Universal Toda brackets of ring spectra

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Universal Toda brackets of ring spectra

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Abstract

We construct and examine the *universal Toda bracket* of a highly structured ring spectrum R. This invariant of R is a cohomology class in the Mac Lane cohomology of the graded ring of homotopy groups of R which carries information about R and the category of R-module spectra. It determines for example all triple Toda brackets of R and the first obstruction to realizing a module over the homotopy groups of R by an R-module spectrum.

For periodic ring spectra, we study the corresponding theory of *higher* universal Toda brackets. The real and complex K-theory spectra serve as our main examples.

Contents

1 Introduction

In this thesis in algebraic topology, we study a question about highly structured ring spectra. More specifically, we construct a cohomological invariant γ_R of a ring spectrum R, called its universal Toda bracket, and examine which information about R is encoded in this class.

We use the term *ring spectrum* for what is called an S-algebra in [EKMM97], a symmetric ring spectrum in [HSS00], or an orthogonal ring spectrum in [MMSS01]. These notions are equivalent in an appropriate way. In all three cases, a ring spectrum R is a monoid object in a symmetric monoidal stable model category that has the sphere spectrum as unit and the stable homotopy category as homotopy category. Therefore, R represents a multiplicative (generalized) cohomology theory.

Many of the multiplicative cohomology theories studied by algebraic topologists are known to be represented by such ring spectra. The notion of a ring spectrum is more restrictive than that of a multiplicative cohomology theory, since it requires a spectrum with a multiplication in a stricter sense. This means that the product is associative and unital on the level of the model category, rather than being only associative and unital in the homotopy category.

The crucial advantage of a ring spectrum in the stricter sense is that it behaves much more like an algebraic object, making it possible to define categories of modules and algebras over it in a meaningful way. Starting with algebraic K-theory of spaces and topological Hochschild homology, which emphasized the need of the invention of strict ring spectra, many concepts from algebra are now successfully applied to the study of ring spectra, including André-Quillen cohomology, Morita theory, or Galois theory. Moreover, building partly on the concept of ring spectra, the new areas of motivic homotopy theory and homotopical algebraic geometry lead to an exchange of ideas between homotopy theory and algebraic geometry.

One basic algebraic feature of a ring spectrum R is that there is an associated module category Mod-R, which is a stable model category and has a triangulated homotopy category Ho(Mod-R). The category Ho(Mod-R) is the analog to the derived category of an ordinary ring. For an object M of Ho(Mod-R), its stable homotopy groups $\pi_*(M)$ form a graded $\pi_*(R)$-module. One of our aims is to achieve a better understanding of the resulting functor $\pi_*(-)\colon \mathrm{Ho}(\text{Mod-}R) \to \text{Mod-}\pi_*(R)$. Particularly, we want to examine under which conditions a $\pi_*(R)$-module M is *realizable*, that is, arises as the homotopy groups of an R-module spectrum.

There is an obstruction theory associated to this problem, with obstructions $\kappa_i(M) \in \mathrm{Ext}^{i,2-i}_{\pi_*(R)}(M,M)$ for $i \geq 3$. The first obstruction $\kappa_3(M)$ is always defined and unique. It vanishes if and only if M is a retract of a realizable module. For $i \geq 4$, $\kappa_i(M)$ is only defined if $\kappa_{i-1}(M)$ vanishes, and there are choices involved. We want to understand these obstructions in a systematic way and show how they depend on the ring spectrum structure of R.

The obstruction theory in fact works in the more general setup of a triangulated category $\mathcal{T}$, where it can be used to find out whether a module M over the graded ring of endomorphisms $\mathcal{T}(N,N)_*$ of a compact object N can be realized as $\mathcal{T}(N,X)_*$ for some object X of $\mathcal{T}$. An algebraic instance of this problem is to realize a module over the cohomology of a differential graded algebra A as the cohomology of a differential graded A-module.

This analogy between ring spectra and differential graded algebras is one reason why the following result serves as an algebraic motivation for our work. For a differential graded algebra A over a field k, Benson, Krause, and Schwede [BKS04] study a class $\gamma_A \in \mathrm{HH}_k^{3,-1}(H^*(A))$ in the Hochschild cohomology of the cohomology ring of A. It determines by evaluation all triple (matric) Massey products of $H^*(A)$. Moreover, by a map $\mathrm{HH}_k^{3,-1}(H^*(A)) \to \mathrm{Ext}^{3,-1}_{H^*(A)}(M,M)$ depending on M, it determines the first realizability obstruction $\kappa_3(M)$ for every $H^*(A)$-module M.

We develop a similar theory for ring spectra. Though the obstruction theory for the realizability problem takes completely place in triangulated categories, the definition of a cohomology class with that property needs information from an underlying 'model'. In the case of the differential graded algebra A, the A_∞-structure of $H^*(A)$ can be used to define γ_A. In the case of ring spectra, there is no such A_∞-structure. The appropriate replacement will be to use that choosing representatives in the model category of maps in the homotopy category is in general not associative with respect to the composition. This non associativity leads to obstructions which assemble to a cohomology class depending only on the ring spectrum.

The formulation of our main results uses Mac Lane cohomology groups, denoted by HML. We define this cohomology theory for graded rings using the normalized cohomology of categories. Its ungraded version is equivalent to Mac Lane's original definition. This theory is, for various reasons, an appropriate replacement of the Hochschild cohomology group in the result of [BKS04]. One reason is that one can, similar to Hochschild cohomology, evaluate a representing cocycle on a sequence of composable maps. If the sequence of maps is a complex, it makes sense to ask the evaluation to be an element of the Toda bracket of the complex.

One main result will be

Theorem 5.1.1. *Let R be a ring spectrum. Then there exists a well defined cohomology class $\gamma_R \in \mathrm{HML}^{3,-1}(\pi_*(R))$ which, by evaluation, determines all triple matric Toda brackets of $\pi_*(R)$. For a $\pi_*(R)$-module M which admits a resolution by finitely generated free $\pi_*(R)$-modules, the product $\mathrm{id}_M \cup \gamma_R \in \mathrm{Ext}^{3,-1}_{\pi_*(R)}(M,M)$ is the first realizability obstruction $\kappa_3(M)$.*

The term *universal Toda bracket* for such a cohomology class, as well as the usage of cohomology of categories, are motivated by Baues' study of universal Toda brackets for subcategories of the homotopy category of topological spaces [Bau97, BD89].

One interesting example for this theorem is the real K-theory spectrum KO. As this spectrum has non vanishing triple Toda brackets, its universal Toda bracket is nontrivial. Moreover, the obstructions determined by γ_{KO} detect non realizable $\pi_*(KO)$-modules. As we will discuss in Remark 5.2.3, this contradicts a claim of Wolbert [Wol98, Theorems 20 and 21].

Many examples of ring spectra have the property that their ring of homotopy groups is concentrated in degrees divisible by n for some $n \geq 2$. In this case, all realizability obstructions κ_3 vanish for degree reasons. The first realizability obstruction not vanishing for degree reasons will be determined by a *higher* universal Toda bracket, which we construct in Theorem 5.1.2.

The higher universal Toda bracket of a ring spectrum R becomes particularly nice if the homotopy groups of R form a graded Laurent polynomial ring, as the class then arises as an element of an ungraded Mac Lane cohomology group.

Corollary 5.1.4. *Let R be a ring spectrum such that $\pi_*(R) \cong (\pi_0(R))[u^{\pm 1}]$ with u a central unit in degree n. Then there is a well defined cohomology class $\gamma_R^{n+2} \in \mathrm{HML}^{n+2}(\pi_0(R))$ in the ungraded Mac Lane cohomology of $\pi_0(R)$. It determines, by evaluation, all $(n+2)$-fold Toda brackets of complexes of $(n+2)$ composable maps between finitely generated free $\pi_*(R)$-modules which are concentrated in degrees divisible by n. For a $\pi_*(R)$-module M which admits a resolution by such modules, the class γ_R^{n+2} determines the unique realizability obstruction $\kappa_{n+2}(M)$ not vanishing for degree reasons.*

The ungraded Mac Lane cohomology groups are equivalent to topological Hochschild cohomology or Ext-groups in certain functor categories. Their computation is known in relevant cases. As an example, we consider the universal Toda bracket of the complex K-theory spectrum KU. Since $\pi_*(KU) \cong \mathbb{Z}[u^{\pm 1}]$ with u of degree 2, its universal Toda bracket is an element of $\mathrm{HML}^4(\mathbb{Z}) \cong \mathbb{Z}/2$, and it turns out to be the non-zero element.

The calculation of γ_{KU}^4 will be a consequence of a different kind of information which is detected by universal Toda brackets. Associated to a ring spectrum R and an integer $q \geq 1$, there is a path connected space $B\,\mathrm{GL}_q\,R$, which is an important building block for the algebraic K-theory of R. If $\pi_*(R)$ is concentrated in degrees divisible by n for some $n \geq 1$, we know that $\pi_k(B\,\mathrm{GL}_q\,R) = 0$ for $1 < k < n+1$.

Theorem 5.1.5. *Let R be a ring spectrum such that $\pi_*(R)$ is concentrated in degrees divisible by n for some $n \geq 1$. For $q \geq 1$, the restriction map*

$$\mathrm{HML}^{n+2,-n}_{n-\mathrm{sp}}(\pi_*(R)) \to \mathrm{HML}^{n+2}(\pi_0(R), \pi_n(R)) \to H^{n+2}(\pi_1(B\,\mathrm{GL}_q\,R), \pi_{n+1}(B\,\mathrm{GL}_q\,R))$$

sends the universal Toda bracket γ_R^{n+2} of R to the first k-invariant of the space $B\,\mathrm{GL}_q\,R$.

This relation between the universal Toda bracket of R and the spaces $B\,\mathrm{GL}_q\,R$ will be used interpret the vanishing of γ_R^{n+2} in terms of the algebraic K-theory of R.

Organization The main results, as stated in the introduction, can be found in the fifth and last section. There we also discuss the examples.

In the second section, we briefly review cohomology of categories and Mac Lane cohomology, including a version for graded rings. We also introduce a map relating the Mac Lane cohomology of a graded ring to Ext-groups over it.

In the third section, we explain the general obstruction theory for the realizability problem described above. Furthermore, we review the definition of (higher) Toda brackets in triangulated categories and explain how Toda brackets determine realizability obstructions. All results of this sections are formulated in terms of triangulated categories.

The fourth section is the technical backbone of this thesis. Using the framework of stable topological model categories, we give a general construction of a universal Toda bracket and show how it is related to k-invariants of certain classifying spaces. In the course of the construction, we encounter different definitions of Toda brackets, which we discuss in Paragraph 4.3.

Notation and conventions The letter $\mathcal{T}$ will always denote a triangulated category. We write $[1]$ for the shift in $\mathcal{T}$, and $[n]$ for the n-fold shift if $n \in \mathbb{Z}$. By setting $\mathcal{T}(X,Y)_i = \mathcal{T}(X[i],Y)$, we obtain a graded abelian group $\mathcal{T}(X,Y)_*$ of homomorphisms from X to Y in $\mathcal{T}$. The term $\mathcal{T}(X,Y)$ without a decoration is its degree 0 part.

When Λ is a graded ring, we also denote its shifts by $[n]$, i.e., $(\Lambda[n])_i = \Lambda_{i-n}$. This is compatible with the shift in the triangulated category, as we have $(\mathcal{T}(X,X)_*)[n] \cong \mathcal{T}(X,X[n])_* \cong \mathcal{T}(X[-n],X)_*$. Sometimes we write $\Lambda(M,M')$ for $\mathrm{Hom}_\Lambda(M,M')$. The Ext-groups of modules over a graded ring Λ are bigraded by setting $\mathrm{Ext}^{s,t}_\Lambda(M,M') = \mathrm{Ext}^s_\Lambda(M,M'[t])$.

For $n \geq 1$, a graded abelian group or a graded ring is called *n-sparse* if it is concentrated in degrees divisible by n. A full subcategory $\mathcal{U}$ of a triangulated category $\mathcal{T}$ is called *n-split* if for each pair of objects X and Y in $\mathcal{U}$, the graded abelian group $\mathcal{T}(X,Y)_*$ is n-sparse.

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2 Mac Lane cohomology

In the first paragraph of this section we recall some facts about Mac Lane cohomology, including a definition and some results about computations. A good background for this is [Lod98, Chapter 13]. The second paragraph is concerned with a map from Mac Lane cohomology to Ext-groups which we will use for next section's Theorem 3.4.5.

2.1 Cohomology of categories and Mac Lane cohomology

Let $\mathcal{C}$ be a small category. A *$\mathcal{C}$-bimodule* is a functor $D\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{A}b$. For a map $f\colon X \to Y$ in $\mathcal{C}$, we denote the abelian group $D(X,Y)$ by D_f. On these abelian groups, the $\mathcal{C}$-bimodule structure induces actions $g^*\colon D_f \to D_{fg}$ and $h_*\colon D_f \to D_{hf}$ for maps $g\colon X' \to X$, $h\colon Y \to Y'$, and $f\colon X \to Y$. If A is a ring and $\mathcal{C}$ is the category of A-modules, the bifunctor $\mathrm{Hom}_A(-,-)$ provides an example for a $\mathcal{C}$-bimodule.

In order to define the cohomology of categories, we introduce the following cochain complex $C^*(\mathcal{C},D)$ associated to a category $\mathcal{C}$ and a $\mathcal{C}$-bimodule D: we set

$$C^n(\mathcal{C},D) = \{c\colon N_n(\mathcal{C}) \to \coprod_{g\in \mathrm{Mor}(\mathcal{C})} D_g \mid c(g_1,\dots,g_n) \in D_{g_1\cdots g_n}\}$$

for $n \geq 1$ and

$$C^0(\mathcal{C},D) = \{c\colon \mathrm{Ob}(\mathcal{C}) \to \coprod_{X\in \mathrm{Ob}(\mathcal{C})} D_{\mathrm{id}_X} \mid c(X) \in D_{\mathrm{id}_X}\}$$

for $n = 0$. Here the simplicial set $N(\mathcal{C})$ is the nerve of the category $\mathcal{C}$, so an element $(g_1,\dots,g_n) \in N_n(\mathcal{C})$ is a sequence

$$X_n \xrightarrow{g_n} X_{n-1} \xrightarrow{g_{n-1}} \dots \xrightarrow{g_2} X_1 \xrightarrow{g_1} X_0$$

of n composable maps in $\mathcal{C}$.

The abelian group structure on $C^n(\mathcal{C},D)$ is given by the pointwise addition in D_g. For $n > 1$, the differential $\delta\colon C^{n-1}(\mathcal{C},D) \to C^n(\mathcal{C},D)$ is defined by

$$\begin{aligned}(\delta c)(g_1,\dots,g_n) =& (g_1)_* c(g_2,\dots,g_n) + \sum_{i=1}^{n} (-1)^i c(g_1,\dots,g_ig_{i+1},\dots,g_n) \\ &+ (-1)^{n+1}(g_n)^* c(g_1,\dots,g_{n-1}).\end{aligned}$$

For $n = 1$, the differential of $c \in C^0(\mathcal{C},D)$ evaluated on $g_1\colon X_1 \to X_0$ is $(\delta c)(g_1) = (g_1)_* c(X_1) - (g_1)^* c(X_0)$. It is easy to verify $\delta^2 = 0$.

Definition 2.1.1. [BW85, Definition 1.4] The *cohomology $H^*(\mathcal{C},D)$ of the category $\mathcal{C}$* with coefficients in the $\mathcal{C}$-bimodule D is defined to be the cohomology of the cochain complex $C^*(\mathcal{C},D)$.

There is a normalized version of the cohomology of categories. We call a category *pointed* if it has a preferred zero object, i.e., an object $*$ which is both initial and terminal. A *zero morphism* in a pointed category is a map which factors through the zero object. If $\mathcal{C}$ is a pointed category, we call a $\mathcal{C}$-bimodule D *normalized* if $D(*,X) = 0 = D(X,*)$ holds for all objects X in $\mathcal{C}$.

For a pointed category $\mathcal{C}$ and a normalized $\mathcal{C}$-bimodule D, we consider the subgroup

$$\overline{C}^n(\mathcal{C}, D) = \{c \in C^n(\mathcal{C}, D) | c(g_1, \ldots, g_n) = 0 \text{ if } g_i \text{ is zero for some } i\}$$

of normalized cochains in $C^n(\mathcal{C}, D)$. Using that D is normalized, it is easy to see that the differential of $C^*(\mathcal{C}, D)$ restricts to $\overline{C}^*(\mathcal{C}, D)$. Therefore, $\overline{C}^*(\mathcal{C}, D)$ is a subcomplex of $C^*(\mathcal{C}, D)$.

Proposition 2.1.2. *[BD89, Theorem 1.1] Let $\mathcal{C}$ be a pointed category and let D be a normalized $\mathcal{C}$-bimodule. Then the inclusion $\overline{C}^*(\mathcal{C}, D) \to C^*(\mathcal{C}, D)$ induces an isomorphism in cohomology.*

When working with the cohomology of a pointed category with coefficients in a normalized bimodule, we can therefore henceforth assume that it arises as the cohomology of the normalized chain complex. That is, representing cocycles can be assumed to be normalized.

Cohomology of categories has good naturality properties. For a functor $F\colon \mathcal{C} \to \mathcal{D}$ and a $\mathcal{D}$-bimodule D, there is an induced $\mathcal{C}$-bimodule F^*D, and F induces an obvious map $F^*\colon C^*(\mathcal{D}, D) \to C^*(\mathcal{C}, F^*D)$.

Proposition 2.1.3. *[BW85, Theorem 1.11] In the situation above, F induces a homomorphism $H^*(\mathcal{D}, D) \to H^*(\mathcal{C}, F^*D)$. If F is an equivalence of categories, this map is an isomorphism.*

For a ring A, we denote the category of finitely generated free right A-modules by $F(A)$. To avoid set theoretic problems, we assume $F(A)$ to be small, i.e., we require it to contain only one element from each isomorphism class of objects. The category $F(A)$ is pointed, and for an A-bimodule M, the functor $\mathrm{Hom}_A(-, - \otimes_A M)$ is a normalized $F(A)$-module.

Definition 2.1.4. Let A be a ring and let M be an A-bimodule. The *Mac Lane cohomology* of A with coefficients in M is defined by

$$\mathrm{HML}^s(A, M) = H^s(F(A), \mathrm{Hom}_A(-, - \otimes_A M)).$$

If M equals A, we adopt the convention $\mathrm{HML}^s(A) = \mathrm{HML}^s(A, A)$.

Remark 2.1.5. In this definition of Mac Lane cohomology, we do not necessarily need to take the category $F(A)$. If we replace $F(A)$ by any small full additive subcategory $\mathcal{C}$ of Mod-A containing $F(A)$, then the induced restriction map on the cohomology groups is an isomorphism [JP91, §2 and Corollary 3.11].

For some fixed infinite ordinal, the full subcategory of Mod-A containing one A-module from each isomorphism class of free A-modules which have rank smaller than the fixed ordinal provides an example for such a $\mathcal{C}$. The possibility of such an enlargement will become relevant in our applications (see Remark 3.4.6 and Remark 5.1.3).

We will also need an equivalent characterization of this cohomology theory, which is due to Jibladze and Pirashvili [JP91]. For a ring A, let $\mathcal{F}(A)$ be the category of functors from $F(A)$ to Mod-A. This is an abelian category with all structure defined object-wise. Examples for objects in $\mathcal{F}(A)$ are the inclusion functor $I\colon F(A) \to$ Mod-A or the functor $- \otimes_A M\colon F(A) \to$ Mod-A with M an A-bimodule.

Proposition 2.1.6. *[JP91, Corollary 3.11] Let $U, T \in \mathcal{F}(A)$ be functors such that U takes values in projective A-modules. Then*

$$H^*(F(A), \mathrm{Hom}_A(U(-), T(-))) \cong \mathrm{Ext}_{\mathcal{F}(A)}(U, T).$$

Corollary 2.1.7. *For a ring A and an A-bimodule M, there is an isomorphism*

$$\mathrm{HML}^*(A, M) \cong \mathrm{Ext}^*_{\mathcal{F}(A)}(I, - \otimes_A M).$$

Remark 2.1.8. Mac Lane cohomology was originally defined by Mac Lane in 1956 [ML57]. The equivalence of his definition to the one in terms of cohomology of categories and with the Ext-groups in functor categories was established by Jibladze and Pirashvili [JP91].

Mac Lane cohomology is also isomorphic to *topological Hochschild cohomology*. The homological version of the latter was invented by Bökstedt [Bök85a]. It should be thought of as Hochschild homology with the sphere spectrum serving as the ground ring. A good account for the equivalence of different definitions of topological Hochschild homology which make this slogan precise is [Shi00]. The equivalence of topological Hochschild homology and Mac Lane homology was proved by Pirashvili and Waldhausen [PW92].

A reference for the equivalence between topological Hochschild cohomology and Mac Lane cohomology is [Sch01, Theorem 6.7]. Here it is important to insist on the Mac Lane cohomology having coefficients in a bimodule, as the more general case of Mac Lane cohomology groups with coefficients in an object of $\mathcal{F}(A)$ is not necessarily isomorphic to the corresponding topological Hochschild cohomology group. This will not become relevant for us, as we only consider Mac Lane cohomology groups with coefficients in a bimodule.

We call a functor U in $\mathcal{F}(A)$ reduced if it satisfies $U(0) = 0$. A functor $U \in \mathcal{F}(A)$ is constant if it sends all objects of $F(A)$ to the same A-module M and all morphisms to id_M. If M is an A-module, we denote the constant functor with value M also by M. The following lemma will be needed later.

Lemma 2.1.9. *Let A be a ring, let T be an object in $\mathcal{F}(A)$, let M be an A-module, and let P be a projective A-module. For $i \geq 1$, there is an isomorphism*

$$H^i(F(A), \mathrm{Hom}_A((I \oplus P)(-), (T \oplus M)(-))) \cong H^i(F(A), \mathrm{Hom}_A(-, T(-))).$$

Proof. By Proposition 2.1.6, this translates to a statement about Ext-groups in $\mathcal{F}(A)$. Since $A = A[F(A)(0, -)]$ is projective in $\mathcal{F}(A)$ [JP91, Proposition 2.5], the constant functor P is projective in $\mathcal{F}(A)$ as well. Together with the additivity of $\mathrm{Ext}_{\mathcal{F}(A)}$ in the first variable, this yields

$$\mathrm{Ext}^i_{\mathcal{F}(A)}((I \oplus P)(-), (T \oplus M)(-)) \cong \mathrm{Ext}^i_{\mathcal{F}(A)}(I, (T \oplus M)(-))$$

for $i \geq 1$.

There is only the zero morphism from a reduced functor in $\mathcal{F}(A)$ to a constant functor. Since every reduced functor admits a projective resolution by reduced functors, $\mathrm{Ext}^*_{\mathcal{F}(A)}(I, M)$ vanishes. This implies $\mathrm{Ext}^*_{\mathcal{F}(A)}(I, (T \oplus M)(-)) \cong \mathrm{Ext}^*_{\mathcal{F}(A)}(I, T)$. □

We state some results about calculations of Mac Lane cohomology groups. The formulation uses the product structure of the graded ring $\mathrm{HML}^*(A)$, which is the Yoneda product on $\mathrm{Ext}^*_{\mathcal{F}(A)}(I,I)$.

Theorem 2.1.10. *([FLS94, FP98]) There are isomorphisms of graded rings*

$$\mathrm{HML}^*(\mathbb{F}_p) \cong (\mathbb{Z}/p\mathbb{Z})[e_0,\dots,e_i,\dots]/(e_i^p, i \geq 0) \text{ with } |e_i| = 2p^i \text{ and}$$
$$\mathrm{HML}^*(\mathbb{Z}) \cong \Gamma(x)/(x) \text{ with } |x| = 2,$$

where $\Gamma(x)$ is the free divided power algebra on one generator x in degree 2. Regarding only the additive structure, this means

$$\mathrm{HML}^k(\mathbb{F}_p) \cong \begin{cases} \mathbb{Z}/p\mathbb{Z} & k \text{ even} \\ 0 & k \text{ odd} \end{cases} \qquad \text{and} \qquad \mathrm{HML}^k(\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/i\mathbb{Z} & k = 2i \\ 0 & k \text{ odd} \end{cases}.$$

Proof. See [FLS94] for the statement about $\mathbb{F}_p$, and [FP98] for the statement about $\mathbb{Z}$. The cohomology groups without the multiplicative structure can also be deduced from earlier results of Bökstedt [Bök85b] or Breen [Bre78]. □

We will also need a graded version of Mac Lane cohomology. In the sequel we will call a graded ring, a graded abelian group, or a graded module *n-sparse* if it is concentrated in degrees divisible by n. If Λ is a graded ring, the morphisms between graded Λ-modules M and N form a graded abelian group by setting

$$\mathrm{Hom}^i_\Lambda(M,N) = \mathrm{Hom}_\Lambda(M,N)_{-i} = \mathrm{Hom}_\Lambda(M[-i],N).$$

We call a full subcategory $\mathcal{C}$ of Mod-Λ n-split if for each pair of objects M, N in $\mathcal{C}$, the graded abelian group $\mathrm{Hom}_\Lambda(M,N)_*$ is n-sparse.

For a graded ring Λ, let $F(\Lambda)$ be the category of finitely generated free graded right Λ-modules. This means that the objects of $F(\Lambda)$ are finite sums of shifted copies of the free module of rank 1. If the ring Λ is n-sparse for some $n \geq 1$, we will also consider $F(\Lambda, n)$, the full subcategory of $F(\Lambda)$ given by the n-sparse Λ-modules. For $n = 1$, the additional condition on objects in $F(\Lambda, 1)$ is empty, hence $F(\Lambda, 1) = F(\Lambda)$. The category $F(\Lambda, n)$ is an example for an n-split subcategory of Mod-Λ.

Definition 2.1.11. Let Λ be an n-sparse graded ring, and let M be a graded right Λ-module. Then the *graded n-split Mac Lane cohomology* of Λ with coefficients in M is defined by

$$\mathrm{HML}^s_{n-\mathrm{sp}}(\Lambda, M) = H^s(F(\Lambda,n), \mathrm{Hom}_\Lambda(-, - \otimes_\Lambda M)).$$

If M equals $\Lambda[t]$, a t-fold shift of Λ for some $t \in \mathbb{Z}$, we adopt the convention

$$\mathrm{HML}^{s,t}_{n-\mathrm{sp}}(\Lambda) = \mathrm{HML}^s_{n-\mathrm{sp}}(\Lambda, \Lambda[t]).$$

For $n = 1$, we drop '1−sp' from the notation and write just $\mathrm{HML}^s(\Lambda, M)$ or $\mathrm{HML}^{s,t}(\Lambda)$.

The graded Mac Lane cohomology is related to the ungraded theory.

Lemma 2.1.12. *Let Λ be an n-sparse graded ring. Then there is a restriction map*

$$\mathrm{HML}^{*,-n}_{n-\mathrm{sp}}(\Lambda) \to \mathrm{HML}^*(\Lambda_0, \Lambda_n).$$

Proof. We apply Proposition 2.1.3 to the functor $- \otimes_{\Lambda_0} \Lambda \colon F(\Lambda_0) \to F(\Lambda, n)$. The resulting restriction map on Mac Lane cohomology groups has values in the Mac Lane cohomology of Λ_0 with coefficients in Λ_n, as

$$(- \otimes_{\Lambda_0} \Lambda)^* \operatorname{Hom}_\Lambda(-, - \otimes_\Lambda \Lambda[-n]) \cong \operatorname{Hom}_{\Lambda_0}(-, - \otimes_{\Lambda_0} \Lambda_n).$$

□

Lemma 2.1.13. *Let Λ be a graded ring. Suppose that Λ has a central unit u of degree n, that is, a homogeneous element u of degree n which is a unit and which is central in the graded sense. Then the restriction map of the last lemma induces an isomorphism* $\mathrm{HML}^{*,-n}_{n-\mathrm{sp}}(\Lambda) \xrightarrow{\cong} \mathrm{HML}^*(\Lambda_0)$.

Proof. In this case, $- \otimes_{\Lambda_0} \Lambda \colon F(\Lambda_0) \to F(\Lambda, n)$ is an equivalence of categories. Hence it induces an isomorphism. Since u is central, Λ_0 is isomorphic to Λ_n as a Λ_0-bimodule, and we have $\mathrm{HML}^*(\Lambda_0, \Lambda_n) \cong \mathrm{HML}^*(\Lambda_0)$. □

Cohomology of categories, and therefore Mac Lane cohomology, is related to group cohomology. For an object X in a category $\mathcal{C}$, we denote its group of automorphisms by $\mathrm{Aut}(X)$. The category with a single object X and $\mathrm{Hom}(X, X) = \mathrm{Aut}(X)$ is denoted by $\mathcal{A}ut(X)$. It comes with a canonical inclusion functor $\mathcal{A}ut(X) \to \mathcal{C}$.

If D is an $\mathcal{A}ut(X)$-bimodule, the automorphism group $\mathrm{Aut}(X)$ acts via the conjugation action $gx = (g^{-1})^*(g_*(x))$ from the left on the abelian group $D(X, X)$.

Proposition 2.1.14. *Let $\mathcal{C}$ be a category, let X be an object of $\mathcal{C}$, and let D be a $\mathcal{C}$-bimodule. Then the inclusion functor $F \colon \mathcal{A}ut(X) \to \mathcal{C}$ induces a restriction map*

$$\Phi \colon H^*(\mathcal{C}, D) \to H^*(\mathcal{A}ut(X), F^*D) \xrightarrow{\cong} H^*(\mathrm{Aut}(X), D(X, X))$$

from the cohomology of $\mathcal{C}$ with coefficients in D to the cohomology of the group $\mathrm{Aut}(X)$ *with coefficients in the* $\mathrm{Aut}(X)$*-module $D(X, X)$.*

Proof. The first map is provided by Proposition 2.1.3. The second map is analogous to the Mac Lane isomorphism between the Hochschild homology of a group ring and group homology [Lod98, Proposition 7.4.2]. It is induced by an isomorphism φ between the cochain complex $C^*(\mathcal{A}ut(X), F^*D)$ to the cochain complex computing $H^*(\mathrm{Aut}(X), D(X, X))$ obtained from the bar resolution. On a cochain c, the isomorphism is given by

$$(\varphi(c))(g_1, \ldots, g_n) = (g_n^{-1} \cdots g_1^{-1})^* c(g_1, \ldots, g_n).$$

□

When A is a ring and M is an A-bimodule, we write as usual $\mathrm{GL}_q A$ for the group of invertible $(q \times q)$-matrices, which acts on the abelian group $\mathrm{Mat}_q M$ of all $(q \times q)$-matrices with entries in M by conjugation. The map of the last proposition specializes to Mac Lane cohomology for graded and ungraded rings.

Corollary 2.1.15. *Let Λ be an n-sparse graded ring, let A be a ring, and let M be an A-bimodule. For $q \geq 1$, there are restriction maps*

$$\mathrm{HML}^{*,-n}_{n-\mathrm{sp}}(\Lambda) \to H^*(\mathrm{GL}_q \Lambda_0, \mathrm{Mat}_q \Lambda_n) \quad \text{and} \quad \mathrm{HML}^*(A, M) \to H^*(\mathrm{GL}_q A, \mathrm{Mat}_q M).$$

If $A = \Lambda_0$ and $M = \Lambda_n$, the first map factors through the second map and the restriction map of Lemma 2.1.12, i.e.,

$$\mathrm{HML}^{*,-n}_{n-\mathrm{sp}}(\Lambda) \to \mathrm{HML}^*(\Lambda_0, \Lambda_n) \to H^*(\mathrm{GL}_q \Lambda_0, \mathrm{Mat}_q \Lambda_n).$$

Proof. We have $\mathrm{Hom}_\Lambda(\Lambda, \Lambda) \cong \mathrm{Hom}_{\Lambda_0}(\Lambda_0, \Lambda_0)$ as in both cases a morphism is determined by the image of 1 in Λ_0. This implies that the automorphism group of Λ^q in the category $F(\Lambda, n)$ is $\mathrm{GL}_q \Lambda_0$. The group $\mathrm{Mat}_q \Lambda_n$ arises in a similar way as the bifunctor $\mathrm{Hom}_{\Lambda_0}(-, - \otimes_{\Lambda_0} \Lambda_n)$ in Lemma 2.1.12.

The factorization is a consequence of $\mathcal{A}ut(\Lambda) \to F(\Lambda, n)$ factoring through $F(\Lambda_0)$. □

Remark 2.1.16. The map of the last corollary is hard to describe in examples, as the source and especially the target are cohomology groups which are very difficult to compute even for not too complicated rings.

Later we will encounter the case of the map $\mathrm{HML}^4(\mathbb{Z}) \to H^4(\mathrm{GL}_q \mathbb{Z}, \mathrm{Mat}_q \mathbb{Z})$. Here we know that for $q = 1$, both cohomology groups are isomorphic to $\mathbb{Z}/2$. Nevertheless, it turns out that the map is trivial. This can be verified by translating the map into the definition of $\mathrm{HML}^4(\mathbb{Z})$ in terms of $\mathrm{Ext}^4_{\mathcal{F}(\mathbb{Z})}(I, I)$. In this description, one can represent the generator by an explicit extension which becomes trivial when restricting it to an extension of $\mathbb{Z}[\mathbb{Z}/2]$-modules.

Though we give only the trivial map as an example here, our application of this map in Theorem 5.1.5 shows that it carries interesting information in general. As we will see in Remark 5.2.6, the vanishing of the map in the case $q = 1$ and $A = M = \mathbb{Z}$ will have a topological interpretation in terms of a certain k-invariant.

2.2 A cup product

In this section, Λ denotes a graded ring, and we work in the category of graded right Λ-modules. Most of the time, Ext-groups are understood in the sense of Yoneda, i. e. , Ext-classes are represented by exact sequences of Λ-modules (see for example [ML67, Chapter III] for details). Shifting of modules gives rise to a bigrading on Ext-groups, that is, $\mathrm{Ext}^{s,t}(M, N) = \mathrm{Ext}^s(M, N[t])$.

