Interlacing Patterns in Exclusion Processes and Random Matrices

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René Frings

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1. Gutachter: Prof. Dr. Patrik L. Ferrari

2. Gutachter: Prof. Dr. Benjamin Schlein

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Abstract

In the last decade, there has been increasing interest in the fields of random matrices, interacting particle systems, stochastic growth models, and the connections between these areas. For instance, several objects that appear in the limit of large matrices also arise in the long-time limit for interacting particles and growth models. Examples of these are the famous Tracy-Widom distribution function and the Airy_2 process.

The objectives of this thesis are threefold: First, we discuss known relations between random matrices and some models in the Kardar-Parisi-Zhang universality class, namely the polynuclear growth model and the totally/partially asymmetric simple exclusion processes. For these models, in the limit of large time t, universality of fluctuations has been previously obtained. We consider the convergence to the limiting distributions and determine the (non-universal) first order corrections, which turn out to be a non-random shift of order $t^{-1/3}$. Subtracting this deterministic correction, the convergence is then of order $t^{-2/3}$. We also determine the strength of asymmetry in the exclusion process for which the shift is zero and discuss to what extend the discreteness of the model has an effect on the fitting functions.

Second, we focus on the Gaussian Unitary Ensemble and its relation to the totally asymmetric simple exclusion process and discuss the appearance of the Tracy-Widom distribution in the two models. For this, we consider extensions of these systems to triangular arrays of interlacing points, the so-called Gelfand-Tsetlin patterns. We show that the correlation functions for the eigenvalues of the matrix minors for complex Dyson's Brownian motion have, when restricted to increasing times and decreasing matrix dimensions, the same correlation kernel as in the extended interacting particle system under diffusion scaling limit. We also analyze the analogous question for a diffusion on complex sample covariance matrices.

Finally, we consider the minor process of Hermitian matrix diffusions with constant diagonal drifts. At any given time, this process is determinantal and we provide an explicit expression for its correlation kernel. This is a measure on Gelfand-Tsetlin patterns that also appears in a generalization of Warren's process, in which Brownian motions have level-dependent drifts. We will also show that this process arises in a diffusion scaling limit from the interacting particle system on Gelfand-Tsetlin patterns with level-dependent jump rates.

Contents

1. Introduction

One of the most famous results in probability theory is the central limit theorem in which one considers the sum of independent and identically distributed random variables with finite variances. This theorem tells us that if we center the variables and divide them by the square root of the sample size, then the sum of these rescaled variables will be approximately normally distributed. The remarkable feature is that the appearance of the Gaussian distribution does not depend on the distribution of the random variables that we started with. In this sense, the normal distribution is universal and this is also the reason why the Gaussian distribution plays such a prominent role in probability theory, and more generally speaking in applied mathematics and physics.

Even if a large part of modern stochastics are based on this Gaussian universality, there is another universality class that has been investigated starting at the end of the 90s. To introduce this class, we consider the following example. Suppose that we are on an airfield and there are n passengers boarding an airplane. For simplicity, let us assume that there is only one single seat in each of the n rows of the airplane and that each passenger needs one minute to stow his hand baggage and sit down. We are interested in the boarding time t_n, i. e., the time it takes until all passengers are seated. If the travelers are queuing in the same order as the order of their seats, then the boarding time is minimal. However, this is usually not the case and passengers with rear seats are blocked by travelers with front seats, i. e., they have to wait until the others have organized their luggage. Supposing that the order of the passengers is random, we consider the uniform distribution on the symmetric group of n symbols. The boarding time t_n is then a random variable and corresponds to the length of the longest increasing subsequence of a given permutation. Asymptotically, the expected value of t_n behaves like $2\sqrt{n}$ for large n. By the law of large numbers, it seems thus reasonable that the fluctuations around the deterministic mean cancel each other out as n grows. To study these fluctuations around the expected value, we consider $t_n - 2\sqrt{n}$ and scale this variable not by $n^{-1/2}$ as in the central limit theorem, but by $n^{-1/6}$. In a seminal work published in 1999, Baik, Deift, and Johansson [7] showed that as n tends to infinity, this rescaled random variable is not Gaussian as one might expect, but the distribution is different. Actually, the distribution was known from random matrix theory where Tracy and Widom [99] had identified it in the mid 90s as describing the fluctuations of the largest eigenvalues of Hermitian Gaussian matrices when the matrix size becomes large.

Soon after, Johansson [57] related the problem of the longest increasing subsequence of a random permutation to the totally asymmetric simple exlusion process (TASEP) in which he also discovered the Tracy-Widom distribution. This was the starting point for a lot of research activities in this field located at the intersection between random matrices and interacting particle systems. Indeed, the totally asymmetric simple exclusion process is seen as belonging to the Kardar-Parisi-Zhang (KPZ) class of stochastic growth models and in the years following Johansson's breakthrough, it turned out that the Tracy-Widom distribution describes the limiting fluctuations in many other models from the KPZ class. The same is true for random matrices for which it was shown during the last 15 years that this probability law governs the fluctuations of the largest eigenvalues for a large class of random matrices. This means that both KPZ models and random matrices show the same limit distribution which distinguishes them from the Gaussian limiting behavior in classical probability theory. Moreover, it seems that the appearance of the Tracy-Widom distribution is somehow characteristic for a large class of random matrices and growth models, and for that reason this phenomenon is often referred to as Tracy-Widom universality.

It is surprising that precisely these two groups, the class of KPZ growth models and the class of random matrices, are related in the way that they share a common feature that is different from the rest of the probabilistic world although a direct connection between these classes is not evident. At least, there is no known one-to-one correspondence that would allow us to translate results from the world of random matrices to the world of KPZ models or vice versa. The present thesis provides some partial explanation why the Tracy-Widom distribution shows up in both kinds of models. Throughout this work, we will mainly focus on continuous time TASEP as a representative of the KPZ class and on the Gaussian Unitary Ensemble (GUE) which is the standard model from random matrix theory. These two specific models can be extended in such a way that they both live on the same pattern of interlacing points, the Gelfand-Tsetlin cone. This is a triangular array consisting of a fixed number N of levels, with n particles at each level $1 \leq n \leq N$, subject to an interlacing condition. The Gelfand-Tsetlin cone is a deep and rather hidden structure from which we can recover each model by an appropriate projection. We will show that on this set, along certain projections, the generalized random matrix model can be obtained as the diffusion scaling limit of the generalized interlacing particle system. The method that we use to compute the relevant quantities is not limited to the Gaussian unitary ensemble, but also applies to another model. Moreover, this connection can be generalized by adding a deterministic diagonal matrix to our random matrix model living on the the interlacing structure. As we will show, these drifts are inherited from the corresponding system of interacting particles where they appear as jump rates on the different levels of the Gelfand-Tsetlin cone.

The thesis is organized as follows: The first two chapters present a very rough overview of the state of the art and give the context in which Results 1 to 13 are embedded, while the remaining chapters provide the proofs of these results. They are based on the research articles [45–48] that the author of this thesis published in collaboration with his adviser Prof. Dr. Patrik L. Ferrari of Bonn University.

In Chapter 2, we introduce the Gaussian Unitary and the Gaussian Orthogonal Ensembles and explain that the Tracy-Widom distribution appears in the study of the fluctuations of the largest eigenvalue. In view of universality, we present how this behavior extends to other matrix ensembles and also to multi-point distributions. Then we turn towards the KPZ models, and after characterizing this class, we define the polynuclear growth and the continuous time

TASEP as being typical models in the KPZ class and thus being governed by the Tracy-Widom law. Finally, we explain where this universality ends and state Results 1 to 4 about the speed of convergence to the Tracy-Widom distribution and give finite time correction for KPZ models. We will prove these results in Chapter 4.

In Chapter 3 we present the notions of random point processes and determinantal correlation functions. This gives us the framework we need in order to study the correlations of the GUE eigenvalues' point process and to define the Airy processes. A small side note about survival probabilities of these objects allows us to come to Results 5 and 6 on spatial persistence for the Airy processes which will be proven in Appendix A.1. Then, we generalize the process on GUE eigenvalues to processes on the corresponding minors and in time, and discuss how we can combine these two evolution types to a process on space-like paths. This Markov process (Result 7) has determinantal correlations (Result 8) and this property along space-like paths also holds for complex Wishart matrices (Results 9 and 10); we will prove these theorems in Chapter 5. In the last part of Chapter 3, we connect these results with an interacting particle model model in $2+1$ dimensions that has been introduced by Borodin and Ferrari and give some hints why the Tracy-Widom distribution shows up in both GUE and TASEP. As mentioned before, this link is still there if we generalize our models to perturbed GUE minors (Result 11) and interacting particles in $2+1$ dimensions on Gelfand-Tsetlin patterns with level-dependent jump rates (Result 12). The measure that we study can be observed in a system of interlacing Brownian motions, Warren's process with drifts (Result 13). The proofs for these last results can be found in Chapter 6.

Chapter 4 is based on [47], Chapter 5 on [46], Chapter 6 on [45], and Appendix A.1 is taken from [48].

2. Tracy-Widom universality

2.1. Edge universality of random matrices

2.1.1. One-point distribution

An $N \times N$ *Wigner matrix* is a complex Hermitian or real symmetric matrix $H = [H_{ij}]_{1 \le i,j \le N}$ where the upper-triangular entries H_{ij}, $1 \le i < j \le N$, are independent and identically distributed complex or real random variables with mean zero and unit variance, and the diagonal entries H_{ii}, $1 \le i \le N$, are independent and identically distributed real variables, independent of the upper-triangular entries, with bounded mean and variance. Such a Hermitian or symmetric matrix H has N real eigenvalues which we denote in increasing order by $\lambda_1 \le \lambda_2 \cdots \le \lambda_N$. Consider the empirical spectral distribution μ_N of the eigenvalues,

$$\mu_N = \frac{1}{N} \sum_{k=1}^{N} \delta_{\lambda_k/\sqrt{N}}$$

for large N. Note that the scaling $\frac{1}{\sqrt{N}} H$ is somehow natural since it ensures the variance to be of order 1. Wigner's famous semicircle law tells us that μ_N converges almost surely to the semicircle distribution μ_{sc},

$$\mu_{\mathrm{sc}}(\mathrm{d}x) = \frac{1}{2\pi} \sqrt{(4 - x^2)_+}\, \mathrm{d}x.$$

We thus expect the largest eigenvalues λ_N of H to be around $2\sqrt{N}$ for large N. Let us focus on the fluctuations of λ_N around its *deterministic* limit. For small $\varepsilon > 0$, the number of eigenvalues in the interval $[2\sqrt{N} - \varepsilon, 2\sqrt{N}]$ is roughly

$$\#\big\{1 \le k \le N : \lambda_k \ge 2\sqrt{N} - \varepsilon\big\} \approx \frac{N}{2\pi} \int_{2-\varepsilon/\sqrt{N}}^{2} \sqrt{4 - x^2}\, \mathrm{d}x \approx \frac{2}{3\pi}\, \varepsilon^{3/2} N^{1/4}.$$

Thus, if we want this quantity to be of order 1, we should choose $\varepsilon = \mathcal{O}(N^{-1/6})$, and the fluctuations around $2\sqrt{N}$ are then given by

$$\lambda_N \approx 2\sqrt{N} + N^{-1/6}\zeta,$$

where the distribution of the random variable ζ has still to be determined.

Gaussian Unitary Ensemble

The first Wigner matrices for which this distribution has been identified were the Gaussian Hermitian matrices, the so called *Gaussian Unitary Ensemble* (GUE). More precisely, the diagonal entries H_{ii}, $1 \leq i, j \leq N$ are independent, centered Gaussian variables with unit variance, and the upper-triangular entries H_{ij}, $1 \leq i < j \leq N$, have independent real and imaginary parts that are centered Gaussian variables with variance $\frac{1}{2}$. Equivalently, the GUE can be described as the Hermitian $N \times N$ matrices equipped with the measure

$$\mathrm{const} \times \exp\left(-\frac{\beta}{4} \operatorname{Tr} H^2\right) \mathrm{d}H, \quad \beta = 2, \tag{2.1}$$

where $\mathrm{d}H = \prod_{i=1}^{N} \mathrm{d}H_{ii} \prod_{1 \leq i < j \leq N} \mathrm{d}\operatorname{Re} H_{ij}\, \mathrm{d}\operatorname{Im} H_{ij}$ is the N^2-dimensional Lebesgue measure and const is the normalization constant. The joint density of the eigenvalues is then

$$\mathrm{const} \times \prod_{1 \leq i < j \leq N} |x_i - x_j|^{\beta} \prod_{i=1}^{N} \mathrm{e}^{-\beta x_i^2/4}, \quad \beta = 2, \tag{2.2}$$

and this explicit formula allows us to calculate the fluctuations of the largest eigenvalue λ_N of a GUE matrix. Tracy and Widom [99] proved that

$$\lim_{N \to \infty} \mathbb{P}\big(\lambda_N \leq 2\sqrt{N} + sN^{-1/6}\big) = F_{\mathrm{GUE}}(s), \quad s \in \mathbb{R}, \tag{2.3}$$

exists and is given by

$$F_{\mathrm{GUE}}(s) = \exp\left(-\int_s^{\infty} \mathrm{d}t\, (t - s) q^2(t)\right), \tag{2.4}$$

where q is the unique solution (the so called *Hastings-McLeod solution*) to the *Painléve II equation*

$$q''(t) = tq(t) + 2q^3(t) \tag{2.5}$$

satisfying the boundary condition $q(t) \sim \mathrm{Ai}(t)$ as $t \to \infty$ with Ai the Airy function. For that reason, we call F_{GUE} nowadays the *GUE Tracy-Widom distribution*. Soon after, the same authors discovered in [100] similar distributions for the Gaussian Orthogonal and the Gaussian Symplectic Ensembles. We restrict ourselves here to the presentation of the orthogonal case.

Gaussian Orthogonal Ensemble

The *Gaussian Orthogonal Ensemble* (GOE) is the subclass of real Wigner matrices with Gaussian entries and normalization $\mathbb{E}\, H_{ii}^2 = 2$ for $1 \leq i \leq N$. As in the unitary case, we can equivalently consider the measure (2.1) with $\beta = 1$ and $\mathrm{d}H = \prod_{1 \leq i \leq j \leq N} \mathrm{d}H_{ij}$. Then, the fluctuations of the largest eigenvalue λ_N are given by

$$\lim_{N \to \infty} \mathbb{P}\big(\lambda_N \leq 2\sqrt{N} + sN^{-1/6}\big) = F_{\mathrm{GOE}}(s), \quad s \in \mathbb{R},$$

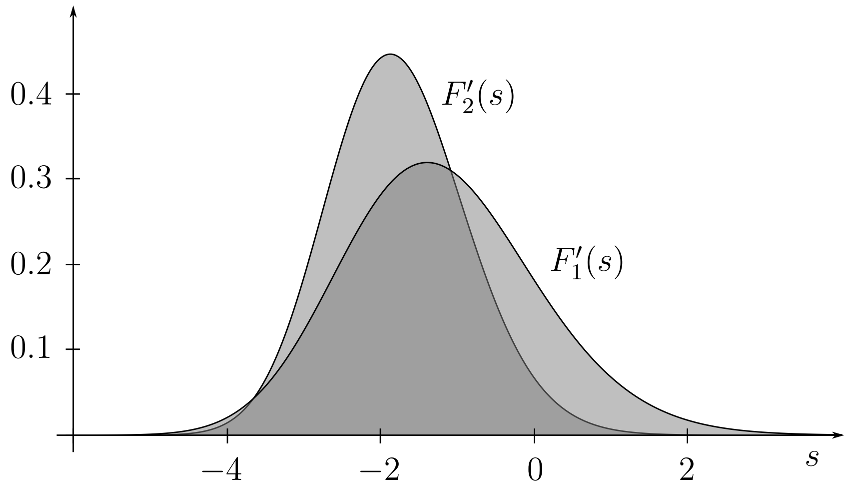


Figure 2.1.: Plots of F'_{GOE} and F'_{GUE}

where F_{GOE} is the *GOE Tracy-Widom distribution* which can also be expressed in terms of the Hastings-McLeod solution q from (2.5),

$$F_{\mathrm{GOE}}(s) = \exp\left(-\frac{1}{2}\int_s^\infty \mathrm{d}t\, q(t)\right)(F_{\mathrm{GUE}}(s))^{1/2}. \tag{2.6}$$

The plots of the probability density functions F'_{GUE} and F'_{GOE} are in Figure 2.1.

Wigner matrices

It is conjectured that these results are not only valid for GUE and GOE, but for a much larger class of random matrices. Since the eigenvalue in question is the largest one and thus located at the edge of the spectrum, the appearance of the Tracy-Widom distribution is called the *Tracy-Widom edge universality*. For Wigner matrices, edge universality was proven by Soshnikov [92] under the additional assumptions that the distribution of the entries is symmetric (which implies that all odd moments vanish) with at least Gaussian decay and the same normalization of the variances as for GOE (for real symmetric Wigner matrices) and GUE (for complex Hermitian Wigner matrices). We then have for the largest eigenvalue λ_N of such a Wigner matrix that

$$\lim_{N\to\infty} \mathbb{P}\big(\lambda_N \leq 2\sqrt{N} + sN^{1/6}\big) = F(s), \quad s \in \mathbb{R},$$

with $F = F_{\mathrm{GOE}}$ for real symmetric and $F = F_{\mathrm{GUE}}$ for complex Hermitian matrices. In the following years, the symmetry assumption could be weakened [98] and was finally removed in [41]. Recently, Lee and Yin [66] proved that Tracy-Widom edge universality holds if and only if $s^4\,\mathbb{P}(|H_{12}| \geq s) \to 0$ as $s \to \infty$.

Invariant ensembles

As we have seen, Wigner matrices are a generalization of the GUE (resp. GOE) in the sense that distributions other than the Gaussian law are permitted, at the expense of unitary (resp.

orthogonal) invariance. Another way to generalize GUE and GOE would be to keep this invariance by replacing the Gaussian measure (2.1) by

$$\text{const} \times \exp\Big(-\frac{\beta}{4}\,\mathrm{Tr}\,V(H)\Big)\mathrm{d}H, \tag{2.7}$$

where V is a polynomial of even degree ($\deg V \neq 0$) and positive leading coefficient. Under this measure, H has the same distribution as UHU^* for all unitary (or orthogonal) matrices U. The joint density of the eigenvalues is then

$$\text{const} \times \prod_{1\le i<j\le N} |x_i - x_j|^\beta \prod_{i=1}^{N} \mathrm{e}^{-\beta V(x_i)/4},$$

with $\beta = 1$ in the real and $\beta = 2$ in the complex case. Using Riemann-Hilbert theory, Deift and Gioev [36] could show that edge universality also holds for invariant ensembles, i. e., there are constants a_β and b_β (depending on V) such that

$$\lim_{N\to\infty} \mathbb{P}(\lambda_N \le a_\beta\sqrt{N} + s b_\beta N^{-1/6}) = F(s), \quad s \in \mathbb{R},$$

again with $F = F_{\text{GOE}}$ for $\beta = 1$ in (2.7) and $F = F_{\text{GUE}}$ in the $\beta = 2$ case.

Wishart matrices

Another class of random matrices are *Wishart* or *sample covariance matrices*. Let M be a $p \times N$ matrix with independent and identically distributed complex (or real) entries M_{ij}, $1 \le i \le p$, $1 \le j \le N$ and consider the $N \times N$ matrix $X = M^*M$ with ordered eigenvalues $\lambda_1 \le \dots \le \lambda_N$. We also assume that $p = p_N$ is a function of N such that $p_N/N \to \vartheta$ for some $\vartheta \in [0,\infty]$ as $N \to \infty$. The joint eigenvalue distribution in the real ($\beta = 1$) and the complex ($\beta = 2$) cases has density

$$\text{const} \times \prod_{1\le i<j\le N} |x_i - x_j|^\beta \prod_{i=1}^{N} x_i^{\beta(p-N+1)/2-1} \mathrm{e}^{-\beta x_i/2},$$

with respect to the Lebesgue measure on $\mathbb{R}_+^N$. If we consider the empirical distribution $\tilde\mu_N$ for the eigenvalues,

$$\tilde\mu_N = \frac{1}{N}\sum_{k=1}^{n} \delta_{\lambda_k/N},$$

then for $\vartheta \in [1,\infty)$, $\tilde\mu_N$ will converge almost surely to the counterpart of the semicircle law for Wishart matrices, the Marčenko-Pastur distribution [28],

$$\mu_{\text{MP}}(\mathrm{d}x) = \frac{1}{2\pi}\,\frac{\sqrt{(c_+ - x)(x - c_-)}}{x}\,\mathbb{1}_{[c_-,c_+]}(x)\,\mathrm{d}x,$$

where $c_\pm = (1 \pm \sqrt{\vartheta})^2$. When studying a random growth model, Johansson [57] found, so to say as a byproduct, the edge fluctuations of complex Wishart matrices,

$$\lim_{N\to\infty} \mathbb{P}\big(\lambda_N \le \mu_{N,p} + s\sigma_{N,p}\big) = F_{\mathrm{GUE}}(s), \quad s \in \mathbb{R}, \tag{2.8}$$

where $\vartheta \in (0,\infty)$ and the constants $\mu_{N,p}$ and $\sigma_{N,p}$ are defined by

$$\mu_{N,p} = (\sqrt{N} + \sqrt{p})^2, \quad \sigma_{N,p} = \frac{(\sqrt{Np})^{1/3}}{(\sqrt{N} + \sqrt{p})^2}.$$

The same result holds true for real Wishart matrices, then with F_{GOE} instead of F_{GUE} in (2.8), which was proved by Johnstone [62]. Later, El Karoui [39] extended these results to the cases where $\vartheta \in \{0, +\infty\}$.

2.1.2. Dyson's Brownian motion

In 1962, Dyson [38] introduced the following diffusion on GUE matrices. Let $(B(t) : t \ge 0)$ be a Brownian motion on the $N \times N$ Hermitian matrices, i. e., $(B(t) : t \ge 0)$ is a stochastic process with almost surely continuous paths such that $H(0)$ is the zero matrix, the increments are independent and for any $0 < s < t$, we have that $H(t) - H(s)$ is $\sqrt{t-s}$ times a GUE matrix drawn from (2.1). Then we define the stationary Ornstein-Uhlenbeck process on the $N \times N$ Hermitian matrices by

$$\mathrm{d}M(t) = -\frac{1}{2} M(t)\mathrm{d}t + \mathrm{d}B(t).$$

The stationary distribution is given by

$$\mathrm{const} \times \exp\Big(-\frac{1}{2} \operatorname{Tr} M^2\Big).$$

The dynamics of the ordered eigenvalues $\lambda_1(t) \le \cdots \le \lambda_N(t)$ of $M(t)$ are described by *Dyson's Brownian motion*, i. e., the satisfy the stochastic differential equations

$$\mathrm{d}\lambda_i(t) = \Bigg(-\frac{1}{2}\lambda_i(t) + \sum_{\substack{j=1 \\ j\ne i}}^{n} \frac{1}{\lambda_i(t) - \lambda_j(t)}\Bigg)\mathrm{d}t + \mathrm{d}b_i(t), \quad 1 \le i \le N, \tag{2.9}$$

where $b_1, \dots, b_N$ are independent standard Brownian motions. The rescaled largest eigenvalue process $(\lambda_N^{\mathrm{resc}}(t) : t \ge 0)$ of the stationary solution $(\lambda_N(t) : t \ge 0)$ to (2.9),

$$\lambda_N^{\mathrm{resc}}(t) = N^{1/6}\big(\lambda_N(2N^{-1/3}t) - 2\sqrt{N}\big), \quad t \ge 0,$$

will then converge to the *Airy process* [58],

$$\lim_{N\to\infty} \lambda_N^{\mathrm{resc}} = \mathcal{A}_2 \tag{2.10}$$

in the sense of finite-dimensional distributions. The Airy process was introduced by Prähofer and Spohn [81] when studying polynuclear growth models. They showed that this process is stationary with almost surely continuous sample paths, and the one-point distribution is given by the GUE Tracy-Widom distribution,

$$\mathbb{P}(\mathcal{A}_2(t) \leq s) = F_{\mathrm{GUE}}(s), \quad s, t \in \mathbb{R}.$$

A precise definition of the Airy process will be presented in Chapter 3.1.3.

2.2. Kardar-Parisi-Zhang universality

We now present some results about the Kardar-Parisi-Zhang universality class of stochastic growth models. Let us consider the growth of a surface, like the burning front of a piece of paper or the propagation of a bacterial colony, and describe this surface by a random function $h : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}$, the *height function*, which gives the surface height for a space position $x \in \mathbb{R}^d$ and a time $t \geq 0$. Suppose that there is a local growth, whereas macroscopically, due to some smoothing effects, the surface growth will be described by a deterministic growth velocity function v, see also Figure 2.2. This means that v only depends on the slope ∇h of the interface, and thus we expect on a macroscopic scale that

$$\partial_t h = v(\nabla h).$$

However, on a mesoscopic scale we should see the randomness. In their seminal paper [64], Kardar, Parisi and Zhang argued that the smoothing effect should be related to the surface tension and enters as $\nu \Delta h$, while the local random growth is modeled by a space-time white noise η,

$$\partial_t h = \nu \Delta h + v(\nabla h) + \eta.$$

If we expand v around 0, then we have

$$v(u) = v(0) + \big\langle \nabla v(0), u \big\rangle + \frac{1}{2} \big\langle u, \operatorname{Hess} v(0) u \big\rangle + o\big(\|u\|^2\big), \tag{2.11}$$

where Hess denotes the Hessian matrix. Note that the constant and the linear term in (2.11) can be removed from the equation by applying a shift and a rotation. Anyway, the second term should vanish, since v is usually assumed to be symmetric. The first non-trivial contribution is thus the quadratic term, which should be different from zero, because otherwise we would be in the so-called Edwards-Wilkinson class and the effects of the non-linearity in the equation would disappear.

From now on, we only consider the one-dimensional case. Setting $\lambda = v''(0) \neq 0$, the Kardar-Parisi-Zhang equation then finally reads

$$\partial_t h = \nu \, \partial_x^2 h + \frac{\lambda}{2} (\partial_x h)^2 + \eta. \tag{2.12}$$

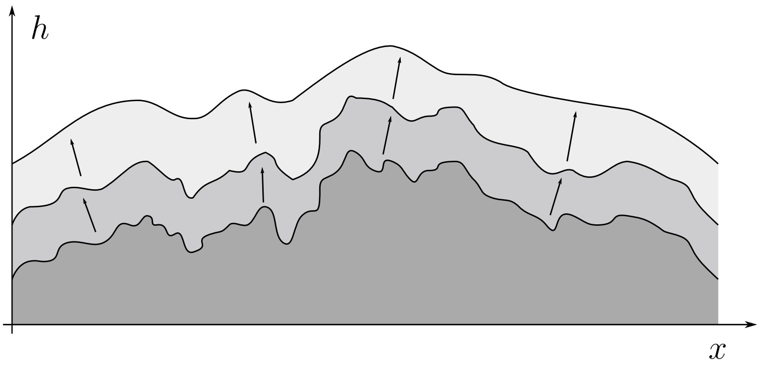


Figure 2.2.: Lateral growth and smoothing mechanism for growth models in the KPZ class

The problem about this reasoning is that $|\partial_x h|$ is not expected to be small, but very large. However, this heuristic derivation gives us a rough idea about the equation. To summarize, a model in the KPZ class should have (a) a deterministic limit shape, (b) local growth dynamics, (c) satisfy $v''(0) \neq 0$.

Let us denote the deterministic limit shape by h_{ma},

$$h_{\mathrm{ma}}(\xi) = \lim_{t\to\infty} \frac{h(\xi t, t)}{t}.$$

The fluctuations around this limit shape should be of order $t^{1/3}$ and the spatial correlation length scales as $t^{2/3}$, i. e., the rescaled height function h_t^{resc} at time t around a macroscopic position ξ,

$$h_t^{\mathrm{resc}}(u) = \frac{h(\xi t + ut^{2/3}, t) - t\, h_{\mathrm{ma}}((\xi t + ut^{2/3})/t)}{t^{1/3}} \tag{2.13}$$

should converge, as $t \to \infty$, to a well-defined, non-trivial stochastic process.

Some solvable models in the KPZ class have been analyzed in great detail. Two of the best studied models are the polynuclear growth (PNG) model and the (totally/partially) asymmetric simple exclusion process (TASEP/PASEP).

2.2.1. Polynuclear growth

The polynuclear growth model describes the growth of an interface on a one-dimensional substrate. The height function $h : \mathbb{R} \times [0, \infty) \to \mathbb{Z}$ takes values in the integers and, to make it well-defined, we assume that h is upper semi-continuous, i. e., the set $\{x \in \mathbb{R} : h(x,t) \geq n\}$ is closed for every $n \in \mathbb{Z}$. Let x be a discontinuity point of $h(\,\cdot\,, t)$. Then we say that there is an *up-step* (_|‾) at x if $h(x^-, t) < h(x^+, t)$, a *down-step* (‾|_) if $h(x^-, t) > h(x^+, t)$ and a *nucleation event* ($\perp$) if there is both an up-step and a down-step. The growth dynamics of this model have a deterministic and a stochastic part.

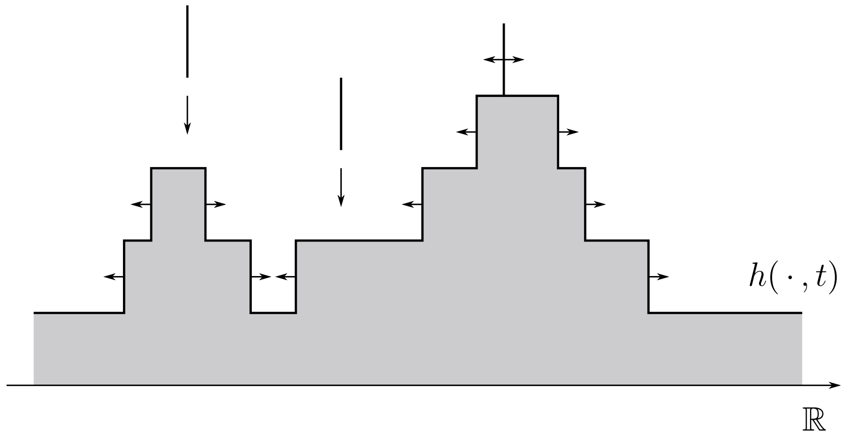


Figure 2.3.: Polynuclear growth. The islands spread deterministically with unit speed while nucleations are created randomly according to a space-time Poisson process

(a) *Deterministic part.* When time increases, the "islands" spread, i. e., the up-steps move to the left and the down-steps move to the right, each with unit speed. If an up-step and a down-step meet, then they merge to a single island.

(b) *Stochastic part.* The nucleation events are drawn from a Poisson process in space-time, and once such an up-down-step pair is created, the steps move symmetrically apart from each other following the deterministic dynamics.

See Figure 2.3 for an illustration.

PNG droplet

For the PNG droplet, we start with the initial condition $h(x, 0) = 0$ for all $x \in \mathbb{R}$, and the rate function $\varrho : \mathbb{R} \times [0, \infty) \to [0, \infty)$ of the space-time Poisson process is defined as

$$\varrho(x,t) = \begin{cases} 2, & \text{for } |x| \leq t, \\ 0, & \text{for } |x| > t. \end{cases} \tag{2.14}$$

The macroscopic limit shape h_{ma} in the PNG droplet model is a semi-circle, hence the name "droplet",

$$h_{\mathrm{ma}}(\xi) = \lim_{t\to\infty} \frac{h(\xi t, t)}{t} = 2\sqrt{(1-\xi^2)_+}.$$

Thus, for $\xi = 0$, we expect $h(\xi, t)$ to be around $2t$ for large t. The fluctuations live on a $t^{1/3}$ vertical scale and are governed by the GUE Tracy-Widom distribution [80],

$$\lim_{t\to\infty} \mathbb{P}(h(0,t) \leq 2t + t^{1/3}s) = F_{\mathrm{GUE}}(s) \tag{2.15}$$

with F_{GUE} as defined in (2.4). If we look away from 0 at some $\xi \in (-1, 1)$, then we simply replace t by $t\sqrt{1-\xi^2}$ in (2.15).

This means that the Tracy-Widom distribution does not only appear in random matrix theory, but also in the study of interacting particle systems. At this level, this common feature of GUE and TASEP is rather unexpected, since there is no direct link between these two models. To understand if this just an incident or if there are structural reasons for this behavior, we study the multi-point distribution of PNG droplet and apply the scaling from (2.13),

$$h_{t,\mathrm{resc}}^{\mathrm{curvPNG}}(u) = \frac{h(ut^{2/3},t) - 2t\sqrt{1-u^2t^{-2/3}}}{t^{1/3}}.$$

Then, Prähofer and Spohn [81] showed that

$$\lim_{t\to\infty} h_{t,\mathrm{resc}}^{\mathrm{curvPNG}} = \mathcal{A}_2$$

in the sense of finite-dimensional distributions, where $\mathcal{A}_2$ is the Airy process that we already met in the random matrix context in (2.10). This is a first hint that there should be some connection between TASEP and GUE.

Flat PNG

Instead of taking nucleations from the cone $\{(x,t)\in\mathbb{R}\times[0,\infty) : |x|\leq t\}$ as in (2.14), we can also consider translation-invariant nucleations, i. e., we take a Poisson process with rate $\varrho(x,t) = 2$ for all $(x,t)\in\mathbb{R}\times[0,\infty)$. Choosing again $h(\,\cdot\,0) = 0$ as initial condition, the resulting deterministic limit profile h_{ma} will be flat, $h_{\mathrm{ma}}(\xi) = 2$ for all $\xi\in\mathbb{R}$. Mapping this model to a point-to-line last passage directed percolation model, it was known that [9, 79, 80]

$$\lim_{t\to\infty} \mathbb{P}\big(h(0,t)\leq 2t + \tfrac{s}{2}(2t)^{1/3}\big) = F_{\mathrm{GOE}}(s), \quad s\in\mathbb{R},$$

with F_{GOE} the GOE Tracy-Widom distribution defined in (2.6), and it was conjectured in [17] that the rescaled height function

$$h_{t,\mathrm{resc}}^{\mathrm{flatPNG}}(u) = 2^{-1/3}h_t^{\mathrm{resc}}(2^{2/3}u) = \frac{h\big(u(2t)^{2/3},t\big) - 2t}{(2t)^{1/3}}$$

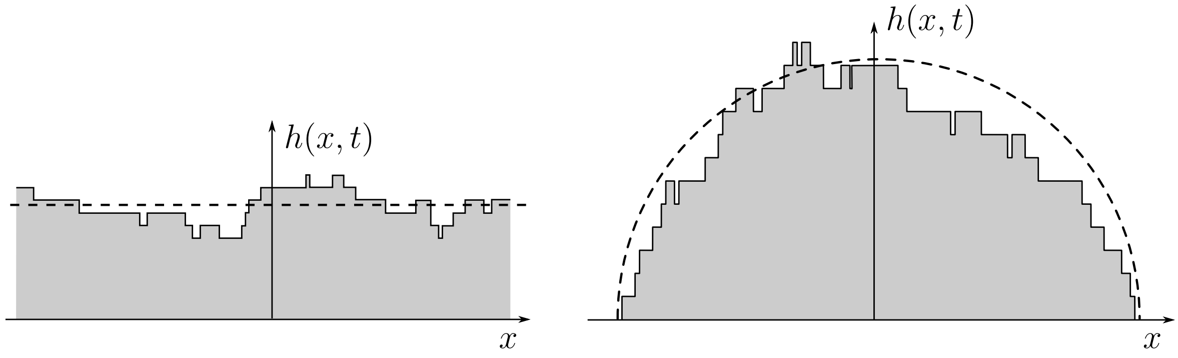
converges to a process $\mathcal{A}_1$ that is also defined in terms of Airy functions. This was finally proven by Borodin, Ferrari, and Sasamoto [18],

$$\lim_{t\to\infty} h_{t,\mathrm{resc}}^{\mathrm{flatPNG}} = \mathcal{A}_1$$

in the sense of finite-dimensional distributions. To distinguish this process from the previous Airy process, we call from now on $\mathcal{A}_2$ the *Airy$_2$ process* and $\mathcal{A}_1$ the *Airy$_1$ process*. Again, a precise definition for $\mathcal{A}_1$ will be given in Chapter 3.1.3. Like the Airy$_2$ process, the Airy$_1$ process is stationary and looks locally like a Brownian motion. For its one-point distribution, we have

$$\mathbb{P}\big(\mathcal{A}_1(0)\leq s\big) = F_{\mathrm{GOE}}(2s), \quad s\in\mathbb{R}.$$

Hence, the Airy processes can be seen as the multi-point extensions of the GOE/GUE Tracy-Widom distributions.



(a) Flat PNG (b) PNG droplet

2.2.2. Continuous time TASEP

The totally asymmetric simple exclusion process on $\mathbb{Z}$ in continuous time is an interacting particle system. For all times t, at most one particle can occupy a site in $\mathbb{Z}$ ("simple") and particles try to jump independently to a neighboring site with rate 1, but only to the right one ("totally asymmetric"). The jumps are made only if the arrival sites are free ("exclusion"), otherwise the jumps are blocked. Note that these dynamics leave the order of the particles as it is. We label the particles from right to left so that $x_k(t)$ denotes the position of the k-labeled particle at time t and $x_k(t) > x_{k+1}(t)$ for all k and t.

Formally, the continuous time TASEP is a Markov process defined on the space $\Omega = \{0,1\}^{\mathbb{Z}}$. For a configuration $\eta(t) \in \Omega$, there is a particle at position $j \in \mathbb{Z}$ and time $t \geq 0$ if $\eta_j(t) = 1$, and the position is empty if $\eta_j(t) = 0$. Let $f : \Omega \to \mathbb{R}$ be a function depending on a finite number of η_j. Then, the backward generator L of TASEP is given by

$$Lf(\eta) = \sum_{j\in\mathbb{Z}} \eta_j(1-\eta_{j+1}) \big(f(\eta^{j,j+1}) - f(\eta)\big)$$

where $\eta^{j,j+1}$ is the configuration η with the occupations at sites j and $j+1$ interchanged. The transition probability for TASEP is e^{Lt}, see [67,68] for more details on the construction. There is a one-to-one correspondence between TASEP configurations and height functions defined by setting the origin $h(0,0) = 0$ and the discrete height gradient to be $1-2\eta_j(t)$. Let us denote by N_t the integrated current of particles through the origin, i. e., the number of particles that jumped from 0 to 1 during the time interval $[0,t]$. Then, the height function h is given by

$$h(x,t) = \begin{cases} 2N_t + \sum\limits_{j=1}^{x} \big(1-2\eta_j(t)\big), & \text{for } x \geq 1, \\ 2N_t, & \text{for } x = 0, \\ 2N_t - \sum\limits_{j=x+1}^{0} \big(1-2\eta_j(t)\big), & \text{for } x \leq -1. \end{cases} \tag{2.16}$$

see Figure 2.4 for an illustration. In the following we will discuss results for two specific initial conditions.

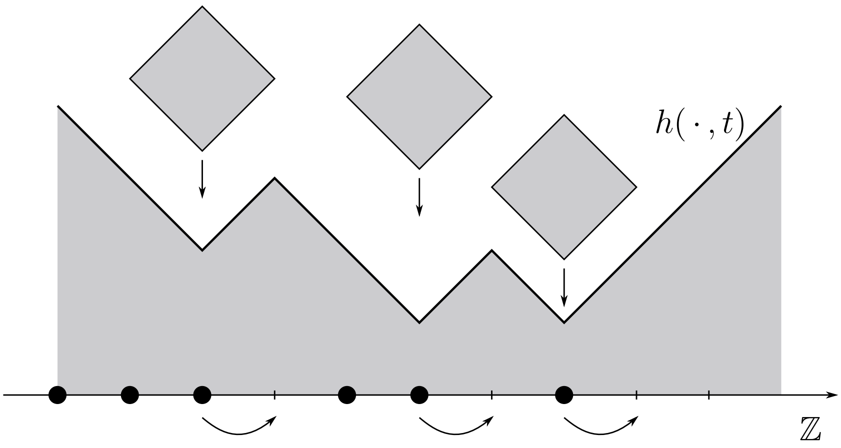


Figure 2.4.: Height function (thick line) corresponding to a particle configuration (black dots). If a particle jumps, a new “corner” will be added to the profile as indicated.

TASEP with step initial condition

Let us choose the initial conditions $\eta_j(0) = 1$ for $j < 0$ and $\eta_j(0) = 0$ for $j \geq 0$. This is called *step initial condition*, see Figure 2.5(a). The macroscopic limit shape h_{ma} for this initial condition is a parabola continued by two straight lines,

$$h_{\mathrm{ma}}(\xi) = \begin{cases} \frac{1}{2}(1+\xi^2), & \text{for } |\xi| \leq 1, \\ |\xi|, & \text{for } |\xi| \geq 1. \end{cases}$$

Now we can scale the height function as in (2.13) and add some constants to avoid them in the limit,

$$h_{t,\mathrm{resc}}^{\mathrm{stepTASEP}}(u) := -2^{1/3} h_t^{\mathrm{resc}}(2^{1/3}u) = \frac{h\big(2u\big(\frac{t}{2}\big)^{2/3}, t\big) - \big(\frac{t}{2} + u^2\big(\frac{t}{2}\big)^{1/3}\big)}{-\big(\frac{t}{2}\big)^{1/3}}.$$

Then, for the one-point distribution, Johansson [57] proved that

$$\lim_{t\to\infty} \mathbb{P}\big(h_{t,\mathrm{resc}}^{\mathrm{stepTASEP}}(0) \leq s\big) = F_{\mathrm{GUE}}(s), \quad s \in \mathbb{R}, \tag{2.17}$$

where F_{GUE} is again the GUE Tracy-Widom distribution. Instead of using the definition of h given in (2.16), we can define the (unrescaled) height function h for TASEP in the case of the step initial condition via

$$\{h(x,t) \geq 2n + x\} = \{x_n(t) \geq x\}, \quad n \geq 1, x \in \mathbb{Z},$$

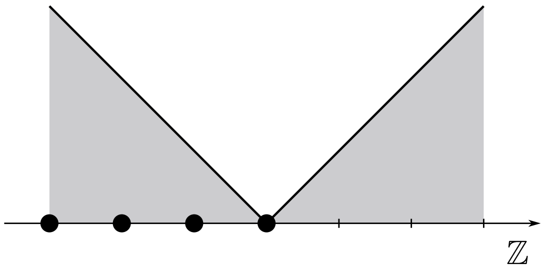
with linear interpolation for non-integer values of x. This allows us to translate Johansson's result (2.17) into the particle picture,

$$\lim_{t\to\infty} \mathbb{P}\big(x_{[t/4]}(t) \geq -s(t/2)^{1/3}\big) = F_{\mathrm{GUE}}(s), \quad s \in \mathbb{R}. \tag{2.18}$$

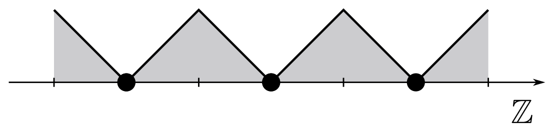
The extension of this result to the multi-point case was done in [16, 19, 58]. It turns out that

$$\lim_{t\to\infty} h_{t,\mathrm{resc}}^{\mathrm{stepTASEP}} = \mathcal{A}_2$$

in the sense of finite dimensional distributions with $\mathcal{A}_2$ being the Airy process from (2.10).



(a) Step profile



(b) Flat profile

Figure 2.5.: TASEP with two different initial conditions

TASEP with flat initial condition

Another type of initial conditions that we present here is the *flat* (or alternating) *initial condition*, i. e., $\eta_j(0) = 1$ for even j and $\eta_j(0) = 0$ for odd j, see Figure 2.5(b) for the associated "flat" height function. In this case, the macroscopic limit shape h_{ma} is just a constant, $h_{\mathrm{ma}}(\xi) = \frac{1}{2}$ for all ξ. The rescaled height functions for the alternating initial condition reads

$$h^{\mathrm{flatTASEP}}_{t,\mathrm{resc}}(u) := -2\, h^{\mathrm{resc}}_t(2u) = \frac{h(2ut^{2/3}, t) - \frac{t}{2}}{-\frac{1}{2}t^{1/3}}$$

and as $t \to \infty$, its one-point distribution is

$$\lim_{t\to\infty} \mathbb{P}(h^{\mathrm{flatTASEP}}_{t,\mathrm{resc}}(0) \leq s) = F_{\mathrm{GOE}}(s),$$

where F_{GOE} is the GOE Tracy-Widom distribution defined in (2.6). The limiting multi-point distribution for $h^{\mathrm{flatTASEP}}_{t,\mathrm{resc}}$ was studied in [17, 85],

$$\lim_{t\to\infty} h^{\mathrm{flatTASEP}}_{t,\mathrm{resc}} = \mathcal{A}_1,$$

where $\mathcal{A}_1$ is again the Airy$_1$ process.

