \sum_{j=2,\ j \text{ even}}^{7} \binom{7}{j} \mathbb{E}\left[\epsilon_5^j\right] \left(\sum_{l=2,\ l \text{ even}}^{7-j} \binom{7-j}{l} \mathbb{E}\left[\epsilon_2^l\right] \left(\frac{k}{2}\right)^{7-j-l} \right) \\ & + \sum_{j=2,\ j \text{ odd}}^{7} \binom{7}{j} \mathbb{E}\left[\epsilon_5^j\right] \left(\sum_{l=2,\ l \text{ odd}}^{7-j} \binom{7-j}{l} \mathbb{E}\left[\epsilon_3^l\right] \left(\frac{k}{2}\right)^{7-j-l} \right) \Bigg\}.\end{aligned}$$

The second sum of the above expression can be computed as

$$\begin{aligned}&\binom{7}{3} \mathbb{E}\left[\epsilon_5^3\right] \left(\binom{4}{3} \mathbb{E}\left[\epsilon_3^3\right] \left(\frac{k}{2}\right)^1 \right) + \binom{7}{5} \mathbb{E}\left[\epsilon_5^5\right] \cdot \left(\binom{2}{1} \cdot 0 \right) \\ &= 4\mathbb{E}\left[\epsilon_3^3\right] \left(\frac{k}{2}\right),\end{aligned}$$

which might be negative such that the whole expression might be negative. However, we again conjecture that this is not possible.

In the remainder of this section, we use our results to present some motivation for higher-order risk preferences of order higher than 4. To this means, consider the edginess lottery pair depicted in Figure 3.3. Clearly, B_5^{S} is skewed to the right as it has a long and lean

Figure 3.3: Example of an edginess lottery pair with symmetric (S) zero-mean risks

The lotteries in Figure 3.3 are constructed with initial wealth $x = 2$, fixed loss $-k = -1$, and the zero-mean risks ϵ_1 and ϵ_2 both yield 1 or -1 with equal probability. Thus, the nested prudence lotteries used in the construction are A_3^{S} and B_3^{S} displayed in Figure 3.1. All outcomes have been aggregated.

right tail due to outcome 4 being far right of the mean of 1.5 and occurring with small probability 1/16. The left tail is shorter and heavier as outcome 0 is closer to the mean and occurs with probability 5/16. Analogous arguments imply that A_5^{S} is left-skewed. As all zero-mean risks used in the construction of $(A_5^{\mathsf{S}}, B_5^{\mathsf{S}})$ are symmetric, all even moments of the two lotteries are equal, i.e., they have the same kurtosis. From Roger (2011), B_5^{S} has larger odd moments of order 5 and higher which, indeed, indicates that it is more skewed to the right. However, the third moments of the lotteries are the same.

The previous example shows two important points. Firstly, it illustrates why the third moment of a distribution can fail as a measure of skewness. Secondly, prudence does not exhaustively describe skewness preference. The right-skewed lottery B_5^{S} is preferred to the left-skewed lottery A_5^{S} if and only if the decision maker exhibits edginess. This illustrates that higher-order risk preferences of any order are important in modeling skewness preference. Analogous arguments show that all higher-order risk preferences of even order imply kurtosis aversion in a complementary way.

3.5 Conclusion to Chapter 3

This chapter builds upon and extends recent results from Roger (2011) in order to present a characterization of higher-order risk preferences in terms of statistical moments. This characterization provides a better understanding of how higher-order risk preferences are related to skewness preference and kurtosis aversion. Further, moments are well understood such that our results should be easily accessible to a wide audience in economics and finance.

Prudence is shown to imply a preference for larger odd moments (skewness seeking) that is robust towards variation in the even moments (kurtosis robustness). In particular, prudence does not only determine preference between distributions that purely differ in their skewness. Generally, it is the asymmetry of the zero-mean risks in Eeckhoudt and Schlesinger's proper risk apportionment model that drives the lotteries' statistical properties. Restriction to symmetric zero-mean risks reduces prudence and all higher-order risk preferences of odd order to "pure" skewness seeking. Thus, our theoretical results are in line with experimental evidence presented in the next chapter, where we show empirically that there is more to prudence than skewness seeking. Analogous results in the present chapter relate temperance to preference for small even moments (kurtosis aversion) that is robust towards variation in the odd moments (skewness robustness).

Moreover, we show that not only prudence and temperance, but all higher-order risk preferences of odd and even order, respectively, are related to skewness preference and kurtosis aversion in a complementary way. This highlights the importance of these concepts which are generally viewed as rather abstract and thus have not received that much attention in the literature yet.

Although not based on EUT, our results have implications for EUT. All of the commonly used utility functions exhibit nth-degree risk aversion for all orders, i.e., mixed risk aversion. Thus, according to the results of this chapter, all commonly used utility functions exhibit both skewness preference and kurtosis aversion with reference to all odd and even moments, respectively.

Another way to look at this is to ask the following question: What are necessary conditions for a preference functional to imply skewness preference and kurtosis aversion? Then, we have answered this question for the EUT preference functional and the notions of skewness and kurtosis being moments. The question is important for realistic modeling of economic behavior because it is a stylized fact that investors are skewness seeking and kurtosis averse. Further research could investigate this question for different preference functionals as well as for different measures of skewness and kurtosis.

Chapter 4

Testing for Prudence and Skewness Seeking

4.1 Introduction

Risk aversion is just one piece in the puzzle when describing individuals' risk preferences. An example is the following lottery pair defined by Mao (1970). Lottery M_A pays zero with a probability of $p = \frac{1}{4}$ and 2000 with the counterprobability of $\frac{3}{4}$. Lottery M_B pays 1000 with a probability of $\frac{3}{4}$ and 3000 with a probability of $\frac{1}{4}$. Statistically, these lotteries have the same mean and variance, but M_B is more skewed to the right. While M_A may seem "riskier," the preference of M_B over M_A is not implied by risk aversion but by prudence. This follows from Menezes et al. (1980) who show that M_A can be obtained from M_B by an increase in downside risk. Such a density transformation leaves mean and variance of a distribution unchanged, but decreases its skewness. They also show that, under expected utility theory (EUT), aversion to increases in downside risk is equivalent to the third derivative of the utility function being positive, i.e., $u''' > 0$.[1] This is the EUT-based definition of prudence given later by Kimball (1990), which will be discussed below. The results of Menezes et al. further imply that prudence, unlike risk aversion, relates to measures of skewness, in particular to the third central moment and to semi-variance; see also Chiu (2005). In the previous chapter, we related prudence to all odd moments, each of which shares reasonable properties of a skewness measure; see van Zwet (1964). Note that all of these results are independent of EUT. Thus, prudence plays a key role when considering preference towards downside and right-skewed risks.

[1]Again, we assume sufficiently smooth von Neumann-Morgenstern utility functions when referring to EUT.

Such risks occur frequently in everyday life. For example, insurance contracts often address downside risks similar to M_A, where probability p might be much smaller than the $\frac{1}{4}$ assumed in our example. Similarly, on the gain side, M_B corresponds to the risk of a typical lottery ticket. The payoff structures of numerous assets exhibit downside risk. For example, the payoff distribution of a (defaultable) bond resembles M_A. Also, numerous risk measures such as value-at-risk, which are employed frequently in the financial industry, address downside risk. In his seminal study, Mao reports an unambiguous preference among surveyed business executives for investments of type M_B over M_A.

In a different, EUT-based strand of the literature, it was discovered that prudence plays a decisive role in analyzing precautionary demand for saving. Although the term "prudence" was coined by Kimball (1990), its relationship to saving behavior was noted earlier by Leland (1968) and Sandmo (1970). These authors showed that the awareness of uncertainty in future payoffs will raise an individual's optimal saving today, if and only if the individual is prudent. The term "prudence" is meant to suggest the propensity to prepare and forearm oneself in the face of uncertainty, in contrast to "risk aversion," which is how much one dislikes uncertainty and would turn away from uncertainty if possible; see Kimball (1990, p. 54).
Meanwhile numerous other implications of prudence on economic behavior have been described within EUT. The broad range of areas within economics and finance where prudence finds application is indicated by the following non-exhaustive list. Eeckhoudt and Gollier (2005) analyze the impact of prudence on prevention, i.e., the action undertaken to reduce the probability of an adverse effect to occur. Similarly, Courbagé and Rey (2006) note that prudence is an important factor in preventive care decisions within a medical decision making context. Esö and White (2004) show that there can be precautionary bidding in auctions when the value of the object is uncertain and when bidders are prudent. Likewise, White (2008) analyzes prudence in bargaining. Treich (forthcoming) shows that prudence can decrease rent-seeking efforts in a symmetric contest model. Fagart and Sinclair-Desgagné (2007) investigate prudence in a principal-agent model with applications to monitoring and optimal auditing. Within a standard macroeconomic consumption and labor model, Eeckhoudt and Schlesinger (2008) analyze the impact of prudence on policy decisions such as changes in the interest rate. Other examples are insurance demand (e.g., Fei and Schlesinger (2008)) or life-cycle investment behavior (e.g., Gomes and Michaelides (2005)). Even in environmental economics prudence plays an important role; Gollier (2010) finds an ecological prudence effect when discounting future environmental impacts.

Prudence is also necessary (but not sufficient) for decreasing absolute risk aversion, proper risk aversion, and standard risk aversion. According to Brockett and Golden (1987), all of the commonly used utility functions exhibit prudence. This is true, in particular, for power and exponential utility, but also for the interesting parametrizations of hyperbolic absolute risk aversion and expo-power utility. Therefore, implicitly, prudence is assumed widely in the economics and finance literature.

While preference of M_B over M_A is necessary but not sufficient for prudence, Eeckhoudt and Schlesinger (2006) presented a more general lottery preference which is equivalent to prudence. Given two equally likely future states, a prudent individual prefers to have an unavoidable zero-mean risk in the state where her wealth is higher. Equivalently, she prefers to have the unavoidable harms of a sure loss and a zero-mean risk in different future states rather than in the same state. More generally, Eeckhoudt and Schlesinger define "proper risk apportionment" of all orders (where prudence corresponds to order 3). This new understanding of risk preferences does not rely on EUT. Further, it can be generalized to the multi-attribute case as shown in Eeckhoudt et al. (2007) or Tsetlin and Winkler (2009).

Despite the substantial amount of theoretical work on prudence, there is little empirical, i.e., experimental research. Some empirical papers trace prudence via the precautionary savings motive relying on Kimball's EUT-based model (e.g., Dynan (1993), Carrol (1994), and Carroll and Kimball (2008)).
To test the theories and behavioral traits based on prudence in a more controlled environment, we need a valid methodology to test individuals for prudence in the laboratory. The first attempt in this direction was made by Tarazona-Gomez (2004), who finds weak evidence for the existence of prudence. Her experiment relies on a certainty equivalent approach involving tabulated trinomial lotteries. It is based on strong assumptions and approximations within EUT. The only other and much more elegant approach to test for prudence is Deck and Schlesinger (2010). Using six pairs of Eeckhoudt and Schlesinger's lotteries, they find some evidence for prudence.

The contribution of this chapter is as follows. Firstly, we propose a method to test for prudence in a laboratory setting. To facilitate the presentation of Eeckhoudt and Schlesinger's prudence lotteries in a way easily accessible for experimental subjects, we propose a novel graphical representation of compound lotteries in experiments. In particular, it allows for a rather general implementation of the zero-mean risks in the prudence lotteries. This fea-

ture is necessary, because prudence is not only a preference for high skewness—just as risk aversion is not only a preference for a low variance, but this preference is robust towards different levels of kurtosis. As was shown in the previous chapter, it is the skewness of the zero-mean risk that drives the statistical properties, in particular the kurtosis, of the prudence lotteries. This distinguishes them from the simpler lotteries of Mao and from the ones of Deck and Schlesinger who considered symmetric risks only. We illustrate this in the theory part of this chapter. More specifically, we analyze the prudence and Mao lotteries in terms of their statistical moments. As a left-skewed zero-mean risk constitutes more harm to a prudent individual, one could conjecture a greater tendency to "apportion the risks properly." Indeed, in the experiment we observe significantly more prudent decisions when the risks to be apportioned are left-skewed. On the aggregate, 65% of choices are prudent, which is close to Deck and Schlesinger's finding of 61%.
Secondly, we show that lotteries as used in Mao's survey purely differ in their skewness and employ them for the first time in an incentivized experiment. That is, we compare skewness seeking, i.e., a preference for M_B over M_A with prudence, i.e., a preference for proper risk apportionment. Theoretically, prudence implies skewness seeking, but not the other way around. Skewness seeking can be motivated by the assumption of third-order moment preferences, where individuals' decisions between two prospects only depend on the first few statistical moments of these prospects.[2] When studying prudence, only prospects with equal mean and variance will be compared, such that *third-order moment preferences* are equivalent to a preference for or against a high third central moment and refer to "the" skewness of the prospect. That is, in this setting prudence is equivalent to skewness seeking. In the experiment, skewness seeking is more widely observed than prudence. There is also a significant positive correlation between the two and, consistent with theory, most individuals we diagnose as prudent prefer M_B over M_A. However, prudence does not boil down to skewness seeking which also leads us to reject the assumption of third-order moment preferences.

The chapter proceeds as follows. Section 4.2 analyzes the lotteries underlying the experiment and motivates the parameter choices. In Section 4.3, the research questions are stated. Section 4.4 describes the experimental design and procedure. In Section 4.5, re-

[2]Although moment preferences, in general, are incompatible with EUT (e.g., Brockett and Kahane (1992)), they are widely assumed in economic and financial modeling due to their simplicity and tractability. For example, they underlie a large number of classical and also modern portfolio choice models, such as Kraus and Litzenberger (1976) or Briec et al. (2007). The experiment of Tarazona-Gomez (2004) relies on moment preferences. In particular, she assumes a utility function which is truncated at third order.

sults from the experiment are provided, and Section 4.6 concludes. Appendix A.3 contains proofs and experimental instructions.

4.2 Prudence and skewness seeking

In this section, we first define the lotteries of Mao (1970) and Eeckhoudt and Schlesinger (2006) employed in the experiment. Then, we analyze and interpret their statistical properties and show how they relate to skewness seeking and prudence.

4.2.1 Mao's lotteries and Eeckhoudt and Schlesinger's prudence lotteries

Let us start with the definition of binary lotteries in general.

Definition 1. Let $x_1, x_0 \in \mathbb{R}$, with $x_1 > x_0$. X is a Bernoulli-distributed random variable with parameter $p \in (0, 1)$. A binary lottery denoted by $L = L(p, x_1, x_0)$ is defined as the random variable

$$L = X \cdot x_1 + (1 - X) \cdot x_0.$$

In recognition of Mao (1970), we define the following class of lottery pairs as well as skewness preference as referred to in the experiment.[3] An example of a Mao lottery pair is given in Figure 4.1.

Definition 2. Let $p \in (\frac{1}{2}, 1)$. Two binary lotteries $M_A = L(p, x_1, x_0)$ and $M_B = L(1 - p, y_1, y_0)$ constitute a *Mao pair* if they have equal means and variances. An individual is said to be *skewness seeking* if, for any given Mao pair, she prefers M_B over M_A.

Intuitively, M_A has its high payoff associated with the *high* probability, whereas M_B has its high payoff associated with the *small* probability, and vice versa. This is just how negative and positive skewness, respectively, manifest in a binary lottery. This is shown formally in Appendix B. Further, in the next subsection we show that the lotteries of a Mao pair essentially only differ in their skewness. Now we define the prudence lotteries of Eeckhoudt and Schlesinger (2006) and give an example in Figure 4.2.

[3] Definition 2 specifies a class of lotteries that characterizes the risks analyzed in Mao's survey. Evidently, the definition of skewness preference given here is not intended to be general. It is suitable in the context of the experiment presented in this chapter.

Figure 4.1: Example of a Mao pair (M_A, M_B)

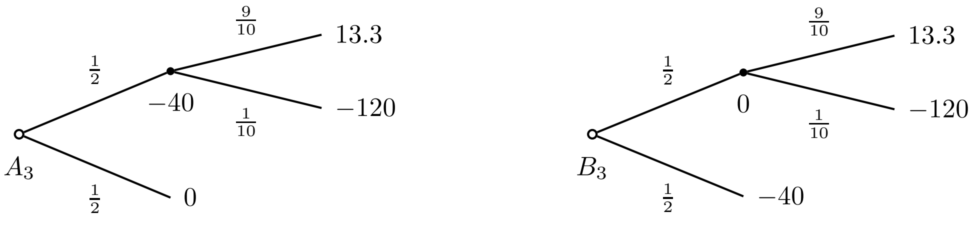
The lotteries above correspond to the Mao pair displayed to subjects in question MAO1 of the experiment. A skewness seeking individual prefers lottery M_B (with a positive skewness) over lottery M_A (with a negative skewness); see Proposition 8.

Definition 3 (Eeckhoudt-Schlesinger Prudence)**.** Let X be a Bernoulli-distributed random variable with parameter $p = \frac{1}{2}$ and let $k > 0$. Let ϵ be a non-degenerate random variable independent of X with $\mathbb{E}[\epsilon] = 0$. The lotteries

$$A_3 = X \cdot (0) + (1 - X) \cdot (-k + \epsilon) \text{ and } B_3 = X \cdot (-k) + (1 - X) \cdot \epsilon$$

as a pair are called *(Eeckhoudt-Schlesinger) prudence lottery pair* or an *ES pair.* An individual is called *prudent* if she prefers B_3 over A_3 for all values of k, for all random variables ϵ, and for all wealth levels x.

Figure 4.2: Example of an ES pair (A_3, B_3)



The lotteries above correspond to the ES pair displayed to subjects in question ES1 of the experiment. In the example, ϵ is left-skewed implying that lottery A_3 has a larger kurtosis than lottery B_3; see Proposition 9.