If E is a graded k-algebra over a field k, the isomorphism

$$\mathrm{HH}^{s,t}_k(E) \cong \mathrm{Ext}^{s,t}_{E-\mathrm{Mod}-E}(E, E)$$

provides an interpretation of Hochschild cohomology as bimodule Ext-groups. The left derived functor of the tensor product of a right module with a bimodule therefore gives a bilinear map

$$- \otimes^L - \colon \mathrm{Hom}_E(P, Q) \times \mathrm{HH}^{s,t}_k(E) \to \mathrm{Ext}^{s,t}_E(P, Q).$$

Next we construct a similar map with Hochschild cohomology replaced by Mac Lane cohomology. We think of it as an analogon to the left derived tensor product.

Construction 2.2.1. Let Λ be a graded ring which is concentrated in degrees divisible by n for some $n \geq 1$. Let M and N be graded Λ-modules such that M admits a resolution by objects in $F(\Lambda, n)$. Then there is a well defined map

$$\mathrm{Hom}_\Lambda(M, N) \times \mathrm{HML}^{s,t}_{n-\mathrm{sp}}(\Lambda) \to \mathrm{Ext}^{s,t}_\Lambda(M, N), \quad (f, \gamma) \mapsto f \cup \gamma$$

which we refer to as the *cup product*. It is bilinear and natural in the sense that $(gf)\cup\gamma = g_*(f\cup\gamma)$ holds for composable maps of Λ-modules f and g.

In order to define the cup product, we first choose a resolution

$$\cdots \to M_n \xrightarrow{\lambda_n} M_{n-1} \xrightarrow{\lambda_{n-1}} \dots \xrightarrow{\lambda_1} M_0 \xrightarrow{\lambda_0} M$$

of M by finitely generated free graded n-sparse Λ-modules M_i and a normalized cocycle

$$c \in \overline{C}^s(F(\Lambda,n), \mathrm{Hom}_\Lambda(-, - \otimes_\Lambda \Lambda[t]))$$

representing the cohomology class $\gamma \in \mathrm{HML}^{s,t}_{n-\mathrm{sp}}(\Lambda)$.

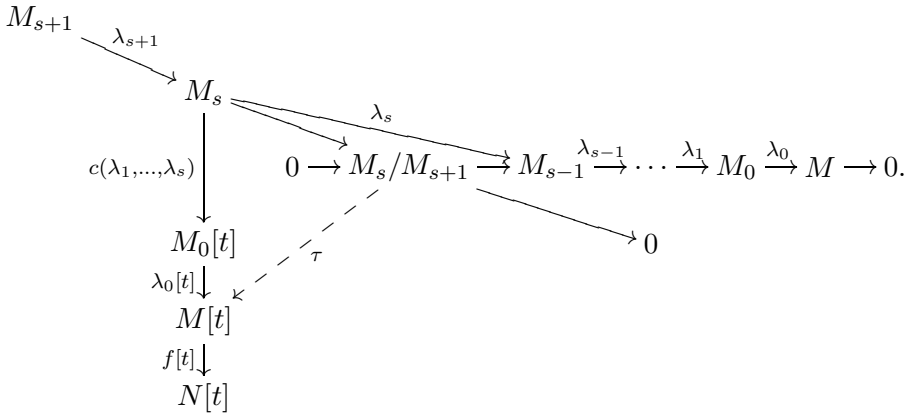
Since $\delta(c) = 0$ and the λ_i form a resolution, we have

$$\begin{aligned} 0 &= (\delta c)(\lambda_1, \lambda_2, \dots, \lambda_s, \lambda_{s+1}) \\ &= (\lambda_1)_* c(\lambda_2, \dots, \lambda_s, \lambda_{s+1}) + (-1)^{s+1}(\lambda_{s+1})^* c(\lambda_1, \lambda_2, \dots, \lambda_s) \\ &= \lambda_1[t] c(\lambda_2, \dots, \lambda_s, \lambda_{s+1}) + (-1)^{s+1} c(\lambda_1, \lambda_2, \dots, \lambda_s)\lambda_{s+1} \end{aligned}$$

and therefore

$$\lambda_0[t] c(\lambda_1, \dots, \lambda_s)\lambda_{s+1} = (-1)^s \lambda_0[t]\lambda_1[t] c(\lambda_2, \dots, \lambda_{s+1}) = 0.$$

This implies that the dotted arrow τ in the following diagram exists



Here we write M_s/M_{s+1} for the Λ-module $\mathrm{coker}\,\lambda_{s+1} \cong \ker\lambda_{s-1}$.

If $\Theta \in \mathrm{Ext}^{s,0}_\Lambda(M, M_s/M_{s+1})$ denotes the Yoneda class of the extension

$$0 \to M_s/M_{s+1} \to M_s \to \cdots \to M_0 \to M \to 0,$$

we define $f\cup\gamma$ to be $(-1)^{\frac{(n+2)(n+1)}{2}+1}((f[t])\tau)_*(\Theta) \in \mathrm{Ext}^{s,t}_\Lambda(M,N)$. The mysterious sign is built in to cancel out with another sign which will arise in proof of Theorem 3.4.5.

The bilinearity and the naturality with respect to compositions of maps are obvious from the definition and the usual bifunctor properties of the Yoneda Ext-groups [ML67, Chapter III]. In Lemma 2.2.4 and Lemma 2.2.5 we will show that the Ext-class of $f\cup\gamma$ does not depend on the choice of the cocycle representing γ and the chosen resolution of M.

Remark 2.2.2. The analogy between the cup product and the derived tensor product in Hochschild cohomology becomes clearer in the definition of Mac Lane cohomology in terms of Ext-groups in functor categories. We sketch the ungraded case.

Let A be a discrete ring, and let M be an A-module. By Remark 2.1.5, we can enlarge the category $F(A)$ to a small additive category $\mathcal{C}$ which contains the module M. If we represent a cohomology class in $\mathrm{HML}^n(A)$ by an extension of functors from $\mathcal{C}$ to Mod-A, we can evaluate it on the module M to get an element of $\mathrm{Ext}^n_A(M, M)$.

To prove that this coincides with the map we described above, one has to go into the construction of the isomorphism between Ext-groups in $\mathcal{F}(A)$ and the cohomology of the category $F(A)$ [JP91, Theorem B]. It uses a bicomplex whose two associated spectral sequences are both concentrated in one line on the E_2-term. One E_2-term is isomorphic to $\mathrm{HML}^*(A)$, and the other to $\mathrm{Ext}^*\,\mathcal{F}(A)(I, I)$. It is possible to define a map from this bicomplex to another bicomplex which induces the cup product on the E_2-term of the first spectral sequence and the evaluation on the E_2-term of the second. We do not go into the details here as we will only use the description of the cup product given in Construction 2.2.1.

Lemma 2.2.3. *Let*

$$\begin{array}{ccccccccc} 0 \longrightarrow & M' & \xrightarrow{g} & M_{n-1} & \longrightarrow \cdots \longrightarrow & M_0 & \longrightarrow M \longrightarrow & 0 \\ & {\scriptstyle f}\downarrow & \swarrow_{h} & & & & & \\ & N & & & & & & \end{array}$$

be a diagram in the category of Λ-modules s.t. the $M_0, \ldots, M_{n-1}$ are free and the upper line is exact and represents $\Theta \in \mathrm{Ext}^n_\Lambda(M, M')$. Then we have $(f + hg)_*(\Theta) = f_*(\Theta)$

Proof. This statement becomes trivial when we define Ext using projective resolutions. □

Lemma 2.2.4. *The cup product of Construction 2.2.1 does not depend on the choice of the cocycle representing γ.*

Proof. By definition, the product map is linear with respect to addition of cocycles. Hence it is enough to show that the extension associated to a coboundary represents the trivial element in $\mathrm{Ext}^{s,t}_\Lambda(M, N)$.

Let $b \in \overline{C}^{s-1}(F(\Lambda, n), \mathrm{Hom}_\Lambda(-, - \otimes_\Lambda \Lambda[t]))$ be a normalized cochain. Then we have

$$\delta(b)(\lambda_1, \ldots, \lambda_s) = \lambda_1[t]b(\lambda_2, \ldots, \lambda_s) + (-1)^s b(\lambda_1, \ldots, \lambda_{s-1})\lambda_s,$$

hence

$$\lambda_0[t]\delta(b)(\lambda_1, \ldots, \lambda_s) = (-1)^s \lambda_0[t]b(\lambda_1, \ldots, \lambda_{s-1})\lambda_s.$$

If we define τ associated to $\delta(b)$ and the chosen resolution of M as in Construction 2.2.1, the last equation implies that this map τ extends to M_{s-1}. An application of Lemma 2.2.3 shows that $((f[t])\tau)_*(\Theta)$ is zero. □

Lemma 2.2.5. *The cup product of Construction 2.2.1 does not depend on the choice of the resolution of M.*

Proof. Suppose we are given another resolution $\cdots \to M'_n \xrightarrow{\lambda'_n} M'_{n-1} \to \dots M'_0 \xrightarrow{\lambda'_0} M$ of M by objects of $F(\Lambda, n)$. Then there exist maps α_n such that the following diagram commutes:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & M'_n & \xrightarrow{\lambda'_n} & M'_{n-1} & \longrightarrow \cdots \longrightarrow & M'_0 & \xrightarrow{\lambda'_0} & M \\
 & & \alpha_n \downarrow & & \alpha_{n-1} \downarrow & & \alpha_0 \downarrow & & = \downarrow \\
\cdots & \longrightarrow & M_n & \xrightarrow{\lambda_n} & M_{n-1} & \longrightarrow \cdots \longrightarrow & M_0 & \xrightarrow{\lambda_0} & M
\end{array}$$

The problem is that the diagram

$$\begin{array}{ccccc}
M'_s & \xrightarrow{c(\lambda'_1,\dots,\lambda'_s)} & M'_0[t] & \xrightarrow{\lambda'_0[t]} & M[t] \\
\alpha_s \downarrow & & & & \downarrow = \\
M_s & \xrightarrow{c(\lambda_1,\dots,\lambda_s)} & M_0[t] & \xrightarrow{\lambda_0[t]} & M[t]
\end{array}$$

will in general not be commutative. As we are are only interested in the induced maps on Ext-groups, it suffices to show that

$$(f[t])(\lambda_0[t])c(\lambda_1, \dots, \lambda_s)\alpha_s \quad \text{and} \quad (f[t])(\lambda'_0[t])c(\lambda'_1, \dots, \lambda'_s)$$

give rise to maps $M'_s/M'_{s+1} \to N[t]$ which induce the same map

$$\mathrm{Ext}^{s,0}_\Lambda(M, M'_s/M'_{s+1}) \to \mathrm{Ext}^{s,t}_\Lambda(M, N).$$

To do so, we show that there is a map $h\colon M'_{s-1} \to M[t]$ such that

$$\lambda_0[t]c(\lambda_1, \dots, \lambda_s)\alpha_s = \lambda'_0[t]c(\lambda'_1, \dots, \lambda'_s) + \lambda_0[t]h\lambda'_s$$

and apply Lemma 2.2.3 again. To find such a map h, we first calculate some coboundaries:

$$\begin{aligned}
0 &= (\delta c)(\alpha_0, \lambda'_1, \dots, \lambda'_s) \\
&= \alpha_0[t]c(\lambda'_1, \dots, \lambda'_s) - c(\alpha_0\lambda'_1, \dots, \lambda'_s) + (-1)^{s+1}c(\alpha_0, \lambda'_1, \dots, \lambda'_{s-1})\lambda'_s
\end{aligned}$$

$$\begin{aligned}
0 &= (\delta c)(\lambda_1, \dots, \lambda_n, \alpha_n, \lambda'_{n+1}, \dots \lambda'_s) \\
&= \lambda_1[t]c(\lambda_2, \dots, \lambda_n, \alpha_n, \lambda'_{n+1}, \dots \lambda'_s) + (-1)^n c(\lambda_1, \dots, \lambda_n\alpha_n, \lambda'_{n+1}, \dots \lambda'_s) \\
&+ (-1)^{n+1}c(\lambda_1, \dots, \lambda_n, \alpha_n\lambda'_{n+1}, \dots \lambda'_s) + (-1)^{s+1}c(\lambda_1, \dots, \lambda_n, \alpha_n, \lambda'_{n+1}, \dots \lambda'_{s-1})\lambda'_s
\end{aligned}$$

$$\begin{aligned}
0 &= (\delta c)(\lambda_1, \dots, \lambda_s, \alpha_s) \\
&= \lambda_1[t]c(\lambda_2, \dots, \lambda_s, \alpha_s) + (-1)^s c(\lambda_1, \dots, \lambda_{s-1}, \lambda_s\alpha_s) + (-1)^{s+1}c(\lambda_1, \dots, \lambda_s)\alpha_s
\end{aligned}$$

Using $\lambda_n\alpha_n = \alpha_{n-1}\lambda'_n$, we can form the alternating sum of all coboundaries calculated above to end up with the following formula in which $h\colon M'_{s-1} \to M[t]$ and $g\colon M'_s \to M_0[t]$ are maps which we don't need to know explicitly:

$$\begin{aligned}
0 &= (\delta c)(\alpha_0, \lambda'_1, \dots, \lambda'_s) + (-1)^s(\delta c)(\lambda_1, \dots, \lambda_s, \alpha_s) \\
&\quad + \sum_{n=1}^{s-1}(-1)^n(\delta c)(\lambda_1, \dots, \lambda_n, \alpha_n, \lambda'_{n+1}, \dots \lambda'_s) \\
&= \alpha_0[t]c(\lambda'_1, \dots, \lambda'_s) - c(\lambda_1, \dots, \lambda_s)\alpha_s \\
&\quad + \lambda_1[t]g + h\lambda'_s
\end{aligned}$$

Composing with $\lambda_0[t]$ yields the desired equation. □

3 Toda brackets and realizability

This section is concerned with Toda brackets in triangulated categories and their relation to a realizability problem which we explain in the first paragraph. We assume familiarity with the axioms and the basic properties of triangulated categories. Background for this can for example be found in Weibel's book [Wei94]. In particular, we assume our triangulated categories to have infinite coproducts and call an object X in a triangulated category $\mathcal{T}$ *compact* if the functor $\mathcal{T}(X,-)$ preserves arbitrary coproducts.

Throughout this section, $\mathcal{T}$ will always be a triangulated category, N will be a compact object in $\mathcal{T}$, and the graded endomorphism ring $\mathcal{T}(N,N)_*$ of N will be denoted by Λ.

3.1 Realizability

The functor $\mathcal{T}(N,-)\colon \mathcal{T} \to \mathcal{A}b$ induces a functor $\mathcal{T}(N,-)_*\colon \mathcal{T} \to \text{Mod-}\Lambda$ from $\mathcal{T}$ to the category of right modules over Λ. The module structure on $\mathcal{T}(N,X)_*$ is given by composition. This functor preserves arbitrary coproducts since $\mathcal{T}(N,-)\colon \mathcal{T} \to \mathcal{A}b$ does. Our grading conventions ensure that $\mathcal{T}(N,-)$ commutes with the shift functors in $\mathcal{T}$ and Mod-Λ, i.e., we have $\mathcal{T}(N,X[k])_* \cong (\mathcal{T}(N,X)_*)[k]$. Furthermore, $\mathcal{T}(N,-)_*$ maps distinguished triangles $X \to Y \to Y \to X[1]$ in $\mathcal{T}$ to exact sequences

$$\mathcal{T}(N,X)_* \to \mathcal{T}(N,Y)_* \to \mathcal{T}(N,Z)_* \to (\mathcal{T}(N,X)_*)[1]$$

in Mod-Λ.

In this context, a Λ-module M is called *realizable* if there exists an object X in $\mathcal{T}$ such that $\mathcal{T}(N,X)_* \cong M$. In the next section we will introduce an obstruction theory which helps to answer the question whether a Λ-module is realizable.

An object of $\mathcal{T}$ is called *N-free* if it is a sum of shifted copies of N. Since N is compact, the functor $\mathcal{T}(N,-)_*$ induces a map

$$\mathcal{T}(X,Y) \to \text{Hom}_\Lambda(\mathcal{T}(N,X)_*, \mathcal{T}(N,Y)_*)$$

which is an isomorphism if X is N-free. This proves the basic but important

Lemma 3.1.1. *Let $\mathcal{T}$ be a triangulated category, let N be a compact object of $\mathcal{T}$, and let Λ be the graded ring $\mathcal{T}(N,N)_*$. Then the functor $\mathcal{T}(N,-)_*\colon \mathcal{T} \to$ Mod-Λ restricts to an equivalence between the full subcategory of $\mathcal{T}$ given by the N-free objects and the category of free graded Λ-modules.*

As we will explain in more detail in Section 5, ring spectra give rise to an interesting class of examples for this situation: if R is a ring spectrum, the homotopy category of R-module spectra is a triangulated category in which R, the free module of rank 1, is a compact object with $\mathcal{T}(R,R)_* \cong \pi_*(R)$. In this case, the general realizability problem introduced above amounts to the question whether a $\pi_*(R)$-module M arises as the homotopy groups of an R-module spectrum.

A more algebraic instance of this is studied in [BKS04]. For a differential graded algebra A over a field k, the authors consider the derived category $\mathcal{D}(A)$ of the category of differential graded A-modules. This is a triangulated category in which A, the free module of rank 1, is a compact object. When using cohomological grading convention, the graded endomorphism object of A in $\mathcal{D}(A)$ is the cohomology algebra $H^*(A)$ of A, and

an $H^*(A)$-module is realizable if it is the cohomology of a differential graded A-module. If G is a finite group and A is the endomorphism dga of a complete resolution of k as a kG-module, the cohomology algebra of A is the Tate Ext-algebra $\widehat{\mathrm{Ext}}^*_{kG}(k,k)$. In this case, the realizability obstructions of the next section can be used to answer the question if a module over $\widehat{\mathrm{Ext}}^*_{kG}(k,k)$ arises as the Tate cohomology of G with coefficients in some kG-module [BKS04, Theorem 6.9].

3.2 Obstruction Theory

In this paragraph, we recall from [BKS04, Appendix A] the obstruction theory for the realizability problem introduced in the last paragraph and extend its study by addressing uniqueness questions.

Remark 3.2.1. Before explaining the general approach, we give the easier definition of the first realizability obstruction: an N-special $\mathcal{T}$-presentation of a Λ-module M is a distinguished triangle $X_1 \to X_0 \to C \to X_1[1]$ in $\mathcal{T}$ together with an epimorphism $\epsilon\colon \mathcal{T}(N,X_0)_* \to M$ such that X_0 and X_1 are N-free and the sequence

$$\mathcal{T}(N,X_1)_* \to \mathcal{T}(N,X_0)_* \xrightarrow{\epsilon} M \to 0$$

is exact. Every Λ-module admits an N-special $\mathcal{T}$-presentation: we can realize the first two modules M_i in a free resolution of M by N-free objects X_i, and we can realize the map $M_1 \to M_0$ by a map $X_1 \to X_0$ in $\mathcal{T}$ as X_1 is N-free. Extending this map to a distinguished triangle in $\mathcal{T}$ gives the required data.

Given an N-special $\mathcal{T}$-presentation of M, there is a monomorphism η such that the following diagram commutes:

$$\begin{array}{ccccc} \mathcal{T}(N,X_1)_* & \longrightarrow & \mathcal{T}(N,X_0)_* & \longrightarrow & \mathcal{T}(N,C)_* \\ & & & \searrow \quad \nearrow_{\eta} & \\ & & & M & \end{array}$$

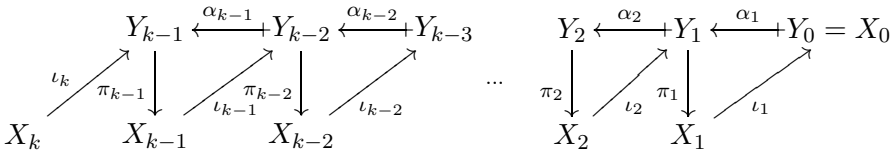
The first obstruction class $\kappa_3(M) \in \mathrm{Ext}^{3,-1}_\Lambda(M,M)$ of M is defined to be the Yoneda class represented by the exact sequence

$$0 \to M[-1] \xrightarrow{\eta[-1]} \mathcal{T}(N,C[-1])_* \to \mathcal{T}(N,X_1)_* \to \mathcal{T}(N,X_0)_* \xrightarrow{\epsilon} M \to 0.$$

In [BKS04, Proposition 3.4, Theorem 3.7] it is shown that $\kappa_3(M)$ is well defined and that $\kappa_3(M) = 0$ holds if and only if M is a direct summand of a realizable module. Since the $\mathcal{T}(N,X_i)_*$ are free Λ-modules, the Yoneda class of the extension is trivial if and only if $\eta[-1]$ splits as a map of Λ-modules.

The construction of the higher obstructions uses the following

Definition 3.2.2. [BKS04, Definition A.6] Let $\mathcal{T}$ be a triangulated category and let N be a compact object in $\mathcal{T}$. For $k \geq 1$, an *N-exact k-Postnikov system* for a Λ-module M consists of an epimorphism $\mathcal{T}(N,X_0)_* \to M$ and a diagram



such that all arrows of the form $\longleftarrow\!\!\!\!+$ denote morphisms of degree 1, all triangles are distinguished triangles in $\mathcal{T}$, each object X_i is N-free, and the maps $d_j = \pi_{j-1}\iota_j$ induce an exact sequence

$$\mathcal{T}(N, X_k)_* \xrightarrow{(d_k)_*} \mathcal{T}(N, X_{k-1}) \xrightarrow{(d_{k-1})_*} \dots \xrightarrow{(d_2)_*} \mathcal{T}(N, X_1)_* \xrightarrow{(d_1)_*} \mathcal{T}(N, X_0)_* \to M \to 0.$$

An *N-exact Postnikov system* is a collection of distinguished triangles as above which extends infinitely to the left.

Given a Λ-module M, we can always find an N-exact 2-Postnikov system for M. Its data is almost given by an N-special $\mathcal{T}$-presentation $X_1 \to X_0 \to C \to X_1[1]$ of M: we can realize the third term in the free resolution of M chosen in the construction of the $\mathcal{T}$-presentation by an N-free object X_2. Then the map $X_2 \to X_1$ which realizes the differential in the resolution lifts to a map $X_2 \to C$. With this we have specified the data of an N-exact 2-Postnikov system.

The following Proposition shows why Postnikov systems are relevant for the realizability problem.

Proposition 3.2.3. *[BKS04, Proposition A.19] If there exists an N-exact Postnikov system of a Λ-module M, then M is realizable.*

Since an N-exact 2-Postnikov system for M always exists, the problem of finding a realization of M can be approached by iteratively extending an N-exact k-Postnikov system to an N-exact $(k+1)$-Postnikov system. To understand this process, we need the following fact which is verified in [BKS04, Lemma A.15(iii)]: an N-exact k-Postnikov system of M induces an exact sequence

$$\begin{aligned}\mathcal{T}(N, X_1)_*[1-k] \xrightarrow{(d_1)_*} \mathcal{T}(N, X_0)_*[1-k] \xrightarrow{\alpha_*} \mathcal{T}(N, Y_{k-1})_* \\ \xrightarrow{(\pi_{k-1})_*} \mathcal{T}(N, X_{k-1})_* \xrightarrow{(d_{k-1})_*} \mathcal{T}(N, X_{k-2})_*\end{aligned}$$

of Λ-modules, where the map $\alpha\colon X_0[1-k] = Y_0[1-k] \to Y_{k-1}$ is the composition $\alpha_{k-1} \cdots \alpha_1$.

Since $\operatorname{coker}(d_1)_* \cong M$, the exactness of this sequences enables us to state

Definition 3.2.4. [BKS04, Definition A.16] Associated to an N-exact k-Postnikov system for M there is an exact sequence

$$\begin{aligned}0 \to M[1-k] \xrightarrow{\eta_{k-1}} \mathcal{T}(N, Y_{k-1})_* \xrightarrow{(\pi_{k-1})_*} \mathcal{T}(N, X_{k-1})_* \xrightarrow{(d_{k-1})_*} \dots \\ \dots \xrightarrow{(d_2)_*} \mathcal{T}(N, X_1)_* \xrightarrow{(d_1)_*} \mathcal{T}(N, X_0)_* \to M \to 0.\end{aligned}$$

We denote the Yoneda class of this extension by $\kappa_{k+1}(M) \in \operatorname{Ext}_\Lambda^{k+1,k-1}(M, M)$.

The key point about the obstruction theory is

Lemma 3.2.5. *[BKS04, Lemma A.18] If the obstruction class $\kappa_{k+1}(M)$ of an N-exact k-Postnikov system of M is trivial, then there exists an N-exact $(k+1)$-Postnikov system for M whose underlying $(k-1)$-Postnikov system agrees with that of the given k-Postnikov system.*

Recall that a graded ring is *n-sparse* if it is concentrated in degrees divisible by n.

Corollary 3.2.6. *Let $\mathcal{T}$ be a triangulated category and let N be a compact object in $\mathcal{T}$. If the graded ring $\Lambda = \mathcal{T}(N,N)_*$ is n-sparse, then there exists an N-exact $(n+1)$-Postnikov system for every graded Λ-module M.*

Proof. As Λ is n-sparse, the module M splits into a sum $M^{(0)} \oplus M^{(1)} \oplus \cdots \oplus M^{(n-1)}$ with $M^{(i)}$ concentrated in degrees $\equiv i \pmod n$. The last lemma provides the existence of an N-exact $(n+1)$-Postnikov system for each $M^{(i)}$ since the groups $\mathrm{Ext}_\Lambda^{j+1,j-1}(M^{(i)}, M^{(i)})$ vanish for $2 \leq j \leq n$. The sum of the N-exact $(n+1)$-Postnikov systems the $M^{(i)}$ provides an N-exact $(n+1)$-Postnikov system of M. □

To study the uniqueness of Postnikov systems and their associated obstruction classes, we need

Definition 3.2.7. Let $(X_j, Y_j, \alpha_j, \iota_j, \pi_j, M)$ and $(X'_j, Y'_j, \alpha'_j, \iota'_j, \pi'_j, M)$ be two N-exact k-Postnikov systems for M. A morphism between them consists of maps $f_j \colon X_j \to X'_j$ and $g_j \colon Y_j \to Y'_j$ such that $f_{k-1}d_k = d'_k f_k$ and the following commutativity relations hold for $1 \leq j \leq k-1$:

$$g_{j-1}\iota_j = \iota'_j f_j \qquad (g_j[1])\alpha_j = \alpha'_j g_{j-1} \qquad f_j \pi_j = \pi'_j g_j$$

In other words, all the obvious squares built from this data commute *except* the square

$$\begin{array}{ccc} X_k & \xrightarrow{\iota_k} & Y_{k-1} \\ {\scriptstyle f_k}\downarrow & & \downarrow{\scriptstyle g_{k-1}} \\ X'_k & \xrightarrow[\iota'_k]{} & Y'_{k-1}. \end{array}$$

More generally, for $1 \leq l \leq k$, an l-map of N-exact k-Postnikov systems for M is a map of the underlying N-exact l-Postnikov systems.

Corollary 3.2.8. *If there is a map between two N-exact k-Postnikov systems for M, then the associated obstruction classes $\kappa_{k+1}(M)$ and $\kappa'_{k+1}(M)$ in $\mathrm{Ext}_\Lambda^{k+1,k-1}(M,M)$ coincide.*

Proof. The data of the map of Postnikov systems can be used to obtain a map between the exact sequences representing the obstruction classes. Since this map is id_M on the outer terms, both exact sequences represent the same Yoneda class in the Ext-group. This does not need the relation $g_{k-1}\iota_k = \iota'_k f_k$ which was left out in the definition of a map of Postnikov systems. □

Given two N-exact 2-Postnikov systems of M, it is easy to see that there is always a map between them. The next lemma gives a criterion which can be used to extend maps of Postnikov systems.

Lemma 3.2.9. *Suppose we are given an l-map between two N-exact k-Postnikov systems with $1 \leq l < k$. Then there exists an element in $\mathrm{Ext}_\Lambda^{l,1-l}(M,M)$ which vanishes if and only if there is an $(l+1)$-map between the Postnikov systems whose underlying $(l-1)$-map coincides with that of the given map.*

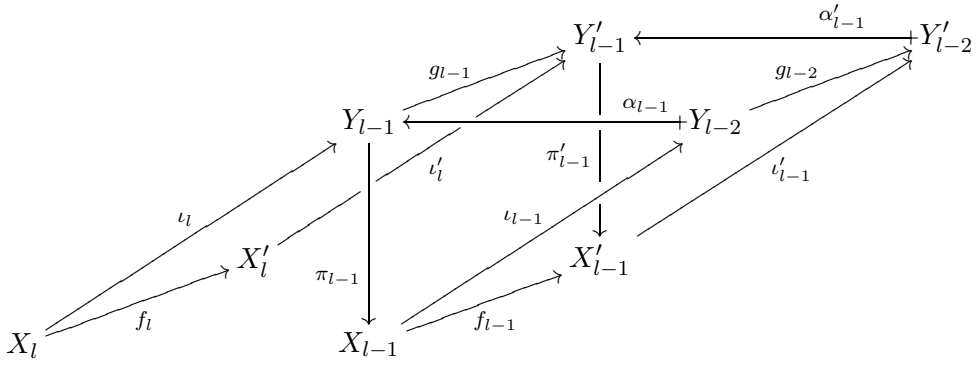


Figure 1: Extending a map of Postnikov systems

Proof. In the proof we denote $\mathcal{T}(N, X_i)_*$ by M_i. Together with the maps $(d_i)_*$, the M_i form a free resolution of M. Similarly, the M'_i and the $(d'_i)_*$ form another resolution of M. The maps f_i induce maps between the free resolutions M_i and M'_i up to stage l. We can extend this to stage $l+1$ and realize the resulting map $M_{l+1} \to M'_{l+1}$ by a map $f_{l+1}\colon X_{l+1} \to X'_{l+1}$. This guarantees that $f_l d_{l+1} = d'_{l+1} f_{l+1}$ holds.

Let us for a moment assume our map of Postnikov systems satisfies $g_{l-1}\iota_l = \iota'_l f_l$. In this case it is easy to extend the map one step further. By the axioms of a triangulated category, we can find a map g_l such that the diagram

$$\begin{array}{ccccccc} X_l & \xrightarrow{\iota_l} & Y_{l-1} & \longrightarrow & Y_l & \longrightarrow & X_l[1] \\ \downarrow{\scriptstyle f_l} & & \downarrow{\scriptstyle g_{l-1}} & & \downarrow{\scriptstyle g_l} & & \downarrow \\ X'_l & \xrightarrow{\iota'_l} & Y'_{l-1} & \longrightarrow & Y'_l & \longrightarrow & X'_l[1] \end{array}$$

commutes. Therefore, the two commutativity relations for an $(l+1)$-map involving g_l are automatically satisfied, and we have succeeded in extending the map.

In general, the additional commutativity relation does not hold, and there is a $\varphi = \iota'_l f_l - g_{l-1}\iota_l$ which may be non zero. For the diagram chase we are about to perform, it is helpful to look at Figure 1. As we have

$$\pi'_{l-1}\varphi = \pi'_{l-1}\iota'_l f_l - \pi'_{l-1} g_{l-1}\iota_l = d'_l f_l - f_{l-1} d_l = 0,$$

the exactness of

$$\mathcal{T}(X_l, X_{l-1}[-1]) \xrightarrow{(\iota'_{l-1})_*} \mathcal{T}(X_l, Y'_{l-2}[-1]) \xrightarrow{(\alpha'_{l-1})_*} \mathcal{T}(X_l, Y'_{l-1}) \xrightarrow{(\pi'_{l-1})_*} \mathcal{T}(X_l, X'_{l-1})$$

tells us that there is an element $\psi \in \mathcal{T}(X_l, Y'_{l-2}[-1])$ with $(\alpha'_{l-1})_*(\psi) = \varphi$.

As X_l is an N-free object, we can apply the functor $\mathcal{T}(N, -)_*$ to the last exact sequence to obtain an isomorphic sequence

$$\Lambda(M_l, M'_{l-1}[-1]) \to \Lambda(M_l, \mathcal{T}(N, Y'_{l-2}[-1])_*) \to \Lambda(M_l, \mathcal{T}(N, Y'_{l-1})_*) \to \Lambda(M_l, M'_{l-1}).$$

Next let P be the Λ-module

$$\ker(\mathcal{T}(N, Y'_{l-1})_* \to M'_{l-1}) \cong \operatorname{coker}(M'_{l-1}[-1] \to \mathcal{T}(N, Y'_{l-2}[-1])_*).$$

Since φ is in the kernel of $(\pi'_{l-1})_*$, it defines an element $\overline{\varphi} \in \Lambda(M_l, P)$.

We show that this $\overline{\varphi}$ represents an element in $\mathrm{Ext}^l(M,P)$. As the M_i form a free resolution of M, it suffices to show that $\overline{\varphi}$ is in the kernel of $\Lambda(M_l,P) \to \Lambda(M_{l+1},P)$. For this is it enough to verify that φ is mapped to zero under

$$(d_{l+1})^* \colon \mathcal{T}(X_l, Y'_{l-1}[-1])_* \to \mathcal{T}(X_{l+1}, Y'_{l-1}[-1])_*.$$

Since both N-exact Postnikov systems have a length $k > l$, [BKS04, Lemma A.12(i)] provides the equalities $\ker(\iota_l)_* = \ker(d_l)_*$ and $\ker(\iota'_l)_* = \ker(d'_l)_*$. This implies $\iota_l d_{l+1} = 0$ and $\iota'_l d'_{l+1} = 0$. Therefore

$$(d_{l+1})^*\varphi = \varphi d_{l+1} = \iota'_l d'_{l+1} f_{l+1} - g_{l-1}\iota_l d_{l+1} = 0$$

holds, where the map f_{l+1} can be constructed as in the introduction to the proof.