PASEP with step initial conditions

We generalize the TASEP in the sense that we drop the restriction of total asymmetry in the jump direction and assume that particles can jump independently to the right with rate p and to the left with rate $q = 1 - p$. However, we keep the exclusion principle which says that a particle can only jump if the (left or right) neighboring site is empty and that there is at most one particle per site. This is called the *partially asymmetric simple exclusion process* (PASEP) or sometimes just *asymmetric simple exclusion process* (ASEP). As before, we label the particles from right to left and choose step initial conditions, $x_n(0) = -n$ for $n \geq 1$. We have to assume that $q < p$ to have a drift to the right which ensures lateral growth. In a series of papers [102–106], Tracy and Widom were able to show that

$$\lim_{t\to\infty} \mathbb{P}\big(x_{[t/4]}(t/(p-q)) \geq -s(t/2)^{1/3}\big) = F_{\mathrm{GUE}}(s), \quad s \in \mathbb{R}. \tag{2.19}$$

Note that for $p = 1$ and $q = 0$, this is same as (2.18).

2.2.3. KPZ equation

Let us consider again the KPZ equation from (2.12), this time we set $\nu = \frac{1}{2}$ and $\lambda = -1$,

$$\partial_t h = \tfrac{1}{2}\partial_x^2 h - \tfrac{1}{2}(\partial_x h)^2 + \eta. \tag{2.20}$$

In the 90s, Bertini and Giacomin [11] proposed solving this equation via the stochastic heat equation

$$\partial_t Z = \tfrac{1}{2}\partial_x^2 Z - \eta Z, \quad Z(x,0) = Z_0(x), \tag{2.21}$$

where η is space-time white noise. The initial condition Z_0 can be random and is assumed to be independent of η. If $Z_0(x) \geq 0$ for all x and $\int \mathrm{d}x\, Z_0(x) > 0$, then $Z(x,t)$ will be strictly positive for all x and $t > 0$. Thus, we may define

$$h(x,t) := -\log Z(x,t), \tag{2.22}$$

which is called the *Hopf-Cole solution* to the KPZ equation (2.20). Moreover, Bertini and Giacomin proposed that this solution can be obtained from PASEP with $p - q \approx 0$, which is then referred to as the *weakly asymmetric simple exclusion process* (WASEP). Based on explicit formulas for the PASEP that Tracy and Widom obtained in order to prove the convergence in (2.19), Amir, Corwin, and Quastel [4] (and, independently, Sasamoto and Spohn [86, 87, 89])[1] were able to make this approach rigorous. Consider PASEP with step initial condition, i. e., $h(x,0) = |x|$ for all $x \in \mathbb{R}$. Let $\varepsilon := (p-q)^2$ and denote by $h_\varepsilon \equiv h_{p,q}$ the corresponding WASEP height function. Then, as $\varepsilon \to 0$,

$$\varepsilon^{1/2} h_\varepsilon(\varepsilon^{-1}x, \varepsilon^{-2}t) - \tfrac{1}{2}\varepsilon^{-1}t - \log\big(\tfrac{1}{2}\varepsilon^{-1/2}\big) \to h(x,t)$$

with h given by (2.22) where Z is the solution of the stochastic heat equation (2.21) with initial data $Z_0(x) = \delta_0(x)$. Moreover, Amir, Corwin, and Quastel showed that

$$F_t(s) := \mathbb{P}\left(-h(x,t) - \frac{x^2}{2t} + \frac{t}{24} \leq \left(\frac{t}{2}\right)^{1/3} s\right), \quad s \in \mathbb{R}, \quad t > 0,$$

does not depend on x, and F_t converges to the GUE Tracy-Widom distribution,

$$\lim_{t\to\infty} F_t(s) = F_{\mathrm{GUE}}(s), \quad s \in \mathbb{R}.$$

Thus, the KPZ equation itself is also in the KPZ universality class!

2.3. Limits of universality

2.3.1. GOE diffusion and Airy$_1$ process

Let us revisit what we have discussed so far about the correspondence between KPZ models and random matrices. On the one hand, we have growth models with curved limit shape such

[1] A replica approach is in [27, 37, 82]; see the review [88] for details.

as PNG droplet or TASEP with step initial condition, and the fluctuations in these models are described by the GUE Tracy-Widom distribution. As their counterpart in the random matrix world, we identified the Hermitian matrices with Gaussian or Wishart distribution and, more generally, complex Wigner matrices and matrices from the invariant ensemble whose largest eigenvalue fluctuations are also distributed according to F_{GUE}. Thus, the conjecture is that this probability law appears in KPZ models with curved limit shape and in certain classes of Hermitian random matrices.

On the other hand, we presented models that give rise to the GOE Tracy-Widom distribution. For the physical models, these are flat PNG and TASEP with alternating initial condition while for random matrices, we have the symmetric Gaussian and real Wishart matrices as well as real Wigner matrices and matrices from the invariant ensemble. In this case, the conjecture would be that this behavior is universal for flat curved models and for symmetric random matrices with, for example, independent entries.

Let us come to multi-point distributions. For KPZ models with curved limit shapes and Hermitian Gaussian matrices, this connection is still there (we mentioned the Airy$_2$ process), and there is reason to believe that this the universal limit object for these models. Now, to make the picture complete, we should consider the multi-point limiting distribution of the fluctuations in the flat/symmetric models. For the KPZ models with flat limit shape, we already know that they are governed by the Airy$_1$ process. Let us now look at the symmetric analogue of Dyson's Brownian motion. In the Hermitian case, one first calculates the joint distribution of a finite number of eigenvalues $\lambda_1(t) \leq \ldots \leq \lambda_n(t)$. For that, one has to solve an integral of the form

$$\int_{\mathcal{U}(n)} \mathrm{e}^{-\operatorname{Tr}(AUBU^*)}\, \mu(\mathrm{d}U), \tag{2.23}$$

where μ is the Haar measure on the unitary group $\mathcal{U}(n)$ and A, B are diagonal matrices, see also Chapter 3.2.1. An explicit formula for (2.23) was given by Itzykson and Zuber [55] in 1980, see Appendix A.7. Later, one found this formula in a paper by Harish-Chandra [52] from the 50s. But in the symmetric case, i. e., if we integrate over the orthogonal group $\mathcal{O}(n)$ instead of $\mathcal{U}(n)$, such a formula is not available. This is the reason why the multi-point analogue for the GOE distribution in the case of symmetric Dyson's Brownian motion has not been identified yet. Unfortunately, as numerical simulations indicate, there is no hope that the Airy$_1$ process will appear here. Indeed, Bornemann, Ferrari, and Prähofer [14] compared the covariances of the Airy$_1$ process and the rescaled largest eigenvalue in the GOE Dyson's Brownian motion and observed in their simulations that the first one decays super-exponentially fast in the argument, while the latter one decay only polynomially. They concluded that "the Airy$_1$ process is not the limit of the largest eigenvalue in GOE matrix diffusion", which means that the link between random matrices and exclusion processes broken at this level.

2.3.2. Speed of convergence

Another issue concerning the question of universality is the speed of convergence to the Tracy-Widom distribution and the nature of the first order corrections. Since this question is motivated by recent results by Takeuchi et al. [95, 97], let us first look at the appearance of the Tracy-Widom distribution in the "real world". The various models from the KPZ class that we have presented so far aim to describe growth models from physics, so one can ask whether the typical features of the KPZ class can also be verified by experiments. Until recently, there were only few experiments giving the fluctuations exponent $1/3$, see e. g. [71, 108]. Besides the difficulties of having good statistics, one of the main issues in the experimental set-up is to have *really a local* dynamic, and the centering in (2.13) has to be obtained experimentally from the measured asymptotic growth velocity. In any case, experimental data were not good enough to have more detailed information on the scaling exponents, until the recent amazing experiments carried out by Takeuchi et al. (see [95] and [97]). Using nematic liquid crystals they were able to get accurate statistics that confirmed not only the fluctuation and correlation exponents, but also the limiting distribution functions and the covariance of the Airy processes previously obtained in solvable models.

A further aspect that was observed in these experiments was that the fit between the predicted density of the Tracy-Widom distributions and the measurements is quite good even for relatively small time t, but a finite size correction is still visible. It is therefore interesting to study, on a theoretical level, the difference between $F_t := \mathbb{P}(h_t^{\mathrm{resc}}(0) \leq s)$ and F_{GUE} (or F_{GOE}) as $t \to \infty$. As noticed in [86], this correction is of order $t^{-1/3}$, which means that on the original scale, the difference between the height function $h_t(\xi t)$ and $t\, h_{\mathrm{ma}}(\xi)$ is of order 1. In their experiments, Takeuchi et al. also measured the decay of mean, variance, skewness and kurtosis. In the scaled variables, the mean has been seen to decay as $t^{-1/3}$, while other moments decay as $t^{-2/3}$. Thus, in the unrescaled variables, the mean has a shift of order 1.

We will now describe some results that have been obtained for the finite size corrections in some KPZ models. Note that, compared to the liquid crystal experiment, the shift of the mean in the solution of the KPZ equation has opposite sign, and in the experiments, the same sign as for TASEP is observed. This means that if we denote by $h_{p,q}^{\mathrm{resc}}$ the rescaled PASEP height function and by ζ a random variable with GUE Tracy-Widom distribution, then the sign of

$$a(p) := \lim_{t\to\infty} \mathbb{E}\big[t^{1/3}\big(h_{p,q}^{\mathrm{resc}}(0) - \zeta\big)\big]$$

is different for $p = 1$ (TASEP) and $p \approx \frac{1}{2}$ (WASEP). Hence, there will be a certain value $p \in \left(\frac{1}{2}, 1\right)$ of asymmetry for which the mean has no shift (up to $\mathcal{O}(t^{-2/3})$). A Monte-Carlo simulation [86] indicates that this happens for the PASEP height function at the origin for the critical value $p = p_c \simeq 0.78$. We can determine this value analytically.

Result 1. *In Corollary 16 we show that the critical value p_c is the solution of*

$$\sum_{\ell=1}^{\infty} \frac{(1-p_c)^\ell}{p_c^\ell - (1-p_c)^\ell} = \frac{1}{2} \qquad \Longleftrightarrow \qquad p_c = 0.78227\,87862\ldots$$

We first determine an analytic formula for the shift of the distribution of a *tagged particle* (see Proposition 14). The shift turns out to be a function of the macroscopic particle number.

However, when we switch back to the height function representation, the shift becomes again independent of the macroscopic position (the ξ in (2.13)).

We will also obtain the first order correction to the limiting distribution function and density. Let us illustrate this result for the PNG droplet. (The other cases are analogue, but instead of the GUE one has for example the GOE Tracy-Widom distribution. Form (2.15) we know that

$$h_{t,\text{resc}}^{\text{curvPNG}}(0) = \frac{h(0,t) - c_1 t}{c_2 t^{1/3}} \xrightarrow{d} \zeta \quad \text{as } t \to \infty, \tag{2.24}$$

where $c_1 = 2$, $c_2 = 1$ and ζ is again a GUE Tracy-Widom distributed random variable, i. e.,

$$F_t(s) := \mathbb{P}\big(h_{t,\text{resc}}^{\text{curvPNG}}(0) \leq s\big) \to F_{\text{GUE}}(s) \quad \text{as } t \to \infty.$$

Since the unrescaled height function h only takes values in the integers, F_t is piecewise constant over intervals of length $\delta_t := 1/(c_2 t^{1/3})$. Thus, we expect that

$$h_{t,\text{resc}}^{\text{curvPNG}}(0) = \zeta + \eta\,\delta_t + \mathcal{O}(\delta_t^2) \quad \text{on } I_t := (\mathbb{Z} - c_1 t)\delta_t, \tag{2.25}$$

where η is another random variable that is a priori not independent from ζ. Note that (2.25) is a shorthand notation of

$$\mathbb{P}\big(h_{t,\text{resc}}^{\text{curvPNG}}(0) \leq s\big) = \mathbb{P}\big(\zeta + \eta\,\delta_t + \mathcal{O}(\delta_t^2) \leq s\big), \quad s \in I_t.$$

What is the nature of this first order coefficient η? The surprising result is that for all the models we consider, η is a *deterministic constant* and therefore *independent* of ζ (see Chapter 4.2 for PNG and TASEP, Chapter 4.3 for PASEP). This implies the following.

Result 2. *Let us denote by $\delta_t := c_2^{-1} t^{-1/3}$ the discrete lattice width where $h_{t,\text{resc}}^{\text{curvPNG}}(0)$ lives. There exists a constant η such that*

$$F_t(s) = \mathbb{P}\big(h_{t,\text{resc}}^{\text{curvPNG}}(0) \leq s\big) = F_{\text{GUE}}(s - \eta\,\delta_t) + \mathcal{O}(\delta_t^2) \tag{2.26}$$

for all $s \in I_t = (\mathbb{Z} - c_1 t)\delta_t$.

For PNG and TASEP the shift $\eta\,\delta_t$ does not depend on the chosen macroscopic position, but this property is not generic and might depend on the chosen observable too, as shown by the result on PASEP. (This non-universality of η is quite intuitive, since η is a correction term on the microscopic scale, thus model-dependent.) Consequently, by shifting the height function h_t by the constant η as in (2.26), the convergence of the distribution function to F_{GUE} is of order $\mathcal{O}(t^{-2/3})$. If η was not independent from ζ, then one would have a convergence only of order $\mathcal{O}(t^{-1/3})$ instead.

In the domain of random matrices, similar results have been obtained by Choup [29–31] and El Karoui [40]. For instance, let λ_N^{GUE} be the largest eigenvalue of an $N \times N$ GUE matrix as in (2.3). Then,

$$\mathbb{P}\left(\frac{\sqrt{N}\lambda_N^{\text{GUE}} - c_1 N}{c_2 N^{1/3}} \leq s\right) = F_{\text{GUE}}(s - \eta\,\delta_t) + \mathcal{O}(\delta_N^2),$$

where $c_1 = 2$, $c_2 = 1$ and $\eta = 0$. Thus, in this case, we do not need any shift to have convergence of order $\mathcal{O}(t^{-2/3})$.

An important difference between the KPZ models and random matrices is that the largest eigenvalues are, even for finite matrix size N, *continuous* random variables, while in the models considered above the random variables live on $\mathbb{Z}$ and after the rescaling (2.13), they still are on discrete lattice of width δ_t. This discreteness is of course irrelevant for the universal limit statements, but at first order it can not be neglected when looking at the fit with the limiting distribution function and density. Indeed, the shift needed to have a fit with accuracy of order $\mathcal{O}(t^{-2/3})$ is not the same for the density as for distribution function. In order to see this feature, consider the slightly modified scaling of the height function

$$\widetilde{h}^{\text{curvPNG}}_{t,\text{resc}}(0) := h^{\text{curvPNG}}_{t,\text{resc}}(0) - a\,\delta_t, \tag{2.27}$$

where $a \in \mathbb{R}$ is a given constant. Further, we set

$$\widetilde{F}_t := \mathbb{P}\Big(\widetilde{h}^{\text{curvPNG}}_{t,\text{resc}}(0) \le s\Big),$$

and define the discrete probability density function as

$$\widetilde{p}_t(s) := \frac{\widetilde{F}_t(s) - \widetilde{F}_t(s-\delta_t)}{\delta_t}. \tag{2.28}$$

If the η is the shift that yields a convergence of order $\mathcal{O}(t^{-2/3})$ for the cumulative distribution function, then we have to shift the probability density function not by η, but by $\eta + \frac{1}{2}$ to have the same order of convergence. This states the following result.

Result 3. *With the choice $a := \eta + \frac{1}{2}$ we have*

$$\widetilde{p}_t(s) = F'_{\text{GUE}}(s) + \mathcal{O}(\delta_t^2).$$

for all $s \in \widetilde{I}_t := (\mathbb{Z} - c_1 t - a)\delta_t$.

These results are discussed in Chapter 4.1 and used for the fits of the simulations of TASEP in Chapter 4.2.

Remark 1. Result 3 does not depend on the concrete representation of our distributions, but is generic in the sense that it is a consequence of the $\mathcal{O}(\delta_t^2)$ error for the centered discrete derivative (2.28).

Remark 2. With the scaling (2.27), Result 2 writes

$$\widetilde{F}_t(s) = F_{\text{GUE}}(s + \tfrac{1}{2}\delta_t) + \mathcal{O}(\delta_t^2), \quad s \in \widetilde{I}_t,$$

while the scaling (2.24) yields

$$F_t(s) = F_{\text{GUE}}(s + \tfrac{1}{2}\delta_t - a\delta_t) + \mathcal{O}(\delta_t^2), \quad s \in I_t, \tag{2.29}$$

and

$$p_t(s) = F'_2(s - a\delta_t) + \mathcal{O}(\delta_t^2), \quad s \in \widetilde{I}_t, \tag{2.30}$$

where $p_t(s) := \delta_t^{-1}(F_t(s) - F_t(s-\delta_t))$.

In view of these results, we carried out a simulation for TASEP with time $t = 1000$. As observable we used a tagged particle. The dots in Figure 4.4 represent $s \mapsto \widetilde{p}_t(s)$ for $s \in I_t$ and $a = \eta + \frac{1}{2}$, which is well approximated by the solid line $s \mapsto F'_{\mathrm{GUE}}(s)$ as predicted by Result 3. As comparison, the dashed line is the unshifted density $s \mapsto F'_{\mathrm{GUE}}(s - a\delta_t)$ (see (2.30)), i.e., the fit obtained with $a = 0$.

The same applies to the distribution function. The dots in Figure 4.3 are the plot of $s \mapsto \widetilde{F}_t(s)$ for $s \in I_t$ and $a = \eta + \frac{1}{2}$. The dashed line is the predicted limiting distribution function with scaling (2.24), be $s \mapsto F_{\mathrm{GUE}}(s) = F_{\mathrm{GUE}}(s + \frac{1}{2}\delta_t - a\delta_t)$ (see (2.29)). The fit suggested by Result 2 is the solid line, $s \mapsto F_{\mathrm{GUE}}(s + \frac{1}{2}\delta_t)$, which indeed is a better fit.

In the same way we fit Figures 4.2 and 4.1 with the difference that the limiting distribution function is $s \mapsto F_{\mathrm{GOE}}(2s)$.

Finally, the shift used in $h_{t,\mathrm{resc}}^{\mathrm{curvPNG}}(0)$ is the same needed to have a convergence of the moments, and consequently of the variance, skewness, kurtosis of order $\mathcal{O}(t^{-2/3})$. The following result will be discussed in Section 4.1.2.

Result 4. *We have*

$$\mathbb{E}\big[\big(h_{t,\mathrm{resc}}^{\mathrm{curvPNG}}(0)\big)^m\big] = \mathbb{E}\big[\zeta^m\big] + \mathcal{O}(\delta_t^2)$$

for all $m \in \mathbb{N}$.

Remark that if η was not independent from ζ, the convergence of the variance, skewness, and kurtosis would still be of order $\mathcal{O}(t^{-1/3})$.

3. At the interface between GUE and TASEP

3.1. Determinantal point processes

3.1.1. Correlation functions and kernels

In this section we will introduce the basic notions that we need to state Results 7 to 13. Since we do not want to develop the whole theory on point processes, determinantal correlation functions and Fredholm determinants, our presentation will be rather sketchy. For more information on these topics, we refer to [33, 34, 90].

The mathematical concept behind both random matrices and growth models are *point processes*, which means that we consider the eigenvalues of a random matrix or the particles in a jump process as randomly placed points on $\mathbb{R}$ or $\mathbb{Z}$. To be more precise, we consider the one-particle space Λ which is a complete separable metric space, equipped with some reference measure λ on the Borel σ-algebra $\mathcal{B}(\Lambda)$ generated by the open sets in Λ. Typically, Λ will be $\mathbb{R}$ or $\mathbb{Z}$, but also $\mathbb{R} \times \{1, \ldots, N\}$ etc. are possible. A *point measure* on Λ is then a positive measure ν on $(\Lambda, \mathcal{B}(\Lambda))$ such that ν is a locally finite sum of Dirac measures, i. e., $\nu = \sum_{i \in I} \delta_{x_i}$ with $x_i \in \Lambda$, $I \subseteq \mathbb{N}$, and for any bounded Borel set $B \in \mathcal{B}(\Lambda)$ we have that $x_i \in B$ only for a finite number of $i \in I$.

Denote by $M(\Lambda)$ the space of point measures on Λ and let $\mathcal{M}(\Lambda)$ be the smallest σ-algebra such that for any Borel set $B \in \mathcal{B}(\Lambda)$, the mapping $M(\Lambda) \to \mathbb{N} \cup \{\infty\}$, $\nu \mapsto \nu(A)$ is measurable. A *point process* η on Λ is a random variable with values in $M(\Lambda)$, i. e., a measurable mapping from some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to $(M(\Lambda), \mathcal{M}(\Lambda))$. The distribution of η is the image of $\mathbb{P}$ by η.

We will only consider *simple* point processes, i. e., $\mathbb{P}\big(\eta(\{x\}) \leq 1\big) = 1$ for all $x \in \Lambda$. Let us now define the correlation functions of a point process. For bounded and disjoint subsets $A_1, \ldots, A_n$ of Λ we define

$$M_n(A_1, \ldots, A_n) = \mathbb{E}\Bigg[\prod_{i=1}^{n} \eta(A_i)\Bigg].$$

Let η be simple point process. If M_n is absolutely continuous with respect to μ, i. e., if there exists a function $\varrho^n : \Lambda^n \to [0,\infty)$ such that

$$M_n(A_1,\dots A_n) = \int_{A_1\times\cdots\times A_n} \mu(\mathrm{d}x_1)\cdots\mu(\mathrm{d}x_n)\,\varrho^n(x_1,\dots,x_n) \tag{3.1}$$

for all bounded and disjoint $A_1,\dots,A_n \subseteq \Lambda$, then we call ϱ^n the *n-point correlation function* of η. Moreover, we assume that $\varrho^n(x_1,\dots,x_n) = 0$ if $x_i = x_j$ for some $i \neq j$.

Informally, the n-point correlation $\varrho^n(x_1,\dots,x_n)$ is the probability of finding particles of η at positions $x_1,\dots,x_n$,

$$\varrho^n(x_1,\dots,x_n) = \lim_{\varepsilon\to 0} \frac{\mathbb{P}(\eta \text{ has a point in } B_\varepsilon(x_i) \text{ for } 1\le i\le n)}{\mu(B_\varepsilon(x_1))\cdots\mu(B_\varepsilon(x_n))},$$

where $B_\varepsilon(x)$ denotes the ball of radius $\varepsilon > 0$ around x.

If $A_1,\dots,A_n$ are not all disjoint, say $A_1 = \cdots = A_n \equiv A$, there is another way to express the correlation functions, which is a consequence of (3.1),

$$\int_{A^n} \mu(\mathrm{d}x_1)\cdots\mu(\mathrm{d}x_n)\,\varrho^n(x_1,\dots,x_n) = \mathbb{E}\left[\frac{\eta(A)!}{(\eta(A)-n)!}\right]. \tag{3.2}$$

To convince ourselves that this is true, note that the case $n=1$ is trivial. For $n=2$, we have

$$\mathbb{E}[\eta(A)^2] = \mathbb{E}\left[\left(\sum_{x\in A}\eta(\{x\})\right)^2\right] = \mathbb{E}\left[\sum_{x\in A}\eta(\{x\})\right] + \mathbb{E}\left[\sum_{x\neq y}\eta(\{x\})\eta(\{y\})\right], \tag{3.3}$$

where we used that η is simple, i. e., $\eta(\{x\})$ is 0 or 1, and $\eta(\{x\}) = 1$ for a finite number of x only. Now, we have that $\mathbb{E}[\eta(\{x\})\eta(\{y\})] = \varrho_2(x,y)$ for $x \neq y$ and $\varrho_2(x,x) = 0$. Thus,

$$\mathbb{E}[\eta(A)^2] = \mathbb{E}[\eta(A)] + \int_{A^2}\mu(\mathrm{d}x)\mu(\mathrm{d}y)\,\varrho_2(x,y),$$

which implies (3.2) for $n=2$.

A point process η on Λ is called *determinantal point process* with kernel K if the n-point correlation functions ϱ^n are given by

$$\varrho^n(x_1,\dots,x_n) = \det[K(x_i,x_j)]_{1\le i,j\le n}$$

for any $n \geq 1$ and $x_1,\dots,x_n \in \Lambda$, where $K : \Lambda\times\Lambda \to \mathbb{C}$ is a measurable function.

To such a kernel we can associate an integral operator $K : L^2(\Lambda) \to L^2(\Lambda)$ by setting

$$(Kf)(x) = \int_\Lambda \mu(\mathrm{d}y)\,K(x,y)f(y), \quad f \in L^2(\Lambda).$$

Let us assume that K is a locally trace class operator, i. e., $K\,\mathbb{1}_B$ is trace class for any compact subset B of Λ. Moreover, let K be Hermitian, i. e., $K(x,y) = \overline{K(y,x)}$ for any $x,y \in \Lambda$. In the 70s, Macchi [69] argued that a sufficient condition on K to define a determinantal point process is $0 \leq K < 1$. Later, Soshnikov [93] proved that K defines a determinantal point process if and only if $0 \leq K \leq 1$, i. e., both K and $\mathbb{1} - K$ are non-negative operators.

The determinantal structure of the correlation functions is very useful to calculate gap probabilities. That is the probability that a random configuration $(x_i)_i$ of a point process in not in a Borel set B of Λ,

$$\begin{aligned}\mathbb{P}\big(\text{no point of } (x_i)_i \text{ is in } B\big) &= \mathbb{E}\Big[\prod_i (1 - \mathbb{1}_B(x_i))\Big] \\ &= \sum_{n\geq 0} \frac{(-1)^n}{n!} \mathbb{E}\Big[\sum_{\substack{i_1,\dots,i_n \\ \text{pairwise distinct}}} \prod_{k=1}^n \mathbb{1}_B(x_{i_k})\Big] \\ &= \sum_{n\geq 0} \frac{(-1)^n}{n!} \int_{B^n} \mu(\mathrm{d}x_1)\dots\mu(\mathrm{d}x_n)\, \varrho^n(x_1,\dots,x_n),\end{aligned}$$

where the last equation follows from a calculation similar to (3.3). As the point process is determinantal with kernel K, we can now replace the correlation functions in the last line by the determinant over K. The resulting expression is the Fredholm determinant $\det(\mathbb{1}-K)_{L^2(B)}$ of the operator K on $L^2(B)$,

$$\det(\mathbb{1} - K)_{L^2(B)} = \sum_{n\geq 0} \frac{(-1)^n}{n!} \int_{B^n} \mu(\mathrm{d}x_1)\dots\mu(\mathrm{d}x_n)\, \det[K(x_i,x_j)]_{1\leq i,j\leq n}.$$

We can take this series as a definition of $\det(\mathbb{1} - K)_{L^2(B)}$. As soon as the series is absolutely convergent, everything is well-defined. Another point of view is to think of $\det(\mathbb{1} - K)_{L^2(B)}$ as the Fredholm determinant of an operator. Since this approach requires some facts from spectral theory, we refer the reader to [90].

3.1.2. Hermite kernel

Usually, it is not trivial to decide whether a given measure induces a determinantal point process. However, if it can be written as a product of determinants, there is a good chance that one can find a correlation kernel. As an example, let us consider the point process of GUE eigenvalues. As mentioned in (2.2), the joint density of the ordered eigenvalues $\lambda_1 \leq \dots \leq \lambda_N$ of an $N \times N$ GUE matrix is given by

$$\text{const} \times \Delta_N^2(\lambda) \prod_{k=1}^N \mathrm{e}^{-(\lambda_k)^2/2}, \tag{3.4}$$

where the repulsion between the eigenvalues is described by

$$\Delta_N(\lambda) = \prod_{1\leq k<\ell\leq N} (\lambda_\ell - \lambda_k) = \det[\lambda_\ell^{k-1}]_{1\leq k,\ell\leq N}$$

which is known as the *Vandermonde determinant*. Since the determinant is a multilinear mapping, we can write

$$\det[\lambda_\ell^{k-1}]_{1\le k,\ell\le N} = \mathrm{const} \times \det[p_{k-1}(\lambda_\ell)]_{1\le k,\ell\le N} \tag{3.5}$$

for any polynomials $p_0, \dots, p_{N-1}$ with $\deg p_j = j$, where const depends on the leading coefficients in the polynomials. In this situation, a more general theorem applies. To employ it, we define Φ_k and Ψ_k as

$$\Phi_k(x) = \Psi_k(x) = \mathrm{e}^{-x^2/4} p_{k-1}(x), \quad 1 \le k \le N, \quad x \in \mathbb{R}. \tag{3.6}$$

Then, denoting by P_N the joint density of $\lambda_1 \le \dots \le \lambda_N$ from (3.4), we can write

$$\begin{aligned} P_N(\lambda_1, \dots, \lambda_N) = \frac{1}{Z_N} & \det[\Phi_k(\lambda_\ell)]_{1\le k,\ell\le N} \det[\Psi_k(\lambda_\ell)]_{1\le k,\ell\le N} \\ & = \frac{1}{Z_N} \det\left[\sum_{j=1}^N \Phi_j(\lambda_k)\Psi_j(\lambda_\ell)\right]_{1\le k,\ell\le N}, \end{aligned} \tag{3.7}$$

and using the symmetry of P_N, we have

$$\varrho^n(\lambda_1, \dots, \lambda_n) = \frac{N!}{(N-n)!} \int_{\mathbb{R}^{N-n}} \mathrm{d}\lambda_{n+1} \cdots \lambda_N \, P_N(\lambda_1, \dots, \lambda_N).$$

For general Φ_k and Ψ_k, if a probability density P_N is given as the product of two determinants as in (3.7) with $Z_N \neq 0$, then one can show that P_N has determinantal correlation functions,

$$\varrho^n(\lambda_1, \dots, \lambda_n) = \det[K(\lambda_i, \lambda_j)]_{1\le i,j\le n}, \quad n \ge 1,$$

with correlation kernel K. For a proof, see e. g. [60]. A representation for K is then given by

$$K(x, x') = \sum_{i,j=1}^N \Psi_i(x)[M^{-1}]_{ij}\Phi_j(x'), \tag{3.8}$$

where the $N \times N$ matrix M is defined by

$$M_{ij} = \Phi_i * \Psi_j = \int_\Lambda \mu(\mathrm{d}x)\, \Phi_i(x)\Psi_j(x).$$

Although (3.8) gives an explicit expression for the kernel, it is often difficult to use this formula for practical issues, because we have to invert the matrix M. However in the special case of (3.6), by choosing the polynomials $p_0, \dots, p_{N-1}$ to be the *Hermite polynomials* (see Appendix A.5 for a definition), the kernel has a simpler representation. Actually, this choice corresponds to a change of basis in which the Hermite polynomials are orthonormal with respect to the Gaussian weight $x \mapsto \mathrm{e}^{-x^2/2}$. Thus, M is then the identity matrix which can be easily inverted. The correlation kernel K^{GUE} for the eigenvalues' point process of an $N \times N$ GUE matrix is then

$$K^{\mathrm{GUE}}(x, x') = \frac{2}{(2\pi \mathrm{i})^2} \oint_{|z|=\varepsilon/2} \mathrm{d}z \int_{\mathrm{i}\mathbb{R}+\varepsilon} \mathrm{d}w \, \frac{\mathrm{e}^{w^2-2xw}}{\mathrm{e}^{z^2-2x'z}} \frac{1}{w-z} \frac{w^N}{z^N} \tag{3.9}$$

for any given $\varepsilon > 0$. Since Hermite polynomials play an important role in this representation, K^{GUE} is also called the *Hermite kernel*.

3.1.3. Airy processes and spatial persistence

Here is a good place to define the Airy processes that we already met in Chapter 2, the definitions are taken from [44]. The *Airy₁ process* $\mathcal{A}_1$ is the process with m-point joint distributions at $u_1 < u_2 < \cdots < u_m$ given by the Fredholm determinant

$$\mathbb{P}\Big(\bigcap_{k=1}^{m}\{\mathcal{A}_1(u_k) \le s_k\}\Big) = \det(\mathbb{1} - \chi_s K_{\mathcal{A}_1}\chi_s)_{L^2(\{u_1,\ldots,u_m\}\times\mathbb{R})} \tag{3.10}$$

where $\chi_s(u_k,x) = \mathbb{1}_{[x>s_k]}$ and the kernel $K_{\mathcal{A}_1}$ is given by

$$\begin{aligned} K_{\mathcal{A}_1}(u,x;u',x') = &-\frac{1}{\sqrt{4\pi(u'-u)}}\exp\Big(-\frac{(x'-x)^2}{4(u'-u)}\Big)\mathbb{1}_{[u<u']} \\ &+ \mathrm{Ai}\big(x+x'+(u'-u)^2\big)\exp\Big((u'-u)(x+x')+\frac{2}{3}(u'-u)^3\Big). \end{aligned}$$

To make this definition more explicit, the Fredholm determinant in (3.10) can be represented by the following expansion,

$$\begin{aligned} &\det(\mathbb{1} - \chi_s K_{\mathcal{A}_1}\chi_s)_{L^2(\{u_1,\ldots,u_m\}\times\mathbb{R})} \\ &\quad= \sum_{n\ge 0}\frac{(-1)^n}{n!}\sum_{i_1,\ldots,i_n=1}^{m}\int_{s_{i_1}}^{\infty}\mathrm{d}x_1\cdots\int_{s_{i_n}}^{\infty}\mathrm{d}x_n\,\det\big[K_{\mathcal{A}_1}(u_{i_k},x_k;u_{i_\ell},x_\ell)\big]_{1\le k,\ell\le n}. \end{aligned}$$

The *Airy₂ process* $\mathcal{A}_2$ is the process with m-point joint distributions at $u_1 < u_2 < \cdots < u_m$ given by the Fredholm determinant

$$\mathbb{P}\Big(\bigcap_{k=1}^{m}\{\mathcal{A}_2(u_k) \le s_k\}\Big) = \det(\mathbb{1} - \chi_s K_{\mathcal{A}_2}\chi_s)_{L^2(\{u_1,\ldots,u_m\}\times\mathbb{R})},$$

where χ_s is defined as above and the kernel $K_{\mathcal{A}_2}$ is given by

$$K_{\mathcal{A}_2}(u,x;u',x') = \begin{cases} \displaystyle\int_0^{\infty}\mathrm{d}\lambda\,\mathrm{e}^{(u'-u)\lambda}\,\mathrm{Ai}(\lambda+x)\,\mathrm{Ai}(\lambda+x'), & \text{for } u \ge u', \\ \displaystyle -\int_{-\infty}^{0}\mathrm{d}\lambda\,\mathrm{e}^{(u'-u)\lambda}\,\mathrm{Ai}(\lambda+x)\,\mathrm{Ai}(\lambda+x'), & \text{for } u < u'. \end{cases}$$

Let us briefly attack the question of persistence (or survival) probability for the Airy processes. This is the probability that a process stays positive (resp. negative), or more generally, above (resp. below) a certain threshold during a time interval $[0,L]$, i. e., for a threshold $c \in \mathbb{R}$ and a time interval $[0,L]$ with $L > 0$, the persistence probabilities are defined by

$$\begin{aligned} P_-(\mathcal{A},c,L) &= \mathbb{P}\big(\mathcal{A}(t) \le c \text{ for all } t \in [0,L]\big), \\ P_+(\mathcal{A},c,L) &= \mathbb{P}\big(\mathcal{A}(t) \ge c \text{ for all } t \in [0,L]\big), \end{aligned}$$

where $\mathcal{A} \in \{\mathcal{A}_1,\mathcal{A}_2\}$. Based on two works on the continuum statistics [32, 83], it is possible to determine analytic formulas for the persistence probability to stay below a threshold c, both for the Airy$_1$ and the Airy$_2$ processes.

Result 5. *For the Airy*$_1$ *process we have*

$$P_-(\mathcal{A}_1, c, L) = \det(\mathbb{1} - K_{1,L})_{L^2(\mathbb{R})}$$

where the kernel $K_{1,L}$ *is given by*

$$K_{1,L}(x, x') = \mathrm{Ai}(|x| + x' + 2c) + \mathbb{1}_{[x\leq 0]}(\widetilde{K}_{1,L}(x, x' + 2c) - \widetilde{K}_{1,L}(-x, x' + 2c)) \tag{3.11}$$

with

$$\widetilde{K}_{1,L}(x, x') = \frac{1}{\sqrt{4\pi L}} \int_{\mathbb{R}_+} \mathrm{d}y\, \mathrm{e}^{-(x-y)^2/4L} \mathrm{e}^{-2L^3/3} \mathrm{e}^{-L(x'+y)}\, \mathrm{Ai}(x' + y + L^2).$$

Result 6. *For the Airy*$_2$ *process we have*

$$P_-(\mathcal{A}_2, c, L) = \det(\mathbb{1} - K_{2,L})_{L^2(\mathbb{R})}$$

where the kernel $K_{2,L}$ *is given by*

$$\begin{aligned} K_{2,L}(x, x') &= K_{\mathrm{Ai}}(x + c, x' + c) \\ &\qquad - \mathbb{1}_{[x\leq 0]} \int_{\mathbb{R}_-} \mathrm{d}y \int_{\mathbb{R}} \mathrm{d}\mu\, \mathrm{e}^{(\mu-c)L} \phi(x, \mu)\phi(y, \mu) K_{\mathrm{Ai},L}(y + c, x' + c) \end{aligned}$$

with

$$\begin{aligned} K_{\mathrm{Ai},L}(x, x') &:= K_{\mathcal{A}_2}(L, x; 0, x') = \int_{\mathbb{R}_+} \mathrm{d}\lambda\, \mathrm{e}^{-L\lambda}\, \mathrm{Ai}(\lambda + x)\, \mathrm{Ai}(\lambda + x'), \\ K_{\mathrm{Ai}}(x, x') &:= K_{\mathrm{Ai},0}(x, x') = \int_{\mathbb{R}_+} \mathrm{d}\lambda\; \mathrm{Ai}(\lambda + x)\, \mathrm{Ai}(\lambda + x') \end{aligned}$$

and[1]

$$\phi(x, \mu) = \frac{\mathrm{Ai}(\mu)\, \mathrm{Bi}(x + \mu) - \mathrm{Ai}(x + \mu)\, \mathrm{Bi}(\mu)}{\sqrt{\mathrm{Ai}^2(\mu) + \mathrm{Bi}^2(\mu)}}, \qquad x \in \mathbb{R}_-, \quad \mu \in \mathbb{R}.$$

For large L, the persistence probabilities decay exponential in L with persistence coefficients $\kappa_\pm$ given by

$$P_\pm(\mathcal{A}, c, L) \simeq C_\pm(\mathcal{A}, c) e^{-\kappa_\pm(\mathcal{A},c)L} \quad \text{for large } L$$

for $\mathcal{A} \in \{\mathcal{A}_1, \mathcal{A}_2\}$.

In Chapter 2.3.2 we already mentioned that in an amazing experiment with turbulent nematic liquid crystals, Takeuchi and Sano [95, 97] were able to verify experimentally the KPZ predictions at the level of distribution functions and covariances (and not only at the level of the scaling exponents). The agreement with the theory is very good. In a more recent paper [96], the same authors measured, among others, the spatial persistence coefficients with respect to

[1] Note that $\mathrm{Ai}^2(x) + \mathrm{Bi}^2(x) > 0$ for all $x \in \mathbb{R}$.

a threshold given by the average of the process. In the case of the Airy$_2$ process, the persistence coefficients have also been measured in an off-lattice Eden model [94] and verified by a numerical simulation of GUE Dyson's Brownian Motion [96].

Using Result 5 and the numerical approach for computing Fredholm determinants developed by Bornemann in [13], it is possible to determine for Airy$_1$ process the associated persistence coefficient and its dependence on the threshold c. The advantage of looking directly at the limit process is that we do not have uncontrolled uncertainties coming from the finite size settings of an experimental setup. The experimental results of [96] fits fairly well with the exact numerical results that can be found in [48].

3.2. Extended kernels

3.2.1. Diffusion on GUE matrices

As a variant of the classical Dyson's Brownian motion let us consider the Brownian motion $(H(t) : t \geq 0)$ on Hermitian matrices that we defined in Chapter 2.1.2, i. e., let $H(t)$ be the $N \times N$ Hermitian matrix defined by

$$H_{ij}(t) = \begin{cases} \frac{1}{\sqrt{2}}\, b_{ii}(t), & \text{for } 1 \leq i \leq N, \\ \frac{1}{2}\big(b_{ij} + \mathrm{i}\,\tilde{b}_{ij}(t)\big), & \text{for } 1 \leq i < j \leq N, \\ \frac{1}{2}\big(b_{ij} - \mathrm{i}\,\tilde{b}_{ij}(t)\big), & \text{for } 1 \leq j < i \leq N, \end{cases} \tag{3.12}$$

where b_{ij} and $\tilde{b}_{ij}$ are independent standard Brownian motions. Then the transition probability from $H(s)$ at time s to $H(t)$ at time t for $0 \leq s < t$ is given by

$$\text{const} \times \exp\bigg(-\frac{\mathrm{Tr}(H(t) - H(s))^2}{2(t-s)}\bigg),$$

where the normalization constant still depends on $t - s$. Then we diagonalize the Hermitian matrices $H(t)$ and $H(s)$ and use the Harish-Chandra/Itzykson-Zuber formula to integrate out the unitary matrices. The induced transition density where starting from the ordered eigenvalues $\lambda_1(s) \leq \cdots \leq \lambda_N(s)$ of $H(s)$ and going to the eigenvalues $\lambda_1(t) \leq \cdots \leq \lambda_N(t)$ of $H(t)$ is then given by

$$\frac{\Delta_N(\lambda(t))}{\Delta_N(\lambda(s))} \det\bigg[\frac{1}{\sqrt{2\pi(t-s)}} \exp\bigg(-\frac{(\lambda_i(t) - \lambda_j(s))^2}{2(t-s)}\bigg)\bigg]_{1\leq i,j\leq N}.$$

For $t > 0$, let us define the Markov kernel

$$P_t^n(x, \mathrm{d}^n y) = \frac{\Delta_n(y)}{\Delta_n(x)} \det\bigg[\frac{1}{\sqrt{2\pi t}} \exp\bigg(-\frac{(y_i - x_j)^2}{2t}\bigg)\bigg]_{1\leq i,j\leq n} \mathrm{d}^n y. \tag{3.13}$$

Thus, starting with $\lambda_1(0) = \cdots = \lambda_N(0) = 0$ at time $t = 0$, the joint density of the eigenvalues $\lambda(t_1), \lambda(t_2), \ldots, \lambda(t_m)$ for times $0 < t_1 < t_2 < \cdots < t_m$ is given by

$$\text{const} \times \prod_{n=1}^{N} \exp\Big(-\frac{(\lambda_n(t_1))^2}{2t_1}\Big)\Delta_N(\lambda(t_1)) \\ \times \prod_{j=1}^{m-1} \det\Big[\exp\Big(-\frac{(\lambda_k(t_{j+1}) - \lambda_\ell(t_j))^2}{2(t_{j+1}-t_j)}\Big)\Big]_{1\le k,\ell\le N} \Delta_N(\lambda(t_m)).$$

As in (3.5), we use that $\Delta_N(x) = \det[x_i^{j-1}]_{1\le i,j\le N}$ and write this formula as

$$\frac{1}{Z_{N,m}} \det[\Phi_k(\lambda_\ell(t_1))]_{1\le k,\ell\le N} \prod_{j=1}^{m-1} \det[\mathcal{T}_{t_j,t_{j+1}}(\lambda_k(t_j), \lambda_\ell(t_{j+1})]_{1\le k,\ell\le N} \\ \times \det[\Psi_k(\lambda_\ell(t_m))]_{1\le k,\ell\le N}] \quad (3.14)$$

with Φ_k and Ψ_k essentially chosen as in (3.6) and the transition densities $\mathcal{T}_{s,t}$ defined as

$$\mathcal{T}_{s,t}(x,y) = \frac{1}{\sqrt{2\pi(t-s)}} \exp\Big(-\frac{(x-y)^2}{2(t-s)}\Big).$$

We now consider the eigenvalues $\{\lambda_n(t_j) : 1 \le j \le m, 1 \le n \le N\}$ as a point process on $\mathbb{R} \times \{t_1, t_2, \ldots, t_m\}$. Eynard and Mehta [42] were the first to prove that a measure of the form (3.14) has determinantal correlation functions ϱ^n, i. e., there is a correlation kernel K such that

$$\varrho^n((\tau_1, x_1), (\tau_2, x_2), \ldots, (\tau_n, x_n)) = \det[K(\tau_i, x_i; \tau_j, x_j)]_{1\le i,j\le n}$$

for all $x_1, x_2 \ldots, x_n \in \mathbb{R}$ and $\tau_1, \tau_2, \ldots, \tau_n \in \{t_1, \ldots, t_m\}$. Later, Borodin and Rains [25] showed the same theorem in the more general setting of L-ensembles. Another proof is due to Johansson [60]. To write down the correlation kernel K define

$$\mathcal{T}^{(t_i,t_j)}(x,y) = \begin{cases} (\mathcal{T}_{t_i,t_{i+1}} * \cdots * \mathcal{T}_{t_{j-1},t_j})(x,y), & \text{for } i < j, \\ 0, & \text{for } i \ge j, \end{cases}$$

and let M be the $N \times N$ matrix with

$$M_{ij} = \Phi_i * \mathcal{T}^{(t_1,t_m)} * \Psi_j.$$

Then, the correlation kernel K is given by

$$K(\tau, x; \tau', x') = -\mathcal{T}^{(\tau,\tau')}(x,x') + \sum_{i,j=1}^{N} (\mathcal{T}^{(\tau,t_m)} * \Psi_i)(x)[M^{-1}]_{ij}(\Phi_j * \mathcal{T}^{(t_1,\tau')})(x'). \quad (3.15)$$

for all $x, x' \in \mathbb{R}$ and $\tau, \tau' \in \{t_1, t_2, \ldots, t_m\}$.