For the prudent option B_3, the additional zero-mean risk ϵ (i.e., the second lottery) occurs in the good state of the 50/50 gamble (i.e., in the state without the sure reduction in wealth, $-k$), whereas for the imprudent option A_3 the zero-mean risk occurs in the bad state. Intuitively, a prudent choice implies *proper risk apportionment* across states of nature. Eeckhoudt and Schlesinger show that this preference is equivalent to prudence within EUT, i.e., $u''' > 0$. Menezes et al. (1980) define an increase in downside risk and show that, under EUT, $u''' > 0$ is equivalent to downside risk aversion. They further reinterpret the results of Mao's survey and show that the lottery M_B has less downside risk than the corresponding lottery M_A. Thus, we can state

Proposition 7 (Menezes, Geiss, and Tressler, 1980)**.** *Let (M_A, M_B) denote a pair of Mao lotteries. Prudence is sufficient (but not necessary) for preferring M_B over M_A.*

4.2.2 Prudence, moments, and skewness seeking

The following two propositions, respectively, will allow to investigate the statistical features of the Mao and ES (prudence) lotteries in greater detail. These will motivate the particular choices of the lottery pairs we implement in the experiment. It will be most convenient to first plainly state both propositions and to discuss the results afterwards by comparison. For experimental calibration reasons, unlike in the previous chapter, in this chapter "moments" refer to *standardized* central moments.[4] Therefore, the nth moment ($n \in \mathbb{N}$ and $n \geq 3$) of a random variable Z is given by $\mathbb{M}_n^S(Z) := \mathbb{E}[(Z - \mathbb{E}[Z])^n] / (\mathbb{V}(Z))^{n/2}$. With $\nu(Z) := \mathbb{M}_3^S(Z)$ and $\kappa(Z) := \mathbb{M}_4^S(Z)$ we denote the third and fourth moment, respectively.

Proposition 8. *Consider an arbitrary Mao pair given by M_A and M_B as in Definition 2. Then,*

(a) $\nu(M_B) - \nu(M_A) > 0$ *and* $\kappa(M_B) - \kappa(M_A) = 0$.

(b) *More generally,* $\mathbb{M}_n^S(M_B) - \mathbb{M}_n^S(M_A) > 0$ *for all n odd, and* $\mathbb{M}_n^S(M_B) - \mathbb{M}_n^S(M_A) = 0$ *for all n even.*

The following proposition gives the corresponding result for the ES lotteries.

Proposition 9. *Consider an arbitrary ES lottery pair in Definition 3. A_3 and B_3 have equal expectation and variance and thus $\mathbb{V}(A_3) = \mathbb{V}(B_3) =: \sigma^2$ is well-defined. Then,*

(a) $\nu(B_3) - \nu(A_3) = \frac{3k\mathbb{E}[\epsilon^2]}{2\sigma^3} > 0$, and $\kappa(B_3) - \kappa(A_3) = \frac{2k\mathbb{E}[\epsilon^3]}{\sigma^4}$ *can be positive, negative, or zero.*

(b) *More generally,* $\mathbb{M}_n^S(B_3) - \mathbb{M}_n^S(A_3) > 0$ *for all n odd. For n even,* $\mathbb{M}_n^S(B_3) - \mathbb{M}_n^S(A_3)$ *can be positive, negative, or zero.*

[4] In the discussion of the Mao lotteries, standardization does not matter, because the two lotteries of a pair do not differ in their variance. The same is true for the prudence lotteries. However, while in the proofs to the previous chapter it would have been cumbersome to standardize, in this chapter, standardization is convenient. This is because the moments of the particular lotteries employed in the experiment are actually calculated and tabulated. They would be extremely large if not standardized.

To interpret these results, note that the third and fourth moments, respectively, are sometimes referred to as "the" skewness and "the" kurtosis. However, there are numerous measures for these properties; see MacGillivray (1986) for an overview. Parts (b) of the above propositions imply the following. While thinking in third- and fourth-order terms will provide the reader with the correct intuition, our arguments actually apply to the very strong notions of skewness and kurtosis that refer to *all* odd and even moments, respectively.

Comparing Propositions 8 and 9, we see that both prudence (i.e., a preference for B_3 over A_3) and a preference for M_B over M_A imply higher skewness to be beneficial to the individual. The Mao lotteries essentially purely differ in their skewness. Prudence further requires that the lottery with the higher skewness is preferred no matter whether it has a smaller or larger kurtosis. That is, prudence implies a preference for skewness, but it also requires this preference to be robust towards variations in kurtosis. This was put on a more rigorous basis in the previous chapter and referred to as the "kurtosis robustness feature of prudence."

What is the origin of this additional statistical freedom of the ES lotteries compared to the Mao lotteries? From Proposition 9, part (a), we see that the prudent choice has the smaller kurtosis if and only if the zero-mean risk that has to be apportioned is left-skewed.[5] The zero-mean risks of the ES lotteries employed in the experiment of Deck and Schlesinger (2010) are symmetric. This constantly implies the same kurtosis for the two ES lotteries. Moreover, from Roger (2011) or our Proposition 9, the signs of all moments of ES lotteries with symmetric ϵ's coincide with those we derived in Proposition 8 for the Mao lotteries. Thus, from a statistical point of view, prudence lotteries with symmetric zero-mean risks are much closer to the Mao lotteries testing for skewness seeking than to the general proper risk apportionment lotteries of Eeckhoudt and Schlesinger (2006). Preference between the former lotteries is solely determined by skewness preference and does not reflect the kurtosis robustness feature of prudence.

In the work presented in this chapter, we not only avoid this restriction, but also evaluate it. This requires a comprehensive experimental presentation of the compound ES lotteries as the skewed risks to be apportioned cannot be presented as a fair coin toss to subjects. In the experiment, subjects will also decide over Mao lotteries to test them for skewness seeking, which theoretically is necessary, but not sufficient to imply prudence.

[5] This is meant in the sense that $\mathbb{E}[\epsilon^3] < 0$. Also this argument applies to the stronger notions of skewness referring to all odd moments; see Chapter 3 for details.

4.3 Research questions

In this chapter, we propose a method to test for prudence, employ it in an experiment, and test it for robustness. A main focus is on whether prudence boils down to skewness seeking or if, on the other hand, we find evidence for the kurtosis robustness feature of prudence. Therefore, we also employ the Mao lotteries in the experiment to test for skewness seeking directly.

Research Question 1. *What is the relationship between prudence and skewness seeking?*

If preferences over ES and Mao pairs are equivalent, then skewness seeking seems to characterize prudence sufficiently well. On the other hand, if there are skewness seeking individuals that do not exhibit prudence, then prudence is a stronger property, not only in theory, but also in practice. In particular, it is not sufficient then to use binary lotteries to test for prudence because such lotteries can not reflect the kurtosis robustness feature of prudence.[6]

Eeckhoudt and Schlesinger's definition of prudence (Definition 3) is very broad in scope. That is, the lottery preference must hold for any random variable ϵ, any loss $-k$, any wealth level x and, of course, is robust towards framing of the decision task. In particular, the fact that the zero-mean risks are arbitrary adds a large amount of stochastic freedom to these lotteries. As explained in Section 4.2, the skewness of the zero-mean risks determines whether the prudent or imprudent lottery choice has the smaller or larger kurtosis. We will test in a systematic way which of these features do significantly influence subjects' decisions.

Concerning the robustness towards framing, we test whether it makes a difference if the task is to add the zero-mean risk ϵ or the fixed amount $-k$ to a state of the 50/50 gamble, given that the other item ($-k$ or ϵ, respectively) is already present in one state. This relates to the intuition of Eeckhoudt and Schlesinger's definition of prudence as "proper risk apportionment." Further, Definition 3 of prudence could be adapted such that the loss $-k$ is replaced by a fixed gain in wealth k. The prudent choice then is the one where k and ϵ appear in the *same* state (a prudent individual prefers the unavoidable additional risk when wealth is higher). Further in-depth explanations are provided in Section 4.4. In short, we state the following research questions.[7]

[6]This is an immediate consequence of Theorem 3 in Appendix B.

[7]Research Questions 2 to 4 have been addressed to some degree in Deck and Schlesinger (2010). We will

Research Question 2. *Are individuals' decisions on ES pairs independent of whether the fixed amount k corresponds to a gain or a loss?*

Research Question 3. *Are individuals' decisions on ES pairs influenced by the wealth level x?*

Research Question 4. *Are individuals' decisions on ES pairs influenced by different framing of the decision task—whether they are asked to add the zero-mean risk ϵ or the fixed amount k to a state of the 50/50 gamble?*

Research Question 5. *Are individuals' decisions on ES pairs influenced by the skewness of the zero-mean risk ϵ and, therewith, the kurtosis of the prudence lotteries?*

4.4 Experimental design and procedure

The computerized experiment was programmed in z-Tree (Fischbacher (2007)). In total, each subject makes 34 individual binary lottery choices. The lottery outcomes are disclosed in "Taler," our experimental currency. One Taler is worth €0.15 (about $0.2). Decisions are incentivized by a random-choice payment technique. That means, one out of 34 decisions is randomly drawn to determine solely a subject's payoff.[8] The lottery chosen by the individual in the randomly determined decision is actually played out at the end of the experiment.

The experiment consisted of three stages. In stage ES, we tested subjects for prudence using 16 ES lottery pairs. Subjects decided on 8 Mao pairs in stage MAO to test them for skewness preference. The remaining 10 questions were used to test for risk aversion. We will not elaborate on the method as well as the data analysis of the third stage in the following. A questionnaire comprising demographic questions followed the experiment. The experimental instructions handed out to subjects are given in Appendix A.3.2. We

compare results in Section 4.5.

[8]It has become increasingly common in economic experiments to elicit a series of choices from participants and then to pay for only one selected at random; see Baltussen et al. (2010) for a fine overview. The random choice payment technique enables the researcher to observe a large number of individual decisions for a given research budget. However, the important question arises whether subjects behave as if each of these choices involves the stated payoffs. This issue has been analyzed, among various other setups, in experiments with pairwise lottery choice problems similar to our experiment. For example, Starmer and Sugden (1991) found clear evidence that under random payment subjects isolate choices as if paid for each task. Similar evidence was reported by Beattie and Loomes (1997) and Cubitt et al. (1998). In a lottery experiment with a multiple price list format, Laury (2005) reports no significant difference in choices between paying for 1 or all 10 decision.

now describe the experimental stages in more detail.

4.4.1 Prudence test embedded in a factorial design: stage ES

In stage ES, we test whether individuals are prudent according to Definition 3. To this end subjects are asked to make preference choices over the 16 ES pairs $\mathrm{ES1}, \mathrm{ES2}, \ldots,$ ES16. We introduce a new ballot box representation to display the compound lotteries of the ES pairs. Figure 4.3 shows, as an example, how question ES1 (that has already been illustrated more formally in Figure 4.2) appears on subjects' decision screens. It must be understood as follows: Option A and Option B are displayed in the left and right panel of Figure 4.3, respectively. For both options, the 50/50 gamble is depicted as a ballot box that contains two balls labeled "Up" and "Down." The displays of both Option A and Option B themselves are spatially separated, each into an upper panel containing the "Up-ball" and into a lower panel containing the "Down-ball." Now consider Option A. If the draw from the first ballot box is "Up," then the subject loses 40 Taler, and a second lottery (the zero-mean risk ϵ) follows. The zero-mean risk ϵ is also displayed in a ballot box format with 10 balls in total. Balls implying a loss (here: -120 Taler) are colored in yellow on subjects' decision screens, and balls implying a gain (here: 13.3 Taler) are colored in white. In situation "Down," no second lottery follows and no loss occurs. For Option B, if the draw from the first ballot box is "Up," no loss occurs and a second lottery follows (the same ϵ as depicted in Option A). If the draw is "Down," a loss of 40 Taler occurs. The order of subjects' 16 decision screens is randomized for each subject, and also the position of the prudent option being either left or right on the screen has been randomized.
This ballot box representation interlinks decisions on the computer screen with the lottery play at the end of the experiment; see Figure 4.4. Further, it visualizes asymmetric zero-mean risks and all probabilities in an intuitive way.

To test Research Questions 2 to 5, we employ a completely randomized factorial design.[9] The factors are as follows: sign of k (Factor A), wealth level x (Factor B), framing (Factor C), and composition of ϵ (Factor D); see columns 6 to 9 in Table 4.1 for a complete design layout. Along the illustration in Figure 4.3, we now explain how the factors of the factorial design translate into subjects' decision screens.

[9]For a detailed description of the factorial design technique see, e.g., Montgomery (2005).

Figure 4.3: Example of the lottery display in stage ES (Question ES1)

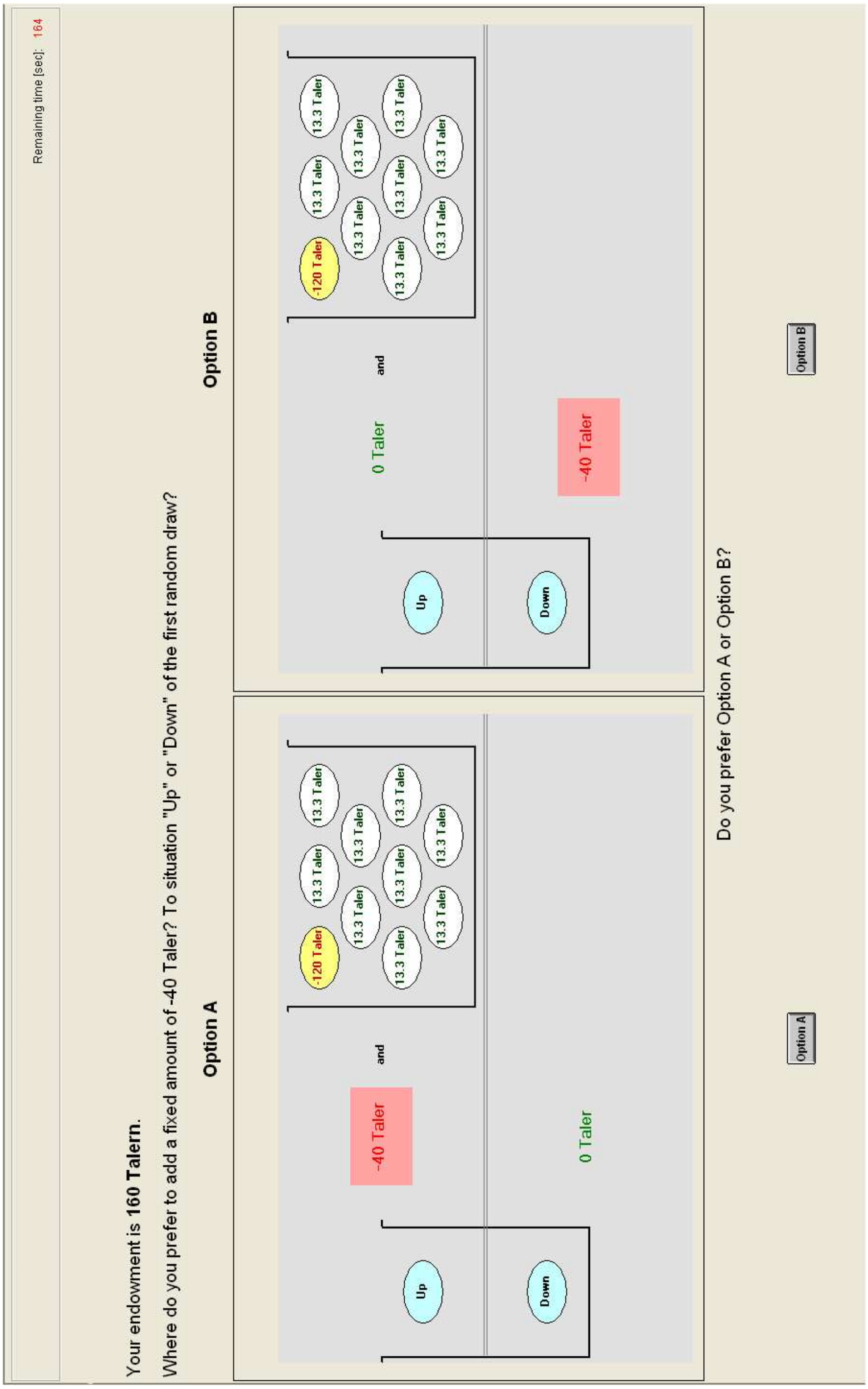


Figure 4.4: Sample of ballot boxes

This photograph shows an example of the ballot boxes used to determine subjects' payoffs at the end of the experiment from a decision made in stage ES, e.g., ES1 (compare to screenshot in Figure 4.3).

When Factor A is at its low level ($k_1 = 40$), the outcomes of the 50/50 gamble are 0 Taler and -40 Taler. That is, the fixed amount added corresponds to a loss. Hence, in the example, the imprudent choice is Option A, as the additional zero-mean risk occurs in the bad state. At the alternative level ($k_2 = -40$) of Factor A the amount 40 Taler is added, which corresponds to a gain and is displayed as a green bill on subjects' screens. With Factor A we test for an experimental framing effect (Research Question 2) and whether individuals really exhibit the intuition of proper risk apportionment. For example, if a subject consistently prefers the option where ϵ is added to outcome 0 Taler (independent of the sign of k), we could conjecture that this is due to framing and conclude that 0 is a so-called focal point.

Factor B tests for a wealth effect according to Research Question 3 and comprises the levels $x_1 = 160$ or $x_2 = 80$ Taler. This test is limited in that wealth levels are presented as endowments to subjects that they receive in order to accommodate possible negative lottery outcomes. The wealth level on subjects' screens is indicated in the upper left corner. In Figure 4.3, it is set to 160 Taler.

Next we consider Factor C. In the example, the decision between the imprudent Option A and the prudent Option B is whether in the up-state or in the down-state a fixed loss of 40 Taler is preferred *given* that the additional risk will be in the up-state. That is, the question on the decision screen is "*Where do you prefer to add a fixed amount of* -40 *Taler? To situation "Up" or "Down" of the first risky event?*" At the other level of Factor C, subjects are asked to which situation—either 0 or $-k$—of the 50/50 gamble to add another risky event (ϵ). Thus, the two levels of Factor C are "add k" (a sure reduction or increase in wealth) or "add ϵ" (a zero-mean risk). Factor C directly relates to the intuition behind Eeckhoudt and Schlesinger's prudence definition of proper risk apportionment. It purely checks for a framing issue as the lotteries across levels of Factor C are identical in distribution.