Using that P is isomorphic to $\mathrm{coker}(M'_{l-1}[-1] \to \mathcal{T}(N, Y'_{l-2}[-1])_*)$, we can apply [BKS04, Lemma A.12(ii)] which yields $P \cong M[1-l]$. Therefore, $\overline{\varphi}$ represents a class in $\mathrm{Ext}^l_\Lambda(M, M[1-l]) \cong \mathrm{Ext}^{l,1-l}_\Lambda(M,M)$.

If the Ext-class represented by $\overline{\varphi}$ vanishes, there has to be a map $\overline{\rho} \in \Lambda(M_{l-1}, P)$ such that $\overline{\rho}(d_l)_* = \overline{\varphi}$. This means that there is an element $\rho \in \mathcal{T}(X_{l-1}, Y'_{l-2}[-1])$ with $\rho d_l = \rho\pi_{l-1}\iota_l = \psi$. Using this map ρ, we change our l-map of Postnikov systems by replacing the map g_{l-1} by $\overline{g_{l-1}} = g_{l-1} + (\alpha'_{l-1}[-1])\rho\pi_{l-1}$.

This map satisfies the relations $(\overline{g_{l-1}}[-1])\alpha_{l-1} = \alpha'_{l-1}g_{l-2}$ and $\pi'_{l-1}\overline{g_{l-1}} = f_{l-1}\pi_{l-1}$ since g_{l-1} does. In addition we have gained that

$$\begin{aligned}\overline{g_{l-1}}\iota_l &= g_{l-1}\iota_l + (\alpha'_{l-1}[-1])\rho\pi_{l-1}\iota_l = g_{l-1}\iota_l + (\alpha'_{l-1}[-1])\rho d_l \\ &= g_{l-1}\iota_l + (\alpha'_{l-1}[-1])\psi = g_{l-1}\iota_l + \varphi = \iota'_l f_l\end{aligned}$$

Hence the resulting modified l-map can be extended by the argument given at the beginning of the proof. □

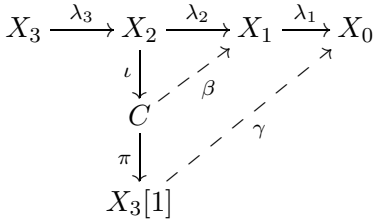
Corollary 3.2.10. *Let $\mathcal{T}$ be a triangulated category and let N be a compact object of $\mathcal{T}$. Assume that the ring $\Lambda = \mathcal{T}(N,N)_*$ is n-sparse and that M is an n-sparse Λ-module. Then there exists an N-exact $(n+1)$-Postnikov system of M, and all N-exact $(n+1)$-Postnikov systems of M give rise to the same obstruction class $\kappa_{n+2}(M) \in \mathrm{Ext}^{n+2,-n}_\Lambda(M,M)$.*

Proof. The existence of the N-exact $(n+1)$-Postnikov system is provided by Corollary 3.2.6. Given two N-exact $(n+1)$-Postnikov systems, Lemma 3.2.9 and the vanishing of $\mathrm{Ext}^{l,1-l}_\Lambda(M,M)$ for $2 \le l \le n$ provide the existence of a map between them. Hence their obstruction classes coincide by Corollary 3.2.8. □

3.3 Toda brackets

Before starting to explain higher Toda brackets, we introduce the more basic concept of a triple Toda bracket in a triangulated category $\mathcal{T}$. For this we consider the following

diagram in $\mathcal{T}$:



Here $(\lambda_1, \lambda_2, \lambda_3)$ is a sequence of maps with $\lambda_1\lambda_2 = 0 = \lambda_2\lambda_3$, and (π, ι, λ_3) is a distinguished triangle in $\mathcal{T}$. If we apply the functor $\mathcal{T}(-, X_1)$ to the triangle, we get an exact sequence

$$\mathcal{T}(X_3[1], X_1) \xrightarrow{\pi^*} \mathcal{T}(C, X_1) \xrightarrow{\iota^*} \mathcal{T}(X_2, X_1) \xrightarrow{(\lambda_3)^*} \mathcal{T}(X_3, X_1).$$

Hence the relation $\lambda_2\lambda_3 = 0$ implies the existence of a $\beta\colon C \to X_1$ with $\beta\iota = \lambda_2$. Similarly, the exact sequence

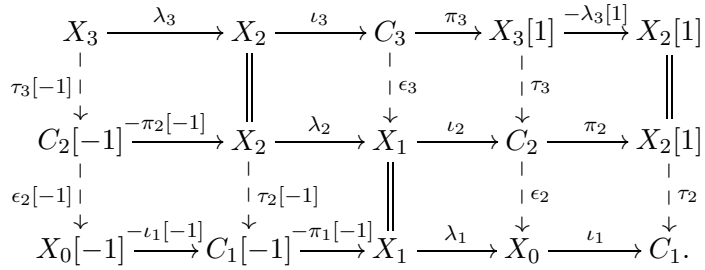
$$\mathcal{T}(X_2[1], X_0) \xrightarrow{(\lambda_3)^*} \mathcal{T}(X_3[1], X_0) \xrightarrow{\pi^*} \mathcal{T}(C, X_0) \xrightarrow{\iota^*} \mathcal{T}(X_2, X_0)$$

and the relation $(\lambda_1\beta)\iota = \lambda_1\lambda_2 = 0$ imply the existence of $\gamma\colon X_3[1] \to X_0$ with $\gamma\pi = \lambda_1\beta$.

There are choices involved in the construction of γ. As one can read off from the exact sequences, we can alter β by an element of $\pi^*(\mathcal{T}(X_3[1], X_1))$ and γ by an element of $(\lambda_3)^*(\mathcal{T}(X_2[1], X_0))$. Putting these choices together, we see that γ is only well defined modulo $(\lambda_3)^*(\mathcal{T}(X_2[1], X_0)) + (\lambda_1)_*(\mathcal{T}(X_3[1], X_1))$.

Definition 3.3.1. Let $X_3 \xrightarrow{\lambda_3} X_2 \xrightarrow{\lambda_2} X_1 \xrightarrow{\lambda_1} X_0$ be a sequence in a triangulated category $\mathcal{T}$ with $\lambda_1\lambda_2 = 0 = \lambda_2\lambda_3$. The *Toda bracket* of $(\lambda_1, \lambda_2, \lambda_3)$ is the set $\langle\lambda_1, \lambda_2, \lambda_3\rangle \subseteq \mathcal{T}(X_3[1], X_0)$ of all maps γ which can be constructed as above. It is a coset of the group $(\lambda_1)_*(\mathcal{T}(X_3[1], X_1)) + (\lambda_3[1])^*(\mathcal{T}(X_2[1], X_0))$, which we refer to as the *indeterminacy* of the Toda bracket.

Remark 3.3.2. There are two other equivalent ways to define the triple Toda bracket of $(\lambda_1, \lambda_2, \lambda_3)$. For this, we have a look at the commutative diagram



Here the horizontal lines are obtained by choosing distinguished triangles containing the λ_i. The vertical maps are constructed by first choosing extensions $\epsilon_3\colon C_3 \to X_1$ and $\epsilon_2\colon C_2 \to X_0$ and then completing them to maps between triangles.

The definition of a Toda bracket given above uses the first line of the diagram and produces a map $\epsilon_2\tau_3\colon X_3[1] \to X_0$. The second possible definition of a Toda bracket uses the distinguished triangle in the middle line: the relations $\lambda_2\lambda_3 = 0$ and $\lambda_1\lambda_2 = 0$ can be used to choose maps $\tau_3[-1]\colon X_3 \to C_2[-1]$ and $\epsilon_2\colon C_2 \to X_0$, which can, after a shift,

be composed to give a map $X_3[1] \to X_0$. The last definition is dual to the first one. It uses the distinguished triangle in the third line and lifts to fibers instead of extensions to cones to obtain a map $X_3 \to X_0[-1]$. One can use the diagram to see that all three definitions are equivalent.

The definition of higher Toda brackets will generalize the second definition which is, as it uses extensions to cones as well as lifts to fibers, the most symmetric one. The next definition will be a main ingredient.

Definition 3.3.3. [Shi02, Definition A.1] Let $\mathcal{T}$ be a triangulated category and let

$$X_{n-1} \xrightarrow{\lambda_{n-1}} X_{n-2} \xrightarrow{\lambda_{n-2}} \dots \xrightarrow{\lambda_1} X_0$$

be a sequence of $(n-1)$ composable maps in $\mathcal{T}$. An *n-filtered object* $X \in \{\lambda_1, \dots, \lambda_{n-1}\}$ consists of a sequence of maps $* = F_0X \xrightarrow{i_0} F_1X \xrightarrow{i_1} \dots \xrightarrow{i_n} F_nX = X$ and choices of distinguished triangles

$$F_jX \xrightarrow{i_j} F_{j+1}X \xrightarrow{p_{j+1}} X_j[j] \xrightarrow{d_j} (F_jX)[1]$$

such that $(p_j[1])(d_j) = \lambda_j[j]$. The maps $X_0 \cong F_1X \to X$ and $X = F_nX \xrightarrow{p_n} X_{n-1}[n-1]$ are denoted by σ'_X and σ_X.

Remark 3.3.4. At the first glance, our definition seems to be more restrictive than the one of Shipley [Shi02, Definition A.1], as we require the objects $X_j[j]$ to *be* the cones of the maps i_j, rather than to be isomorphic to the cones. This does not make a difference since triangles isomorphic to distinguished triangles are distinguished again.

For a map $\lambda_1 \colon X_1 \to X_0$ in $\mathcal{T}$, the cone C of λ_1 is part of a distinguished triangle $X_1 \to X_0 \to C \to X_1[1]$. With the filtration $* \to X_0 \to C$, the cone C is a 2-filtered object in $\{\lambda_1\}$.

If there exists an n-filtered object $X \in \{\lambda_1, \dots, \lambda_n\}$, each twofold composition $\lambda_i\lambda_{i+1}$ has to be zero since it is isomorphic to a composition of maps which contains two consecutive maps in a distinguished triangle.

Remark 3.3.5. The definition of a filtered object is closely related to that of a Postnikov system. It is more general since a Postnikov system always has a resolution as part of its data, while the corresponding maps λ_i of the filtered object only need to form a complex. We will see in Lemma 3.4.1 how in special cases a filtered object gives rise to a Postnikov system.

The next lemma will be our tool for the construction of filtered objects.

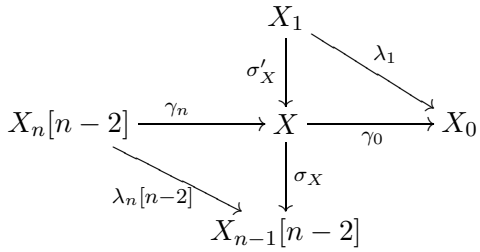
Lemma 3.3.6. *[Shi02, Lemma A.4] Let $\lambda_i \colon X_i \to X_{i-1}$ be a sequence of composable maps in a triangulated category $\mathcal{T}$. An n-filtered object $X \in \{\lambda_2, \dots, \lambda_n\}$ with a map $\alpha \colon X \to X_0$ gives rise to an $(n+1)$-filtered object $C_\alpha \in \{\alpha\sigma'_X, \lambda_2, \dots, \lambda_n\}$, and an n-filtered object $X \in \{\lambda_1, \dots, \lambda_{n-1}\}$ with a map $\alpha \colon X_n[n-1] \to X$ gives rise to an $(n+1)$-filtered object $C_\alpha \in \{\lambda_1, \dots, \lambda_{n-1}, (\sigma_X\alpha)[-n+1]\}$.*

Proof. The first part is a consequence of the octahedral axiom. The relevant diagram will appear in the proof of Proposition 3.3.11 as Figure 2 on page 26. The second part follows immediately from the definition. □

Definition 3.3.7. [Shi02, Definition A.2] Let $\mathcal{T}$ be a triangulated category. A map $\gamma \in \mathcal{T}(X_n[n-2], X_0)$ lies in the *n-fold Toda bracket* of the sequence

$$X_n \xrightarrow{\lambda_n} X_{n-1} \xrightarrow{\lambda_{n-1}} \dots \xrightarrow{\lambda_1} X_0$$

if there exist an $(n-1)$-filtered object $X \in \{\lambda_2, \dots, \lambda_{n-1}\}$ and maps $\gamma_n \colon X_n[n-2] \to X$ and $\gamma_0 \colon X \to X_0$ such that $\gamma = \gamma_0\gamma_n$ holds and the two triangles in the following diagram commute:



We denote the (possibly empty) set of all such maps by $\langle \lambda_1, \dots, \lambda_n \rangle \subseteq \mathcal{T}(X_n[n-2], X_0)$.

For $n = 3$, we can use the fact that the cone of a map is a 2-filtered object to see that this definition specializes to the second definition of the triple Toda bracket mentioned in Remark 3.3.2.

A sequence $(\lambda_1, \dots, \lambda_n)$ of composable maps has to satisfy restrictive conditions for its Toda bracket to be non empty. For example, $0 \in \langle \lambda_2, \dots, \lambda_{n-1} \rangle$ is a necessary condition for the existence of an $(n-1)$-filtered object $X \in \{\lambda_2, \dots, \lambda_{n-1}\}$ (see [Shi02, Proposition A.5]), and the additional requirement $\lambda_1\lambda_2 = 0 = \lambda_{n-1}\lambda_n$ will in general not be sufficient for $\langle \lambda_1, \dots, \lambda_n \rangle$ to be non empty. We now introduce a condition which ensures that higher Toda brackets are always non empty.

Definition 3.3.8. Let $\mathcal{T}$ be a triangulated category and let $n \geq 1$ be a natural number. An *n-split* subcategory $\mathcal{U}$ of $\mathcal{T}$ is a full subcategory such that for all objects X and Y of $\mathcal{U}$ the graded abelian group $\mathcal{T}(X, Y)_*$ is n-sparse, that is, is concentrated in degrees divisible by n. Of course the condition on $\mathcal{U}$ is empty if $n = 1$.

The motivating example for an n-split subcategory of $\mathcal{T}$ is the following: assume that $\mathcal{T}$ has a compact object N such that $\mathcal{T}(N, N)_*$ is n-sparse. Then the category of sums of copies of N which are shifted by integral multiples of n forms an n-split subcategory of $\mathcal{T}$.

Lemma 3.3.9. *Let $\mathcal{U}$ be an n-split subcategory of a triangulated category $\mathcal{T}$ with $n \geq 2$, let*

$$X_{l-1} \xrightarrow{\lambda_{l-1}} X_{l-2} \xrightarrow{\lambda_{l-2}} \dots \xrightarrow{\lambda_1} X_0$$

be a sequence of maps in $\mathcal{U}$ with $2 \leq l \leq n-1$, and let $X \in \{\lambda_1, \dots, \lambda_{l-1}\}$ be an l-filtered object. Then for every object Y in $\mathcal{U}$, we have

$$\mathcal{T}(Y[l], X) = 0 \qquad \text{and} \qquad \mathcal{T}(X, Y[-1]) = 0.$$

Proof. To show the first part, we choose a map $\alpha \colon Y[l] \to X$. Since the composition $Y[l] \to X = F_lX \xrightarrow{\sigma_X} X_{l-1}[l-1]$ is zero, α factors through $F_{l-1}X \to F_lX$. Using

inductively that $\mathcal{T}(Y[l], X_j[j]) = 0$ for $j = l-2, \dots, 0$, we obtain that α factors through $F_0X \to F_lX$. Hence $\alpha = 0$ since $F_0X = *$.

For the second part, we first observe that $\mathcal{T}(F_1X, Y[-1]) \cong \mathcal{T}(X_0, Y[-1]) = 0$. The exact sequence

$$\mathcal{T}(X_j[j], Y[-1]) \to \mathcal{T}(F_{j+1}X, Y[-1]) \to \mathcal{T}(F_jX, Y[-1])$$

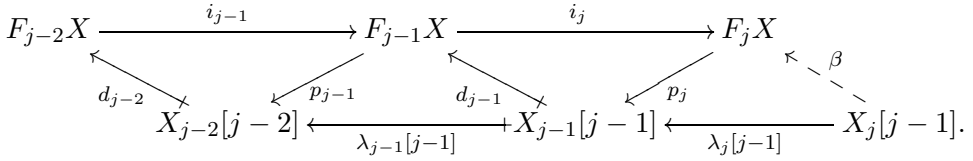
in which the first term is trivial for $j \leq l-2$ can be used to show the assertion by induction. □

Lemma 3.3.10. *Let $\mathcal{U}$ be an n-split subcategory of a triangulated category $\mathcal{T}$. Then a sequence*

$$X_l \xrightarrow{\lambda_l} X_{l-1} \xrightarrow{\lambda_{l-1}} \dots \xrightarrow{\lambda_1} X_0$$

in $\mathcal{U}$ with $\lambda_i\lambda_{i+1} = 0$ admits an $(l+1)$-filtered object $X \in \{\lambda_1, \dots, \lambda_l\}$ if $l \leq n+1$. If $l \leq n$, the $(l+1)$-filtered object is unique up to isomorphism.

Proof. The map from X_0 to the cone of $\lambda_1 \colon X_1 \to X_0$ gives the data of a 2-filtered object in $\{\lambda_1\}$. Inductively, we assume that $X \in \{\lambda_1, \dots, \lambda_{j-1}\}$ is a j-filtered object with $j \leq n$ and consider the diagram



The map $(p_{j-1}d_{j-1})(\lambda_j[j-1])$ is trivial since it is a shift of $\lambda_{j-1}\lambda_j$. Hence $d_{j-1}(\lambda_j[j-1])$ lifts along i_{j-1} and factors through $F_{j-2}X$. Since we have $\mathcal{T}(X_{j-1}[j-1], F_{j-2}X) = 0$ by the last lemma, we obtain $d_{j-1}(\lambda_j[j-1]) = 0$. This provides the existence of the dotted arrow β. By Lemma 3.3.6, the cone of β is a $(j+1)$-filtered object in $\{\lambda_1, \dots, \lambda_j\}$.

Next we prove uniqueness. For 1-filtered objects, the existence of the isomorphism $F_1X \to F_1X'$ follows from the fact that both objects come with isomorphisms to X_0. Now assume we have constructed an isomorphism of $(j-1)$-filtered objects. In order to extend it to an isomorphism of j-filtered objects, we need to construct an isomorphism $F_jX \to F_jX'$ which fits into a commutative diagram

$$\begin{array}{ccccccc}
(X_{j-1}[j-2]) & \longrightarrow & F_{j-1}X & \longrightarrow & F_jX & \longrightarrow & X_{j-1}[j-1] \\
\downarrow{=} & & \downarrow{\cong} & & \downarrow & & \downarrow{=} \\
(X_{j-1}[j-2]) & \longrightarrow & F_{j-1}X' & \longrightarrow & F_jX' & \longrightarrow & X_{j-1}[j-1].
\end{array}$$

The existence would follow immediately from the axioms of the triangulated category $\mathcal{T}$ if we knew that the first square in the following diagram commutes:

$$\begin{array}{ccccc}
X_{j-1}[j-2] & \longrightarrow & F_{j-1}X & \longrightarrow & X_{j-2}[j-2] \\
{=}\downarrow & & \downarrow & & \downarrow{=} \\
X_{j-1}[j-2] & \longrightarrow & F_{j-1}X' & \longrightarrow & X_{j-2}[j-2]
\end{array}$$

We know that the big square commutes. Hence the exact sequence

$$\mathcal{T}(X_{j-2}[j-2], F_{j-2}X') \xrightarrow{(i'_{j-1})_*} \mathcal{T}(X_{j-2}[j-2], F_{j-1}X') \to \mathcal{T}(X_{j-2}[j-2], X_{j-1}[j-2])$$

tells us that the possible deviation from commutativity in the first square is in the image of the map $(i'_{j-1})_*$. Since $\mathcal{T}(X_{j-2}[j-2], F_{j-2}X')$ is zero for $j \leq n+1$ by Lemma 3.3.9, the diagram does commute and we get an isomorphism of j-filtered objects. □

Proposition 3.3.11. *Let $\mathcal{U}$ be an n-split subcategory of a triangulated category $\mathcal{T}$ and let*

$$X_{n+2} \xrightarrow{\lambda_{n+2}} X_{n+1} \xrightarrow{\lambda_{n+1}} \dots \xrightarrow{\lambda_1} X_0$$

be a sequence of maps in $\mathcal{U}$ with $\lambda_i\lambda_{i+1} = 0$. Then the Toda bracket $\langle\lambda_1, \dots, \lambda_{n+2}\rangle$ is defined, is non-empty, and has the indeterminacy

$$(\lambda_1)_*(\mathcal{T}(X_{n+2}[n], X_1)) + (\lambda_{n+2}[n])^*(\mathcal{T}(X_{n+1}[n], X_0)).$$

Proof. The $(n+1)$-filtered object $X \in \{\lambda_2, \dots, \lambda_{n+1}\}$ needed for the construction of the Toda bracket exists and is unique by Lemma 3.3.10. To construct the map γ_{n+2}, we apply $\mathcal{T}(X_{n+2}[n], -)$ to the distinguished triangle $F_nX \to F_{n+1}X \to X_{n+1}[n]$. Since $F_{n+1}X$ equals X, we get an exact sequence

$$\mathcal{T}(X_{n+1}[n], X) \xrightarrow{(\sigma_X)_*} \mathcal{T}(X_{n+2}[n], X_{n+1}[n]) \to \mathcal{T}(X_{n+1}[n], F_nX[1]).$$

The last term is zero by Lemma 3.3.9. Hence there exists a γ_{n+2} with $\sigma_X\gamma_{n+2} = \lambda_{n+2}[n]$.

To obtain γ_0, we first use the isomorphism $F_1X \to X_1$ to get a map $F_1X \to X_0$. This map can be extended to F_2X since $\lambda_1\lambda_2 = 0$. Inductively, we can extend it to a map $\gamma_0 \colon X = F_{n+1}X \to X_0$: the obstruction for extending a map $F_{j-1}X \to X_0$ to F_jX lies in the group $\mathcal{T}(X_{j-1}[j-2], X_0)$, which is zero for $3 \leq j \leq n+1$. The extension of λ_1 to X is the map γ_0.

Next we compute the indeterminacy. Since we have an exact sequence

$$\mathcal{T}(X_{n+2}[n], F_nX) \xrightarrow{(i_n)_*} \mathcal{T}(X_{n+2}[n], F_{n+1}X) \xrightarrow{(\sigma_X)_*} \mathcal{T}(X_{n+2}[n], X_{n+1}[n]),$$

we know that two different choices of γ_{n+2} differ by an element in the image of $(i_n)_*$. Using the same argument as in Lemma 3.3.9, we see that every map $X_{n+2}[n] \to F_nX$ factors through $\sigma'_X \colon X_1 \cong F_1X \to F_nX$. Therefore, the possible difference is in the image $(\sigma'_X)_*$, and after composing with any choice for γ_0 we obtain that this part of the indeterminacy is $(\lambda_1)_*\mathcal{T}(X_{n+2}[n], X_1[n])$.

To examine the other part of the indeterminacy, we first construct an n-filtered object $F'_nX \in \{\lambda_3[1], \dots, \lambda_{n+1}[1]\}$ in the following way. For $0 \leq j \leq n$, we define F'_jX to be the cone in a distinguished triangle $X_1 \to F_{j+1}X \to F'_jX$. The maps id_{X_1} and $i_{j+1} \colon F_{j+1}X \to F_{j+2}X$ induce maps $i'_j \colon F'_j \to F'_{j+1}X$, and the octahedral axioms ensures that F'_nX is an n-filtered object in $\{\lambda_3[1], \dots, \lambda_{n+1}[1]\}$. The last step of this construction and the application of the octahedral axiom are displayed in Figure 2.

Next we use that the distinguished triangle $X_1 \to F_{n+1}X \to F'_nX$ induces an exact sequence

$$\mathcal{T}(F'_nX, X_0) \to \mathcal{T}(X, X_0) \to \mathcal{T}(X_1, X_0).$$

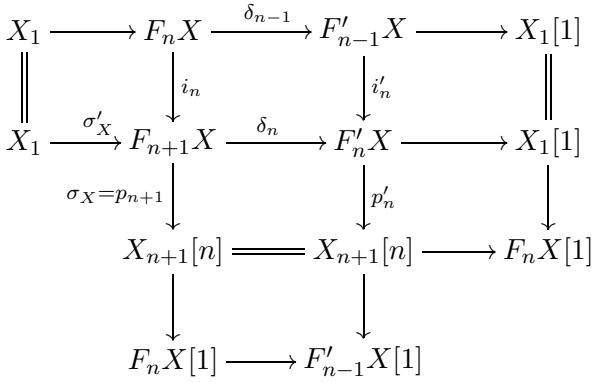


Figure 2: Forming the quotient of a filtered object

Hence the difference $\overline{\gamma}$ of two possible choices for γ_0 is in the image of $\mathcal{T}(F'_nX, X_0)$. Since $\mathcal{T}(F'_{n-1}X, X_0)$ vanishes by Lemma 3.3.9, there is an $\omega\colon X_{n+1}[n] \to X_0$ with $\omega\sigma_X = \overline{\gamma}$. If we apply $(\gamma_{n+2})^*$ to $\omega\sigma_X$, we see that this part of the indeterminacy is given by $(\lambda_{n+2})^*(\mathcal{T}(X_{n+1}[n], X_0))$. □

3.4 Toda brackets and obstructions

In this section we exhibit the link between Toda brackets and realizability obstructions. More precisely, we use the cup product of Construction 2.2.1 to turn the slogan 'the Toda brackets of the resolution are realizability obstructions' into a theorem. The first step is the relation between filtered objects in the sense of Definition 3.3.3 and Postnikov systems in the sense of Definition 3.2.2.

Lemma 3.4.1. *Let $X_{n+1} \xrightarrow{\lambda_{n+1}} X_n \xrightarrow{\lambda} \dots \xrightarrow{\lambda_1} X_0$ be a sequence of maps in $\mathcal{T}$ such that each X_i is N-free and $\mathcal{T}(N,-)$ maps it to an exact sequence of Λ-modules, and let M be the cokernel of the map $(\lambda_1)_*\colon \mathcal{T}(N,X_1)_* \to \mathcal{T}(N,X_0)$.*

Then an $(n+1)$-filtered object $X \in \{\lambda_1, \dots, \lambda_n\}$ determines all data of an N-exact $(n+1)$-Postnikov system of M except the map $X_{n+1} \to Y_n$. In particular, we have $Y_n = (F_{n+1}X)[-n]$, and the map $\alpha\colon X_0 \to Y_n[n]$ of the Postnikov system is, up to the sign $(-1)^{\frac{(n+2)(n+1)}{2}+1}$, given by the map $\sigma'_X\colon X_0 \to F_{n+1}X$ which is part of the data of the filtered object. The underlying resolution of the Postnikov system is induced by the maps $(-1)(\lambda_i)_*$.

Proof. This is just a rephrasing of the definitions. We set $Y_l = (F_{l+1}X)[-l]$ for $0 \le l \le n$ and

$$\begin{aligned}
&\pi'_l\colon Y_l = F_{l+1}X[-l] \xrightarrow{p_{l+1}[-l]} (F_{l+1}X/F_lX)[-l] \xrightarrow{\cong} X_l,\\
&\iota'_l\colon X_l \xrightarrow{\cong} (F_{l+1}X/F_lX)[-l] \xrightarrow{d_l[-l]} F_lX[-l+1] = Y_{l-1}, \text{ and}\\
&\alpha'_l\colon Y_{l-1} = F_lX[-l+1] \xrightarrow{i_l[-l+1]} F_{l+1}[-l+1] = Y_l[1]
\end{aligned}$$

for $1 \le l \le n$. If we bring in signs by defining

$$\pi_l = (-1)^l\pi'_l, \qquad \iota_l = (-1)^l\iota'_l, \text{ and} \qquad \alpha_l = (-1)^{l+1}\alpha'_l$$

it follows that the triangles $(\alpha_l, \iota_l, \pi_l)$ are distinguished as we know that the triangles (d_l, p_{l+1}, i_l) are. Hence we have specified all data of a Postnikov system but the map $\iota_{n+1}\colon X_{n+1} \to Y_n$. The differentials of the underlying resolution of the Postnikov system are given by

$$\pi_{j-1}\iota_j = ((-1)^{j-1}\pi'_{j-1})((-1)^j\iota'_j) = -(p_j[-j+1])(d_j[-j]) = -\lambda_j.$$

Our definitions also imply that $\alpha = \alpha_n \cdots \alpha_1$ differs from σ'_X by the sign

$$\prod_{i=1}^{n}(-1)^{i+1} = (-1)^{\frac{(n+2)(n+1)}{2}+1}.$$

□

Before stating the main theorem of this section, we explain why the Mac Lane cohomology groups of Definition 2.1.11 provide an appropriate tool for the systematic study of Toda brackets. Again, we let $\mathcal{T}$ be a triangulated category and N be a compact object of $\mathcal{T}$ such that $\Lambda = \mathcal{T}(N,N)_*$ is n-sparse, and we define $\mathcal{U}$ to be the full subcategory of $\mathcal{T}$ consisting of finite sums of copies of N which are shifted by integral multiples of n. Then $\mathcal{U}$ is n-split in the sense of Definition 3.3.8, and by Lemma 3.1.1 the functor $\mathcal{T}(N,-)_*\colon \mathcal{T} \to \text{Mod-}\Lambda$ restricts to an equivalence $\mathcal{T}(N,-)\colon \mathcal{U} \to F(\Lambda, n)$. This leads to

Definition 3.4.2. The $(n+2)$-*fold Toda bracket* of a complex

$$M_{n+2} \xrightarrow{\lambda_{n+2}} M_{n+1} \xrightarrow{\lambda_{n+1}} \dots \xrightarrow{\lambda_1} M_0$$

of free Λ-modules is the subset $\langle \lambda_1, \dots, \lambda_{n+2}\rangle \subseteq \text{Hom}_\Lambda(M_{n+2}[n], M_0)$ defined as follows: first realize the complex by a complex in $\mathcal{U}$, then form the Toda bracket in $\mathcal{T}$, and let $\langle \lambda_1, \dots, \lambda_{n+2}\rangle \subseteq \text{Hom}_\Lambda(M_{n+2}[n], M_0)$ be the image of the Toda bracket in $\mathcal{T}$ under $\mathcal{T}(N,-)_*$.

It is clear that statements about a Toda bracket being non empty or its indeterminacy translate from the definition for triangulated categories to the one for chain complexes in $F(\Lambda, n)$.

Remark 3.4.3. In fact, the last definition specializes to a more popular definition of Toda brackets or Massey products in examples. The input is in this case given by elements of homotopy groups (or cohomology groups) of appropriate objects. If $\mathcal{T}$ is for example the derived category of a differential graded algebra A and $n = 1$, a complex of length 3 in which all 4 modules are isomorphic to the free module of rank 1 is the same data as 3 elements $\lambda_1, \lambda_2, \lambda_3 \in H^*(A)$ with $\lambda_1\lambda_2 = 0$ and $\lambda_2\lambda_3 = 0$. One can check that the Toda bracket in this situation is the same as the classical Massey product of $(\lambda_1, \lambda_2, \lambda_3)$. By allowing all finitely generated free $H^*(A)$-modules in the complex, our definition specializes to the one of matric Massey products.

Remark 3.4.4. Coming back to the general setup with $\mathcal{T}$ triangulated and $\mathcal{U}$ a small n-split subcategory of $\mathcal{T}$, we observe the following simple but important fact. If

$$X_{n+2} \xrightarrow{\lambda_{n+2}} X_{n+1} \xrightarrow{\lambda_{n+1}} \dots \xrightarrow{\lambda_1} X_0$$

is a complex in $\mathcal{U}$ and $c \in \overline{C}^{n+2}(\mathcal{U}, \mathcal{T}(-,-)_n)$ is a normalized cocycle representing a cohomology class $\gamma \in H^{n+2}(\mathcal{U}, \mathcal{T}(-,-)_n)$, the evaluation of c on the complex is an element in $\mathcal{T}(X_{n+2}[n], X_0)$. This element of course depends on the representing cocycle. But since $(\lambda_1, \dots, \lambda_{n+2})$ is a complex, we know that the evaluation of a coboundary of a normalized $(n+1)$-cocycle is an element in

$$(\lambda_{n+2})^* \mathcal{T}(X_{n+1}[n], X_0) + (\lambda_1)_* \mathcal{T}(X_{n+2}[n], X_1).$$

Therefore, the evaluation of a *cohomology class* (as opposed to a cocycle) is a well defined object of the quotient of $\mathcal{T}(X_{n+2}[n], X_0)$ by $(\lambda_{n+2})^* \mathcal{T}(X_{n+1}[n], X_0) + (\lambda_1)_* \mathcal{T}(X_{n+2}, X_1)$. By Proposition 3.3.11, the latter group coincides with the indeterminacy of the Toda bracket $\langle \lambda_1, \dots, \lambda_{n+2} \rangle$. Hence it makes sense to ask the evaluation of a cohomology class $\gamma \in H^{n+2}(\mathcal{U}, \mathcal{T}(-,-)_n)$ to *be* the Toda bracket $\langle \lambda_1, \dots, \lambda_{n+2} \rangle$. The fact that the indeterminacies of a Toda bracket and the evaluation of a cohomology class coincide is one reason why the normalized cohomology of categories is the suitable cohomology theory for our purposes.

If $\mathcal{T}$ is a triangulated category with a compact object N such that $\Lambda = \mathcal{T}(N,N)_*$ is n-sparse, this can be reformulated in terms of Mac Lane cohomology groups and Definition 3.4.2. We choose $\mathcal{U}$ to be the category of finite sums of copies of N which are shifted by integral multiples of N again, and observe that the functor $\mathcal{T}(N,-)$ induces an isomorphism

$$\mathcal{T}(X,Y)_n \to \mathrm{Hom}_\Lambda(\mathcal{T}(N,X)_*, \mathcal{T}(N,Y)_* \otimes_{\mathcal{T}(N,N)_*} (\mathcal{T}(N,N)_*[-n])).$$

if X and Y are objects of $\mathcal{U}$. Therefore, the equivalence $\mathcal{T}(N,-)$ between $\mathcal{U}$ and $F(\Lambda, n)$ induces an isomorphism $H^{n+2}(\mathcal{U}, \mathcal{T}(-,-)_n) \to \mathrm{HML}^{n+2,-n}_{n-\mathrm{sp}}(\Lambda)$. Hence it makes sense to ask the evaluation of a cohomology class $\gamma \in \mathrm{HML}^{n+2,-n}_{n-\mathrm{sp}}(\Lambda)$ on a complex of n-split Λ-modules $(\lambda_1, \dots, \lambda_{n+2})$ to *be* the Toda bracket $\langle \lambda_1, \dots, \lambda_{n+2} \rangle$. For $n = 3$, this observation was used for the study of (triple) universal Toda brackets in [BD89].