The representation (3.15) is a general statement which takes a more feasible form in the special case of evolving GUE matrices. Indeed, by a subtle choice for Φ_k and Ψ_k, the matrix M is the identity matrix, and after some calculations, one finds

$$K^{\mathrm{GUE}}_{\mathrm{time}}(\tau,x;\tau',x') = -\frac{2}{2\pi\mathrm{i}}\int_{\mathrm{i}\mathbb{R}+\varepsilon}\mathrm{d}w\,\frac{\mathrm{e}^{\tau w-2xw}}{\mathrm{e}^{\tau' w-2x'w}}\,\mathbb{1}_{[\tau<\tau']} + \frac{2}{(2\pi\mathrm{i})^2}\oint_{|z|=\varepsilon/2}\mathrm{d}z\int_{\mathrm{i}\mathbb{R}+\varepsilon}\mathrm{d}w\,\frac{\mathrm{e}^{\tau w^2-2xw}}{\mathrm{e}^{\tau' z^2-2x'z}}\,\frac{1}{w-z}\,\frac{w^N}{z^N},$$

for any given $\varepsilon>0$, see e. g. [59] for a proof. Note that K^{GUE} from (3.9) is the special case $K^{\mathrm{GUE}}(x,x') = K^{\mathrm{GUE}}_{\mathrm{time}}(\tau,x;\tau',x')$ for $\tau=\tau'=1$. This is why $K^{\mathrm{GUE}}_{\mathrm{time}}$ is called the *extended Hermite kernel*.

3.2.2. GUE minor process

Let us forget about the time evolution for a moment and focus on the GUE minor process. As before, let H be an $N\times N$ Hermitian matrix and for $n=1,\dots,N$, denote by H^n the submatrix of H that is obtained by keeping only the first n rows and columns. In particular, H^N is nothing else but H, and H^1 is the same as the upper-left entry H^N_{11}. Further, denote by $\lambda^n_1\le\dots\le\lambda^n_n$ the ordered eigenvalues of H^n and by W_n the closure of the Weyl chamber of type A,

$$W_n = \{x\in\mathbb{R}^n : x_1\le x_2\le\dots\le x_n\},$$

thus $\lambda^n = (\lambda^n_1,\dots,\lambda^n_n)\in W_n$. Then it is a classical fact of linear algebra (Cauchy's interlacing theorem) that the eigenvalues of H^n and H^{n+1} interlace,

$$\begin{array}{ccccccccc} \lambda^{n+1}_1 & & \lambda^{n+1}_2 & & \lambda^{n+1}_3 & \dots & \lambda^{n+1}_{n+1} \\ & \rotatebox{-45}{$\le$}\ \ \rotatebox{45}{$\le$} & & \rotatebox{-45}{$\le$}\ \ \rotatebox{45}{$\le$} & & & \rotatebox{45}{$\le$} \\ & \lambda^n_1 & & \lambda^n_2 & \dots & \lambda^n_n & \end{array}$$

which we denote by $\lambda^n\preceq\lambda^{n+1}$ and define

$$\begin{aligned} W^{n+1,n} &= \big\{(x,y)\in\mathbb{R}^{n+1}\times\mathbb{R}^n : y\preceq x\big\} \\ &= \big\{(x,y)\in\mathbb{R}^{n+1}\times\mathbb{R}^n : x_k\le y_k\le x_{k+1}\text{ for all }1\le k\le n\big\}. \end{aligned}$$

Now we endow the space of Hermitian matrices with the measure (2.1), i. e., H is a GUE matrix. Given λ^{n+1}, what is the distribution of λ^n? In the GUE case, the answer is simple, see e. g. [10, 35, 49]. Its probability density is given by

$$p(\lambda^n\mid\lambda^{n+1}) = n!\,\frac{\Delta_n(\lambda^n)}{\Delta_{n+1}(\lambda^{n+1})}\,\mathbb{1}_{W^{n+1,n}}(\lambda^n).$$

Note that we have the following recursion formula,

$$\int_{W^{n+1,n}}\mathrm{d}\lambda^n\,\Delta_n(\lambda^n) = \frac{\Delta_{n+1}(\lambda^{n+1})}{n!}. \tag{3.16}$$

For later use, we can thus define the Markov kernel

$$\Lambda_n^{n+1}(x, \mathrm{d}^n y) = n! \, \frac{\Delta_n(y)}{\Delta_{n+1}(x)} \, \mathbb{1}_{W^{n+1,n}}(y) \, \mathrm{d}^n y, \tag{3.17}$$

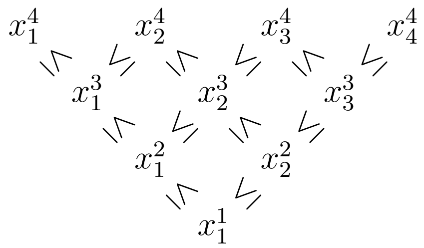
and combining all this, the probability density of $\lambda^1, \dots, \lambda^{N-1}$ given λ^N is

$$\frac{1}{\Delta_N(\lambda^N)} \prod_{n=1}^{N-1} \big(n! \, \mathbb{1}_{[\lambda^n \preceq \lambda^{n+1}]} \big) = \prod_{1 \le i < j \le N} \frac{j - i}{\lambda_j^N - \lambda_i^N} \, \mathbb{1}_{\mathbb{G}_N}, \tag{3.18}$$

where $\mathbb{G}_N$ is the *Gelfand-Tsetlin cone* of depth N,

$$\mathbb{G}_N = \big\{ (x^1, x^2, \dots, x^N) \in \mathbb{R}^1 \times \mathbb{R}^2 \times \cdots \times \mathbb{R}^N : x^1 \preceq x^2 \preceq \cdots \preceq x^N \big\}. \tag{3.19}$$

An element $x \in \mathbb{G}_N$ is called a Gelfand-Tsetlin pattern. Here is a graphical representation of $x \in \mathbb{G}_4$, which illustrates the interlacing condition on x,



Measures on Gelfand-Tsetlin patterns naturally appear in several fields of mathematics like (a) random matrix theory [10, 50, 61, 77] where the question of universality was recently approached in [72], (b) random tiling problems [21, 75, 77], (c) representation theory [22, 23], and (d) interacting particle systems [15, 74, 75] and diffusions [109, 110]. Probably the most famous example which belongs to more than one of these classes is the Aztec diamond.

Let us come back to our problem. Using (3.16), we obtain that

$$\int_{\mathbb{G}_N} \mathrm{d}\lambda^{N-1} \cdots \mathrm{d}\lambda^2 \, \mathrm{d}\lambda^1 = \prod_{1 \le i < j \le N} \frac{\lambda_j^N - \lambda_i^N}{j - i},$$

and hence, the probability (3.18) is nothing else but the uniform distribution on $\mathbb{G}_N$.

To determine the correlation functions of the point process on the GUE minors, a first step would be to write the measure as a product of determinants. Since the distribution of λ^N is basically $\Delta_N^2(\lambda^N)$, we only have to consider $\mathbb{1}_{\mathbb{G}_N}$ from (3.18). Using Sasamoto's trick originally employed for TASEP [85], one can replace the interlacing condition by a product of determinants,

$$\mathbb{1}_{\mathbb{G}_N} = \prod_{n=1}^{N-1} \det \big[\phi_n(\lambda_k^n, \lambda_\ell^{n+1}) \big]_{1 \le k, \ell \le n+1},$$

where $\phi_n(x, y) = \mathbb{1}_{[x \leq y]}$ and λ_n^{n+1} should be read as $+\infty$. Formally, we take $\lambda_n^{n+1} \equiv \mathrm{virt}$ as *virtual variables* such that $\phi_n(\mathrm{virt}, y) = 1$. Then, the joint distribution of $\lambda^1, \lambda^2, \ldots, \lambda^N$ becomes

$$\frac{1}{Z_N} \left(\prod_{n=1}^{N-1} \det\big[\phi_n(\lambda_k^n, \lambda_\ell^{n+1})\big]_{1 \leq k,\ell \leq n+1} \right) \det\big[\Psi_k(\lambda_\ell^N)\big]_{1 \leq k,\ell \leq N} \,\mathrm{d}\lambda, \tag{3.20}$$

where $\mathrm{d}\lambda = \prod_{1 \leq k \leq n \leq N} \mathrm{d}\lambda_k^n$ and Z_N is the normalization constant. Borodin, Ferrari, Prähofer, and Sasamoto [17] showed that the correlation functions are determinantal if $Z_N \neq 0$. To write down the correlation kernel K, define

$$\phi^{(n,n')}(x, x') = \begin{cases} (\phi_n * \cdots * \phi_{n'-1})(x, x'), & \text{for } n < n', \\ 0, & \text{for } n \geq n', \end{cases}$$

where $(f * g)(x, x') = \int_{\mathbb{R}} \mathrm{d}y\, f(x, y) g(y, x')$. Further, let M be the $N \times N$ matrix defined by $M_{k\ell} = (\phi_{k-1} * \phi^{(k,N)} * \Psi_{N-\ell}^N)(x_k^{k-1})$. Then,

$$K(n, x; n', x') = -\phi^{(n,n')}(x, x') + \sum_{k=1}^{n'} \Psi_{n-k}^n(x) \sum_{\ell=1}^{N} [M^{-1}]_{k,\ell} (\phi_\ell * \phi^{(\ell,n')})(x_\ell^{\ell-1}, x').$$

Using biorthogonal ensembles, it is possible to do change of basis such that M is the identity matrix. For the GUE minors, we then find

$$\begin{aligned} K_{\text{minors}}^{\text{GUE}}(n, x; n', x') = &-\frac{2}{2\pi\mathrm{i}} \int_{\mathrm{i}\mathbb{R}+\varepsilon} \mathrm{d}w\, \frac{\mathrm{e}^{-2xw}}{\mathrm{e}^{-2x'w}} \frac{w^n}{w^{n'}} \mathbb{1}_{[n<n']} \\ &+ \frac{2}{(2\pi\mathrm{i})^2} \oint_{|z|=\varepsilon/2} \mathrm{d}z \int_{\mathrm{i}\mathbb{R}+\varepsilon} \mathrm{d}w\, \frac{\mathrm{e}^{w^2-2xw}}{\mathrm{e}^{z^2-2x'z}} \frac{1}{w-z} \frac{w^n}{z^{n'}} \end{aligned}$$

for any given $\varepsilon > 0$.

3.2.3. Evolution on space-like paths

Evolution of GUE minors

In Section 3.2.1 we considered the dynamics of GUE matrices in time, and in Section 3.2.2 we studied their evolution on principal submatrices. We combine these two ways of letting evolve a GUE matrix and define for an $N \times N$ GUE matrix $H(t)$ from (3.12) the principal submatrices $H^1(t), H^2(t), \ldots, H^N(t)$ as in the previous section. We denote again by $\lambda_1^n(t) \leq \cdots \leq \lambda_n^n(t)$ the ordered eigenvalues of $H^n(t)$ at time t. Then for any time t, the collection of all minors $(\lambda^1(t), \ldots, \lambda^N(t))$ is in the Gelfand-Tsetlin cone $\mathbb{G}_N$ of depth N that we defined in (3.19).

As we have seen before, for fixed t, the evolution on minors $n \mapsto \lambda^n(t)$ is Markovian with kernel Λ_n^{n+1} from (3.17), and also for a submatrix of fixed size n, the diffusion $t \mapsto \lambda^n(t)$ is

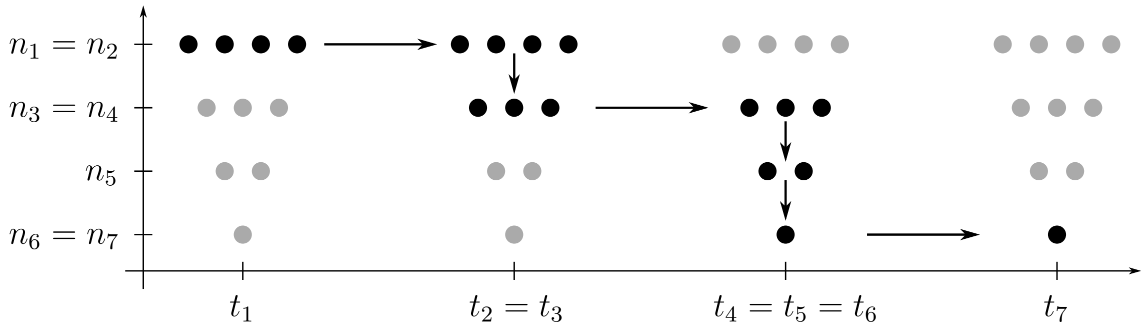


Figure 3.1.: Illustration of a space-like path on diffusive GUE minors

Markovian with kernel P_t^n from (3.13). We aim to combine these two ways of letting evolve a GUE matrix and ask whether an evolution like

$$P_{t_1}^{n_1} \Lambda_{n_2}^{n_1} P_{t_2-t_1}^{n_2} \Lambda_{n_3}^{n_2} \cdots \Lambda_{n_m}^{n_{m-1}} P_{t_m-t_{m-1}}^{n_m} \tag{3.21}$$

with $n_1, \ldots, n_m \in \mathbb{N}$ and $t_1, \ldots, t_m \in [0, \infty)$ is Markovian. If the set $\{n_1, \ldots, n_m\}$ is of the form $\{n, n+1\}$ for some $n \in \mathbb{N}$, i. e., if we restrict the evolution to two consecutive minors, then the answer is affirmative. Adler, Nordenstam and van Moerbeke [2] showed that in this case we have a Markov process, and they also provided an SDE for this diffusion. However, the restriction to three consecutive minors $\lambda^n(t) \preceq \lambda^{n+1}(t) \preceq \lambda^{n+2}(t)$ is *not* Markovian, which has also been proven in [2].

Having this last result in mind, it seems hopeless to consider transitions both in time and on minors. Surprisingly, if we restrict ourselves to so-called *space-like* paths, then we have a Markov process. Introduce the notation

$$(n,t) \prec (n',t') \quad :\Longleftrightarrow \quad n \leq n' \quad \text{and} \quad t \geq t' \quad \text{and} \quad (n,t) \neq (n',t').$$

We say that (n,t) and (n',t') are space-like if either $(n,t) \prec (n',t')$ or $(n',t') \prec (n,t)$. Then, a space is called space-like if any two points on it are space-like, see also Figure 3.1 for an illustration. The two extreme cases of space-like paths are (a) fixed level n and increasing time t and (b) fixed time t and decreasing level n. In Chapter 5.3.1 we will show the following result.

Result 7. *Along space-like paths, the eigenvalues' process is Markovian.*

We are also able to calculate the correlation functions for this point process. For this, we note that P_t^n and Λ_n^{n+1} satisfy an *intertwining property*,

$$\Lambda_n^{n+1} P_t^n = P_t^{n+1} \Lambda_n^{n+1}.$$

This allows us to write (3.21) as

$$(P_{t_1}^{n_1} P_{t_2-t_1}^{n_1} \cdots P_{t_m-t_{m-1}}^{n_1})(\Lambda_{n_2}^{n_1} \Lambda_{n_3}^{n_2} \cdots \Lambda_{n_m}^{n_{m-1}}),$$

and since both $P_{t_1}^{n_1} P_{t_2-t_1}^{n_1} \cdots P_{t_m-t_{m-1}}^{n_1}$ and $\Lambda_{n_2}^{n_1} \Lambda_{n_3}^{n_2} \cdots \Lambda_{n_m}^{n_{m-1}}$ have determinantal correlation, it possible to show (see Chapter 5.1) that (3.21) has determinantal correlations.

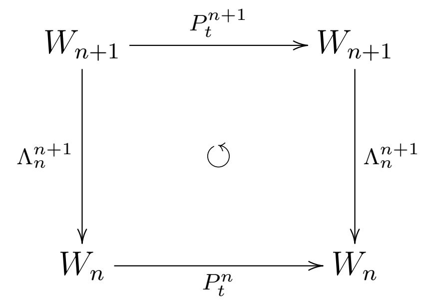


Figure 3.2.: Intertwining property for Λ_n^{n+1} and P_t^n

Result 8. *For any $m \geq 1$ pick m (distinct) triples*

$$\varkappa_j = (x_j, n_j, t_j) \in \mathbb{R} \times \mathbb{N} \times [0, \infty)$$

such that

$$t_1 \leq t_2 \leq \cdots \leq t_m, \qquad n_1 \geq n_2 \geq \cdots \geq n_m.$$

Then, the m-point correlation function of the eigenvalues' point process is given by

$$\rho^{(m)}(\varkappa_1, \ldots, \varkappa_m) = \det \left[K^{\mathrm{GUE}}_{\text{space-like}}(\varkappa_i, \varkappa_j)\right]_{1 \leq i,j \leq m},$$

where

$$K^{\mathrm{GUE}}_{\text{space-like}}(\varkappa_1; \varkappa_2) = -\frac{2}{2\pi \mathrm{i}} \int_{\mathrm{i}\mathbb{R}+\varepsilon} \mathrm{d}w \, \frac{\mathrm{e}^{(t_1-t_2)w^2 - 2(x_1-x_2)w}}{w^{n_2-n_1}} \mathbb{1}_{[(n_1,t_1) \prec (n_2,t_2)]} + \frac{2}{(2\pi \mathrm{i})^2} \oint_{|z|=\varepsilon/2} \mathrm{d}z \int_{\mathrm{i}\mathbb{R}+\varepsilon} \mathrm{d}w \, \frac{\mathrm{e}^{w^2 t_1 - 2x_1 w}}{\mathrm{e}^{z^2 t_2 - 2x_2 z}} \frac{1}{w-z} \frac{w^{n_1}}{z^{n_2}} \tag{3.22}$$

with $\varepsilon > 0$.

Evolution of Wishart minors

To obtain Result 8, we used only a few properties that are specific to GUE matrices. Thus, it seems reasonable that there are other random matrix ensembles with determinantal correlations on space-like paths. For instance, let us take complex Wishart matrices that we already met in Chapter 2.1.1.

Let $B_1, B_2, \ldots, B_N$ be independent p-dimensional complex standard Brownian motions, i. e., the real and imaginary parts of B_n, $1 \leq n \leq N$, are independent real Brownian motions with mean 0 and variance $t/2$. We think of $B_1, \ldots, B_n$ as column vectors and use them to define a $p \times n$ complex valued matrix A^n,

$$A^n(t) = (B_1(t), B_2(t), \ldots, B_n(t)), \quad 1 \leq n \leq N, \quad t \geq 0.$$

Then, we set

$$H^n(t) = A^n(t)^* A^n(t), \quad 1 \le n \le N, \quad t \ge 0,$$

to be the (complex) $n \times n$ Wishart (or sample covariance) matrix. Moreover, we denote by $0 \le \lambda_1^n(t) \le \cdots \le \lambda_n^n(t)$ the ordered eigenvalues of $H^n(t)$. As in the GUE case, orthogonal polynomials show up when analyzing the measure on these eigenvalues, but this time we have to deal with Laguerre instead of Hermite polynomials. That is why for fixed n, the process $t \mapsto H^n(t)$ is often called the *Laguerre process*.

Both the evolution in t and the evolution in n are Markov processes, but also the evolution on space-like paths is Markovian, as we will show in Chapter 5.3.2.

Result 9. *Along space-like paths, the evolution of Wishart minors is a Markov process.*

This process is a determinantal point process and the correlation kernel is given in the following result, see Chapter 5.2 for the proof.

Result 10. *For any $m \ge 1$, pick m (distinct) triples*

$$\varkappa_j = (x_j, n_j, t_j) \in \mathbb{R} \times \{1, \dots, p\} \times [0, \infty)$$

such that

$$t_1 \le t_2 \le \cdots \le t_m, \qquad n_1 \ge n_2 \ge \cdots \ge n_m.$$

Then, the m-point correlation function of the eigenvalues' point process is given by

$$\rho^{(m)}(\varkappa_1, \dots, \varkappa_m) = \det\left[K^{\mathrm{LUE}}_{\text{space-like}}(\varkappa_i; \varkappa_j)\right]_{1 \le i,j \le m},$$

where

$$\begin{aligned} K^{\mathrm{LUE}}_{\text{space-like}}(\varkappa_1; \varkappa_2) = &-\frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} \mathrm{d}z \, \frac{\mathrm{e}^{x_1/(z-t_1)}}{\mathrm{e}^{x_2/(z-t_2)}} \frac{(z-t_1)^{p-1-n_1}}{(z-t_2)^{p+1-n_2}} \mathbb{1}_{[(n_1,t_1) \prec (n_2,t_2)]} \\ &+ \frac{-1}{(2\pi \mathrm{i})^2} \oint_{\Gamma_0} \mathrm{d}z \oint_{\Gamma_{z,t_2}} \mathrm{d}w \, \frac{\mathrm{e}^{x_2/(z-t_1)}}{\mathrm{e}^{x_2/(w-t_2)}} \frac{(z-t_1)^{p-1-n_1}}{(w-t_2)^{p+1-n_2}} \frac{w^p}{z^p} \frac{1}{w-z}. \end{aligned}$$

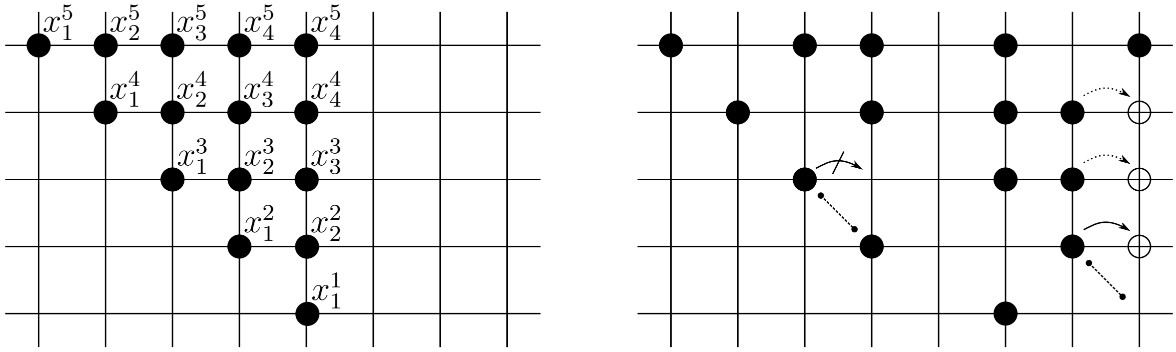
For a set S, the notation $\frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_S} \mathrm{d}w \, f(w)$ means that the integral is taken over any positively oriented simple contour that encloses only the poles of f belonging to S.

3.3. Connecting TASEP and GUE

3.3.1. Dynamics on interlaced particle systems

Result 8 has an interesting relation to interacting particle systems. To explain this connection, we extend TASEP with step initial condition to a process on the discrete Gelfand-Tsetlin cone $\widetilde{\mathbb{G}}_N$ of depth N,

$$\widetilde{\mathbb{G}}_N = \{(x^1, x^2, \dots, x^N) \in \mathbb{Z}^1 \times \mathbb{Z}^2 \times \cdots \times \mathbb{Z}^N : x_k^{n+1} < x_k^n \le x_{k+1}^{n+1}, 1 \le k \le n \le N\}.$$



(a) Initial particle configuration

(b) Possible configuration after some time

Figure 3.3.: The $2+1$ dynamics for the interlacing particle system. In (b), if particle $(1,3)$ tries to jump, the move will be blocked by particle $(1,2)$ while if particle $(2,2)$ jumps, then also particles $(3,3)$ and $(4,4)$ will move by one unit to the right.

Borodin and Ferrari [16] (see also [85]) introduced the following particle system on $\widetilde{\mathbb{G}}_N$. Denote by $x_k^n(t)$ the position at time t of the kth leftmost particle at level n in $\widetilde{\mathbb{G}}_N$ for all $1 \leq k \leq n \leq N$. As initial condition we choose $x_k^n(0) = k - n - 1$, see Figure 3.3(a). The dynamics is as follows: Each particle x_k^n has an independent exponential clock of rate one, and when the x_k^n-clock rings, the particle attempts to jump to the right by one. If at that moment $x_k^n = x_k^{n-1} - 1$, then the jump is blocked (see Figure 3.3(b)). If that is not the case, we take the largest $\ell \geq 1$ such that $x_k^n = x_{k+1}^{n+1} = \cdots = x_{k+\ell-1}^{n+\ell-1}$, and all ℓ particles in this string jump to the right by one. This pushing of particles on higher levels ensures that the interlacing relation between the particles is kept for all times t.

Both the evolution on $\widetilde{\mathbb{G}}_N$ and its projection onto $\{x_1^n : 1 \leq n \leq N\}$ are Markov processes, and the second is nothing else but TASEP with step initial conditions described above. The whole space-time correlations for this model are not yet completely known, but if we restrict ourselves to space-like paths, the correlations are available, see [15]: For any $m \geq 1$ pick m (distinct) triples $\varkappa_j = (x_j, n_j, t_j) \in \mathbb{Z} \times \mathbb{N} \times [0, \infty)$ such that $t_1 \leq t_2 \leq \cdots \leq t_m$ and $n_1 \geq n_2 \geq \cdots \geq n_m$. Then, the m-point correlation function of the point process for the Borodin-Ferrari model is given by

$$\varrho^m(\varkappa_1, \ldots, \varkappa_m) = \det[K^{\mathrm{BF}}_{\text{space-like}}(\kappa_i, \kappa_j)]_{1 \leq i,j \leq m},$$

where

$$\begin{aligned} K^{\mathrm{BF}}_{\text{space-like}}(\varkappa_1; \varkappa_2) = &-\frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_{0,1}} \mathrm{d}w \, \frac{(w-1)^{n_1-n_2} \mathrm{e}^{(t_1-t_2)w}}{w^{x_1+n_1-x_2-n_2+1}} \mathbb{1}_{[(n_1,t_1) \prec (n_2,t_2)]} \\ &+ \frac{1}{(2\pi \mathrm{i})^2} \oint_{\Gamma_1} \mathrm{d}z \oint_{\Gamma_{0,z}} \mathrm{d}w \, \frac{\mathrm{e}^{t_1 w}(1-w)^{n_1}}{w^{x_1+n_1+1}} \frac{z^{x_2+n_2}}{\mathrm{e}^{t_2 z}(1-z)^{n_2}} \frac{1}{w-z}. \end{aligned}$$

Now comes the interesting part. Under the diffusion scaling limit

$$\lambda_k^n(\tau) := \lim_{t \to \infty} \frac{x_k^n\big(\frac{1}{2}\tau t\big) - \frac{1}{2}\tau t}{\sqrt{t}}$$

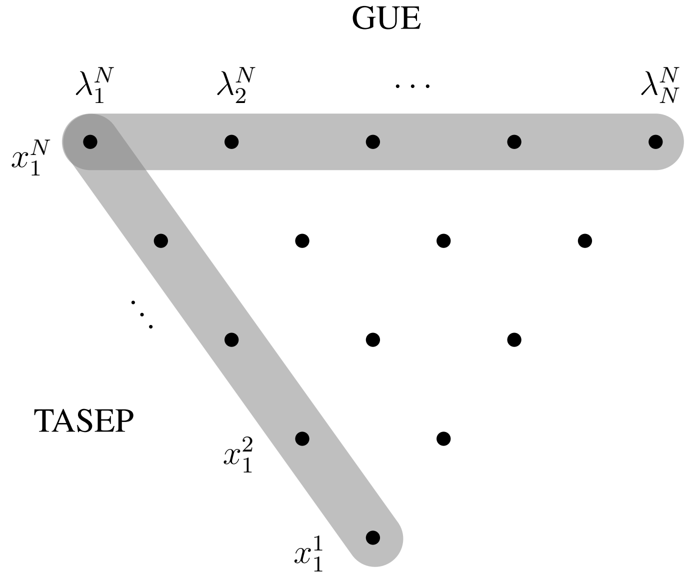


Figure 3.4.: Connection between TASEP and GUE on Gelfand-Tsetlin pattern

one readily obtains that the correlation functions for λ_k^n, $1 \leq k \leq n \leq N$, are still determinantal along space-like paths with correlation kernel

$$K(x,n,\tau;x',n',\tau') = -\frac{2}{2\pi\mathrm{i}}\int_{\mathrm{i}\mathbb{R}+\varepsilon}\mathrm{d}w\,\frac{\mathrm{e}^{(\tau-\tau')w^2-2(\xi-\xi')w}}{w^{n'-n}}\mathbb{1}_{[(n,\tau)\prec(n',\tau')]} + \frac{2}{(2\pi\mathrm{i})^2}\oint_{|z|=\varepsilon/2}\mathrm{d}z\int_{\mathrm{i}\mathbb{R}+\varepsilon}\mathrm{d}w\,\frac{\mathrm{e}^{w^2\tau-2xw}}{\mathrm{e}^{z^2\tau'-2x'z}}\frac{1}{w-z}\frac{w^n}{z^{n'}}$$

for any given $\varepsilon > 0$. But this is exactly the kernel that we obtained for the diffusion on GUE minors on space-like paths in (3.22)! Thus, along space-like paths, we may identify the particles x_k^n in the Borodin-Ferrari model and the eigenvalues λ_k^n in the GUE diffusion on minors because the latter arise from the first by taking a diffusion scaling limit. Moreover, as we mentioned before, the projection on $\{x_1^n : 1 \leq n \leq N\}$ is the continuous time TASEP with step initial conditions and the projection on $\{\lambda_k^N : 1 \leq k \leq N\}$ are the eigenvalues of an $N \times N$ GUE matrix. Both projections share one common point, namely the particle/eigenvalue with label $(1, N)$, see Figure 3.4. In the $N \to \infty$ limit, the fluctuations of this common point are described by the same limiting object, the GUE Tracy-Widom distribution,

$$\lim_{N\to\infty}\mathbb{P}\Big(-\frac{x_1^N(4N)}{(2N)^{1/3}} \geq s\Big) = F_{\mathrm{GUE}}(s), \quad s \in \mathbb{R},$$

and

$$\lim_{N\to\infty}\mathbb{P}\Big(-N^{1/6}\big(\lambda_1^N(0)+2\sqrt{N}\big) \leq s\Big) = F_{\mathrm{GUE}}(s), \quad s \in \mathbb{R}.$$

Thus, this picture explains why F_{GUE} shows up in both interacting particle systems and random matrices.

3.3.2. Interlacing and drifts

The picture given in the previous section can be generalized to processes on Gelfand-Tsetlin pattern where we add a kind of deterministic drift.

Matrix diffusions

The first model we study on $\mathbb{G}_N$ is a variant of the GUE minor process which has been introduced in [61]. Consider an $N \times N$ Hermitian matrix H with eigenvalues $\lambda_1^N \leq \cdots \leq \lambda_N^N$. Denote by H^n the submatrix obtained by keeping the first n rows and columns of H, and its ordered eigenvalues by $\lambda_1^n \leq \cdots \leq \lambda_n^n$. The collection of all these eigenvalues $(\lambda^1, \dots, \lambda^N)$ then forms a Gelfand-Tsetlin pattern, with $\lambda^n = (\lambda_1^n, \dots, \lambda_n^n)$. In this paper we take $H(t)$ to be a GUE matrix diffusion perturbed by a deterministic drift matrix $M = \operatorname{diag}(\mu_1, \dots, \mu_N)$, i.e., we consider $G(t) = H(t) + tM$ with H evolving as standard GUE Dyson's Brownian Motion starting from 0. The eigenvalues' point process ξ has support on $\mathbb{R} \times \{1, \dots, N\}$,

$$\xi(\mathrm{d}x, m) = \sum_{1 \leq k \leq n \leq N} \delta_{n,m} \delta_{\lambda_k^n}(\mathrm{d}x)$$

and its correlation function is given as follows, see Chapter 6.1 for the proof.

Result 11. *For a fixed time $t > 0$ consider the eigenvalues' point process on the N submatrices of $H(t)$. Then, its m-point correlation function ϱ_t^m is given by*

$$\varrho_t^m((x_1, n_1), \dots, (x_m, n_m)) = \det[K_t((x_i, n_i), (x_j, n_j))]_{1 \leq i,j \leq m}, \tag{3.23}$$

with $(x_i, n_j) \in \mathbb{R} \times \{1, \dots, N\}$ and correlation kernel

$$K_t((x,n),(x',n')) = -\phi^{(n,n')}(x,x') + \sum_{k=1}^{n'} \Psi_{n-k}^{n,t}(x) \Phi_{n'-k}^{n',t}(x'), \tag{3.24}$$

where

$$\phi^{(n,n')}(x,x') = \frac{(-1)^{n'-n}}{2\pi \mathrm{i}} \int_{\mathrm{i}\mathbb{R}+\mu_-} \mathrm{d}z \, \frac{\mathrm{e}^{z(x'-x)}}{(z-\mu_{n+1}) \cdots (z-\mu_{n'})} \mathbb{1}_{[n<n']}, \tag{3.25}$$

$$\Psi_{n-k}^{n,t}(x) = \frac{(-1)^{n-k}}{2\pi \mathrm{i}} \int_{\mathrm{i}\mathbb{R}+\mu_-} \mathrm{d}z \, \mathrm{e}^{tz^2/2 - xz} \frac{(z-\mu_1) \cdots (z-\mu_n)}{(z-\mu_1) \cdots (z-\mu_k)}, \tag{3.26}$$

$$\Phi_{n-\ell}^{n,t}(x) = \frac{(-1)^{n-\ell}}{2\pi \mathrm{i}} \oint_{\Gamma_{\mu_1,\dots,\mu_N}} \mathrm{d}w \, \mathrm{e}^{-tw^2/2 + xw} \frac{(w-\mu_1) \cdots (w-\mu_{\ell-1})}{(w-\mu_1) \cdots (w-\mu_n)} \tag{3.27}$$

with $\mu_- < \min\{\mu_1, \dots, \mu_N\}$.

Remark 3. *The integral for $\phi^{(n,n')}$ in (3.25) is only well-defined for $n' - n > 1$. For $n' - n = 1$ we set $\phi^{(n-1,n)}(x,x') := \phi_n(x,x') = \mathrm{e}^{\mu_n(x'-x)} \mathbb{1}_{[x>x']}$ instead.*

In an independent work [3] on minors of random matrices by Adler, van Moerbeke, and Wang appeared on the arXiv after this work, the same kernel is computed and a double integral expression is also provided.

Interacting particle system

Our second model is a generalization of TASEP with particle-dependent jump rates [16] to the $2+1$ dimensional particle system with Markov dynamics introduced in [15]. We denote by $x_k^n \in \mathbb{Z}$ the position of a particle labeled by (k,n), with $1 \leq k \leq n \leq N$, and call n the "level" of the particle. Particle (k,n) performs a continuous time random walk with one-sided jumps (to the right) and with rate v_n. Particles with smaller level evolve independently from the ones with higher level. More precisely, the interaction between levels is the following: (a) if particle (k,n) tries to jump to x and $x_{k-1}^{n-1} = x$, then the jump is suppressed, and (b) when particle (k,n) jumps from $x-1$ to x, then all particles labeled by $(k+\ell, n+\ell)$ (for some $\ell \geq 1$) which were at $x-1$ are forced to jump to x, too. This is a particle system with state space in a discrete Gelfand-Tsetlin pattern.

Consider the diffusion scaling with appropriate scaled jump rates

$$t = \tau T, \quad x_k^n = \tau T - \sqrt{T}\lambda_k^n, \quad v_n = 1 - \frac{\mu_n}{\sqrt{T}}. \tag{3.28}$$

Then, in the $T \to \infty$ limit, the particle process $\{x_k^n(t)\}$ converges to the GUE minor process with drift $\{\lambda_k^n(\tau)\}$.

More precisely, let us denote by $\widetilde{\mathbb{P}}^v$ the probability measure on these particles with jump rates $v = (v_1, \ldots, v_N)$ given in (3.28). We fix $\tau > 0$ and set

$$\nu_T(A) = \widetilde{\mathbb{P}}^v\left(-\frac{x_k^n(\tau T) - \tau T}{\sqrt{T}} \in A_k^n \text{ for all } 1 \leq k \leq n \leq N\right)$$

where $A_k^n \subseteq \mathbb{R}$ are Borel sets, $A = \prod_{1 \leq k \leq n \leq N} A_k^n$. Moreover, we define

$$\nu(A) = \mathbb{P}^\mu\big(\lambda_k^n(\tau) \in A_k^n \text{ for all } 1 \leq k \leq n \leq N\big)$$

where $\mathbb{P}^\mu$ is the GUE minor measure with drift $\mathrm{diag}(\mu_1, \ldots, \mu_N)$. In Chapter 6.2 we show the following result.

Result 12. *As $T \to \infty$, ν_T converges to ν in total variation, i.e.,*

$$\lim_{T\to\infty} \sup_{\substack{A \subseteq \mathbb{R}^{N(N+1)/2}, \\ A \text{ Borel}}} |\nu_T(A) - \nu(A)| = 0. \tag{3.29}$$

In particular, $\nu_T \to \nu$ weakly.

Warren's process with drifts

Under a diffusion scaling limit, the discrete model described above gives rise to our third model which is Warren's process with drifts. It describes the dynamics of a system of Brownian motions $\{B_k^n, 1 \leq k \leq n \leq N\}$ on $\mathbb{G}_N$, where B_1^1 is a standard Brownian motion with drift μ_1 starting from the origin. The Brownian motions B_1^2 and B_2^2 are Brownian motions with drifts μ_2 conditioned to start at the origin and, whenever they touch B_1^1, they are reflected off B_1^1. Similarly for $n \geq 2$, B_k^n is a Brownian motion with drift μ_n conditioned to start at the origin and being reflected off B_k^{n-1} (for $k \leq n-1$) and B_{k-1}^{n-1} (for $k \geq 2$). The process with $\mu_1 = \cdots = \mu_N = 0$ was introduced and studied by Warren in [109].

The correlation functions of this process at a fixed time agree with those of the perturbed GUE minor process. The proof is in Chapter 6.3.

Result 13. *For fixed $t > 0$, the m-point correlation function ϱ_t^m of the point process of the positions of the Brownian motions $\{B_k^n(t) : 1 \leq k \leq n \leq N\}$ is also given by* (3.23).

4. Finite time corrections

This chapter provides the proofs of results 1 to 4 presented in Chapter 2.3.2. It is based on [47].

4.1. Strategy and effects of the discreteness

In this section we present the strategy used to get the results. We will discuss the effects of the intrinsic discreteness of the models on the fitting functions and on the moments, since it is relevant at first order. Finally, we explain how to fit data coming from the experiment by Takeuchi et al.

4.1.1. On the fitting functions

Let us consider Results 2 and 3. For the PNG model and the TASEP, the strategy of getting (4.5) is the following[1]. In these cases, the distribution function of h_t can be expressed as a (discrete) Fredholm determinant with kernel K_t,

$$\mathbb{P}(h_t \leq x) = \det(\mathbb{1} - K_t)_{\ell^2(\{x+1,x+2,\ldots\})}, \quad x \in \mathbb{Z}.$$

For some constant $a \in \mathbb{R}$, the rescaled random variable[2]

$$h_{t,\mathrm{resc}} := (h_t - c_1 t - a)\delta_t \quad \text{with} \quad \delta_t = c_2^{-1} t^{-1/3} \tag{4.1}$$

lives on $I_t := (\mathbb{Z} - c_1 t - a)\delta_t$. According to the scaling in (4.1), we define the rescaled kernel $K_{t,\mathrm{resc}}$ as

$$K_{t,\mathrm{resc}}(s_1, s_2) := \delta_t^{-1} K_t(c_1 t + a + s_1 \delta_t^{-1}, c_1 t + a + s_2 \delta_t^{-1}) \tag{4.2}$$

so that the distribution function F_t defined by

$$F_t(s) := \mathbb{P}(h_t \leq c_1 t + s\delta_t^{-1} + a), \quad s \in \mathbb{R},$$

[1] Mathematically, we get a weaker result, but to illustrate what really happens let us assume that one has (4.5). What is missing are explicit bounds on the decay of the kernels, which can be obtained by standard asymptotic analysis; the ingredients like the steep descent paths are all already contained in previous papers. For TASEP we illustrate the results with a simulation for time $t = 1000$.

[2] Note that we change the notation for the shifted variables. In (2.27) we denoted it by a tilde which we will drop from now on.

can be written as a Fredholm determinant on $\ell^2([s+\delta_t,\infty)\cap I_t,\delta_t\nu)$ with ν the point measure on I_t,

$$\begin{aligned} F_t(s) &= \det(\mathbb{1}-K_{t,\mathrm{resc}})_{\ell^2([s+\delta_t,\infty)\cap I_t,\delta_t\nu)} \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\sum_{x_1,\ldots,x_n\in J_t}\delta_t^n\det(K_{t,\mathrm{resc}}(x_i,x_j))_{1\leq i,j\leq n}. \end{aligned} \tag{4.3}$$

Note that F_t and the Fredholm determinant in (4.3) are piecewise constant functions, with jumps for values of s in the lattice I_t. The next step is to show that for $s_1,s_2\in I_t$ and a well-chosen $a\in\mathbb{R}$,

$$K_{t,\mathrm{resc}}(s_1,s_2)=K(s_1,s_2)+\delta_t K_{\mathrm{asym}}(s_1,s_2)+\mathcal{O}(\delta_t^2) \tag{4.4}$$

where K is a symmetric and K_{asym} an antisymmetric kernel. Then, it follows that[3]

$$\begin{aligned} F_t(s) &= \det(\mathbb{1}-K_{t,\mathrm{resc}})_{\ell^2([s+\delta_t,\infty)\cap I_t,\delta_t\nu)} \\ &= \det(\mathbb{1}-K)_{\ell^2([s+\delta_t,\infty)\cap I_t,\delta_t\nu)} \\ &\quad\times\left(1+\delta_t\,\mathrm{Tr}((\mathbb{1}-\chi_s K\chi_s)^{-1})\chi_s K_{\mathrm{asym}}\chi_s)+\mathcal{O}(\delta_t^2)\right) \end{aligned}$$

where χ_s is the projection onto $[s+\delta_t,\infty)\cap I_t$. The operator under the trace is antisymmetric, therefore its trace is zero and

$$F_t(s)=\det(\mathbb{1}-K)_{\ell^2([s+\delta_t,\infty)\cap I_t,\delta_t\nu)}\left(1+\mathcal{O}(\delta_t^2)\right),\quad s\in\mathbb{R}. \tag{4.5}$$

If we denote by

$$F(s):=\det(\mathbb{1}-K)_{L^2((s,\infty))}$$

the limiting distribution of $F_t(s)$ taken as a Fredholm on $L^2((s,\infty))$, then by Lemma 4 below, we get

$$F_t(s)=F(s+\tfrac{1}{2}\delta_t)+\mathcal{O}(\delta_t^2),$$

for $s\in I_t$, and by the argument below (that gives Result 3), one finally obtains

$$p_t(s)=F'(s)+\mathcal{O}(\delta_t^2).$$

In Section 4.2 we derive (4.4) for the PNG model and the TASEP.