With Factor D we test if prudence is invariant under variation of the ϵ's (Research Question 5) or *equivalently*, for the kurtosis robustness feature of prudence. According to Proposition 9, the prudent lottery choice B_3 has always the higher skewness compared to the imprudent choice A_3. It has the smaller kurtosis, i.e., $\mathbb{M}_n^S(B_3) - \mathbb{M}_n^S(A_3) < 0$ for all $n \geq 4$ even, if and only if ϵ is left-skewed. Thus, when varying the zero-mean risks, it is natural to vary their skewness systematically as this is the significant driver of the statistical differences between the ES lotteries. As shown in Appendix B, the skewness of a binary lottery depends only on its probability parameter. In our example, ϵ is left-skewed such that the prudent lottery choice has the smaller kurtosis. If ϵ in the example had the signs of the outcomes switched, it would be right-skewed, and the prudent option had the higher kurtosis. As ϵ has a mean of zero, skewness has the following interpretation. A left-skewed ϵ yields a small gain with high probability and a large loss with a small probability. Further, as we display ϵ as a ballot box containing 10 balls, skewness translates one-to-one to the number of draws implying losses or gains, respectively. Indeed, in the example, ϵ implies a loss of 120 Taler with a 10% chance and a gain of 13.3 Taler with a 90% chance.
We denote the levels of Factor D by "$\kappa(B_3) - \kappa(A_3) > 0$" (positive kurtosis difference) and "$\kappa(B_3) - \kappa(A_3) < 0$" (negative kurtosis difference). However, any of the mentioned equivalent interpretations (kurtosis difference, skewness of the zero-mean risk, composition of the ballot box) is captured by Factor D. These practical interpretations of kurtosis difference support our theoretical argument that restricting to symmetric ϵ's is a somewhat severe limitation for a procedure that aims to test for prudence.

To sum up, by specifying the four factors above, we manipulate the requirements in

Table 4.1: ES pairs with their underlying factors and their statistical properties

	ϵ				Factors				Statistical properties			
ES pair	p	z_1	$1-p$	z_0	A	B	C	D	$\mathbb{E}[A_3] = \mathbb{E}[B_3]$	$\mathbb{V}(A_3) = \mathbb{V}(B_3)$	$\nu(B_3) - \nu(A_3)$	$\kappa(B_3) - \kappa(A_3)$
ES1	0.90	13.33	0.10	−120.00	40	160	add $-k$	$\kappa(B_3) - \kappa(A_3) < 0$	−20.00	1,200.00	2.30	−9.48
ES2	0.10	120.00	0.90	−13.33	40	160	add $-k$	$\kappa(B_3) - \kappa(A_3) > 0$	−20.00	1,200.00	2.30	9.48
ES3	0.80	12.00	0.20	−48.00	40	160	add ϵ	$\kappa(B_3) - \kappa(A_3) < 0$	−20.00	688.00	1.92	−3.50
ES4	0.20	48.00	0.80	−12.00	40	160	add ϵ	$\kappa(B_3) - \kappa(A_3) > 0$	−20.00	688.00	1.92	3.50
ES5	0.70	12.00	0.30	−28.00	40	80	add $-k$	$\kappa(B_3) - \kappa(A_3) < 0$	−20.00	568.00	1.48	−1.33
ES6	0.30	28.00	0.70	−12.00	40	80	add $-k$	$\kappa(B_3) - \kappa(A_3) > 0$	−20.00	568.00	1.48	1.33
ES7	0.60	8.00	0.40	−12.00	40	80	add ϵ	$\kappa(B_3) - \kappa(A_3) < 0$	−20.00	448.00	0.60	−0.15
ES8	0.40	12.00	0.60	−8.00	40	80	add ϵ	$\kappa(B_3) - \kappa(A_3) > 0$	−20.00	448.00	0.60	0.15
ES9	0.90	13.33	0.10	−120.00	−40	160	add $-k$	$\kappa(B_3) - \kappa(A_3) > 0$	20.00	1,200.00	2.30	9.48
ES10	0.10	120.00	0.90	−13.33	−40	160	add $-k$	$\kappa(B_3) - \kappa(A_3) < 0$	20.00	1,200.00	2.30	−9.48
ES11	0.80	12.00	0.20	−48.00	−40	160	add ϵ	$\kappa(B_3) - \kappa(A_3) > 0$	20.00	688.00	1.92	3.50
ES12	0.20	48.00	0.80	−12.00	−40	160	add ϵ	$\kappa(B_3) - \kappa(A_3) < 0$	20.00	688.00	1.92	−3.50
ES13	0.70	12.00	0.30	−28.00	−40	80	add $-k$	$\kappa(B_3) - \kappa(A_3) > 0$	20.00	568.00	1.48	1.33
ES14	0.30	28.00	0.70	−12.00	−40	80	add $-k$	$\kappa(B_3) - \kappa(A_3) < 0$	20.00	568.00	1.48	−1.33
ES15	0.60	8.00	0.40	−12.00	−40	80	add ϵ	$\kappa(B_3) - \kappa(A_3) > 0$	20.00	448.00	0.60	0.15
ES16	0.40	12.00	0.60	−8.00	-40	80	add ϵ	$\kappa(B_3) - \kappa(A_3) < 0$	20.00	448.00	0.60	−0.15

This table describes the prudence lottery pairs ES1, ES2, . . . ES16 in stage ES. ϵ is the binary zero-mean risk with its up-state z_1, its down-state z_0, and the respective probabilities p and $1-p$ shown in columns 2 to 5. The explicit arrangement of factors A, B, C, and D is given in columns 6 to 9. The remaining columns provide information on moments of the ES lotteries.

Eeckhoudt and Schlesinger's definition of prudence and for framing issues. We can test which factors have a severe impact on individuals' decisions such that they should be accounted for when testing for prudence. A complete overview of the 16 ES pairs, their statistical properties, and the arrangement of factors is provided in Table 4.1.

4.4.2 Skewness seeking test: stage MAO

In this stage, we test subjects for skewness seeking in order to answer Research Question 1. Therefore, we construct 8 different Mao pairs for which subjects have to state their preference. These are shown in Table 4.2 and are matched with the lotteries from stage ES according to their first three moments. For the third moment only differences can be matched. For the details of this calibration procedure see Appendix B. There are only 8 such pairs, because the lotteries of a Mao pair cannot differ in their kurtosis; see Proposition 8. Thus, lottery pair MAO1 is matched to lottery pairs ES1 and ES2, lottery pair MAO2 is matched to ES3 and ES4, and so on. As the Mao lotteries imply negative outcomes, subjects are endowed with a certain amount of money equal to the wealth level x in the matched ES pairs.[10]

For the Mao lotteries we choose a graphical representation similar to the one proposed by Camerer (1989). An example of a decision screen can be found in the instructions to stage II in Appendix A.3.

4.4.3 Procedural details

The experiment was conducted at the *B*onn*E*con*L*ab. Overall 72 students of University of Bonn from various fields participated in 9 experimental sessions in December 2008, January, and February 2009. The stage order was varied systematically across sessions. Each session lasted for about 90 minutes. Subjects earned on average €18.50 (about $24.7).

The procedure of the experiment was as follows: firstly, experimenters extensively introduced the decision task and the entire procedure of the experiment to subjects. Secondly, before each experimental stage started, subjects were asked to answer control questions testing their understanding of the decision task. In particular, they were familiarized with the illustration of lotteries and their outcomes as well as probabilities. Only when subjects

[10] Analogous to stage ES, the order of subjects' decision screens is randomly permutated in stage MAO, and also the position of the lottery with the higher skewness (left or right on the decision screen) is randomized.

Table 4.2: Mao pairs and their statistical properties

	M_A				M_B				Statistical properties		
									$\mathbb{E}[M_A]$	$\mathbb{V}(M_A)$	$\nu(M_B)$
Mao pair	p	x_1	$1-p$	x_0	p	y_1	$1-p$	y_0	$= \mathbb{E}[M_B]$	$= \mathbb{V}(M_B)$	$= -\nu(M_A)$
MAO1	0.75	0.00	0.25	−80.00	0.75	−40.00	0.25	40.00	−20.00	1200.00	1.15
MAO2	0.72	−3.48	0.28	−61.64	0.72	−36.52	0.28	21.64	−20.00	688.00	0.96
MAO3	0.67	−3.44	0.33	−54.30	0.67	−36.56	0.33	14.30	−20.00	568.00	0.74
MAO4	0.58	−1.81	0.42	−44.62	0.58	−38.19	0.42	4.62	−20.00	448.00	0.30
MAO5	0.75	40.00	0.25	−40.00	0.75	0.00	0.25	80.00	20.00	1200.00	1.15
MAO6	0.72	36.52	0.28	−21.64	0.72	3.48	0.28	61.64	20.00	688.00	0.96
MAO7	0.67	36.56	0.33	−14.30	0.67	3.44	0.33	54.30	20.00	568.00	0.74
MAO8	0.58	38.19	0.42	−4.62	0.58	1.81	0.42	44.62	20.00	448.00	0.30

This table shows the eight Mao pairs, i.e., $\text{MAO1}, \text{MAO2}, \ldots, \text{MAO8}$, employed in stage MAO. These are matched to the ES pairs in terms of moments as explained in Appendix B. The final three columns provide information on moments of the Mao lotteries.

had answered these questions correctly, they were allowed to proceed to the decision stages of the experiment. Then, thirdly, subjects made the decisions in the experimental stages. Afterwards, subjects answered a questionnaire for which they received €4.00 ($5.3) in addition to their earnings from the experiment (comparable to a show-up fee). Finally, each subject's payoff was determined by a random-choice payment technique. To this end, for each subject one ball was chosen out of a set of balls numbered between 1 and 34 from a ballot box referring to a lottery pair from the experiment. The subject's lottery choice in this randomly drawn lottery pair was then actually played out. In both stages MAO and ES, the outcome was allocated to the subjects' wealth level in that decision, i.e., subjects could charge the coupon they obtained in the beginning. The ES lotteries were played out using ballot boxes resembling the lotteries displayed on subjects' decision screens; see the photograph in Figure 4.4. The binary lotteries in stage MAO and the risk aversion stage were played out using a ballot box with 100 balls numbered from 1 to 100. If, e.g., the up-state had a likelihood of 90%, a draw of the balls numbered $1, 2, \ldots, 90$ implied the corresponding up-payoff.

4.5 Experimental results

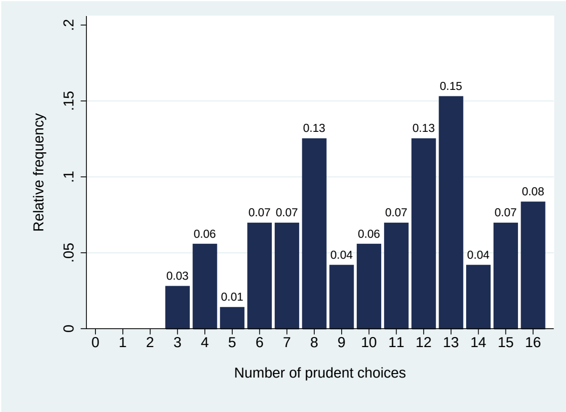
4.5.1 Preliminary analysis

There is evidence for both prudence and skewness seeking at an aggregate level.[11] Figure 4.5 plots the relative frequencies of subjects' prudent choices. Overall, 65.10% of subjects' responses are prudent. This fraction is slightly higher than the 61% of prudent choices reported by Deck and Schlesinger (2010). In our sample, on average 10.42 out of 16 choices are prudent with a standard deviation of 3.65. The median (mode) of prudent choices is 11 (13). The observed behavior in stage ES differs significantly from arbitrary behavior. Formally, we can reject the null hypothesis that the median of subjects' prudent choices is equal to 8 as it would be for arbitrary choices ($p = 0.0000$, two-sided one-sample Wilcoxon signed-rank test).[12]

[11] To rule out possible stage order effects, we compare responses from sessions with stage order MAO-ES with responses from sessions with stage order ES-MAO. The null hypothesis that both samples are drawn from the same distribution cannot be rejected (for ES-responses: $p = 0.413$ and for MAO-responses: $p = 1.000$, two-sided two-sample Kolmogorov-Smirnov test).

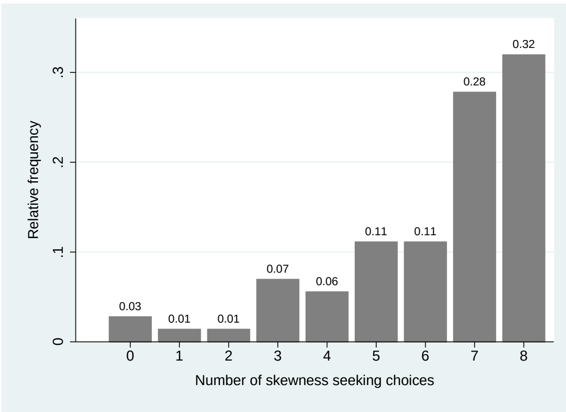
[12] In the following, all statistical tests are *two-sided* if not indicated differently. Under the null hypothesis, the Wilcoxon signed-rank procedure assumes that the sample (of frequencies per individual) is randomly taken from a population with a symmetric (but not necessarily normal) frequency distribution. As an alternative, a two-sided one-sample sign-test with the same null and alternative hypothesis (but without

Figure 4.5: Distribution of the number of prudent choices by subjects



In stage MAO, 77.08% of all choices imply skewness seeking. Figure 4.6 illustrates the rel-

Figure 4.6: Distribution of the number of skewness seeking choices by subjects



ative frequencies of subjects' skewness seeking choices. Each subject has been, on average, skewness seeking in 6.16 out of 8 questions with a standard deviation of 2.01. The median (mode) of skewness seeking choices is 7 (8). Also, this behavior differs significantly from arbitrary choices ($p = 0.0000$.)

The age of the 72 participants is, on average, 24.25 years; the youngest individual is 19, the oldest is 42 years of age. 41 are female and 31 are male. According to Mann-Whitney U-tests, neither age nor gender have a significant influence on the number of prudent answers observed in our experiment; see Ebert and Wiesen (2009) for details.

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the symmetry assumption), would also lead us to reject the null ($p = 0.0004$).

4.5.2 Within subject analysis

Our preliminary analysis suggests substantial evidence for prudence and skewness seeking. This subsection is concerned with their relationship at an individual level (Research Question 1). For starters, we observe a significant positive correlation (which is a symmetric measure of association) of $\rho = 0.2844$ between prudent and skewness seeking choices ($p = 0.0155$, Spearman rank correlation test).

We now show that the actual relationship is asymmetric. To this end, we categorize subjects' responses in stages ES and MAO according to the frequency of prudent and skewness seeking choices, respectively. These categorizations are somewhat arbitrary. However, the qualitative conclusions stay the same when changing the categorizations by plus or minus one. Subjects who answered 12 or more (4 or less) out of 16 questions prudently are said to be *prudent* (*imprudent*). Those subjects who answered 5 to 11 questions prudently are classified as *indifferent*. Similarly, subjects are classified as *skewness seeking* (*not skewness seeking*) if they have answered 7 or 8 (0 or 1) out of 8 questions in favor of the lottery with the positive (negative) skewness. When 2 to 6 questions implied skewness seeking, subjects are allotted to the category *indifferent*.
Table 4.3 cross-tabulates the absolute frequencies of subjects according to the categories.

Table 4.3: Contingency table on categories

	Not skewness seeking	Indifferent	Skewness seeking	Total
Imprudent	0	3	3	6
Indifferent	2	13	17	32
Prudent	1	10	23	34
Total	3	26	43	72

Let us first analyze prudence and skewness seeking separately. 34 (47.22%) of all 72 subjects are prudent, whereas only 6 (8.33%) are imprudent. Note again that this gives a very different picture compared to looking at the aggregate responses only. Deck and Schlesinger (2010) report that very few subjects always decided imprudently (2%) and only 14% were always prudent in their six decision tasks. Skewness seeking is more widely observed than prudence, as 43 (59.72%) of all subjects exhibit it, whereas only 3 (4.17%) do not.[13] This complies with our arguments made in Section 4.2 as it shows that, empirically, skewness seeking is a weaker preference than prudence. The difference in prudent and skewness seeking observations immediately indicates that Mao lotteries

[13]Tarazona-Gomez (2004) finds 63% of the subjects to be "prudent" under the assumption of third-order moment preferences.

are not suitable to test for prudence.

The conditional frequency f(skewness seeking |prudent) that a prudent individual exhibits skewness seeking is 67.65%, whereas f(not skewness seeking |prudent) is only 2.94%.[14] The chance for a prudent individual to be skewness seeking is thus about 23 times higher than not being skewness seeking. The analysis of the reverse statement does not provide such a clear-cut picture. The conditional frequency f(prudent |skewness seeking) is given by 53.49% whereas f(imprudent |skewness seeking) equals 6.98%. Thus, the chance of being prudent given that the individual is skewness seeking is about 8 times higher than for an individual that is not skewness seeking. This result, however, is not very reliable as there are only 3 subjects who were not skewness seeking. In short, we see that knowing about an individual's preference towards the Mao lotteries gives some information about whether the individual is prudent. The result also hints in the "right" direction as being skewness seeking increases the probability of being prudent.

Result 1. *Most prudent individuals are skewness seeking, whereas skewness seeking individuals may not be prudent.*

Result 1 shows that skewness seeking is not sufficient to make conclusions whether an individual is prudent. Thus, there seems to be more to prudence than skewness seeking. Result 1 can also be interpreted as a robustness check for our method to test for prudence. Most subjects it diagnoses as prudent, consistently with theory, are skewness seeking.

4.5.3 Influences on prudent behavior

We now investigate what types of ES questions are more likely to be answered prudently. In general, we find that the particular choice of the prudence lottery pair has a strong impact on the 72 subjects' decisions. Relative frequencies range from 50.00% to 75.00% with a standard deviation of 8.11.% from the reported mean of 65.10%. In order to determine what particular elements in the definition of prudence cause these differences, we investigate Factors A, B, C, and D according to Research Questions 2 to 5.

As formulated in Research Question 2, we are interested whether the fixed amount k being a gain or a loss (Factor A) influences subjects' decisions. When k is a loss, 66.32% of responses are prudent, whereas slightly less responses are prudent (63.89%) when k is

[14]If we exclude subjects who were indifferent at least at one stage, these numbers become 95.6% and 4.2%, respectively.

a gain. Test statistics of a Wilcoxon signed-rank test and a Fisher-Pitman permutation test for paired replicates in Table 4.4 show that this difference is insignificant ($p = 0.5253$ and $p = 0.5008$, respectively).