Theorem 3.4.5. *Let $\mathcal{T}$ be a triangulated category, and let N be a compact object such that $\Lambda = \mathcal{T}(N,N)_*$ is n-sparse. Let M be a Λ-module which admits a resolution*

$$\dots \xrightarrow{\lambda'_{i+1}} M_i \xrightarrow{\lambda'_i} \dots \xrightarrow{\lambda'_1} M_0 \xrightarrow{\lambda'_0} M \to 0$$

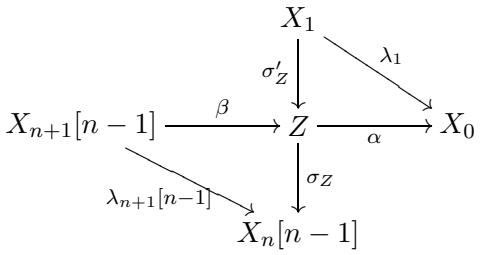
by finitely generated free n-sparse Λ-modules. Let $\gamma \in \mathrm{HML}^{n+2,-n}_{n-\mathrm{sp}}(\Lambda)$ be a cohomology class such that the evaluation $\gamma(\lambda'_1, \dots, \lambda'_{n+2})$ is the Toda bracket $\langle \lambda'_1, \dots, \lambda'_{n+2} \rangle$. Then the product $\mathrm{id}_M \cup \gamma \in \mathrm{Ext}^{n+2,-n}_\Lambda(M,M)$ coincides with the unique obstruction class $\kappa_{n+2}(M)$ of Corollary 3.2.10.

Proof. We denote the realization of the resolution of M by N-free objects by

$$X_{n+2} \xrightarrow{\lambda_{n+2}} X_{n+1} \xrightarrow{\lambda_{n+1}} \dots \xrightarrow{\lambda_1} X_0,$$

that is, $(\lambda_i)_* = \lambda'_i$. By Lemma 3.3.10, there is an unique n-filtered object $Z \in \{\lambda_2, \dots, \lambda_n\}$. Since the $(n+1)$-fold Toda bracket of $(\lambda_1, \dots, \lambda_{n+1})$ contains only zero for degree reasons,

we can find maps α and β such that the following diagram commutes

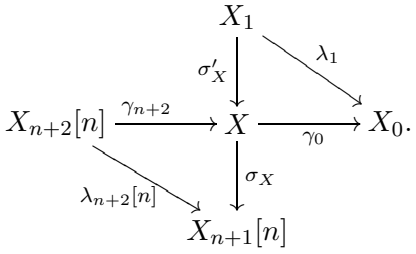


and the composition $\alpha\beta$ is zero. We use α and β to choose distinguished triangles

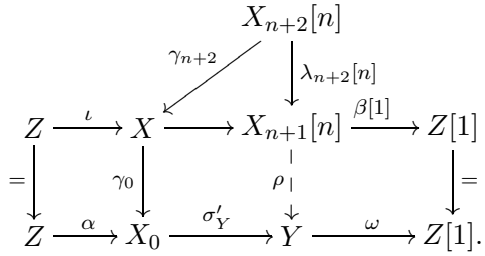
$$Z \xrightarrow{\alpha} X_0 \to Y \xrightarrow{\omega} Z[1] \qquad \text{and} \qquad X_{n+1}[n-1] \xrightarrow{\beta} Z \xrightarrow{\iota} X \to X_{n+1}[n].$$

Lemma 3.3.6 tells us that X is an $(n+1)$-filtered object in $\{\lambda_2, \dots, \lambda_{n+1}\}$ and that Y is an $(n+1)$-filtered object in $\{\lambda_1, \dots, \lambda_n\}$.

The Toda bracket of $(\lambda_1, \dots, \lambda_{n+2})$ is non empty by Proposition 3.3.11 and can be defined using the n-filtered object X. Therefore, we have a diagram

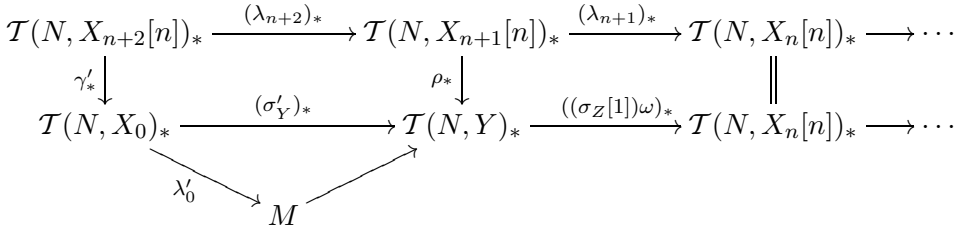


such that $\gamma' = \gamma_0\gamma_{n+2}$ is an element of $\langle \lambda_1, \dots, \lambda_{n+2} \rangle$. Looking at the distinguished triangle defining X, we see that the map γ_0 can be constructed by extending $\alpha\colon Z \to X_0$ to a map $X \to X_0$. The relation $\gamma_0\iota = \alpha$ enables us to construct the map ρ in the following commutative diagram:



Here we use that the map $X_0 \to Y$ from the distinguished triangle defining Y coincides with the map σ'_Y which is part of the data of the n-filtered object Y.

Applying $\mathcal{T}(N, -)_*$ to the last diagram, we obtain the following commutative diagram of Λ-modules:



The lower sequence starting with M in this diagram represents $\mathrm{id}_M \cup \gamma$ up to sign. Inspecting Definition 3.2.4 and Lemma 3.4.1, we observe that it, up to signs, represents as well the exact sequence associated to the $(n+1)$-Postnikov system obtained from Y. This uses that the map $((\sigma_Z[1])\omega)$ equals the map p_{n+1} of the $(n+1)$-filtered object Y, and therefore the map $(-1)^n \pi_n[n]$ of the associated Postnikov system. The sign of the latter map cancels with the n factors (-1) by which the maps $(\lambda_i)_*$ differ from the differentials of the resolution induced by the Postnikov system. The remaining sign $(-1)^{\frac{(n+2)(n+1)}{2}+1}$ of the map σ'_Y cancels with the sign built into the cup product. □

Remark 3.4.6. In view of Remark 2.1.5, the restriction to modules with a resolution by finitely generated free modules is unnecessary. For a given Λ-module M, we only need to replace the category $F(\Lambda, n)$ by a larger full small n-split additive subcategory $\mathcal{C}$ of Mod-Λ which contains all modules of a given free resolution of M. This does not change the cohomology group $HML^{n+2,-n}_{n-\mathrm{sp}}(\Lambda)$, and the proof of the theorem applies in the same way.

Nevertheless, we need *some* finiteness condition on the objects of $\mathcal{C}$ to ensure smallness. Since we do not want to obscure the exposition by taking an ordinal which restricts the size of $\mathcal{C}$ into the statement of our theorems, we continue to use $F(\Lambda, n)$ as in the last theorem.

Applications of this theorem will be given in the last section. We point out that for $n = 1$, the last theorem also leads to an interpretation of the product of a Λ-module homomorphism $f\colon M \to M'$ with γ, provided that M satisfies the hypothesis of the theorem: by [BKS04, Proposition 3.4(iv) and Theorem 3.7] and the naturality of the cup product, $f \cup \gamma$ vanishes if and only if f factors through a realizable Λ-module.

4 Universal Toda brackets for stable model categories

In this section, we construct the *universal Toda bracket* $\gamma_{\mathcal{U}}$ of an n-split subcategory $\mathcal{U}$ of the homotopy category of a stable topological model category $\mathcal{C}$. The class $\gamma_{\mathcal{U}}$ will be an element in a certain cohomology group of the category $\mathcal{U}$ which determines the Toda bracket of every complex of $(n+2)$ composable maps in $\mathcal{U}$. When we specialize to module categories over ring spectra in the next section, $\gamma_{\mathcal{U}}$ can be defined as an element of a Mac Lane cohomology group to which the theory of the last section, namely Theorem 3.4.5, applies.

Though the definition of Toda brackets and their relation to realizability obstructions takes completely place in triangulated categories, the construction of $\gamma_{\mathcal{U}}$ needs additional information from an underlying 'model' of the triangulated category. The point is that Toda brackets are only defined for complexes of maps, while a Mac Lane cohomology class is represented by a cocycle which can be evaluated on arbitrary sequences of composable maps. We explain in the last paragraph of this section how the evaluation of $\gamma_{\mathcal{U}}$ on sequences of isomorphisms can be interpreted in terms of k-invariants of classifying spaces.

The characteristic Hochschild cohomology class γ_A of a differential graded algebra A over a field k studied by Benson, Krause, and Schwede in [BKS04] should be considered as the algebraic counterpart of our construction. In their theory, the derived category $\mathcal{D}(A)$ of the differential graded algebra A plays the role of $\mathrm{Ho}(\mathcal{C})$ in our case. Similarly to $\gamma_{\mathcal{U}}$, the class γ_A determines all triple (matric) Massey products in the cohomology ring of A. Since A is assumed to be a dga over a field k, this class can be defined as an element of a Hochschild cohomology group rather than of a Mac Lane cohomology group. A 'model' for the triangulated category $\mathcal{D}(A)$ is needed for the construction of γ_A as well, since [BKS04, Example 5.15] shows that γ_A cannot be recovered from $\mathcal{D}(A)$. In fact, the construction of γ_A uses the first piece of the A_∞-structure of $H^*(A)$. We will come back to the relation between the Hochschild class and the universal Toda bracket in Remark 5.1.11, where we also outline how *higher* characteristic Hochschild classes can be defined.

Our construction will use information from the model category $\mathcal{C}$ which cannot be recovered from its triangulated homotopy category. We will particularly exploit the presence of mapping spaces, which exist since we require $\mathcal{C}$ to be topological. It would be nice if we could skip the technical assumption of $\mathcal{C}$ being topological, as the topological structure is typically not present in algebraic examples. This is likely be possible by considering either simplicial model categories or, more generally, framings on stable model categories [Hov99]. As this would make our construction considerably more difficult and we do not need this extra generality for the examples we have in mind, we do not attempt to do this.

Another motivation for our construction (and its name) is Baues' work on universal triple Toda brackets [Bau97, BD89]. He is working mainly in an unstable context, considering subcategories of H-group or H-cogroup objects in the homotopy category of topological spaces, though he points out that these constructions generalize to 'cofibration categories' [Bau97, Remark on p. 271]. We will only work in a stable context, in order to provide the link to triangulated categories. This also avoids certain difficulties in the unstable case arising from maps which are not suspensions (see the correction of [BD89] in [Bau97, Remark on p. 270]). We also do not use Baues language of 'linear track extensions', as these seem to be only appropriate for the study of triple universal Toda brackets. Nevertheless, the $n = 1$ case of the isomorphism we construct in Proposition

4.1.4 below is basically what Baues encodes in a linear Track extension.

A motivation for the actual construction of the representing cocycle is the approach of Blanc and Markl to higher homotopy operations [BM03]. For a directed category Γ, the authors use the bar resolution $W\Gamma$ in the sense of Boardman and Vogt [BV73, III, §1] to define general higher homotopy operations. If Γ is the category generated by $n+2$ composable morphisms, this specializes to the higher Toda brackets we would like to construct. In this case, $W\Gamma$ is just an $(n+1)$-dimensional cube. As we are not interested in other indexing categories, we will just use the cubes and do not make use of the bar resolution in our construction.

4.1 Coherent change of basepoints

In what follows, we assume familiarity with model categories. Hovey's book [Hov99] provides a good reference. Other than in Quillen's original treatment of model categories [Qui67], we will follow Hovey in assuming our model categories to have all small limits and colimits as well as functorial factorizations.

Let $\mathcal{T}op_*$be the category of pointed compactly generated weak Hausdorff spaces. This category can be equipped with a model structure [Hov99, Theorem 2.4.25]. It is Quillen equivalent to the category of usual topological spaces with the model structure in which weak equivalences are the weak homotopy equivalences. The reason for working with $\mathcal{T}op_*$ is that it is a closed symmetric monoidal model category [Hov99, Corollary 4.2.12].

A *pointed topological model category* $\mathcal{C}$ is a pointed model category which is enriched, tensored, and cotensored over $\mathcal{T}op_*$ in a way that certain axioms, partly involving the model structures on $\mathcal{C}$ and $\mathcal{T}op_*$, are satisfied. The categorical data of $\mathcal{C}$ consists of functors

$$\begin{aligned} -\wedge -\colon \mathcal{T}op_*\times\mathcal{C}\to\mathcal{T}, &\qquad (K,X)\mapsto K\wedge X\\ \mathrm{Map}_{\mathcal{C}}(-,-)\colon \mathcal{C}^{\mathrm{op}}\times\mathcal{C}\to\mathcal{T}op_*, &\qquad (X,Y)\mapsto \mathrm{Map}_{\mathcal{C}}(X,Y)\\ (-)^{(-)}\colon \mathcal{C}\times\mathcal{T}op_*^{\mathrm{op}}\to\mathcal{C}, &\qquad (X,K)\mapsto X^K, \end{aligned}$$

natural adjunction isomorphisms

$$\mathcal{C}(X,Y^K)\cong\mathcal{C}(K\wedge X,Y)\cong\mathcal{T}op_*(K,\mathrm{Map}_{\mathcal{C}}(X,Y)),$$

and the enriched composition

$$\mathrm{Map}_{\mathcal{C}}(Y,Z)\wedge\mathrm{Map}_{\mathcal{C}}(X,Y)\to\mathrm{Map}_{\mathcal{C}}(X,Z).$$

The data is asked to satisfy the usual associativity and unit conditions with respect to the monoidal structure of $\mathcal{T}op_*$. The compatibility with the model structure can be encoded in the pushout product axiom. Two maps $f\colon K\to L$ in $\mathcal{T}op_*$ and $g\colon X\to Y$ in $\mathcal{C}$ induce a map $f\square g\colon K\wedge Y\coprod_{K\wedge X}L\wedge X\to L\wedge Y$. The pushout product axiom asks $f\square g$ to be a cofibration if f and g are cofibrations, and in addition $f\square g$ has to be an acyclic cofibration if f or g is one. More details can be found in [Hov99, 4.2].

A *stable* topological model category $\mathcal{C}$ is a pointed topological model category in which the suspension functor $S^1\wedge -\colon\mathcal{C}\to\mathcal{C}$ and the loop functor $(-)^{S^1}\colon\mathcal{C}\to\mathcal{C}$ form a Quillen equivalence. The homotopy category of a stable model category is an additive category. We denote the set of morphisms from X to Y in $\mathrm{Ho}(\mathcal{C})$ by $[X,Y]^{\mathrm{Ho}(\mathcal{C})}$ or just $[X,Y]$.

One way to define the addition of maps is to replace X by an isomorphic object $S^1 \wedge X'$, which is possible as $\mathcal{C}$ is stable, and to use the H-cogroup structure of S^1. Later we use that $\mathrm{Ho}(\mathcal{C})$ is in fact a triangulated category [Hov99, Chapter 7].

If $\mathcal{C}$ is a model category and X an object in $\mathcal{C}$, we denote by $(X \downarrow \mathcal{C})$ the category of objects under X. This category inherits a model structure from $\mathcal{C}$ in which a map from $X \to Y$ to $X \to Y'$ is a cofibration, fibration, or a weak equivalence if the underlying map $Y \to Y'$ is one in $\mathcal{C}$ [Hir03, Theorem 7.6.5.(1)]. Since the initial object of $(X \downarrow \mathcal{C})$ is $\mathrm{id}_X \colon X \to X$, an object is cofibrant in $(X \downarrow \mathcal{C})$ if and only if the structure map $X \to Y$ is a cofibration.

If K is an object in $\mathcal{T}op_*$, we write K_+ for the space obtained from K by first forgetting the basepoint and then adding a new one. We either consider K_+ as an object of $\mathcal{T}op_*$, which is pointed by the 'new' basepoint, or as an object of $(S^0 \downarrow \mathcal{T}op_*)$, where the basepoint of S^0 is mapped to the 'new' basepoint of K_+ and the other point of S^0 is mapped to the 'former' basepoint of K.

Let K be an object of $\mathcal{T}op_*$. If X is an object in a pointed topological model category $\mathcal{C}$, applying the functor $- \wedge X$ to $S^0 \to K_+$ gives an object $X \xrightarrow{\cong} S^0 \wedge X \to K_+ \wedge X$ of $(X \downarrow \mathcal{C})$ which we denote simply by $K_+ \wedge X$. This notation is compatible with considering K_+ both as an object of $(S^0 \downarrow \mathcal{T}op_*)$ and $\mathcal{T}op_*$: the first notion is needed to turn $K_+ \wedge X$ into an object under X, and its underlying object in $\mathcal{C}$ is the product of K_+ in $\mathcal{T}op_*$ and X in $\mathcal{C}$. When we write (K, x, y) for an object of $(S^0 \downarrow \mathcal{T}op_*)$, y is understood as the image of the basepoint of S^0 and x as the image of the other point.

If X is cofibrant in $\mathcal{C}$ and K is cofibrant in $\mathcal{T}op_*$, the map $S^0 \to K_+$ is a cofibration, and $K_+ \wedge X$ is cofibrant in $(X \downarrow \mathcal{C})$ by the pushout product axiom. Given another object $f \colon X \to Y$ in $(X \downarrow \mathcal{C})$, we will denote the set of morphisms from $K_+ \wedge X$ to $f \colon X \to Y$ in $\mathrm{Ho}(X \downarrow \mathcal{C})$ by $[K_+ \wedge X, Y]^f_{\mathrm{Ho}(X \downarrow \mathcal{C})}$. We are particularly interested in the case $K = S^n$, where we study $[S^n_+ \wedge X, Y]^f_{\mathrm{Ho}(X \downarrow \mathcal{C})}$

After setting up our notation, we can formulate the aim of this paragraph. Let $\mathcal{C}$ be again a pointed topological model category. If $f \colon X \to Y$ is a zero map, we have a canonical isomorphism

$$(X \downarrow \mathcal{C})(S^n_+ \wedge X, f \colon X \to Y) \cong \mathcal{C}(S^n \wedge X, Y).$$

It is obtained from the universal property of the right pushout square below, which itself is obtained by applying $- \wedge X$ to the left one.

$$\begin{array}{ccc} S^0 & \longrightarrow & S^n_+ \\ \downarrow & \ulcorner & \downarrow \\ * & \longrightarrow & S^n \end{array} \qquad \xrightarrow{-\wedge X} \qquad \begin{array}{ccc} X & \longrightarrow & S^n_+ \wedge X \\ \downarrow & \ulcorner & \downarrow \\ * & \longrightarrow & S^n \wedge X \end{array}$$

Our aim is to construct an analogous isomorphism for the homotopy category of a stable topological model category for f being not necessarily a zero map. This will be done in Proposition 4.1.4.

Lemma 4.1.1. *Let $\mathcal{C}$ be a pointed topological model category and let X be a cofibrant object of $\mathcal{C}$. The functors $- \wedge X$ and $\mathrm{Map}_{\mathcal{C}}(X, -)$ induce a Quillen adjunction between $(S^0 \downarrow \mathcal{T}op_*)$ and $(X \downarrow \mathcal{C})$. For a cofibrant object K in $\mathcal{T}op_*$ and a map $f \colon X \to Y$ in $\mathcal{C}$*

with Y fibrant, the derived adjunction isomorphism has the form

$$[K_+ \wedge X, Y]^f_{\mathrm{Ho}(X\downarrow\mathcal{C})} \cong [K_+, (\mathrm{Map}_{\mathcal{C}}(X,Y), f, 0)]_{\mathrm{Ho}(S^0\downarrow\mathcal{T}op_*)}$$

Proof. The adjunction extends to the undercategories by naturality. It is a Quillen adjunction since $-\wedge X$ and $\mathrm{Map}_{\mathcal{C}}(X,-)$ form a Quillen adjunction and the weak equivalences and fibrations in $(X\downarrow\mathcal{C})$ are defined via the underlying maps in $\mathcal{C}$. With this, the derived adjunction isomorphism is a consequence of our notation conventions. □

Lemma 4.1.2. *Let $\mathcal{C}$ be a pointed topological model category, let $f\colon X \to Y$ be a map in $\mathcal{C}$, and let K be an object of $\mathcal{T}op_*$. Then a map $g\colon X' \to X$ in $\mathcal{C}$ induces a map*

$$g^*\colon (X\downarrow\mathcal{C})(K_+\wedge X, f) \to (X'\downarrow\mathcal{C})(K_+\wedge X', fg).$$

Given another map $h\colon X'' \to X'$, we have $(gh)^* = (h^*)(g^*)$. If K is cofibrant, Y fibrant and g a weak equivalence of cofibrant objects, g^* induces an isomorphism

$$[K\wedge X, Y]^f_{\mathrm{Ho}(X\downarrow\mathcal{C})} \xrightarrow{\cong} [K\wedge X', Y]^{fg}_{\mathrm{Ho}(X'\downarrow\mathcal{C})}.$$

on the level of homotopy categories.

Proof. In view of the adjunction of Lemma 4.1.1, the induced map is the adjoint of

$$K_+ \to (\mathrm{Map}_{\mathcal{C}}(X,Y), f, 0) \xrightarrow{g^*} (\mathrm{Map}_{\mathcal{C}}(X',Y), fg, 0).$$

If Y is fibrant, $\mathrm{Map}_{\mathcal{C}}(-,Y)$ is a left Quillen functor [Hov99, 4.2]. By [Hir03, Theorem 7.7.2.], it preserves weak equivalences between cofibrant objects. After adjunction, this proves the statement about the induced map in the homotopy category. It is clear that $(gh)^* = (h^*)(g^*)$ holds. □

Lemma 4.1.3. *Let $\mathcal{C}$ be a pointed topological model category, let $f, f'\colon X \to Y$ be two maps from a cofibrant object X to a fibrant object Y in $\mathcal{C}$, and let $I_+\wedge X$ be the cylinder object for X obtained by the product of I_+ in $\mathcal{T}op_*$ with X, where I denotes the unit interval. If $H\colon I_+\wedge X \to Y$ is a left homotopy from f to f', it induces an isomorphism*

$$(-)^H\colon [S^n_+\wedge X, Y]^f_{\mathrm{Ho}(X\downarrow\mathcal{C})} \to [S^n_+\wedge X, Y]^{f'}_{\mathrm{Ho}(X\downarrow\mathcal{C})}.$$

Proof. After taking the adjunction of Lemma 4.1.1 and forgetting the basepoint, we only need to show that there is an isomorphism

$$[S^n, (\mathrm{Map}_{\mathcal{C}}(X,Y), f)] \cong [S^n, (\mathrm{Map}_{\mathcal{C}}(X,Y), f')].$$

As the adjoint of H is a path from f to f', we take the isomorphism between the homotopy groups of $\mathrm{Map}_{\mathcal{C}}(X,Y)$ with different basepoints which is induced by this path. □

The set $[S^n_+\wedge X, Y]^f_{\mathrm{Ho}(X\downarrow\mathcal{C})}$ has a group structure, which can be defined using the H-cogroup structure of S^n: since we have $S^n_+ \coprod_{(S^0\downarrow\mathcal{T}op_*)} S^n_+ \cong (S^n\vee S^n)_+$, we get an isomorphism

$$[(S^n\vee S^n)_+\wedge X, Y]^f_{\mathrm{Ho}(X\downarrow\mathcal{C})} \cong [S^n_+\wedge X, Y]^f_{\mathrm{Ho}(X\downarrow\mathcal{C})} \times [S^n_+\wedge X, Y]^f_{\mathrm{Ho}(X\downarrow\mathcal{C})},$$

and the comultiplication $S^n \to S^n\vee S^n$ induces the addition. The group is abelian if $n>1$, or if $\mathcal{C}$ is stable as we will obtain as a byproduct of the next

Proposition 4.1.4. *Let $\mathcal{C}$ be a stable topological model category, and let $f\colon X \to Y$ be a map in $\mathcal{C}$ from a cofibrant and fibrant object X to a fibrant object Y. Then there is an isomorphism*

$$\sigma_f\colon [S^n_+ \wedge X, Y]^f_{\mathrm{Ho}(X\downarrow\mathcal{C})} \xrightarrow{\cong} [S^n \wedge X, Y]_{\mathrm{Ho}(\mathcal{C})}$$

of abelian groups. For a map $g\colon X' \to X$ of cofibrant fibrant objects and a map $h\colon Y \to Y'$ of fibrant object, we have

$$(h_*)(\sigma_f) = (\sigma_{hf})(h_*) \quad \textit{and} \quad (\sigma_{gf})(g^*) = (g^*)(\sigma_f).$$

If $H\colon I_+ \wedge X \to Y$ is a left homotopy from f to f', the isomorphisms satisfy $\sigma_f = (\sigma_{f'})(-)^H$. If f is the zero map, σ_f coincides with the canonical isomorphism.

Proof. Since $\mathcal{C}$ is a stable topological model category, the two endofunctors $S^1 \wedge -$ and $(-)^{S^1}$ of $\mathcal{C}$ form a Quillen equivalence. We define a functor $G\colon \mathcal{C} \to \mathcal{C}$ by $G(X) = (X^{S^1})^{\mathrm{cof}}$, where $(-)^{\mathrm{cof}}$ is the functorial cofibrant replacement. Then the Quillen equivalence property gives us a natural transformation $\tau\colon S^1 \wedge G(X) \to \mathrm{id}_{\mathcal{C}}$ such that τ_X is a weak equivalence if X is fibrant [Hov99, 1.3.13(b)]. By Lemma 4.1.2, we obtain an isomorphism

$$[S^n_+ \wedge X, Y]^f_{\mathrm{Ho}(X\downarrow\mathcal{C})} \xrightarrow{\cong} [S^n_+ \wedge S^1 \wedge G(X), Y]^{f\tau_X}_{\mathrm{Ho}(S^1\wedge G(X)\downarrow\mathcal{C})}.$$

If we apply the functor $- \wedge G(X)$ to the weak equivalence $S^n_+ \wedge S^1 \xrightarrow{\sim} S^{n+1} \vee S^1$ under S^1 to be constructed in Lemma 4.1.5, we get a weak equivalence of cofibrant objects in $(S^1 \wedge G(X) \downarrow \mathcal{C})$ which induces the isomorphism

$$[S^n_+ \wedge S^1 \wedge G(X), Y]^{f\tau_X}_{\mathrm{Ho}(S^1\wedge G(X)\downarrow\mathcal{C})} \xrightarrow{\cong} [(S^{n+1} \vee S^1) \wedge G(X), Y)]^{f\tau_X}_{\mathrm{Ho}(S^1\wedge G(X)\downarrow\mathcal{C})}.$$

The next step exploits the isomorphism $(S^{n+1}\vee S^1)\wedge G(X) \cong (S^{n+1}\wedge G(X))\vee(S^1\wedge G(X))$. Together with the derived adjunction isomorphism of the Quillen adjunction of

$$\mathcal{C} \to (X \downarrow \mathcal{C}), Z \mapsto (Z \to Z \vee X) \quad \text{and} \quad (X \downarrow \mathcal{C}) \to \mathcal{C}, (X \to Y) \mapsto Y,$$

we obtain an isomorphism

$$[(S^{n+1} \vee S^1) \wedge G(X), Y)]^{f\tau_X}_{\mathrm{Ho}(S^1\wedge G(X)\downarrow\mathcal{C})} \xrightarrow{\cong} [S^n \wedge S^1 \wedge G(X), Y]_{\mathrm{Ho}(\mathcal{C})}.$$

Finally, the weak equivalence of cofibrant objects $\tau_X\colon S^1 \wedge G(X) \to X$ induces

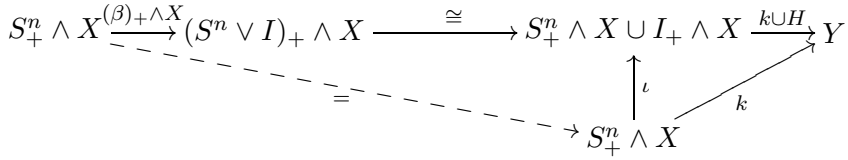
$$[S^n \wedge S^1 \wedge G(X), Y]_{\mathrm{Ho}(\mathcal{C})} \xleftarrow{\cong} [S^n \wedge X, Y]_{\mathrm{Ho}(\mathcal{C})}.$$

We define σ_f to be the composition of these four isomorphisms.

It is additive since the addition on the right hand side can as well be defined using the H-cogroup structure of S^n. Naturality with respect to a map $h\colon Y \to Y'$ is clear. The naturality with respect to $g\colon X' \to X$ is deduced from the naturality of the induced map of Lemma 4.1.2 and the existence of a commutative square

$$\begin{array}{ccc} S^1 \wedge G(X') & \xrightarrow{\tau_X} & X' \\ {\scriptstyle S^1\wedge G(g)}\downarrow & & \downarrow{\scriptstyle g} \\ S^1 \wedge G(X) & \xrightarrow{\tau_{X'}} & X. \end{array}$$

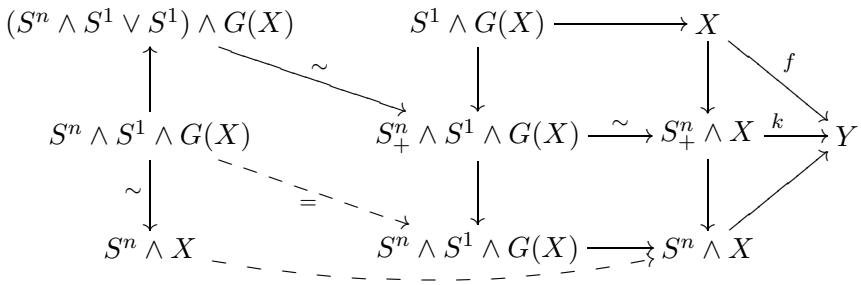
Next we choose an $k \in (X \downarrow \mathcal{C})(S^n_+ \wedge X, Y)$ that represents an $\alpha \in [S^n_+ \wedge X, Y]^f_{\mathrm{Ho}(X\downarrow\mathcal{C})}$. Then the action of a left homotopy $H\colon I_+ \wedge X \to Y$ from f to f' is represented by the composition of the maps in the first line of the diagram



Here $\beta\colon S^n \to S^n \vee I$ is a homotopy equivalence which sends the basepoint of S^n to the 'outer' point of the interval I. It is the kind of map one usually employs to obtain an isomorphism between homotopy groups with different basepoints that are connected by a path.

In the diagram, the right triangle commutes, and the left triangle commutes up to homotopy in $\mathcal{C}$, but not in $(X \downarrow \mathcal{C})$. Composing with the maps which induce σ_f, we see that this is enough obtain that $\sigma_f(\alpha^H)$ and $\sigma_f(\alpha)$ coincide in $[S^n \wedge X, Y]_{\mathrm{Ho}(\mathcal{C})}$.

Now suppose that f is the zero map in the pointed category $\mathcal{C}$. To see that σ_f coincides with the canonical isomorphism, we use Lemma 4.1.5 again to see that the following diagram commutes.



Passing over to the homotopy category and running from the lower left corner through the upper left corner and the middle line to Y gives a representative of $\sigma_f(\alpha)$. Running from the lower left corner using the lowest arrow to Y gives the image of k under the canonical isomorphism which exists since $f = 0$. □

The following lemma was used in the proof of the previous proposition:

Lemma 4.1.5. *For $n \geq 1$, there is a homotopy equivalence $\mu\colon (S^n \wedge S^1) \vee S^1 \xrightarrow{\sim} S^n_+ \wedge S^1$ of topological spaces under S^1. Here the structure map $S^1 \to S^n_+ \wedge S^1$ is given by smashing the map $S^0 \to S^n_+$ with S^1, and the structure map of $S^{n+1} \vee S^1$ is the inclusion of the second summand. If $p\colon S^n_+ \to S^n$ is the map which identifies the two basepoints of S^n_+ specified by $S^0 \to S^n_+$, the map*

$$S^n \wedge S^1 \xrightarrow{incl} (S^n \wedge S^1) \vee S^1 \xrightarrow{\mu} S^n_+ \wedge S^1 \xrightarrow{p \wedge S^1} S^n \wedge S^1$$

is the identity.

Proof. To construct the homotopy equivalence, we consider S^n as a CW-complex with one 0-cell and one n-cell. The complex $S^n \times S^1$ has 4 cells, a 0-cell, a 1-cell, an n-cell and an $n+1$-cell. Since $S^n_+ \wedge S^1 \cong S^n \times S^1/(S^n \times \{s_0\})$, this space has a CW-structure

which is obtained from the one of $S^n \times S^1$ by collapsing the n-cell to the 0-cell. Now the attaching map of the $n+1$-cell of $S^n_+ \wedge S^1$ is a map $S^n \to S^1$. This map is nullhomotopic for $n > 1$ since $\pi_n S^1 = 0$. For $n = 1$ it is nullhomotopic for a different reason: the attaching map of the 2-cell of $S^1 \times S^1$ is the attaching map of the 2-cell of a torus, and if we collapse one 1-cell, this map becomes nullhomotopic.

Since we know that the attaching map of $n+1$-cell of $S^{n+1}_+ \wedge S^1$ is nullhomotopic, this CW-complex is homotopy equivalent to the CW-complex which has also S^1 as 1-skeleton and a further $n+1$-cell attached using the constant map $S^n \to S^1$. This space is $S^n \vee S^1$.