Let us explain how to get Result 3 without the need of Fredholm determinant representations. Assume that there exists a constant γ such that

$$F_t(s)=F(s+\gamma\delta_t)+\delta_t^2 Q(s)+\mathcal{O}(\delta_t^3),\quad s\in I_t,$$

with $F\in C^2$ and $Q\in C^1$. Then using Taylor expansion we readily obtain

$$p_t(s)=F'(s)+\frac{\delta_t}{2}(\gamma^2-(1-\gamma)^2)F''(s)+\mathcal{O}(\delta_t^2),\quad s\in I_t.$$

[3] On a rigorous level, one needs to verify that (a) the $\mathcal{O}(\delta_t^2)$ in (4.4) is an operator with 1-norm of order $\mathcal{O}(\delta_t^2)$ and (b) $(\mathbb{1}-\chi_s K\chi_s)^{-1}\chi_s K_{\mathrm{asym}}\chi_s$ is trace-class.

Therefore, if $\gamma = 1/2$, then $p_t(s) - F'(s) = \mathcal{O}(\delta_t^2)$ for $s \in I_t$, while the approximation would be only of order δ_t if $\gamma \neq 1/2$.

In our case, see Corollary 5, we have $\gamma = 1/2$ which is a consequence of the following lemma.

Lemma 4. *Assume that the kernel K satisfies*[4]

$$\max\{|K(x_1,x_2)|, |\partial_i K(x_1,x_2)|, |\partial_i\partial_j K(x_1,x_2)|\} \leq C\mathrm{e}^{-c(x_1+x_2)} \tag{4.6}$$

for some constants $C, c > 0$, for all $x_1, x_2 \in (s,\infty)$ and $i,j \in \{1,2\}$. Let δ_t be as above the lattice width. Then[5]

$$\left|\det(\mathbb{1} - K)_{L^2((s+\delta_t/2,\infty))} - \det(\mathbb{1} - K)_{\ell^2([s+\delta_t,\infty)\cap I_t,\delta_t\nu)}\right| = \mathcal{O}(\delta_t^2 \mathrm{e}^{-cs}). \tag{4.7}$$

Proof. Let us set $J_t := I_t \cap [s+\delta_t, \infty)$. Then, we have

$$\det(\mathbb{1} - K)_{\ell^2(J_t,\delta_t\nu)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \sum_{x_1,\ldots,x_n \in J_t} \delta_t^n \det(K(x_i,x_j))_{1\leq i,j\leq n}$$

and

$$\det(\mathbb{1} - K)_{L^2((s+\delta_t/2,\infty))} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{(s+\delta_t/2,\infty)^n} \mathrm{d}^n x \, \det(K(x_i,x_j))_{1\leq i,j\leq n}.$$

Equation (4.7) then follows from Lemma 18 with

$$f(x_1,\ldots,x_n) := \det(K(s+\delta_t+x_i, s+\delta_t+x_j))_{1\leq i,j\leq n}$$

together with Lemma 19. □

A straightforward corollary is the following.

Corollary 5. *Assume* (4.5) *and* (4.6) *to hold. Then, for large t, we have (remember that $\delta_t = c_2^{-1} t^{-1/3}$)*

$$F_t(s) = F(s + \tfrac{1}{2}\delta_t) + \mathcal{O}(\delta_t^2) \tag{4.8}$$

for $s \in I_t$.

[4] With ∂_i we mean the derivative with respect to the ith entry of the function. The assumption (4.6) holds for the Airy kernels, see Lemma 20.

[5] For the Airy kernels it is easy to improve $\mathcal{O}(t^{-2/3}\mathrm{e}^{-s})$ to $\mathcal{O}(t^{-2/3}\mathrm{e}^{-\max\{s,0\}})$. However, getting a rigorous *good* bound for the error as $s \to -\infty$ is a much more difficult task (this would be needed for a rigorous proof of the convergence of the moments).

Remark 6. An equivalent way would be to consider the scaling (4.2) without the shift by a, i.e.,

$$\widetilde{K}_{t,\text{resc}}(s_1,s_2) := \delta_t^{-1} K_t(c_1 t + s_1\delta_t^{-1}, c_1 t + s_2\delta_t^{-1}) = K_{t,\text{resc}}(s_1 - a\delta_t, s_2 - a\delta_t)$$

Then, instead of (4.4) we would have obtained

$$\begin{aligned}\widetilde{K}_{t,\text{resc}}(s_1,s_2) =& K(s_1,s_2) - a\delta_t(\partial_1 K(s_1,s_2) + \partial_2 K(s_1,s_2)) \\ &+ \delta_t \widetilde{K}_{\text{asym}}(s_1,s_2) + \mathcal{O}(\delta_t^2).\end{aligned}$$

In the specific case of the Airy kernels $K_{\mathcal{A}_2}(x,y) := \int_{\mathbb{R}_+} \mathrm{d}\lambda\, \mathrm{Ai}(x+\lambda)\,\mathrm{Ai}(y+\lambda)$ and $K_{\mathcal{A}_1}(x,y) := \mathrm{Ai}(x+y)$,

$$\partial_1 K_{\mathcal{A}_2}(s_1,s_2) + \partial_2 K_{\mathcal{A}_2}(s_1,s_2) = -\,\mathrm{Ai}(s_1)\,\mathrm{Ai}(s_2),$$

and

$$\partial_1 K_{\mathcal{A}_1}(s_1,s_2) + \partial_2 K_{\mathcal{A}_1}(s_1,s_2) = 2\mathrm{Ai}'(s_1+s_2).$$

4.1.2. On the moments

Let us now turn to the proof of Result 4. Another consequence of the constant shift by a is that all finite moments of F_t converge as fast as $t^{-2/3}$. Without the shift, the first moment would converge only as fast as $t^{-1/3}$, while the variance, skewness, kurtosis would of course not be affected by the shift.

Lemma 7. *Assume that*[6] $F_t(s) = F(s + \frac{\delta_t}{2}) + \mathcal{O}(\delta_t^2)G_t(s)$ *for* $s \in I_t$ *such that* F *has finite* m*th moment (with* $F'' \in L^1 \cap C^0$*) and* G_t *satisfying* $\int_{\mathbb{R}} \mathrm{d}s\, |s|^m |G_t(s)| < \infty$ *uniformly in* t*. Then,*

$$\int_{\mathbb{R}} s^m\, \mathrm{d}F_t(s) = \int_{\mathbb{R}} s^m\, \mathrm{d}F(s) + \mathcal{O}(\delta_t^2)$$

for all $m \in \mathbb{N}$.

Proof. Let us set $I_t = I_t^+ \cup I_t^-$ where $I_t^\pm = I_t \cap \mathbb{R}_\pm$, $w = \sup I_t^- + \delta_t = \inf I_t^+$, and $\tilde{I}_t^\pm := I_t^\pm + \delta_t/2$. Then, for any $m \geq 1$,

$$\begin{aligned}\int_{\mathbb{R}} s^m\, \mathrm{d}F_t(s) &\equiv \delta_t \sum_{s\in I_t} s^m\, p_t(s) = \sum_{s\in I_t} s^m (F_t(s) - F_t(s-\delta_t)) \\ &= \sum_{s\in I_t^+} s^m (F_t(s) - 1) - \sum_{s\in I_t^+} (s+\delta_t)^m (F_t(s) - 1) + \sum_{s\in I_t^-} s^m F_t(s) \\ &\quad - \sum_{s\in I_t^-} (s+\delta_t)^m F_t(s) + w^m F_t(w-\delta_t) - w^m (F_t(w-\delta_t) - 1) \\ &= w^m + \sum_{s\in I_t^+} (s^m - (s+\delta_t)^m)(F_t(s) - 1) + \sum_{s\in I_t^-} (s^m - (s+\delta_t)^m) F_t(s).\end{aligned} \tag{4.9}$$

[6]Note that this condition is stronger than (4.8) and in general not so easy to obtain rigorously.

Now we set $\tilde{s} = s + \frac{\delta_t}{2}$ so that

$$(s^m - (s+\delta_t)^m) = (\tilde{s} - \tfrac{\delta_t}{2})^m - (\tilde{s} + \tfrac{\delta_t}{2})^m = -m\,\delta_t\,\tilde{s}^{m-1} + \mathcal{O}(\delta_t^3).$$

Then using $F_t(\tilde{s} - \frac{\delta_t}{2}) = F(\tilde{s}) + \mathcal{O}(\delta_t^2) G_t(\tilde{s})$ we obtain

$$\begin{aligned}
(4.9) &= w^m - m\delta_t \sum_{\tilde{s}\in \tilde{I}_t^+} \tilde{s}^{m-1}(F(\tilde{s}) - 1) - m\delta_t \sum_{\tilde{s}\in \tilde{I}_t^-} \tilde{s}^{m-1} F(\tilde{s}) + \mathcal{O}(\delta_t^2)\\
&= w^m - m\delta_t \int_w^\infty \mathrm{d}s\, s^{m-1}(F(s)-1) - m\delta_t \int_{-\infty}^w \mathrm{d}s\, s^{m-1} F(s) + \mathcal{O}(\delta_t^2)\\
&= \int_{\mathbb{R}} s^m \,\mathrm{d}F(s) + \mathcal{O}(\delta_t^2).
\end{aligned}$$

where we used Lemma 18 to approximate the sums by the integrals. □

4.1.3. How to fit the experimental data

For completeness, we explain shortly how to fit the experimental data. We partially follow the description of the Supplementary Notes of [97] and use their notations. Let us assume that we observed a growth process which is thought to belong to the KPZ class. Let $S = \varepsilon\,\mathbb{Z}$, $\varepsilon > 0$, be a discrete subset of $\mathbb{R}$, where the values of the height function at time t, denoted by h_t, lives. Let $N \gg 1$ the number of experimental measurements and denote by $\langle \cdot \rangle$ the empirical average over the N experiments. Having (2.25) in mind, we expect to have

$$h_t \simeq v_\infty\, t + (\Gamma t)^{1/3}\zeta + a$$

where ζ is a GUE (resp. GOE) Tracy-Widom distributed random variable for curved (resp. flat) limit shape, v_∞ the asymptotic growth velocity and a a constant.

(1) *Determine the asymptotic growth velocity* v_∞. Using

$$\frac{\mathrm{d}\langle h_t\rangle}{\mathrm{d}t} \simeq v_\infty + b\,t^{-2/3}, \quad b = \Gamma^{1/3}\,\mathbb{E}(\zeta)/3.$$

one obtains v_∞ from the plot $(t^{-2/3}, \frac{\mathrm{d}\langle h_t\rangle}{\mathrm{d}t})$.

(2) *Verify the fluctuation scaling exponent and the fluctuation amplitude* Γ. With a log-log plot we can verify if the power $2/3$ in

$$\langle (h_t - \langle h_t\rangle)^2\rangle \simeq (\Gamma t)^{2/3}\,\mathrm{Var}(\zeta)$$

holds and at the same time measure the constant $\Gamma \neq 0$.

(3) *Determine the shift parameter* a. Consider $\widetilde{h}_{t,\mathrm{resc}} := (h_t - v_\infty\, t)/(\Gamma\, t)^{1/3}$, the standard KPZ scaling. Then, a is measured according to the relation

$$\langle \widetilde{h}_{t,\mathrm{resc}}\rangle - \mathbb{E}(\zeta) \simeq a\,(\Gamma t)^{-1/3}.$$

Now that we have determined v_∞, Γ and a, we fit the data vs. the theoretical predictions. We set $h_{t,\mathrm{resc}} := \widetilde{h}_{t,\mathrm{resc}} - a\,(\Gamma t)^{-1/3}$.

(4.1) *Density*: We do a plot of the frequencies of $h_{t,\mathrm{resc}}$ with the set $(S - v_\infty t - a)/(\Gamma t)^{1/3}$ in the abscissa axis. Then we compare this with the graph of the Tracy-Widom densites $s \mapsto F'(s)$.

(4.2) *Distribution function*: We do a plot of the cumulated frequencies of $h_{t,\mathrm{resc}}$ with the set $(S - v_\infty t - a)/(\Gamma t)^{1/3}$ in the abscissa axis. Then we compare this with the graph of the (shifted) Tracy-Widom distribution function $s \mapsto F(s + \frac{1}{2}\varepsilon/(\Gamma t)^{1/3}))$.

4.2. PNG and TASEP

In this section we determine the value of the order 1 shifts for the PNG and TASEP models, both with flat and curved geometry.

4.2.1. Flat PNG

In [18] the formula for the height function h_t at time t for the flat PNG was obtained[7]. It is shown that

$$\mathbb{P}(h_t(0) \leq H) = \det(\mathbb{1} - K_t^{\mathrm{flatPNG}})_{\ell^2(\{H+1,H+2,\ldots\})}$$

with

$$K_t^{\mathrm{flatPNG}}(x_1,x_2) = J_{x_1+x_2}(4t) = \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma_0} \mathrm{d}z\, \frac{\mathrm{e}^{2t(z-z^{-1})}}{z^{x_1+x_2+1}},$$

where J_n is the standard Bessel function (we use the conventions of [1])[8]. As $t \to \infty$, we consider the scaling

$$H(s) = 2t + s(2t)^{1/3} \in \mathbb{N} \quad \Rightarrow \quad s \in I_t = (\mathbb{N} - 2t)(2t)^{-1/3}$$

Under this scaling it is known that [18]

$$K_{t,\mathrm{resc}}^{\mathrm{flatPNG}}(s_1,s_2) := (2t)^{1/3} K_t^{\mathrm{flatPNG}}(H(s_1),H(s_2)) \to \mathrm{Ai}(s_1+s_2) = K_{\mathcal{A}_1}(s_1,s_2)$$

as $t \to \infty$ and uniformly for s_1, s_2 in bounded sets. Moreover, there are exponential bounds for the decay of $K_{t,\mathrm{resc}}^{\mathrm{flatPNG}}$ (see, e.g., Appendix A.2 of [43]) which ensures that we can take the limit $t \to \infty$ inside the Fredholm determinant, leading to[9]

$$\lim_{t\to\infty} \mathbb{P}(h_t(0) \leq 2t + s(2t)^{1/3}) = \det(\mathbb{1} - K_{\mathcal{A}_1})_{L^2((s,\infty))} = F_{\mathrm{GOE}}(2s),$$

[7] For the one-point distribution there exists also a formulation in terms of Fredholm Pfaffian.

[8] With the notation Γ_S, with S a set, we mean any simple counterclockwise oriented path encircling the set S.

[9] In [18] the result is for joint distributions of the height function at different positions. The one-point distribution was also obtained through its relation with symmetrized permutations [8].

where $F_{\rm GOE}$ is the GOE Tracy-Widom distribution function [100].

Here we focus on the first order correction of $K_{t,\rm resc}^{\rm flatPNG}$ with respect to $K_{\mathcal{A}_1}$ and show that it is zero. Since the asymptotic analysis is quite standard (see e.g. Lemma 6.1 of [15] for the explanation of general strategy), here and in the next sections we indicate only the important steps.

Proposition 8. *Uniformly for s_1, s_2 in a bounded subset of I_t,*

$$K_{t,\rm resc}^{\rm flatPNG}(s_1,s_2) = K_{\mathcal{A}_1}(s_1,s_2) + \mathcal{O}(t^{-2/3}).$$

Proof. We have

$$K_{t,\rm resc}^{\rm flatPNG}(s_1,s_2) = \frac{(2t)^{1/3}}{2\pi{\rm i}} \oint_{\Gamma_0} {\rm d}z\, \frac{{\rm e}^{2t(z-z^{-1})}}{z^{4t+(s_1+s_2)(2t)^{1/3}+1}}. \tag{4.10}$$

The function $z \mapsto z - z^{-1} - 2\ln z$ has a double critical point at $z_c = 1$. The steepest descent path can be taken to be coming into z_c with an angle ${\rm e}^{-\pi{\rm i}/3}$, leaving with an angle ${\rm e}^{\pi{\rm i}/3}$, and completed by a piece of a circle around zero with a radius strictly larger than 1. Then, the leading term in the asymptotic of $K_{t,\rm resc}^{\rm flatPNG}$ comes from a $t^{-1/3}$-neighborhood of z_c. Setting $z = 1 + Z(2t)^{-1/3}$ and doing the large t expansion of the integrand in (4.10), one obtains

$$\begin{aligned} K_{t,\rm resc}^{\rm flatPNG}(s_1,s_2) = \frac{1}{2\pi{\rm i}} \int_{\infty{\rm e}^{-\pi{\rm i}/3}}^{\infty{\rm e}^{\pi{\rm i}/3}} {\rm d}Z\, \exp\left(\frac{Z^3}{3} - \zeta Z\right) \\ \times \left(1 - t^{-1/3}\left(\frac{Z}{2^{1/3}} - \frac{\zeta Z^2}{2^{4/3}} + \frac{Z^4}{2^{4/3}}\right) + \mathcal{O}(t^{-2/3})\right), \end{aligned}$$

where we have set for simplicity $\zeta := s_1 + s_2$. Using the contour integral representation of the Airy function (4.22) we have

$$K_{t,\rm resc}^{\rm flatPNG}(s_1,s_2) = {\rm Ai}(\zeta) + t^{-1/3}\left[\frac{{\rm Ai}'(\zeta)}{2^{1/3}} + \frac{\zeta\,{\rm Ai}''(\zeta)}{2^{4/3}} - \frac{{\rm Ai}^{(4)}(\zeta)}{2^{4/3}}\right] + \mathcal{O}(t^{-2/3}).$$

Finally, using the identity ${\rm Ai}''(\zeta) = \zeta\,{\rm Ai}(\zeta)$ of the Airy function one readily gets that the square bracket is equal to zero. □

4.2.2. PNG droplet

The formula for the height function h_t at time t for the PNG droplet was determined in [81]. Let us fix $c \in (-1,1)$. Then,

$$\mathbb{P}(h_t(ct) \le H) = \det(\mathbb{1} - K_{t,c}^{\rm curvPNG})_{\ell^2(\{H+1,H+2,\dots\})}$$

with

$$K_{t,c}^{\mathrm{curvPNG}}(x_1,x_2)=\sum_{\ell\geq 0}J_{x_1+\ell}(2t\sqrt{1-c^2})J_{x_2+\ell}(2t\sqrt{1-c^2}).$$

We consider the case $c=0$ (and drop the index c) since the general case is simply obtained by replacing t by $t\sqrt{1-c^2}$. An integral representation of the kernel is given by

$$K_t^{\mathrm{curvPNG}}(x_1,x_2)=\frac{1}{(2\pi\mathrm{i})^2}\oint_{\Gamma_0}\mathrm{d}w\oint_{\Gamma_{0,w}}\mathrm{d}z\,\frac{\mathrm{e}^{2t(z-z^{-1})}}{\mathrm{e}^{2t(w-w^{-1})}}\frac{w^{x_2-1}}{z^{x_1}}\frac{1}{z-w}.$$

As $t\to\infty$, we consider the scaling

$$H(s)=2t+st^{1/3}+a\in\mathbb{N}\quad\Rightarrow\quad s\in I_t=(\mathbb{N}-2t-a)t^{-1/3}$$

for a t-independent constant a to be specified later. In [81] it is proven that the rescaled kernel converges to the Airy kernel, namely

$$\begin{aligned}K_{t,\mathrm{resc}}^{\mathrm{curvPNG}}(s_1,s_2)&:=t^{1/3}K_t^{\mathrm{curvPNG}}(H(s_1),H(s_2))\\&\to\int_{\mathbb{R}_+}\mathrm{d}\lambda\,\mathrm{Ai}(s_1+\lambda)\,\mathrm{Ai}(s_2+\lambda)=K_{\mathcal{A}_2}(s_1,s_2),\end{aligned}$$

as $t\to\infty$ and uniformly for s_1,s_2 in bounded sets. Moreover, exponential bounds for the decay of $K_{t,\mathrm{resc}}^{\mathrm{curvPNG}}$ ensure that[10]

$$\lim_{t\to\infty}\mathbb{P}(h_t(0)\leq 2t+st^{1/3}+a)=\det(\mathbb{1}-K_{\mathcal{A}_2})_{L^2((s,\infty))}=F_{\mathrm{GUE}}(s),$$

where F_{GUE} is the GUE Tracy-Widom distribution function [99].

The first order correction of $K_{t,\mathrm{resc}}^{\mathrm{curvPNG}}$ with respect to $K_{\mathcal{A}_2}$ is the following.

Proposition 9. *Uniformly for s_1,s_2 in a bounded subset of I_t, with the choice $a=1/2$,*

$$K_{t,\mathrm{resc}}^{\mathrm{curvPNG}}(s_1,s_2)=K_{\mathcal{A}_2}(s_1,s_2)+\mathcal{O}(t^{-2/3}).$$

Proof. The rescaled kernel is

$$K_{t,\mathrm{resc}}^{\mathrm{curvPNG}}(s_1,s_2)=\frac{t^{1/3}}{(2\pi\mathrm{i})^2}\oint_{\Gamma_0}\mathrm{d}w\oint_{\Gamma_{0,w}}\mathrm{d}z\,\frac{\mathrm{e}^{t(z-z^{-1})}}{\mathrm{e}^{t(w-w^{-1})}}\frac{w^{2t+s_2t^{1/3}+a-1}}{z^{2t+s_1t^{1/3}+a}}\frac{1}{z-w}.$$

Here, we have to integrate over two contours, the one in the z-variable enclosing the contour in the w-variable. The steepest descent path for z can be taken as in the flat case, the one for w leaves the critical points with an angle $\mathrm{e}^{2\pi\mathrm{i}/3}$ and then is completed by a piece of a circle of radius strictly smaller than 1. Doing the change of variables

$$z=1+t^{-1/3}Z,\quad w=1+t^{-1/3}W,$$

[10]The extension to joint distributions was obtained in [81], while the one-point result is reported in [80] using a mapping to the Poissonized longest increasing subsequence problem, which was already solved in [7].

we eventually get

$$K^{\mathrm{curvPNG}}_{t,\mathrm{resc}}(s_1,s_2)=\frac{1}{(2\pi\mathrm{i})^2}\int_{\infty\mathrm{e}^{-2\pi\mathrm{i}/3}}^{\infty\mathrm{e}^{2\pi\mathrm{i}/3}}\mathrm{d}W\int_{\infty\mathrm{e}^{-\pi\mathrm{i}/3}}^{\infty\mathrm{e}^{\pi\mathrm{i}/3}}\mathrm{d}Z\frac{\mathrm{e}^{Z^3/3-s_1Z}}{\mathrm{e}^{W^3/3-s_2W}}\frac{1}{Z-W}$$
$$\times\left(1+t^{-1/3}\left[\frac{s_1Z^2-s_2W^2}{2}-\frac{Z^4-W^4}{4}-aZ+(a-1)W\right]+\mathcal{O}(t^{-2/3})\right),$$

where the integration paths do not intersect. At this point we see that setting $a=1/2$ the first order term is antisymmetric. In particular we can choose the paths to satisfy $\operatorname{Re}Z>\operatorname{Re}W$ and use $\frac{1}{Z-W}=\int_0^\infty\mathrm{d}\lambda\mathrm{e}^{-\lambda(Z-W)}$ to get

$$K^{\mathrm{curvPNG}}_{t,\mathrm{resc}}(s_1,s_2)=K_{\mathcal{A}_2}(s_1,s_2)+t^{-1/3}\big(P(s_1,s_2)-P(s_2,s_1)\big)+\mathcal{O}(t^{-2/3})$$

with P given by

$$P(s_1,s_2)=\frac{1}{2}\int_0^\infty\mathrm{d}\lambda\,\mathrm{Ai}(s_2+\lambda)\left[\frac{\mathrm{d}}{\mathrm{d}s_1}+s_1\frac{\mathrm{d}^2}{\mathrm{d}s_1^2}-\frac{\mathrm{d}^4}{\mathrm{d}s_1^4}\right]\mathrm{Ai}(s_1+\lambda).$$

Using $\mathrm{Ai}''(x)=x\,\mathrm{Ai}(x)$ and integration by parts one then shows $P(s_1,s_2)=P(s_2,s_1)$. □

Without the shift by a in the scaling, the result would have been

$$K^{\mathrm{curvPNG}}_{t,\mathrm{resc}}(s_1,s_2)=K_{\mathcal{A}_2}(s_1,s_2)-\tfrac{1}{2}t^{-1/3}(-\mathrm{Ai}(s_1)\,\mathrm{Ai}(s_2)))+\mathcal{O}(t^{-2/3})$$

from which we can read off the shift $a=1/2$, compare with Remark 6.

Remark 10. The shift by $a=1/2$ is actually independent of $c\in(-1,1)$, which is due to the fact that it is built up during the first stages of the growth process and for large t it converges to $1/2$. Therefore for large t, the shift at time $t\sqrt{1-c^2}$ is the same as for the model at time t.

4.2.3. TASEP with alternating initial condition

Now consider TASEP with alternating initial condition, $x_k(t=0)=-2k$, $k\in\mathbb{Z}$. The joint distribution of particle positions for this initial condition has been determined in [17]. For one particle (here $-x_n(t)$ plays the role of h_t), we have

$$\mathbb{P}(x_n(t)\geq X)=\det(\mathbb{1}-K^{\mathrm{flatTASEP}}_{t,n})_{\ell^2(\{\dots,X-2,X-1\})}$$

with

$$K^{\mathrm{flatTASEP}}_{t,n}(x_1,x_2)=\frac{-1}{2\pi\mathrm{i}}\oint_{\Gamma_1}\mathrm{d}z\,\mathrm{e}^{t(1-2z)}\frac{z^{2n+x_2}}{(1-z)^{2n+x_1+1}}.$$

As $t\to\infty$, we consider the scaling

$$X(s)=-2n+\tfrac{1}{2}t-st^{1/3}-a\in\mathbb{Z},\quad\text{thus}\quad s\in I_t=(\mathbb{Z}-t/2+a)t^{-1/3}\tag{4.11}$$

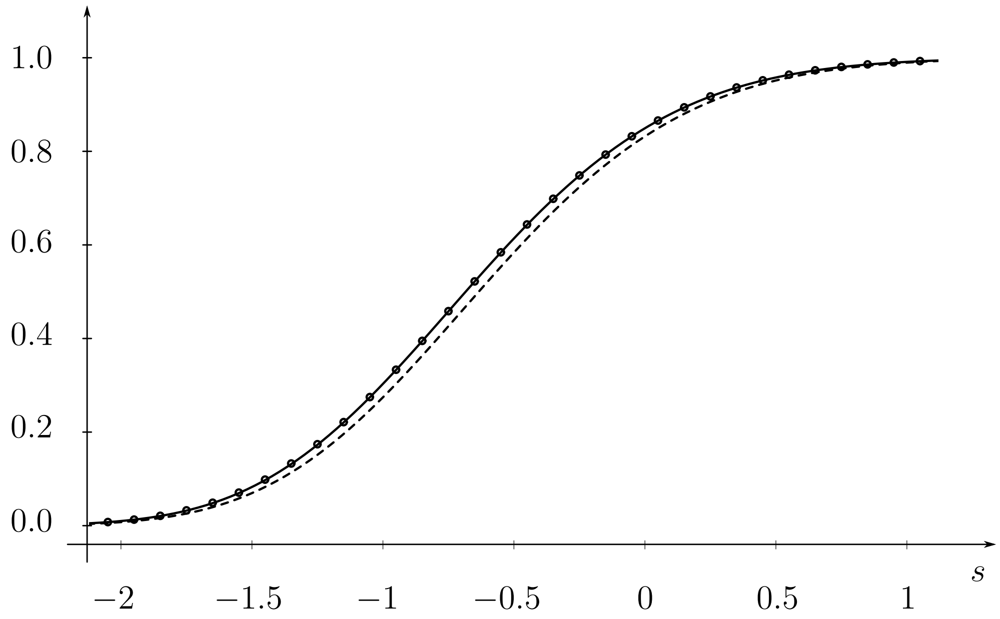


Figure 4.1.: Distribution function of $x_{n=[t/4]}(t)$ for TASEP with alternating initial conditions and $t = 1000$. The number of runs is 10^6. The dots are the plot of $(s \in I_t, \mathbb{P}(x_{[t/4]}(t) \geq X(s)))$ with $\delta_t = t^{-1/3}$ and $a = 1/2$. The solid line is $(s, F_{\rm GOE}(2s + \frac{1}{2}\delta_t))$ while the dashed line is $(s, F_{\rm GOE}(2s + \frac{1}{2}\delta_t - a\delta_t)))$, where the shift by $a\delta_t$ for dashed line follows from the definition I_t, see (4.11).

for a t-independent constant a to be specified later. It is known[11] that [17]

$$K_{t,\rm resc}^{\rm flatTASEP}(s_1, s_2) := 2^{X(s_2)-X(s_1)} t^{1/3} K_t^{\rm flatTASEP}(X(s_1), X(s_2)) \to K_{\mathcal{A}_1}(s_1, s_2)$$

as $t \to \infty$ and uniformly for s_1, s_2 in bounded sets. Moreover, exponential bounds for the decay of $K_{t,\rm resc}^{\rm flatTASEP}$ ensure that

$$\lim_{t\to\infty} \mathbb{P}(x_0(t) \geq t/2 - st^{1/3} - a) = \det(\mathbb{1} - K_{\mathcal{A}_1})_{L^2((s,\infty))} = F_{\rm GOE}(2s).$$

The first order correction of $K_{t,\rm resc}^{\rm flatTASEP}$ with respect to $K_{\mathcal{A}_1}$ is given as follows.

Proposition 11. *Uniformly for s_1, s_2 in a bounded subset of I_t, with the choice $a = 1/2$, it holds*

$$K_{t,\rm resc}^{\rm flatTASEP}(s_1, s_2) = K_{\mathcal{A}_1}(s_1, s_2) + t^{-1/3} K_{\rm asym}(s_1, s_2) + \mathcal{O}(t^{-2/3}),$$

where $K_{\rm asym}(s_1, s_2) = \frac{1}{2}(s_2^2 - s_1^2)\,\mathrm{Ai}(s_1 + s_2)$.

Proof. The rescaled kernel reads

$$K_{t,\rm resc}^{\rm flatTASEP}(s_1, s_2) = -\frac{t^{1/3}}{2\pi \mathrm{i}} \frac{2^{s_1 t^{1/3}}}{2^{s_2 t^{1/3}}} \oint_{\Gamma_1} \mathrm{d}z\, \mathrm{e}^{t(1-2z)} \frac{z^{t/2 - s_2 t^{1/3} - a}}{(1-z)^{t/2 - s_1 t^{1/3} - a + 1}}$$

[11] The prefactor $2^{X(s_2)-X(s_1)}$ is just a conjugation, which does not change the underlying determinantal point process, but it is needed to have a well-defined limit.

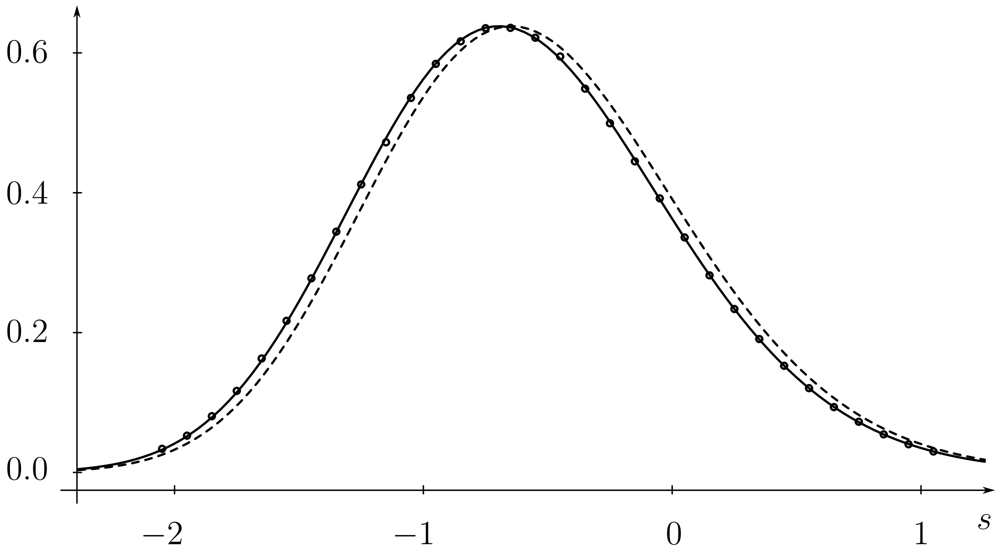


Figure 4.2.: Probabilities for $x_{n=[t/4]}(t)$ for TASEP with alternating initial conditions and time $t = 1000$. The number of runs is 10^6. The dots are the plot of $(s \in I_t, \mathbb{P}(x_{[t/4]}(t) = X(s))\delta_t^{-1})$ with $\delta_t = t^{-1/3}$ and $a = 1/2$. The solid line is $(s, 2F_1'(2s))$ while the dashed line is $(s, 2F_1'(2s - a\delta_t))$.

The function $z \mapsto 1 - 2z + \frac{1}{2} \ln \frac{z}{1-z}$ has a double critical point at $z_c = 1/2$. We choose as steep descent path the one coming into z_c with angle $e^{i\pi/3}$, leaving with angle $e^{-i\pi/3}$, and continued by a piece of a circle around 1 with radius $1/2$. Setting $Z = t^{1/3}(2z - 1)$, we get

$$K_{t,\mathrm{resc}}^{\mathrm{flatTASEP}}(s_1, s_2) = \frac{1}{2\pi i} \int_{\infty e^{-i\pi/3}}^{\infty e^{i\pi/3}} dZ\, e^{Z^3/3 - (s_1+s_2)Z} \times \left(1 + t^{-1/3}\left((1-2a)Z + \frac{s_2 - s_1}{2} Z^2\right) + \mathcal{O}(t^{-2/3})\right).$$

Thus we see that in order to make the first order correction antisymmetric we need to choose $a = 1/2$. With this choice,

$$K_{t,\mathrm{resc}}^{\mathrm{flatTASEP}}(s_1, s_2) = \mathrm{Ai}(s_1 + s_2) + \tfrac{1}{2}(s_2 - s_1)\,\mathrm{Ai}''(s_1 + s_2) t^{-1/3} + \mathcal{O}(t^{-2/3})$$

and the statement follows using $\mathrm{Ai}''(s_1 + s_2) = (s_1 + s_2)\,\mathrm{Ai}(s_1 + s_2)$. □

4.2.4. TASEP with step initial condition

Now consider TASEP with step initial condition, $x_k(0) = -k$, $k = 1, 2, \ldots$ The joint distribution of particle positions for this initial condition can be found for example (as special case) in [16]. For one particle, we have[12]

$$\mathbb{P}(x_n(t) \geq X) = \det(\mathbb{1} - K_{t,n}^{\mathrm{stepTASEP}})_{\ell^2(\{\ldots, X-2, X-1\})}$$

[12] The formula for the one-point distribution can be also given by a last passage percolation model, which can be analyzed by determinantal line ensembles leading to the Laguerre kernel [57]. Joint distributions for the related last passage model can be determined via Schur process [57, 58, 76].

with

$$K_{t,n}^{\text{stepTASEP}}(x_1,x_2) = \frac{1}{(2\pi\mathrm{i})^2}\oint_{\Gamma_0}\mathrm{d}z\oint_{\Gamma_1}\mathrm{d}w\,\frac{\mathrm{e}^{tz}}{\mathrm{e}^{tw}}\left(\frac{1-z}{1-w}\right)^n \frac{w^{n+x_2}}{z^{n+x_1+1}}\frac{1}{z-w}$$

Now consider a particle that at time t is in the "rarefaction fan", i.e., it is in the region with decreasing density strictly between 0 and 1. Such particles have a particle number $n=\sigma t$ for some $\sigma\in(0,1)$. Thus, we define for a couple (n,t) the value of $\sigma := n/t$ and we assume that this value for large n and t is clearly away both from 0 and 1. Then, the scaling for $t\to\infty$ is given by

$$n=\sigma t\in\mathbb{N},\quad X(s) = -n+(1-\sqrt{\sigma})^2 t - s c_2 t^{1/3} - a\in\mathbb{Z},$$

so that

$$I_t = (\mathbb{Z}-(1-\sqrt{\sigma})^2 t + a)c_2^{-1}t^{-1/3}$$

with $c_2 = \sigma^{-1/6}(1-\sqrt{\sigma})^{2/3}$, and for a t-independent constant a to be specified later. It is also known that [16]

$$K_{t,\sigma,\text{resc}}^{\text{stepTASEP}}(s_1,s_2) := c_2^{-1}t^{1/3}(1-\sqrt{\sigma})^{X(s_1)-X(s_2)}K_{t,\sigma t}^{\text{stepTASEP}}(X(s_1),X(s_2)) \to K_{\mathcal{A}_2}(s_1,s_2)$$

as $t\to\infty$ and uniformly for s_1,s_2 in bounded sets. Moreover, exponential bounds for the decay of $K_{t,\text{resc}}^{\text{stepTASEP}}$ ensure that

$$\lim_{t\to\infty}\mathbb{P}(x_n(t)\geq X(s)) = \det(\mathbb{1}-K_{\mathcal{A}_2})_{L^2((s,\infty))} = F_{\text{GUE}}(s).$$

Now let us focus on the first order correction.

Proposition 12. *Uniformly for s_1,s_2 in a bounded set, with the choice $a=1/2$, it holds*

$$K_{t,\sigma,\text{resc}}^{\text{stepTASEP}}(s_1,s_2) = K_{\mathcal{A}_2}(s_1,s_2) + c_2^{-1}t^{-1/3}K_{\text{asym}}(s_1,s_2) + \mathcal{O}(t^{-2/3}), \tag{4.12}$$

where $K_{\text{asym}}(s_1,s_2) = P(s_1,s_2)-P(s_2,s_1)$, with

$$P(s_1,s_2) = \frac{1}{2}\int_{\mathbb{R}_+}\mathrm{d}\lambda\ \mathrm{Ai}(s_1+\lambda) \times\left(\mathrm{Ai}'(s_2+\lambda) + s_2\,\mathrm{Ai}''(s_2+\lambda) - \frac{1-2\sqrt{\sigma}}{2\sqrt{\sigma}}\,\mathrm{Ai}^{(4)}(s_2+\lambda)\right).$$

Proof. With $c_1 = 1-2\sqrt{\sigma}$ we can write $X(s) = c_1 t - c_2 s t^{1/3} - a$. The rescaled kernel then reads

$$K_{t,\sigma,\text{resc}}^{\text{stepTASEP}}(s_1,s_2) = \frac{c_2 t^{1/3}(1-\sqrt{\sigma})^{c_2(s_1-s_2)t^{1/3}}}{(2\pi\mathrm{i})^2} \times\oint_{\Gamma_0}\mathrm{d}z\oint_{\Gamma_1}\mathrm{d}w\,\frac{\mathrm{e}^{tz}}{\mathrm{e}^{tw}}\left(\frac{1-z}{1-w}\right)^{\sigma t}\frac{w^{(\sigma+c_1)t-c_2 s_2 t^{1/3}-a}}{z^{(\sigma+c_1)t-c_2 s_1 t^{1/3}-a+1}}\frac{1}{z-w}.$$

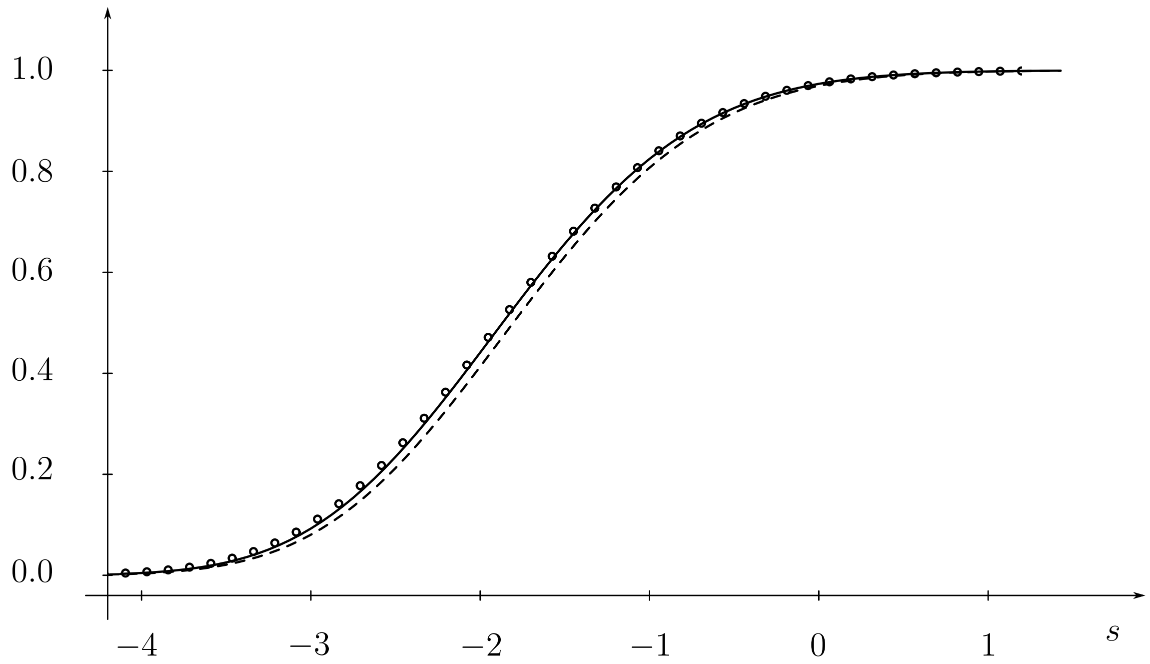


Figure 4.3.: Distribution function of $x_{n=[t/4]}(t)$ for TASEP with step initial conditions and $t = 1000$. The number of runs is 10^6. The dots are the plot of $(s \in I_t, \mathbb{P}(x_{[t/4]}(t) \geq X(s)))$ with $\delta_t = (t/2)^{-1/3}$ and $a = 1/2$. The solid line is $(s, F_{\mathrm{GUE}}(s + \frac{1}{2}\delta_t))$ while the dashed line is $(s, F_{\mathrm{GUE}}(s + \frac{1}{2}\delta_t - a\delta_t)))$.

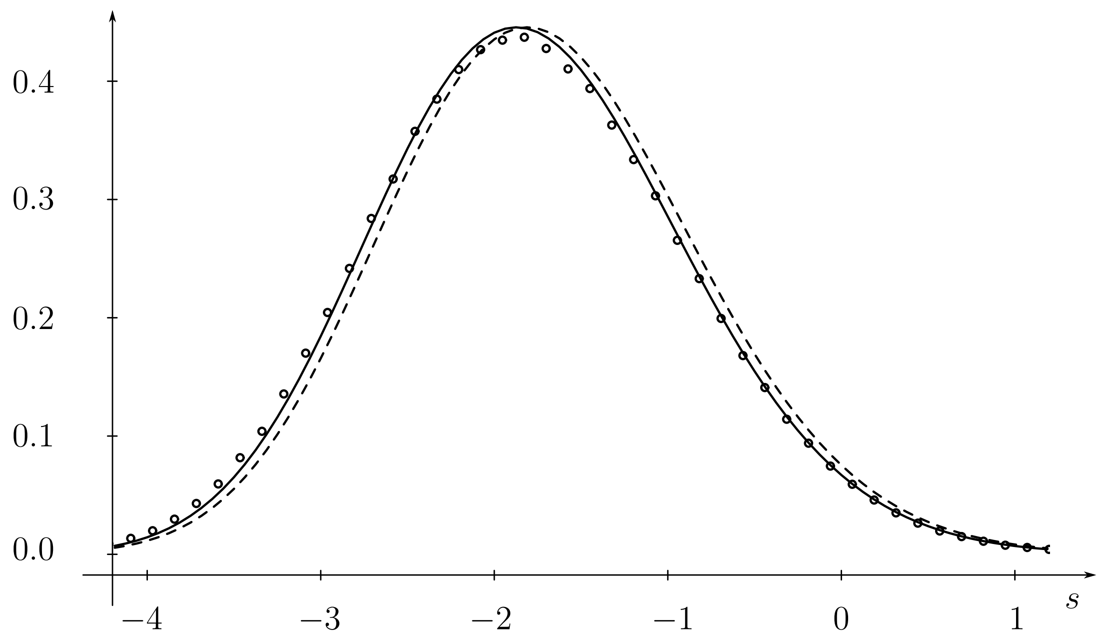


Figure 4.4.: Probabilities for $x_{[t/4]}(t)$ for TASEP with step initial conditions and $t = 1000$. The number of runs is 10^6. The dots are the plot of $(s \in I_t, \mathbb{P}(x_{[t/4]}(t) = X(s))\delta_t^{-1})$ with $a = 1/2$ and $\delta_t = (t/2)^{-1/3}$. The solid line is $(s, F_2'(s))$ while the dashed line is $(s, F_2'(s - a\delta_t))$.