Result 2. *Subjects' decisions are robust towards different outcomes of the 50/50 gamble, i.e., whether the fixed amount k is a gain or a loss. Implicitly, 0 as a focal point did not influence behavior.*

Considering Factor B, 64.76% of choices are prudent if the wealth level x is high ($x_1 = 160$), and 65.45% of choices are prudent if x is low ($x_2 = 80$). This indicates an insignificant difference; see the test results in Table 4.4.

Result 3. *Subjects' decisions are robust towards different wealth levels.*

Research Question 4 asks whether a framing of the decision task (Factor C) influences subjects' decisions. The level of Factor C influences prudent choices substantially, as 67.36% of the choices are prudent if the level is "add ϵ" and 62.85% if the level is "add $-k$." Test statistics show that differences are weakly significant.

Result 4. *Framing of the decision task influences subjects' decisions. Weakly significant more subjects answer questions prudently if the zero-mean risk (ϵ) has to be added to the 50/50 gamble compared to the fixed amount (k).*

In essence, Result 4 shows that the decision task involving subjects' conscious consideration about another risky event leads to more prudent choices, whereas when asked to add a fixed amount subjects make slightly more imprudent choices. When looking at the interaction of Factors A and C, weakly significantly more choices are prudent whenever i) the fixed amount is a loss ($k_1 = 40$) and subjects are asked to "add ϵ" and ii) the fixed amount is a gain ($k_2 = -40$) and they are asked to "add $-k$" ($p = 0.0690$).

In short, our analysis of factors A, B, and C suggests that subjects' decisions are neither influenced by the fixed amount being a gain or a loss nor by the wealth level. These results are in line with behavioral patterns reported by Deck and Schlesinger (2010). They also find that the relative size of the zero-mean risk is not influential. In contrast to their findings, our behavioral data evidence that framing of the decision task weakly influences subjects' choices.

Table 4.4: Analysis of prudent choices for different factor levels

Factor	Level	Relative frequency of prudent choices	p-value (Fisher-Pitman permutation test)
A	$k_1 = 40$	0.6632	0.5008
	$k_2 = -40$	0.6389	
B	$x_1 = 160$	0.6476	0.8362
	$x_2 = 80$	0.6545	
C	add $-k$	0.6285	0.0677
	add ϵ	0.6736	
D	$\kappa(B_3) - \kappa(A_3) < 0$	0.6858	0.0121
	$\kappa(B_3) - \kappa(A_3) > 0$	0.6163	

Factor D considered in Research Question 5 is most significant; see Table 4.4. At its low level (negative kurtosis difference), 68.58% of subjects' choices are prudent. If Factor D is at its high level (positive kurtosis difference), 61.63% of the choices are prudent. For questions ES9 (largest positive kurtosis difference in the experiment) and ES10 (largest negative kurtosis difference, other factors like in ES9), 50.00% and 75.00% of answers are prudent, respectively. Note again that for the prudence lotteries a negative kurtosis difference is equivalent to ϵ being left-skewed, i.e., the ballot box displayed on subjects' screens contains more white balls (implying a small gain) than yellow balls (implying a high loss).

Result 5. *The particular choice of the zero-mean risk ϵ strongly influences subjects' decisions. Significantly more subjects decide prudently if ϵ is left-skewed.*

One intuition supporting Result 5 is that a prudent individual may consider a negatively skewed zero-mean risk as a "bigger" harm. Hence, there is a greater tendency for apportioning the harms of the sure loss and the zero-mean risk properly. In Section 4.2, we showed that ϵ being left-skewed implies a smaller kurtosis for the prudent than for the imprudent choice. An interpretation is that in this case the prudent choice implies a smaller likelihood of extreme events to occur. A prudent individual, however, would seek the higher skewness of the prudent lottery choice irrespectively of its kurtosis. She must not deviate from her preference if the additional risk is not too harmful to her. This was shown in Proposition 9 and was referred to as the kurtosis robustness feature of prudence in Chapter 3. Thus, Result 5 is a major finding of our experiment as it confirms its relevance empirically. It emphasizes the importance to use several lotteries to test for prudence in order to reflect the statistical diversity which is implicit in Eeckhoudt and Schlesinger's definition of prudence. As the kurtosis of the prudence lotteries matters, the significance of Factor D also shows that there is more to prudence than skewness seeking.

4.6 Conclusion to Chapter 4

Currently, the share between theoretical and empirical literature on prudence is very unbalanced. Numerous behavioral implications of prudence have been pointed out, but there is very little empirical, i.e., experimental, research on prudence to support the relevance and validity of these theories. To get there, in this last chapter of this thesis we propose, implement, and check for robustness a method testing for prudence in a laboratory setting.

We construct a set of 16 lottery pairs to test for prudence (Eeckhoudt and Schlesinger (2006)) that not only reflect skewness seeking, but also the kurtosis robustness feature of prudence. As shown in Chapter 3, the latter is also a characteristic feature of prudence. Its origin lies in the skewness of the zero-mean risks and we show how to implement such risks in the experiment. To this end, we propose a new ballot box representation of compound lotteries for application in experiments. It is very easy to understand and translates naturally from subjects' decision screens to the real-world draw of the lotteries.

In the experiment, indeed, the choice of the zero-mean risk significantly affects subjects' decisions. Thus, we find that prudence does not boil down to skewness seeking. This is also evidenced by testing for skewness seeking directly using the lotteries of Mao (1970) which, as we show, only differ in their skewness. Consistently with theory, most subjects we diagnose as prudent are skewness seeking, but not vice versa.

Prudence is observed on the aggregate as well as at the individual level. 65% of responses are prudent and we classify 47% of individuals as prudent and 8% as imprudent. The number of prudent responses varies substantially from 50% to 75% for different prudence lottery pairs. This should be taken into account when testing for prudence.

Given the observed presence of prudence, further experimental research could focus on the empirical validation of prudent behavior. For example, the probably most famous prediction that prudent people exhibit larger precautionary saving has received little attention yet. Moreover, the method proposed in this chapter could be easily adapted to test for temperance and associated theories.

In our opinion, many more experiments should be conducted in order to validate, consolidate, and improve experimental methodology to measure higher-order risk preferences. As indicated in the introduction to this thesis, the robust measurement of risk aversion has been a major topic in economic research for decades. And the problem is far from being solved.

Note that our experiment that aims to test for prudence as well as that of Deck and Schlesinger (2010) test for the *direction* of preference, i.e., they are based on "yes-or-no questions." In Ebert and Wiesen (2010), we propose a method to *measure* the *intensity* of higher-order risk preferences and employ it in an experiment. That is, we measure compensating risk premia (see, e.g., Kimball (1990) or Pope and Chavas (1985)) that individuals demand for making the "risky" choice. This is not only done for prudence, but also for risk aversion and temperance. Further, we propose consistent framing of methods to measure these traits jointly. This methodology builds upon and extends the one presented in this last chapter of this treatise. We hope that it can serve as an attachment to other experiments on economic behavior in order to investigate the relationship to risk attitudes in a refined way, i.e., in a way that clearly distinguishes risk attitudes from risk aversion.

Appendix A

Appendices to Chapters 1, 3, and 4

A.1 Appendix to Chapter 1

Proof of Proposition 1. In the situation of a single partial hedge, the portfolio loss $L_{1,N-1}$ is given by equation (1.3.1). The conditional expectation of the loss ratio of the composite instrument $\hat{U}_1$ is given by

$$\hat{\mu}_1(x) = \mathbb{E}[\hat{U}_1|x] = \mathrm{ELGD}_1\,\mathrm{ELGD}_2\,\mathrm{PD}_1(x)\,\mathrm{PD}_2(x) = \mu_1(x)\mu_2(x). \tag{A.1.1}$$

Equation (1.3.1) and equation (A.1.1) imply that the conditional mean of the portfolio loss is

$$\mu_{1,N-1}(x) = \mu_{0,N-1}(x) + \lambda s_1 \hat{\mu}_1(x) + (1-\lambda)s_1\mu_1(x). \tag{A.1.2}$$

Taking the derivative yields

$$\mu'_{1,N-1}(x) = \mu'_{0,N-1}(x) + \lambda s_1\left(\mu'_1(x)\mu_2(x) + \mu_1(x)\mu'_2(x)\right) + (1-\lambda)s_1\mu'_1(x)$$

and for the second derivative we obtain

$$\mu''_{1,N-1}(x) \quad = \quad 2\lambda s_1 \mu'_1(x)\mu'_2(x)$$

since the second derivative of $\mu_n(x)$ vanishes for any $n = 1, \ldots, N$. Using the CreditRisk$^+$ notation of Section 1.3, the conditional expectation of the portfolio loss ratio and its derivatives can be expressed as

$$\begin{aligned}
\mu_{1,N-1}(x_q) &= \mu_{0,N-1}(x_q) + s_1\lambda(\hat{\mathcal{K}}_1 + \hat{\mathcal{R}}_1) + s_1(1-\lambda)(\mathcal{K}_1 + \mathcal{R}_1) \\
\mu'_{1,N-1}(x_q) &= \frac{\mathcal{K}_{1,N-1}(\lambda)}{x_q - 1} \\
\mu''_{1,N-1}(x_q) &= \frac{2s_1\lambda}{(x_q-1)^2}\mathcal{K}_1\mathcal{K}_2
\end{aligned} \tag{A.1.3}$$

where $\mathcal{K}_{1,N-1}(\lambda)$ is defined in Proposition 1. Hence, it remains to compute the conditional variance of the portfolio loss and its derivative. For the conditional variance of the portfolio loss ratio, we obtain

$$\begin{aligned}
& \mathbb{V}[s_2U_2 + \lambda s_1\hat{U}_1 + (1-\lambda)s_1U_1|x] \\
= \; & \mathbb{V}[s_2U_2 + (1-\lambda)s_1U_1|x] + \mathbb{V}[\lambda s_1\hat{U}_1|x] + 2\,\mathrm{Cov}[s_2U_2 + (1-\lambda)s_1U_1, \lambda s_1\hat{U}_1|x]
\end{aligned} \tag{A.1.4}$$

and the last term can be written as

$$2s_1s_2\lambda\,\mathrm{Cov}[U_2, U_1U_2|x] + 2s_1^2\lambda(1-\lambda)\,\mathrm{Cov}[U_1, U_1U_2|x].$$

Since U_1 and U_2 are conditionally independent one can show that

$$\mathrm{Cov}[U_2, U_1U_2|x] = \mu_1(x)\sigma_2^2(x) \quad \text{and} \quad \mathrm{Cov}[U_1, U_1U_2|x] = \mu_2(x)\sigma_1^2(x). \tag{A.1.5}$$

Recall that, for independent random variables Y_1 and Y_2, the following relation holds

$$\mathbb{V}[Y_1Y_2] = \mathbb{V}[Y_1]\mathbb{V}[Y_2] + \mathbb{V}[Y_1]\mathbb{E}[Y_2]^2 + \mathbb{V}[Y_2]\mathbb{E}[Y_1]^2. \tag{A.1.6}$$

Using these results, equation (A.1.4) can be written as

$$\begin{aligned}
& \mathbb{V}[s_2U_2 + s_1\lambda U_1U_2 + s_1(1-\lambda)U_1|x] \\
= \; & s_2\sigma_2^2(x) + s_1^2(1-\lambda)^2\sigma_1^2(x) + s_1^2\lambda^2\left(\sigma_1^2(x)\sigma_2^2(x) + \sigma_1^2(x)\mu_2^2(x) + \sigma_2^2(x)\mu_1^2(x)\right) \\
& +2s_1s_2\lambda\mu_1(x)\sigma_2^2(x) + 2s_1^2\lambda(1-\lambda)\mu_2(x)\sigma_1^2(x)
\end{aligned}$$

and therefore the conditional variance of the portfolio loss ratio is

$$
\begin{aligned}
\sigma^2_{1,N-1}(x) &= \sigma^2_{0,N-1}(x) + s_1^2(1-\lambda)^2\sigma_1^2(x) \\
&\quad +2s_1s_2\lambda\mu_1(x)\sigma_2^2(x) + 2s_1^2\lambda(1-\lambda)\mu_2(x)\sigma_1^2(x) \\
&\quad +s_1^2\lambda^2\left(\sigma_1^2(x)\sigma_2^2(x)+\sigma_1^2(x)\mu_2^2(x)+\sigma_2^2(x)\mu_1^2(x)\right).
\end{aligned}
\tag{A.1.7}
$$

Evaluating at x_q and inserting equations (1.2.11) and (1.2.12) gives an expression in $\mathcal{K}_n$ and $\mathcal{R}_n$. These quantities are typically quite small so that products of these contribute little to the GA.[1] As double default effects will be second-order effects, i.e., of order $\mathcal{O}(1/N^2)$ as discussed in Remark 3, throughout this chapter we will neglect third- and higher-order terms in $\mathcal{K}_n$ and $\mathcal{R}_n$. For this argument, note that, due to relations (1.2.11) and (1.2.12), the terms $\mu_n(x_q)$ and $\sigma_n^2(x_q)$ and their derivatives are all of order 1 in $\mathcal{K}_n$ and $\mathcal{R}_n$. Moreover, if an expression for the conditional variance of the loss ratio involves a product of three or more of these terms, it will also yield products of three or more of these terms in the derivative. Finally, when computing the GA using formula (1.2.6), third- and higher-order terms in $\mathcal{K}_n$ and $\mathcal{R}_n$ can never turn into more significant lower order terms. This is obvious from derivations that follow. Therefore, in the following we will always compute the expressions for the conditional variance of the portfolio loss and its derivative without third- and higher-order terms in $\mathcal{K}_n$ and $\mathcal{R}_n$ since these terms are of order $\mathcal{O}(1/N^2\cdot \mathrm{PD}^3\cdot \mathrm{ELGD}^3)$ or even higher, and thus would yield negligible terms anyway. Thus, with these simplifications, we obtain

$$
\begin{aligned}
\sigma^2_{1,N-1}(x_q) &\approx \sigma^2_{0,N-1}(x_q) + s_1^2(1-\lambda)^2\left(\mathcal{C}_1(\mathcal{K}_1+\mathcal{R}_1)+(\mathcal{K}_1+\mathcal{R}_1)^2\frac{\mathrm{VLGD}_1^2}{\mathrm{ELGD}_1^2}\right) \\
&\quad +2s_1s_2\lambda\mathcal{C}_2(\hat{\mathcal{K}}_1+\hat{\mathcal{R}}_1)+s_1^2\left[\lambda^2\mathcal{C}_1\mathcal{C}_2+2\lambda(1-\lambda)\mathcal{C}_1\right](\hat{\mathcal{K}}_1+\hat{\mathcal{R}}_1) \\
\tfrac{d}{dx}\sigma^2_{1,N-1}(x_q) &\approx \frac{d}{dx}\sigma^2_{0,N-1}(x_q)+\frac{s_1^2(1-\lambda)^2}{x_q-1}\left(\mathcal{C}_1\mathcal{K}_1+2\mathcal{K}_1(\mathcal{K}_1+\mathcal{R}_1)\frac{\mathrm{VLGD}_1^2}{\mathrm{ELGD}_1^2}\right) \\
&\quad +\frac{2s_1s_2\lambda\mathcal{C}_2}{x_q-1}\left(\mathcal{K}_1(\mathcal{K}_2+\mathcal{R}_2)+\mathcal{K}_2(\mathcal{K}_1+\mathcal{R}_1)\right) \\
&\quad +\frac{s_1^2\left[\lambda^2\mathcal{C}_1\mathcal{C}_2+2\lambda(1-\lambda)\mathcal{C}_1\right]}{x_q-1}\left(\mathcal{K}_1(\mathcal{K}_2+\mathcal{R}_2)+\mathcal{K}_2(\mathcal{K}_1+\mathcal{R}_1)\right).
\end{aligned}
$$

We define the variance of the unhedged part of the portfolio as

$$
\bar{\sigma}^2_{0,N}(x_q) := \sigma^2_{0,N-1}(x_q)+s_1^2(1-\lambda)^2\left[\mathcal{C}_1(\mathcal{K}_1+\mathcal{R}_1)+(\mathcal{K}_1+\mathcal{R}_1)^2\frac{\mathrm{VLGD}_1^2}{\mathrm{ELGD}_1^2}\right]. \tag{A.1.8}
$$

[1] $\mathcal{K}_n$ and $\mathcal{R}_n$ are essentially products of $\mathrm{PD}_n\in[0,1]$ and $\mathrm{ELGD}_n\in[0,1]$.