Mapping from $S^n_+ \wedge S^1$ to $S^n \wedge S^1$ with $p \wedge S^1$ means that we also collapse the 1-cell of $S^n_+ \wedge S^1$ to the point. Therefore we do not see the effect of the nullhomotopy of the attaching map of the $n+1$-cell after mapping to $S^n \wedge S^1$. This verifies the last assertion of the lemma. □

We will also need the adjoint version of Proposition 4.1.4:

Corollary 4.1.6. *Let $\mathcal{C}$ be a stable topological model category and let $f\colon X \to Y$ be a map in $\mathcal{C}$ with X cofibrant fibrant and Y fibrant. Then there is an isomorphism*

$$\widetilde{\sigma}_f\colon [S^n, (\mathrm{Map}_{\mathcal{C}}(X,Y), f)]_{\mathrm{Ho}(\mathcal{T}op_*)} \xrightarrow{\cong} [S^n, (\mathrm{Map}_{\mathcal{C}}(X,Y), 0)]_{\mathrm{Ho}(\mathcal{T}op_*)}$$

of abelian groups.

Proof. This follows from Lemma 4.1.1, Proposition 4.1.4 and the isomorphism

$$[S^n, (\mathrm{Map}_{\mathcal{C}}(X,Y), f)]_{\mathrm{Ho}(\mathcal{T}op_*)} \cong [S^n_+, (\mathrm{Map}_{\mathcal{C}}(X,Y), f, 0)]_{\mathrm{Ho}(S^0\downarrow\mathcal{T}op_*)}$$

□

4.2 The construction of the class

We set up some notation. For $n \geq 1$, we will denote the n-fold cartesian product of the interval $[0,1]$ by W_n, and W_0 will denote the one point space.

We will use the set $T^n = \{0, 1, -1\}^n$ to index the subcubes of the cube W_n. The subcube of W_n associated to $t = (t_i)_{1 \leq i \leq n} \in T^n$ is

$$\{(a_1, \ldots, a_n) \in W_n | a_i = t_i \text{ if } t_i \neq -1\}.$$

Consequently, $|t| = |(t_i)_{1 \leq i \leq n}| = |\{t_i | t_i = -1\}|$ is the dimension of the subcube indexed by t, and there is a canonical embedding $\iota_t\colon W_{|t|} \to W_n$. The vertex $(1, \ldots, 1)$ of W_n will serve as the basepoint of W_n, turning it into an object of $\mathcal{T}op_*$. For $\delta \in \{0,1\}$ and k with $1 \leq k \leq n$, the sequence $t(n,k,\delta) \in T^n$ with $t(n,k,\delta)_i = \delta$ for $i = k$ and $t(n,k,\delta)_i = -1$ otherwise denotes a codimension 1 subcube of W_n.

We will also use projections $p_{i,j}\colon W_n \to W_{j-i+1}$ which are defined by $p_{i,j}(a_1, \ldots, a_n) = (a_i, \ldots, a_j)$ for $i \leq j$. If $i = j+1$, we write $p_{i,j}\colon W_n \to W_0$ for the unique map to the one point space.

Each n-dimensional cube has an obvious CW-structure with cells given by the subcubes described above. By $\mathrm{sk}_k W_n$ we will denote the k-skeleton of the CW-complex W_n. In particular, $\mathrm{sk}_{n-1} W_n$ is the boundary of the n-cube, which we sometimes denote by ∂W_n.

Definition 4.2.1. Let $\mathcal{U}$ be a small full subcategory of the homotopy category of a stable topological model category. For $n \geq 1$, an *n-cube system* for $\mathcal{U}$ consists of the following data. For every object X of $\mathcal{U}$, there is a chosen cofibrant and fibrant object of $\mathcal{C}$ representing it. The isomorphism between X and the representing object is part of the data. However, by abuse of notation, we denote the representing object by X as well. The zero object of $\mathrm{Ho}(\mathcal{C})$ is required to be represented by the zero object in $\mathcal{C}$.

Furthermore, for every j with $1 \leq j \leq n$ and every sequence

$$X_{j+1} \xrightarrow{f_{j+1}} X_j \xrightarrow{f_j} \dots \xrightarrow{f_1} X_0$$

of $(j+1)$ composable maps in $\mathcal{U}$, there is a map $b^j(f_1, \dots, f_{j+1})\colon (W_j)_+ \wedge X_{j+1} \to X_0$ in $\mathcal{C}$ such that the following conditions are satisfied.

(i) For $j = 0$, the map $(W_0)_+ \wedge X_1 \xrightarrow{b^0(f_1)} X_0$ in $\mathcal{C}$ represents f_1 in $\mathcal{U}$. The implicitly used natural isomorphism $(W_0)_+ \wedge X_1 \cong X_1$ which is part of the topological structure of $\mathcal{C}$ and the chosen isomorphisms to the representing objects are suppressed in the notation and will be suppressed in the sequel.

(ii) If one of the maps $f_1, \dots, f_{j+1}$ is a zero map in $\mathrm{Ho}(\mathcal{C})$, the map $b^j(f_1, \dots, f_{j+1})$ is the zero map in $\mathcal{C}$. In particular, the trivial map f_1 in $\mathrm{Ho}(\mathcal{C})$ is represented by the zero map $b^0(f_1)$ in $\mathcal{C}$.

(iii) For $j \geq 1$ and an i with $1 \leq i \leq j$, the following diagram commutes:

$$\begin{array}{ccc} (W_{j-1})_+ \wedge X_{j+1} & & \\ \downarrow{\scriptstyle (\iota_{t(j,i,1)})_+ \wedge X_{j+1}} & \searrow^{b^{j-1}(f_1,\dots,f_if_{i+1},\dots,f_{j+1})} & \\ (W_j)_+ \wedge X_{j+1} & \xrightarrow[b^j(f_1,\dots,f_{j+1})]{} & X_0 \end{array}$$

(iv) For $j \geq 1$ and an i with $1 \leq i \leq j$, the following diagram commutes:

$$\begin{array}{ccc} (W_{j-1})_+ \wedge X_{j+1} & & \\ \downarrow{\scriptstyle (\iota_{t(j,i,0)})_+ \wedge X_{j+1}} & \searrow^{b_i^{j-1}(f_1,\dots,f_{j+1})} & \\ (W_j)_+ \wedge X_{j+1} & \xrightarrow[b^j(f_1,\dots,f_{j+1})]{} & X_0 \end{array}$$

Here $b_i^{j-1}(f_1, \dots, f_{j+1})$ denotes the map

$$\begin{array}{ccc} (W_{j-1})_+ \wedge X_{j+1} & \xrightarrow{p_{1,i-1} \wedge p_{i,j-1} \wedge X_{j+1}} & (W_{i-1})_+ \wedge (W_{j-i})_+ \wedge X_{j+1} \\ & & \swarrow^{(W_{i-1})_+ \wedge b^{j-i}(f_{i+1},\dots,f_{j+1})} \\ (W_{i-1})_+ \wedge X_i & \xrightarrow[b^{i-1}(f_1,\dots,f_i)]{} & X_0. \end{array}$$

From the definition of an n-cube system it is easy to see that we have

$$b^n(f_1, \dots, f_{n+1})((\iota_{(1,\dots,1)})_+ \wedge X_{n+1}) = b^0(f_1 \cdots f_{n+1}).$$

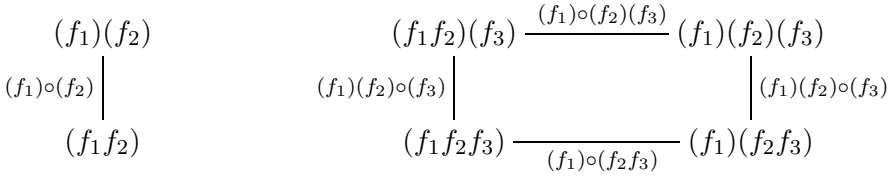


Figure 3: A 1-cube ... ... and a 2-cube.

As we have chosen the basepoint of W_n to be $(1,\dots,1)$, this means that $b^n(f_1,\dots,f_{n+1})$ is a map from $X_{n+1} \to (W_n)_+ \wedge X_{n+1}$ to $b^0(f_1\cdots f_{n+1})\colon X_{n+1} \to X_0$ in $(X_{n+1}\downarrow\mathcal{C})$.

An n-cube system for $\mathcal{U}$ contains choices of maps in $\mathcal{C}$ (on the model category level) which represent maps in $\mathcal{U}$ (on the homotopy category level). In general, it is not possible to choose representing maps such that $b^0(f_1)b^0(f_2) = b^0(f_1f_2)$ holds. If they exist, the maps $b^j(f_1,\dots,f_{j+1})$ for $j \geq 1$ encode coherence homotopies between different choices for representatives for sequences of composable maps. Figures 3 and 4 (compare [BM03, Figures 2.10 and 2.12]) illustrate the cases $n = 1, n = 2$, and $n = 3$. In the pictures, we write $(f_j\cdots f_k)$ for $b^0(f_j\cdots f_k)$ and $(f_j\cdots f_{k-1})\circ(f_k\cdots f_l)$ for $b^1(f_j\cdots f_{k-1}, f_k\cdots f_l)$.

Definition 4.2.2. Let $\mathcal{U}$ be a small full subcategory of the homotopy category of a stable topological model category $\mathcal{C}$. A *pre n-cube system* for $\mathcal{U}$ consists of the same data as an n-cube system for sequences of composable maps of length $\leq n$. For a sequence $(f_1,\dots,f_{n+1})$ of $(n+1)$ composable maps in $\mathcal{U}$, we only require to have maps $\widehat{b^n}(f_1,\dots,f_{n+1})\colon (\mathrm{sk}_{n-1}W_n)_+ \wedge X_{n+1} \to X_0$. The data is asked to satisfy the same compatibility conditions as that of an n-cube system. This makes sense since the conditions only involve $\mathrm{sk}_{n-1}W_n$ in the top dimension.

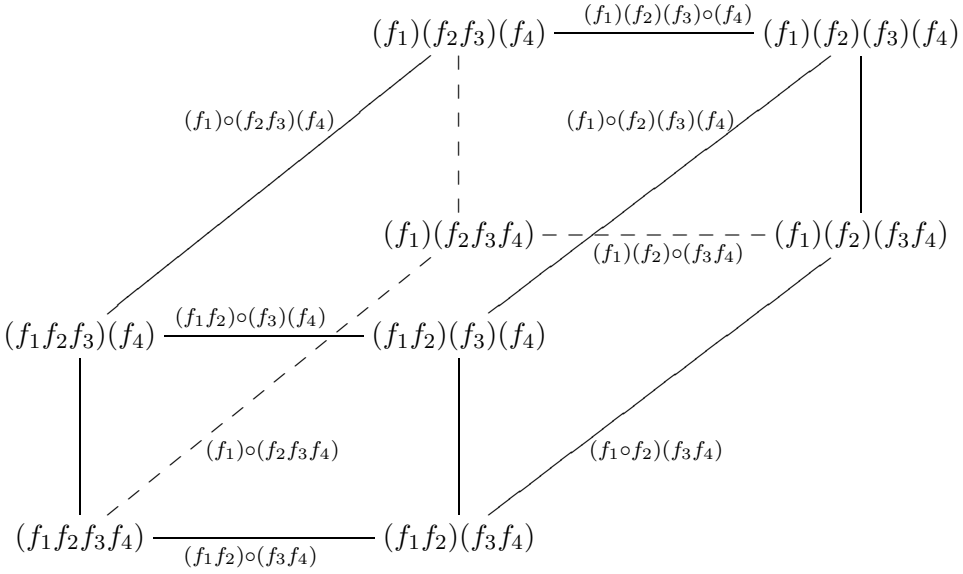


Figure 4: A 3-cube.

Similarly to the map $b^n(f_1,\dots,f_{n+1})$ of an n-cube system, $\widehat{b}^n(f_1,\dots,f_{n+1})$ can be interpreted as a map in $(X_{n+1}\downarrow\mathcal{C})$ from $X_{n+1}\to(W_n)_+\wedge X_{n+1}$ to $b^0(f_1\cdots f_{n+1})\colon X_{n+1}\to X_0$.

Lemma 4.2.3. *An $(n-1)$-cube system for $\mathcal{U}$ can be extended to a pre n-cube system. The restriction of the maps $\widehat{b}^n(f_1,\dots,f_{n+1})\colon(\mathrm{sk}_{n-1}W_n)_+\wedge X_{n+1}\to X_0$ to the $(n-1)$-dimensional subcubes of $\mathrm{sk}_{n-1}W_n$ that are indexed by $t(n,i,0)$ with $1<i<n$ is already determined by the underlying $(n-2)$-cube system.*

Proof. For a sequence of $(n+1)$ composable maps $(f_1,\dots,f_{n+1})$ we have to define a map $(\mathrm{sk}_{n-1}W_n)_+\wedge X_{n+1}\to X_0$. Since $\mathrm{sk}_{n-1}W_n$ is the union of the codimension 1 subcubes of W_n, we define the map on each of these and check that the choices coincide on subcubes of codimension 2.

On the subcube of W_n of codimension 1 indexed by $t(n,i,1)$ with $1\le i\le n$, we choose the map

$$b^{n-1}(f_1,\dots,f_if_{i+1},\dots,f_{n+1})\colon(W_{n-1})_+\wedge X_{n+1}\to X_0.$$

On the subcube of W_n of codimension 1 indexed by $t(n,i,0)$ with $1\le i\le n$, we choose the map

$$b_i^{n-1}(f_1,\dots,f_{n+1})\colon(W_{n-1})_+\wedge X_{n+1}\to X_0$$

which was introduced in Definition 4.2.1. Inspecting the definition of b_i^n, it is easy to see that the additional assumption on the restriction to the cubes $t(n,i,0)$ with $1<i<n$ is satisfied. Though it may be obvious that they assemble to a well defined map, we give the details.

We have to check that these maps coincide on the intersection of the $(n-1)$-cubes, which are $(n-2)$-dimensional subcubes of $(\mathrm{sk}_{n-1}W_n)$. In a similar fashion as known from the simplicial identities, we have a commutative diagram

$$\begin{array}{ccc} W_{n-2} & \xrightarrow{\iota_{t(n-1,j,\delta)}} & W_{n-1} \\ {\scriptstyle\iota_{t(n-1,k-1,\epsilon)}}\downarrow & & \downarrow{\scriptstyle\iota_{t(n,k,\epsilon)}} \\ W_{n-1} & \xrightarrow[\iota_{t(n,j,\delta)}]{} & W_n \end{array}$$

for $1\le j<k\le n$ and $\epsilon,\delta\in\{0,1\}$.

First let us consider the case of an $(n-2)$-cube specified by $t\in T^n$ with $t_j=1=t_k$ and $t_i=-1$ for $i\notin\{j,k\}$. Here we have to check the commutativity of the following diagram:

$$\begin{array}{ccc} (W_{n-2})_+\wedge X_{n+1} & \xrightarrow{(\iota_{t(n-1,j,0)})_+\wedge X_{n+1}} & (W_{n-1})_+\wedge X_{n+1} \\ {\scriptstyle(\iota_{t(n-1,k-1,1)})_+\wedge X_{n+1}}\downarrow & & \downarrow{\scriptstyle b^{n-1}(f_1,\dots,f_kf_{k+1},\dots,f_{n+1})} \\ (W_{n-1})_+\wedge X_{n+1} & \xrightarrow{b^{n-1}(f_1,\dots,f_jf_{j+1},\dots,f_{n+1})} & X_0 \end{array}$$

It is easy to see that both twofold compositions in the diagram equal

$$b^{n-2}(f_1,\dots,f_jf_{j+1},\dots,f_kf_{k+1},\dots f_{n+1}).$$

Next we check the case of an $(n-2)$ cube specified by a $t \in T^n$ with $t_j = 0$ and $t_k = 1$ for $1 \leq j < k \leq n$. This time we have to verify the commutativity of

$$\begin{array}{ccc} (W_{n-2})_+ \wedge X_{n+1} & \xrightarrow{(\iota_{t(n-1,j,0)})_+ \wedge X_{n+1}} & (W_{n-1})_+ \wedge X_{n+1} \\ \downarrow{\scriptstyle (\iota_{t(n-1,k-1,0)})_+ \wedge X_{n+1}} & & \downarrow{\scriptstyle b^{n-1}(f_1,\dots,f_k f_{k+1},\dots,f_{n+1})} \\ (W_{n-1})_+ \wedge X_{n+1} & \xrightarrow{b_j^{n-1}(f_1,\dots,f_{n+1})} & X_0. \end{array}$$

In this case, both twofold compositions equal $b_j^{n-2}(f_1, \dots, f_k f_{k+1}, \dots, f_{n+1})$. The case of a $t \in T^n$ with $t_j = 1$, $t_k = 0$, and $t_i = -1$ for $i \notin \{j, k\}$ is similar.

For $t \in T^n$ with $t_j = t_k = 0$ and $t_i = -1$ for $i \notin \{j, k\}$ we have to check the commutativity of an obvious diagram in which both twofold compositions turn out to be

$$\begin{array}{ccccc} (W_{n-2})_+ \wedge X_{n+1} & \xrightarrow{(p_{1,j-1} \times p_{j,k-1} \times p_{k,n-2}) \wedge X_{n+1}} & (W_{j-1})_+ \wedge (W_{k-j-1})_+ \wedge (W_{n-k})_+ \wedge X_{n+1} & & \\ & \xleftarrow{(W_{j-1})_+ \wedge (W_{k-j-1})_+ \wedge b^{n-k}(f_{k+1},\dots,f_{n+1})} & & & \\ (W_{j-1})_+ \wedge (W_{k-j-1})_+ \wedge X_k & \xrightarrow[(W_{j-1})_+ \wedge b^{k-j-1}(f_{j+1},\dots,f_k)]{} & (W_{j-1})_+ \wedge X_j & \xrightarrow[b^{j-1}(f_1,\dots,f_j)]{} & X_0. \end{array}$$

□

Lemma 4.2.4. *Let $f\colon X \to Y$ be a map from a cofibrant object X to a fibrant object Y in $\mathcal{C}$. A map $b\colon (\mathrm{sk}_{n-1} W_n)_+ \wedge X \to Y$ under X which represents the trivial map in $[(\mathrm{sk}_{n-1} W_n)_+ \wedge X, Y]^f_{\mathrm{Ho}(X\downarrow\mathcal{C})}$ can be extended to a map $(W_n)_+ \wedge X \to Y$ in $(X \downarrow \mathcal{C})$.*

Proof. This is an easy consequence of Lemma 4.1.1. □

Proposition 4.2.5. *Let $\mathcal{U}$ be a small n-split subcategory of the homotopy category of a stable topological model category $\mathcal{C}$. Then there exists an n-cube system for $\mathcal{U}$.*

Proof. First we choose for every object of $\mathcal{U}$ a cofibrant and fibrant object in $\mathcal{C}$ which represents it. In the next step, we choose for every map $f_1\colon X_1 \to X_0$ in $\mathcal{U}$ a map $b^0(f_1)\colon (W_0)_+ \wedge X_1 \to X_0$ which represents f_1. We choose the zero map $b^0(f_1)$ in $\mathcal{C}$ if f_1 is a trivial map in $\mathrm{Ho}(\mathcal{C})$. Extending these data to a 1-cube system amounts to choosing a homotopy $b^1(f_1, f_2)\colon (W_1)_+ \wedge X_2 \to X_1$ between the two maps $b^0(f_1 f_2)$ and $b^0(f_1)b^0(f_2)$, which is always possible. Again, we choose $b^1(f_1, f_2)$ to be the trivial map if either f_1 or f_2 is trivial.

Now suppose we have constructed a j-cube system for some $j < n$. We want to extend it to a $(j+1)$-cube system. By Lemma 4.2.3, we can extend it to a pre $(j+1)$-cube system. Hence for each sequence $(f_1, \dots, f_{j+2})$ of $j+2$ composable maps in $\mathcal{U}$, we have to extend the map $\widehat{b}^{j+1}(f_1, \dots, f_{j+2})\colon (\mathrm{sk}_j W_{j+1})_+ \wedge X_{j+2} \to X_0$ of the pre $(j+1)$-cube system to a map $(W_{j+1})_+ \wedge X_{j+2} \to X_0$. If one of the maps $(f_1, \dots, f_{j+2})$ is trivial, $\widehat{b}^{j+1}(f_1, \dots, f_{j+2})$ is the trivial map in $\mathcal{C}$, and we extend it to $(W_{j+1})_+ \wedge X_{j+2}$ by taking the zero map. If no f_i happens to be zero, we know by Lemma 4.2.4 that it is enough to show that the homotopy class of $\widehat{b}^{j+1}(f_1, \dots, f_{j+2})$ in

$$[(\mathrm{sk}_j W_{j+1})_+ \wedge X_{j+2}, X_0]^{b^0(f_1 \cdots f_{j+2})}_{\mathrm{Ho}(X_{j+2}\downarrow\mathcal{C})}$$

vanishes. By Proposition 4.1.4, this follows from $\mathcal{U}$ being n-split. □

Now we are ready to introduce our main object of study.

Construction 4.2.6. Let $\mathcal{C}$ be a stable topological model category and let $\mathcal{U}$ be a small n-split subcategory of $\mathrm{Ho}(\mathcal{C})$. Then there is a well defined cohomology class $\gamma_{\mathcal{U}} \in H^{n+2}(\mathcal{U}, [-,-]_n^{\mathrm{Ho}(\mathcal{C})})$ which determines by evaluation all $(n+2)$-fold Toda brackets of complexes of $n+2$ composable maps in $\mathcal{U}$.

We choose an n-cube system for $\mathcal{U}$ which is possible by Proposition 4.2.5, and extend it to a pre $(n+1)$-cube system by Lemma 4.2.3. Then we define a normalized cochain $c \in \overline{C}^{n+2}(\mathcal{U}, [-,-]_n^{\mathrm{Ho}(\mathcal{C})})$ as follows. Its evaluation on a sequence of $(n+2)$ composable maps

$$X_{n+2} \xrightarrow{f_{n+2}} X_{n+1} \xrightarrow{f_{n+1}} \dots \xrightarrow{f_1} X_0$$

in $\mathcal{U}$ is the image of the homotopy class of the map $\widehat{b}^{n+1}(f_1, \dots, f_{n+2})$ under the isomorphism

$$\begin{array}{c} [(\mathrm{sk}_n W_{n+1})_+ \wedge X_{n+2}, X_0]^{b^0(f_1 \cdots f_{n+2})}_{\mathrm{Ho}(X_{n+2}\downarrow\mathcal{C})} \\ \Big\downarrow {\scriptstyle \sigma_{b^0(f_1 \cdots f_{n+2})}} \\ [\mathrm{sk}_n W_{n+1} \wedge X_{n+2}, X_0]^{\mathrm{Ho}(\mathcal{C})} \cong [X_{n+2}[n], X_0]^{\mathrm{Ho}\,\mathcal{C}} \end{array}$$

In Lemma 4.2.7, we will show that this cochain is a cocycle, and in Lemma 4.2.8 we will verify that the cohomology class of this cocycle does not depend on the choice of the cube system. We observed in Remark 3.4.4 that the Toda bracket of a complex has the same indeterminacy as the evaluation of a cohomology class in the (normalized) cohomology of categories. Hence it is enough to show that the evaluation of our cocycle on a complex is an element of the Toda bracket of the complex. This will be proved in Proposition 4.3.8.

We will use the Homotopy Addition Theorem [Bre97, VII.9.6] to prove that the cochain constructed in the last theorem is a cocycle. For this we need to choose orientations of the attaching maps of the $(n+1)$-cells of the CW-complex $\mathrm{sk}_{n+1} W_{n+2}$. This space is homeomorphic to an $(n+1)$-sphere, and we will choose an orientation on the $(n+1)$-cells such that they are oriented coherently with $\mathrm{sk}_{n+1} W_{n+2}$.

We start with fixing the vertex $e = (0,1,0,1,\dots)$ of W_{n+2} and the opposite vertex $e' = (1,0,1,0,\dots)$. Each $(n+1)$-dimensional subcube of W_{n+2} contains either e or e'. We write $T_e \subset T^{n+2}$ for the indexing set of those containing e, and $T_{e'}$ for the indexing set of those containing e'.

The union of all n-dimensional subcubes of $\mathrm{sk}_{n+1} W_{n+2}$ containing neither e nor e' is homeomorphic to an n-sphere. Changing the CW-structure for a moment, $\mathrm{sk}_{n+1} W_{n+2}$ can be obtained from this space by attaching two $(n+1)$-cells. For $\mathrm{sk}_{n+1} W_{n+2}$ to be oriented, the attaching maps of these two cells have to have opposite orientations. Subdividing to the CW-structure of $\mathrm{sk}_{n+1} W_{n+2}$ given by the subcubes, this means that we can choose all $(n+1)$-cells containing e to have the same, say positive, orientation, and all $(n+1)$-cell containing e' to have negative orientation.

Let K be a pointed space. The Homotopy Addition Theorem [Bre97, VII.9.6] now says that for every based map $f\colon \mathrm{sk}_n W_{n+2} \to K$, we have

$$\sum_{t \in T_e} [f|_{\iota_t}] - \sum_{t \in T_{e'}} [f|_{\iota_t}] = 0 \quad \text{in } \widetilde{\pi}_n(K).$$

Here $f|_{\iota_t}$ denotes the restriction of f to the copy of $\mathrm{sk}_n W_{n+1}$ in $\mathrm{sk}_n W_{n+2}$ indexed by t, and $\widetilde{\pi}_n(K)$ is $\pi_n(K)$ if $n > 1$ and the abelianized $\pi_1(K)$ for $n = 1$. The use of $\widetilde{\pi}_n$ is necessary for the following reason: if the cube indexed by t does not contain the basepoint $(1, \dots, 1)$ of $\mathrm{sk}_n W_{n+2}$, we do have to use the action of a path to the basepoint on $f|_{\iota_t}$ to get an element of $\pi_n(K)$, and this may depend on the homotopy class of the path for $n = 1$ (compare [Bre97]).

Lemma 4.2.7. *The cochain of Construction 4.2.6 is a cocycle.*

Proof. In a similar way as in the construction of a pre cube system from a cube system, we can use the data of the chosen n-cube system to construct a map

$$d = d(f_1, \dots, f_{n+3}) \colon (\mathrm{sk}_n W_{n+2})_+ \wedge X_{n+3} \to X_0$$

with the following properties: for each k with $1 \leq k \leq n+2$, the restriction of d along $\iota_{t(n+2,k,\delta)}$ satisfies

$$(\iota_{t(n+2,k,1)})^* d = \widehat{b}^n(f_1, \dots, f_k f_{k+1}, \dots, f_{n+3})$$

and

$$(\iota_{t(n+2,k,0)})^* d = \begin{cases} (b^0(f_{n+3}))^*(\widehat{b}^n(f_1, \dots, f_{n+2})) & \text{if } k = n+2 \\ b_k^{n+1}(f_1, \dots, f_{n+3})|_{\mathrm{sk}_n W_{n+1}} & \text{if } 1 < k < n+2 \\ (b^0(f_1))_*(\widehat{b}^n(f_2, \dots, f_{n+3})) & \text{if } k = 0 \end{cases}$$

Here the map $b_k^{n+1}(f_1, \dots, f_{n+3}) \colon (W_{n+1})_+ \wedge X_{n+3} \to X_0$ is the one introduced in Definition 4.2.1. It is, in a similar way as pointed out in Lemma 4.2.3, already determined by the n-cube system. As it is implicit in the conditions listed above, d is a map under $b^0(f_1 \cdots f_{n+3})$.

After adjoining and forgetting the additional basepoint, we get a map

$$\widehat{d} \colon \mathrm{sk}_n W_{n+2} \to (\mathrm{Map}_{\mathcal{C}}(X_{n+3}, X_0), b^0(f_1, \dots, f_{n+3})).$$

As explained before the lemma, the Homotopy Addition Theorem yields

$$0 = \sum_{t \in T_e} [\widehat{d}|_{\iota_t(\mathrm{sk}_n W_{n+1})}] - \sum_{t \in T_{e'}} [\widehat{d}|_{\iota_t(\mathrm{sk}_n W_{n+1})}]$$

in $\widetilde{\pi}_n(\mathrm{Map}_{\mathcal{C}}(X_{n+3}, X_0))$. The homotopy classes of those maps indexed by a $t(n+2, k, 0)$ with $1 < k < n+2$ vanish since these maps can be extended to W_{n+1}.

To interpret this equation in $\pi_n(\mathrm{Map}_{\mathcal{C}}(X_{n+3}, X_0), b^0(f_1 \cdots f_{n+3}))$, we only have to change the homotopy classes of the maps of the terms corresponding to $t(n+2, n+2, 0)$ and $t(n+2, 1,))$ with paths from $b^0(f_1)b^0(f_2 \cdots f_{n+3})$ and $b^0(f_1 \cdots f_{n+2})b^0(f_{n+3})$ to $b^0(f_1 \dots f_{n+3})$ in order to have the right basepoints. We do not loose information when passing to the abelianized fundamental group in the case $n = 1$, as it is a consequence of Proposition 4.1.4, Lemma 4.1.1, and $\mathcal{C}$ being stable that $\mathrm{Map}_{\mathcal{C}}(X_{n+3}, X_0)$ has an abelian fundamental group.

After adjoining, we get a sum of homotopy classes of maps $(\mathrm{sk}_n W_{n+1})_+ \wedge X_3 \to X_1$. Under the adjunction, the action of the paths correspond to actions of homotopies.

Now we can apply the isomorphism $\sigma_{b^0(f_1\cdots f_{n+3})}$ of Proposition 4.1.4 to this formula. Exploiting that it is natural, additive, and invariant under the action of homotopies, we obtain the formula

$$\begin{aligned} 0 = (f_1)_* c(f_1,\dots,f_{n+3}) + \sum_{i=1}^{n+2} (-1)^i c(f_1,\dots,f_i f_{i+1},\dots,f_{n+3}) \\ + (-1)^{n+3} (f_{n+3})^* c(f_1,\dots,f_{n+3}). \end{aligned}$$

The signs in this formula are a consequence of our orientation conventions. For example, the sign of $c(f_1,\dots,f_if_{i+1},\dots,f_{n+3})$ is positive if and only if $e=(0,1,0,\dots)$ lies in the $(n+1)$-dimensional cube indexed by $t(n+2,i,1)$, and this is the case if and only if the i^{th} entry of e is 1. □

For the $n=1$ case of the last lemma, it is again helpful to have a look at Figure 4. The first and the last term of the sum correspond to the back face and the upper face of the cube. The three middle terms correspond to the 3 faces containing $(f_1f_2f_3f_4)$, and the term belonging to the right face vanishes as the 'product homotopy' $b^1(f_1,f_2)b^1(f_3,f_4)$ needed to fill the 2-cube is already given by the data of the 1-cube system.

Lemma 4.2.8. *The cohomology class of Construction 4.2.6 does not depend on the choice of a cube system.*

Proof. We have to show that the cocycle associated to another n-cube system for $\mathcal{U}$ gives a cocycle representing the same cohomology class as the one associated to our original choice.

First assume that we are only given a different $(n-1)$-cube system associated to $\mathcal{U}$. Then we will show that we can extend it to an n-cube system which yields the same cohomology class as our first choice.

For this we fix an n-cube system (X_j, b^j) and an $(n-1)$-cube system $(\underline{X}_j, \underline{b}^j)$. For every object of $\mathcal{U}$, the data of the cube system gives an isomorphism between the representing objects X and $\underline{X}$ which we can realize by a weak equivalence $g\colon X \to \underline{X}$ since both objects are fibrant and cofibrant. For a map f_1 in $\mathcal{U}$, the diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{b^0(f_1)} & X_0 \\ {\scriptstyle g_1}\downarrow & & \downarrow{\scriptstyle g_0} \\ \underline{X}_1 & \xrightarrow{\underline{b}^0(f_1)} & \underline{X}_0 \end{array}$$

will in general not be commutative in $\mathcal{C}$, but it commutes in $\mathrm{Ho}(\mathcal{C})$. The homotopy can be considered as a map $h^0(f_1)\colon I_+ \wedge (W_0)_+ \wedge X_1 \to \underline{X}_0$ with $(\iota_0)^* h^0(f_1) = g_0 b^0(f_1)$ and $(\iota_1)^* h^0(f_1) = \underline{b}^0(f_1) g_1$. Here ι_0, ι_1 denote the two inclusions of the endpoints in the unit interval I.

Using the same arguments as in the construction of a cube system in Proposition 4.2.5, we can inductively find maps $h^j(f_1,\dots,f_{j+1})\colon I_+ \wedge (W_j)_+ \wedge X_j \to \underline{X}_0$ for $j<n$ such that

$$\begin{aligned} (\iota_0)^* h^j(f_1,\dots,f_{j+1}) &= g_0 b^j(f_1,\dots,f_{j+1}) \qquad \text{and} \\ (\iota_1)^* h^j(f_1,\dots,f_{j+1}) &= \underline{b}^j(f_1,\dots,f_{j+1}) g_{j+1}. \end{aligned}$$

For $j = n$, we construct a map $h^n(f_1, \dots, f_{n+1})$ as follows: the cube system (X_j, b^j) specifies its values on $\{0\} \times W_n$. We can extend it to $\partial(I \times W_n) \setminus \{1\} \times W_n$ since $\mathcal{U}$ is n-split. Using the homotopy extension property of

$$\partial(I \times W_n) \setminus (\{1\} \times W_n) \to I \times W_n,$$

we get a map $h^n(f_1, \dots, f_{n+1})\colon I_+ \wedge (W_n)_+ \wedge X_{n+1} \to \underline{X}_0$. The restriction of h^n to $\{1\} \times W_n$ is an extension of the $(n-1)$-cube system $(\underline{X}_j, \underline{b}^j)$ to an n-cube system. When we pass over to the associated pre n-cube systems, we obtain a homotopy

$$h\colon I_+ \wedge (\mathrm{sk}_n W_{n+1})_+ \wedge X_{n+2} \to \underline{X}_0$$

between the maps $g_0\widehat{b}^{n+1}(f_1, \dots, f_{n+2})$ and $\widehat{\underline{b}}^{n+1}(f_1, \dots, f_{n+2})g_{n+2}$. Therefore, the latter map represents $[g_0\widehat{b}^{n+1}(f_1, \dots, f_{n+2})]^{h^0(f_1 \cdots f_{n+2})}$ in

$$[(\mathrm{sk}_n W_{n+1})_+ \wedge X_{n+2}, \underline{X}_0]^{b^0(f_1 \cdots f_{n+2})g_{n+2}}_{\mathrm{Ho}(\underline{X}_{n+2}\downarrow\mathcal{C})}.$$

Hence the isomorphism σ sends both maps to the same homotopy class in $[X_{n+2}, X_0]_n^{\mathrm{Ho}(\mathcal{C})}$.