TASEP, $t = 1000$	Mean	Variance	Skewness	Kurtosis
Alternating IC	-0.60495	0.4027	0.282	0.143
$s \mapsto F_{\rm GOE}(2s)$	-0.60327	$0.4019(5)$	0.293	0.165
Relative error	0.28 %	0.18 %	-3.6 %	-13 %
Step IC	$-1.7949(7)$	$0.842(2)$	$0.19(0)$	$0.06(7)$
$s \mapsto F_{\rm GUE}(s)$	-1.77109	0.8132	0.224	0.094
Relative error	1.3 %	3.6 %	-15 %	-29 %

Table 4.1.: Comparison between alternating and step initial conditions. The data comes from the simulation used for the previous figures.

The function $z \mapsto z+\sigma\ln(1-z)-(\sigma+c_1)\ln z$ has a double critical point at $\xi = 1-\sqrt{\sigma}$. The steepest descent path for z can be taken such that it comes into ξ with an angle $e^{-2\pi i/3}$, leaves with an angle $e^{2\pi i/3}$, and is completed by a piece of a circle around zero of radius strictly larger than ξ. The steepest descent path for w comes into ξ with an angle $e^{\pi i/3}$, leaves it with an angle $e^{-\pi i/3}$, and is completed by a piece of a circle around zero of radius strictly larger than 1. By the change of variables

$$z = \xi + c_3^{-1}t^{-1/3}Z, \quad w = \xi + c_3^{-1}t^{-1/3}W$$

with $c_3 = \sigma^{-1/6}(1-\sqrt{\sigma})^{-1/3}$ and a large t expansion of the integrand, we have

$$\begin{aligned} K_{t,\sigma,\rm resc}^{\rm stepTASEP}(s_1,s_2) &= \frac{1}{(2\pi i)^2}\int_{\infty e^{-2\pi i/3}}^{\infty e^{2\pi i/3}} dZ \int_{\infty e^{-\pi i/3}}^{\infty e^{\pi i/3}} dW \, \frac{e^{W^3/3-s_2W}}{e^{Z^3/3-s_1Z}}\frac{1}{W-Z} \\ &\times \left(1+c_2^{-1}t^{-1/3}\left((a-1)Z-aW+\frac{s_2W^2-s_1Z^2}{2}+c_4(Z^4-W^4)\right)+\mathcal{O}(t^{-2/3})\right), \end{aligned}$$

with $c_4 = (1-2\sqrt{\sigma})/(4\sqrt{\sigma})$. The choice $a = 1/2$ makes the first order correction of the kernel antisymmetric. Finally, we can choose the paths satisfying $\mathrm{Re}\, Z < \mathrm{Re}\, W$ and use $\frac{1}{W-Z} = \int_0^\infty d\lambda\, e^{-\lambda(W-Z)}$ to obtain (4.12). □

Remark 13. Looking at Figures 4.1–4.4 one has the impression that TASEP with alternating initial conditions is already "closer" than with step initial conditions to its asymptotics at time $t = 1000$, which is confirmed by the data in Table 4.1. This is to remind the reader that although in both cases the error is $\mathcal{O}(t^{-2/3})$, depending on the prefactor one still might see some differences of the accuracy for not too large times t. The slower convergence for curved vs. flat geometry holds also for the PNG model as verified numerically by Richter in his diploma thesis [84] adapting the numerical approach of Borneman [13].

4.3. PASEP

Consider the partially asymmetric simple exclusion process on $\mathbb{Z}$ in continuous time with step initial condition. A formula for the one-point distribution of the nth particle from the right has been derived in [102, 103]. The expression is this time not just a Fredholm determinant, but an integral in the complex plane of a Fredholm determinant. A rigorous large time asymptotic analysis is in [104], in which it is shown that particles in the rarefaction fan fluctuate asymptotically according to the GUE Tracy-Widom distribution $F_{\rm GUE}$. The scaling limit to be considered is[13]

$$n = \sigma t \in \mathbb{Z}, \quad X(s) = c_1(\sigma)t - sc_2(\sigma)t^{1/3} - a \in \mathbb{Z}$$

where

$$c_1(\sigma) = 1 - 2\sqrt{\sigma}, \quad c_2(\sigma) = \sigma^{-1/6}(1-\sqrt{\sigma})^{2/3}.$$

The t-independent constant a will be specified later. Then in [104] it is proven that

$$\lim_{t\to\infty} \mathbb{P}(x_n(t/\gamma) \geq X(s)) = F_{\rm GUE}(s) \quad \text{with} \quad \gamma = p - q > 0.$$

Our result on the first order correction is the following.

Proposition 14. *Let $p \in (\frac{1}{2}, 1]$, $q = 1 - p$, and set*

$$a_{p,q} = \sum_{\ell=1}^{\infty} \frac{q^\ell}{p^\ell - q^\ell} \quad \text{and} \quad a = \frac{1}{2} - \frac{1}{\sqrt{\sigma}} a_{p,q}.$$

Then for large time t it holds

$$\mathbb{P}\left(x_n(t/\gamma) \geq X(s)\right) = F_{\rm GUE}(s + \tfrac{1}{2}\delta_t)(1 + \mathcal{O}(t^{-2/3})), \tag{4.13}$$

for s in a bounded subset of $I_t = (\mathbb{Z} - c_1(\sigma)t + a)\delta_t$ with $\delta_t = c_2(\sigma)^{-1}t^{-1/3}$.

Remark 15. Note that the previously discussed TASEP with step initial conditions is a special case of this result, with $p = 1 - q = 1$, since $a_{1,0} = 0$

From this result one can easily get the corresponding result for the height function. Let $\eta_x(t)$ be 1 if there is a particle at site x at time t, and zero otherwise. Then the height function is defined by

$$h(x,t) = \begin{cases} 2J(0,t) + \sum_{y=0}^{x-1}(1 - 2\eta_y(t)), & \text{for } x \geq 1, \\ 2J(0,t), & \text{for } x = 0, \\ 2J(0,t) - \sum_{y=x}^{-1}(1-2\eta_y(t)), & \text{for } x \leq -1, \end{cases}$$

where $J(0,t) = \sum_{y\geq 0} \eta_y(t)$. To get the result for the h from Proposition 14 one simply uses the identity

$$\mathbb{P}(x_j(t) \geq x) = \mathbb{P}(h(x,t) \geq 2j + x) \tag{4.14}$$

with the following result.

[13] There is a minor difference with respect to the papers of Tracy and Widom. To get their framework we need to apply the transformation $x \to -x$.

Corollary 16. *Let $x = \xi t \in \mathbb{Z}$ and set*

$$H(s) = \frac{1}{2}(1+\xi^2)t - s\frac{(1-\xi^2)^{2/3}}{2^{1/3}}t^{1/3} + \tilde{a}, \quad \tilde{a} = 2a_{p,q} - 1.$$

Then, for large t it holds

$$\mathbb{P}(h(x,t/\gamma) \geq H(s)) = F_{\rm GUE}(s + \tfrac{1}{2}\delta_t^h)(1+\mathcal{O}(t^{-2/3})),$$

for $s \in I_t^h$, where I_t^h is defined by the requirement $H(s) \in 2\mathbb{Z}$ if x is even and $H(s) \in 2\mathbb{Z}+1$ if x is odd. Since the lattice width of h is 2, we have $\delta_t^h = 2^{4/3}(1-\xi^2)^{-2/3}t^{-1/3}$.

From this result it follows that the critical value p_c of the asymmetry in PASEP such that the density (and the moments) of the rescaled integrated current are correct up to order $\mathcal{O}(t^{-2/3})$ is the solution of

$$a_{p_c,1-p_c} = \frac{1}{2} \quad \Longleftrightarrow \quad p_c = 0.78227\,87862\ldots$$

In Figure 4.5 we plot the function $2a_{p,1-p} - 1$.

Proof of Corollary 16. Let us define a linearization of the distribution functions by

$$\widetilde{F}_t(s) := \begin{cases} \mathbb{P}(x_n(t/\gamma) \geq c_1(\sigma)t - sc_2(\sigma)t^{1/3} - \frac{1}{2} + \frac{1}{\sqrt{\sigma}}a_{p,q}), & \text{if } s \in I_t, \\ \text{linear interpolation}, & \text{otherwise}, \end{cases}$$

and similarly

$$\widetilde{F}_t^h(s) := \begin{cases} \mathbb{P}(h(\xi t, t/\gamma) \geq H(s), & \text{if } s \in I_t^h, \\ \text{linear interpolation}, & \text{otherwise}. \end{cases}$$

Then, Proposition 14 tell us that

$$\widetilde{F}_t(s - \tfrac{1}{2}\delta_t) = F_{\rm GUE}(s)(1+\mathcal{O}(t^{-2/3})), \quad s \in \mathbb{R}, \tag{4.15}$$

and we want to show that

$$\widetilde{F}_t^h(s - \tfrac{1}{2}\delta_t^h) = F_{\rm GUE}(s)(1+\mathcal{O}(t^{-2/3})), \quad s \in I_t^h + \tfrac{1}{2}\delta_t^h.$$

For $s \in I_t^h + \frac{1}{2}\delta_t^h$, using (4.14) we have

$$\begin{aligned} \widetilde{F}_t^h(s - \tfrac{1}{2}\delta_t^h) &= \mathbb{P}(h(\xi t, t/\gamma) \geq H(s - \tfrac{1}{2}\delta_t^h)) \\ &= \mathbb{P}(h(\xi t, t/\gamma) \geq H(s) + 1) = \mathbb{P}(x_{\sigma t}(t/\gamma) \geq \xi t) \end{aligned} \tag{4.16}$$

with $\sigma = (H(s) + 1 - \xi t)/(2t)$. With this value of σ, an algebraic computation gives

$$\xi t = c_1(\sigma)t - sc_2(\sigma)t^{1/3} - (a - 1/2) + \mathcal{O}(t^{-1/3}).$$

Since $sc_2(\sigma)t^{1/3} + (a - 1/2) = (s - \frac{1}{2}\delta_t)c_2(\sigma)t^{1/3} + a$ we get that

$$\widetilde{F}_t^h(s - \tfrac{1}{2}\delta_t^h) = (4.16) = \widetilde{F}_t(s - \tfrac{1}{2}\delta_t + \mathcal{O}(t^{-2/3})) = F_{\rm GUE}(s)(1+\mathcal{O}(t^{-2/3}))$$

where in the last step we used (4.15) coming from Proposition 14. □

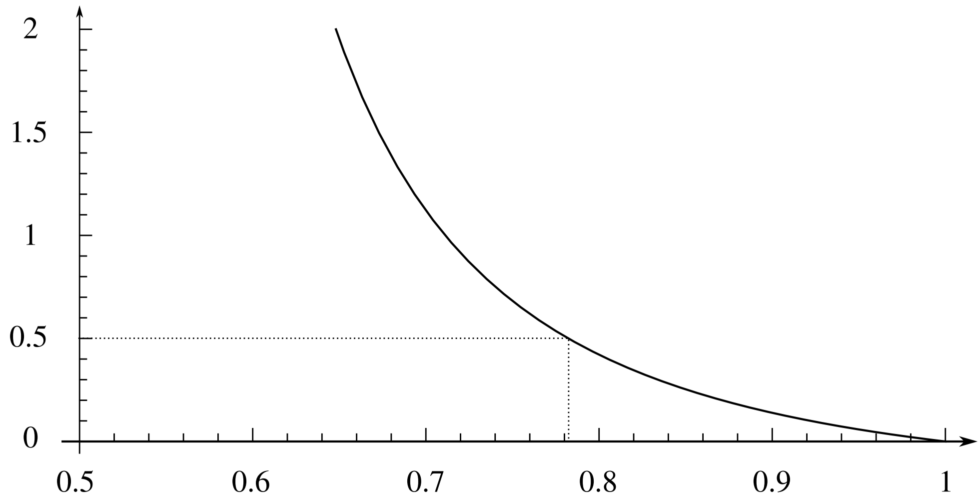


Figure 4.5.: The function $p \mapsto a_{p,1-p}$ for $p \in (1/2, 1]$. It attains the value 0.5 at the point $p = p_c = 0.78227\,87862\ldots$

Remark 17. As $p \to 1/2$ the model becomes close to the WASEP studied in [4, 86, 87, 89]. In particular, our result matches the limit behavior of [86]. Indeed, when the asymmetry $\beta := 2p - 1 \to 0$, the shift for the fitting of the density behaves as

$$a_{p,1-p} \simeq \frac{\gamma_{\rm E} - \ln(2\beta)}{2\beta} + \frac{1}{4} + \mathcal{O}(\beta),$$

with $\gamma_{\rm E} = -\partial_x \ln(\Gamma(x))\big|_{x=1} = -0.57721\,56649\ldots$ the Euler constant, so that for the height function the shift is then $\tilde{a} = \beta^{-1}(\ln(2\beta) - \gamma_{\rm E}) - \frac{1}{2} + \mathcal{O}(\beta)$.

Proof of Proposition 14. As for PNG and TASEP, we indicate the main steps of the asymptotic analysis to get (4.13) for s in a bounded set, but we will derive bounds for $|s| \to \infty$ needed to determine moment convergence. Set $u = c_1 t - c_2 s t^{1/3} - a$ and $\tau = q/p < 1$. As shown in [106],

$$\mathbb{P}\big(x_n(t/\gamma) \geq u\big) = \frac{1}{2\pi\mathrm{i}} \oint \frac{\mathrm{d}\mu}{\mu} \, (\mu;\tau)_\infty \det\big(\mathbb{1} + \mu J_\mu\big),$$

where $(\mu;\tau)_\infty$ is the q-Pochhammer symbol (see Appendix A.4 for identities) and the integral is taken over a circle around the origin with radius in the interval $(0, \tau)$. The operator J_μ has kernel

$$J_\mu(\eta, \eta') = \frac{1}{2\pi\mathrm{i}} \oint \mathrm{d}\zeta \, \frac{\exp\big(\frac{t\zeta}{1-\zeta}\big)(1-\zeta)^u \zeta^n}{\exp\big(\frac{t\eta'}{1-\eta'}\big)(1-\eta')^u (\eta')^{n+1}} \, \frac{f\big(\mu, \frac{\zeta}{\eta'}\big)}{\zeta - \eta},$$

where η, η' are on a circle around 0 with radius $r \in (\tau, 1)$ and ζ runs on a circle around 0 with radius in $(1, r/\tau)$. For $1 < |z| < \tau^{-1}$, the function f is given by

$$f(\mu, z) = \sum_{k=-\infty}^{\infty} \frac{\tau^k}{1 - \tau^k \mu} \, z^k,$$

which extends analytically to $\mathbb{C}^* \setminus \{\tau^k : k \in \mathbb{Z}\}$.

The function $\zeta \mapsto \frac{\zeta}{1-\zeta} + \sigma \ln \zeta + c_1 \ln(1-\zeta)$ has a double critical point at $\xi = -\frac{\sqrt{\sigma}}{1-\sqrt{\sigma}}$, and the steepest descent path can be taken such that the η-contour for J_μ is a pair of rays from ξ in the directions $\pm\pi/3$ completed by a circle around zero of radius strictly smaller than 1, and the ζ-contour is a pair of rays from $\xi - t^{-1/3}$ in the directions $\pm 2\pi/3$ completed by a circle around zero of radius strictly larger than 1. We then do the transformations

$$\zeta = \xi + c_3^{-1} t^{-1/3} z, \quad \eta = \xi + c_3^{-1} t^{-1/3} w, \quad \eta' = \xi + c_3^{-1} t^{-1/3} \tilde{w},$$

with $c_3 = \sigma^{-1/6}(1-\sqrt{\sigma})^{5/3}$. Expanding f around $z = 1$ yields

$$\mu\, f(\mu, z) = \frac{1}{1-z} + g(\mu) + \mathcal{O}\big(|z-1|\big),$$

where

$$g(\mu) = \sum_{k=0}^{\infty} \frac{\mu\tau^k}{1-\mu\tau^k} + \sum_{k=1}^{\infty} \frac{\tau^k}{\tau^k - \mu}.$$

This expansion is obtained by dividing the series into $\{0, 1, \dots\}$ and $\{\dots, -2, -1\}$, using

$$\frac{\mu\tau^k}{1-\tau^k\mu} = \frac{1}{1-\tau^k\mu} - 1,$$

and then change the variable $k \to -k$ in one of the sum. After a large t expansion, the kernel μJ_μ can be written as

$$\begin{aligned} &\frac{1}{2\pi\mathrm{i}} \int_{\mathrm{e}^{-2\pi\mathrm{i}/3}\infty}^{\mathrm{e}^{2\pi\mathrm{i}/3}\infty} \mathrm{d}z\, \frac{\mathrm{e}^{\tilde{w}^3/3 - \tilde{w}(s+\delta_t/2)}}{\mathrm{e}^{z^3/3 - z(s+\delta_t/2)}} \frac{\mathrm{e}^{(\tilde{w}-z)\delta_t/2}}{(z-w)(\tilde{w}-z)} \\ &\qquad \times \frac{\exp\Big(\big(c_4\tilde{w}^4 - \frac{1}{2}s\tilde{w}^2 - \big(\frac{g(\mu)}{\sqrt{\sigma}} + a\big)\tilde{w}\big)c_2^{-1}t^{-1/3} + \mathcal{O}(t^{-2/3})\Big)}{\exp\Big(\big(c_4 z^4 - \frac{1}{2}s z^2 - \big(\frac{g(\mu)}{\sqrt{\sigma}} + a\big)z\big)c_2^{-1}t^{-1/3} + \mathcal{O}(t^{-2/3})\Big)} \end{aligned} \tag{4.17}$$

with $c_4 = \frac{1+2\sqrt{\sigma}}{4\sqrt{\sigma}}$. Since $\mathrm{Re}(z - \tilde{w}) < 0$, we have

$$\frac{\mathrm{e}^{(z-\tilde{w})(s+\delta_t/2)}}{\tilde{w} - z} = \int_{s+\delta_t/2}^{\infty} \mathrm{d}x\, \mathrm{e}^{(z-\tilde{w})x}, \tag{4.18}$$

and plugging (4.18) into (4.17) gives

$$\begin{aligned} &\frac{1}{2\pi\mathrm{i}} \int_{s+\delta_t/2}^{\infty} \mathrm{d}x \int_{\mathrm{e}^{-2\pi\mathrm{i}/3}\infty}^{\mathrm{e}^{2\pi\mathrm{i}/3}\infty} \mathrm{d}z\, \frac{\mathrm{e}^{\tilde{w}^3/3 - \tilde{w}x}}{\mathrm{e}^{z^3/3 - zx}} \frac{1}{z-w} \\ &\qquad \times \frac{\exp\Big(\big(c_4\tilde{w}^4 - \frac{1}{2}s\tilde{w}^2 - \big(\frac{g(\mu)}{\sqrt{\sigma}} - \frac{1}{2} + a\big)\tilde{w}\big)c_2^{-1}t^{-1/3} + \mathcal{O}(t^{-2/3})\Big)}{\exp\Big(\big(c_4 z^4 - \frac{1}{2}s z^2 - \big(\frac{g(\mu)}{\sqrt{\sigma}} - \frac{1}{2} + a\big)z\big)c_2^{-1}t^{-1/3} + \mathcal{O}(t^{-2/3})\Big)} \end{aligned}$$

The operator J_μ is the product $C_1\, C_2\, C_3$, where the factors have kernels

$$\begin{aligned} C_1(w,z) &= \frac{1}{2\pi \mathrm{i}} \frac{\mathrm{e}^{-z^3/3}}{z-w}, \\ C_2(z,x) &= \frac{1}{2\pi \mathrm{i}} \mathrm{e}^{xz}\, \mathrm{e}^{-c_2^{-1} t^{-1/3} \left[c_4 z^4 - s z^2/2 - z(g(\mu)/\sqrt{\sigma} - 1/2 + a)\right] + \mathcal{O}(t^{-2/3})}, \\ C_3(x,\tilde{w}) &= \mathrm{e}^{\tilde{w}^3/3 - \tilde{w}x}\, \mathrm{e}^{c_2^{-1} t^{-1/3} \left[c_4 \tilde{w}^4 - s\tilde{w}^2/2 - \tilde{w}(g(\mu)/\sqrt{\sigma} - 1/2 + a)\right] + \mathcal{O}(t^{-2/3})}. \end{aligned}$$

The operator $C_3\, C_1\, C_2$, which has the same Fredholm determinant, acts on $L^2(s+\delta_t/2,\infty)$ and has kernel with (x,y) entry given by

$$\begin{aligned} &\frac{1}{(2\pi \mathrm{i})^2} \int_{\mathrm{e}^{-2\pi \mathrm{i}/3}\infty}^{\mathrm{e}^{2\pi \mathrm{i}/3}\infty} \mathrm{d}z \int_{\mathrm{e}^{-\pi \mathrm{i}/3}\infty}^{\mathrm{e}^{\pi \mathrm{i}/3}\infty} \mathrm{d}w\, \frac{\mathrm{e}^{w^3/3 - wx}}{\mathrm{e}^{z^3/3 - zy}} \frac{1}{z-w} \Big(1 + \mathcal{O}(t^{-2/3}) \\ &\qquad + \big((c_4(w^4 - z^4) - \tfrac{1}{2} s(w^2 - z^2) - \big(\tfrac{g(\mu)}{\sqrt{\sigma}} - \tfrac{1}{2} + a\big)(w-z)\big) c_2^{-1} t^{-1/3}\Big). \quad (4.19) \end{aligned}$$

Using again that $\mathrm{Re}(z-w) < 0$, we can write $\frac{1}{w-z} = \int_0^\infty \mathrm{d}\lambda \mathrm{e}^{-\lambda(w-z)}$. Thus, (4.19) equals

$$-K_{\mathcal{A}_2}(x,y) + c_2^{-1} t^{-1/3}\big(K_{\text{asym}}(x,y) + K_{\text{sym}}(x,y)\big) + \mathcal{O}(t^{-2/3}),$$

where $K_{\text{asym}}(x,y) = P(x,y) - P(y,x)$ with

$$P(x,y) = \int_0^\infty \mathrm{d}\lambda\, \mathrm{Ai}(x+\lambda) \left[c_4 \frac{\mathrm{d}^4}{\mathrm{d}y^4} - \frac{s}{2} \frac{\mathrm{d}^2}{\mathrm{d}y^2}\right] \mathrm{Ai}(y+\lambda)$$

is asymmetric, and $K_{\text{sym}}(x,y) = -\big(\frac{g(\mu)}{\sqrt{\sigma}} - \frac{1}{2} + a\big)\, \mathrm{Ai}(x)\, \mathrm{Ai}(y)$ is symmetric. Hence, we have

$$\begin{aligned} &\mathbb{P}\big(x_m(t/\gamma) \geq u\big) = F_{\mathrm{GUE}}(s+\delta_t/2) \\ &\quad \times \Big(1 - \big(\tfrac{G}{\sqrt{\sigma}} - \tfrac{1}{2} + a\big) \operatorname{Tr}\big((\mathbb{1} - \chi_s K_{\mathcal{A}_2} \chi_s)^{-1} \chi_s (\mathrm{Ai} \otimes \mathrm{Ai}) \chi_s\big) t^{-1/3} + \mathcal{O}(t^{-2/3})\Big). \quad (4.20) \end{aligned}$$

with χ_s the projection onto $(s+\delta_t/2,\infty)$ and

$$G = \frac{1}{2\pi \mathrm{i}} \oint_{\tau<|\mu|<1} \frac{\mathrm{d}\mu}{\mu} g(\mu) (\mu;\tau)_\infty.$$

We will show that $G = a_{p,q}$ so that by choosing $a = \frac{1}{2} - \frac{1}{\sqrt{\sigma}} a_{p,q}$ the prefactor of the first order correction vanishes. First note that

$$\frac{1}{2\pi \mathrm{i}} \oint_{\tau<|\mu|<1} \frac{\mathrm{d}\mu}{\mu} (\mu;\tau)_\infty \sum_{k=0}^\infty \frac{\mu \tau^k}{1 - \tau^k \mu} = 0,$$

as the integrand has no poles inside the unit circle. So, we have

$$G = \frac{1}{2\pi \mathrm{i}} \oint_{\tau<|\mu|<1} \frac{\mathrm{d}\mu}{\mu} (\mu;\tau)_\infty \sum_{k=1}^\infty \frac{\tau^k}{\tau^k - \mu}.$$

We use

$$\frac{1}{\mu}\frac{\tau^k}{\tau^k-\mu}=\frac{1}{\mu}+\frac{1}{\tau^k-\mu},$$

the fact that $(\mu;\tau)_\infty$ is analytic inside the integration domain, so that the sum of the contributions of the simple poles gives

$$G=\sum_{k=1}^{\infty}\left(1-(\tau^k;\tau)_\infty\right).$$

There is a simpler expression for G. Using the identity (A.9) we get

$$G=-\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{(-1)^\ell\tau^{\ell(\ell-1)/2}}{(\tau;\tau)_\ell}\tau^{k\ell}=\sum_{\ell=1}^{\infty}\frac{(-1)^\ell\tau^{\ell(\ell-1)/2}}{(\tau;\tau)_\ell}\frac{\tau^\ell}{\tau^\ell-1}$$

Then we use (A.10) and $(\alpha;\tau)_\ell/(\alpha\tau;\tau)_\ell=(\alpha-1)/(\alpha\tau^\ell-1)$ to get

$$\begin{aligned}G&=\lim_{\alpha\to1}\frac{1}{\alpha-1}\sum_{\ell=1}^{\infty}\frac{(-1)^\ell\tau^{\ell(\ell-1)/2}(\alpha;\tau)_\ell}{(\tau;\tau)_\ell(\alpha\tau;\tau)_\ell}\tau^\ell\\&=\lim_{\alpha\to1}\frac{1}{\alpha-1}\left({}_1\phi_1\left(\begin{matrix}\alpha\\ \alpha\tau\end{matrix}\middle|\tau;\tau\right)-1\right)\\&=\lim_{\alpha\to1}\frac{1}{\alpha-1}\lim_{\beta\to\infty}\left({}_2\phi_1\left(\begin{matrix}\alpha,\beta\\ \alpha\tau\end{matrix}\middle|\tau;\frac{\tau}{\beta}\right)-1\right)\end{aligned}$$

where we used (A.11) in the last equality. Finally, the q-Gauss identity (A.12) leads to

$$\begin{aligned}G&=\lim_{\alpha\to1}\frac{1}{\alpha-1}\lim_{\beta\to\infty}\left(\frac{(\alpha\tau/\beta;\tau)_\infty(\tau;\tau)_\infty}{(\alpha\tau;\tau)_\infty(\tau/\beta;\tau)_\infty}-1\right)=-\frac{\partial_\alpha(\alpha\tau;\tau)_\infty}{(\tau;\tau)_\infty}\bigg|_{\alpha=1}\\&=-\partial_\alpha\ln[(\alpha\tau;\tau)_\infty]\big|_{\alpha=1}=-\partial_\alpha\sum_{\ell=0}^{\infty}\ln(1-\alpha\tau^{\ell+1})\bigg|_{\alpha=1}=\sum_{\ell=1}^{\infty}\frac{\tau^\ell}{1-\tau^\ell}.\end{aligned}$$

Replacing $\tau=q/p$ leads to $G=a_{p,q}$. This and (4.20) shows that

$$\mathbb{P}\big(x_n(t/\gamma)\geq u\big)=F_{\mathrm{GUE}}(s)(1+\mathcal{O}(t^{-2/3})).$$

□

4.4. Discrete sums versus integrals

Lemma 18. *Let $f:\mathbb{R}^n\to\mathbb{R}$ be a smooth function such that $\partial_j\partial_k f\in L^1(\mathbb{R}^n)$ for any $j,k=1,\ldots,n$. Then, for $\delta>0$ (small)*

$$\left|\delta^n\sum_{x\in(\mathbb{Z}_+^*)^n}f(x\delta)-\int_{(-\frac{\delta}{2},\infty)^n}\mathrm{d}^nx\,f(x)\right|=\mathcal{O}(\delta^2)\sum_{j,k=1}^{n}\int_{\mathbb{R}_+^n}\mathrm{d}^nx\,|\partial_j\partial_k f(x)|.$$

Proof. We first rewrite

$$\int_{(-\frac{\delta}{2},\infty)^n} \mathrm{d}^n x\, f(x) = \sum_{x\in(\mathbb{Z}_+^*)^n} \int_{[-\frac{\delta}{2},\frac{\delta}{2}]^n} \mathrm{d}^n y\, f(x\delta+y)$$

and use Taylor development

$$f(x\delta+y) = f(x\delta) + \sum_{j=1}^{n} \partial_j f(x\delta) y_j + \frac{1}{2}\sum_{j,k=1}^{n} y_j y_k R_{j,k}(\delta,y),$$

with

$$|R_{j,k}(\delta,y)| \le \max_{u\in[-\frac{\delta}{2},\frac{\delta}{2}]^2} |\partial_j\partial_k f(x\delta+u)|, \quad \text{for} \quad y\in[-\tfrac{\delta}{2},\tfrac{\delta}{2}]^n$$

to obtain (after integrating over y),

$$\left| \int_{(-\frac{\delta}{2},\infty)^n} \mathrm{d}^n x\, f(x) - \sum_{x\in(\mathbb{Z}_+^*)^n} \delta^n f(x\delta) \right| \le \frac{\delta^2}{12} \sum_{x\in(\mathbb{Z}_+^*)^n} \sum_{j,k=1}^{n} \delta^n \max_{u\in[-\frac{\delta}{2},\frac{\delta}{2}]^2} |\partial_j\partial_k f(x\delta+u)|.$$

The statement then follows because the sum over x converges, as $\delta\to 0$, to

$$\sum_{j,k=1}^{n} \int_{\mathbb{R}_+^n} \mathrm{d}^n x\, |\partial_j\partial_k f(x)|,$$

which finishes the proof. □

Lemma 19. *Let* $f(x_1,\ldots,x_n) = \det(K(x_i,x_j))_{1\le i,j\le n}$ *with the kernel* K *satisfying*

$$\max\{|K(x_1,x_2)|, |\partial_i K(x_1,x_2)|, |\partial_i\partial_j K(x_1,x_2)|\} \le C\mathrm{e}^{-c(x_1+x_2)} \tag{4.21}$$

for all $x_1,x_2\in(s,\infty)$*,* $i,j\in\{1,2\}$ *and some positive constants c,* C*. Then,*

$$|\partial_i\partial_j f(x_1,\ldots,x_n)| \le 4\, C^n n^{n/2} \prod_{k=1}^{n} \mathrm{e}^{-2cx_k}$$

for all $1\le i,j\le n$*.*

Proof. Let $K_{i,}$ (resp. $K_{,j}$) be the matrix $(K(x_i,x_j))_{1\le i,j\le n}$ with the ith row (resp. the jth column) replaced by its derivative w.r.t. the first (resp. the second) variable. Then,

$$\partial_i f(x_1,\ldots,x_n) = \det K_{i,} + \det K_{,i}$$

and from this

$$\partial_i\partial_j f(x_1,\ldots,x_n) = \det K_{ij,} + \det K_{i,j} + \det K_{j,i} + \det K_{,ji}$$

with $K_{ij,} = (K_{i,})_{j,}$. By Hadamard's bound, the absolute value of an $n\times n$ determinant with entries in the closed unit disk is bounded by $n^{n/2}$. It then follows

$$|\partial_i\partial_j f(x_1,\ldots,x_n)| \le 4\, C^n n^{n/2} \prod_{k=1}^{n} \mathrm{e}^{-2cx_k}$$

for any $1\le i,j\le n$. □

Lemma 20. *For the the Airy*$_2$ *kernel,* $K_{\mathcal{A}_2}(x,y) = \int_{\mathbb{R}_+} \mathrm{d}\lambda \ \mathrm{Ai}(x+\lambda)\,\mathrm{Ai}(y+\lambda)$*, and the Airy*$_1$ *kernel,* $K_{\mathcal{A}_1}(x,y) = \mathrm{Ai}(x+y)$*, assumption* (4.21) *of Lemma 19 is satisfied.*

Proof. It is easy to see from the integral representation

$$\mathrm{Ai}(x) = \frac{1}{2\pi\mathrm{i}} \int_{\infty\mathrm{e}^{-\pi\mathrm{i}/3}}^{\infty\mathrm{e}^{\pi\mathrm{i}/3}} \mathrm{d}z\, \mathrm{e}^{z^3/3 - z\,x} = \frac{1}{2\pi\mathrm{i}} \int_{\mathrm{i}\mathbb{R}+\delta} \mathrm{d}z\, \mathrm{e}^{z^3/3 - z\,x}, \quad \mathrm{e} > 0, \tag{4.22}$$

that for any $\delta > 0$, there exists a constant $C_\delta \in (0,\infty)$ so that

$$\max\{|\mathrm{Ai}(x)|, |\mathrm{Ai}'(x)|, |\mathrm{Ai}''(x)|\} \le C_\delta \mathrm{e}^{-\delta x} \tag{4.23}$$

uniformly in $x \in \mathbb{R}$.

In the case $K(x_1,x_2) = \int_{\mathbb{R}_+} \mathrm{d}\lambda \ \mathrm{Ai}(x_1+\lambda)\,\mathrm{Ai}(x_2+\lambda)$ we get from the bounds (4.23) with $\varepsilon = \frac{1}{2}$, after integration with respect to λ, that

$$\max\{|K(x_1,x_2)|, |\partial_i K(x_1,x_2)|, |\partial_i \partial_j K(x_1,x_2)|\} \le C\mathrm{e}^{-(x_1+x_2)/2} \tag{4.24}$$

for some constant $C > 0$ and all $i,j \in \{1,2\}$.

The case $K(x_1,x_2) = \mathrm{Ai}(x_1+x_2)$ is even easier, since the bound (4.24) comes directly from (4.23). □

5. Random matrices and space-like paths

In this chapter we will prove Results 7 to 10 of Chapter 3.2.3. The presentation is taken from [46].

5.1. Evolution of GUE minors

First, we will prove the theorem that we referred to as Result 8. The issue of the Markov property is discussed in Section 5.3 below and therefore we assume it to hold in this section. For $0 < t_1 < t_2$, the joint distribution of $H_1 = H(n, t_1)$ and $H_2 = H(n, t_2)$ is given by

$$\text{const} \times \exp\left(-\frac{\operatorname{Tr}(H_1^2)}{t_1}\right) \exp\left(-\frac{\operatorname{Tr}((H_2 - H_1)^2)}{t_2 - t_1}\right) \mathrm{d}H_1 \, \mathrm{d}H_2. \tag{5.1}$$

The measure on eigenvalues is obtained using Eynard-Mehta formula [42] for coupled random matrices, which on its turn is based on the Harish-Chandra/Itzykson-Zuber formula [52, 55] (see Appendix A.7). It results in the following formula.

Lemma 21. *Let n be fixed. Denote by $\lambda_k^n(t)$, $1 \leq k \leq n$, the eigenvalues of $H(n,t)$. Their joint distribution at $0 < t_1 < t_2$ is given by*

$$\begin{aligned} \text{const} \times \Delta(\lambda^n(t_1)) \det\left(\mathrm{e}^{-(\lambda_i^n(t_1) - \lambda_j^n(t_2))^2/(t_2 - t_1)}\right)_{1 \leq i,j \leq n} \Delta(\lambda^n(t_2)) \\ \times \prod_{i=1}^{N} \mathrm{e}^{-(\lambda_i^n(t_1))^2/t_1} \, \mathrm{d}\lambda_i^n(t_1) \, \mathrm{d}\lambda_i^n(t_2), \end{aligned}$$

with Δ the Vandermonde determinant and $\lambda^n(t) = (\lambda_1^n(t), \ldots, \lambda_n^n(t))$.

The second formula concerns the joint distribution of the eigenvalues at two different levels. This result is a special case of the formula (3.20) discussed above. (It is enough to reintegrate out the lower levels, which gives a Vandermonde determinant).

Lemma 22. *Let t be fixed. Denote by $\lambda_k^n(t)$, $1 \leq k \leq n$, the eigenvalues of $H(n,t)$. Their joint distribution at levels n and $n+1$ is given by*

$$\begin{aligned} \text{const} \times \Delta(\lambda^n(t)) \det[\phi(\lambda_i^n(t), \lambda_j^{n+1}(t))]_{1 \leq i,j \leq n+1} \Delta(\lambda^{n+1}(t)) \\ \times \prod_{i=1}^{n+1} \mathrm{e}^{-(\lambda_i^{n+1}(t))^2/t} \, \mathrm{d}\lambda_i^n(t) \, \mathrm{d}\lambda_i^{n+1}(t), \end{aligned}$$

where $\lambda_{n+1}^n \equiv \mathrm{virt}$ *are* virtual variables, $\phi(x,y) = \mathbb{1}_{[x\leq y]}$, $\phi(\mathrm{virt},y) = 1$ *(and* Δ *the Vandermonde determinant).*

The eigenvalues' process is a Markov process (see Section 5.3 for details) for both fixed matrix dimension n and increasing time t, as well as for fixed time t and decreasing matrix dimension n. The combination of the formulas in Lemma 21 and Lemma 22 leads to Proposition 23:

Proposition 23. *Let* $N_1 \geq \cdots \geq N_m = 1$ *be integers and* $0 < t_1 < \cdots < t_m$ *be reals. We denote by* $\lambda_1^n(t) < \cdots < \lambda_n^n(t)$ *the eigenvalues of* $H(n,t)$ *and set* $N_0 = N_1$, $N_{m+1} = 0$*. Then the joint density of*

$$\{\lambda_k^n(t_j) : 1 \leq j \leq m, N_j \leq n \leq N_{j-1}, 1 \leq k \leq n\}$$

is given by

$$\begin{aligned}
\mathrm{const} \times \det\big[\Psi_{N_1-\ell}^{N_1,t_1}(\lambda_k^{N_1}(t_1))\big]_{1\leq k,\ell\leq N_1} \\
\times \prod_{j=1}^{m-1}\Bigg[\det\big[\mathcal{T}_{t_{j+1},t_j}(\lambda_k^{N_j}(t_{j+1}),\lambda_\ell^{N_j}(t_j))\big]_{1\leq k,\ell\leq N_j} \\
\times \prod_{n=N_{j+1}+1}^{N_j} \det\big[\phi(\lambda_k^{n-1}(t_{j+1}),\lambda_\ell^n(t_{j+1}))\big]_{1\leq k,\ell\leq n}\Bigg],
\end{aligned} \tag{5.2}$$

where

$$\begin{aligned}
\phi(x,y) &= \mathbb{1}_{[x\leq y]}, \quad \phi(x_n^{n-1},y) = 1, \\
\mathcal{T}_{t,s}(x,y) &= \frac{1}{\sqrt{\pi(t-s)}}\exp\left(-\frac{(x-y)^2}{t-s}\right)\mathbb{1}_{[t\geq s]}, \\
\Psi_k^{N_1,t_1}(x) &= \frac{1}{t_1^{k/2}}\,p_k\left(\frac{x}{\sqrt{t_1}}\right)\frac{1}{\sqrt{\pi t_1}}\exp\left(-\frac{x^2}{t_1}\right),
\end{aligned}$$

for $k = 0,\ldots,N_1-1$*. Here* p_k *is the standard Hermite polynomial of degree* k *(see Appendix A.5 for details).*

We could have chosen any polynomials of degree k multiplied by the Gaussian weight without changing the probability measure (5.2) since the modifications would just affect the normalization constant. However, this choice allows a huge simplification of the computations, because of the properties of Lemma 24 below.

To determine the kernel, we first slightly rewrite (5.2). Let $c(n) = \#\{i : N_i = n\}$ for $1 \leq n \leq N_1$, and we denote the consecutive times for such a level by $t_1^n < \cdots < t_{c(n)}^n$. Then, the measure (5.2) can be rewritten as

$$\begin{aligned}
\mathrm{const} \times \prod_{n=2}^{N_1}\Bigg(\det\big[\phi(\lambda_k^{n-1}(t_1^{n-1}),\lambda_\ell^n(t_{c(n)}^n))\big]_{1\leq k,\ell\leq n} \\
\times \prod_{a=2}^{c(n)}\det\big[\mathcal{T}_{t_a^n,t_{a-1}^n}(\lambda_k^n(t_a^n),\lambda_\ell^n(t_{a-1}^n))\big]_{1\leq k,\ell\leq n}\Bigg)\det[\Psi_{N_1-\ell}^{N_1,t_1^{N_1}}(\lambda_k^{N_1}(t_1^{N_1}))\big]_{1\leq k,\ell\leq N_1}.
\end{aligned}$$

It is known that a measure of this form has determinantal correlations and the correlation kernel is computed by means of Theorem 4.2 of [16], which we report in Appendix A.3 for the reader.

For any given $k \in \mathbb{Z}$ we set

$$\Psi_k^{n,t}(x) = \frac{2^{k+1}}{t^{(k+1)/2}} \frac{1}{2\pi \mathrm{i}} \int_{\mathrm{i}\mathbb{R}+\varepsilon} \mathrm{d}w \, \mathrm{e}^{w^2 - 2wx/\sqrt{t}} w^k, \quad \varepsilon > 0. \tag{5.3}$$

For $n = N_1, t = t_1$ and $k = 0, \ldots, N_1 - 1$, this function is the one in the measure (5.2), which is obtained from the first representation of Hermite polynomials in (A.14).

Lemma 24. *It holds, for* $0 < r < s < t$ *and* $k \geq 1$*,*

(i) $\phi * \Psi_{n-k}^{n,t} = \Psi_{n-1-k}^{n-1,t}$,

(ii) $\mathcal{T}_{t,s} * \Psi_{n-k}^{n,s} = \Psi_{n-k}^{n,t}$,

(iii) $\phi * \mathcal{T}_{t,s} = \mathcal{T}_{t,s} * \phi$,

(iv) $\mathcal{T}_{t,s} * \mathcal{T}_{s,u} = \mathcal{T}_{t,u}$.

Proof. For the first relation, we use $\mathrm{Re}(w) = \varepsilon > 0$ so that we can exchange the two integrals,

$$\begin{aligned} (\phi * \Psi_k^{n,t})(x) &= \int_x^\infty \mathrm{d}y \, \frac{2^{k+1}}{t^{(k+1)/2}} \frac{1}{2\pi \mathrm{i}} \int_{\mathrm{i}\mathbb{R}+\varepsilon} \mathrm{d}w \, \mathrm{e}^{w^2 - 2wy/t^{1/2}} w^k \\ &= \frac{2^{k+1}}{t^{(k+1)/2}} \frac{1}{2\pi \mathrm{i}} \int_{\mathrm{i}\mathbb{R}+\varepsilon} \mathrm{d}w \, \mathrm{e}^{w^2} w^k \int_x^\infty \mathrm{d}y \, \mathrm{e}^{-2wy/t^{1/2}} \\ &= \frac{2^k}{t^{k/2}} \frac{1}{2\pi \mathrm{i}} \int_{\mathrm{i}\mathbb{R}+\varepsilon} \mathrm{d}w \, \mathrm{e}^{w^2 - 2wx/t^{1/2}} w^{k-1} = \Psi_{k-1}^{n-1,t}(x). \end{aligned}$$

For the second identity, we first do the change of variable $w = z(s/t)^{1/2}$ in the integral representation (5.3) of $\Psi_k^{n,s}$ and then perform a Gaussian integration:

$$\begin{aligned} (\mathcal{T}_{t,s} * \Psi_k^{n,s})(x) &= \frac{2^{k+1}}{t^{(k+1)/2}} \frac{1}{2\pi \mathrm{i}} \int_{\mathrm{i}\mathbb{R}+\varepsilon} \mathrm{d}z \, \mathrm{e}^{z^2 s/t} z^k \int_{\mathbb{R}} \mathrm{d}y \, \frac{\exp\left(-\frac{(x-y)^2}{t-s} - \frac{2yz}{\sqrt{t}}\right)}{\sqrt{\pi(t-s)}} \\ &= \frac{2^{k+1}}{t^{(k+1)/2}} \frac{1}{2\pi \mathrm{i}} \int_{\mathrm{i}\mathbb{R}+\varepsilon} \mathrm{d}z \, \mathrm{e}^{z^2} \mathrm{e}^{-2xz/\sqrt{t}} z^k = \Psi_k^{n,t}(x). \end{aligned}$$

The third relation is also easy to verify. Indeed,

$$\begin{aligned} (\phi * \mathcal{T}_{t,s})(x,z) &= \int_{\mathbb{R}} \mathrm{d}y \, \phi(x,y) \mathcal{T}_{t,s}(y,z) = \int_{\mathbb{R}_+} \mathrm{d}y \, \mathcal{T}_{t,s}(y+x,z) \\ &= \int_{\mathbb{R}} \mathrm{d}y \, \mathcal{T}_{t,s}(x,z-y) \phi(z-y,z) = \int_{\mathbb{R}} \mathrm{d}y \, \mathcal{T}_{t,s}(x,y) \phi(y,z) = (\mathcal{T}_{t,s} * \phi)(x,z). \end{aligned}$$

The last relation is the standard heat kernel semigroup identity. $\square$

By Theorem 44 and Remark 45, there is a simple way of getting the kernel if the matrix M with

$$M_{k,\ell} = (\phi * \mathcal{T}^k * \cdots * \phi * \mathcal{T}^{N_1} * \Psi^{N_1,t_1^{N_1}}_{N_1-\ell})(x_k^{k-1}),$$

is upper triangular, where $\mathcal{T}^n := \mathcal{T}_{t^n_{c(n)},t^n_1}$. The identities in Lemma 24 give, for $k \geq \ell$,

$$M_{k,\ell} = (\phi * \Psi^{k,t^k_{c(k)}}_{k-\ell})(x_k^{k-1}) = \int_{\mathbb{R}} \mathrm{d}x\, \Psi^{k,t^k_{c(k)}}_{k-\ell}(x) \begin{cases} = 0, & \text{for } \ell < k, \\ \neq 0, & \text{for } \ell = k, \end{cases}$$

because the last expression is (after a rescaling in x) proportional to the orthogonal relation (A.13) for $n = 0$ and $m = k - \ell$.