Applying further the notation of $\hat{\mathcal{C}}_1(\lambda)$ in Proposition 1, we can reformulate the conditional variance of the portfolio loss and its derivative at x_q as

$$\begin{aligned}\sigma^2_{1,N-1}(x_q) &\approx \bar{\sigma}^2_{0,N}(x_q) + s_1^2\hat{\mathcal{C}}_1(\lambda)(\hat{\mathcal{K}}_1+\hat{\mathcal{R}}_1) + 2s_1s_2\lambda\mathcal{C}_2(\hat{\mathcal{K}}_1+\hat{\mathcal{R}}_1)\\ \frac{d}{dx}\sigma^2_{1,N-1}(x_q) &\approx \frac{d}{dx}\bar{\sigma}^2_{0,N}(x_q) + \frac{s_1^2\hat{\mathcal{C}}_1(\lambda)+2s_1s_2\lambda\mathcal{C}_2}{x_q-1}\left(\mathcal{K}_1(\mathcal{K}_2+\mathcal{R}_2)+\mathcal{K}_2(\mathcal{K}_1+\mathcal{R}_1)\right).\end{aligned} \tag{A.1.9}$$

We now use these representations to compute the GA in the case of one hedged position. Therefore, first note that the formula for the "full" GA, equation (1.2.6), can be reformulated as

$$\begin{aligned}GA_{1,N-1} &= \frac{1}{2\mathcal{K}_{1,N-1}(\lambda)}\Bigg(\delta\sigma^2_{1,N-1}(x_q) - (x_q-1)\frac{d}{dx}\sigma^2_{1,N-1}(x_q)\\ &\qquad +(x_q-1)\frac{\sigma^2_{1,N-1}(x_q)\mu''_{1,N-1}(x_q)}{\mu'_{1,N-1}(x_q)}\Bigg).\end{aligned} \tag{A.1.10}$$

Rearranging and using the simplified expressions for the conditional variance and its derivative, equation (A.1.9), this becomes

$$\begin{aligned}\widetilde{GA}_{1,N-1} &= \frac{1}{2\mathcal{K}_{1,N-1}(\lambda)}\left(\delta\bar{\sigma}^2_{0,N}(x_q) - (x_q-1)\frac{d}{dx}\bar{\sigma}^2_{0,N}(x_q)\right)\\ &+ \frac{\left(s_1^2\hat{\mathcal{C}}_1(\lambda)+2s_1s_2\lambda\mathcal{C}_2\right)}{2\mathcal{K}_{1,N-1}(\lambda)}\left(\delta(\hat{\mathcal{K}}_1+\hat{\mathcal{R}}_1) - (\mathcal{K}_1(\mathcal{K}_2+\mathcal{R}_2)+\mathcal{K}_2(\mathcal{K}_1+\mathcal{R}_1))\right)\\ &+ \frac{1}{2\mathcal{K}_{1,N-1}(\lambda)}\left((x_q-1)\frac{\sigma^2_{1,N-1}(x_q)\mu''_{1,N-1}(x_q)}{\mu'_{1,N-1}(x_q)}\right).\end{aligned} \tag{A.1.11}$$

Unlike in the case without hedging, the last summand of equation (A.1.11) does not vanish since

$$\mu''_{1,N-1}(x_q) = 2\lambda s_1\mu'_1(x_q)\mu'_2(x_q) = 2\lambda s_1\mathcal{K}_1\mathcal{K}_2/(x_q-1)^2$$

is in general not zero. We have

$$\begin{aligned}&\frac{\sigma^2_{1,N-1}(x_q)\mu''_{1,N-1}(x_q)}{\mu'_{1,N-1}(x_q)}\\ =\;& \frac{2\lambda s_1\mathcal{K}_1\mathcal{K}_2}{\mathcal{K}_{1,N-1}(\lambda)(x_q-1)}\left(\bar{\sigma}^2_{0,N}(x_q) + s_1^2\hat{\mathcal{C}}_1(\lambda)(\hat{\mathcal{K}}_1+\hat{\mathcal{R}}_1) + 2s_1s_2\lambda\mathcal{C}_2(\hat{\mathcal{K}}_1+\hat{\mathcal{R}}_1)\right).\end{aligned}$$

The last two summands are very small and can be neglected.[2] Using this result, inserting the GA formula for the portfolio with $N-1$ ordinary obligors, equation (1.2.15), and using

[2] The expression $\mathcal{K}_n/\mathcal{K}_{1,N-1}$ should be reasonably close to 1 so that the neglected terms are of order $\mathcal{O}(1/N^3)$.

the notation $\overline{GA}_{0,N}$, we obtain from equation (A.1.11) the GA formula of Proposition 1. $\square$

Proof of Proposition 2. We start with the situation where two different guarantors hedge two different obligors. Therefore, we consider a portfolio with two partially hedged obligors (1 and 2) and $N-2$ ordinary obligors $(3,\dots,N)$ where $g_1 \neq g_2$. The portfolio loss is then given by equation (1.3.7). Similarly to equation (A.1.3) we obtain for the conditional expectation of the portfolio loss and its derivatives

$$\begin{aligned}
\mu_{2,N-2}(x_q) &= \mu_{0,N-2}(x_q) + s_1\lambda_1(\hat{\mathcal{K}}_1 + \hat{\mathcal{R}}_1) + s_1(1-\lambda_1)(\mathcal{K}_1 + \mathcal{R}_1) \\
&\qquad + s_2\lambda_2(\hat{\mathcal{K}}_2 + \hat{\mathcal{R}}_2) + s_2(1-\lambda_2)(\mathcal{K}_2 + \mathcal{R}_2) \\
\mu'_{2,N-2}(x_q) &= \frac{\mathcal{K}_{2,N-2}(\lambda)}{x_q - 1} \\
\mu''_{2,N-2}(x_q) &= \frac{2}{(x_q-1)^2}\left(s_1\lambda_1\mathcal{K}_1\mathcal{K}_{g_1} + s_2\lambda_2\mathcal{K}_2\mathcal{K}_{g_2}\right).
\end{aligned} \tag{A.1.12}$$

Note that, in the equation for the portfolio loss, terms referring to the hedged obligor 1 are conditionally independent to those referring to the hedged obligor 2. This is why we can compute the contributions to the variance of the portfolio loss separately for obligor 1 and obligor 2. Each component is obtained as in the proof of Proposition 1. Thus, for the conditional variance of the portfolio loss ratio and its derivative, we obtain the natural extensions of equation (A.1.9), namely

$$\begin{aligned}
\sigma^2_{2,N-2}(x_q) &\approx \bar{\sigma}^2_{0,N}(x_q) + s_1^2\hat{\mathcal{C}}_1(\lambda_1)(\hat{\mathcal{K}}_1 + \hat{\mathcal{R}}_1) + 2s_1 s_{g_1}\lambda_1\mathcal{C}_{g_1}(\hat{\mathcal{K}}_1 + \hat{\mathcal{R}}_1) \\
&\qquad + s_2^2\hat{\mathcal{C}}_2(\lambda_2)(\hat{\mathcal{K}}_2 + \hat{\mathcal{R}}_2) + 2s_2 s_{g_1 2}\lambda_2\mathcal{C}_{g_2}(\hat{\mathcal{K}}_2 + \hat{\mathcal{R}}_2) \\
\frac{d}{dx}\sigma^2_{2,N-2}(x_q) &\approx \frac{d}{dx}\bar{\sigma}^2_{0,N}(x_q) \\
&\quad + \frac{s_1^2\hat{\mathcal{C}}_1(\lambda_1) + 2s_1 s_{g_1}\lambda_1\mathcal{C}_{g_1}}{x_q - 1}(\mathcal{K}_1(\mathcal{K}_{g_1} + \mathcal{R}_{g_1}) + \mathcal{K}_{g_1}(\mathcal{K}_1 + \mathcal{R}_1)) \\
&\quad + \frac{s_2^2\hat{\mathcal{C}}_2(\lambda_2) + 2s_2 s_{g_2}\lambda_2\mathcal{C}_{g_2}}{x_q - 1}(\mathcal{K}_2(\mathcal{K}_{g_2} + \mathcal{R}_{g_2}) + \mathcal{K}_{g_2}(\mathcal{K}_2 + \mathcal{R}_2)).
\end{aligned}$$

Here we naturally extended the definition (A.1.8) of $\bar{\sigma}^2_{0,N}(x)$ to the case with two guarantees. Thus, in case of two partially hedged positions, the equivalent to equation (1.3.2) is given by equation (1.3.9), the result of Proposition 2.

Now consider the case where one guarantor hedges two different obligors. Similarly to the previous case, we consider a portfolio with two hedged obligors (1 and 2) and $N-2$ ordinary obligors $(3, 4, \ldots, N)$. However, the obligors now have the same guarantor $g_1 = g_2 = 3$. Then, the portfolio loss is given by equation (1.3.8). It is obvious that the conditional expectation of the portfolio loss and its derivatives are also given by equation (A.1.12), where terms referring to the composite instrument of course have to be adjusted to the current situation. The conditional variance of the portfolio loss can be written as

$$\begin{aligned}
&\mathbb{V}[L_{2,N-2}|x] \\
= \; & \mathbb{V}[L_{0,N-3}|x] + \mathbb{V}[s_1(1-\lambda_1)U_1 + s_2(1-\lambda_2)U_2|x] \\
&+\mathbb{V}\left[s_{g_1}U_{g_1} + s_1\lambda_1 U_1 U_{g_1} + s_2\lambda_2 U_2 U_{g_1}|x\right] \\
&+2\operatorname{Cov}\Big[s_{g_1}U_{g_1} + s_1\lambda_1 U_1 U_{g_1} + s_2\lambda_2 U_2 U_{g_1}, s_1(1-\lambda_1)U_1 + s_2(1-\lambda_2)U_2|x\Big].
\end{aligned} \tag{A.1.13}$$

We can compute the individual terms further using the same technique as in the case of a single partial hedge. Applying formula (A.1.5) then reduces the covariance term to

$$\begin{aligned}
&2\operatorname{Cov}\left[s_{g_1}U_{g_1} + s_1\lambda_1 U_1 U_{g_1} + s_2\lambda_2 U_2 U_{g_1}, s_1(1-\lambda_1)U_1 + s_2(1-\lambda_2)U_2|x\right] \\
= \; & 2s_1^2\lambda_1(1-\lambda_1)\sigma_1^2(x)\mu_{g_1}(x) + 2s_2^2\lambda_2(1-\lambda_2)\sigma_2^2(x)\mu_{g_1}(x),
\end{aligned}$$

and the second variance term equals

$$\mathbb{V}[s_1(1-\lambda_1)U_1 + s_2(1-\lambda_2)U_2|x] = s_1^2(1-\lambda_1)^2\sigma_1^2(x) + s_2^2(1-\lambda_2)^2\sigma_2^2(x).$$

The third variance term in equation (A.1.13) can be computed using formula (A.1.6). Again, neglecting higher-order terms in capital contributions, one can show that

$$\begin{aligned}
&\mathbb{V}[U_{g_1}\left(s_{g_1} + s_1\lambda_1 U_1 + s_2\lambda_2 U_2\right)|x_q] \\
= \; & \sigma_{g_1}^2(x_q)\left(\lambda_1^2 s_1^2\sigma_1^2(x_q) + \lambda_2^2 s_2^2\sigma_2^2(x_q) + 2s_{g_1}\lambda_1 s_1\mu_1(x_q) + 2s_{g_1}\lambda_2 s_2\mu_1(x_q) + s_{g_1}^2\right).
\end{aligned}$$

Then, the conditional variance of the portfolio loss can be expressed as

$$\begin{aligned}
\sigma_{2,N-2}^2(x_q) \;=\; & \bar{\sigma}_{0,N}^2(x_q) \\
&+\mu_{g_1}(x_q)\mu_1(x_q)\left(\lambda_1^2 s_1^2\mathcal{C}_1\mathcal{C}_{g_1} + 2s_{g_1}\lambda_1 s_1\mathcal{C}_{g_1} + 2s_1^2\lambda_1(1-\lambda_1)\mathcal{C}_1\right) \\
&+\mu_{g_1}(x_q)\mu_2(x_q)\left(\lambda_2^2 s_2^2\mathcal{C}_{g_1}\mathcal{C}_2 + 2s_{g_1}\lambda_2 s_2\mathcal{C}_{g_1} + 2s_2^2\lambda_2(1-\lambda_2)\mathcal{C}_2\right).
\end{aligned}$$

Inserting the definition (1.3.6) for $\hat{\mathcal{C}}_n(\lambda_n)$ and for the EL and UL capital $\hat{\mathcal{R}}_n$ and $\hat{\mathcal{K}}_n$, respectively, we obtain

$$\begin{aligned}\sigma^2_{2,N-2}(x_q) &= \bar{\sigma}^2_{0,N}(x_q) + s_1^2\hat{\mathcal{C}}_1(\lambda_1)(\hat{\mathcal{K}}_1 + \hat{\mathcal{R}}_1) + 2s_{g_1}s_1\lambda_1\mathcal{C}_{g_1}(\hat{\mathcal{K}}_1 + \hat{\mathcal{R}}_1) \\ &\quad + s_2^2\hat{\mathcal{C}}_2(\lambda_2)(\hat{\mathcal{K}}_2 + \hat{\mathcal{R}}_2) + 2s_{g_1}s_2\lambda_2\mathcal{C}_{g_1}(\hat{\mathcal{K}}_2 + \hat{\mathcal{R}}_2)\end{aligned} \tag{A.1.14}$$

which coincides with the expression for $\sigma^2_{2,N-2}(x_q)$ in the previous case. That is, if higher-order terms in EL and UL capital contributions are neglected, the expressions for $\mu_{2,N-2}(x_q)$ and $\sigma^2_{2,N-2}(x_q)$ and their derivatives do not depend on whether both obligors have different guarantors or the same guarantor. Obviously, the formula for the GA also has to be the same as in the case with different guarantors. Thus, it is given by equation (1.3.9). □

Proof of Theorem 1. The generalization to the case of several guarantees uses the same techniques as the proof of Proposition 2 since no further interactions will appear. We omit the proof here because the computations become rather tedious and do not provide any additional insight. □

A.2 Appendix to Chapter 3

The following lemma is proven in Roger (2011) and will be used several times in our proofs.

Lemma 2 (Roger's Lemma)**.** *Let X be Bernoulli-distributed with parameter* 0.5 *and be independent from Y_1 and Y_2. Then:*

$$\mathbb{E}\left[(XY_1 + (1-X)Y_2)^p\right] = \frac{1}{2}\left(\mathbb{E}\left[Y_1^p\right] + \mathbb{E}\left[Y_2^p\right]\right).$$

We also will make frequent use of the following lemma.

Lemma 3. *There exists a binary zero-mean risk with all odd moments of order three and higher being positive (negative, zero), referred to as a right-skewed (left-skewed, symmetric) zero-mean risk.*

Proof of Lemma 3. The result is an immediate consequence of Theorems 3 and 4 proven in Appendix B. □

Proof of Proposition 3. We first define auxiliary lotteries

$$\begin{aligned}\hat{A}_3 &:= A_3 + \frac{k}{2} = X \cdot \frac{k}{2} + (1-X)\left(-\frac{k}{2} + \epsilon_1\right)\\ \hat{B}_3 &:= B_3 + \frac{k}{2} = X\left(-\frac{k}{2}\right) + (1-X)\left(\epsilon_1 + \frac{k}{2}\right).\end{aligned}$$

These lotteries can be understood as the prudence lotteries shifted such that they have mean zero. Because the operator $\mathbb{M}_p(\cdot)$ is translation invariant we have

$$\mathbb{M}_p(A_3) = \mathbb{M}_p(\hat{A}_3) = \mathbb{E}[\hat{A}_3^p], \tag{A.2.1}$$

which analogously holds for B_3. Thus, it suffices to focus on the computation of the non-central moments $\mathbb{E}[\hat{A}_3^p]$ and $\mathbb{E}[\hat{B}_3^p]$. In the second equality below, we apply Roger's Lemma and obtain

$$\begin{aligned}\mathbb{M}_p(A_3) &= \mathbb{E}\left[\left\{X \cdot \frac{k}{2} + (1-X)\left(\epsilon_1 - \frac{k}{2}\right)\right\}^p\right]\\ &= \frac{1}{2}\mathbb{E}\left[\left(\epsilon_1 + \left(-\frac{k}{2}\right)\right)^p\right] + \frac{1}{2}\left(\frac{k}{2}\right)^p\\ &= \frac{1}{2}\mathbb{E}\left[\sum_{j=0}^{p}\binom{p}{j}\epsilon_1^j\left(-\frac{k}{2}\right)^{p-j}\right] + \frac{1}{2}\left(\frac{k}{2}\right)^p\\ &= \frac{1}{2}\sum_{j=2}^{p}\binom{p}{j}\left(-\frac{k}{2}\right)^{p-j}\mathbb{E}\left[\epsilon_1^j\right] + \frac{1}{2}\left(\left(-\frac{k}{2}\right)^p + \left(\frac{k}{2}\right)^p\right),\end{aligned} \tag{A.2.2}$$

where we used that the summand for $j=1$ is zero since $\mathbb{E}[\epsilon_1] = 0$. This argument will be used several times in the proofs of this appendix. Similarly, for B_3 we get

$$\mathbb{M}_p(B_3) = \frac{1}{2}\sum_{j=2}^{p}\binom{p}{j}\left(\frac{k}{2}\right)^{p-j}\mathbb{E}\left[\epsilon_1^j\right] + \frac{1}{2}\left(\left(-\frac{k}{2}\right)^p + \left(\frac{k}{2}\right)^p\right). \tag{A.2.3}$$

To prove (1) and (2), if p is odd we have

$$\begin{aligned}\mathbb{M}_p(A_3) &= \frac{1}{2}\sum_{j=2}^{p}\binom{p}{j}\mathbb{E}\left[\epsilon_1^j\right]\left(-\frac{k}{2}\right)^{p-j} + \frac{1}{2}\left(-\left(\frac{k}{2}\right)^p + \left(\frac{k}{2}\right)^p\right)\\ &= \frac{1}{2}\sum_{j=2}^{p}\binom{p}{j}\mathbb{E}\left[\epsilon_1^j\right]\left(-\frac{k}{2}\right)^{p-j}\end{aligned}$$

and analogously

$$\mathbb{M}_p(B_3) = \frac{1}{2}\sum_{j=2}^{p}\binom{p}{j}\mathbb{E}\left[\epsilon_1^j\right]\left(\frac{k}{2}\right)^{p-j}.$$

If p is even

$$\begin{aligned}\mathbb{M}_p(A_3) &= \frac{1}{2}\sum_{j=2}^{p}\binom{p}{j}\mathbb{E}\left[\epsilon_1^j\right]\left(-\frac{k}{2}\right)^{p-j}+\frac{1}{2}\left(\left(\frac{k}{2}\right)^p+\left(\frac{k}{2}\right)^p\right)\\ &= \left(\frac{k}{2}\right)^p+\frac{1}{2}\sum_{j=2}^{p}\binom{p}{j}\mathbb{E}\left[\epsilon_1^j\right]\left(-\frac{k}{2}\right)^{p-j}\end{aligned}$$

and analogously

$$\mathbb{M}_p(B_3) = \left(\frac{k}{2}\right)^p+\frac{1}{2}\sum_{j=2}^{p}\binom{p}{j}\mathbb{E}\left[\epsilon_1^j\right]\left(\frac{k}{2}\right)^{p-j}.$$

Proof of part (3). For odd p, using the expressions proven in (1) and (2), the difference can be computed as

$$\begin{aligned}\mathbb{M}_p(B_3)-\mathbb{M}_p(A_3) &= \frac{1}{2}\sum_{j=2}^{p}\binom{p}{j}\mathbb{E}\left[\epsilon_1^j\right]\underbrace{\left(\left(\frac{k}{2}\right)^{p-j}-\left(-\frac{k}{2}\right)^{p-j}\right)}_{=\begin{cases}2\left(\frac{k}{2}\right)^{p-j} & ,\ p-j \text{ odd} \Leftrightarrow j \text{ even}\\ 0 & ,\ \text{o.w.}\end{cases}}\\ &= \sum_{j=2,\ j\text{ even}}^{p}\binom{p}{j}\mathbb{E}\left[\epsilon_1^j\right]\left(\frac{k}{2}\right)^{p-j}.\end{aligned}$$