The second step is similar to the proof of the last lemma and will use the Homotopy Addition Theorem again. We can now assume that we are given two n-cube systems $(X_j, b^j), (\underline{X}_j, \underline{b}^j)$ with the same underlying $(n-1)$-cube system. As observed in Lemma 4.2.3, the maps $\widehat{b}^{n+1}$ and $\widehat{\underline{b}}^{n+1}$ associated to their pre $(n+1)$-cube systems coincide on all n-dimensional subcubes of $\mathrm{sk}_n W_{n+1}$ indexed by $t(n+1, k, 0)$ with $1 < k < n+1$.

We denote the set of all n-dimensional subcubes of $\mathrm{sk}_n W_{n+1}$ on which the two pre $(n+1)$-cube systems possibly deviate by $T_d \subset T^{n+1}$, i.e.,

$$T_d = \{t(n+1, k, 1) | 1 \le k \le n+1\} \cup \{t(n+1, k, 0) | k \in \{1, n+1\}\}.$$

Now let A be the space obtained by gluing for each $t \in T_d$ one copy of $W_n \cup_{\partial W_n} W_n$ to $\mathrm{sk}_n W_{n+1}$, using the right copy of W_n in the first term and the copy of W_n associated to t in the second. Then A has two inclusion $i, \underline{i}\colon \mathrm{sk}_n W_{n+1} \to A$, the first being the canonical inclusion and the second being the inclusion using the left 'new' copy of W_n on all subcubes indexed by a $t \in T_d$.

For a given sequence of composable maps $(f_1, \dots, f_{n+2})$, the two pre cube systems together yield a map $a = a(f_1, \dots, f_{n+2})\colon A_+ \wedge X_{n+2} \to X_0$ with $a(i_+ \wedge X_{n+2}) = \widehat{b}^{n+1}$ and $a(\underline{i}_+ \wedge X_{n+2}) = \widehat{\underline{b}}^{n+1}$. The Homotopy Addition Theorem yields the formula

$$\widehat{\underline{b}}^{n+1} = \widehat{b}^{n+1} + \sum_{t \in T_d \cap T_e} (i_t^*(a)) - \sum_{t \in T_d \cap T_e} (i_t^*(a)),$$

where $i_t\colon W_n \cup_{\partial W_n} W_n \to A$ is the inclusion which belongs to $t \in T_d$. The signs arise in the same way as in the last lemma.

Next let $\widehat{a} = \widehat{a}(f_1, \dots, f_{n+1})\colon (W_n \cup_{\partial W_n} W_n)_+ \wedge X_{n+1} \to X_0$ be the map which is $b^n(f_1, \dots, f_{n+1})$ on the right copy of W_n and $\underline{b}^n(f_1, \dots, f_{n+1})$ on the left copy. Then we can define an $(n+1)$-cochain $\overline{a} \in \overline{C}^{n+1}(\mathcal{U}, [-,-]_n^{\mathrm{Ho}(\mathcal{C})})$ whose value on a sequence of $(n+1)$ composable maps $(f_1, \dots, f_{n+1})$ is

$$\overline{a}(f_1, \dots, f_{n+1}) = \sigma_{b^0(f_1 \cdots f_{n+1})}[\widehat{a}(f_1, \dots, f_{n+1})].$$

If we now apply the isomorphism $\sigma_{b^0(f_1 \cdots f_{n+2})}$ to the formula obtained above from the Homotopy Addition Theorem and exploit its additivity and naturality again, we obtain

$$c(f_1, \ldots, f_{n+2}) = \underline{c}(f_1, \ldots, f_{n+2}) + (\delta\overline{a})(f_1, \ldots, f_{n+2}),$$

where c and $\underline{c}$ are the two cocycles belonging to the two cube systems and $\delta\overline{a}$ is the boundary of $\overline{a}$. Hence the cohomology class of these two cocycles coincide. □

The last lemma completes the construction of the class $\gamma_{\mathcal{U}}$. For later use, we prove two more lemmas closely related to this construction.

Lemma 4.2.9. *Let $\mathcal{C}$ be a stable topological model category and let $\mathcal{U}$ be a small n-split subcategory of* $\mathrm{Ho}(\mathcal{C})$. *Suppose that $\gamma_{\mathcal{U}} \in H^{n+2}(\mathcal{U}, [-,-]_n^{\mathrm{Ho}(\mathcal{C})})$ is zero. Then we can change the maps b^n of any n-cube system for $\mathcal{U}$ such that the resulting modified n-cube system has the zero cochain in $\overline{C}^{n+2}(\mathcal{U}, [-,-]_n^{\mathrm{Ho}(\mathcal{C})})$ as the associated cocycle representing $\gamma_{\mathcal{U}}$. In particular, the modified n-cube system can be extended to an $(n+1)$-cube system.*

Proof. Let (X_j, b^j) be any n-cube system for $\mathcal{U}$, and let $c \in \overline{C}^{n+2}(\mathcal{U}, [-,-]_n^{\mathrm{Ho}(\mathcal{C})})$ be the associated cocycle representing $\gamma_{\mathcal{U}}$. Our assumption $\gamma_{\mathcal{U}} = 0$ implies the existence of a cochain $e \in \overline{C}^{n+1}(\mathcal{U}, [-,-]_n^{\mathrm{Ho}(\mathcal{C})})$ with $\delta(e) = c$.

We use e to change the n-cube system. This should be interpreted as the inverse to our construction in the proof of Lemma 4.2.8.

Let

$$X_{n+1} \xrightarrow{f_{n+1}} X_n \xrightarrow{f_n} \ldots \xrightarrow{f_1} X_0$$

be a sequence of composable maps in $\mathcal{U}$. As in the last lemma, we model the n-sphere by gluing two copies of the n-cube W_n together along their boundaries. Using the isomorphism of Corollary 4.1.6 and the homotopy extension property, we can represent $e(f_1, \ldots, f_{n+1})$ by a map

$$\widehat{e}(f_1, \ldots, f_{n+1})\colon (W_n \cup_{\partial W_n} W_n) \to \mathrm{Map}_{\mathcal{C}}(X_{n+1}, X_0)$$

which coincides with the adjoint of $b^n(f_1, \ldots, f_{n+1})$ on the first copy of W_n. If one of the maps f_i is the zero map, we can choose $\widehat{e}(f_1, \ldots, f_{n+1})$ to be the trivial map to the basepoint. Now we define $\underline{b}^n(f_1, \ldots, f_{n+1})$ to be the restriction of $\widehat{e}(f_1, \ldots, f_{n+1})$ to the second copy of W_n.

The maps $\underline{b}^n$, together with the maps b^j for $j < n$, specify the data of an n-cube system, as $\underline{b}^n$ coincides with b^n on $\mathrm{sk}_{n-1} W_n$. Let $\underline{c} \in \overline{C}^{n+2}(\mathcal{C}, [-,-]_n^{\mathrm{Ho}(\mathcal{C})})$ be the cochain associated to the new cube system. With the same arguments as in Lemma 4.2.8, we see that the cochains c and $\underline{c}$ differ by $\delta\overline{a}$. Hence $\underline{c}$ is zero. □

Lemma 4.2.10. *Let $G\colon \mathcal{C} \to \mathcal{D}$ be a left Quillen functor between stable topological model categories $\mathcal{C}$ and $\mathcal{D}$ which commutes up to natural isomorphism with the action of $\mathcal{T}op_*$ on $\mathcal{C}$ and $\mathcal{D}$. If $\mathcal{U}$ and $\mathcal{W}$ are small n-split subcategories of* $\mathrm{Ho}(\mathcal{C})$ *and* $\mathrm{Ho}(\mathcal{D})$ *such that G induces an equivalence $\mathcal{U} \to \mathcal{W}$, then the induced isomorphism*

$$H^{n+2}(\mathcal{W}, [-,-]_n^{\mathrm{Ho}(\mathcal{D})}) \to H^{n+2}(\mathcal{U}, [-,-]_n^{\mathrm{Ho}(\mathcal{C})})$$

sends $\gamma_{\mathcal{W}}$ to $\gamma_{\mathcal{U}}$.

Proof. Since G is a left Quillen functor, it induces a functor on the homotopy categories. In particular, it preserves colimits. When we apply G to the data of an n-cube system (X_j, b^j) for $\mathcal{U}$, we almost get an n-cube system for $\mathcal{W}$. The only missing part is that the objects $G(X_j)$ are not fibrant. If $p_j \colon G(X_j) \to \underline{Y}_j$ denotes a map to a fibrant replacement for every j, we can extend these maps to a map of cube systems in a similar fashion as in the last lemma, using the homotopy extension property to obtain a cube system for $\mathcal{W}$ with the $\underline{Y}_j$ as the chosen objects in $\mathcal{D}$. It is clear that the cocycle defined in terms of the cube system for $\mathcal{U}$ is mapped to the cocycle defined in terms of this cube system for $\mathcal{W}$. Since the associated cohomology classes do not depend the choices, we are done. □

4.3 Comparing definitions of Toda brackets

Triple Toda bracket where introduced by Toda [Tod52, Tod62] to study the stable homotopy groups of spheres. Higher Toda brackets where introduced in the 60's, and there are different approaches in the literature. One of them is Cohen's definition using *filtered objects* [Coh68, §2]. It was originally introduced in the context of the homotopy category of spaces or the stable homotopy category and generalizes easily to triangulated categories [Shi02, App A]. We used this definition in the last section in order to show how Toda brackets determine realizability obstructions. As pointed out in Remark 3.4.3, this definition also specializes to the definition of higher Massey products when applied to the derived category of a differential graded algebra.

Another approach is Spanier's definition of higher Toda brackets [Spa63] using the concept of a *carrier*, which is a functor from a simplex to the category of spaces (see for example [Spa63, 4.8] for his definition of a 4-fold Toda bracket). A related concept is Klaus' definition of a pyramid [Kla01, 3.4], which is as well a system of higher coherence homotopies. This is linked to Spanier's definition by [Kla01, Proposition 3.6].

The perhaps most general approach to Toda brackets and other higher homotopy operations is that of Blanc and Markl [BM03], who define them as obstructions to realizing a homotopy commutative diagram by a strictly commutative one. The case of Toda brackets is linked to Spanier's definition by [BM03, Example 3.12].

In Lemma 4.3.1 below we will see that the evaluation of the universal Toda bracket can be interpreted as something similar to a pyramid in the sense of Klaus. Proposition 4.3.8 will then show that this is in fact equivalent to the Toda bracket in the context of triangulated categories defined in terms of filtered objects. Therefore, our comparison can be interpreted as a link between these different approaches, which does not seem to be covered by the literature. Nevertheless, we point out that we do not claim that all these approaches are equivalent in general, and that it is not our aim to prove this here.

Let $\mathcal{C}$ be a stable topological model category and let $\mathcal{U}$ be an n-split subcategory of $\mathrm{Ho}(\mathcal{C})$. Throughout this section, we fix an n-cube system for $\mathcal{U}$ which exists by Proposition 4.2.5. We also fix a sequence of maps

$$X_{n+2} \xrightarrow{f_{n+2}} X_{n+1} \xrightarrow{f_{n+1}} \dots \xrightarrow{f_1} X_0$$

in $\mathcal{U}$ which satisfies $f_i f_{i+1} = 0$ for $1 \leq i \leq n+1$. Our aim is to prove that the evaluation of the cocycle defined in Construction 4.2.6 on this sequence is an element of the Toda bracket $\langle f_1, \dots, f_{n+2} \rangle \subseteq [X_{n+2}, X_0]_n^{\mathrm{Ho}(\mathcal{C})}$ in the sense of Definition 3.3.7.

We denote by $\widetilde{\partial W_{n+1}}$ the space obtained from $\mathrm{sk}_n W_{n+1}$ by collapsing all those n-dimensional subcubes of $\mathrm{sk}_n W_{n+1}$ to the basepoint $(1, 1, \dots, 1)$ which are indexed by

$t(n+1,i,1)$ with $1 \leq i \leq n+1$. This space is pointed again, and it is homeomorphic to an n-sphere.

Lemma 4.3.1. *The map $\widehat{b}^{n+1}(f_1,\dots,f_{n+2})$ of the pre cube system associated to our chosen cube system induces a map $\widetilde{b}^{n+1}(f_1,\dots,f_{n+2})\colon \widetilde{\partial W_{n+1}}\wedge X_{n+2} \to X_0$ in a canonical way. This map represents the evaluation of the cocycle of Construction 4.2.6 on the complex $(f_1,\dots f_{n+2})$ in $\mathcal{U}$.*

Proof. As defined in 4.2.6, the evaluation of c on $(f_1,\dots f_{n+2})$ is the image of the homotopy class of the map $\widehat{b}^{n+1}(f_1,\dots,f_{n+1})\colon (\mathrm{sk}_n W_{n+1})_+ \wedge X_{n+2} \to X_0$ under the isomorphism $\sigma_{b^0(f_1\cdots f_{n+2})}$ of Proposition 4.1.4. Since the composition $(f_1\cdots f_{n+2})$ is a zero map in $\mathrm{Ho}(\mathcal{C})$, the representing map $b^0(f_1,\dots,f_{n+2})$ in $\mathcal{C}$ is also the zero map. Proposition 4.1.4 tells us that in this case the evaluation of the cocycle is given by the homotopy class of the induced map $(\mathrm{sk}_n W_{n+1})\wedge X_{n+2} \to X_0$, where $(1,1,\dots,1)$ is taken as the basepoint of $\mathrm{sk}_n W_{n+1}$.

The restriction of this map to the cubes indexed by $t(n+1,i,1)$ with $1 \leq i \leq n+1$ is trivial: it is given by $b^n(f_1,\dots,f_if_{i+1},\dots,f_{n+2})\colon W_n\wedge X_{n+2} \to X_0$, and this map is trivial since $f_if_{i+1}=0$. So we get an induced map $\widetilde{b}^{n+1}(f_1,\dots,f_{n+1})\colon \widetilde{\partial W_{n+1}}\wedge X_{n+2} \to X_0$ which represents the evaluation of the cocycle. □

Depending on our chosen n-cube system and the sequence of maps $(f_1,\dots,f_{n+2})$, we now construct an object $F_j = F_j(f_2,\dots,f_{j+1})$ in $\mathcal{C}$ for $j \leq n+1$. Set

$$A_j = A'_j = \coprod_{1\leq r<s\leq j+1} (W_{j-1})_+ \wedge X_s \qquad \text{and} \qquad B_j = \coprod_{1\leq i\leq j+1} (W_j)_+ \wedge X_i.$$

The object F_j is the coequalizer of two maps $h,k\colon A_j \coprod A'_j \to B_j$ we describe next.

We think of the copies of W_j in B_j as the $j+1$ subcubes of dimension j of W_{j+1} which contain the vertex $(0,\dots,0)$. The copies of W_{j-1} in A_j are thought of as those $(j-1)$-dimensional subcubes of W_{j+1} which are indexed by a $t\in T^{j+1}$ with $t_r = 0 = t_s$, and the copies of W_{j-1} in A'_j are thought of as those $(j-1)$-dimensional subcubes which are indexed by a $t\in T^{j+1}$ with $t_r = 1, t_s = 0$.

The map h is given as follows: on the copy of $(W_{j-1})_+\wedge X_s$ in A_j indexed by (r,s), it is the inclusion $(\iota_{t(j,r,0)})_+\wedge X_s\colon (W_{j-1})_+\wedge X_s \to (W_j)_+\wedge X_s$ into the summand of B_j indexed by s. On the copy of $(W_{j-1})_+\wedge X_s$ in A'_j, the map h is

$$(\iota_{t(j,r,1)})_+\wedge X_s\colon (W_{j-1})_+\wedge X_s \to (W_j)_+\wedge X_s.$$

The map k is the trivial map to the basepoint on A'_j. On the copy of $(W_{j-1})_+\wedge X_s$ in A_j indexed by (r,s), it is given by the composition

$$\begin{array}{ccc} (W_{j-1})_+\wedge X_s & \xrightarrow{\mathrm{diag}\wedge X_s} & (W_{j-1})_+\wedge(W_{j-1})_+\wedge X_s \\ & \overset{\iota_{t(j,s-1,0)}\wedge p_{j-s+r+1,j-1}\wedge X_s}{\swarrow} & \\ (W_j)_+\wedge(W_{s-r-1})_+\wedge X_s & \xrightarrow[b^{s-r-1}(f_{r+1},\dots,f_s)]{} & (W_j)_+\wedge X_r \end{array}$$

In other words, it is the map $b^{s-r-1}(f_{r+1},\dots,f_s)$ on X_s and the last $s-r-1$ coordinates of the cube W_{j-1}.

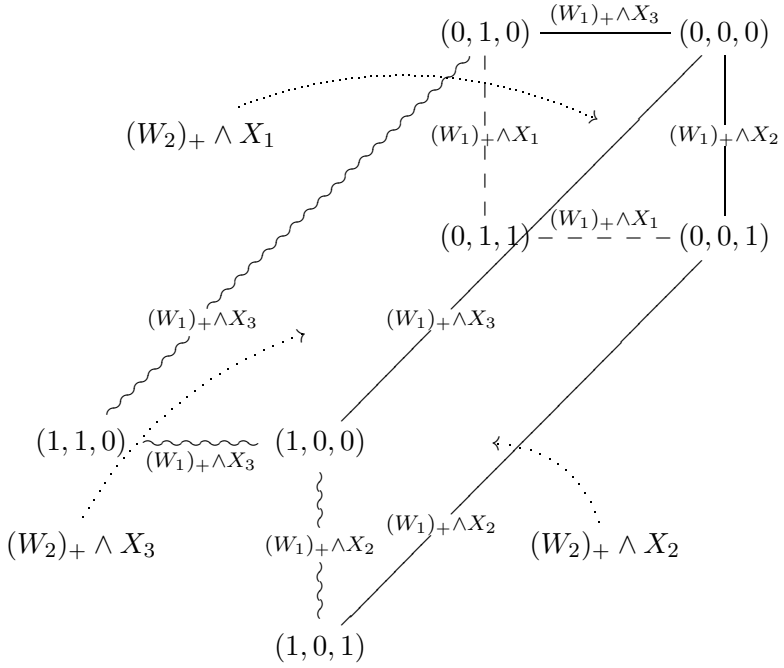


Figure 5: The object $F_2(f_2, f_3)$.

Example 4.3.2. The case $j = 2$ is displayed in Figure 5. In the diagram, the lines of the shape $\sim\sim\sim$ mark the part which is collapsed to the basepoint. Thinking of all cubes as subcubes of W_3, we glue the 3 objects $(W_2)_+ \wedge X_3, (W_2)_+ \wedge X_2$, and $(W_2)_+ \wedge X_1$ (indexed by $(-1,-1,0), (-1,0,-1)$ and $(0,-1,-1)$) together along two copies of $(W_1)_+ \wedge X_3$ (indexed by $(-1,0,0)$ and $(0,-1,0)$) and one copy of $(W_1)_+ \wedge X_2$ (indexed by $(0,0,-1)$). Furthermore, we collapse two copies of $(W_1)_+ \wedge X_3$) (indexed by $(-1,1,0)$ and $(1,-1,0)$) and one copy of $(W_1)_+ \wedge X_2$ (indexed by $(1,0,-1)$) to the basepoint.

Lemma 4.3.3. *The data of the cube system induces maps*

$$\xi_j \colon \widetilde{\partial W_{j+1}} \wedge X_{j+2} \to F_j(f_2, \dots, f_{j+1}) \quad \textit{and} \quad \zeta_j \colon F_j(f_2, \dots, f_{j+1}) \to X_0,$$

If $j = n$, the composition $\zeta_n \xi_n$ coincides with the map $\widetilde{b}^{n+1}(f_1, \dots, f_{n+2})$ of Lemma 4.3.1.

Proof. On the j-dimensional subcube $(W_j)_+ \wedge X_{j+2}$ of $\mathrm{sk}_j W_{j+1}$ indexed by $t(j+1, i, 0)$, we consider the map

$$\begin{array}{ccccc} (W_j)_+ \wedge X_{j+2} & \xrightarrow{p_{1,j} \wedge p_{i,j} \wedge X_{j+2}} & (W_j)_+ \wedge (W_{j+1-i})_+ \wedge X_{j+2} & & \\ & & \overset{(W_j)_+ \wedge b^{j+1-i}(f_{i+1}, \dots, f_{j+2})}{\swarrow} & & \\ (W_j)_+ \wedge X_i & \longrightarrow & B_j & \longrightarrow & F_j(f_2, \dots, f_{j+1}) \end{array}$$

The restriction of this map along $(\iota_{t(j,k-1,1)})_+ \wedge X_{j+2}$ is trivial for $k > i$, as we can replace $b^{j+1-i}(f_{i+1}, \dots, f_{j+2})$ by the trivial map $b^{j-i}(f_{i+1}, \dots, f_k f_{k+1}, \dots, f_{j+2})$ there. It is also trivial on the subcubes indexed by $t(j, k, 1)$ for $k < i$, since these subcubes are mapped to the part of $F_j(f_2, \dots, f_{j+1})$ which gets collapsed.

The same arguments as in Lemma 4.2.3 show that the maps for different i coincide on the $(j-1)$-dimensional subcubes of $\mathrm{sk}_j W_{j+1}$ indexed by a t with $t_r = 0 = t_s$. Therefore, we get an induced map $\xi_j \colon \widetilde{\partial W_{j+1}} \wedge X_{j+2} \to F_j(f_2, \dots, f_{j+1})$.

We now come to the map ζ_j. On the copy $(W_j)_+ \wedge X_i$ of B_j indexed by i with $1 \leq i \leq j+1$, we consider the map

$$(W_j)_+ \wedge X_i \xrightarrow{p_{1,i-1} \wedge X_i} (W_{i-1})_+ \wedge X_i \xrightarrow{b^{i-1}(f_1, \dots, f_i)} X_0.$$

The restriction of this map along the $\iota_{t(j,k,1)}$ for $k < i$ is trivial as it can be expressed using $b^{i-1}(f_1, \dots, f_k f_{k+1}, \dots, f_i)$. This ensures the compatibility with the part of the coequalizer coming from A'_j. The compatibility with the other part follows from the axioms of a cube system.

To see $\zeta_n \xi_n = \widetilde{b}^{n+1}(f_1, \dots, f_{n+2})$, we check its behavior on the n-dimensional subcubes $(W_n)_+ \wedge X_{n+2}$ indexed by $t(j+1, i, 0)$. Here $\zeta_n \xi_n$ is the map $b^{n+1-i}(f_{i+1}, \dots, f_{n+2})$ using the last $(n+1-i)$ coordinates of the cube, composed with $b^{i-1}(f_1, \dots, f_i)$ using the first $(i-1)$-coordinates. This coincides with $b^n_i(f_1, \dots, f_{n+2})$, which was used for the construction of the maps of the pre cube system and hence of $\widetilde{b}^{n+1}(f_1, \dots, f_{n+2})$. □

Lemma 4.3.4. *The object $F_{j+1}(f_2, \dots, f_{j+2})$ can be constructed from $F_j(f_2, \dots, f_{j+1})$ as the mapping cylinder of the map from $F_j(f_2, \dots, f_{j+1})$ to the cone C of the map $\xi_j \colon \widetilde{\partial W_{j+1}} \wedge X_{j+2} \to F_j(f_2, \dots, f_{j+1})$. The inclusion of $F_j(f_2, \dots f_{j+1})$ into the mapping cylinder therefore gives a map $\iota_j \colon F_j(f_2, \dots f_{j+1}) \to F_{j+1}(f_2, \dots f_{j+2})$.*

Proof. Let $\widetilde{W_{j+1}}$ denote the quotient of W_{j+1} by the equivalence relation which identifies all j-dimensional subcubes indexed by $t(j+1, i, 1) \in T^{j+1}$ with $1 \leq i \leq j+1$. Then there is a canonical map $\widetilde{\partial W_{j+1}} \to \widetilde{W_{j+1}}$, and we can interpret $\widetilde{W_{j+1}}$ as a cone on $\widetilde{\partial W_{j+1}}$. Hence we can model the mapping cone of ξ_j by the pushout of

$$\widetilde{W_{j+1}} \wedge X_{j+2} \leftarrow \widetilde{\partial W_{j+1}} \wedge X_{j+2} \xrightarrow{\xi_j} F_j(f_2, \dots, f_{j+1}).$$

In order to replace the map from F_j to the cone by a cofibration, we need a cylinder object for F_j. A possible choice for this is $(W_1)_+ \wedge F_j$, which amounts to adding one additional coordinate to each $(W_i)_+ \wedge X_k$ that occurred in the construction of F_j. We choose it to be the last coordinate. Hence the mapping cylinder of $F_j \to C$ is weakly equivalent to the pushout of

$$\widetilde{W_{j+1}} \wedge X_{j+2} \leftarrow \widetilde{\partial W_{j+1}} \wedge X_{j+2} \xrightarrow{((\iota_0)_+ \wedge F_j)(\xi_j)} (W_1)_+ \wedge F_j$$

The pushout of this diagram is isomorphic to F_{j+1} as defined above. The case $j = 1$ can again easily be deduced from Figure 5. □

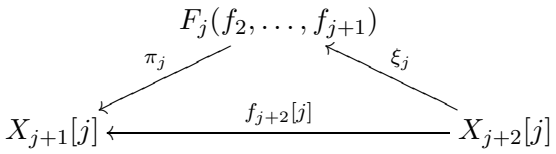
Corollary 4.3.5. *For $j \leq n$, there is a distinguished triangle*

$$X_{j+2}[j] \xrightarrow{\xi_j} F_j(f_2, \dots, f_{j+1}) \xrightarrow{\iota_j} F_{j+1}(f_2, \dots, f_{j+2}) \xrightarrow{\pi_{j+1}} X_{j+2}[j+1]$$

in $\mathrm{Ho}(\mathcal{C})$.

Proof. This follows from the last lemma and the definition of the distinguished triangles in the homotopy category of a stable model category [Hov99, Chapter 7]. □

Lemma 4.3.6. *For $0 \leq j \leq n$, the following diagram commutes in* $\mathrm{Ho}(\mathcal{C})$*:*



Proof. The last lemma says that we have a cofibration sequence

$$F_{j-1}(f_2, \ldots, f_j) \xrightarrow{\iota_{j-1}} F_j(f_2, \ldots, f_{j+1}) \xrightarrow{\pi_j} X_{j+1}[j].$$

Hence π_j is up to homotopy the map from F_j to its quotient obtained by collapsing every subcube $(W_j)_+ \wedge X_i$ of B_j indexed by $2 \leq i \leq j$ to the $(j-1)$-dimensional subcube along which it is glued to $(W_j)_+ \wedge X_{j+1}$. To examine the homotopy class of $\pi_j \xi_j$, we hence only need to know what ξ_j does on the subcube $(W_j) \wedge X_{j+2}$ indexed by $j+1$. As it is defined to be the map f_{j+2} on this one, we are done. □

Lemma 4.3.7. *If we consider the $F_j(f_2, \ldots, f_{j+1})$ as objects of* $\mathrm{Ho}(\mathcal{C})$*, the sequence*

$$* \to X_1 \xrightarrow{\iota_0} F_1(f_2) \xrightarrow{\iota_1} \ldots \xrightarrow{\iota_{n-1}} F_n(f_2, \ldots, f_{n+1})$$

gives $F_n(f_2, \ldots, f_{n+1})$ the structure of an $(n+1)$-filtered object in $\{f_2, \ldots, f_{n+1}\}$.

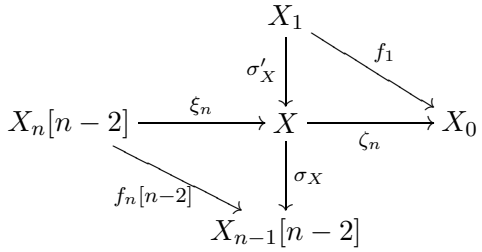
Proof. We prove that $F_j(f_2, \ldots, f_{j+1})$ is a $(j+1)$-filtered object in $\{f_2, \ldots, f_{j+1}\}$ by induction. This is clear for $j = 1$. Using that $\pi_j \colon F_j(f_2, \ldots, f_{j+1}) \to X_{j+1}[j]$ plays the role of the map σ_X for X being the $(j+1)$-filtered object $F_j(f_2, \ldots, f_{j+1})$, we can use Lemma 3.3.6 and Corollary 4.3.5 to see that $F_{j+1}(f_2, \ldots, f_{j+2})$ is a $(j+2)$-filtered object in $\{f_2, \ldots, f_{j+1}, \pi_j \xi_j[-j]\}$. The last lemma provides the remaining fact $(\pi_j \xi_j)[-j] = f_{j+2}$. □

Proposition 4.3.8. *Let $\mathcal{C}$ be a stable topological model category, let $\mathcal{U}$ be an n-split subcategory of* $\mathrm{Ho}(\mathcal{C})$*, and let*

$$X_{n+2} \xrightarrow{f_{n+2}} X_{n+1} \xrightarrow{f_{n+1}} \ldots \xrightarrow{f_1} X_0$$

be a sequence of maps in $\mathcal{U}$ with $f_i f_{i+1} = 0$ for $1 \leq i \leq n+1$. If c is a representing cocycle of the cohomology class $\gamma_{\mathcal{U}}$ of Construction 4.2.6, the evaluation of c on $(f_1, \ldots, f_{n+2})$ is an element of the Toda bracket $\langle f_1, \ldots, f_{n+2} \rangle$.

Proof. As we have seen in Lemma 4.3.3, the composition $\zeta_n \xi_n$ is the map $\widetilde{b}^n(f_1, \ldots, f_{n+2})$. Hence it represents by the evaluation of the cocycle c associated to our chosen cube system by Lemma 4.3.1. On the other hand, we have the following commutative diagram in $\mathrm{Ho}(\mathcal{C})$:



The left triangle commutes up to isomorphism by Lemma 4.3.6. The commutativity of the right triangle is an immediate consequence of the definition of the map ζ_n and the fact that σ'_X is the composition

$$X_1 \cong (W_0)_+ \wedge X_0 \xrightarrow{\iota_{(1,1,\dots,1)}} (W_{n+1})_+ \wedge X_1 \to B_{n+1} \to F_n(f_2, \dots, f_{n+1}).$$

As $F_n(f_2, \dots, f_{n+2})$ is an $(n+1)$-filtered object, this shows that $\zeta_n \xi_n$ is an element of $\langle f_1, \dots, f_{n+2} \rangle$. □

4.4 The relation to k-invariants of classifying spaces

In the last paragraph, we saw that the evaluation of the universal Toda bracket on a complex is the Toda bracket of the complex. Since it may as well be evaluated on arbitrary sequences of maps, it will carry more information than just that about the Toda bracket in general. We will now exhibit how its evaluation on a sequences of automorphisms can be expressed. When we apply our theory to ring spectra in the next section, this will give us information about the units of ring spectra (and the units of their matrix rings), rather than only about their zero divisors (and the zero divisors of their matrix rings).

A motivation for this comes from Igusa's results [Igu82] about the first k-invariant of the space $B\,\mathrm{GL}_\infty(Q\Omega X_+)$, which is related to Waldhausen's algebraic K-theory of spaces [Wal78]: Igusa shows that the first k-invariant of a connected space X is determined by a cohomology class $k_1^H(\Omega X)$ in the cohomology of the monoid $\pi_0(\Omega X)$ with coefficients in $H_1(X)$, where the class $k_1^H(\Omega X)$ is constructed from the A_4-part of the A_∞-structure of ΩX [Igu82, B, Property 1.1.]. This observation is also used in [BD89, Example 4.9, Theorem 3.10].

The first k-invariant of a path connected pointed topological space K with $\pi_i(K) = 0$ for $1 < i < m$ is a class $k^{m+1}(K) \in H^{m+1}(\pi_1(K), \pi_m(K))$ in the cohomology of the group $\pi_1(K)$ with coefficients in $\pi_m(K)$. If K satisfies $\pi_i(K) = 0$ for $i > m$ in addition, $k^{m+1}(K)$ is the obstruction to K having an Eilenberg-Mac Lane space $K(\pi_1(K), 1)$ as a retract up to homotopy.

We sketch the definition of the k-invariant we are going to work with, as it does not seem to be the most common one. It uses the explicit construction of a representing cocycle and was introduced by Eilenberg and Mac Lane in [EML49, §19], who were probably the first to study this k-invariant.

The group $\pi_1(K)$ has an associated simplicial set $B\pi_1(K)$, defined by the bar construction, whose geometric realization is an $K(\pi_1(K), 1)$. We denote by $\mathrm{sk}_i\, B\pi_1(K)$ the sub simplicial set generated by the non degenerated simplices of degree $\leq i$, and by $|\,\mathrm{sk}_i\, B\pi_1(K)|$ its geometric realization. Now we can define a map $|\,\mathrm{sk}_1\, B\pi_1(K)| \to K$ by sending the 1-simplex associated to $g \in \pi_1(K)$ to a path representing g. Inductively, this map can be extended to a map $|\,\mathrm{sk}_i\, B\pi_1(K)| \to K$ as long as $i \leq m$. In degree i, we have for every $(i+1)$-tuple $(g_1, \dots, g_{i+1})$ of elements $g_i \neq 1$ in $\pi_1(K)$ to extend a map $\partial\Delta^{i+1} \to K$ to Δ^{i+1}. Since $\pi_i(K) = 0$ for $1 < i < m$, this is possible for $i < m$ and leads to obstructions $c(g_1, \dots, g_{m+1}) \in \pi_m(K)$ when we try to extend the map to $|\,\mathrm{sk}_{m+1}\, B\pi_1(K)|$

It turns out that this c represents a cohomology class in $H^{m+1}(\pi_1(K), \pi_m(K))$ which defines $k^{m+1}(K)$. The fact that c is a cocycle can be proved by applying the Homotopy Addition Theorem in a similar fashion as in Lemma 4.2.7. Changing the cochain by a

boundary amounts to changing the chosen map $|\operatorname{sk}_m B\pi_1(K)| \to K$ on the m-skeleton. This is relevant and may be necessary as two different maps $|\operatorname{sk}_m B\pi_1(K)| \to K$ need not to be homotopic: the problem to extend a homotopy on the $(m-1)$-skeleton to an m-simplex leads to an obstruction in $\pi_m(K)$. More details and an equivalence of this definition to a more common one in terms of universal covering spaces can also be found in [EML49, §19].