Next we need to determine the polynomials $\Phi^{n,t}_\ell(x)$, $\ell = 0, \ldots, n-1$, which are biorthogonal to the functions $\Psi^{n,t}_k(x)$, $k = 0, \ldots, n-1$, i.e., polynomials satisfying

$$\int_{\mathbb{R}} \mathrm{d}x\, \Psi^{n,t}_k(x) \Phi^{n,t}_\ell(x) = \delta_{k,\ell}, \quad 1 \leq k, \ell \leq n-1. \tag{5.4}$$

Lemma 25. *The functions*

$$\Phi^{n,t}_\ell(x) = \frac{1}{\ell!} \frac{t^{\ell/2}}{2^\ell} p_\ell\Big(\frac{x}{\sqrt{t}}\Big) = \frac{t^{\ell/2}}{2^\ell} \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma_0} \mathrm{d}z\, \frac{\mathrm{e}^{-z^2+2zx/t^{1/2}}}{z^{\ell+1}} \tag{5.5}$$

satisfy the relation (5.4).

Proof. Do the change of variable $x \mapsto x\sqrt{t}$ and then use the orthogonal relation (A.13). □

Let us compute the last term in (A.8). To simplify the notations, we set $t_1 = t^{n_1}_{a_1}$ and $t_2 = t^{n_2}_{a_2}$. First, we do the changes of variables $w = \sqrt{t_1}\tilde{w}$ and $z = \sqrt{t_2}\tilde{z}$ in (5.3) and (5.5). We obtain

$$\sum_{k=1}^{n_2} \Psi^{n_1,t_1}_{n_1-k}(x_1) \Phi^{n_2,t_2}_{n_2-k}(x_2) = \sum_{k=1}^{n_2} \frac{2^{n_1}}{2^{n_2}} \frac{2}{(2\pi\mathrm{i})^2} \int_{\mathrm{i}\mathbb{R}+\varepsilon} \mathrm{d}\tilde{w} \oint_{\Gamma_0} \mathrm{d}\tilde{z} \frac{\mathrm{e}^{\tilde{w}^2 t_1 - 2\tilde{w}x_1}}{\mathrm{e}^{\tilde{z}^2 t_2 - 2\tilde{z}x_2}} \frac{\tilde{w}^{n_1-k}}{\tilde{z}^{n_2+1-k}}$$

Now, we take the integral over $\tilde{z}$ to satisfy $|\tilde{z}| < |\tilde{w}|$, say $|\tilde{z}| = \varepsilon/2$. This allows us to take the sum inside and extend it to $+\infty$ (because for $k > n_2$ the pole at zero for $\tilde{z}$ vanishes). The sum over k gives

$$\sum_{k\geq 1} \frac{\tilde{z}^{k-1}}{\tilde{w}^k} = \frac{1}{\tilde{w} - \tilde{z}},$$

so that we obtain

$$\sum_{k=1}^{n_2} \Psi^{n_1,t_1}_{n_1-k}(x_1) \Phi^{n_2,t_2}_{n_2-k}(x_2) = \frac{2^{n_1}}{2^{n_2}} \frac{2}{(2\pi\mathrm{i})^2} \int_{\mathrm{i}\mathbb{R}+\varepsilon} \mathrm{d}\tilde{w} \oint_{|z|=\varepsilon/2} \mathrm{d}\tilde{z} \frac{\mathrm{e}^{\tilde{w}^2 t_1 - 2\tilde{w}x_1}}{\mathrm{e}^{\tilde{z}^2 t_2 - 2\tilde{z}x_2}} \frac{\tilde{w}^{n_1}}{\tilde{z}^{n_2}} \frac{1}{\tilde{w} - \tilde{z}}. \tag{5.6}$$

The last term we have to compute is $\phi^{(t_{a_1}^{n_1},t_{a_2}^{n_2})}$. We set $\phi^{(t_{a_1}^{n_1},t_{a_2}^{n_2})}(x,y) = \phi^{(n_1,t_1;n_2,t_2)}(x,y)$ to simplify the notations. We have

$$\phi^{(n_1,t_1;n_2,t_2)} = \begin{cases} \phi^{*(n_2-n_1)} * \mathcal{T}_{t_2,t_1}, & \text{if } (n_1,t_1) \prec (n_2,t_2), \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to verify that $\phi(x,y)$ has the integral representation

$$\phi(x,y) = \frac{2}{2\pi \mathrm{i}} \int_{\mathrm{i}\mathbb{R}+\varepsilon} \mathrm{d}w \, \frac{\mathrm{e}^{2w(y-x)}}{2w}, \quad \varepsilon > 0.$$

and similarly,

$$\phi^{*n}(x,y) = \frac{2}{2\pi \mathrm{i}} \int_{\mathrm{i}\mathbb{R}+\varepsilon} \mathrm{d}w \, \frac{\mathrm{e}^{2w(y-x)}}{(2w)^n}, \quad \varepsilon > 0.$$

Then, for $(n_1,t_1) \prec (n_2,t_2)$, a Gaussian integration gives us

$$\phi^{(n_1,t_1;n_2,t_2)}(x_1,x_2) = \frac{2^{n_1}}{2^{n_2}} \frac{2}{2\pi \mathrm{i}} \int_{\mathrm{i}\mathbb{R}+\varepsilon} \mathrm{d}w \, \frac{\mathrm{e}^{w^2(t_1-t_2)-2w(x_1-x_2)}}{w^{n_2-n_1}}. \tag{5.7}$$

Equations (5.6) and (5.7) yield a kernel which is, up to the conjugation factor[1] $2^{n_1-n_2}$, the same as (3.22). Thus the proof of Result 8 is completed.

5.2. Evolution on Wishart minors

In this section we prove Result 10 on Wishart matrices. As for the GUE case, the issue of the Markov property is discussed in Section 5.3 below. For $0 < t_1 < t_2$, the joint distribution of $A_1 = A(n,t_1)$ and $A_2 = A(n,t_2)$ is given by

$$\text{const} \times \exp\left(-\frac{\mathrm{Tr}(A_1^* A_1)}{t_1}\right) \exp\left(-\frac{\mathrm{Tr}((A_2^*-A_1^*)(A_2-A_1))}{t_2-t_1}\right) \mathrm{d}A_1 \, \mathrm{d}A_2. \tag{5.8}$$

The measure on eigenvalues is obtained (as in the Ornstein-Uhlenbeck case studied in [101]) by the Harish-Chandra/Itzykson-Zuber formula for rectangular matrices [56, 111] (see Appendix A.7). It results in the following formula.

Lemma 26. *Let n be fixed. Denote by $\lambda_k^n(t)$, $1 \leq k \leq n \leq p$, the eigenvalues of the matrix $H(n,t) = A(n,t)^* A(n,t)$. Their joint distribution at $0 < t_1 < t_2$ is given by*

$$\text{const} \times \det\left[I_{p-n}\left(\tfrac{2\sqrt{\lambda_i^n(t_1)\lambda_j^n(t_2)}}{t_2-t_1}\right) \left(\tfrac{\lambda_j^n(t_2)}{\lambda_i^n(t_1)}\right)^{(p-n)/2} \mathrm{e}^{-(\lambda_i^n(t_1)+\lambda_j^n(t_2))/(t_2-t_1)} \right]_{1\leq i,j\leq n}$$
$$\times \Delta(\lambda^n(t_1))\Delta(\lambda^n(t_2)) \prod_{i=1}^n (\lambda_i(t_1))^{p-n} \mathrm{e}^{-\lambda_i^n(t_1)/t_1} \, \mathrm{d}\lambda_i^n(t_1) \, \mathrm{d}\lambda_i^n(t_2),$$

where I_m is the modified Bessel function of order m, see (A.18).

[1] A determinantal point process is defined by its correlation kernel, which is defined up to conjugations.

The second formula concerns the joint distributions of the eigenvalues at two different levels. This is studied in [49] with the following result.

Lemma 27. *Let t be fixed. Denote by $\lambda_k^n(t)$, $1 \leq k \leq n < p$, the eigenvalues of $H(n,t)$. Their joint distribution at levels n and $n+1$ is given by*

$$\begin{aligned} \mathrm{const} \times \Delta(\lambda^n(t)) \det\big[\phi(\lambda_i^n(t), \lambda_j^{n+1}(t))\big]_{1\leq i,j\leq n+1} \Delta(\lambda^{n+1}(t)) \\ \times \prod_{i=1}^{n+1} (\lambda_i^{n+1}(t))^{p-(n+1)} \mathrm{e}^{-\lambda_i^{n+1}(t)/t}\, \mathrm{d}\lambda_i^n(t)\, \mathrm{d}\lambda_i^{n+1}(t), \end{aligned}$$

where $\lambda_{n+1}^n \equiv \mathrm{virt}$ are virtual variables, $\phi(x,y) = \mathbb{1}_{[x\geq y]}$ and $\phi(\mathrm{virt}, y) = 1$.

Putting together the formulas in lemmata 26 and 27 leads to the next proposition.

Proposition 28. *Let $p \geq N_1 \geq \dots \geq N_m = 1$ be integers and $0 < t_1 < \dots < t_m$ be real numbers. We denote by $\lambda_1^n(t) < \dots < \lambda_n^n(t)$ the eigenvalues of $H(n,t)$ and set $N_0 = N_1$, $N_{m+1} = 0$. Then the joint density of*

$$\{\lambda_k^n(t_j) : 1 \leq j \leq m, N_j \leq n \leq N_{j-1}, 1 \leq k \leq n\}$$

is given by

$$\begin{aligned} \mathrm{const} \times \det\big[\Psi_{N_1-\ell}^{p-N_1,t_1}\big(\lambda_k^{N_1}(t_1)\big)\big]_{1\leq k,\ell\leq N_1} \prod_{j=1}^{m-1} \Bigg[\det\big[\mathcal{T}_{t_{j+1},t_j}^{p-N_j}\big(\lambda_k^{N_j}(t_{j+1}), \lambda_\ell^{N_j}(t_j)\big)\big]_{1\leq k,\ell\leq n_j} \\ \times \prod_{\ell=N_{j+1}+1}^{N_j} \det\big[\phi\big(\lambda_k^{n-1}(t_{j+1}), \lambda_\ell^n(t_{j+1})\big)\big]_{1\leq k,\ell\leq n}\Bigg], \end{aligned} \tag{5.9}$$

where

$$\begin{aligned} \phi(x,y) &= \mathbb{1}_{[x\geq y]} \quad \textit{and} \quad \phi(\lambda_{n+1}^n, y) = 1, \\ \mathcal{T}_{t,s}^n(x,y) &= \left(\frac{x}{y}\right)^{n/2} I_n\left(\frac{2\sqrt{xy}}{t-s}\right) \frac{1}{t-s} \exp\left(-\frac{x+y}{t-s}\right) \mathbb{1}_{[x,y>0]} \mathbb{1}_{[s\leq t]}, \\ \Psi_k^{p-N_1,t_1}(x) &= \frac{k!}{(p-N_1+k)!\,t_1^{k+1}} \left(\frac{x}{t_1}\right)^{p-N_1} \exp\left(-\frac{x}{t_1}\right) L_k^{p-N_1}\left(\frac{x}{t_1}\right) \mathbb{1}_{[x>0]}, \end{aligned} \tag{5.10}$$

for $k = 0, \dots, N_1 - 1$. Here L_k^n are the generalized Laguerre polynomials of order n and degree k, see Appendix A.6.

Comparing the mathematical structure of (5.2) and (5.9), we see that the only difference is that the transition kernel for time depends also on the level. However, this does not pose any problem, see Remark 46.

For $k \in \mathbb{Z}$ and $x \in \mathbb{R}_+$ we set

$$\Psi_k^{n,t}(x) = \frac{t^{-(k+1)}}{2\pi \mathrm{i}} \oint_{\Gamma_0} \mathrm{d}z \, \frac{(z-1)^k}{z^{n+k+1}} \mathrm{e}^{x(z-1)/t}. \tag{5.11}$$

For $n = p - N_1$, $t = t_1$ and $k = 0, \ldots, N_1 - 1$ the above defined function coincides with (5.10). Moreover, the prefactors are chosen such that the following nice recursion relations hold.

Lemma 29. *It holds, for $t > s > r > 0$, $n \leq p$, and $k \geq 1$*

(i) $\phi * \Psi_{n-k}^{p-n,t} = \Psi_{(n-1)-k}^{p-(n-1),t}$,

(ii) $\mathcal{T}_{t,s}^{p-n} * \Psi_{n-k}^{p-n,s} = \Psi_{n-k}^{p-n,t}$,

(iii) $\phi * \mathcal{T}_{t,s}^{p-n} = \mathcal{T}_{t,s}^{p-(n-1)} * \phi$,

(iv) $\mathcal{T}_{t,s}^{p-n} * \mathcal{T}_{s,r}^{p-n} = \mathcal{T}_{t,r}^{p-n}$.

To prove this lemma, we first obtain a different integral representation for (5.11). Namely, after the change of variable $z = \tilde{z}/(\tilde{z} - t)$ we get

$$\Psi_k^{n,t}(x) = \frac{-1}{2\pi \mathrm{i}} \oint_{\Gamma_0} \mathrm{d}\tilde{z} \, \frac{(\tilde{z}-t)^{n-1}}{\tilde{z}^{n+k+1}} \mathrm{e}^{x/(\tilde{z}-t)}. \tag{5.12}$$

Proof of Lemma 29. Using the representation (5.12), we have

$$\begin{aligned} (\phi * \Psi_k^{n,t})(x) &= \frac{-1}{2\pi \mathrm{i}} \oint_{\Gamma_0} \mathrm{d}z \, \frac{(z-t)^{n-1}}{z^{n+k+1}} \int_0^x \mathrm{d}y \, \mathrm{e}^{y/(z-t)} \\ &= \frac{-1}{2\pi \mathrm{i}} \oint_{\Gamma_0} \mathrm{d}z \, \frac{(z-t)^n}{z^{n+k+1}} \left(\mathrm{e}^{x/(z-t)} - 1 \right) = \Psi_{k-1}^{n+1,t}(x) \end{aligned}$$

because for $k \geq 0$ the term independent of x has residue equal to zero.

Using the integral representation (A.18) of the modified Bessel function I_n in (5.9), we get (for $x, y > 0, t > s > 0$)

$$\mathcal{T}_{t,s}^n(x,y) = \frac{1}{2\pi \mathrm{i}(t-s)} \oint_{\Gamma_0} \frac{\mathrm{d}z}{z^{n+1}} \exp\left(-\frac{x(1-z) + y(1-z^{-1})}{t-s} \right), \tag{5.13}$$

and the change of variable $z = (w-s)/(w-t)$ leads to

$$\mathcal{T}_{t,s}^n(x,y) = \frac{-1}{2\pi \mathrm{i}} \oint_{\Gamma_s} \mathrm{d}w \, \frac{(w-t)^{n-1}}{(w-s)^{n+1}} \mathrm{e}^{x/(w-t) - y/(w-s)}.$$

We choose the integration path for w large and z small such that $\mathrm{Re}(1/(z-s)-1/(w-t)) < 0$ (in particular, z is contained in Γ_s, so that we write it explicitly as $\Gamma_{s,z}$). Then, we can exchange the integral over y with the integral over z and w,

$$\begin{aligned}
&(\mathcal{T}^n_{t,s} * \Psi^{n,s}_k)(x) \\
&= \frac{1}{(2\pi\mathrm{i})^2} \oint_{\Gamma_0} \mathrm{d}z \frac{(z-s)^{n-1}}{z^{n+k+1}} \oint_{\Gamma_{s,z}} \mathrm{d}w \frac{(w-t)^{n-1}}{(w-s)^{n+1}} \mathrm{e}^{x/(w-t)} \int_{\mathbb{R}_+} \mathrm{d}y\, \mathrm{e}^{y/(z-s)-y/(w-s)} \\
&= \frac{1}{(2\pi\mathrm{i})^2} \oint_{\Gamma_0} \mathrm{d}z\, \frac{(z-s)^{n}}{z^{n+k+1}} \oint_{\Gamma_{s,z}} \mathrm{d}w\, \frac{(w-t)^{n-1}}{(w-s)^{n}} \mathrm{e}^{x/(w-t)}\, \frac{1}{z-w}.
\end{aligned}$$

Now we enlarge the path of z so that encloses the path of w. This can be made at the expense of the residue at $z = w$. Thus we get

$$\begin{aligned}
(\mathcal{T}^n_{t,s} * \Psi^{n,s}_k)(x) &= \frac{1}{(2\pi\mathrm{i})^2} \oint_{\Gamma_s} \mathrm{d}w\, \frac{(w-t)^{n-1}}{(w-s)^{n}} \mathrm{e}^{x/(w-t)} \oint_{\Gamma_{0,w}} \mathrm{d}z\, \frac{(z-s)^{n}}{z^{n+k+1}}\, \frac{1}{z-w} \\
&\quad - \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma_0} \mathrm{d}w\, \frac{(w-t)^{n-1}}{w^{n+k+1}} \mathrm{e}^{x/(w-t)} = \Psi^{n,t}_k(x),
\end{aligned}$$

because the first term is zero, since the residue of z at infinity is zero ($k \geq 0$).

For the third identity, we use the representation (5.13) in which we take the path Γ_0 for w to satisfy $|w| > 1$. Then,

$$\begin{aligned}
(\phi * \mathcal{T}^{n-1}_{t,s})(x,y) &= \frac{1}{2\pi\mathrm{i}(t-s)} \oint_{\Gamma_0} \frac{\mathrm{d}w}{w^n}\, \mathrm{e}^{-\frac{y}{t-s}(1-w^{-1})} \int_0^x \mathrm{d}z\, \mathrm{e}^{-\frac{z}{t-s}(1-w)} \\
&= \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma_0} \frac{\mathrm{d}w}{(w-1)w^n}\, \mathrm{e}^{-\frac{y}{t-s}(1-w^{-1})} \left(\mathrm{e}^{-\frac{x}{t-s}(1-w)} - 1\right).
\end{aligned}$$

The last term (the integrand independent of x) is zero, because the integrand has residue zero at infinity, whenever $n-1 \geq 0$. Thus,

$$\begin{aligned}
(\phi * \mathcal{T}^{n-1}_{t,s})(x,y) &= \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma_0} \frac{\mathrm{d}w}{(w-1)w^n}\, \mathrm{e}^{-\frac{x}{t-s}(1-w)-\frac{y}{t-s}(1-w^{-1})} \\
&= \frac{1}{2\pi\mathrm{i}(t-s)} \oint_{\Gamma_0} \frac{\mathrm{d}w}{w^{n+1}}\, \mathrm{e}^{-\frac{x}{t-s}(1-w)} \int_y^\infty \mathrm{d}z\, \mathrm{e}^{-\frac{z}{t-s}(1-w^{-1})} \\
&= (\mathcal{T}^n_{t,s} * \phi)(x,y),
\end{aligned}$$

The final identity is true because $\mathcal{T}^n$ it is the transition density of a $2n+1$ dimensional Bessel process. □

We proceed as in the proof of Theorem 8 to show that the matrix M is upper triangular. Indeed, with $\mathcal{T}^n := \mathcal{T}^{p-n}_{t^n_{c(n)},t^n_1}$ and Lemma 29 we find

$$M_{k,\ell} = (\phi * \Psi^{p-k,t^k_{c(k)}}_{k-\ell})(x^{k-1}_k) = \int_{\mathbb{R}_+} \mathrm{d}x\, \Psi^{p-k,t^k_{c(k)}}_{k-\ell}(x) = \begin{cases} 0, & \text{if } \ell < k, \\ 1, & \text{if } \ell = k, \end{cases}$$

because of the orthogonality between $\Psi^{n,t}_k$, $k \geq 1$, and the constant function.

Lemma 30. *Define, for* $\ell = 0, \dots, n-1$, *the polynomial* $\Phi_\ell^{n,t}$ *of degree* ℓ *by*

$$\Phi_\ell^{n,t}(x) = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_{0,t}} \mathrm{d}w \, \frac{w^{n+\ell}}{(w-t)^{n+1}} \, \mathrm{e}^{-x/(w-t)}. \tag{5.14}$$

These polynomials satisfy the orthogonal relation

$$\int_{\mathbb{R}_+} \mathrm{d}x \, \Psi_k^{n,t}(x) \Psi_\ell^{n,t}(x) = \delta_{k,\ell} \tag{5.15}$$

for $k, \ell = 0, \dots, n-1$.

Proof. By the integral representation in Appendix A.6 for Laguerre polynomials, we have

$$t^\ell L_\ell^n(x/t) = \frac{t^\ell}{2\pi \mathrm{i}} \oint_{\Gamma_1} \mathrm{d}w \, \frac{w^{n+\ell}}{(w-1)^{\ell+1}} \, \mathrm{e}^{-x(w-1)/t}$$

and, after change of variable $w = \tilde{w}/(\tilde{w} - t)$, we get $t^\ell L_\ell^n(x/t) = \Phi_\ell^{n,t}(x)$ as defined in (5.14). The orthogonality relation (5.15) holds because after the change of variable $x \to xt$, the left-hand side becomes

$$t \int_{\mathbb{R}_+} \mathrm{d}x \, \Psi_k^{n,t}(xt) \Phi_\ell^{n,t}(xt) = \frac{k! t^\ell}{(n+k)! t^k} \int_{\mathbb{R}_+} \mathrm{d}x \, x^n \mathrm{e}^{-x} L_\ell^n(x) L_k^n(x) = \delta_{k,\ell},$$

which is the orthogonal relation (A.15) for Laguerre polynomials. □

We now compute the kernel and start with the sum in (A.8). Let us use the notations $t_1 = t_{a_1}^{n_1}$, $t_2 = t_{a_2}^{n_2}$. Then we get

$$\begin{aligned} &\sum_{k=1}^{n_2} \Psi_{n_1-k}^{p-n_1,t_1}(x_1) \Phi_{n_2-k}^{p-n_2,t_2}(x_2) \\ &\qquad = \frac{-1}{(2\pi \mathrm{i})^2} \oint_{\Gamma_0} \mathrm{d}z \oint_{\Gamma_{0,t_2}} \mathrm{d}w \, \frac{\mathrm{e}^{x_1/(z-t_1)}}{\mathrm{e}^{x_2/(w-t_2)}} \, \frac{(z-t_1)^{p-n_1-1}}{(w-t_2)^{p-n_2+1}} \, \frac{w^p}{z^{p+1}} \sum_{k=1}^{n_2} \left(\frac{z}{w}\right)^k . \end{aligned}$$

We choose Γ_0 and Γ_{0,t_2} such that they do not intersect, i.e., $|z| < |w|$. For $k > n_2$ the pole at $w = \infty$ vanishes and we can thus extend the summation over k to ∞ with the result

$$\frac{-1}{(2\pi \mathrm{i})^2} \oint_{\Gamma_0} \mathrm{d}z \oint_{\Gamma_{z,t_2}} \mathrm{d}w \, \frac{\mathrm{e}^{x_1/(z-t_1)}}{\mathrm{e}^{x_2/(w-t_2)}} \, \frac{(z-t_1)^{p-n_1-1}}{(w-t_2)^{p-n_2+1}} \, \frac{w^p}{z^p} \, \frac{1}{w-z},$$

which is the second term in the kernel in Result 10. It remains to compute $\phi^{(t_{a_1}^{n_1}, t_{a_2}^{n_2})}$. To simplify the notations, we set $\phi^{(t_{a_1}^{n_1}, t_{a_2}^{n_2})}(x,y) = \phi^{(n_1,t_1;n_2,t_2)}(x,y)$. By Lemma 29 we have

$$\phi^{(n_1,t_1;n_2,t_2)} = \begin{cases} \mathcal{T}_{t_2,t_1}^{p-n_1} * \phi^{*(n_2-n_1)}, & \text{if } (n_1,t_1) \prec (n_2,t_2), \\ 0, & \text{otherwise.} \end{cases}$$

The integral representation (5.13) for $\mathcal{T}$ and $\phi^{*n}(x,y) = \frac{(x-y)^{n-1}}{(n-1)!}\phi(x,y)$ lead to

$$\begin{aligned}\phi^{(n_1,t_1;n_2,t_2)}&(x,y)\\ &= \frac{(t_1-t_2)^{-1}}{2\pi\mathrm{i}} \oint_{\Gamma_0} \mathrm{d}w\, \frac{\mathrm{e}^{-(1-w)x/(t_1-t_2)}}{w^{p+1-n_1}} \int_y^\infty \mathrm{d}z\, \mathrm{e}^{-z(1-w^{-1})/(t_1-t_2)} \frac{(z-y)^{n_2-n_1-1}}{(n_2-n_1-1)!}\\ &= \frac{(t_1-t_2)^{n_2-n_1-1}}{2\pi\mathrm{i}} \oint_{\Gamma_0} \mathrm{d}w\, \frac{\mathrm{e}^{-x(1-w)/(t_1-t_2)-y(1-w^{-1})/(t_1-t_2)}}{w^{p+1-n_2}(w-1)^{n_2-n_1}}.\end{aligned}$$

Finally, the change of variable $w = (z-t_2)/(z-t_1)$ gives the first term in Result 10.

5.3. Markov property on space-like paths

What remains to prove is the Markov property on space-like paths that we claimed in Results 7 and 9. The process on matrices is clearly a Markov process along space-like paths. What we have to see is that the Markov property still holds for the eigenvalues. The key ingredients are that the measure on matrices is invariant under choice of basis, and that the choice of basis at an observation point (n,t) depends neither on the eigenvalues at that the previous point ($(n+1,t)$ or (n,t') with $t'<t$) nor on the eigenvalues at (n,t).

5.3.1. Diffusion on GUE minors

Let us start with the proof of Result 7. We first consider Dyson's diffusion. Here we denote by $H(n,t)$ the $n\times n$ minor at time t and by $\Lambda(n,t)$ the diagonalized matrix of $H(n,t)$ which is obtained from conjugation by the unitary matrix $U(n,t)$,

$$H(n,t) = U(n,t)\Lambda(n,t)U^*(n,t).$$

The Jacobian of the transformation $H(n,t)\to(\Lambda(n,t),U(n,t))$ gives

$$\mathrm{d}H(n,t) = \Delta(\Lambda(n,t))^2\,\mathrm{d}\Lambda(n,t)\,\mathrm{d}\mu_n(U(n,t)) \tag{5.16}$$

where $\mathrm{d}\mu_n$ denotes the Haar measure on the unitary group $\mathcal{U}(n)$.

Consider a measure at (n,t) which is invariant under unitary transformations, i.e. with respect to the group $\mathcal{U}(n)$. It has the form

$$f_1(\Lambda(n,t))\,\mathrm{d}\Lambda(n,t)\,\mathrm{d}\mu_n(U(n,t)) \tag{5.17}$$

for some explicit function f_1 (e.g. $f_1(H) = \exp(-\operatorname{Tr}(H^2)/t)$).

Next fix n and consider times $t' > t$. Then, the probability measure on the matrices has the form (see (5.1))

$$\begin{aligned} & f_1(\Lambda(n,t))\mathrm{e}^{-b\,\mathrm{Tr}((H(n,t')-H(n,t)))^2}\,\mathrm{d}\Lambda(n,t)\,\mathrm{d}\mu_n(U(n,t))\,\mathrm{d}H(n,t') \\ = & f_1(\Lambda(n,t))\mathrm{e}^{-b\,\mathrm{Tr}((\Lambda(n,t')-\tilde{U}(n,t)\Lambda(n,t)\tilde{U}^*(n,t)))^2}\,\mathrm{d}\Lambda(n,t)\,\mathrm{d}\mu_n(\tilde{U}(n,t))\,\mathrm{d}H(n,t') \end{aligned}$$

because of the unitary invariance of the Haar measure. (Note that here we have set $\tilde{U}(n,t) = U(n,t')^*U(n,t)$). The integration with respect to $\mathrm{d}\mu_n(\tilde{U}(n,t))$ is made by the well-known Harish-Chandra/Itzykson-Zuber (A.16) and the result is as in Lemma 21. We are left with a probability density that depends only on the eigenvalues times $\mathrm{d}H(n,t')$, that is, *the projection onto eigenvalues at time t did not restrict the complete freedom of choice of basis at (n,t').* Otherwise stated, by the decomposition (5.16), after integration over $\mathrm{d}\mu_n(\tilde{U}(n,t))$ we have a measure on eigenvalues times $\mathrm{d}\mu_n(U(n,t'))$ of the form (for some explicit f_2, which can be easily computed)

$$f_2(\Lambda(n,t),\Lambda(n,t'))\,\mathrm{d}\Lambda(n,t)\,\mathrm{d}\Lambda(n,t')\,\mathrm{d}\mu_n(U(n,t')). \tag{5.18}$$

The other choice is to consider t fixed and look at the measure at (n,t) and $(n-1,t)$. The result explained in Proposition 4.2 of [10] is actually a conditional measure on eigenvalues given the eigenvalues of the minor of size n, thus it is not restricted to GUE, but it holds for any measure which is invariant under $\mathcal{U}(n)$, see Theorem 3.4 of [35]. The projection on the eigenvalues at (n,t) and $(n-1,t)$ leads to Lemma 22. We can also decide to project on the eigenvalues at (n,t) and $(n-1,t)$ and the eigenvectors at $(n-1,t)$. This means that we do not integrate out the variables corresponding to the unitary transformations of the $(n-1)\times(n-1)$ minor given by

$$\begin{pmatrix} U(n-1,t) & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{with } U(n-1,t) \text{ distributed as } \mathrm{d}\mu_{n-1},$$

which form a subgroup of $\mathcal{U}(n)$. The eigenvalues $\Lambda(n-1,t)$ are independent of the eigenvectors (thus of the choice of basis $U(n-1,t)$) and the measure on $U(n-1,t)$ is then $\mathrm{d}\mu_{n-1}(U(n-1,t))$ (see e.g. Corollary 2.5.4 in [5]). The measure on $H(n,t)$ is invariant under $\mathcal{U}(n)$, so are the eigenvalues $\Lambda(n,t)$ independent of the choice of $U(n-1,t)$ (this last property follows also from the direct computation in Section 3.1 of [49]; Section 3.2 for Wishart matrices). Thus, the projection on the eigenvalues at (n,t) and $(n-1,t)$ and the eigenvectors at $(n-1,t)$ leads to a measure of the form

$$f_3(\Lambda(n,t),\Lambda(n-1,t))\,\mathrm{d}\Lambda(n,t)\,\mathrm{d}\Lambda(n-1,t)\,\mathrm{d}\mu_{n-1}(U(n-1,t)). \tag{5.19}$$

for some explicit function f_3 (compare with Lemma 22).

To resume, (5.18) and (5.19) tell us that starting from a measure of the form (5.17), in which the choice of basis is completely free, the projection onto the eigenvalues obtained by integration over the angular variables does not fix the basis at the next step in the basic steps of space-like paths. This implies that the eigenvalues' process along space-like paths is a Markov process.

5.3.2. Diffusion on Wishart minors

We now show Result 9 and consider diffusion on Wishart matrices. As before, we define $H(n,t) = A^*(n,t)A(n,t)$ to be the $n \times n$ minor at time t, where $A(n,t)$ is the $p \times n$ matrix with singular value decomposition $A(n,t) = U(p,t)\Sigma(n,t)V^*(n,t)$, where $U(p,t)$ is a $p \times p$ Haar-distributed on $\mathcal{U}(p)$, $V(n,t)$ is a $n \times n$ Haar-distributed on $\mathcal{U}(n)$, and $\Sigma(n,t)$ is a $p \times n$ matrix with entries zeros except on the diagonal, where we find the singular values of $A(n,t)$. Also, let $\Lambda(n,t) = \Sigma^*(n,t)\Sigma(n,t)$ the matrix of the eigenvalues of $H(n,t)$. Thus we have

$$H(n,t) = A^*(n,t)A(n,t) = V(n,t)\Lambda(n,t)V^*(n,t).$$

The Jacobian of the transformation $A(n,t) \to (\Sigma(n,t), U(n,t), U(p,t))$ gives (see e.g. [73])

$$\begin{aligned} \mathrm{d}A(n,t) = \mathrm{const} \times (\det(\Sigma^*(n,t)\Sigma(n,t)))^{p-n+1/2}\Delta^2(\Sigma^*(n,t)\Sigma(n,t)) \\ \times \mathrm{d}\Sigma(n,t)\,\mathrm{d}\mu_n(V(n,t))\,\mathrm{d}\mu_p(U(p,t)), \end{aligned}$$

or, using that $\Lambda(n,t) = \Sigma^*(n,t)\Sigma(n,t)$,

$$\begin{aligned} \mathrm{d}A(n,t) = \mathrm{const} \times (\det(\Lambda(n,t)))^{p-n}\Delta^2(\Lambda(n,t)) \\ \times \mathrm{d}\Lambda(n,t)\,\mathrm{d}\mu_n(V(n,t))\,\mathrm{d}\mu_p(U(p,t)). \end{aligned} \tag{5.20}$$

Therefore, the starting measure at (n,t) has the form

$$g_1(\Lambda(n,t))\,\mathrm{d}\Lambda(n,t)\,\mathrm{d}\mu_n(V(n,t))\,\mathrm{d}\mu_p(U(p,t)) \tag{5.21}$$

for some explicit function g_1.

Next consider fixed n and time $t' > t$. Then, the probability measure on the matrices has the form (see (5.8))

$$\begin{aligned} &g_1(\Lambda(n,t))\,\mathrm{e}^{-b\,\mathrm{Tr}((A^*(n,t')-A^*(n,t))(A(n,t')-A(n,t)))} \\ &\qquad\qquad \times \mathrm{d}\Lambda(n,t)\,\mathrm{d}\mu_n(V(n,t))\,\mathrm{d}\mu_p(U(p,t))\,\mathrm{d}A(n,t') \\ &= g_1(\Lambda(n,t))\,\mathrm{e}^{-b\,\mathrm{Tr}\left([\Sigma^*(n,t')-\tilde{V}^*(n,t)\Sigma^*(n,t)\tilde{U}(p,t)]\,[\Sigma(n,t')-\tilde{U}(p,t)\Sigma(n,t)\tilde{V}^*(n,t)]\right)} \\ &\qquad\qquad \times\ \mathrm{d}\Lambda(n,t)\,\mathrm{d}\mu_n(\tilde{V}(n,t))\,\mathrm{d}\mu_p(\tilde{U}(p,t))\,\mathrm{d}A(n,t') \end{aligned}$$

because of unitary invariance of the Haar measure (we set $\tilde{V}(n,t) = V(n,t')^*V(n,t)$ and $\tilde{U}(p,t) = U(p,t')^*U(p,t)$). An integration with respect to $\mathrm{d}\mu_n(\tilde{V}(n,t))\,\mathrm{d}\mu_p(\tilde{U}(p,t))$ according to (A.17) results in the formula of Lemma 26. We are left with a probability density that depends only on the eigenvalues times $\mathrm{d}A(n,t')$, that is, the projection onto eigenvalues at time t did not restrict the complete freedom of choice of basis at (n,t'). Otherwise stated, by (5.20) we have a measure on eigenvalues times $\mathrm{d}\mu_n(V(n,t'))\,\mathrm{d}\mu_p(U(p,t'))$ of the form (for some explicit g_2, which can be easily computed)

$$g_2(\Lambda(n,t),\Lambda(n,t'))\,\mathrm{d}\Lambda(n,t)\,\mathrm{d}\Lambda(n,t')\,\mathrm{d}\mu_n(V(n,t'))\,\mathrm{d}\mu_p(U(p,t')). \tag{5.22}$$

The other choice is to consider t fixed and look at the measure at (n,t) and $(n-1,t)$. This works as for the Hermitian case and we get a measure of the form

$$g_3(\Lambda(n,t),\Lambda(n-1,t))\,\mathrm{d}\Lambda(n,t)\,\mathrm{d}\Lambda(n-1,t)\,\mathrm{d}\mu_{n-1}(V(n-1,t))\,\mathrm{d}\mu_p(U(p,t)). \tag{5.23}$$

for some explicit function g_3 (compare with Lemma 27).

Therefore (5.22) and (5.23) tell us that starting from a measure of the form (5.21), in which the choice of basis (in which the matrix A is represented) is completely free, the projection onto the eigenvalues obtained by integration over the angular variables does not fix the basis at the next step in the basic steps of space-like paths. This implies that the eigenvalues' process along space-like paths is a Markov process.

6. Perturbed GUE Minor Process and Warren's Process with Drifts

6.1. GUE minor process with drift

This last chapter provides the proofs of Results 11 to 13 from Section 3.3.2 and is based on [45].

6.1.1. Model and measure

Let $(H(t) : t \geq 0)$ be a process on the $N \times N$ Hermitian matrices defined by

$$H_{k\ell}(t) = \begin{cases} b_{kk}(t) + \mu_k t, & \text{if } 1 \leq k \leq N, \\ \frac{1}{\sqrt{2}}(b_{k\ell}(t) + \mathrm{i}\tilde{b}_{k\ell}(t)), & \text{if } 1 \leq k < \ell \leq N, \\ \frac{1}{\sqrt{2}}(b_{k\ell}(t) - \mathrm{i}\tilde{b}_{k\ell}(t)), & \text{if } 1 \leq \ell < k \leq N, \end{cases}$$

where $\{b_{kk}, b_{k\ell}, \tilde{b}_{k\ell}\}$ are independent one-dimensional standard Brownian motions[1]. Denote by $M = \mathrm{diag}(\mu_1, \ldots, \mu_N)$ the diagonal drift matrix added to the matrix H. Then, the probability measure on these matrices at time t is given by

$$\mathbb{P}(H \in \mathrm{d}H) = \mathrm{const} \times \exp\left(-\frac{\mathrm{Tr}(H - tM)^2}{2t}\right) \mathrm{d}H \tag{6.1}$$

where $\mathrm{d}H = \prod_{i=1}^N \mathrm{d}H_{ii} \prod_{1 \leq j < k \leq N} \mathrm{d}\,\mathrm{Re}(H_{j,k})\mathrm{d}\,\mathrm{Im}(H_{j,k})$ and const is the normalization constant.

Since we are interested in the statistics of the eigenvalues' minors at time t, we first determine the measure on the eigenvalues of the $N \times N$ matrix.

Lemma 31. *Assume that $\mu_1, \ldots, \mu_N$ are all distinct. Then under* (6.1)*, the joint probability measure of the eigenvalues $\lambda_1, \ldots, \lambda_N$ of H is given by*

$$\begin{aligned} &\mathbb{P}(\lambda_1 \in \mathrm{d}\lambda_1, \ldots \lambda_N \in \mathrm{d}\lambda_N) \\ &\qquad = \mathrm{const} \times \det\left[\mathrm{e}^{-(\lambda_i - t\mu_j)^2/(2t)}\right]_{1 \leq i,j \leq N} \frac{\Delta(\lambda_1, \ldots, \lambda_N)}{\Delta(\mu_1, \ldots, \mu_N)} \mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_N \end{aligned} \tag{6.2}$$

[1] Here, standard Brownian motions start from 0 and are normalized to have variance t at time t.

with const *a normalization constant and* $\Delta(x_1,\dots,x_m) = \prod_{1\le i<j\le m}(x_j - x_i)$ *the Vandermonde determinant.*

Remark 32. If $\mu_1,\dots,\mu_N$ are not all distinct, we have to take limits in (6.2). For instance, if $\mu_1 = \cdots = \mu_N \equiv \mu$, then

$$\mathbb{P}(\lambda_1 \in \mathrm{d}\lambda_1, \dots \lambda_N \in \mathrm{d}\lambda_N) = \mathrm{const} \times \left(\prod_{k=1}^{N} \mathrm{e}^{-(\lambda_k - t\mu)^2/(2t)}\right)\Delta^2(\lambda_1,\dots,\lambda_N)\,\mathrm{d}\lambda_1\cdots\mathrm{d}\lambda_N.$$

Proof of Lemma 31. We diagonalize $H = U\Lambda U^*$ with a unitary matrix U and the diagonal matrix $\Lambda = \mathrm{diag}(\lambda_1,\dots,\lambda_N)$. Then,

$$\mathrm{e}^{-\operatorname{Tr}(H-tM)^2/(2t)}\mathrm{d}H = \mathrm{const} \times \mathrm{e}^{-\operatorname{Tr}(U\Lambda U^* - tM)^2/(2t)}\Delta^2(\lambda)\,\mathrm{d}U\,\mathrm{d}\lambda, \tag{6.3}$$

where $\mathrm{d}U$ is the Haar measure on the unitary group $\mathcal{U}$. Moreover, since

$$\operatorname{Tr}(U\Lambda U^* - tM)^2 = \operatorname{Tr}\Lambda^2 + t^2 \operatorname{Tr} M^2 - 2t\operatorname{Tr}(U\Lambda U^* M)$$

by integrating over $\mathcal{U}$ in (6.3) and using the Harish-Chandra-Itzykson-Zuber formula, we obtain the desired expression. □

Now we focus on the minor process. For $1 \le n \le N$ let us denote by $H^n(t)$ the $n\times n$ principal submatrix of $H(t)$ which is obtained from $H(t)$ by keeping only the n first rows and columns. In particular, $H^1(t) = H_{11}(t)$ and $H^N(t) = H(t)$. We denote by $\lambda_1^n(t) \le \cdots \le \lambda_n^n(t)$ the ordered eigenvalues of $H^n(t)$. It is then a classical fact of linear algebra that at any time t, the process $(\lambda^1,\dots,\lambda^N)(t)$ lies in the Gelfand-Tsetlin cone of order N,

$$\mathbb{G}_N = \{(x^1,\dots,x^N) \in \mathbb{R}^1 \times \cdots \times \mathbb{R}^N : x^n \preceq x^{n+1} \text{ for all } 1 \le k \le N-1\},$$

where $x^n \preceq x^{n+1}$ means that x^n and x^{n+1} interlace, i.e.,

$$x_k^{n+1} \le x_k^n \le x_{k+1}^{n+1} \quad \text{for all } 1 \le k \le n.$$

The induced measure on $\{\lambda_k^n : 1 \le k \le n \le N\}$ is the following.

Proposition 33. *Fix* $t > 0$. *Then, under the measure* (6.1)*, the joint density of the eigenvalues of* $\{H^n : 1 \le n \le N\}$ *on* $\mathbb{G}_N$ *is given by*

$$\mathrm{const} \times \prod_{k=1}^{N} \mathrm{e}^{-t\mu_k^2/2} \prod_{k=1}^{N} \mathrm{e}^{-(\lambda_k^N)^2/(2t)}\Delta(\lambda^N) \prod_{\substack{1\le n\le N\\ 1\le k\le n}} \mathrm{e}^{\mu_n \lambda_k^n} \prod_{\substack{2\le n\le N\\ 1\le k\le n-1}} \mathrm{e}^{-\mu_n\lambda_k^{n-1}}, \tag{6.4}$$

where the normalization constant does not depend on $\mu_1,\dots,\mu_N$.