Similarly, for even p we obtain

$$\begin{aligned}\mathbb{M}_p(B_3)-\mathbb{M}_p(A_3) &= \frac{1}{2}\sum_{j=2}^{p}\binom{p}{j}\mathbb{E}\left[\epsilon_1^j\right]\underbrace{\left(\left(\frac{k}{2}\right)^{p-j}-\left(-\frac{k}{2}\right)^{p-j}\right)}_{=\begin{cases}2\left(\frac{k}{2}\right)^{p-j} & ,\ p-j \text{ odd} \Leftrightarrow j \text{ odd}\\ 0 & ,\ \text{o.w.}\end{cases}}\\ &= \sum_{j=2,\ j\text{ odd}}^{p}\binom{p}{j}\mathbb{E}\left[\epsilon_1^j\right]\left(\frac{k}{2}\right)^{p-j}.\end{aligned}$$

Summarizing these results proves part (3). For the claims on the sign of $\mathbb{M}_p(B_3)-\mathbb{M}_p(A_3)$, consider the cases where ϵ_1 is one of the three risks that exist according to Lemma 3. For these three risks, respectively, all summands in the expression of part (3) are throughout positive, negative, and zero. Thus, for p even, the whole expression in part (3) can be positive, negative, or zero. For p odd, all summands in the expressions must be strictly positive because, by definition, $\mathbb{M}_j(\epsilon^j) > 0$ for j even. $\square$

Proof of Proposition 4. By application of Roger's Lemma, we have

$$\begin{aligned}\mathbb{M}_p(A_4) &= \mathbb{E}\left[(1-X)^p(\epsilon_1+\epsilon_2)^p\right] = \frac{1}{2}\mathbb{E}\left[(\epsilon_1+\epsilon_2)^p\right] \\ &= \frac{1}{2}\sum_{j=0}^{p}\binom{p}{j}\mathbb{E}\left[\epsilon_1^j\right]\mathbb{E}\left[\epsilon_2^{p-j}\right]\end{aligned}$$

and

$$\mathbb{M}_p(B_4) = \frac{1}{2}\left(\mathbb{E}\left[\epsilon_2^p\right]+\mathbb{E}\left[\epsilon_1^p\right]\right).$$

Claim (3) follows immediately by substraction. For the last statement, consider the products $\mathbb{E}\left[\epsilon_1^j\right]\cdot\mathbb{E}\left[\epsilon_2^{p-j}\right]$ and risks as in Lemma 3. Obviously, if both zero-mean risks are right-skewed, then all these products are positive such that $\mathbb{M}_p(B_4)-\mathbb{M}_p(A_4)<0$. If both zero-mean risks are symmetric, we have that the difference is zero (as shown by Roger). Finally, as p is odd, $p-j$ is odd if and only if j is even. Thus, if both zero-mean risks are left-skewed, we have that $\mathbb{M}_p(B_4)-\mathbb{M}_p(A_4)>0$. □

Proof of Lemma 1. First, let n be even. By Roger's Lemma, we have

$$\begin{aligned}\mathbb{M}_p(A_n) = \mathbb{E}\left[A_n^p\right] &= \mathbb{E}\left[\left(XB_{n-2}+(1-X)\left(A_{n-2}+\epsilon_{\llcorner n/2\lrcorner}\right)\right)^p\right] \\ &= \frac{1}{2}\left(\mathbb{E}\left[B_{n-2}^p\right]+\mathbb{E}\left[\left(A_{n-2}+\epsilon_{\llcorner n/2\lrcorner}\right)^p\right]\right) \\ &= \frac{1}{2}\left(\mathbb{E}\left[B_{n-2}^p\right]+\mathbb{E}\left[A_{n-2}^p\right]+\sum_{j=2}^{p}\binom{p}{j}\mathbb{E}\left[\epsilon_{\llcorner n/2\lrcorner}^j\right]\mathbb{E}\left[A_{n-2}^{p-j}\right]\right) \\ &= \frac{1}{2}\left(\mathbb{E}\left[B_{n-2}^p\right]+\mathbb{E}\left[A_{n-2}^p\right]+\sum_{j=2}^{p}\binom{p}{j}\mathbb{E}\left[\epsilon_{\llcorner n/2\lrcorner}^j\right]\mathbb{M}_{p-j}(A_{n-2})\right) \quad \text{(A.2.4)}\end{aligned}$$

and similarly

$$\begin{aligned}\mathbb{M}_p(B_n) &= \frac{1}{2}\left(\mathbb{E}\left[A_{n-2}^p\right]+\mathbb{E}\left[\left(B_{n-2}+\epsilon_{\llcorner n/2\lrcorner}\right)^p\right]\right) \\ &= \frac{1}{2}\left(\mathbb{E}\left[B_{n-2}^p\right]+\mathbb{E}\left[A_{n-2}^p\right]+\sum_{j=2}^{p}\binom{p}{j}\mathbb{E}\left[\epsilon_{\llcorner n/2\lrcorner}^j\right]\mathbb{M}_{p-j}(B_{n-2})\right). \quad \text{(A.2.5)}\end{aligned}$$

Thus, we get

$$\mathbb{M}_p(B_n)-\mathbb{M}_p(A_n) = \frac{1}{2}\left(\sum_{j=2}^{p}\binom{p}{j}\left[\epsilon_{\llcorner n/2\lrcorner}^j\right]\left(\mathbb{M}_{p-j}(B_{n-2})-\mathbb{M}_{p-j}(A_{n-2})\right)\right),$$

which is part (3) of Lemma 1. Now assume n is odd. Like in the proof of Proposition 3, define $\hat{A}_3 = A_3 + \frac{k}{2}$ and $\hat{B}_3 = B_3 + \frac{k}{2}$. For $n \geq 5$, we naturally extend this definition, i.e., let

$$\begin{aligned}\hat{A}_n &= X\hat{B}_{n-2} + (1-X)\left(\epsilon_{\llcorner n/2 \lrcorner} + \hat{A}_{n-2}\right)\\ \hat{B}_n &= X\hat{A}_{n-2} + (1-X)\left(\epsilon_{\llcorner n/2 \lrcorner} + \hat{B}_{n-2}\right).\end{aligned}$$

Then, like in the proof of Proposition 3, we have

$$\begin{aligned}\mathbb{M}_p\left(A_n\right) = \mathbb{M}_p\left(\hat{A}_n\right) = \mathbb{E}\left[\hat{A}_n^p\right] &= \frac{1}{2}\left(\mathbb{E}\left[\hat{B}_{n-2}^p\right] + \mathbb{E}\left[\left(\epsilon_{\llcorner n/2 \lrcorner} + \hat{A}_{n-2}\right)^p\right]\right)\\ &= \frac{1}{2}\left(\mathbb{E}\left[\hat{B}_{n-2}^p\right] + \sum_{j=0}^{p}\binom{p}{j}\mathbb{E}\left[\epsilon_{\llcorner n/2 \lrcorner}^j\right]\mathbb{E}\left[\hat{A}_{n-2}^{p-j}\right]\right)\\ &= \frac{1}{2}\left(\mathbb{E}\left[\hat{B}_{n-2}^p\right] + \mathbb{E}\left[\hat{A}_{n-2}^p\right] + \sum_{j=2}^{p}\binom{p}{j}\mathbb{E}\left[\epsilon_{\llcorner n/2 \lrcorner}^j\right]\mathbb{E}\left[\hat{A}_{n-2}^{p-j}\right]\right)\\ &= \frac{1}{2}\left(\mathbb{M}_p(B_{n-2}) + \mathbb{M}_p(A_{n-2}) + \sum_{j=2}^{p}\binom{p}{j}\mathbb{E}\left[\epsilon_{\llcorner n/2 \lrcorner}^j\right]\mathbb{M}_{p-j}(A_{n-2})\right) \qquad \text{(A.2.6)}\end{aligned}$$

and analogously

$$\mathbb{M}_p\left(B_n\right) = \frac{1}{2}\left(\mathbb{M}_p(A_{n-2}) + \mathbb{M}_p(B_{n-2}) + \sum_{j=2}^{p}\binom{p}{j}\mathbb{E}\left[\epsilon_{\llcorner n/2 \lrcorner}^j\right]\mathbb{M}_{p-j}(B_{n-2})\right). \qquad \text{(A.2.7)}$$

Equations (A.2.6) and (A.2.7), respectively, are identical to equations (A.2.4) and (A.2.5). Thus, the subtraction of equation (A.2.6) from equation (A.2.7) also yields the equation in part (3) of Lemma 1. □

Proof of Proposition 5. We prove part (1) by induction. For $n = 4$, we show that for $p = 1, 2, 3$ the summands in the equation of Proposition 4, part (3), are zero. Indeed, the only effective summand is for $p = 3$ which is zero because $\mathbb{E}[\epsilon_2^{3-2}] = 0$. Now assume the claim is true for $n - 2$ where n is even. Let $p < n$. For $j = 2, 3, ..., n$, we have $p - j < n - j \leq n - 2$ and thus $\mathbb{M}_{p-j}(B_{n-2}) - \mathbb{M}_{p-j}(A_{n-2}) = 0$ by the induction assumption. Then, the claim follows from repeated application of part (3) of Lemma 1. Also part (2) is proven by induction. For $n = 4$, the claim can easily be inferred from Proposition 4, part (3). Now assume the claim is true for $n - 2$ where n is even. Part (3) of Lemma 1 for $p = n$ is

$$\mathbb{M}_n(B_n) - \mathbb{M}_n(A_n) = \frac{1}{2}\left(\sum_{j=2}^{n}\binom{n}{j}\mathbb{E}[\epsilon_{\llcorner n/2 \lrcorner}^j]\left(\mathbb{M}_{n-j}(B_{n-2}) - \mathbb{M}_{n-j}(A_{n-2})\right)\right). \qquad \text{(A.2.8)}$$

For $j = 2$, we have $\mathbb{M}_{n-2}(B_{n-2}) - \mathbb{M}_{n-2}(A_{n-2}) > 0$ by the induction assumption, further $\mathbb{E}[\epsilon^2_{\llcorner n/2 \lrcorner}] > 0$, and thus this summand is strictly positive. For $j > 2$, all summands are zero by part (1) of this proposition, and the claim follows. To prove the last statement, suppose that $\epsilon_1, \epsilon_2, \ldots, \epsilon_{\llcorner n/2 \lrcorner - 1}$ are symmetric. Then, from Roger (2011), Proposition 3, we have that $\mathbb{M}_k(B_{n-2}) - \mathbb{M}_k(A_{n-2})$ is strictly positive for $k \geq n$ even and zero otherwise. We want to show that for $p > n$ odd $\mathbb{M}_p(B_n) - \mathbb{M}_p(A_n)$ can be positive, negative, or zero. In order to do this, we consider the summands in the equation of part (3) of Lemma 1 and start with those summands for which j is even. As p is odd, $p - j$ is odd and thus $\mathbb{M}_{p-j}(B_{n-2}) - \mathbb{M}_{p-j}(A_{n-2})$ is zero always. If j is odd, then $p - j$ is even and thus $\mathbb{M}_{p-j}(B_{n-2}) - \mathbb{M}_{p-j}(A_{n-2})$ is zero if $p - j < n - 2$ and strictly positive otherwise. Now, if $\epsilon_{\llcorner n/2 \lrcorner}$ is symmetric, i.e., $\mathbb{E}[\epsilon^j_{\llcorner n/2 \lrcorner}] = 0$ for all j odd, all summands are zero and we have (as proven by Roger) that $\mathbb{M}_p(B_n) - \mathbb{M}_p(A_n) = 0$. If $\epsilon_{\llcorner n/2 \lrcorner}$ is right-skewed and binary (see Lemma 3), then all summands are positive and, as $p > n$, at least one summand is strictly positive, such that $\mathbb{M}_p(B_n) - \mathbb{M}_p(A_n) > 0$. Similarly, if $\epsilon_{\llcorner n/2 \lrcorner}$ is left-skewed and binary, we obtain that $\mathbb{M}_p(B_n) - \mathbb{M}_p(A_n) < 0$. □

Proof of Proposition 6. By induction. For prudence, i.e., $n = 3$, both claims (1) and (2) could be verified using part (3) of Proposition 3. However, the results are also given in Crainich and Eeckhoudt (2008). Suppose the claim is true for $n-2$ where n is odd. For part (1), the induction assumption is that $p < n-2$ implies that $\mathbb{M}_p(B_{n-2}) - \mathbb{M}_p(A_{n-2}) = 0$. If $p < n$, then for $j = 2, 3, \ldots, p$ we have $p-j < n-j \leq n-2$. Thus, $\mathbb{M}_p(B_{n-2}) - \mathbb{M}_p(A_{n-2}) = 0$ for $j = 2, 3, \ldots, p$ such that each summand on the right hand side of the equation in part (3) of Lemma 1 is zero. This proves part (1) of the proposition. For part (2), the induction assumption is $\mathbb{M}_{n-2}(B_{n-2}) - \mathbb{M}_{n-2}(A_{n-2}) > 0$ for n odd. Consider equation (A.2.8) which likewise holds for n odd. Similarly to the proof of Proposition 5, the summand for $j = 2$ is strictly positive by the induction assumption and all other summands are zero by part (1) of this proposition, and the claim follows. To prove the last statement, suppose that $\epsilon_1, \epsilon_2, \ldots, \epsilon_{\llcorner n/2 \lrcorner - 1}$ are symmetric. Then, from Roger (2011), Proposition 4, we have that $\mathbb{M}_k(B_{n-2}) - \mathbb{M}_k(A_{n-2})$ is strictly positive for $k \geq n$ odd and zero otherwise. We want to show that for $p > n$ even $\mathbb{M}_p(B_n) - \mathbb{M}_p(A_n)$ can be positive, negative, or zero. In order to do this, we consider the summands in the equation in part (3) of Lemma 1 and start with those summands for which j is even. As p is even, $p - j$ is even and thus $\mathbb{M}_{p-j}(B_{n-2}) - \mathbb{M}_{p-j}(A_{n-2})$ is zero always. If j is odd, then $p - j$ is odd and thus $\mathbb{M}_{p-j}(B_{n-2}) - \mathbb{M}_{p-j}(A_{n-2})$ is zero if $p - j < n - 2$ and strictly positive otherwise. Now, if $\epsilon_{\llcorner n/2 \lrcorner}$ is symmetric, all summands are zero and we have (as proven by Roger) that

$\mathbb{M}_p(B_n) - \mathbb{M}_p(A_n) = 0$. If $\epsilon_{\llcorner n/2 \lrcorner}$ is right-skewed and binary (see Lemma 3), then all summands are positive and, as $p > n$, at least one summand is strictly positive such that $\mathbb{M}_p(B_n) - \mathbb{M}_p(A_n) > 0$. Similarly, if $\epsilon_{\llcorner n/2 \lrcorner}$ is left-skewed and binary, we obtain that $\mathbb{M}_p(B_n) - \mathbb{M}_p(A_n) < 0$. □

A.3 Appendix to Chapter 4

A.3.1 Proofs to Section 4.2

Proof of Proposition 8. We prove part (b) which is a generalization of part (a). Denote the lotteries by $M_A = L(p_X, x_1, x_0)$ and $M_B = L(p_Y, y_1, y_0)$. For a random variable Z and $n \in \mathbb{N}$, the nth non-standardized central moment is defined as $\mathbb{M}_n(Z) := \mathbb{E}[(Z - \mathbb{E}[Z])^n]$, and it is well known that the $\mathbb{M}_n(\cdot)$-operator is homogeneous of degree n and translation invariant. By definition $p_X = 1 - p_Y$, which implies that X is equal to $1 - Y$ in distribution. Therefore, $\mathbb{M}_n(X) = (-1)^n \mathbb{M}_n(Y)$ which for $n = 2$ implies that $\mathbb{V}(X) = \mathbb{V}(Y)$. Note that we can write $M_A = X \cdot x_1 + (1 - X) \cdot x_0 = (x_1 - x_0)X + x_0$ and thus $\mathbb{V}(M_A) = (x_1 - x_0)^2 \mathbb{V}(X)$. Analogously, we have $\mathbb{V}(M_B) = (y_1 - y_0)^2 \mathbb{V}(Y)$. Since the Mao lotteries have equal variances we obtain $(x_1 - x_0)^2 = (y_1 - y_0)^2$ and, because of the unique representation of binary lotteries (see Definition 1), this is equivalent to $x_1 - x_0 = y_1 - y_0$. Using once more homogeneity and translation invariance of the $\mathbb{M}_n(\cdot)$-operator and plugging in yields

$$\mathbb{M}_n(M_A) = (x_1 - x_0)^n \mathbb{M}_n(X) = ((y_1 - y_0))^n (-1)^n \mathbb{M}_n(Y) = (-1)^n \mathbb{M}_n(M_B). \quad \text{(A.3.1)}$$

Because the Mao lotteries have equal variances we also have that $\mathbb{M}_n^S(M_A) = (-1)^n \mathbb{M}_n^S(M_B)$. The claim for the even moments follows immediately. It is easy to check that M_B must have a positive third moment (see also equation B.1.3 in the proof of Theorem 3) and thus the claim for all odd moments also follows from the previous equation. □

Proof of Proposition 9. The proposition is a restatement of Proposition 3 in Chapter 3 in terms of standardized central moments. Because lotteries A_3 and B_3 have equal variances (see Proposition 3), the claim follows immediately. □

A.3.2 Instructions

[Starting from the next page, this appendix contains instructions handed out to participants in the experiment presented in Chapter 4. They are translated from German for sessions with order ES-MAO-RA. RA refers to the risk aversion stage that we did not further elaborate on.]

Thank you very much for participating in this decision experiment!

General Information

In the following experiment, you will make a couple of decisions. Following the instructions and depending on your decisions, you can earn money. It is therefore very important that you read the instructions carefully.

You will make your decisions anonymously on your computer screen in your cubicle. During the experiment, you are not allowed to talk to the other participants. Whenever you have a question, please raise your hand. The experimenter will answer your question in private in your cubicle. If you disregard these rules, you can be excluded from the experiment. Then, you receive no payment.

During the experiment, all amounts are stated in Taler, the experimental currency. At the end of the experiment, your achieved earnings will be converted into Euro at an exchange rate of 1 Taler = € 0.15 and paid to you in cash.