Coming back to the setup of Paragraph 4.2, we fix a stable topological model category $\mathcal{C}$, an n-split subcategory $\mathcal{U}$ of $\mathrm{Ho}(\mathcal{C})$ for some $n \geq 1$, and an n-cube system which we use to define the class $\gamma_{\mathcal{U}}$. We also fix an object of $\mathcal{U}$, or more specifically a cofibrant and fibrant object X of $\mathcal{C}$ representing it. Without loss of generality, it is the same representing object as our cube system chooses. For this X, we consider the pointed topological space $\operatorname{Map}_{\mathcal{C}}(X,X)$. Its basepoint is given by the zero map in $\mathcal{C}$, and its homotopy groups can be expressed as

$$\pi_i(\operatorname{Map}_{\mathcal{C}}(X,X),0) \cong [S^i, \operatorname{Map}_{\mathcal{C}}(X,X)]^{\mathrm{Ho}(\mathcal{T}op_*)} \cong [S^i \wedge X, X]^{\mathrm{Ho}(\mathcal{C})} \cong [X,X]_i^{\mathrm{Ho}(\mathcal{C})}.$$

As X is an object of an n-split subcategory of $\mathrm{Ho}(\mathcal{C})$, we know that $\pi_i(\operatorname{Map}_{\mathcal{C}}(X,X),0)$ is concentrated in degrees divisible by n.

The enriched composition in the category $\mathcal{C}$ equips $\operatorname{Map}_{\mathcal{C}}(X,X)$ with the structure of a topological monoid, and we refer to the composition as the multiplication. Under the adjunction given above, the composition of maps in $\mathrm{Ho}(\mathcal{C})$ corresponds to the multiplication of $\operatorname{Map}_{\mathcal{C}}(X,X)$.

The set $\pi_0(\operatorname{Map}_{\mathcal{C}}(X,X))$ of path components of $\operatorname{Map}_{\mathcal{C}}(X,X)$ inherits a monoid structure from $\operatorname{Map}_{\mathcal{C}}(X,X)$, and we will denote by $\operatorname{Map}_{\mathcal{C}}(X,X)^\times$ the union of all path components of $\operatorname{Map}_{\mathcal{C}}(X,X)$ which are invertible with respect to the multiplication on $\pi_0(\operatorname{Map}_{\mathcal{C}}(X,X))$. Therefore, $\operatorname{Map}_{\mathcal{C}}(X,X)^\times$ is a group-like topological monoid.

When considering $\operatorname{Map}_{\mathcal{C}}(X,X)^\times$ as a pointed space, we take the unit id_X of the multiplication as its basepoint. This is relevant as the zero map, serving as the basepoint of $\operatorname{Map}_{\mathcal{C}}(X,X)$, is not an element of $\operatorname{Map}_{\mathcal{C}}(X,X)^\times$. Nevertheless, we have isomorphisms

$$\pi_i(\operatorname{Map}_{\mathcal{C}}(X,X),0) \cong \pi_i(\operatorname{Map}_{\mathcal{C}}(X,X),\mathrm{id}_X) \cong \pi_i(\operatorname{Map}_{\mathcal{C}}(X,X)^\times,\mathrm{id}_X)$$

for $i \geq 1$. The second isomorphism is the restriction to the path component. For the first one, one could appeal to the additive structure on $\mathrm{Ho}(\mathcal{C})$ and take the isomorphism induced by $\mathrm{id}_X \oplus(-)$. Instead of this, we will just use the isomorphism $\widetilde{\sigma}_{\mathrm{id}_X}$ of Corollary 4.1.6, which we already constructed and which has good properties we will use later.

A topological monoid G has a classifying space BG, defined via the bar construction. It comes with a map $\omega\colon G \to \Omega BG$. If the topological monoid G is group-like, that is, the monoid $\pi_0(G)$ is a group, then ω is a weak equivalence. The space ΩBG is called the group completion of G in general. In our example we get a space $B\operatorname{Map}_{\mathcal{C}}(X,X)^\times$ with

$$\pi_i(B\operatorname{Map}_{\mathcal{C}}(X,X)^\times) \cong \begin{cases} ([X,X]^{\mathrm{Ho}(\mathcal{C})})^\times & i=1 \\ 0 & 1<i\leq n \\ [X,X]_n^{\mathrm{Ho}(\mathcal{C})} & i=n+1. \end{cases}$$

Under this isomorphism, the left action of the fundamental group $\pi_1(B\operatorname{Map}_{\mathcal{C}}(X,X)^\times)$ on $\pi_{n+1}(B\operatorname{Map}_{\mathcal{C}}(X,X)^\times)$ group corresponds to the conjugation action of $[X,X]^\times$ on $[X[n],X]$, which is given by $g\cdot\lambda = (g^{-1})^*(g)_*\lambda$.

Theorem 4.4.1. *Let $\mathcal{C}$ be a stable topological model category, let $\mathcal{U}$ be a small n-split subcategory of $\mathrm{Ho}(\mathcal{C})$, and let X be a cofibrant and fibrant object of $\mathcal{C}$ representing an object in $\mathcal{U}$. Then the restriction map*

$$H^{n+2}(\mathcal{U}, [-,-]_n^{\mathrm{Ho}(\mathcal{C})}) \xrightarrow{\Phi} H^{n+2}(\pi_1(B\,\mathrm{Map}_{\mathcal{C}}(X,X)^{\times}), \pi_{n+1}(B\,\mathrm{Map}_{\mathcal{C}}(X,X)^{\times}))$$

of Proposition 2.1.14 sends the universal Toda bracket of $\mathcal{U}$ to the first k-invariant of the space $B\,\mathrm{Map}_{\mathcal{C}}(X,X)^{\times}$.

Before we give the proof of the theorem, we need to observe a fact about the relation between the homotopy groups of a group-like topological monoid G and those of its classifying space BG. Let $\omega\colon G \to \Omega BG$ be the weak equivalence to the group completion of G, and let $\varphi\colon S^n \to G$ be a map sending the basepoint s_0 of S^n to a point g in G. Then there are two ways to associate an element of $\pi_{n+1}(BG)$ to φ.

The first way works as follows: we choose an element h of G such that the product gh is in the component of the unit of G, and we also choose a path v from gh to 1_G in G. Then we have a map $r_h\varphi$, given by φ followed by right multiplication with h. It represents an element of $\pi_n(G, gh)$. Letting the path v act on $[r_h\varphi]$ gives an element $[r_h\varphi]^v \in \pi_n(G, 1_G)$. Note that this is well defined as G being a topological monoid implies that the $\pi_1(G,1_G)$-action on $\pi_n(G,1_G)$ is trivial. Applying ω_* to $[r_h\varphi]^v$ gives an element in $\pi_n(\Omega BG, \mathrm{const}_{\mathrm{pt}})$, which we can adjoin to get an element of $\pi_{n+1}(BG, \mathrm{pt})$.

For the second way, we consider the space $(S^n \times I)/\sim$. Here I is the unit interval, and $\sim$ is the equivalence relation which collapses $S^n \times \{1\}$ to one point and $S^n \times \{0\}$ to another point, with the latter serving as the basepoint of $(S^n \times I)/\sim$. This space is homeomorphic to S^{n+1}, and we can adjoin the map $\omega\varphi\colon S^n \to \Omega BG$ to get a map $(S^n \times I)/\sim \to BG$. For this we do not need the basepoint of S^n to be sent to the constant path in ΩBG.

Lemma 4.4.2. *These two ways to associate an element of $\pi_{n+1}(BG)$ to $\varphi\colon S^n \to G$ are equivalent.*

Proof. We set

$$\begin{aligned}
P &= \{*_0\} \cup_{S^n\times\{0\}} S^n \times I \cup_{S^n\times\{1\}} \{*_1\} = (S^n \times I)/\sim,\\
Q &= \{*_0\} \cup_{(S^n\times\{0\})\cup\{0\}} (S^n \times I) \cup I \cup_{(S^n\times\{1\})\cup\{1\}} \{*_1\}, \quad \text{and}\\
R &= \{*_0\} \cup_{(\{s_0\}\times\{0\})\cup\{0\}} (\{s_0\} \times I) \cup I \cup_{(\{s_0\}\times\{1\})\cup\{1\}} \{*_1\}.
\end{aligned}$$

Then both P and R come with injections into Q. We can define a map $Q \to BG$ by taking the adjoint of $\omega\varphi$ on $S^n \times I$ and the path $\omega(h)$ on I. The restriction of this map to $P = (S^n \times I)/\sim$ is the map of our second construction. The restriction of $Q \to BG$ to R is the concatenation of the paths $\omega(g)$ and $\omega(h)$. Since $\omega(g)\omega(h) \simeq \omega(gh)$ is homotopic to the constant path, there is a homotopy $R \times I \to R$ from this restriction to the constant map.

The homotopy extension property yields the dotted arrow in

$$\begin{array}{ccc}
Q \cup_R R \times I & \longrightarrow & BG \\
\downarrow & \nearrow & \\
Q \times I. & &
\end{array}$$

We restrict it to a map $P \times I \to BG$. On $P \times \{0\}$, it is still the map of our second construction. On $P \times \{1\} \cong (S^n \times I)/\sim$ it is a map which is constant on $\{s_0\} \times I$ and, by construction, a possible representative for our first way to associate an element of $\pi_{n+1}(BG)$ to φ. □

Proof of Theorem 4.4.1. The universal Toda bracket of $\mathcal{U}$ is represented by a cocycle c which is built from a cube system. To prove the theorem, we will examine the image of the cocycle under the restriction map. For this, we fix a sequence of $(n+2)$ automorphisms $(f_1, \dots, f_{n+2})$ of X in $\mathcal{U}$. Let $f = f_1 \cdots f_{n+2}$ be their composition.

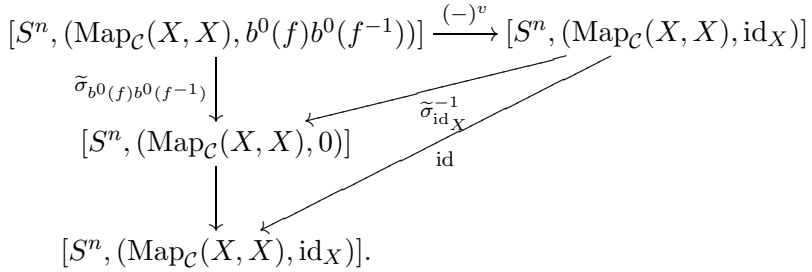
Without loss of generality, we can assume our cocycle c to be constructed from a cube system which uses our chosen X to represent the corresponding object of $\mathcal{U}$ on the model category level. By the definitions of the cocycle c in Construction 4.2.6 and the restriction map Φ in Proposition 2.1.14, the evaluation of Φ on $(f_1, \dots, f_{n+2})$ is given by

$$(f^{-1})^* \sigma_{b^0(f)} (\widehat{b}(f_1, \dots, f_{n+2})) = \sigma_{b^0(f) b^0(f^{-1})} (\widehat{b}(f_1, \dots, f_{n+2}) b^0(f^{-1})) \in [X, X]_n^{\mathrm{Ho}(\mathcal{C})}.$$

We have to understand where the homotopy class of this map goes to under the chain of isomorphisms

$$\begin{aligned} [X, X]_n^{\mathrm{Ho}(\mathcal{C})} &\cong [S^n, (\mathrm{Map}_{\mathcal{C}}(X, X), 0)] \cong [S^n, (\mathrm{Map}_{\mathcal{C}}(X, X)^\times, \mathrm{id}_X)] \\ &\cong [S^{n+1}, B\,\mathrm{Map}_{\mathcal{C}}(X, X)^\times]. \end{aligned}$$

For this, we let $\widetilde{b}\colon \mathrm{sk}_n W_{n+1} \to (\mathrm{Map}_{\mathcal{C}}(X, X), b^0(f))$ be the adjoint of $\widehat{b}^{n+1}(f_1, \dots, f_{n+2})$. If we choose a path v from $b^0(f) b^0(f^{-1})$ to id_X in $\mathrm{Map}_{\mathcal{C}}(X, X)$ and recall that the isomorphism $\widetilde{\sigma}$ of Corollary 4.1.6 was constructed as an adjoint of σ, the naturality of σ gives us a commutative diagram



Therefore, the image of $\Phi(c(f_1, \dots, f_{n+2}))$ in $\pi_n(\mathrm{Map}_{\mathcal{C}}(X, X)^\times, \mathrm{id}_x)$ can be represented by $[\widetilde{b} b^0(f^{-1})]^v$.

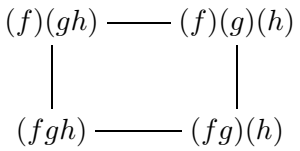
For the next step we use Lemma 4.4.2. It tells us that the associated element of $\pi_{n+1}(B\,\mathrm{Map}_{\mathcal{C}}(X, X)^\times)$ can be represented by the map

$$(\mathrm{sk}_n W_{n+1} \times I)/\sim \to B\,\mathrm{Map}_{\mathcal{C}}(X, X)^\times$$

which we get from adjoining $\omega\widetilde{b}$. Here $\sim$ is the equivalence relation which collapses $\mathrm{sk}_n W_{n+1} \times \{1\}$ to one point and $\mathrm{sk}_n W_{n+1} \times \{0\}$ to another one.

The space $(\mathrm{sk}_n W_{n+1} \times I)/\sim$ is homotopy equivalent to $\partial\Delta^{n+2}$, the boundary of an $(n+2)$-simplex, and we will now explain the resulting map

$$a = a(f_1, \dots, f_{n+2})\colon \partial\Delta^{n+2} \to B\,\mathrm{Map}_{\mathcal{C}}(X, X)^\times.$$



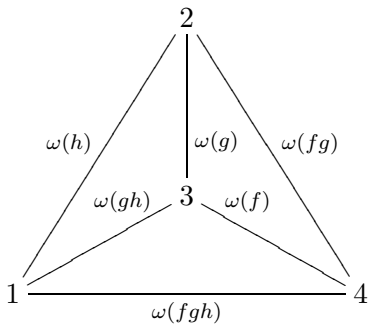


Figure 6: The square ... ... and the associated simplex.

If we denote the set of vertices of $\partial\Delta^{n+2}$ by $\{1,\dots,n+2\}$, then a maps every vertex i of $\partial\Delta^{n+2}$ to the basepoint. The 1-simplex of $\partial\Delta^{n+2}$ containing the two vertices $i<j$ is mapped to $B\operatorname{Map}_{\mathcal{C}}(X,X)^{\times}$ using the path associated to $b^0(f_i\cdots f_{j-1})$ via the map $\omega\colon \operatorname{Map}_{\mathcal{C}}(X,X)^{\times}\to\Omega B\operatorname{Map}_{\mathcal{C}}(X,X)^{\times}$. Accordingly, every 0-dimensional subcube of $\operatorname{sk}_n W_{n+1}$ specifies a path from the initial to the terminal vertex of $\partial\Delta^{n+2}$. This path runs through the vertex containing $i<j$ if the map $b^0(f_i\cdots f_{j-1})$ occurs in the restriction of the cube system to that particular vertex given by the 0-dimensional cube.

The 2-simplices of $\partial\Delta^{n+2}$ containing $i<j<k$ are mapped to $B\operatorname{Map}_{\mathcal{C}}(X,X)^{\times}$ by the coherence homotopy between the paths associated to $b^0(f_i\cdots f_{j-1})$, $b^0(f_j\cdots f_{k-1})$ and $b^0(f_i\cdots f_{k-1})$ which we get from $b^1(f_i\cdots f_{j-1},f_j\cdots f_{k-1})$. This time, the 1-dimensional subcubes of $\operatorname{sk}_n W_{n+1}$ correspond to the 2-simplices of $\partial\Delta^{n+2}$.

The case $n=1$ is displayed in Figure 6, whose right part also appears in [Igu82, B.2.2]. The situation gets a little bit more involved if $n>1$, since an $(n+1)$-cube has $2(n+1)$ subcubes of dimension n, but the $(n+2)$-simplex has only $(n+3)$ sub $(n+1)$-simplices. In this case, the $2(n+1)-(n+3)=n-1$ codimension 1 subcubes of $\operatorname{sk}_n W_{n+1}$ which are indexed by $t(n+1,k,0)$ with $2\le k\le n$ do not contribute new information to the map defined on the boundary of the $(n+2)$-simplex. The reason for this is that the restriction of the pre $(n+1)$-cube system to these subcubes is already determined by the underlying $(n-1)$-cube system. We recall that the restriction to the subcube indexed by $t(n+1,k,0)$ with $2\le k\le n$ is given by the map $b^n_k(f_1,\dots,f_{n+2})$ built from $b^{k-1}(f_1,\dots,f_k)$ and $b^{n+1-k}(f_{k+1},\dots,f_{n+2})$. Accordingly, it corresponds to the restriction of the map $a\colon\partial\Delta^{n+2}\to B\operatorname{Map}_{\mathcal{C}}(X,X)^{\times}$ to the two simplices with the vertices $\{1,\dots,k\}$ and $\{k+1,\dots,n+2\}$. The maps on all other n-dimensional subcubes induce maps on one of the $(n+1)$-simplices of $\partial\Delta^{n+2}$.

As described above, this is a representing cocycle for the k-invariant used in [EML49, §19]. □

The last theorem tells us that the vanishing of the class $\gamma_{\mathcal{U}}$ implies the vanishing of the first k-invariant of the space $B\operatorname{Map}_{\mathcal{C}}(X,X)^{\times}$ for every cofibrant and fibrant object X of $\mathcal{C}$ representing an object of $\mathcal{U}$. For our applications, we need a slightly stronger statement in a special case.

We assume for the rest of this section that $\mathcal{C}$ is a stable topological model category in which all objects are fibrant. Furthermore, we assume $\mathcal{U}$ to be a small n-split subcategory of $\operatorname{Ho}(\mathcal{C})$ with a fixed object X^1 such that all other objects of $\mathcal{U}$ are finite sums of copies

of X^1. Such a q-fold sum will be denoted by X^q.

We choose a cofibrant (and automatically fibrant) object of $\mathcal{C}$ representing X^1 and denote it also by X^1. Then the object X^q in $\mathcal{U}$ is represented by the q-fold coproduct $X^1 \vee \ldots \vee X^1$ of copies of X^1 in $\mathcal{C}$. We denote the resulting object of $\mathcal{C}$, which is still cofibrant and fibrant, also by X^q. The difference between objects in the homotopy category and the model category will be emphasized by writing $\vee$ for the coproduct in $\mathcal{C}$ and $\oplus$ for the coproduct in $\mathrm{Ho}(\mathcal{C})$.

By adding the identity id_{X^1} on the last summand, we get maps

$$\mathrm{Map}_{\mathcal{C}}(X^q, X^q) \to \mathrm{Map}_{\mathcal{C}}(X^{q+1}, X^{q+1}).$$

The restriction of these maps to the set of invertible path components is multiplicative with respect to the monoid structure induced by composition. Therefore, we get for every q an induced map

$$t_q \colon B\,\mathrm{Map}_{\mathcal{C}}(X^q, X^q)^\times \to B\,\mathrm{Map}_{\mathcal{C}}(X^{q+1}, X^{q+1})^\times.$$

The reason for working in a setup with all objects fibrant is that otherwise there are difficulties in the construction of these maps: we would have to replace the sum $X^q \vee X^1$ fibrantly in this case, and this would mean that we only get a homotopy class of maps $\mathrm{Map}_{\mathcal{C}}(X^q, X^q) \to \mathrm{Map}_{\mathcal{C}}(X^{q+1}, X^{q+1})$, rather than an actual map.

We denote the homotopy colimit, i.e., the mapping telescope, of

$$B\,\mathrm{Map}_{\mathcal{C}}(X^1, X^1)^\times \to B\,\mathrm{Map}_{\mathcal{C}}(X^2, X^2)^\times \to \ldots$$

by $B\,\mathrm{Map}^\infty_{\mathcal{C}}(X, X)^\times$. The vanishing of the first k-invariant of this space does not follow from the vanishing of the first k-invariant of all spaces $B\,\mathrm{Map}_{\mathcal{C}}(X^q, X^q)^\times$ in general, since this vanishing does not have to be compatible with the maps t_q. The next lemma provides a sufficient condition for this.

Lemma 4.4.3. *Let $\mathcal{C}$ be a stable topological model category in which all objects are fibrant. Let $\mathcal{U}$ be a small n-split subcategory of $\mathrm{Ho}(\mathcal{C})$ such that*

(i) there is a fixed object X^1 in $\mathcal{U}$ such that all objects are finite sums of copies of X^1,

(ii) $\gamma_{\mathcal{U}} \in H^{n+2}(\mathcal{U}, [-,-]_n^{\mathrm{Ho}(\mathcal{C})})$ vanishes,

(iii) $[X, X]_i^{\mathrm{Ho}(\mathcal{C})} = 0$ for all objects X of $\mathcal{U}$ and all $i > n$, and

(iv) $H^{n+1}(\mathcal{U}, [(-) \oplus X^q, (-) \oplus X^q]_n^{\mathrm{Ho}(\mathcal{C})}) = 0$ for all $q \geq 1$.

Then the space $B\,\mathrm{Map}^\infty_{\mathcal{C}}(X, X)^\times$ has a vanishing first k-invariant, i.e., it has the Eilenberg-Mac Lane space $|B\pi_1(B\,\mathrm{Map}^\infty_{\mathcal{C}}(X, X)^\times)|$ as a retract up to homotopy.

Proof. Inductively, we construct a family of homotopy commutative diagrams

$$\begin{array}{ccc} B\,\mathrm{Map}_{\mathcal{C}}(X^q, X^q)^\times & \xrightarrow{\quad t_q \quad} & B\,\mathrm{Map}_{\mathcal{C}}(X^{q+1}, X^{q+1})^\times \\ {\scriptstyle s_q}\uparrow & & \uparrow{\scriptstyle s_{q+1}} \\ |\,\mathrm{sk}_{n+2}\, B\pi_1(B\,\mathrm{Map}_{\mathcal{C}}(X^q, X^q)^\times)| & \xrightarrow{(t_q)_*} & |\,\mathrm{sk}_{n+2}\, B\pi_1(B\,\mathrm{Map}_{\mathcal{C}}(X^{q+1}, X^{q+1})^\times)| \end{array}$$

in which the maps s_q are isomorphisms on π_1. Since the groups $\pi_i(B\operatorname{Map}_{\mathcal{C}}(X^q, X^q)^\times)$ vanish for $i \geq n+2$, these maps extend to $|B\pi_1(B\operatorname{Map}_{\mathcal{C}}(X^q, X^q)^\times)|$ and induce the desired splitting on the mapping telescope.

As in the proof of the last theorem, we see that the map $b^j(f_1, \dots, f_{j+1})$ of the n-cube system for $\mathcal{U}$ induces a map from a $(j+1)$-simplex of $|B\pi_1(B\operatorname{Map}_{\mathcal{C}}(X^q, X^q)^\times)|$ to $B\operatorname{Map}_{\mathcal{C}}(X^q, X^q)^\times$. By the compatibility axioms of the cube system, the maps on the simplices assemble to a map

$$|\operatorname{sk}_{n+1} B\pi_1(B\operatorname{Map}_{\mathcal{C}}(X^q, X^q)^\times)| \to B\operatorname{Map}_{\mathcal{C}}(X^q, X^q)^\times$$

with the desired behavior on the fundamental group.

In general, these maps will not be compatible with the maps induced by

$$t_q \colon B\operatorname{Map}_{\mathcal{C}}(X^q, X^q)^\times \to B\operatorname{Map}_{\mathcal{C}}(X^{q+1}, X^{q+1})^\times.$$

We ensure the compatibility by building it into our cube system. Its chosen objects are requested to be the same as explained before the lemma. For a map $f_1 \colon X^{k_1} \to X^{k_0}$ in $\mathcal{U}$, we require $b^1(f_1 \oplus X^1)$ to be $b^1(f_1) \vee \mathrm{id}_{X^1} \colon X^{k_1} \vee X^1 \to X^{k_0} \vee X^1$ in $\mathcal{C}$. Inductively, we require

$$b^j(f_1 \oplus X^1, \dots, f_{j+1} \oplus X^1) \colon (\operatorname{sk}_j W_{j+1})_+ \wedge (X^{k_{j+1}} \vee X^1) \to X^{k_0} \vee X^1$$

to be the coproduct of $b^j(f_1, \dots, f_{j+1})$ and the projection to X^1.

An n-cube system with this property always exists, as these conditions can be forced in every step of its inductive construction. If $(f_1, \dots, f_{j+1})$ is a sequence of automorphisms of X^q in $\mathcal{U}$, the adjoints of the b^j fit into the commutative diagram

$$\begin{array}{ccc} W_j & & \\ {\scriptstyle \widetilde{b}^j(f_1,\dots,f_{j+1})}\downarrow & \searrow^{\widetilde{b}^j(f_1\oplus X^1,\dots,f_{j+1}\oplus X^1)} & \\ (\operatorname{Map}_{\mathcal{C}}(X^q, X^q), b^0(f_1 \cdots f_{j+1})) & \xrightarrow[-\vee X^1]{} & \operatorname{Map}_{\mathcal{C}}(X^{q+1}, X^{q+1}), b^0(f_1 \cdots f_{j+1}) \vee X^1). \end{array}$$

This ensures that we get maps $|\operatorname{sk}_{n+1} B\pi_1(B\operatorname{Map}_{\mathcal{C}}(X^q, X^q)^\times)| \to B\operatorname{Map}_{\mathcal{C}}(X^q, X^q)^\times$ compatible with t_q.

We could directly extend these maps to $|\operatorname{sk}_{n+2} B\pi_1(B\operatorname{Map}_{\mathcal{C}}(X^q, X^q)^\times)|$ if we knew that our n-cube system extends to an $(n+1)$-cube system. By Lemma 4.2.9, the vanishing of $\gamma_{\mathcal{U}}$ does almost imply this. Here 'almost' means that we have to change the last stage of our n-cube system. So we have to ensure that this change does not destroy the compatibility with t_q.

In Lemma 4.2.9 we changed the n-cube system with a cochain $e \in \overline{C}^{n+1}(\mathcal{U}, [-,-]_n^{\mathrm{Ho}(\mathcal{C})})$ to obtain an n-cube system for which the associated cocycle $c \in \overline{C}^{n+2}(\mathcal{U}, [-,-]_n^{\mathrm{Ho}(\mathcal{C})})$ is trivial. The manipulation of the cube system performed with the cocycle e will in general change the maps

$$b^n(f_1, \dots, f_{n+1}) \vee \mathrm{pr}_{X^1} \quad \text{and} \quad b^n(f_1 \oplus X^1, \dots, f_{n+1} \oplus X^1)$$

in a different way. We were done if we knew that these maps are homotopic relative $(\operatorname{sk}_{n-1} W_n)_+ \wedge (X^{k_1} \vee X_1)$, since this homotopy would induce a homotopy of the associated maps from the $(n+1)$-simplices into the classifying space.

To measure the difference between these two maps, we introduce two cochains $e_1, e_1' \in \overline{C}^{n+1}(\mathcal{U}, [(-)\oplus X^1, (-)\oplus X^1]_n^{\mathrm{Ho}(\mathcal{C})})$ defined by

$$\begin{aligned} e_1(f_1,\dots,f_{n+1}) &= e(f_1\oplus X^1,\dots,f_{n+1}\oplus X^1) \qquad \text{and} \\ e_1'(f_1,\dots,f_{n+1}) &= e(f_1,\dots,f_{n+1}) \vee \mathrm{pr}_{X^1}. \end{aligned}$$

The evaluation of the difference $e_1 - e_1'$ on $(f_1,\dots,f_{n+1})$ is the obstruction to our two maps being homotopic. Though it is non zero in general, $e_1 - e_1'$ is a cocycle in the complex $\overline{C}^{n+1}(\mathcal{U}, [(-)\oplus X^1, (-)\oplus X^1]_n^{\mathrm{Ho}(\mathcal{C})})$: if $c \in \overline{C}^{n+2}(\mathcal{U}, [-,-]_n^{\mathrm{Ho}(\mathcal{C})})$ is the cocycle associated to the cube system which we are about to change with e, we know

$$0 = \delta(e_i)(f_1,\dots,f_{n+2}) + c(f_1\oplus X^1,\dots,f_{n+1}\oplus X^1)$$

for $i = 1, 2$, as both e_1 and e_1' belong to a way to change $\widehat{b}^{n+1}(f_1\oplus X^1,\dots,f_{n+2}\oplus X^1)$ such that it represents the trivial homotopy class. Hence $\delta(e_1 - e_1') = 0$.

As we have assumed that $H^{n+1}(\mathcal{U}, [(-)\oplus X^1, (-)\oplus X^1]_n^{\mathrm{Ho}(\mathcal{C})})$ vanishes, it follows that $e_1 - e_1'$ is a coboundary. So there is an $a_1 \in \overline{C}^n(\mathcal{U}, [(-)\oplus X^1, (-)\oplus X^1]_n^{\mathrm{Ho}(\mathcal{C})})$ with $\delta a_1 = e_1 - e_1'$. We can now use a_1 to change our modified cube system once more. For a sequence $(f_1,\dots,f_{n+1})$, we change $b^n(f_1\oplus X^1,\dots,f_{n+1}\oplus X^{n+1})$ by $\delta(a_1)(f_1,\dots,f_{n+1})$. Then the resulting n-cube system still extends to an $(n+1)$-cube system as $\delta^2(a_1) = 0$.

To analyze the resulting cube system, we say that a sequence of $(n+1)$ composable maps $(f_1,\dots,f_{n+1})$ has *filtration* k if k is the smallest integer such that there exist maps $(f_1',\dots,f_{n+1}')$ with $f_i = f_i' \oplus X^k$. The cube system we constructed in the last step is compatible on all sequences of filtration 0, as we changed the cube system on all sequences of filtration at least 1 in order to achieve this.

The cube system may not be compatible with adding X^1 to a sequence of filtration 1, since we possibly changed it on sequences of filtration ≥ 1 in a non compatible way. In a next step, we define cochains $e_2, e_2' \in \overline{C}^{n+1}(\mathcal{U}, [(-)\oplus X^2, (-)\oplus X^2]_n^{\mathrm{Ho}(\mathcal{C})})$ such that $e_2 - e_2'$ measures the difference of the maps that the cube system associates to $(f_1,\dots,f_{n+1})$ and $(f_1\oplus X^2,\dots,f_{n+1}\oplus X^2)$. As above, we get an $a_2 \in \overline{C}^n(\mathcal{U}, [(-)\oplus X^2, (-)\oplus X^2]_n^{\mathrm{Ho}(\mathcal{C})})$ which can be used to change the maps $b^n(f_1\oplus X^2,\dots,f_{n+1}\oplus X^2)$. This does not change the value of the maps b^n on the sequences of filtration ≤ 1, and it makes it compatible on those of filtration 2.

Inductively, we use this procedure to get an n-cube system which extends to an $(n+1)$-cube system and is compatible on all sequences of maps of filtration $\leq k$. This is enough to define the desired maps on all spaces $|\,\mathrm{sk}_{n+2}\, B\pi_1(B\,\mathrm{Map}_{\mathcal{C}}(X^q, X^q)^{\times})|$ with $q \leq k$ in the homotopy colimit system. As the step to filtration $k+1$ does not change the map on sequences of lower filtration, we obtain an induced map on the telescope. □

Though the last lemma is a little bit involved, its conditions to ensure the coherent vanishing of the first k-invariants of the spaces $B\,\mathrm{Map}_{\mathcal{C}}(X^q, X^q)^{\times}$ are relatively easy to check in examples, as we will see in Propositions 5.1.10 and 5.2.5.

5 Applications to ring spectra

For us, a ring spectrum will be a monoid in one of the categories of spectra with a symmetric monoidal smash product, as S-modules [EKMM97], symmetric spectra [HSS00], or orthogonal spectra [MMSS01]. In order to apply the results of the last section, it will be important that these spectra are built on topological spaces rather than on simplicial sets. In the case of symmetric spectra, this means that we have to use the version defined in [MMSS01] instead of the original version of [HSS00]. When we write $\pi_*(R)$ for the homotopy groups of R this is always understood in the derived sense, i.e., we replace R fibrantly if necessary.

A ring spectrum R has a category of modules Mod-R. This category inherits a model structure from the underlying category of spectra, and the resulting stable model category Mod-R is topological again [MMSS01, Proposition 5.13]. If $\mathcal{C}$ denotes the underlying category of spectra, the free R-module spectrum functor $R \wedge -\colon \mathcal{C} \to \text{Mod-}R$ and the forgetful functor $U\colon \text{Mod-}R \to \mathcal{C}$ form a Quillen adjunction. This yields an isomorphism

$$\pi_n(R) = [\mathbb{S}[n], R]^{\mathrm{Ho}(\mathcal{C})} \cong [R, R]_n^{\mathrm{Ho}(\text{Mod-}R)}.$$

Hence the graded ring of homotopy groups of R is isomorphic to the graded endomorphism ring of the free module of rank 1 in Ho(Mod-R). As the left adjoint $R \wedge -\colon \mathcal{C} \to \text{Mod-}R$ maps the sphere spectrum to R, the object R is compact in Ho(Mod-R).