Proof. We first derive (6.4) under the assumption that the $\mu_1, \dots, \mu_N$ are all distinct; the case where some of the μ_i are equal is then recovered by taking the limit. We prove the statement inductively and follow the presentation in [49]. For $N = 1$, the density is clearly proportial to $\exp(-(\lambda_1^1 - \mu_1 t)^2/(2t))$. For $N \geq 2$, we consider an $N \times N$ matrix H^N distributed according to (6.1) which we write as

$$H^N - tM = \begin{pmatrix} H^{N-1} & w \\ w^* & x \end{pmatrix} - t \begin{pmatrix} M^{N-1} & 0 \\ 0 & \mu_N \end{pmatrix},$$

where M^{N-1} denotes the $(N-1)\times(N-1)$ principal submatrix of M, $w \in \mathbb{C}^{N-1}$ is a Gaussian vector and $x \in \mathbb{R}$ is a Gaussian variable. Then we diagonalize H^{N-1}, i.e., we choose a unitary matrix U such that $H^{N-1} = U\Lambda U^*$ with $\Lambda = \mathrm{diag}(\lambda_1^{N-1}, \dots, \lambda_{N-1}^{N-1})$ the diagonal matrix for the eigenvalues. Since the Gaussian distribution is invariant under unitary rotations and w is independent of H^{N-1}, we have

$$\begin{pmatrix} U^* & 0 \\ 0 & 1 \end{pmatrix} (H^N - tM) \begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} \Lambda & w \\ w^* & x \end{pmatrix} - t \begin{pmatrix} U^* M^{N-1} U & 0 \\ 0 & \mu_N \end{pmatrix},$$

where $\stackrel{d}{=}$ denotes equality in distribution. Applying the map $H^{N-1} \mapsto (\Lambda, U)$, we get that measure (6.1) on H^N is proportional to

$$\exp\left(-\frac{1}{2t} \mathrm{Tr}\left[\begin{pmatrix} \Lambda & w \\ w^* & x \end{pmatrix} - t \begin{pmatrix} U^* M^{N-1} U & 0 \\ 0 & \mu_N \end{pmatrix}\right]^2\right) \Delta^2(\lambda^{N-1}) \times \mathrm{d}U\, \mathrm{d}w\, \mathrm{d}x\, \mathrm{d}\lambda^{N-1}, \quad (6.5)$$

where $\mathrm{d}U$ is the Haar measure on the unitary group $\mathcal{U}(N-1)$. We consider only the part of (6.5) that depends on U and integrate over $\mathcal{U}(N-1)$, using the Harish-Chandra-Itzykson-Zuber formula,

$$\int_{\mathcal{U}_{N-1}} \mathrm{d}U\, \mathrm{e}^{\mathrm{Tr}(\Lambda U^* M^{N-1} U)} = \mathrm{const} \times \frac{\det[\mathrm{e}^{\lambda_i^{N-1}\mu_j}]_{1\leq i,j\leq N-1}}{\Delta(\lambda^{N-1})\Delta(\mu_1, \dots, \mu_{N-1})}.$$

After this integration the measure (6.5) reads

$$\mathrm{const} \times \mathbb{P}(\lambda^{N-1} \in \mathrm{d}\lambda^{N-1})\, \mathrm{e}^{x\mu_N - t\mu_N^2/2} \prod_{k=1}^{N-1} \mathrm{e}^{-|w_k|^2/t} \mathrm{d}x\, \mathrm{d}w. \quad (6.6)$$

We focus on the measure on w_k and represent the variables in polar coordinates, $w_k = r_k \mathrm{e}^{\mathrm{i}\varphi_k}$ with $r_k \in \mathbb{R}_+$ and $\varphi_k \in [0, 2\pi)$. Since the Jacobian of this transformation is given by $r_1 \cdots r_{N-1}$, we get

$$\prod_{k=1}^{N-1} \mathrm{e}^{-|w_k|^2/t} \mathrm{d}w_k = \prod_{k=1}^{N-1} r_k \mathrm{e}^{-r_k^2/t} \mathrm{d}r_k\, \mathrm{d}\varphi_k,$$

where $\mathrm{d}r_k$ and $\mathrm{d}\varphi_k$ are Lebesgue measures on $\mathbb{R}_+$ and $[0, 2\pi)$. Then we can express r_k and x in terms of the eigenvalues of H^{N-1} and H^N, see e.g. [49] for details,

$$r_k^2 = -\frac{\prod_{j=1}^{N}(\lambda_k^{N-1} - \lambda_j^N)}{\prod_{j=1,j\neq k}^{N-1}(\lambda_k^{N-1} - \lambda_j^{N-1})}\, \mathbb{1}_{[\lambda^{N-1}\preceq\lambda^N]},$$
$$x = \mathrm{Tr}(H^N - H^{N-1}) = \sum_{i=1}^{N}\lambda_k^N - \sum_{k=1}^{N-1}\lambda_k^{N-1}.$$

The Jacobian of the transformation $T : (r_1, \ldots, r_{N-1}, x) \mapsto \lambda^N$ is then given by

$$r_1 \cdots r_{N-1}|\det T'| = \frac{\Delta(\lambda^N)}{\Delta(\lambda^{N-1})}\, \mathbb{1}_{[\lambda^{N-1}\preceq\lambda^N]},$$

and hence, given λ^{N-1}, we have

$$\mathrm{e}^{x\mu_N}\prod_{k=1}^{N-1}\mathrm{e}^{-|w_k|^2/t}\mathrm{d}x\,\mathrm{d}w = \prod_{k=1}^{N}\mathrm{e}^{-(\lambda_k^N)^2/(2t)+\mu_N\lambda_k^N}\prod_{k=1}^{N-1}\mathrm{e}^{(\lambda_k^{N-1})^2/(2t)-\mu_N\lambda_k^{N-1}} \times \frac{\Delta(\lambda^N)}{\Delta(\lambda^{N-1})}\,\mathbb{1}_{[\lambda^{N-1}\preceq\lambda^N]}\,\mathrm{d}\lambda^N\,\mathrm{d}\varphi. \quad (6.7)$$

Here we used that $2(r_1^2 + \cdots + r_{N-1}^2) = \mathrm{Tr}(H^N)^2 - \mathrm{Tr}(H^{N-1})^2$. Moreover, by the induction assumption for $N-1$ we have

$$\mathbb{P}(\lambda^{N-1} \in \mathrm{d}\lambda^{N-1}) = \mathrm{const} \times \prod_{k=1}^{N-1}\mathrm{e}^{-t\mu_k^2/2}\prod_{k=1}^{N-1}\mathrm{e}^{-(\lambda_k^{N-1})^2/(2t)}\Delta(\lambda^{N-1}) \times \prod_{\substack{1\le n\le N-1\\1\le k\le n}}\mathrm{e}^{\mu_n\lambda_k^n}\prod_{\substack{2\le n\le N-1\\1\le k\le n}}\mathrm{e}^{-\mu_n\lambda_k^{n-1}}\prod_{n=1}^{N-1}\mathrm{d}\lambda^n. \quad (6.8)$$

Finally, inserting (6.7) and (6.8) into (6.6) and integrating out φ (which multiplies the measure by a finite constant) results in the claimed formula (6.4). □

6.1.2. Correlation functions

Now we determine the correlation functions of the point process on the eigenvalues $\{\lambda_k^n : 1 \le k \le n \le N\}$, and for that purpose, we rewrite the density in (6.4) as a product of determinants. We set $\phi_n(x, y) = \mathrm{e}^{\mu_n(y-x)}\mathbb{1}_{\{x>y\}}$ and introduce "virtual" variables $\lambda_n^{n-1} = \mathrm{virt}$ with the property that $\phi_n(\mathrm{virt}, y) = \mathrm{e}^{\mu_n y}$. Then in (6.4) we have, up to a set of measure zero,

$$\det[\phi_n(\lambda_i^{n-1}, \lambda_j^n)]_{1\le i,j\le n} = \prod_{j=1}^{n}\mathrm{e}^{\mu_n\lambda_j^n}\prod_{j=1}^{n-1}\mathrm{e}^{-\mu_n\lambda_j^{n-1}}\mathbb{1}_{[\lambda^n\preceq\lambda^{n+1}]}.$$

Moreover, for $k=1,\dots,N$ we set

$$\Psi^{N,t}_{N-k}(x)=\frac{\mathrm{e}^{-x^2/(2t)}}{\sqrt{2\pi t}}\,t^{-(N-k)/2}\,p_{N-k}\left(\frac{\mu_{k+1}t-x}{\sqrt{t}},\dots,\frac{\mu_N t-x}{\sqrt{t}}\right),$$

where p_n are symmetric polynomials of degree n in n variables defined by $p_0\equiv 1$ and

$$p_n(x_1,\dots,x_n)=\frac{(-1)^n}{\mathrm{i}\sqrt{2\pi}}\int_{\mathrm{i}\mathbb{R}}\mathrm{d}w\,\mathrm{e}^{w^2/2}(w-x_1)\cdots(w-x_n)\quad \text{for } n\geq 1.$$

Hence we have that

$$\prod_{k=1}^N \mathrm{e}^{-(\lambda_k^N)^2/(2t)}\Delta(\lambda^N)=\mathrm{const}\times\det\left[\Psi^{N,t}_{N-k}(\lambda^N_\ell)\right]_{1\leq k,\ell\leq N},$$

which means that we can rewrite (6.4) as

$$\mathrm{const}\times\prod_{n=1}^N\det\left[\phi_n(\lambda_i^{n-1},\lambda_j^n)\right]_{1\leq i,j\leq n}\prod_{k=1}^N\mathrm{e}^{-t\mu_k^2/2}\det\left[\Psi^{N,t}_{N-k}(\lambda^N_\ell)\right]_{1\leq k,\ell\leq N}. \tag{6.9}$$

Note that by a change of variable $w=(tz-x)/\sqrt{t}$ we have

$$\Psi^{N,t}_{N-k}(x)=\frac{(-1)^{N-k}}{2\pi\mathrm{i}}\int_{\mathrm{i}\mathbb{R}}\mathrm{d}z\,\mathrm{e}^{tz^2/2-xz}(z-\mu_{k+1})\cdots(z-\mu_N). \tag{6.10}$$

A measure of the form (6.9) has determinantal correlation functions and the kernel can be computed with Lemma 3.4 of [17], see Appendix A.2.

6.1.3. Perturbed GUE matrices

In this section we give a proof of Result 11. We show it first for $\mu_1<\dots<\mu_N$ and then use analytic continuation. Note that for $n=N$, the function $\Psi^{n,t}_{n-k}$ in (3.26) is the same as $\Psi^{N,t}_{N-k}$ in (6.10).

Lemma 34. *The following identities hold.*

(i) For all $n\in\{1,\dots,N\}$, $k\in\mathbb{Z}$ and $t>0$, we have $\phi_n*\Psi^{n,t}_{n-k}=\Psi^{n-1,t}_{n-1-k}$.

(ii) For $n<n'$, we have $\phi_{n+1}*\dots*\phi_{n'}=\phi^{(n,n')}$ with $\phi^{(n,n')}$ given in (3.25).

Proof. Because of $\operatorname{Re} z<\mu_n$ we can exchange the two integrals,

$$\begin{aligned}
&(\phi_n*\Psi^{n,t}_{n-k})(x)\\
&=\int_{-\infty}^x\mathrm{d}y\,\mathrm{e}^{\mu_n(y-x)}\frac{(-1)^{n-k}}{2\pi\mathrm{i}}\int_{\mathrm{i}\mathbb{R}+\mu_-}\mathrm{d}z\,\mathrm{e}^{tz^2/2-yz}\frac{(z-\mu_1)\dots(z-\mu_n)}{(z-\mu_1)\dots(z-\mu_k)}\\
&=\frac{(-1)^{n-k}}{2\pi\mathrm{i}}\int_{\mathrm{i}\mathbb{R}+\mu_-}\mathrm{d}z\,\mathrm{e}^{tz^2/2-\mu_n x}\frac{(z-\mu_1)\dots(z-\mu_n)}{(z-\mu_1)\dots(z-\mu_k)}\int_{-\infty}^x\mathrm{d}y\,\mathrm{e}^{y(\mu_n-z)}\\
&=\frac{(-1)^{n-1-k}}{2\pi\mathrm{i}}\int_{\mathrm{i}\mathbb{R}+\mu_-}\mathrm{d}z\,\mathrm{e}^{tz^2/2-xz}\frac{(z-\mu_1)\dots(z-\mu_{n-1})}{(z-\mu_1)\dots(z-\mu_k)}\\
&=\Psi^{n-1,t}_{n-1-k}(x).
\end{aligned}$$

This proves the first statement. To show (ii), we first consider the case $n' - n = 2$. A simple calculation gives

$$\phi^{(n-2,n)}(x, x') = (\phi_{n-1} * \phi_n)(x, x') = -\left(\frac{e^{\mu_n(x'-x)}}{\mu_n - \mu_{n-1}} + \frac{e^{\mu_{n-1}(x'-x)}}{\mu_{n-1} - \mu_n}\right) \mathbb{1}_{[x>x']},$$

which has the following contour integral representation,

$$\phi^{(n-2,n)}(x, x') = \frac{1}{2\pi i} \int_{i\mathbb{R}+\mu_-} dz \, \frac{e^{z(x'-x)}}{(z - \mu_{n-1})(z - \mu_n)}.$$

For $n' - n > 2$, we get inductively that

$$\begin{aligned}
&(\phi_n * \phi^{(n,n')})(x, x') \\
&= \frac{(-1)^{n'-n}}{2\pi i} \int_{-\infty}^{x} dy \, e^{\mu_n(y-x)} \int_{i\mathbb{R}+\mu_-} dz \, \frac{e^{z(x'-y)}}{(z - \mu_{n+1}) \cdots (z - \mu_{n'})} \\
&= \frac{(-1)^{n'-n}}{2\pi i} \int_{i\mathbb{R}+\mu_-} dz \, \frac{e^{zx'-\mu_n x}}{(z - \mu_{n+1}) \cdots (z - \mu_{n'})} \int_{-\infty}^{x} dy \, e^{y(\mu_n - z)} \\
&= \frac{(-1)^{n'-n+1}}{2\pi i} \int_{i\mathbb{R}+\mu_-} dz \, \frac{e^{z(x'-x)}}{(z - \mu_n)(z - \mu_{n+1}) \cdots (z - \mu_{n'})} \\
&= \phi^{(n-1,n')}(x, x'),
\end{aligned}$$

where, as before, we could exchange the integrals because of $\operatorname{Re} z < \mu_n$. □

Next we consider the n-dimensional space V_n spanned by the set of functions

$$\{\phi_1 * \phi^{(1,n)}(x_1^0, \cdot), \ldots, \phi_{n-1} * \phi^{(n-1,n)}(x_{n-1}^{n-2}, \cdot), \phi_n(x_n^{n-1}, \cdot)\}.$$

According to Lemma 3.4 of [17] we need to find a basis $\{\Phi_{n-k}^{n,t} : 1 \le k \le n\}$ of V_n that is biorthogonal to the set $\{\Psi_{n-k}^{n,t} : 1 \le k \le n\}$, i.e.,

$$\int_{\mathbb{R}} dx \, \Psi_{n-k}^{n,t}(x) \Phi_{n-\ell}^{n,t}(x) = \delta_{k\ell}, \quad 1 \le k, \ell \le n.$$

The form of the biorthogonal functions can be guessed, with some experience, from the form of the kernel [24].

Lemma 35. *We have:*

(i) V_n *is spanned by* $\{x \mapsto e^{\mu_k x} : 1 \le k \le n\}$.

(ii) The functions $\{\Phi_{n-k}^{n,t} : 1 \le k \le n\}$ *are given by* (3.27).

Proof. For any $\varepsilon > 0$ we have

$$(\phi_k * \phi^{(k,n)})(x_k^{k-1}, x) = \int_{\mathbb{R}} \mathrm{d}y\, \mathrm{e}^{\mu_k y} \frac{(-1)^{n-k}}{2\pi \mathrm{i}} \int_{\mathrm{i}\mathbb{R}+\mu_{k+1}-\varepsilon} \mathrm{d}z\, \frac{\mathrm{e}^{z(x-y)}}{(z-\mu_{k+1})\cdots(z-\mu_n)}.$$

We split the y-integral into one over $\mathbb{R}_+$ and one over $\mathbb{R}_-$. Then we can exchange the integrals over $\mathbb{R}_-$ and the imaginary axis provided that $\mathrm{Re}\, z < \mu_k$ and use $\int_{\mathbb{R}_-} \mathrm{d}y\, \mathrm{e}^{y(\mu_k - z)} = \frac{1}{z-\mu_k}$. In the same way we integrate over $\mathbb{R}_+$ taking z such that $\mu_k < \mathrm{Re}\, z < \mu_{k+1}$. This gives $\int_{\mathbb{R}_+} \mathrm{d}y\, \mathrm{e}^{y(\mu_k - z)} = -\frac{1}{z-\mu_k}$. Putting these two integrals together we get

$$(\phi_k * \phi^{(k,n)})(x_k^{k-1+1}, x) = \frac{(-1)^{n-k}}{2\pi \mathrm{i}} \oint_{\Gamma_{\mu_k}} \mathrm{d}z\, \frac{\mathrm{e}^{xz}}{(z-\mu_k)(z-\mu_{k+1})\cdots(z-\mu_n)},$$

which is a constant multiple of $\mathrm{e}^{\mu_k x}$. This proves (i). For (ii) we proceed similarly. Using that $1 \leq k, \ell \leq n$ we have

$$\int_{\mathbb{R}} \mathrm{d}x\, \Psi_{n-k}^{n,t}(x) \Phi_{n-\ell}^{n,t}(x) = \frac{(-1)^{k+\ell}}{(2\pi \mathrm{i})^2} \int_{\mathbb{R}} \mathrm{d}x \int_{\mathrm{i}\mathbb{R}} \mathrm{d}z \oint_{\Gamma_{\mu_\ell,\dots,\mu_n}} \mathrm{d}w\, \frac{\mathrm{e}^{tz^2 - xz}}{\mathrm{e}^{tw^2 - xw}} \frac{(z-\mu_{k+1})\cdots(z-\mu_n)}{(w-\mu_\ell)\cdots(w-\mu_n)}. \quad (6.11)$$

When integrating x over $\mathbb{R}_-$, we take the z-integral such that $\mathrm{Re}\, z < \mathrm{Re}\, w$, and when we integrate x over $\mathbb{R}_+$, we choose $\mathrm{Re}\, z > \mathrm{Re}\, w$. Thus, (6.11) reduces to

$$\frac{(-1)^{k+\ell}}{2\pi \mathrm{i}} \oint_{\Gamma_{\mu_\ell,\dots,\mu_n}} \mathrm{d}w\, \frac{(w-\mu_{k+1})\cdots(w-\mu_n)}{(w-\mu_\ell)\cdots(w-\mu_n)} = \delta_{k\ell}.$$

Finally, note that

$$\Phi_{n-\ell}^{n,t}(x) = \sum_{i=\ell}^{n} b_i \mathrm{e}^{\mu_i x} \quad \text{with} \quad b_i = \prod_{\substack{j=\ell \\ j\neq i}}^{n} \frac{\mathrm{e}^{-t\mu_i^2/2}}{\mu_i - \mu_j}.$$

which shows that the set $\{\Phi_{n-k}^{n,t} : 1 \leq k \leq n\}$ spans V_n. □

Next we verify Assumption (A) from Lemma 3.4 in [17]. Indeed,

$$\Phi_0^{n,t}(x) = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_{\mu_n}} \mathrm{d}w\, \frac{\mathrm{e}^{-tw^2/2 + xw}}{w - \mu_n} = c_n \phi_n(x_n^{n-1}, x)$$

with $c_n = \mathrm{e}^{-t\mu_n^2/2} \neq 0$ for $n = 1, \dots, N$.

Finally, we can also determine the value of the normalization constant in (6.9), since it is given by $1/\det[M_{k\ell}]_{1\leq k,\ell \leq N}$ with

$$M_{k\ell} = (\phi_k * \cdots * \phi_N * \Psi_{N-\ell}^{N,t})(x_k^{k-1}).$$

Lemma 36. *We have* $\det M = \prod_{n=1}^{N} \mathrm{e}^{t\mu_n^2/2}$*, in particular* $\det M > 0$*.*

Proof. By Lemma 34 (i) we may write $M_{k\ell} = (\phi_k * \Psi_{k-\ell}^{k,t})(x_k^{k-1})$. Thus, for $k \geq \ell$,

$$M_{k\ell} = \int_{\mathbb{R}} \mathrm{d}y \, \mathrm{e}^{\mu_k y} \frac{(-1)^{k-\ell}}{2\pi\mathrm{i}} \int_{\mathrm{i}\mathbb{R}} \mathrm{d}z \, \mathrm{e}^{tz^2/2 - yz} (z - \mu_{\ell+1}) \cdots (z - \mu_k).$$

Once again, we let run the y-integral over $\mathbb{R}_-$ and $\mathbb{R}_+$ separately. In the first case we take the z-integral such that $\operatorname{Re} z < \mu_k$, in the second case such that $\operatorname{Re} z > \mu_k$. This allows us to exchange the integrals, which gives

$$M_{k\ell} = \frac{(-1)^{k-\ell}}{2\pi\mathrm{i}} \oint_{\Gamma_{\mu_k}} \mathrm{d}z \, \mathrm{e}^{tz^2/2} \frac{(z - \mu_{\ell+1}) \cdots (z - \mu_k)}{z - \mu_k}.$$

Since the integrand has no poles for $k > \ell$, we have $M_{k\ell} = 0$ in this case, while for $k = \ell$ we get $M_{kk} = \mathrm{e}^{-t\mu_k^2/2}$. Thus, M is upper triangular and the claim follows. □

With the results of Lemma 34 and Lemma 35, Theorem 11 follows directly from Lemma 3.4 of [17].

We have shown that Theorem 11 holds when we impose $\mu_1 < \cdots < \mu_N$. In particular, the joint density (6.4) is given by an $(N(N+1)/2)$-point correlation function: With $m = N(N+1)/2$ we have

$$(6.4) = m! \, \rho^{(m)}(\{(\lambda_k^n, n), 1 \leq k \leq n \leq N\}). \tag{6.12}$$

Let $M > 0$ be any fixed real number. The density (6.4) is analytic in each of the μ_j in $[-M, M]$, $j = 1, \ldots, N$. The same holds for the correlation kernel (take e.g., $\mu_- = -M - 1$). From this it follows that also the r.h.s. of (6.12) is analytic in each of the variables $\mu_1, \ldots, \mu_N$. Since this holds for any M, by analytic continuation it follows that Theorem 11 holds for any given drift vector $(\mu_1, \ldots, \mu_N) \in \mathbb{R}^N$.

6.2. $2 + 1$ dynamics with different jump rates

In this section we show that the correlation functions (3.23) that we obtained for the GUE matrix diffusion with drifts can be obtained as a limit from an $\mathbb{G}_N$-extension of TASEP with particle-dependent jump rates. This latter process was introduced in [15]. Before we come to the convergence result, let us describe the model.

At a fixed time t, let us denote by $x(t) = (x_k^n(t) : 1 \leq k \leq n \leq N) \in \mathbb{G}_N$ the positions of the $N(N+1)/2$ particles at time t. We choose initial conditions $x_k^n(0) = k - n - 1$ and let the particles evolve as follows: Each particle x_k^n has an independent exponential clock of rate $v_n > 0$, i.e., particles on the same level have the same jump rates. When the x_k^n-clock rings, the particle jumps to the right by one, provided that $x_k^n < x_k^{n-1} - 1$, otherwise we say that x_k^n is blocked by x_k^{n-1}. If the x_k^n-particle can jump, we take the largest $c \geq 1$ such that $x_k^n = x_{k+1}^{n+1} = \cdots = x_{k+c-1}^{n+c-1}$, and all c particles in this string jump to the right by one, see Figure 3.3 for an example). This ensures that at any time t, all the particles are in $\mathbb{G}_N$. More precisely, these dynamics imply that the particles stay in a discrete version of $\mathbb{G}_N$, namely

$$\widetilde{\mathbb{G}}_N = \{(x^1, x^2, \dots, x^N) \in \mathbb{Z}^1 \times \mathbb{Z}^2 \times \cdots \times \mathbb{Z}^N : x_k^{n+1} < x_k^n \leq x_{k+1}^{n+1}\}.$$

The joint distribution of the particles has been calculated in Theorem 4.1 of [16], and the result is

$$\text{const} \times \det\big[\widetilde{\Psi}_{N-k}^{N,t}(x_\ell^N)\big]_{1\leq k,\ell\leq N} \prod_{n=1}^{N} \det\big[\widetilde{\phi}_n(x_i^{n-1}, x_j^n)\big]_{1\leq i,j\leq n}, \tag{6.13}$$

where

$$\widetilde{\Psi}_{N-k}^{N,t}(x) = \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma_0} \mathrm{d}z\, \mathrm{e}^{t/z} z^{x+N-1}(1-v_{k+1}z)\cdots(1-v_N z),$$
$$\widetilde{\phi}_n(x,y) = (v_n)^{y-x}\mathbb{1}_{[y\geq x]} \quad \text{and} \quad \widetilde{\phi}_n(x_n^{n-1}, y) = (v_n)^y.$$

Actually, Theorem 4.1 of [16] is a statement about the marginal of a (possibly signed) measure. However, this model is the continuous time limit of a generic Markov chain introduced in Section 2 of [15], from which it follows that the measure with fully packed initial conditions $y_n = x_1^n(0) = -n$ for $1 \leq n \leq N$ is actually a probability distribution. The formulation of (6.13) follows then from the theorem by taking $a(t) = t$ and $b(t) = 0$ for all $t \geq 0$. Also note that we put the transition from time $t = 0$ to time t (which is encoded by $\mathcal{T}_{t,0}$ in the theorem) into Ψ_{N-k}^N. As shown in [16], the correlation functions of this point process are determinantal, so what remains to do is the biorthogonalization for the generic jump rates.

Proposition 37. *Consider a system of particles on $\widetilde{\mathbb{G}}_N$ with fully packed initial conditions and dynamics described above. Then, at fixed time t, the corresponding point process has m-point correlation function $\widetilde{\varrho}_t^m$ given by*

$$\widetilde{\varrho}_t^m((x_1,n_1),\dots,(x_m,n_m)) = \det[\widetilde{K}_t^v((x_i,n_i),(x_j,n_j))]_{1\leq i,j\leq m}$$

with $(x_i, n_i) \in \mathbb{R} \times \{1,\dots,N\}$ and correlation kernel

$$\widetilde{K}_t^v((x,n),(x',n')) = -\widetilde{\phi}^{(n,n')}(x,x') + \sum_{k=1}^{n'} \widetilde{\Psi}_{n-k}^{n,t}(x)\widetilde{\Phi}_{n'-k}^{n',t}(x'),$$

where

$$\widetilde{\phi}^{(n,n')}(x,x') = \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma_{0,v}} \mathrm{d}z\, \frac{1}{z^{x-x'+1}} \frac{z^{n'-n}}{(z-v_{n+1})\cdots(z-v_{n'})} \mathbb{1}_{[n<n']}, \tag{6.14}$$
$$\widetilde{\Psi}_{n-k}^{n,t}(x) = \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma_{0,v}} \mathrm{d}z\, \frac{\mathrm{e}^{tz}}{z^{x+n+1}} \frac{(z-v_1)\cdots(z-v_n)}{(z-v_1)\cdots(z-v_k)}, \tag{6.15}$$
$$\widetilde{\Phi}_{n-\ell}^{n,t}(x) = \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma_v} \mathrm{d}w\, \frac{w^{x+n}}{\mathrm{e}^{tw}} \frac{(w-v_1)\cdots(w-v_{\ell-1})}{(w-v_1)\cdots(w-v_n)}.$$

Proof. By Proposition 3.1 of [16], we have

$$\widetilde{\Psi}^{n,t}_{n-k}(x) = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} \mathrm{d}w\, w^{x+k-1} \mathrm{e}^{t/w} \frac{(1-v_1 w)\cdots(1-v_n w)}{(1-v_1 w)\cdots(1-v_k w)}$$

for $k \geq 1$. A change of variable $z = 1/w$ then yields (6.15). Next we need to verify that $\{\widetilde{\Psi}^{n,t}_{n-k} : 1 \leq k \leq n\}$ is biorthogonal to $\{\widetilde{\Phi}^{n,t}_{n-\ell} : 1 \leq \ell \leq n\}$ (see Eq. (3.5) of [16]). We split the sum over $\mathbb{Z}$ into two parts, one over $x \geq 0$ and one over $x < 0$. Then,

$$\sum_{x\geq 0} \widetilde{\Psi}^{n,t}_{n-k}(x)\widetilde{\Phi}^{n,t}_{n-k}(x)$$
$$= \sum_{x\geq 0} \frac{1}{(2\pi\mathrm{i})^2} \oint_{\Gamma_v} \mathrm{d}w \oint_{\Gamma_0} \mathrm{d}z \frac{\mathrm{e}^{tz}}{\mathrm{e}^{tw}} \frac{w^{x+n}}{z^{x+n+1}} \frac{(z-v_{k+1})\cdots(z-v_n)}{(w-v_\ell)\cdots(w-v_n)}.$$

We choose Γ_0 and Γ_v such that $|w| \leq |z|$ which allows us to put the sum inside the integrals. This gives

$$\sum_{x\geq 0} \widetilde{\Psi}^{n,t}_{n-k}(x)\widetilde{\Phi}^{n,t}_{n-k}(x)$$
$$= \frac{1}{(2\pi\mathrm{i})^2} \oint_{\Gamma_v} \mathrm{d}w \oint_{\Gamma_{0,w}} \mathrm{d}z \frac{\mathrm{e}^{tz}}{\mathrm{e}^{tw}} \frac{w^n}{z^n} \frac{(z-v_{k+1})\cdots(z-v_n)}{(w-v_k)\cdots(w-v_\ell)} \frac{1}{z-w}.$$

For $x < 0$ we choose Γ_0 and Γ_v such that they satisfy $|w| > |z|$ which gives

$$\sum_{x<0} \widetilde{\Psi}^{n,t}_{n-k}(x)\widetilde{\Phi}^{n,t}_{n-k}(x)$$
$$= -\frac{1}{(2\pi\mathrm{i})^2} \oint_{\Gamma_0} \mathrm{d}z \oint_{\Gamma_{v,z}} \mathrm{d}w \frac{\mathrm{e}^{tz}}{\mathrm{e}^{tw}} \frac{w^n}{z^n} \frac{(z-v_{k+1})\cdots(z-v_n)}{(w-v_k)\cdots(w-v_\ell)} \frac{1}{z-w}.$$

Thus,

$$\sum_{x\in\mathbb{Z}} \widetilde{\Psi}^{n,t}_{n-k}(x)\widetilde{\Phi}^{n,t}_{n-k}(x) = \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma_v} \mathrm{d}w \frac{(w-v_{k+1})\cdots(w-v_n)}{(w-v_\ell)\cdots(w-v_n)} = \delta_{k\ell}.$$

Finally, we show that $\{\widetilde{\Phi}^{n,t}_{n-\ell} : 1 \leq \ell \leq n\}$ spans the space of functions V_n. Let us denote by $u_1 < \cdots < u_\nu$ the different values of $v_1, \ldots, v_n$ and α_k the multiplicity of u_k, i.e., we have $\alpha_1 + \cdots + \alpha_\nu = n$. Then, we may write

$$\begin{aligned}
\widetilde{\Phi}^{n,t}_{n-1}(x) &= \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma_v} \mathrm{d}w \frac{w^{x+n}}{\mathrm{e}^{tw}} \frac{1}{(w-u_1)^{\alpha_1}\cdots(w-u_\nu)^{\alpha_\nu}} \\
&= \sum_{i=1}^{\nu} \frac{1}{(\alpha_i-1)!} \frac{\mathrm{d}^{\alpha_i-1}}{\mathrm{d}w^{\alpha_i-1}}\bigg|_{w=u_i} \left(\frac{w^{x+n}}{\mathrm{e}^{tw}} \prod_{j\neq i} \frac{1}{(w-u_j)^{\alpha_j}} \right) \\
&= \sum_{i=1}^{\nu} (u_i)^x \sum_{j=1}^{\alpha_i} c_{i,j} x^{j-1}.
\end{aligned}$$

For $\ell = 2,\dots,n$, we can represent $\widetilde{\Phi}^{n,t}_{n-\ell}$ in the same way, but with exponents $\alpha_{\ell,i} \le \alpha_i$, $1 \le i \le \nu$. Since $(\alpha_{k,1},\dots,\alpha_{k,\nu}) \ne (\alpha_{\ell,1},\dots,\alpha_{\ell,\nu})$ for $k \ne \ell$, this shows that

$$\mathrm{span}\{\widetilde{\Phi}^{n,t}_{n-\ell} : 1 \le \ell \le n\} = \mathrm{span}\{x \mapsto (u_i)^x x^{j-1} : 1 \le u \le \nu, 1 \le j \le \alpha_i\},$$

which is V_n. □

We continue by establishing the convergence result under the scaling (3.28). Correspondingly, we rescale (and conjugate) the kernel $\widetilde{K}_t$ and define the rescaled kernel as

$$K^{\mu}_{\tau,T,\mathrm{resc}}((\xi,n),(\xi',n')) = \frac{T^{n'/2}}{T^{n/2}} \sqrt{T}\, \widetilde{K}^{\mu_T}_{\tau T}\big(([\tau T - \xi\sqrt{T}],n),([\tau T - \xi'\sqrt{T}],n')\big)$$

where $[\,\cdot\,]$ denotes the integer part, and the drift v is now $\mu_T = 1 - \mu/\sqrt{T}$. Of course, T is assumed to be so large that $\mu_T > 0$ is satisfied.

Proposition 38. *For any fixed $L > 0$, the rescaled kernel $K^{\mu}_{\tau,T,\mathrm{resc}}$ converges, uniformly for $\xi,\xi' \in [-L,L]$, as*

$$\lim_{T\to\infty} K^{\mu}_{\tau,T,\mathrm{resc}}((\xi,n),(\xi',n')) = K^{\mu}_{\tau}((\xi,n),(\xi',n'))$$

with $K^{\mu}_{\tau} \equiv K_\tau$ given in (3.24).

Proof. Let us define the rescaled functions

$$\begin{aligned}
\phi^{(n,n')}_{T,\mathrm{resc}}(\xi,\xi') &= T^{-(n'-n+1)/2}\, \widetilde{\phi}^{(n,n')}(\tau T + \xi\sqrt{T}, \tau T + \xi'\sqrt{T}),\\
\Psi^{n,\tau}_{n-k,T,\mathrm{resc}}(\xi) &= T^{(n-k+1)/2} \mathrm{e}^{-\tau T}\, \widetilde{\Psi}^{n,\tau T}_{n-k}(\tau T + \xi\sqrt{T}),\\
\Phi^{n,\tau}_{n-k,T,\mathrm{resc}}(\xi') &= T^{-(n-k)/2} \mathrm{e}^{\tau T}\, \widetilde{\Phi}^{n,\tau T}_{n-k}(\tau T + \xi'\sqrt{T}),
\end{aligned}$$

where we also rescale the jump rates as in (3.28). We have to show that these functions converge to their analogues from (3.25)–(3.27). We first verify that $\phi^{(n,n')}_{T,\mathrm{resc}}(\xi,\xi') \to \phi^{(n,n')}(\xi,\xi')$ with $n < n'$. For $y \ge y'$, the integrand of $\widetilde{\phi}^{(n,n')}(y,y')$ in (6.14) has residue 0 at infinity and thus the whole integral vanishes, while for $y < y'$, there is no pole at $z = 0$ and therefore

$$\widetilde{\phi}^{(n,n')}(y,y') = \sum_{i=n+1}^{n'} v_i^{(y'-y)+(n'-n)-1} \prod_{j\ne i} \frac{1}{v_i - v_j} \mathbb{1}_{[y<y']}.$$

Hence, for its rescaled version,

$$\phi^{(n,n')}_{T,\mathrm{resc}}(\xi,\xi') = \sum_{i=n+1}^{n'} \left(1 - \frac{\mu_i}{\sqrt{T}}\right)^{(\xi-\xi')\sqrt{T}+(n'-n)-1} \prod_{j\ne i} \frac{1}{\mu_j - \mu_i} \mathbb{1}_{[\xi>\xi']},$$

which, as $T \to \infty$, converges to

$$\sum_{i=n+1}^{n'} \mathrm{e}^{\mu_i(\xi'-\xi)} \prod_{j\ne i} \frac{1}{\mu_j - \mu_i} \mathbb{1}_{\{\xi>\xi'\}} = \frac{(-1)^{n'-n}}{2\pi\mathrm{i}} \int_{\mathrm{i}\mathbb{R}+\mu_-} \mathrm{d}z\, \frac{\mathrm{e}^{z(\xi'-\xi)}}{(z-\mu_{n+1})\cdots(z-\mu_{n'})}.$$

Next we show that $\Psi^{n,\tau}_{n-k,T,\mathrm{resc}}(\xi) \to \Psi^{n,\tau}_{n-k}(\xi)$ uniformly for $\xi \in [-L, L]$. We have

$$\begin{aligned}\Psi^{n,\tau,T}_{n-k,\mathrm{resc}}(\xi) &= \frac{\sqrt{T}}{2\pi\mathrm{i}} \oint_{\Gamma_{0,v}} \mathrm{d}z \, \frac{\mathrm{e}^{\tau T(z-1)}}{z^{\tau T - \xi\sqrt{T}+n+1}} \, g(z) \\ &= \frac{\sqrt{T}}{2\pi\mathrm{i}} \oint_{\Gamma_{0,v}} \mathrm{d}z \, \mathrm{e}^{\tau T f_0(z) + \sqrt{T} f_1(z) + f_2(z)} g(z)\end{aligned} \tag{6.16}$$

with $f_0(z) = z - 1 - \ln z$, $f_1(z) = \xi \ln z$, $f_2(z) = -(n+1)\ln z$, and

$$g(z) = \frac{(\sqrt{T}(z-1)+\mu_1)\cdots(\sqrt{T}(z-1)+\mu_n)}{(\sqrt{T}(z-1)+\mu_1)\cdots(\sqrt{T}(z-1)+\mu_k)}.$$

A Taylor expansion around the double critical point of f_0, i.e., around $z_c = 1$ gives

$$\begin{aligned} f_0(z) &= \frac{1}{2}(z-1)^2 + \mathcal{O}((z-1)^3), \\ f_1(z) &= \xi(z-1) + \mathcal{O}((z-1)^2), \\ f_2(z) &= 0 + \mathcal{O}(z-1). \end{aligned}$$

Fix $r > 1 - \mu_-/\sqrt{T}$ and deform $\Gamma_{0,v}$ to the contour $\gamma = \gamma_1 \cup \gamma_2$ with

$$\gamma_1 = 1 - \mu_-/\sqrt{T} + \mathrm{i}[-r, r], \quad \gamma_2 = \{|z| = r\} \cap \{\operatorname{Re} z < 1 - \mu_-/\sqrt{T}\}.$$

Let us verify that γ is a steep descent path for f_0. We have $\operatorname{Re} f_0(x+\mathrm{i}y) = x - 1 - \frac{1}{2}\ln(x^2+y^2)$ on the segment γ_1, so

$$\frac{\mathrm{d}\operatorname{Re} f_0(x+\mathrm{i}y)}{\mathrm{d}y} = -\frac{y}{x^2+y^2}, \qquad y \in [-r, r],$$

with $x = 1 - \mu_-/\sqrt{T}$. Thus f_0 is strictly increasing on $1 - \frac{\mu_-}{\sqrt{T}} + \mathrm{i}[-r, 0)$ and strictly decreasing on $1 - \frac{\mu_-}{\sqrt{T}} + \mathrm{i}(0, r]$. On the segment γ_2, we compute

$$\operatorname{Re} f_0(r\mathrm{e}^{\mathrm{i}\varphi}) = r\cos\varphi - 1 - \ln r, \qquad \varphi \in (\arccos\tfrac{1}{r}, 2\pi - \arccos\tfrac{1}{r}),$$

which means that f_0 is strictly decreasing on $\gamma_2 \cap \{\operatorname{Im} z > 0\}$ and strictly increasing on $\gamma_2 \cap \{\operatorname{Im} z < 0\}$. Thus γ is a steepest descent path for f_0 and the major contribution comes from a line segment $\gamma_\delta = 1 - \frac{\mu_-}{\sqrt{T}} + \mathrm{i}[-\delta, \delta]$ for any $\delta \in (0, 1)$. Indeed, the error we make when we integrate along γ_δ instead of γ is of order $\mathcal{O}(\mathrm{e}^{-cT})$ with $c \sim \delta^2$. We therefore consider the integral on γ_δ only,

$$\frac{\sqrt{T}}{2\pi\mathrm{i}} \int_{\gamma_\delta} \mathrm{d}z \, g(z) \mathrm{e}^{\xi\sqrt{T}(z-1) + \frac{\tau T}{2}(z-1)^2} \mathrm{e}^{\mathcal{O}\left(z-1, \sqrt{T}(z-1)^2, T(z-1)^3\right)}. \tag{6.17}$$

Using $|\mathrm{e}^x - 1| \leq |x|\mathrm{e}^{|x|}$, the difference between (6.17) and the same integral without the error term can be bounded by

$$\frac{\sqrt{T}}{2\pi}\int_{\gamma_\delta} \mathrm{d}z \left| \mathrm{e}^{c_1 \xi\sqrt{T}(z-1)+c_2\frac{\tau T}{2}(z-1)^2} \right.$$
$$\left. \times\, \mathcal{O}\left(z-1, \sqrt{T}(z-1)^2, T(z-1)^3, T^{(n-k)/2}(z-1)^{n-k}\right)\right|$$

for some constants c_1 and c_2 that can be chosen arbitrarily close to 1 as $\delta \to 0$. By a change of variable $Z = \sqrt{T}(1-z)$ one then sees that this error is of order $\mathcal{O}(T^{-1/2})$. Hence we can consider the integral in (6.17) without the error term, which simplifies to

$$\frac{\sqrt{T}}{2\pi\mathrm{i}}\int_{\gamma_\delta} \mathrm{d}z\, \mathrm{e}^{\tau T(z-1)^2/2+\xi\sqrt{T}(z-1)} g(z).$$

The error we make if we extend γ_δ to $1-\frac{\mu_-}{\sqrt{T}}+\mathrm{i}\mathbb{R}$ is of order $\mathcal{O}(\mathrm{e}^{-cT})$. All together the integral from (6.16) agrees, up to an error $\mathcal{O}(\mathrm{e}^{-cT}, T^{-1/2})$ uniform in $\xi \in [-L, L]$, with

$$\frac{\sqrt{T}}{2\pi\mathrm{i}}\int_{1-\frac{\mu_-}{\sqrt{T}}+\mathrm{i}\mathbb{R}} \mathrm{d}z\, \mathrm{e}^{\tau T(z-1)^2/2+\xi\sqrt{T}(z-1)} g(z),$$

where the poles of g lie on the left of the integration axis. After applying the change of variable $Z = -\sqrt{T}(z-1)$, this integral can be identified as $\Psi^{n,\tau}_{n-k}(\xi)$.

Finally we show that $\Phi^{n,\tau}_{n-k,T,\mathrm{resc}}(\xi') \to \Phi^{n,\tau}_{n-k}(\xi')$. We have

$$\Phi^{n,\tau}_{n-k,T,\mathrm{resc}}(\xi') = \frac{\sqrt{T}}{2\pi\mathrm{i}}\oint_{\Gamma_v} \mathrm{d}w\, \mathrm{e}^{\tau T(\ln w - w+1)-\sqrt{T}\xi'\ln w + n\ln w}$$
$$\times \frac{(\sqrt{T}(w-1)+\mu_1)\cdots(\sqrt{T}(w-1)+\mu_{k-1})}{(\sqrt{T}(w-1)+\mu_1)\cdots(\sqrt{T}(w-1)+\mu_n)},$$

and by a change of variable $W = -\sqrt{T}(w-1)$ and a Taylor expansion in the exponent we get

$$\Phi^{n,\tau}_{n-k,T,\mathrm{resc}}(\xi')$$
$$= \frac{(-1)^{n-k+1}}{2\pi\mathrm{i}}\oint_{\Gamma_a} \mathrm{d}W\, \mathrm{e}^{-\tau W^2/2+\xi' W+\mathcal{O}(T^{-1/2})} \frac{(w-\mu_1)\cdots(w-\mu_{k-1})}{(w-\mu_1)\cdots(w-\mu_n)},$$

which converges uniformly for $\xi' \in [-L, L]$ to $\Phi^{n,\tau}_{n-k}(\xi')$. □

With the above results we can now prove the theorem that we named Result 12.