Structure of the Experiment and Your Decisions

In total, you will make 34 decisions throughout the experiment. In each decision, you will decide upon which of **two different risky events—either Option A or Option B**—you prefer.

An example of Option A could be as follows: With 50% chance you will lose 10 Taler or with 50% chance you will receive 20 Taler. Option B could be: With 20% chance you will receive 0 Taler and with 80% chance you will receive 10 Taler.

The experiment consists of three stages that will be explained in detail in the following. To determine your payoff in the experiment, one of your decisions will be randomly chosen. This takes place after you have completed all your decisions. To this end, the experimenter picks one of 34 balls, marked with numbers from 1 to 34, out of a ballot box. Each number occurs only once in the ballot box, whereby the draw of a particular number is equally likely. The outcome of the risky event that you have opted for at the randomly chosen decision will afterwards be determined by another random draw. This procedure will be explained extensively when the stages of the experiment are described.

Keep in mind that only one of your 34 decisions determines your payoff in the experiment. Therefore, each of your single decisions can determine your entire payoff in the experiment.

You make your decisions at the computer screen in the computer lab. For each decision,

you have a maximum of 3 minutes. After the experiment, the decision relevant for every participant's payoff and the outcome of the risky event will be determined by random draws for each participant in the seminar room. To this end, the experimenter will call upon participants one by one.
Note that some risky events comprise negative outcomes. For these questions you receive coupons indicating an endowment (in Taler). You can charge the coupons when the outcome of the risky event is determined.

Stage I

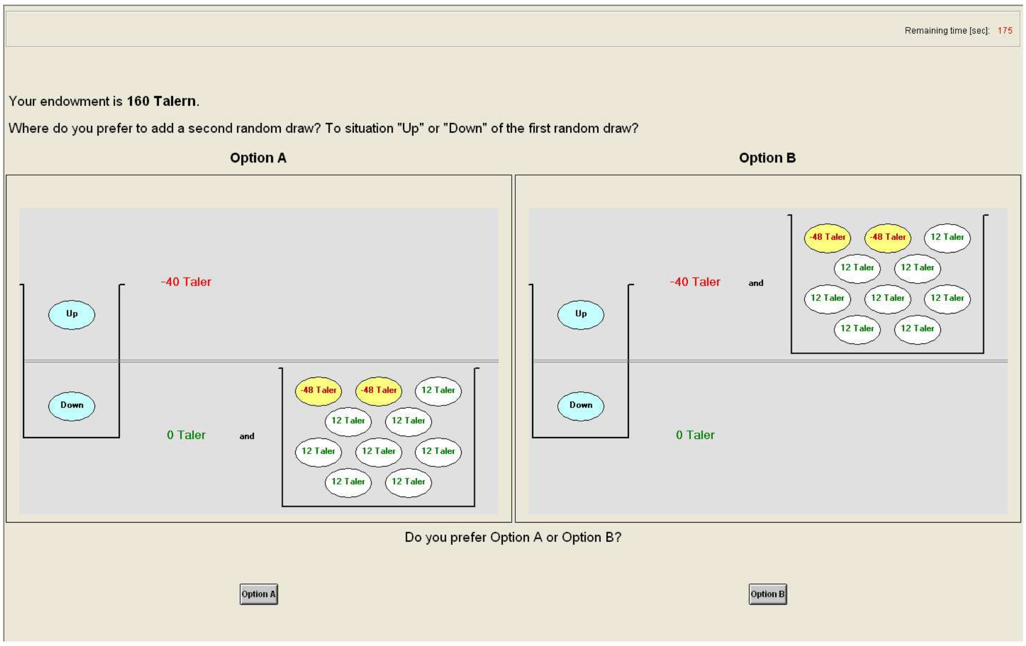
In the first stage of the experiment, you make 16 decisions. On each of the 16 sequent decision screens you decide which of the two risky events—either Option A or Option B—you prefer. In this stage, risky events (may) comprise two random draws.
For each decision, one random draw is given. This draw is as follows: With 50% chance the situation "Up" occurs, or with 50% chance the situation "Down" occurs.
For your decisions, you receive an endowment in Taler, because outcomes of risky events in this stage can also comprise losses. Accordingly, your payoff in this stage consists of two components:

Endowment and **Outcome of the Chosen Risky Event**

How is the outcome of the (chosen) risky event determined in Stage I? For the first random draw there are two balls in a ballot box—one marked with "Up" and the other with "Down." The draw of a particular ball is equally likely. To determine your payoff in this stage, **two** random draws may be necessary. For the second random draw one ball is drawn from another ballot box with 10 balls. The balls are either yellow or white. Note that the composition of yellow and white balls may change for different decisions in this stage. But within one decision, i.e., for Option A and Option B, the composition of yellow and white balls remains the same.

Decision Type 1

For 8 out of 16 decisions in stage I, you are asked the following: Given what situation of the first random draw—either "Up" or "Down"—do you prefer a **second random draw**? An example is provided by the following screen:



In Option A, you lose 40 Taler if situation "Up" occurs in the first random draw. If situation "Down" occurs, you receive 0 Taler and a second random draw succeeds. This second random draw is as follows: With 20% chance you lose 48 Taler and with 80% chance you receive 12 Taler. In Option B, you lose 40 Taler if in the first random draw the situation "Up" occurs, and a second random draw succeeds (the second random draw is the same as in Option A). When situation "Down" occurs, you receive 0 Taler. For this decision, you are endowed with 160 Taler.

Now suppose the decision from the example above is randomly drawn to determine your payoff. Suppose you have chosen ***Option A****.*

- *If in the first random draw the ball "Up" is drawn, you lose 40 Taler. After allocating your endowment of 160 Taler for this decision to the lottery outcome, your payoff is 120 Taler.*

- *If in the first random draw the ball "Down" is drawn, you receive 0 Taler, and a second random draw succeeds.*

  – *If in the second random draw a yellow ball is drawn, you lose 48 Taler, and your payoff after allocating your endowment is 112 Taler.*

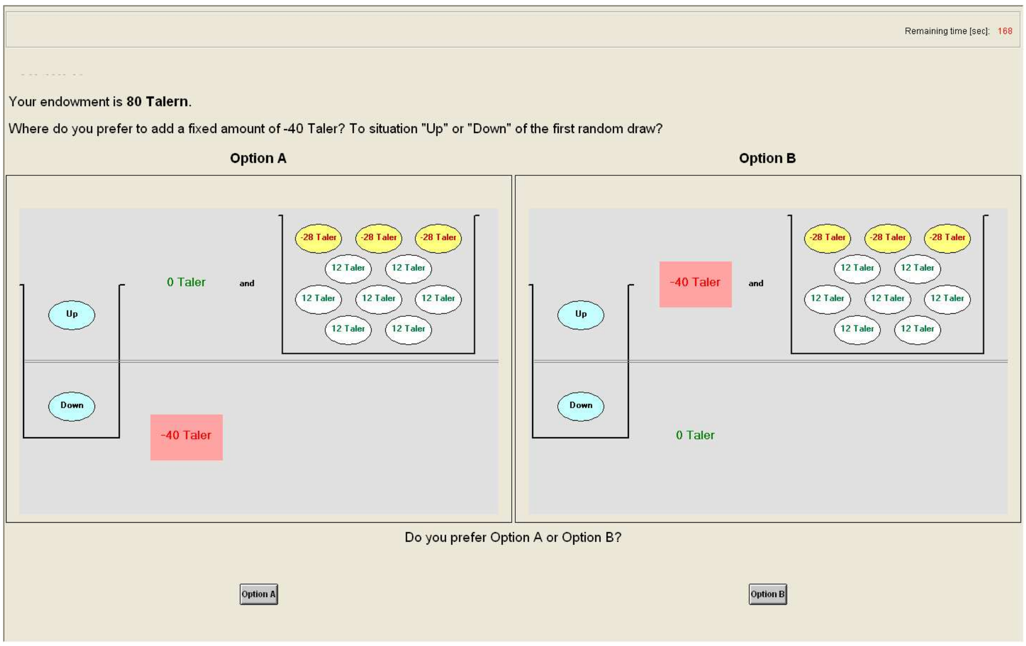
  – *If in the second random draw a white ball is drawn, you receive 12 Taler, and your payoff after allocating your endowment is 172 Taler.*

Suppose you have chosen ***Option B****.*

- *If in the first random draw ball "Up" is drawn, you lose 40 Taler, and a second random draw succeeds.*
  - *If in the second random draw a* <u>*yellow*</u> *ball is drawn, you lose 48 Taler, and your payoff after allocating your endowment is 72 Taler.*
  - *If in the second random draw a* <u>*white*</u> *ball is drawn, you receive 12 Taler, and your payoff after allocating your endowment is 132 Taler.*
- *If in the first random draw "Down" is drawn, you receive 0 Taler. After allocating your endowment of 160 Taler for this decision to the lottery outcome, your payoff is 160 Taler.*

Decision Type 2

For the remaining 8 out of 16 decisions in stage I, you are asked the following: To what situation do you prefer to add a **(fixed) amount**—either to situation "Up" where a second random draw succeeds or to situation "Down" where no second random draw succeeds. Note that the fixed amount can either be positive or negative. An example is provided by the following screen:



In Option A, if situation "Up" occurs in the first random draw, you receive 0 Taler, and

a second random draw succeeds. The second random draw is as follows: With 30% chance you lose 28 Taler and with 70% chance you receive 12 Taler. When situation "Down" occurs in the first random draw, you lose 40 Taler, and no second random draw succeeds. In Option B, if situation "Up" occurs in the first random draw, you lose 40 Taler, and a second random draw succeeds (the second random draw is the same as in Option A). When situation "Down" occurs, you receive 0 Taler, and no second random draw succeeds. For this decision, you are endowed with 80 Taler.
Now suppose the decision from the example above is randomly drawn to determine your payoff. Suppose you have chosen ***Option A****.*

- *If in the first random draw the ball "Up" is drawn, you receive 0 Taler, and a second random draw succeeds.*
  - *If in the second random draw a yellow ball is drawn, you lose 28 Taler, and your payoff after allocating your endowment is 52 Taler.*
  - *If in the second random draw a white ball is drawn, you receive 12 Taler, and your payoff after allocating your endowment is 92 Taler.*
- *If in the first random draw the ball "Down" is drawn, you lose 40 Taler. After allocating your endowment of 80 Taler for this decision to the lottery outcome, your payoff is 40 Taler.*

Suppose you have chosen ***Option B****.*

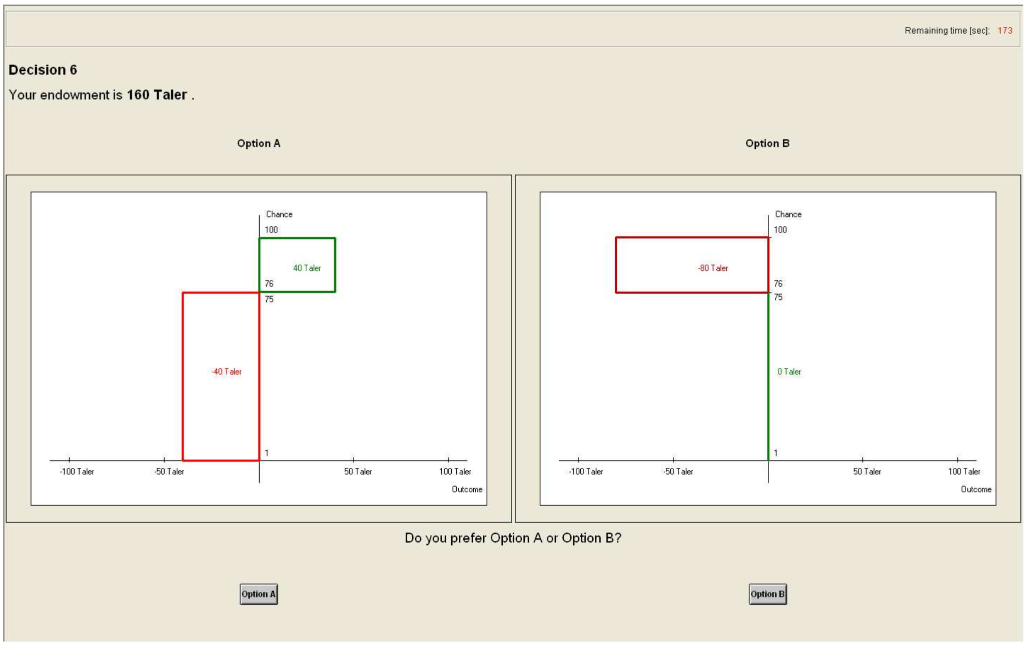
- *If in the first random draw the ball "Up" is drawn, you lose 40 Taler, and a second random draw succeeds.*
  - *If in the second random draw a yellow ball is drawn, you lose 28 Taler, and your payoff after allocating your endowment is 12 Taler.*
  - *If in the second random draw a white ball is drawn, you receive 12 Taler, and your payoff after allocating your endowment is 52 Taler.*
- *If in the first random draw the ball "Down" is drawn, you receive 0 Taler. After allocating your endowment of 80 Taler for this decision to the lottery outcome, your payoff is 80 Taler.*

Stage II

In the second stage of the experiment, you are asked to make eight decisions. On each of the 8 sequent decision screens you decide which of the two risky events—Option A or Option B—you prefer. For your decisions, you receive an endowment in Taler, because outcomes of risky events in this stage can comprise losses. Accordingly, your payoff in this stage consists of two components:

Endowment and **Outcome of the Chosen Risky Event**

How is the outcome of the (chosen) risky event determined in Stage II? To this end, there is another ballot box. This ballot box contains 100 balls with numbers from 1 to 100. Each number occurs only once, thus the draw of a particular number is equally likely. An example of a decision screen provides the following screen:



In Option A, you will lose 40 Taler with 75% chance (balls 1 to 75), or with 25% chance you will receive 40 Taler (balls 76 to 100). In Option B, you receive 0 Taler with 75% chance (balls 1 to 75), or you will lose 80 Taler with 25% chance (balls 76 to 100). Your endowment is 160 Taler in this example.

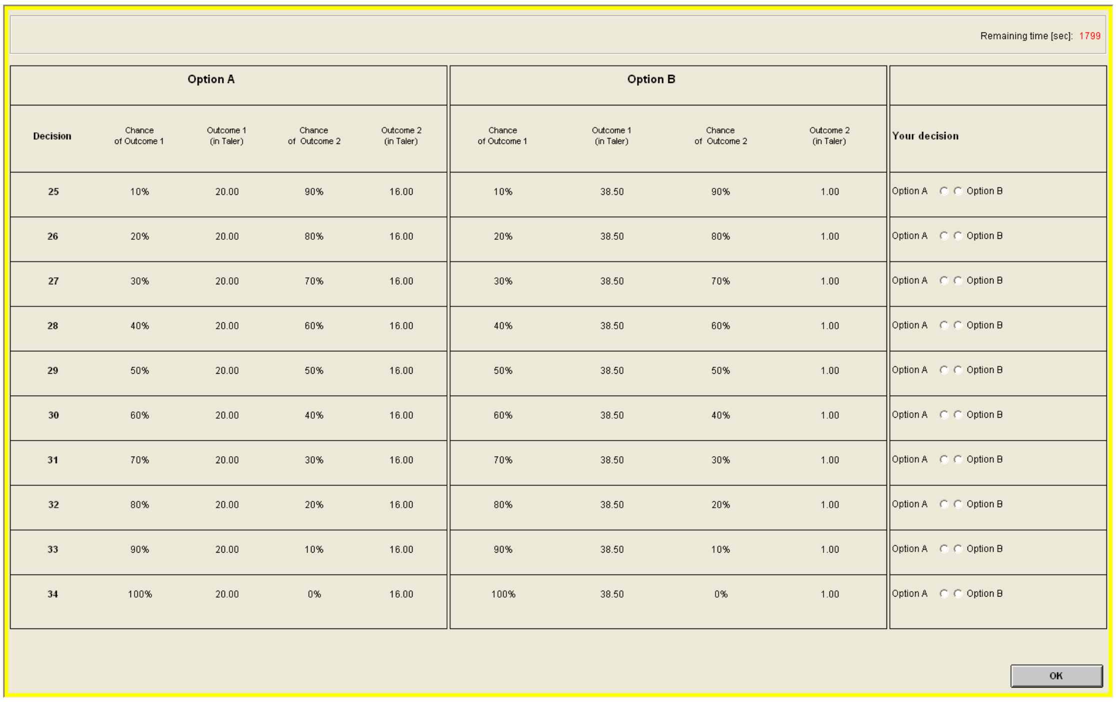
Now suppose that this decision was randomly drawn to determine your payoff.

- *Suppose you have chosen **Option A** and assume that a ball is drawn from the ballot box with a number between 1 and 75. That means, you lose 40 Taler. Your resulting payoff, after allocating the endowment of 160 Taler for this decision to the lottery* *outcome, is 120 Taler. If a ball with a number between 76 and 100 is drawn, you receive 40 Taler. Under consideration of your endowment your payoff is 200 Taler.*

- *Suppose you have chosen **Option B** and assume that a ball is drawn from the ballot box with a number between 1 and 75. That means, you receive 0 Taler. Your resulting payoff after allocating the endowment of 160 Taler for this decision to the lottery outcome is 160 Taler. If a ball with a number between 76 and 100 is drawn, you lose 80 Taler. Under consideration of your endowment, your payoff is 80 Taler.*

Stage III

In stage III, you are asked to make 10 decisions on a single decision screen. The risky events between which you have to decide in this stage are displayed in a table format. In each row of the table, you make one decision. For an illustration see the following figure:



Remaining time [sec]: 1799

	Option A				Option B				
Decision	Chance of Outcome 1	Outcome 1 (in Taler)	Chance of Outcome 2	Outcome 2 (in Taler)	Chance of Outcome 1	Outcome 1 (in Taler)	Chance of Outcome 2	Outcome 2 (in Taler)	Your decision
25	10%	20.00	90%	16.00	10%	38.50	90%	1.00	Option A Option B
26	20%	20.00	80%	16.00	20%	38.50	80%	1.00	Option A Option B
27	30%	20.00	70%	16.00	30%	38.50	70%	1.00	Option A Option B
28	40%	20.00	60%	16.00	40%	38.50	60%	1.00	Option A Option B
29	50%	20.00	50%	16.00	50%	38.50	50%	1.00	Option A Option B
30	60%	20.00	40%	16.00	60%	38.50	40%	1.00	Option A Option B
31	70%	20.00	30%	16.00	70%	38.50	30%	1.00	Option A Option B
32	80%	20.00	20%	16.00	80%	38.50	20%	1.00	Option A Option B
33	90%	20.00	10%	16.00	90%	38.50	10%	1.00	Option A Option B
34	100%	20.00	0%	16.00	100%	38.50	0%	1.00	Option A Option B

OK

Each risky event comprises two possible outcomes and two corresponding probabilities. You make your decision at the end of each row by indicating the risky event you prefer (either Option A or Option B). When making your decisions you do not have to follow a particular order, and you can change your decisions as often as desired within the time permitted.