With this identification and Definition 3.4.2, the (matric) Toda brackets in $\pi_*(R)$ can be expressed in terms of Toda brackets in the triangulated category Ho(Mod-R).

5.1 The universal Toda bracket of a ring spectrum

In order to obtain our main results, we will now apply Construction 4.2.6 to different subcategories of Ho(Mod-R), the homotopy category of the modules over a ring spectrum R. In all the different cases, we call the resulting cohomology class a *universal Toda bracket*.

Theorem 5.1.1. *Let R be a ring spectrum. Then there exists a well defined cohomology class $\gamma_R \in \mathrm{HML}^{3,-1}(\pi_*(R))$ which, by evaluation, determines all triple matric Toda brackets of $\pi_*(R)$. For a $\pi_*(R)$-module M which admits a resolution by finitely generated free $\pi_*(R)$-modules, the product $\mathrm{id}_M \cup \gamma_R \in \mathrm{Ext}^{3,-1}_{\pi_*(R)}(M, M)$ is the first realizability obstruction $\kappa_3(M)$.*

Proof. Let $\mathcal{U}$ be the full subcategory of Ho(Mod-R) given by finite sums of shifted copies of the free module of rank 1. An application of Construction 4.2.6 to $\mathcal{U}$ provides a cohomology class $\gamma_{\mathcal{U}} \in H^3(\mathcal{U}, [-,-]_1)^{\mathrm{Ho}(\text{Mod-}R)}$. Lemma 3.1.1 yields an equivalence $\mathcal{U} \to F(\pi_*(R))$, which induces an isomorphism $H^3(\mathcal{U}, [-,-]_1^{\mathrm{Ho}(\mathcal{C})}) \cong \mathrm{HML}^{3,-1}(\pi_*(R))$ by Proposition 2.1.3. The image of $\gamma_{\mathcal{U}}$ in the latter group is defined to be the class γ_R. By Construction 4.2.6 and Definition 3.4.2, it determines all Toda brackets. Finally, Theorem 3.4.5 shows that γ_R determines the realizability obstructions. □

The corresponding theorem for higher Toda brackets is

Theorem 5.1.2. *Let R be a ring spectrum such that $\pi_*(R)$ is n-sparse for some $n \geq 1$. Then there exists a well defined cohomology class $\gamma_R^{n+2} \in \mathrm{HML}^{n+2,-n}_{n-\mathrm{sp}}(\pi_*(R))$ which, by*

evaluation, determines the $(n+2)$-fold Toda bracket of every complex of $(n+2)$ composable maps between finitely generated free n-sparse $\pi_*(R)$-modules. For a $\pi_*(R)$-module M which admits a resolution by such modules, the product $\mathrm{id}_M \cup \gamma_R^{n+2}$ is the unique realizability obstruction $\kappa_{n+2}(M) \in \mathrm{Ext}^{n+2,-n}_{\pi_*(R)}(M,M)$.

Proof. The proof is almost the same as for the last theorem. The only change is that this time we apply Construction 4.2.6 to the full subcategory of Ho(Mod-R) given by finite sums of copies of the free module of rank 1 which are shifted by integral multiples of n. □

Remark 5.1.3. The restriction to modules with a resolution by finitely generated free $\pi_*(R)$-modules in the last two theorems can be avoided. As discussed in Remarks 3.4.6 and 2.1.5, this can be achieved by enlarging the category category $\mathcal{U}$ which serves as an input for Construction 4.2.6. We keep the restriction to finitely generated free modules in the formulation, as some restriction of the cardinality is needed and this seems to be the most natural choice.

Corollary 5.1.4. *Let R be a ring spectrum such that $\pi_*(R) \cong (\pi_0(R))[u^{\pm 1}]$ with u a central unit in degree n. Then there is a well defined cohomology class $\gamma_R^{n+2} \in \mathrm{HML}^{n+2}(\pi_0(R))$ in the ungraded Mac Lane cohomology of $\pi_0(R)$. It determines, by evaluation, all $(n+2)$-fold Toda brackets of complexes of $(n+2)$ composable maps between finitely generated free $\pi_*(R)$-modules which are concentrated in degrees divisible by n. For a $\pi_*(R)$-module M which admits a resolution by such modules, the class γ_R^{n+2} determines the unique realizability obstruction $\kappa_{n+2}(M)$ not vanishing for degree reasons.*

Proof. This follows from the isomorphism $\mathrm{HML}^{n+2}(\pi_0(R)) \cong \mathrm{HML}^{n+2,-n}_{n-\mathrm{sp}}(\pi_*(R))$ of Lemma 2.1.13 and the last theorem. □

Let R^q denote a cofibrant and fibrant object of Mod-R representing the free R-module spectrum of rank q. We write $\mathrm{GL}_q R$ for the space $\mathrm{Map}_{\text{Mod-}R}(R^q, R^q)^\times$ considered in Paragraph 4.4. This definition of the 'general linear group' of a ring spectrum R is an important ingredient for construction of the algebraic K-theory of R in the sense of Waldhausen [Wal78], if his definition is interpreted in the modern language of ring spectra [EKMM97, VI.7]. We will come back to the algebraic K-theory of R in Proposition 5.1.10.

Theorem 5.1.5. *Let R be a ring spectrum such that $\pi_*(R)$ is concentrated in degrees divisible by n for some $n \geq 1$. For $q \geq 1$, the restriction map*

$$\mathrm{HML}^{n+2,-n}_{n-\mathrm{sp}}(\pi_*(R)) \to \mathrm{HML}^{n+2}(\pi_0(R), \pi_n(R)) \to H^{n+2}(\pi_1(B\,\mathrm{GL}_q R), \pi_{n+1}(B\,\mathrm{GL}_q R))$$

sends the universal Toda bracket γ_R^{n+2} of R to the first k-invariant of the space $B\,\mathrm{GL}_q R$.

Proof. Since $B\,\mathrm{GL}_q R = B\,\mathrm{Map}_{\text{Mod-}R}(R^q, R^q)^\times$, this follows from Theorem 4.4.1 and the construction of the restriction map in Corollary 2.1.15. □

If we were only interested in the k-invariants of the spaces $B\,\mathrm{GL}_q$, we could have constructed the image of γ_R^{n+2} under the restriction map to $\mathrm{HML}^{n+2}(\pi_0(R), \pi_n(R))$ directly by applying Construction 4.2.6 to a smaller subcategory of free R-modules in Ho(Mod-R), as will become clear with the next theorem. The resulting class in the ungraded Mac Lane cohomology group would have the disadvantage that it only determines Toda brackets

of complexes of maps of degree 0 in Mod-$\pi_*(R)$, that is, Toda brackets of (matrices) of elements of $\pi_0(R)$. This would be not enough to determine the realizability obstructions.

Beside the periodic ring spectra appearing in Corollary 5.1.4, there is another class of examples in which the universal Toda brackets are elements of ungraded Mac Lane cohomology groups in a canonical way:

Theorem 5.1.6. *Let R be a ring spectrum with $\pi_*(R)$ concentrated in degrees 0 and n for some $n \geq 1$. Then there exists a universal Toda bracket $\gamma_R^{n+2} \in \mathrm{HML}^{n+2}(\pi_0(R), \pi_n(R))$ which determines all $(n+2)$-fold Toda brackets in $\pi_*(R)$, the realizability obstruction $\kappa_{n+2}(M)$ of a $\pi_*(R)$-module M which admits a resolution by finitely generated free n-sparse $\pi_*(R)$-modules, and the first k-invariant*

$$k^{n+2}(B\,\mathrm{GL}_q\,R) \in H^{n+2}(\pi_1(B\,\mathrm{GL}_q\,R), \pi_{n+1}(B\,\mathrm{GL}_q\,R))$$

of the space $B\,\mathrm{GL}_q\,R$.

Proof. This time we take the subcategory $\mathcal{U}$ of Ho(Mod-R) given by the finitely generated free R-modules (without any shifts) as an input for Construction 4.2.6. By Lemma 3.1.1, the category $\mathcal{U}$ is equivalent to the category $F_0(\pi_*(R))$ of finite sums of (unshifted) copies of $\pi_*(R)$. Next we use that the functor

$$- \otimes_{\pi_0(R)} \pi_*(R) \colon F(\pi_0(R)) \to F_0(\pi_*(R))$$

is an equivalence of categories. This holds since we have an isomorphism

$$\mathrm{Hom}_{\pi_0(R)}(\pi_0(R), \pi_0(R)) \to \mathrm{Hom}_{\pi_*(R)}(\pi_*(R), \pi_*(R)),$$

as in both cases a map is determined by the image of 1 in $\pi_0(R)$.

After verifying the easy fact that the pullback of $[-,-]_n^{\mathrm{Ho}(\mathcal{C})}$ along this equivalence is $\mathrm{Hom}_{\pi_0(R)}(-, - \otimes_{\pi_0(R)} \pi_n(R))$, we end up with an equivalence of categories $F(\pi_0(R)) \to \mathcal{U}$, which induces an isomorphism

$$H^{n+2}(\mathcal{U}, [-,-]_n^{\mathrm{Ho}(\mathcal{C})}) \cong \mathrm{HML}^{n+2}(\pi_0(R), \pi_n(R)).$$

Hence Construction 4.2.6 provides the desired class γ_R^{n+2} and the fact that it determines all Toda brackets. The link to realizability obstructions and k-invariants is provided by Theorem 3.4.5 and Theorem 4.4.1 again. □

The next Proposition shows how the universal Toda bracket of a ring spectrum R is related to the one of the first Postnikov section of its connective cover. In the proof we will also encounter the universal Toda bracket of a connective ring spectrum. A general theorem about universal Toda bracket for connective ring spectra can be stated and proved in the same way as the last Theorem.

Proposition 5.1.7. *Let R be a ring spectrum such that $\pi_*(R)$ is n-sparse. Let $R_{\geq 0}$ be its connective cover and let $P_n(R_{\geq 0})$ be the first nontrivial Postnikov section of $R_{\geq 0}$. Then the restriction map*

$$\mathrm{HML}_{n-\mathrm{sp}}^{n+2,-n}(\pi_*(R)) \to \mathrm{HML}^{n+2}(\pi_0(R), \pi_n(R))$$

sends the universal Toda bracket of R to the one of $P_n(R_{\geq 0})$.

Proof. Let $\mathcal{U}$ be the subcategory of Ho(Mod-R) given by the finite sums of copies of R which are shifted by integral multiples of n. The class γ_R^{n+2} was defined by applying Construction 4.2.6 to $\mathcal{U}$. If $\mathcal{U}_0$ is the subcategory of $\mathcal{U}$ of finite *unshifted* copies of R, the map from the graded to the ungraded Mac Lane cohomology is induced by the restriction along the inclusion $\mathcal{U}_0 \to \mathcal{U}$.

Let $\mathcal{U}_{\geq 0}$ be the subcategory of Ho(Mod-$R_{\geq 0}$) which is given by the finite sums of unshifted copies of $R_{\geq 0}$. Then the left Quillen functor

$$- \wedge_{R_{\geq 0}} R \colon \text{Mod-}R_{\geq 0} \to \text{Mod-}R$$

induces an equivalence between $\mathcal{U}_{\geq 0}$ and $\mathcal{U}_0$. The reason for this is that the induced map on homotopy groups

$$\text{Mod-}\pi_*(R_{\geq 0}) \to \text{Mod-}\pi_*(R), \qquad M \mapsto M \otimes_{\pi_*(R_{\geq 0})} \pi_*(R)$$

restricts to an equivalence between between the subcategories of *unshifted* copies of the free module of rank 1. Lemma 4.2.10 shows that this equivalence maps the universal Toda bracket of $\mathcal{U}_0$ to the one of $\mathcal{U}_{\geq 0}$.

A similar argument applied to the left Quillen functor

$$- \wedge_{R_{\geq 0}} P_n(R_{\geq 0}) \colon \text{Mod-}R_{\geq 0} \to \text{Mod-}P_nR_{\geq 0}$$

shows that $\gamma_{\mathcal{U}_0}$ equals the universal Toda bracket of the subcategory of Ho($P_nR_{\geq 0}$) given by the finite sums of unshifted copies of $P_nR_{\geq 0}$. By Theorem 5.1.6, this is $\gamma^{n+2}_{P_nR_{\geq 0}}$. □

The last proposition explains also why the map sending γ_R^{n+2} to the first k-invariant of $B\,\mathrm{GL}_q R$ factors through $\mathrm{HML}^{n+2}(\pi_0(R), \pi_n(R))$: as $P_{n+1}B\,\mathrm{GL}_q R$ is homotopy equivalent to $B\,\mathrm{GL}_q P_nR_{\geq 0}$, the first k-invariant of $B\,\mathrm{GL}_q R$ is already determined by $\gamma^{n+2}_{P_nR_{\geq 0}}$, and the latter class is the image of γ_R^{n+2} in the intermediate step.

We can use the last proposition also to compare the universal Toda brackets of Corollary 5.1.4 and Theorem 5.1.6, which both arise in ungraded Mac Lane cohomology groups.

Corollary 5.1.8. *Let R be a ring spectrum with $\pi_*(R) \cong (\pi_0(R))[u^{\pm 1}]$ for a central unit u in degree n. Then the universal Toda bracket $\gamma_R^{n+2} \in \mathrm{HML}^{n+2}(\pi_0(R))$ coincides with the one of the first nontrivial Postnikov section of its connective cover.*

Proof. In Corollary 5.1.4, we defined γ_R^{n+2} as an element in an ungraded Mac Lane cohomology group using the isomorphism $\mathrm{HML}^{n+2,-n}_{n-\mathrm{sp}}(\pi_*(R)) \to \mathrm{HML}^{n+2}(\pi_0(R))$ of Lemma 2.1.13. The last proposition shows that this isomorphism maps it as well to $\gamma^{n+2}_{P_nR_{\geq 0}}$ in the sense of Theorem 5.1.6. □

Remark 5.1.9. A ring spectrum R with only two homotopy groups $\pi_0(R)$ and $\pi_n(R)$ has a first k-invariant in the group $\mathrm{Der}^{n+1}(\pi_0R, \pi_nR) \cong \mathrm{THH}^{n+2}(\pi_0R, \pi_nR)$ [Laz01]. Since $\mathrm{THH}^{n+2}(\pi_0R, \pi_nR) \cong \mathrm{HML}^{n+2}(\pi_0R, \pi_nR)$ (see Remark 2.1.8), we expect the universal Toda bracket of such a ring spectrum to coincide with this k-invariant. The difficult point is that these two groups are only related by a chain of isomorphisms, and we do not know how to identify the k-invariant or the universal Toda bracket in the intermediate steps of the chain of isomorphisms.

A proof of such an equivalence would not only be interesting for the computation of universal Toda brackets. It would also give a relation between the first k-invariants of a ring spectrum and its Toda brackets, which does not seem to be known.

For a connective ring spectrum R, there is a map $R \to H(\pi_0(R))$ from R to the Eilenberg-Mac Lane spectrum of $\pi_0(R)$ which is the identity on π_0. In view of the last remark, we expect the map $R \to H(\pi_0(R))$ to split in the homotopy category of ring spectra if R has only two nontrivial homotopy groups and a vanishing universal Toda bracket. Though we are not able prove this statement, the following proposition will provide a weaker result.

We briefly recall the definition of the algebraic K-theory of a ring spectrum R, following [EKMM97, VI]. To avoid technical difficulties, we assume our ring spectrum R to be an S-algebra in the sense of [EKMM97]. Since all objects in the category of R-modules are fibrant in this case, we obtain maps $B\,\mathrm{GL}_q\,R \to B\,\mathrm{GL}_{q+1}\,R$ as described before Lemma 4.4.3.

Let $B\,\mathrm{GL}\,R$ be the (homotopy) colimit of the spaces $B\,\mathrm{GL}_q\,R$ with respect to these maps. We apply Quillen's plus construction to the space $B\,\mathrm{GL}\,R$ to obtain $(B\,\mathrm{GL}\,R)^+$. For $i \geq 1$, algebraic K-groups of R can be defined as $K_i(R) = \pi_i((B\,\mathrm{GL}\,R)^+)$. We will not need $K_0(R)$, which has to be defined separately. If R is an Eilenberg-Mac Lane spectrum of a discrete ring A, this definition recovers the algebraic K-groups $K_*(A)$ of A in the sense of Quillen [EKMM97, VI, Theorem 4.3].

We will later need that the algebraic K-theory construction increases connectivity by 1. Recall that map $R \to R'$ of ring spectra is *n-connected* if the induced map $\pi_i(R) \to \pi_i(R')$ is an isomorphism for $i < n$ and an epimorphism for $i = n$. If $R \to R'$ is n-connected, the induced map $K_i(R) \to K_i(R')$ is an isomorphism for $i \leq n$ and an epimorphism for $i = n + 1$. This fact is due to the appearance of the bar construction in the definition of the algebraic K-theory and can be proved in a similar way as the corresponding statement about simplicial rings in [Wal78, Proposition 1.1].

Proposition 5.1.10. *Let R be a ring spectrum with homotopy groups concentrated in degrees 0 and n. Suppose that the universal Toda bracket γ_R^{n+2} of R is trivial and that $\mathrm{HML}^{n+1}(\pi_0(R), \pi_n(R))$ vanishes. Then the map $K_i(R) \to K_i(\pi_0(R))$ induced by the ring spectra map $R \to H(\pi_0(R))$ splits for all i.*

Proof. It is enough to show that $B\,\mathrm{GL}\,R \to B\,\mathrm{GL}(H(\pi_0(R))$ splits up to homotopy, as this property is preserved by the plus construction in this case (see for example [Ber82] for details on the plus construction). This is equivalent to the splitting of the map $B\,\mathrm{GL}_q\,R \to |B\pi_1(B\,\mathrm{GL}_q\,R)|$, since both maps are isomorphisms on the fundamental group and map into an Eilenberg-Mac Lane space.

We prove this using Lemma 4.4.3. The first three conditions are obviously satisfied, so it remains to show that $H^{n+1}(\mathcal{U}, [(-) \oplus R^q, (-) \oplus R^q]_n)$ vanishes.

As we have seen in the proof of Theorem 5.1.6, $\mathcal{U}$ is equivalent to $F(\pi_0(R))$. If we set $A = \pi_0(R)$ and $M = \pi_n(R)$, this equivalence induces an isomorphism

$$H^{n+1}(F(A), \mathrm{Hom}_A((-) \oplus A^q, (-) \oplus M^q)) \cong H^{n+1}(\mathcal{U}, [(-) \oplus R^q, (-) \oplus R^q]_n^{\mathrm{Ho(Mod\text{-}}R)}).$$

Lemma 2.1.9 provides an isomorphism between the first group and

$$\mathrm{HML}^{n+1}(A, M) = \mathrm{HML}^{n+1}(\pi_0(R), \pi_n(R)),$$

which vanishes by assumption. □

Remark 5.1.11. As mentioned before, Benson, Krause, and Schwede studied a characteristic cohomology class $\gamma_A \in \mathrm{HH}_k^{3,-1}(H^*(A))$ for a differential graded algebra A over a field k, which is a motivation for our universal Toda brackets. Though they are only concerned with the case of a class which determines all triple Massey products, their theory easily generalizes to higher classes when we assume $H^*(A)$ to be n-sparse.

In this case, the Hochschild cochain m_{n+2}, which is part of the A_∞-structure of $H^*(A)$ [Kad80], happens to be a Hochschild cocycle, as one can easily deduce from the A_∞-relations. Similarly to the class $\gamma_A \in \mathrm{HH}_k^{3,-1}(H^*(A))$, the cohomology class $[m_{n+2}] \in \mathrm{HH}_k^{n+2,-n}(H^*(A))$ is well defined and determines all $(n+2)$-fold Massey products in $H^*(A)$.

One may ask whether it is possible to define the higher classes under weaker assumptions, that is, without $H^*(A)$ or $\pi_*(R)$ being n-sparse. To some extend, this is possible if the lower universal classes vanish. We begin by sketching the first step in the case of a dga A. Suppose that $\gamma_A = [m_3] \in \mathrm{HH}_k^{3,-1}(H^*(A))$ vanishes. Then it is possible to find an equivalent A_∞-structure (m_i') on $H^*(A)$ such that the *cocycle* m_3 is zero. This uses the same kind of argument needed to show that every A_∞-structure on $H^*(A)$ is trivial if $\mathrm{HH}_k^{n+2,-n}(H^*(A)) = 0$ for $n \geq 1$ [Kad88]. As a consequence, m_4' is a Hochschild cocycle which can be used to define a cohomology class in $\mathrm{HH}_k^{4,-2}(H^*(A))$. This class is the candidate for the higher Hochschild class. However, we point out that it is not unique.

In the case of ring spectra, Lemma 4.2.9 is the tool for a similar kind of argument. If for example $\gamma_R \in \mathrm{HML}^{3,-1}(\pi_*(R))$ vanishes, the lemma says that we can find a 1-cube system which extends to a 2-cube system. This cube system can be used to define $\gamma_R^4 \in \mathrm{HML}^{4,-2}(\pi_*(R))$ without requiring $\pi_*(R)$ to be 2-sparse. As in the algebraic case, there may me different choices for this class γ_R^4. Moreover, the relation to the Toda brackets becomes more involved since the indeterminacy is not as easy to control as in the 2-sparse case. This also affects the obstruction theory, as there is no unique obstruction class. We reserve the study of the classes arising in this way for later work.

5.2 Computations in examples

Real K-theory

By [EKMM97, VIII, Theorem 4.2] or [Joa01], the real K-theory spectrum KO is a ring spectrum. Its graded ring of homotopy groups is given by

$$\pi_*(KO) = \mathbb{Z}[\eta, \omega, \beta^{\pm 1}]/(2\eta, \eta^3, \eta\omega, \omega^2 - 4\beta) \quad \text{with} \quad |\eta| = 1, |\omega| = 4, \text{ and } |\beta| = 8.$$

There are several non vanishing Toda brackets in $\pi_*(KO)$. One well known example is

Lemma 5.2.1. *The Toda bracket $\langle 2, \eta, 2\rangle$ in $\pi_*(KO)$ is defined, has trivial indeterminacy, and contains η^2.*

Proof. As $2\eta = 0 = \eta 2$ and $\pi_2(KO)$ is 2-torsion, the first two statements hold. The ring spectra map $\mathbb{S} \to KO$ is a π_i-isomorphism for $0 \leq i \leq 2$, so it suffices to calculate the corresponding Toda bracket for the sphere spectrum. This can be either taken from [Tod62] or computed directly, following an argument of [Tod71, Theorem 6.1]: Suppose $0 \in \langle 2, \eta, 2\rangle$. This would imply the existence of a 4-cell complex X with 2, η and 2 as attaching maps. We consider $H^*(X, \mathbb{Z}/2)$. Since Sq^1 detects 2 and Sq^2 detects η, the

existence of X implies that $\mathrm{Sq}^1\mathrm{Sq}^2\mathrm{Sq}^1$ acts nontrivial on the bottom dimensional class in $H^*(X, \mathbb{Z}/2)$. But $\mathrm{Sq}^1\mathrm{Sq}^2\mathrm{Sq}^1 = \mathrm{Sq}^2\mathrm{Sq}^2 = 0$. □

It follows that the universal Toda bracket γ_{KO} of KO is nontrivial. There is little hope to identify this element in $\mathrm{HML}^{3,-1}(\pi_*(KO))$, as this group seems to be rather difficult to compute. Nevertheless, the following lemma shows that γ_{KO} detects a nontrivial realizability obstruction.

Lemma 5.2.2. *The first realizability obstruction κ_3 of the $\pi_*(KO)$-module $\pi_*(KO)\otimes\mathbb{Z}/2$ does not vanish. Hence $\pi_*(KO)\otimes\mathbb{Z}/2$ cannot be the homotopy of a KO-module spectrum.*

Proof. After extending the map $\cdot 2\colon KO \to KO$ to a distinguished triangle

$$KO \xrightarrow{\cdot 2} KO \to C(2) \to KO[1],$$

we obtain a diagram

$$\begin{array}{ccccc} \pi_*(KO) & \xrightarrow{\cdot 2} & \pi_*(KO) & \longrightarrow & \pi_*(C(2)) \\ & & & \searrow & \uparrow{\scriptstyle \iota} \\ & & & & \pi_*(KO)\otimes\mathbb{Z}/2 \end{array}$$

with a monomorphism ι. In view of the description of κ_3 in Remark 3.2.1, we have to show that ι does not split. For this it is enough to verify $\pi_2(C(2)) \cong \mathbb{Z}/4$, as this means that ι does not split in degree 2 as a map of abelian groups $\mathbb{Z}/2 \cong \pi_2(KO) \to \pi_2(C(2)) \cong \mathbb{Z}/4$.

Since multiplication with 2 is trivial both on $\pi_1(KO)$ and $\pi_2(KO)$, long exact sequence associated to the distinguished triangle gives rise to a short exact sequence

$$0 \to \pi_2(KO) \to \pi_2(C(2)) \to \pi_1(KO) \to 0.$$

So we know that $\pi_2(C(2))$ is either $\mathbb{Z}/4$ or $\mathbb{Z}/2 \oplus \mathbb{Z}/2$. Let $\rho \in \pi_2(C(2))$ be a lift of $\eta \in \pi_1(KO)$ along the epimorphism in the short exact sequence. Then we can consider ρ as a map $KO[1] \to C(2)[-1]$ which fits into the commutative solid arrow diagram

$$\begin{array}{ccccccc} KO[-1] & & & & & & \\ \uparrow & \searrow & & & & & \\ {\scriptstyle\tau}\,| & & C(2)[-1] & & & & \\ | & & \uparrow{\scriptstyle\rho} & \searrow & & & \\ KO[1] & \xrightarrow[\cdot 2]{} & KO[1] & \xrightarrow[\eta]{} & KO & \xrightarrow{\cdot 2} & KO. \end{array}$$

By the definition of the triple Toda bracket (see Remark 3.3.2), a map τ such that the left square commutes is an element of $\langle 2, \eta, 2\rangle$. Hence $2\rho = 0$ would imply the contradiction $0 = \tau \in \langle 2, \eta, 2\rangle$. Therefore, ρ cannot be 2-torsion, and $\pi_2(C(2)) \cong \mathbb{Z}/4$. □

Remark 5.2.3. The same argument as in the last lemma shows the corresponding statement about the connective real K-theory spectrum. This contradicts [Wol98, Theorem 20]. The reason is an error in [Wol98, 14.1]. In this construction, the author assumes ku_* to be flat as a ko_*-module, which does not hold. Accordingly, the generalization [Wol98, Theorem 21] is false as well.

The first Postnikov section of the sphere spectrum

Let $\mathbb{S}$ be the sphere spectrum. As $\pi_0(\mathbb{S}) \cong \mathbb{Z}$ and $\pi_1(\mathbb{S}) \cong \mathbb{Z}/2$, the universal Toda bracket of $P_1\mathbb{S}$ is an element of $\mathrm{HML}^3(\mathbb{Z}, \mathbb{Z}/2)$. This group is isomorphic to $\mathbb{Z}/2$ [Lod98, Proposition 13.4.23]. From computations of Igusa [Igu82], we deduce the following

Proposition 5.2.4. *The universal Toda bracket $\gamma_{P_1\mathbb{S}}$ of the first Postnikov system of the sphere spectrum is the non zero element in* $\mathrm{HML}^3(\mathbb{Z}, \mathbb{Z}/2)$.

Proof. Let $\mathcal{H}_q^m$ be the topological monoid of self homotopy equivalences of q copies of the m-sphere. Suspension induces a map $\mathcal{H}_q^m \to \mathcal{H}_q^{m+1}$, which is $(m-1)$-connected by the Freudenthal suspension theorem.

Let $B\mathcal{H}_q^m$ be the classifying space of $\mathcal{H}_q^m$. The map $\mathrm{colim}_m B\mathcal{H}_q^m \to B\,\mathrm{GL}_q\,\mathbb{S}$ is a homotopy equivalence by [EKMM97, Proposition VI.8.3].

From a result of Igusa [Igu82] (compare also [BD89, (7.6)]), we know that the first k-invariant of $B\mathcal{H}_q^m$ is nontrivial for $q \geq 4$ and $m \geq 3$. The increasing connectivity of the maps in the colimit system therefore implies that the first k-invariant of $B\,\mathrm{GL}_q\,\mathbb{S}$ does not vanish for $q \geq 4$. Hence the first k-invariant of $B\,\mathrm{GL}_q\,P_1\mathbb{S}$ is non trivial as well. By Theorem 5.1.5, $\gamma_{P_1\mathbb{S}}$ has to be nontrivial since the map

$$\mathrm{HML}^3(\mathbb{Z}, \mathbb{Z}/2) \to H^3(\pi_1(B\,\mathrm{GL}_q\,P_1(\mathbb{S})), \pi_2(B\,\mathrm{GL}_q\,P_1(\mathbb{S})))$$

sends it to this k-invariant. □

This recovers in some sense the non vanishing of the universal Toda bracket of the full homotopy category of finite one point unions of n-spheres proved by Baues and Dreckmann [BD89, (3.7)]. Since the first Postnikov section of $\mathbb{S}$ is the same as that of ko, the proposition also shows that the image of the universal Toda bracket of KO under the homomorphism $\mathrm{HML}^{3,-1}(\pi_*(KO) \to \mathrm{HML}^3(\pi_0(KO), \pi_1(KO))$ is the non zero element.

Complex K-theory

By [EKMM97, VIII, Theorem 4.2], the complex K-theory spectrum KU can be represented by a ring spectrum. Since $\pi_*(KU) \cong \mathbb{Z}[u^{\pm 1}]$ with u of degree 2, the universal Toda bracket of KU is an element $\gamma_{KU}^4 \in \mathrm{HML}^4(\mathbb{Z})$, which is $\mathbb{Z}/2$ by the result of Franjou and Pirashvili [FP98] stated in Theorem 2.1.10.

Proposition 5.2.5. *The universal Toda bracket of KU is the non zero element of* $\mathrm{HML}^4(\mathbb{Z}) \cong \mathbb{Z}/2$.

Proof. In view of Proposition 5.1.8 it is enough to show that $\gamma_{P_2ku}^4$, the universal Toda bracket of the first Postnikov section of the connective complex K-theory spectrum, is nontrivial. We assume $\gamma_{P_2ku}^4 = 0$ and show that this leads to a contradiction.

As $\mathrm{HML}^3(\mathbb{Z}) = 0$, Proposition 5.1.10 would imply that $K_3(P_2ku) \to K_3(\mathbb{Z})$ is split. This map is onto since $P_2ku \to H\mathbb{Z}$ is 2-connected. Since $ku \to P_2ku$ is 4-connected, $K_3(ku) \cong K_3(P_2ku)$, and our assumption implies that $K_3(ku) \to K_3(\mathbb{Z})$ is split. As the author learned from Ch. Ausoni and J. Rognes, there is a commutative diagram

$$\begin{array}{ccc} K_3(ku) & \twoheadrightarrow & \mathrm{THH}_3(ku) \cong \mathbb{Z} \\ \downarrow & & \downarrow \\ \mathbb{Z}/48 \cong K_3(\mathbb{Z}) & \longrightarrow & \mathrm{THH}_3(\mathbb{Z}) \cong \mathbb{Z}/2 \end{array}$$

in which the upper and the right arrow are epimorphisms [Aus06]. Here the horizontal maps are the Bökstedt trace maps from algebraic K-theory to topological Hochschild homology, and the vertical maps are induced by $ku \to H(\pi_0(ku)) = H(\mathbb{Z})$. It follows that the lower map is an epimorphism as well. If the left map was split, this would mean that $\mathbb{Z}/48 \twoheadrightarrow \mathbb{Z}/2$ factors through $\mathbb{Z}$. This is the contradiction implying $\gamma^4_{P_2ku} \neq 0$. □

Remark 5.2.6. At a first glance, one may think that the last proposition shows that the first k-invariant of $B\,\mathrm{GL}_q\,KU$ is non trivial for q large enough. But this is not a consequence, since we made a stronger assumption than the vanishing of these k-invariants. To obtain information about the k-invariants, one would need to know that the restriction map $\mathrm{HML}^4(\mathbb{Z}) \to H^4(\mathrm{GL}_q\,\mathbb{Z}, \mathrm{Mat}_q\,\mathbb{Z})$ is nontrivial for q large enough.

However, as outlined in Remark 2.1.16, this map is trivial for $q = 1$, so the vanishing of the first k-invariant of $B\,\mathrm{GL}_1\,KU$ is a consequence of Theorem 5.1.5. Of course, there are more direct proofs for this fact.

Morava K-theory

The n-th Morava K-theory spectrum at a prime p can be represented by a ring spectrum $K(n)$ [Laz01, §11]. Here 'can be represented' means that there are different choices for this structure, i.e., there are different ring spectra which are not equivalent as ring spectra but which represent the same multiplicative cohomology theory.

Since $\pi_*(K(n)) \cong \mathbb{F}_p[v_n^{\pm 1}]$ with $|v_n| = 2(p^n - 1)$, we obtain a universal Toda bracket $\gamma^{2p^n}_{K(n)} \in \mathrm{HML}^{2p^n}(\mathbb{F}_p) \cong \mathbb{Z}/p$. Hence for fixed p and varying n, the universal Toda brackets of the $K(n)$ are elements of $\mathrm{HML}^*(\mathbb{F}_p)$ lying in the same degrees as the multiplicative generators (see the result of Franjou, Lannes, and Schwartz quoted in Theorem 2.1.10).

The ring $\pi_*(K(n))$ is a graded field, that is, all $\pi_*(K(n))$-modules are free. Hence it follows that all $\pi_*(K(n))$-modules are realizable. Therefore, we cannot detect $\gamma^{2p^n}_{K(n)}$ by finding a non vanishing realizability obstruction.

We do not know if the universal Toda bracket $\gamma^{2p^n}_{K(n)}$ depends on the choice of a model for $K(n)$, and we do also not know whether it is nontrivial or not. However, in view of Corollary 5.1.8 and Remark 5.1.9, we expect $\gamma^{2p^n}_{K(n)}$ to be nontrivial, as the connective Morava K-theory spectrum $k(n)$ has a non vanishing first k-invariant.

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