Proof of Result 12. Set $m = N(N+1)/2$ and define $n_1, \dots, n_m$ by

$$n_1 = 1,\ \ n_2 = n_3 = 2,\ \ n_4 = n_5 = n_6 = 3,\ \ \dots,\ \ n_{m-N+1} = \dots = n_m = N.$$

For $A \subseteq \mathbb{R}^m$ we set $A_T = (\tau T - \sqrt{T}A) \cap \mathbb{Z}$. Then, we have

$$\begin{aligned}
\nu_T(A) &= \sum_{(x_1,\dots,x_m)\in A_T} \det\big[\widetilde{K}^{\mu_T}_{\tau T}((x_i,n_i),(x_j,n_j))\big]_{1\le i,j\le m} \\
&= T^{m/2} \int_A \mathrm{d}^m x \det\big[\widetilde{K}^{\mu_T}_{\tau T}((\tau T - [x_i]\sqrt{T}, n_i), (\tau T - [x_j]\sqrt{T}, n_j))\big]_{1\le i,j\le m} \\
&= \int_A \mathrm{d}^m x \det\big[\widetilde{K}_{\tau,T,\mathrm{resc}}(([x_i],n_i),([x_j],n_j))\big]_{1\le i,j\le m}
\end{aligned}$$

and

$$\nu(A) = \int_A \mathrm{d}^m x \det\big[K^{\mu}_{\tau}((x_i,n_i),(x_j,n_j))\big]_{1\le i,j\le m}.$$

Since the determinants are continuous functions of the kernels, we have by Proposition 38 that

$$\lim_{T\to\infty} \det\big[\widetilde{K}^{\mu}_{\tau,T,\mathrm{resc}}(([x_i],n_i),([x_j],n_j))\big]_{1\le i,j\le m} = \det\big[K^{\mu}_{\tau}((x_i,n_i),(x_j,n_j))\big]_{1\le i,j\le m}$$

for all $x_1,\dots,x_m \in \mathbb{R}$. Thus we have shown that the densities of the probability measures in question converge pointwise to each other. Then, (3.29) is a direct consequence of Scheffé's theorem, see e.g. [12]. □

6.3. Warren's process with drifts

We have seen in Section 6.1 that the eigenvalues' density can be written as a product of determinants, and, in Lemma 36, we calculated the normalization constant, so that the probability measure on the eigenvalues reads

$$\mathbb{P}\Bigg(\bigcap_{1\le k\le n\le N} \{\lambda^n_k \in \mathrm{d}\lambda^n_k\}\Bigg) = \tilde{p}_t(\lambda)\,\mathrm{d}\lambda$$

with $\mathrm{d}\lambda = \prod_{1\le k\le n\le N} \mathrm{d}\lambda^n_k$, and

$$\tilde{p}_t(\lambda) = \det\big[\Psi^{N,t}_{N-k}(\lambda^N_\ell)\big]_{1\le k,\ell\le N} \prod_{n=1}^{N} \mathrm{e}^{-t\mu_n^2/2} \prod_{n=1}^{N} \det\big[\phi_n(\lambda^{n-1}_i, \lambda^n_j)\big]_{1\le i,j\le n} \tag{6.18}$$

In this section we explain the connection to a system of Brownian motions in $\mathbb{G}_N$. More precisely, we consider Brownian motions $\{B^n_k, 1\le k\le n\le N\}$ in $\mathbb{G}_N$ starting from 0, with drift μ_n, and interacting as follows:

- The evolution of B^n_k does not depend on the Brownian motions with higher upper index (B^m_ℓ for $m \ge n+1$, and any ℓ);

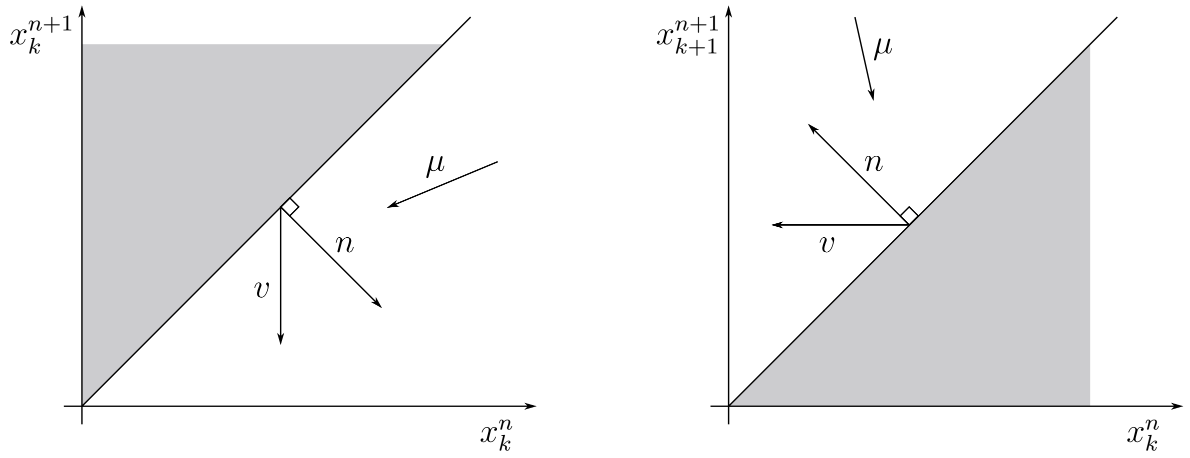


Figure 6.1.: The two reflection types in our system. They correspond to the boundary condition (6.19).

- B_k^n is reflected off B_k^{n-1} and B_{k-1}^{n-1}.

These reflections are sometimes called oblique reflections [107], since in the (x_k^{n-1}, x_k^n)-plane (resp. (x_{k-1}^{n-1}, x_k^n)-plane) the reflection directions are not normal, but oblique as indicated in Figure 6.1. Note that the projection on $\{B_1^n, 1 \leq n \leq N\}$ differs from the process studied in [78], where the reflections are in the normal direction.

Let us now describe the system of Brownian motions. Denote by p_t be the probability density of the Brownian motions in $\mathbb{G}_N$ (its existence will be a consequence of our result). Following [53], where Brownian motions with oblique reflections were studied, for a Brownian motion with drift μ reflected at the boundary in the direction v, the boundary conditions on the density function may be expressed as follows. Denote by n the normal vector of the boundary, let v be normalized such that $n \cdot v = 1$ and let $q = v - n$. Moreover, set $\nabla_T = \nabla - n(n \cdot \nabla)$, $D^* = n \cdot \nabla - q \cdot \nabla_T$. Then, the boundary condition can be written as

$$D^* p_t = (\nabla_T \cdot q + 2\mu \cdot n) p_t \quad \text{on the boundary.}$$

Specializing to our case, we get

$$\frac{\partial}{\partial x_k^n} p_t(x) + (\mu_{n+1} - \mu_n) p_t(x) = 0, \tag{6.19}$$

whenever $x_k^n = x_k^{n+1}$ or $x_k^n = x_{k+1}^{n+1}$, for $1 \leq k \leq N-1$.

This process, without drifts, was introduced by Warren in [109], where he determined the transition probability for any initial condition and also showed that the process is well-defined when starting from zero. We here consider a system of Brownian motions with constant (bounded) drifts, which can be expressed as follows,

$$\begin{aligned}
&B_1^1(t) = \mu_1 t + b_1^1(t),\\
&B_1^n(t) = \mu_n t + b_1^n(t) - L_{B_1^{n-1}-B_1^n}(t), \quad n = 2, \dots, N,\\
&B_k^n(t) = \mu_n t + b_k^n(t) - L_{B_k^{n-1}-B_k^n}(t) + L_{B_k^n - B_{k-1}^{n-1}}(t), \quad 2 \le k < n \le N,\\
&B_n^n(t) = \mu_n t + b_n^n(t) + L_{B_n^n - B_{n-1}^{n-1}}(t), \quad n = 2, \dots, N,
\end{aligned}$$

where the b_k^n, $1 \le k \le n \le N$, are independent standard Brownian motions and $L_{X-Y}(t)$ is twice the semimartingale local time at zero of $X(t) - Y(t)$. The question of well-definedness was related to the, a priori possible, presence of triple collisions. Bounded drifts do not influence this property as can be seen by applying Girsanov's theorem like in the works [54, 63].

A standard one-dimensional reflected Brownian motion can also be defined as the image under the Skorokhod map of standard Brownian motion. More precisely, one defines a Brownian motion B starting from $y \in \mathbb{R}$ and being reflected at some continuous function f satisfying $f(0) < y$ via the Skorokhod representation [6, 91] for $t \ge 0$,

$$\begin{aligned}
B(t) &= y + b(t) - \min\big\{0, \inf_{0 \le s \le t} (y + b(s) - f(s))\big\}\\
&= \max\big\{y + b(t), \sup_{0 \le s \le t} (f(s) + b(t) - b(s))\big\},
\end{aligned}$$

where b is a standard Brownian motion starting at 0. In this work we take Warren's process with drifts as being the image of independent Brownian motions under the extended Skorokhod map introduced by Burdzy, Kang and Ramanan, see Theorem 2.6 of [26] for an explicit formula.

Proof of Result 13. Consider a particle system as in Section 6.2, but let the particles evolve independently, i. e., $\tilde{x}_k^n(0) = -n + k - 1$ for $1 \le k \le n \le N$ and the evolution of $\tilde{x}_k^n(t)$ is a continuous time random walk with jump rate v_n. Consider now the scaling (3.28)

$$t = \tau T, \quad \tilde{B}_k^n = \frac{\tilde{x}_k^n - \tau T}{-\sqrt{T}}, \quad v_n = 1 - \frac{\mu_n}{\sqrt{T}}.$$

The $\tilde{x}_k^n$ are independent, so in the $T \to \infty$ limit, $(\tilde{B}_k^n, 1 \le k \le n \le N)$ will converge weakly to a Brownian motion $(B_k^n, 1 \le k \le n \le N)$ in $N(N+1)/2$ dimensions, where B_k^n has drift μ_n (see Donsker's theorem). As shown in [51] by Gorin and Shkolnikov, the particle with the blocking/pushing dynamics converges wearkly, as T tends to infinity, to the Warren process with level-dependent drifts. To be precise, the proof was written down for the drift-less case. However, as they mention in Remark 10 of [51], the same proof applies to more general cases, in particular to our case.

In Proposition 37 we have proven that the correlation functions hava a $T \to \infty$ limit. Further, the integral of the density is one, so that no mass is lost at infinity or localized in some Dirac mass. Thus, the n-point correlation function of the reflected Brownian motion is the $T \to \infty$ limit of the n-point correlation function for the interacting particle system. □

For completeness, let us remark that the transition density p_t in $\mathbb{G}_N$ satisfies:

(1) the Fokker-Planck equation (or Kolmogorov forward equation)

$$\frac{\partial}{\partial t} p_t(x) = \sum_{n=1}^{N} \sum_{k=1}^{n} \left(\frac{1}{2} \frac{\partial^2}{\partial (x_k^n)^2} - \mu_n \frac{\partial}{\partial x_k^n} \right) p_t(x), \tag{6.20}$$

(2) the initial condition

$$\lim_{t \searrow 0} p_t(x) \mathrm{d}x = \prod_{1 \le k \le n \le N} \delta_{x_k^n}, \tag{6.21}$$

(3) the boundary condition (6.19).

Proposition 39. *Denote by $p_t : \mathbb{G}_N \to [0,1]$ be the probability density defined in* (6.18)*. Inside $\mathbb{G}_N$, this density satisfies the Fokker-Planck equation* (6.20)*, the initial condition* (6.21)*, and the boundary condition* (6.19)*.*

Proof. First observe that by setting $\tilde{\Psi}_{N-k}^{N,t}(x) = \mathrm{e}^{\mu_N x} \Psi_{N-k}^{N,t}(x)$, we can rewrite (6.18) as a probability measure on $\mathbb{G}_N$ with density

$$\tilde{p}_t(x) = \det \left[\tilde{\Psi}_{N-k}^{N,t}(x_\ell^N) \right]_{1 \le k, \ell \le N} \prod_{k=1}^{N} \mathrm{e}^{-t\mu_k^2/2} \prod_{n=1}^{N-1} \prod_{k=1}^{n} \mathrm{e}^{(\mu_n - \mu_{n+1}) x_k^n}.$$

for $x = (x_k^n)_{1 \le k \le n \le N} \in \mathbb{G}_N$. The double product only depends on $(x_k^n)_{1 \le k \le n \le N-1}$, while the determinant is a function of $(x_k^N)_{1 \le k \le N}$. We have

$$\frac{1}{2} \frac{\partial^2}{\partial x^2} \tilde{\Psi}_{N-k}^{N,t}(x) = \frac{\partial}{\partial t} \tilde{\Psi}_{N-k}^{N,t}(x) + \mu_N \frac{\partial}{\partial x} \tilde{\Psi}_{N-k}^{N,t}(x) - \frac{\mu_N^2}{2} \tilde{\Psi}_{N-k}^{N,t}(x),$$

from which follows that

$$\frac{1}{2} \sum_{\ell=1}^{N} \frac{\partial^2}{\partial (x_\ell^N)^2} \tilde{p}_t(x) = \frac{\partial}{\partial t} \tilde{p}_t(x) + \mu_N \sum_{\ell=1}^{N} \frac{\partial}{\partial x_\ell^N} \tilde{p}_t(x) + \frac{1}{2} \left(\sum_{n=1}^{N} \mu_n^2 - N\mu_N^2 \right) \tilde{p}_t(x). \tag{6.22}$$

For $k = 1, \dots, N-1$, we have

$$\frac{\partial}{\partial x_k^n} \tilde{p}_t(x) = (\mu_n - \mu_{n+1}) \tilde{p}_t(x), \qquad \frac{\partial^2}{\partial (x_k^n)^2} \tilde{p}_t(x) = (\mu_n - \mu_{n+1})^2 \tilde{p}_t(x), \tag{6.23}$$

and thus, putting (6.22) and (6.23) together,

$$\begin{aligned} \frac{1}{2} \sum_{n=1}^{N} \sum_{k=1}^{n} \frac{\partial^2}{\partial (x_k^n)^2} \tilde{p}_t(x) &= \frac{\partial}{\partial t} \tilde{p}_t(x) + \mu_N \sum_{k=1}^{N} \frac{\partial}{\partial x_k^N} \tilde{p}_t(x) \\ &\quad + \frac{1}{2} \left(\sum_{n=1}^{N} \mu_n^2 - N\mu_N^2 + \sum_{n=1}^{N-1} n(\mu_n - \mu_{n+1})^2 \right) \tilde{p}_t(x). \end{aligned} \tag{6.24}$$

Using that

$$N\mu_N^2 - \sum_{n=1}^{N} \mu_n^2 = \sum_{n=1}^{N-1} n(\mu_{n+1}^2 - \mu_n^2) = \sum_{n=1}^{N-1} n(\mu_n - \mu_{n+1})^2 - 2\sum_{k=1}^{N-1} n\mu_n(\mu_n - \mu_{n+1}) \quad (6.25)$$

the expression between the brackets in (6.24) simplifies to $2\sum n\mu_n(\mu_n - \mu_{n+1})$. On the other hand,

$$\sum_{n=1}^{N}\sum_{k=1}^{n} \mu_n \frac{\partial}{\partial x_k^n}\, \tilde{p}_t(x) = \sum_{n=1}^{N-1} n\mu_n(\mu_n - \mu_{n+1})\tilde{p}_t(x) + \mu_N \sum_{k=1}^{N} \frac{\partial}{\partial x_k^N}\, \tilde{p}_t(x). \quad (6.26)$$

Then, (6.20) follows from (6.24), (6.25) and (6.26). The initial condition (6.21) is satified because as $t \searrow 0$, we obtain the Dirac measure at $x_k^N = 0$ for $1 \leq k \leq N$ and since we consider $\tilde{p}_t$ on $\mathbb{G}_N$, this immediately implies that $x_n^k = 0$ for all $1 \leq k \leq n \leq N-1$. Finally, the boundary condition (6.19) holds trivially by (6.23). □

Remark 40. The three conditions in Proposition 39 are, in general, not enough to prove that $\tilde{p}_t = p_t$. For that, one would need the backwards equation.

A. Appendix

A.1. Spatial persistence for the Airy processes

At this place we provide the proofs for Results 5 and 6 from Section 3.1.3. Our presentation is taken from [48]. The starting point of our analysis are two formulas on the continuum statistics for the Airy_1 process [83] and for the Airy_2 process [32]. Let us start with the Airy_1 process.

Theorem 41 (Theorem 4 of [83])**.** *It holds*

$$\mathbb{P}\big(\mathcal{A}_1(t) \leq g(t) \text{ for all } t \in [0, L]\big) = \det(\mathbb{1} - B_0 + \Lambda_{L,g}\mathrm{e}^{-L\Delta}B_0)_{L^2(\mathbb{R})}$$

where g is a function in $H^1([0, L])$, Δ is the Laplacian, $B_0(x, y) = \mathrm{Ai}(x + y)$, and

$$\Lambda_{L,g}(x, y) = \frac{\mathrm{e}^{-(x-y)^2/(4L)}}{\sqrt{4\pi L}}\, \mathbb{P}_{b(0)=x,b(L)=y}(b(s) \leq g(s), 0 \leq s \leq L)$$

with b a Brownian Bridge from x at time 0 to y at time L and with diffusion coefficient 2.

To get the persistence probabilities, we have to determine the explicit kernel for the function $g(s) = c$.

Proof of Result 5. We have to determine a formula for the Fredholm determinant of the operator $\mathbb{1} - B_0 + \Lambda_{L,c}\mathrm{e}^{-L\Delta}B_0$. Since the Fredholm determinant is on all $\mathbb{R}$, we can shift the variables by c and obtain the kernel

$$B_0(x + c, y + c) - \int_{\mathbb{R}} \mathrm{d}z\, \Lambda_{L,c}(x + c, y + c)(\mathrm{e}^{-L\Delta}B_0)(z + c, y + c). \tag{A.1}$$

Clearly, $\Lambda_{L,c}(x, y) = \Lambda_{L,0}(x - c, y - c)$, therefore

$$\text{(A.1)} = \mathrm{Ai}(x + y + 2c) - \int_{\mathbb{R}} \Lambda_{L,0}(x, z)(\mathrm{e}^{-L\Delta}B_0)(z + c, y + c). \tag{A.2}$$

By the reflection principle we have

$$\begin{aligned}\Lambda_{L,0}(x, z) &= \mathbb{P}_{b(0)=x,b(L)=z}(b(s) \leq 0, 0 \leq s \leq L)\\ &= \frac{1}{\sqrt{4\pi L}}\left(\mathrm{e}^{-(x-z)^2/(4L)} - \mathrm{e}^{-(x+z)^2/(4L)}\right)\mathbb{1}_{[x,z<0]}.\end{aligned}$$

Moreover, it is known (see e.g. the review [44]) that

$$e^{-L\Delta} B_0(z+c, y+c) = e^{-2L^3/3-(z+y+2c)L} \operatorname{Ai}(z+y+2c+L^2).$$

Putting all together we have that (A.2) is equal to

$$\operatorname{Ai}(x+y+2c) - \mathbb{1}_{[x<0]} \left(\widehat{K}_{1,L}(x, y+2c) - \widehat{K}_{1,L}(-x, y+2c) \right), \tag{A.3}$$

where

$$\widehat{K}_{1,L}(x,y) = \frac{1}{\sqrt{4\pi L}} \int_{\mathbb{R}_-} dz \, e^{-(x-z)^2/4L} e^{-2L^3/3} e^{-L(y+z)} \operatorname{Ai}(y+z+L^2).$$

Finally, using the identity (see below)

$$\frac{1}{\sqrt{4\pi L}} \int_{\mathbb{R}} dz \, e^{-(x-z)^2/4L} e^{-2L^3/3} e^{-L(y+z)} \operatorname{Ai}(y+z+L^2) = \operatorname{Ai}(x+y) \tag{A.4}$$

we get

$$\widehat{K}_{1,L}(x,y) = \operatorname{Ai}(x+y) - \widetilde{K}_{1,L}(x,y).$$

Replacing this into (A.3) gives the desired result (3.11).

Finally, let us verify (A.4). By the integral representation of the Airy function,

$$\operatorname{Ai}(b^2+c) e^{2b^3/3+bc} = \frac{1}{2\pi i} \int_{e^{-i\pi/3}\infty}^{e^{i\pi/3}\infty} dw \, e^{w^3/3+bw^2-cw} \tag{A.5}$$

and a Gaussian integration we get

$$\begin{aligned} &\frac{1}{\sqrt{4\pi L}} \int_{\mathbb{R}} dz \, e^{-(x-z)^2/4L} e^{-2L^3/3} e^{-L(y+z)} \operatorname{Ai}(y+z+L^2) \\ &\qquad = e^{-L(x+y)} e^{L^3/3} \frac{1}{2\pi i} \int_{e^{-i\pi/3}\infty}^{e^{i\pi/3}\infty} dw \, e^{w^3/3+Lw^2-w(x+y-L^2)} = \operatorname{Ai}(x+y), \end{aligned}$$

where we used again (A.5). □

Now we consider the Airy$_2$ process. The analogue of Theorem 41 for the Airy$_1$ process is given by

Theorem 42 (Theorem 2 of [32])**.** *It holds*

$$\mathbb{P}\big(\mathcal{A}_2(t) \leq g(t) \textit{ for all } t \in [0,L]\big) = \det(\mathbb{1} - K_{\mathrm{Ai}} + \Lambda_{L,g} e^{LH_{\mathrm{Ai}}} K_{\mathrm{Ai}})_{L^2(\mathbb{R})}$$

where g is a function in $H^1([0,L])$, $K_{\mathrm{Ai}}(x,y) = \int_{\mathbb{R}_+} d\lambda \, \operatorname{Ai}(x+\lambda) \operatorname{Ai}(y+\lambda)$ is the Airy kernel, $H_{\mathrm{Ai}} = -\Delta + x$ is the Airy operator, and

$$\Lambda_{L,g}(x,y) = e^{-Ly-L^3/3} \frac{e^{-(x-y)^2/(4L)}}{\sqrt{4\pi L}} \mathbb{P}_{b(0)=x, b(L)=y-L^2}(b(s) \leq g(s) - s^2, 0 \leq s \leq L)$$

with b a Brownian Bridge from x at time 0 to $y-L^2$ at time L and with diffusion coefficient 2.

We have to determine the kernel for the special function $g(s) = c$.

Proof of Result 6. We have to compute the Fredholm determinant of $\mathbb{1} - K_{\mathrm{Ai}} + \Lambda_{L,c}\mathrm{e}^{-LH_{\mathrm{Ai}}}K_{\mathrm{Ai}}$ over $L^2(\mathbb{R})$. As in the proof of Proposition 5, we first do a shift in the variables by c and obtain the kernel

$$K_{\mathrm{Ai}}(x+c, y+c) - \int_{\mathbb{R}} \mathrm{d}z\, \Lambda_{L,c}(x+c, z+c)(\mathrm{e}^{LH_{\mathrm{Ai}}}K_{\mathrm{Ai}})(z+c, y+c) \tag{A.6}$$

It is easy to verify that

$$\Lambda_{L,c}(x,y) = \Lambda_{L,0}(x-c, y-c)\mathrm{e}^{-Lc}.$$

Therefore, the kernel becomes

$$\text{(A.6)} = K_{\mathrm{Ai}}(x+c, y+c) - \mathrm{e}^{-Lc}\int_{\mathbb{R}} \mathrm{d}z\, \Lambda_{L,0}(x, z)(\mathrm{e}^{LH_{\mathrm{Ai}}}K_{\mathrm{Ai}})(z+c, y+c).$$

Thus, the desired formula follows if we can show that

$$\begin{aligned}\Lambda_{L,0}(x,z) &= \mathrm{e}^{-Lz-L^3/3}\frac{\mathrm{e}^{-(x-z)^2/(4L)}}{\sqrt{4\pi L}}\,\mathbb{P}_{b(0)=x,b(L)=z-L^2}(b(s)\leq -s^2, 0\leq s\leq L)\\ &= \mathbb{1}_{[x,z\leq 0]}\int_{\mathbb{R}} \mathrm{d}\mu\, \mathrm{e}^{\mu L}\phi(x,\mu)\phi(z,\mu).\end{aligned} \tag{A.7}$$

To this end we use another representation of the kernel $\Lambda_{L,0}$, that can also be found in [32] and that follows from (A.7) by applying the Girsanov theorem and the Feynman-Kac formula. According to this characterization, $\Lambda_{L,0}(x,z) = u(L; x, z)\mathbb{1}_{[z<0]}$ is the solution at time $t = L$ of the boundary value problem

$$\begin{aligned}\partial_t u + H_{\mathrm{Ai}}u = 0 &\quad \text{for } x<0 \text{ and } t\in(0,L),\\ u(0;x,z) = \delta_{x-z}, &\\ u(t;x,z) = 0 &\quad \text{for } x\geq 0.\end{aligned}$$

The solution of this problem can be found in [70, eq. (40)],

$$u(t;x,z) = \mathbb{1}_{[x<0]}\int_{\mathbb{R}} \mathrm{d}\mu\, \mathrm{e}^{\mu t}\phi(x,\mu)\phi(z,\mu).$$

Note that in [32] the boundary value problem describes the action of the operator $\Lambda_{L,0}$ while our formulation considers the kernel of this operator. □

A.2. Determinantal correlations

Since we refer several times to Lemma 3.4 of [17], we report it here.

Lemma 43 (Lemma 3.4 of [17]). *Assume we have a signed measure on* $\{x_i^n : 1 \le i \le n \le N\}$ *given in the form,*

$$\frac{1}{Z_N} \prod_{n=1}^{N-1} \det[\phi_n(x_i^n, x_j^{n+1})]_{1\le i,j\le n+1} \det[\Psi_{N-i}^N(x_j^N)]_{1\le i,j\le N},$$

where x_{n+1}^n *are some "virtual" variables and* Z_N *is a normalization constant. If* $Z_N \neq 0$*, then the correlation functions are determinantal.*

To write down the kernel we need to introduce some notations. Define

$$\phi^{(n_1,n_2)}(x,y) = \begin{cases} (\phi_{n_1} * \cdots * \phi_{n_2-1})(x,y), & n_1 < n_2, \\ 0, & n_1 \ge n_2, \end{cases}.$$

where $(a*b)(x,y) = \sum_{z\in\mathbb{Z}} a(x,z)b(z,y)$*, and, for* $1 \le n < N$*,*

$$\Psi_{n-j}^n(x) := (\phi^{(n,N)} * \Psi_{N-j}^N)(y), \quad j = 1,\dots,N.$$

Set $\phi_0(x_1^0, x) = 1$*. Then the functions*

$$\{(\phi_0 * \phi^{(1,n)})(x_1^0,x),\dots,(\phi_{n-2} * \phi^{(n-1,n)})(x_{n-1}^{n-2},x), \phi_{n-1}(x_n^{n-1},x)\}$$

are linearly independent and generate the n*-dimensional space* V_n*. Define a set of functions* $\{\Phi_j^n(x), j = 0,\dots,n-1\}$ *spanning* V_n *defined by the orthogonality relations*

$$\sum_x \Phi_i^n(x)\Psi_j^n(x) = \delta_{i,j}$$

for $0 \le i,j \le n-1$.

Under Assumption (A): $\phi_n(x_{n+1}^n, x) = c_n\Phi_0^{(n+1)}(x)$*, for some* $c_n \neq 0$*,* $1 \le n \le N-1$*, the kernel takes the simple form*

$$K(n_1,x_1;n_2,x_2) = -\phi^{(n_1,n_2)}(x_1,x_2) + \sum_{k=1}^{n_2} \Psi_{n_1-k}^{n_1}(x_1)\Phi_{n_2-k}^{n_2}(x_2).$$

A.3. Space-like determinantal correlations

For convenience we report here the statement of Theorem 4.2 of [16].

Let $\mathfrak{X}_1,\dots,\mathfrak{X}_N$ be finite sets and $c(1),\dots,c(N)$ be arbitrary nonnegative integers. Consider the set

$$\mathfrak{X} = (\mathfrak{X}_1 \sqcup \cdots \sqcup \mathfrak{X}_1) \sqcup \cdots \sqcup (\mathfrak{X}_N \sqcup \cdots \sqcup \mathfrak{X}_N)$$

with $c(n)+1$ copies of each $\mathfrak{X}_n$. We want to consider a special form of the weight $W(X)$ for any subset $X \subset \mathfrak{X}$, which turns out to have determinantal correlations.

To define the weight we need a bit of notations. Let

$$\begin{aligned} \phi_n(\,\cdot\,,\,\cdot\,) : \mathfrak{X}_{n-1} \times \mathfrak{X}_n \to \mathbb{C}, \qquad & n = 2, \dots, N, \\ \phi_n(\mathrm{virt},\,\cdot\,) : \mathfrak{X}_n \to \mathbb{C}, \qquad & n = 1, \dots, N, \\ \Psi_j^N(\,\cdot\,) : \mathfrak{X}_N \to \mathbb{C}, \qquad & j = 0, \dots, N-1, \end{aligned}$$

be arbitrary functions on the corresponding sets. Here the symbol virt stands for a "virtual" variable, which is convenient to introduce for notational purposes. In applications virt can sometimes be replaced by $+\infty$ or $-\infty$. The ϕ_n represents the transitions from $\mathfrak{X}_{n-1}$ to $\mathfrak{X}_n$.

Also, let

$$t_0^N \le \dots \le t_{c(N)}^N = t_0^{N-1} \le \dots \le t_{c(N-1)}^{N-1} = t_0^{N-2} \le \dots \le t_{c(2)}^2 = t_0^1 \le \dots \le t_{c(1)}^1$$

be real numbers. In applications, these numbers refer to time moments. Finally, let

$$\mathcal{T}_{t_a^n, t_{a-1}^n}(\,\cdot\,,\,\cdot\,) : \mathfrak{X}_n \times \mathfrak{X}_n \to \mathbb{C}, \qquad n = 1, \dots, N, \quad a = 1, \dots, c(n),$$

be arbitrary functions. The $\mathcal{T}_{t_a^n, t_{a-1}^n}$ represents the transition between two copies of $\mathfrak{X}_n$ associated to "times" t_{a-1}^n and t_a^n.

Then, to any subset $X \subset \mathfrak{X}$ we assign its weight $W(X)$ as follows. $W(X)$ is zero unless X has exactly n points in each copy of $\mathfrak{X}_n$, $n = 1, \dots, N$. In the latter case, denote the points of X in the mth copy of $\mathfrak{X}_n$ by $x_k^n(t_m^n)$, $k = 1, \dots, n$, $m = 0, \dots, c(n)$. Thus,

$$X = \{x_k^n(t_m^n) \mid k = 1, \dots, n;\ m = 0, \dots, c(n);\ n = 1, \dots, N\}.$$

Set

$$\begin{aligned} W(X) = \prod_{n=1}^{N} \Bigg[& \det\left[\phi_n(x_k^{n-1}(t_0^{n-1}), x_l^n(t_{c(n)}^n))\right]_{1\le k,l\le n} \\ & \times \prod_{a=1}^{c(n)} \det\left[\mathcal{T}_{t_a^n, t_{a-1}^n}(x_k^n(t_a^n), x_l^n(t_{a-1}^n))\right]_{1\le k,l\le n} \Bigg] \det\left[\Psi_{N-l}^N(x_k^N(t_0^N))\right]_{1\le k,l\le N}, \end{aligned}$$

where $x_n^{n-1}(\,\cdot\,) = \mathrm{virt}$ for all $n = 1, \dots, N$.

In what follows we assume that the partition function of our weights does not vanish,

$$Z := \sum_{X \subset \mathfrak{X}} W(X) \neq 0.$$

Under this assumption, the normalized weights $\widetilde{W}(X) = W(X)/Z$ define a (generally speaking, complex valued) measure on $2^{\mathfrak{X}}$ of total mass 1. One can say that we have a (complex valued) random point process on $\mathfrak{X}$, and its correlation functions are defined accordingly, see e.g., [25]. We are interested in computing these correlation functions.

Let us introduce the compact notation for the convolution of several transitions. For any $n = 1, \dots, N$ and two time moments $t_a^n > t_b^n$ we define

$$\mathcal{T}_{t_a^n, t_b^n} = \mathcal{T}_{t_a^n, t_{a-1}^n} * \mathcal{T}_{t_{a-1}^n, t_{a-2}^n} * \cdots * \mathcal{T}_{t_{b+1}^n, t_b^n}, \qquad \mathcal{T}^n = \mathcal{T}_{t_{c(n)}^n, t_0^n},$$

where we use the notation $(f * g)(x, y) := \sum_z f(x, z) g(z, y)$. For any time moments $t_{a_1}^{n_1} \geq t_{a_2}^{n_2}$ with $(a_1, n_1) \neq (a_2, n_2)$, we denote the convolution over all the transitions between them by $\phi^{(t_{a_1}^{n_1}, t_{a_2}^{n_2})}$:

$$\phi^{(t_{a_1}^{n_1}, t_{a_2}^{n_2})} = \mathcal{T}_{t_{a_1}^{n_1}, t_0^{n_1}} * \phi_{n_1+1} * \mathcal{T}^{n_1+1} * \cdots * \phi_{n_2} * \mathcal{T}_{t_{c(n_2)}^{n_2}, t_{a_2}^{n_2}}.$$

If there are no such transitions, i.e., if $t_{a_1}^{n_1} < t_{a_2}^{n_2}$ or $(a_1, n_1) = (a_2, n_2)$, we set $\phi^{(t_{a_1}^{n_1}, t_{a_2}^{n_2})} = 0$.

Furthermore, define the matrix $M = \|M_{k,l}\|_{k,l=1}^N$ by

$$M_{k,l} = \big(\phi_k * \mathcal{T}^k * \cdots * \phi_N * \mathcal{T}^N * \Psi_{N-l}^N\big)(\mathrm{virt})$$

and the vector

$$\Psi_{n-l}^{n, t_a^n} = \phi^{(t_a^n, t_0^N)} * \Psi_{N-l}^N, \qquad l = 1, \dots, N.$$

The following statement describing the correlation kernel is a part of Theorem 4.2 of [16].

Theorem 44 (Part of Theorem 4.2 of [16])**.** *Assume that the matrix M is invertible. Then $Z = \det M \neq 0$, and the (complex valued) random point process on $\mathfrak{X}$ defined by the weights $\widetilde{W}(X)$ is determinantal. Its correlation kernel can be written in the form*

$$K(n_1, t_{a_1}^{n_1}, x_1; n_2, t_{a_2}^{n_2}, x_2) = -\phi^{(t_{a_1}^{n_1}, t_{a_2}^{n_2})}(x_1, x_2) + \sum_{k=1}^{N} \sum_{l=1}^{n_2} \Psi_{n_1-k}^{n_1, t_{a_1}^{n_1}}(x_1) [M^{-1}]_{k,l} (\phi_l * \phi^{(t_{c(l)}^l, t_{a_2}^{n_2})})(\mathrm{virt}, x_2).$$

Remark 45. As stated in the complete statement of Theorem 4.2 of [16], there is one situation where the kernel takes a simple formula. Namely, when the matrix M is upper-triangular, then

$$\Phi_{n_2-k}^{n_2, t_{a_2}^{n_2}}(x) := \sum_{l=1}^{n_2} [M^{-1}]_{k,l} (\phi_l * \phi^{(t_{c(l)}^l, t_{a_2}^{n_2})})(\mathrm{virt}, x)$$

are the function biorthogonal to $\Psi_{n_2-k}^{n_2, t_{a_2}^{n_2}}(x)$ obtained for the non-extended kernel (i.e., at fixed level and fixed time). In the case of random matrices which we consider, the functions $\Phi_k^{n,t}$, $k = 0, \dots, n-1$, have to be polynomials of degree k because $\det(\Phi_k^{n,t}(x_j))_{1 \leq j,k \leq n}$ must be proportional to $\Delta(x) = \det(x_j^{k-1})_{1 \leq j,k \leq n}$, the Vandermonde determinant. Then, the kernel is simply written as

$$K(n_1, t_{a_1}^{n_1}, x_1; n_2, t_{a_2}^{n_2}, x_2) = -\phi^{(t_{a_1}^{n_1}, t_{a_2}^{n_2})}(x_1, x_2) + \sum_{k=1}^{n_2} \Psi_{n_1-k}^{n_1, t_{a_1}^{n_1}}(x_1) \Phi_{n_2-k}^{n_2, t_{a_2}^{n_2}}(x_2). \tag{A.8}$$

Remark 46. Looking at the proof of the above theorem in [16] one also sees that the time evolutions $\mathcal{T}$ can be taken to be level-inhomogeneous, i.e., the $\mathcal{T}_{t_a^n, a_0^n}$ can be a function of t_a^n, t_0^n and also of the level n. Such a situation occurs for the Wishart matrices case.

The proof of Theorem 44 given in [16] is based on the algebraic formalism of [25]. Another proof can be found in Section 4.4 of [50]. Although we stated Theorem 44 for the case when all sets $\mathfrak{X}_n$ are finite, one easily extends it to a more general setting. Indeed, the determinantal formula for the correlation functions is an algebraic identity, and the limit transition to the case when the $\mathfrak{X}_n$'s are allowed to be countably infinite is immediate, under the assumption that all the sums needed to define the $*$-operations above are absolutely convergent.

A.4. q-Pochhammer symbols, q-hypergeometric functions

Here we collect some identities on q-Pochhammer symbols and q-hypergeometric functions, used for the PASEP. We use the standards as in [65]. The q-Pochhammer symbol is defined by

$$(\mu; q)_\infty = \prod_{k=0}^{\infty}(1-\mu q^k) \quad \text{and} \quad (\mu; q)_n = \prod_{k=0}^{n-1}(1-\mu q^k).$$

They satisfies the following identities:

$$(\mu; q)_n = \frac{(\mu; q)_\infty}{(\mu q^n; q)_\infty}, \quad (\mu; q)_\infty = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2}}{(q; q)_n} \mu^n \tag{A.9}$$

so that in particular $(0; q)_\infty = 1$ and $(1; q)_\infty = 0$.

The q-hypergeometric function is defined by

$${}_r\phi_s\left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} \middle| q; z\right) = \sum_{n=0}^{\infty} \frac{(a_1; q)_n \cdots (a_r; q)_n}{(b_1; q)_n \cdots (b_s; q)_n} \frac{z^n}{(q; q)_n} \left((-1)^n q^{n(n-2)/2}\right)^{1+s-r}. \tag{A.10}$$

In particular, it holds

$${}_{r-1}\phi_s\left(\begin{matrix} a_1, \dots, a_{r-1} \\ b_1, \dots, b_s \end{matrix} \middle| q; z\right) = \lim_{a_r \to \infty} {}_r\phi_s\left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} \middle| q; \frac{z}{a_r}\right). \tag{A.11}$$

The q-Gauss identity is

$${}_2\phi_1\left(\begin{matrix} \alpha, \beta \\ \gamma \end{matrix} \middle| q; \gamma\right) = \frac{(\gamma/\alpha; q)_\infty (\gamma/\beta; q)_\infty}{(\gamma; q)_\infty (\gamma/(\alpha\beta); q)_\infty}. \tag{A.12}$$

A.5. Hermite polynomials

The Hermite polynomial of degree n is denoted here $p_n(x)$. We use the normalization of [65],

$$\int_{\mathbb{R}} \mathrm{d}x\, \mathrm{e}^{-x^2} p_n(x) p_m(x) = \delta_{m,n} \sqrt{\pi} 2^n n!. \tag{A.13}$$

There are two useful integral representations for the Hermite polynomials $p_n(x)$,

$$\begin{aligned} p_n(x) &= \frac{2^n}{\mathrm{i}\sqrt{\pi}} \mathrm{e}^{x^2} \int_{\mathrm{i}\mathbb{R}+\mathrm{e}} \mathrm{d}w\, \mathrm{e}^{w^2-2xw} w^n, \\ p_n(x) &= \frac{n!}{2\pi\mathrm{i}} \oint_{\Gamma_0} \mathrm{d}z\, \mathrm{e}^{-(z^2-2xz)} z^{-(n+1)}, \end{aligned} \tag{A.14}$$

as well as the identities (with $0 < q < 1$) which can be found in [59, 65]

$$\begin{aligned} \frac{1}{\sqrt{\pi(1-q^2)}} \exp\left(-\frac{(x-qy)^2}{1-q^2}\right) &= \mathrm{e}^{-x^2} \sum_{k=0}^{\infty} \frac{p_k(x) p_k(y) q^k}{\sqrt{\pi} 2^k k!}, \\ \int_x^\infty \mathrm{d}y\, \mathrm{e}^{-y^2} p_n(y) &= \mathrm{e}^{-x^2} p_{n-1}(x), \\ p_n(x) &= (-1)^n p_n(-x). \end{aligned}$$

These identities can be useful to rewrite the double integral representation into an expression in terms of Hermite polynomials (as it was made e.g. in Lemma 24 of [20] for the antisymmetric GUE minor kernel).

A.6. Laguerre polynomials

The generalized Laguerre polynomials L_k^p of degree k and order p are polynomials on $\mathbb{R}_+$ defined by

$$L_k^p(x) = \frac{x^{-p}\mathrm{e}^x}{k!} \frac{\mathrm{d}^k}{\mathrm{d}x^k} (x^{p+k}\mathrm{e}^{-x}).$$

They satisfy the orthogonal relation

$$\int_{\mathbb{R}_+} \mathrm{d}x\, x^p \mathrm{e}^{-x} L_k^p(x) L_\ell^p(x) = \frac{(p+k)!}{k!} \delta_{k,\ell} \tag{A.15}$$

and have integral representations,

$$\begin{aligned} L_k^p(x) &= \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma_1} \mathrm{d}w\, \frac{\mathrm{e}^{-x(w-1)} w^{p+k}}{(w-1)^{k+1}}, \\ L_k^p(x) &= \frac{(p+k)!}{k! x^p} \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma_0} \mathrm{d}z\, \frac{\mathrm{e}^{xz}(z-1)^k}{z^{p+k+1}}. \end{aligned}$$

A.7. Harish-Chandra/Itzykson-Zuber formulas

Here we report the Harish-Chandra/Itzykson-Zuber formula as well as its generalization for rectangular matrices.

Let $A = \mathrm{diag}(a_1, \ldots, a_N)$ and $B = \mathrm{diag}(b_1, \ldots, b_N)$ two diagonal $N \times N$ matrices. Let $\mathrm{d}\mu$ denote the Haar measure on the unitary group $\mathcal{U}(N)$. Then,

$$\int_{\mathcal{U}(N)} \mathrm{d}\mu(U) \exp\left(\mathrm{Tr}(AUBU^*)\right) = \prod_{p=1}^{N-1} p! \, \frac{\det\left(\mathrm{e}^{a_i b_j}\right)_{1 \le i,j \le N}}{\Delta(a)\Delta(b)}, \tag{A.16}$$

where $\Delta(a)$ is the Vandermonde determinant of the vector $a = (a_1, \ldots, a_N)$.

The extension to rectangular matrices can be found in section 3.2 of [111] and was derived in [56]. Let A be a complex $N_1 \times N_2$ matrix, B a complex $N_2 \times N_1$ matrix so that the $N_2 \times N_2$ matrices A^*A and BB^* are diagonal with (real positive) eigenvalues $a = (a_1, \ldots, a_{N_2})$ and $b = (b_1, \ldots, b_{N_2})$ respectively. W.l.o.g. we assume $N_1 \ge N_2$. Then,

$$\begin{aligned}\int_{\mathcal{U}(N_2)} \mathrm{d}\mu(U) \int_{\mathcal{U}(N_1)} \mathrm{d}\mu(V) \exp\left(\mathrm{Tr}(AUBV^* + B^*U^*A^*V)\right)\\ = \frac{\prod_{p=1}^{N_2-1} p! \prod_{q=1}^{N_1-1} q!}{\prod_{r=1}^{N_1-N_2-1} r!} \frac{\det\left(I_{N_1-N_2}(2\sqrt{a_i b_j})\right)_{1 \le i,j \le N_2}}{\Delta(a)\Delta(b)\prod_{i=1}^{N_2}(a_i b_i)^{(N_1-N_2)/2}},\end{aligned} \tag{A.17}$$

where I_n is the modified Bessel function defined by

$$I_n(2x) = \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma_0} \mathrm{d}z \, \frac{\mathrm{e}^{x(z+z^{-1})}}{z^{n+1}} = \sum_{k=0}^{\infty} \frac{x^k}{k!} \frac{x^{k+|n|}}{(k+|n|)!}, \tag{A.18}$$

for $n \in \mathbb{Z}$.

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