The outcomes of the risky events in this stage do not comprise losses. Thus, for the decisions in this stage you do not receive an endowment. Accordingly, your payoff is as follows:

Outcome of the Risky Event

How is the outcome of the chosen risky event determined in Stage III? To determine the outcome, there is a ballot box with 100 balls marked with numbers from 1 to 100 (analogously to stage II). Each number occurs exactly once in the ballot box, i.e., the draw of a particular number is equally likely.

Before the experiment will start now, please note: You are asked comprehension questions before each stage starts. These questions should familiarize you with the decision task in each stage.
After the experiment, you are asked to answer a questionnaire. For answering the questionnaire you receive, independently from your earnings during the experiment, € 4.

[This is the end of the experimental instructions.]

Appendix B

Calibration of Binary Lotteries in Experiments and a Result on their Skewness

In this appendix, we present two results on binary lotteries. Firstly, we show how to calibrate binary lotteries in terms of moments. This might be useful for economists or psychologists, who frequently employ these items in experiments. Secondly, we present a theorem on how skewness manifests in binary lotteries. Both Chapters 3 and 4 refer several times to this appendix.
For convenience of the reader we first reintroduce some notations.

Definition 4. Let $x_1, x_0 \in \mathbb{R}$, with $x_1 > x_0$. X is a Bernoulli-distributed random variable with parameter $p \in (0, 1)$. A binary lottery denoted by $L = L(p, x_1, x_0)$ is defined as the random variable

$$L = X \cdot x_1 + (1 - X) \cdot x_0.$$

To understand the results and their proofs presented in the following, it is important to take notice of the delicacies of this representation we chose for a binary lottery. It excludes degenerate lotteries because the cases $p = 0$, $p = 1$, and $x_1 = x_0$ are excluded. Further, probability p is always associated with x_1, i.e, with the larger of the two different outcomes. Note that the above definition guarantees uniqueness of representation. Let us also repeat the definition of moments. For $n \geq 3$, we denote the nth standardized central moment of lottery L by $\mathbb{M}_n^S(L) := \mathbb{E}[(L - \mathbb{E}[L])^n] / (\mathbb{V}(L))^{n/2}$. With $\nu(L) := \mathbb{M}_3^S(L)$ we denote the third standardized central moment. Throughout this appendix, if not noted otherwise, “moments” (of order three or higher) refer to standardized central moments.

B.1 Lottery calibration in experiments

Our first result is a theorem stating that a binary lottery with non-trivial variance and otherwise arbitrary first three moments always exists, and the moments uniquely determine the lottery. It implies that every non-degenerate probability distribution with finite first three moments can be matched up to the third moment with a binary lottery, and this lottery is unique.

Theorem 3. *For constants $E \in \mathbb{R}$, $V \in \mathbb{R}_+^*$, and $S \in \mathbb{R}$, there exists exactly one binary lottery $L = L(p, x_1, x_0)$ such that $\mathbb{E}[L] = E$, $\mathbb{V}[L] = V$, and $\nu[L] = S$. Its parameters are given by*

$$p = \begin{cases} \frac{4+S^2+\sqrt{S^4+4S^2}}{8+2S^2} & \text{if } S < 0 \\ \frac{1}{2} & \text{if } S = 0 \\ \frac{4+S^2-\sqrt{S^4+4S^2}}{8+2S^2} & \text{if } S > 0, \end{cases}$$

$$x_1 = E + \sqrt{\frac{V \cdot (1-p)}{p}}, \quad \textit{and}$$

$$x_0 = E - \sqrt{\frac{V \cdot p}{1-p}}.$$

Proof of Theorem 3. After calculating expectation, variance, and third moment[1] of a binary lottery as in Definition 4, we find that $L = L(p, x_1, x_0)$ has to suffice the following system of equations

$$E = px_1 + (1-p)x_0 \tag{B.1.1}$$

$$V = (x_1 - x_0)^2 p(1-p) \tag{B.1.2}$$

$$S = \frac{1-2p}{\sqrt{p(1-p)}} \tag{B.1.3}$$

[1]For the computation of the third moment the reader may consult the proof of Theorem 4.

with $x_1 > x_0$ and $0 < p < 1$. It is natural to start with solving equation (B.1.3) for p. After squaring and some rearrangement one obtains

$$p^2(-S^2 - 4) + p(4 + S^2) - 1 = 0.$$

Setting $\tilde{S} := 4 + S^2$, the solutions to this quadratic equation are given by

$$p_{1/2} = \frac{\tilde{S} \overset{+}{-} \sqrt{\tilde{S}^2 - 4\tilde{S}}}{2\tilde{S}}, \tag{B.1.4}$$

where p_1 is the solution associated with the addition. It is easy to see that the expression under the square root is always positive. If $S = 0$, there is one solution, namely $p = \frac{1}{2}$. Otherwise, there are two solutions. Both these solutions are strictly positive since $\sqrt{\tilde{S}^2 - 4\tilde{S} + 4 - 4} = \sqrt{(\tilde{S} - 2)^2 - 4} \leq \tilde{S} - 2$, and thus

$$p_{1/2} \geq p_2 \geq \frac{\tilde{S} - (\tilde{S} - 2)}{2\tilde{S}} = \frac{1}{\tilde{S}} > 0.$$

All solutions are smaller than 1 since

$$p_1 < 1 \Longleftrightarrow \sqrt{\tilde{S}^2 - 4\tilde{S}} < \tilde{S}$$

which can be shown to be true for all $\tilde{S}$ (and thus for all S) by doing the quadratic expansion as in the previous calculation. Note that equation (B.1.3) is a square root equation and thus we have to verify the solutions. Obviously, if $S = 0$, then $p = 0.5$ is the unique solution. Otherwise, from equation (B.1.4) it follows that $p_1 > p_2$ and $p_1 + p_2 = 1$, i.e., $p_1 > 0.5$ and $p_2 < 0.5$. Thus, if $S < 0$, then p_2 does not solve equation (B.1.3) because $1 - 2p_2 > 0$, but p_1 does. Similarly, if $S > 0$, only p_2 solves equation (B.1.3). Thus, in any case, equation (B.1.3) has a unique solution in $(0, 1)$ such that it is a probability. This solution will be denoted by p in the following.
For any p obtained from equation (B.1.3), the system of equations (B.1.1) and (B.1.2) can be shown to have two solutions in (x_1, x_0). However, exactly one of them satisfies $x_1 > x_0$, and this solution is given by the expressions stated in the claim. $\square$

To motivate the calibration issue, consider the following example with reference to the experiment of Chapter 4. If decisions on Mao lotteries involved hundreds of dollars, while those over ES lotteries only involved a few dollars, it could be reasonably argued that results between stages MAO and ES are not comparable. Likewise, if lotteries in stage

MAO throughout had higher variance than in stage ES, this could distort comparability of results. While it is rather trivial to match the means of some lottery with the mean of a binary lottery (like a Mao lottery), with the above theorem we can also match variances and third moments. Indeed, with Theorem 3 such a calibration is quite easy to obtain. The given equations conveniently allow to construct exactly the lottery an experimenter is looking for. Further, Theorem 3 implies that, in general, more than three moments cannot be matched.

We now detail the calibration of Mao and ES lotteries in Chapter 4. In fact, things are slightly more complicated than just indicated. This is because we not only have to match two lotteries, but two pairs of lotteries. Further, while the two lotteries of each given pair (MAO or ES) do not differ in their variances, they naturally differ in their third moments; see Propositions 8 and 9. The Mao pair in Figure 4.1 and the ES pair in Figure 4.2 are an example of the calibration we employed in the experiment. All four lotteries depicted have equal mean and variance, and the *differences* in the third moments between the ES pair and the Mao pair, respectively, are also equal. In this sense, every ES pair is matched with one Mao pair as can be seen from Tables 4.1 and 4.2. The following proposition gives an existence and uniqueness result for such a calibration. In Ebert and Wiesen (2009), we show that this calibration indeed has an effect on subjects' decisions in the experiment. That is, roughly speaking, responses to matched lottery pairs correlate more than responses to non-matched pairs.

Proposition 10 (Calibration)**.** *Consider a prudence lottery pair* (A_3, B_3) *with finite first three moments. For every* $S > 0$, *there exists exactly one Mao lottery pair* (M_A, M_B) *such that*

$$\begin{aligned}
&\mathbb{E}[M_A] = \mathbb{E}[A_3] \textit{ and } \mathbb{E}[M_B] = \mathbb{E}[B_3],\\
&\mathbb{V}[M_A] = \mathbb{V}[A_3] \textit{ and } \mathbb{V}[M_B] = \mathbb{V}[B_3], \textit{ as well as}\\
&\nu[M_A] = -S \textit{ and } \nu[M_B] = S.
\end{aligned}$$

For $S = 0.5(\nu[B_3] - \nu[A_3])$, *the difference in third moments of the prudence pair equals the difference in third moments of the Mao pair, and the quadratic error* $\Delta := (\nu[B_3] - \nu[M_B])^2 + (\nu[A_3] - \nu[M_A])^2$ *is minimized.*

Proof of Proposition 10. By Theorem 3, there exists exactly one binary lottery $L_A \equiv M_A$

with $\mathbb{E}[M_A] = \mathbb{E}[A_3]$, $\mathbb{V}[M_A] = \mathbb{V}[A_3]$, and $\nu[M_A] = -S$. By Theorem 3, there also exists exactly one lottery $L_B \equiv M_B$ whose expectation and variance equal that of B_3 (i.e., they also equal that of A_3 and thus that of M_A), and further $\nu[M_B] = +S$. From equation (B.1.3), one immediately obtains that for the probabilities p_A and p_B of $L_A = L(p_A, x_1, x_0)$ and $L_B = (p_B, x_1, x_0)$, respectively, we have $p_A = 1 - p_B$. Therefore, by Definition 2, (M_A, M_B) constitutes a Mao lottery pair that fulfills the requested moment conditions. For the second part, note that by taking derivatives,

$$\Delta = (\nu[B_3] - S)^2 + (\nu[A_3] - (-S))^2$$

indeed achieves its minimum at $S = 0.5(\nu[B_3] - \nu[A_3])$. The difference in third moments of the Mao pair is $2S$ and, for the specification of S as above, this indeed equals the skewness difference $\nu[B_3] - \nu[A_3]$ of the ES pair. $\square$

B.2 Skewness in binary lotteries

Now we present the second result of this appendix, which is on the skewness of binary lotteries. The following theorem contains five statements, each of which indicates that a binary lottery is left-skewed. The theorem then says that each of these statements is necessary and sufficient for the others. Therefore, the theorem illustrates in a compact way how skewness manifests in binary lotteries. It is straightforward to formulate analogous versions for right-skewed or symmetric binary lotteries. For convenient access to the theorem, it might be insightful to exemplarily validate the statements with reference to a particular left-skewed binary lottery. Therefore, Figure B.1 plots the probability mass function of the left-skewed lottery $L = L(0.75, 2, 0)$.

Theorem 4 (Skewness in binary lotteries)**.** *For any binary lottery $L = L(p, x_1, x_0)$, the following statements are equivalent.*

(i) The right tail of L is shorter than its left tail, i.e., $|x_1 - \mathbb{E}[L]| < |x_0 - \mathbb{E}[L]|$.

(ii) The right tail of L is heavier than its left tail, i.e., $\mathbb{P}(L \in [\mathbb{E}[L], x_1]) > \mathbb{P}(L \in [x_0, \mathbb{E}[L]])$.

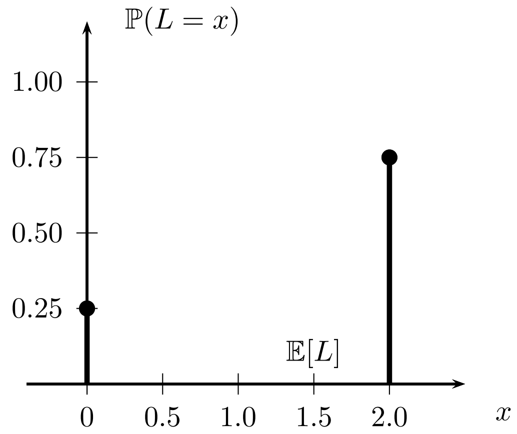
(iii) L has its high probability associated with the high outcome, whereas its low probability

is associated with the low outcome, i.e., $p > 0.5$.

(iv) *There exists at least one odd moment of order three or higher which is strictly negative, i.e.,* $\exists\ n \geq\ 3$ odd: $\mathbb{M}_n^S(L) < 0$.

(v) *All odd moments of order three and higher are strictly negative, i.e.,* $\mathbb{M}_n^S(L) < 0\ \forall\ n\ \geq\ 3$ odd.

Figure B.1: Probability mass function of a left-skewed binary lottery



This graph shows the probability mass function of lottery $L = L(0.75, 2, 0)$, whose expectation is 1.5. The high outcome 2 occurs with the high probability 0.75, whereas the low outcome 0 occurs with the low probability 0.25. The right tail of L is both shorter (length 0.5) and heavier (mass 0.75) than its left tail, which has length 1.5 and mass 0.25. Note that lottery L has been previously illustrated in tree form as lottery A_3^S in Figure 3.1.

Proof of Theorem 4. We prove the two equivalences $(i) \iff (iii)$, $(ii) \iff (iii)$, and the circle $(iii) \implies (iv) \implies (v) \implies (iii)$.

$(i) \iff (iii)$ According to Theorem 3, we can write the outcomes of a binary lottery as a function of its mean, variance, and the probability p associated with the high outcome. From these expressions, it immediately follows that

$$|x_1 - \mathbb{E}[L]| < |x_0 - \mathbb{E}[L]| \iff \sqrt{\frac{\mathbb{V}[L](1-p)}{p}} < \sqrt{\frac{\mathbb{V}[L]p}{1-p}}.$$

By use of elementary algebra, it can be shown that the latter equation is equivalent to $p > 0.5$.

$(ii) \iff (iii)$ This is straightforward because a binary lottery has probability mass only at its two outcomes. Formally, from Definition 4 it is easily seen that $x_0 < E < x_1$. Therefore, $\mathbb{P}\left(L \in [\mathbb{E}[L], x_1]\right) = \mathbb{P}\left(L \in \{x_1\}\right) = p$ and $\mathbb{P}\left(L \in [x_0, \mathbb{E}[L]]\right) = \mathbb{P}\left(L \in \{x_0\}\right) = 1 - p$.

$(iii) \implies (iv)$ It is easily inferred from equation (B.1.3) in the proof of Theorem 3 that the third moment is strictly negative if $p > 0.5$.

$(iv) \implies (v)$ We prove the evidently equivalent claim for non-standardized central moments $\mathbb{M}_n(L) := \mathbb{E}[(L - \mathbb{E}[L])^n]$. Using translation invariance, we can write the nth central moment of L as

$$\mathbb{M}_n(L) = \mathbb{M}_n(L - x_0) = \mathbb{M}_n\left(X(x_1 - x_0)\right) = \mathbb{E}\left[(X(x_1 - x_0) - p(x_1 - x_0))^n\right].$$

Because X is Bernoulli-distributed with parameter p, this can be explicitly computed as

$$\mathbb{M}_n(L) = p \cdot (1 \cdot (x_1 - x_0) - p(x_1 - x_0))^n + (1 - p) \cdot (0 \cdot (x_1 - x_0) - p(x_1 - x_0))^n.$$

Using that n is odd, this simplifies to

$$\mathbb{M}_n(L) = (x_1 - x_0)^n \cdot (p\,(1 - p)^n - (1 - p)p^n).$$

It is easily seen that $(p(1 - p)^n - (1 - p)p^n) < 0 \iff p > 0.5$, and since by definition $(x_1 - x_0)^n > 0$ we have

$$\mathbb{M}_n(L) < 0 \iff p > 0.5. \tag{B.2.1}$$

Therefore, if some odd moment $M_n(L)$ of L is strictly negative, then $p > 0.5$ by the sufficiency of equation (B.2.1). Then, the claim follows by the necessity of equation (B.2.1).

$(v) \implies (iii)$ The claim follows from equation (B.2.1). □

B.3 Concluding remarks to Appendix B

In this appendix, we present a characterization of binary lotteries in terms of its first three moments. Theorem 3 further shows how to construct binary lotteries with arbitrary first three moments. We argue that this might be useful for economists and psychologists, who frequently employ these items in experiments on decision making under uncertainty. As an example, we detailed the calibration procedure for the experiment of Chapter 4 of this thesis. The second result of this appendix, Theorem 4, shows how skewness manifests in binary lottery distributions. Specifically, it presents five interpretations for skewness in binary lotteries, and shows that each of them is necessary and sufficient for the others.

Some closing remarks might be interesting. Firstly, the result on how skewness manifests in a binary lottery provides additional intuition to the recent result of Chiu (2010). He shows that all binary lotteries are *generalized skewness comparable* implying that third-order moment preferences over such risks are consistent with EUT. Theorem 4 shows that the sign of the third moment is a sufficient indicator of skewness in the case of binary lotteries. Secondly, note that the result on odd moments in Proposition 8 immediately follows from Theorems 3 and 4. Thirdly, equation (A.3.1) from the proof of Proposition 8 gives some more insight into how Mao lotteries differ in their skewness. All higher odd moments of the lottery of a Mao pair are equal in absolute terms, but differ in their sign. This is not true for ES lotteries, where it could be, for example, that both lotteries are right-skewed, but the prudent choice is more right-skewed. This is the reason why concerning calibration in the experiment, only differences in third moments (rather than third moments themselves) can be matched. Fourthly, Lemma 3 is a corollary to Theorems 3 and 4 that is used to show that certain results in Roger (2011) are limited to symmetric zero-mean risks within Eeckhoudt and Schlesinger's proper risk apportionment model.

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