Sharp interface limit for the stochastic Allen-Cahn equation

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Abstract

The behavior of the Allen-Cahn equation

$$\frac{\mathrm{d}}{\mathrm{dt}} u^\varepsilon(x,t) = \Delta u^\varepsilon(x,t) - \frac{1}{\varepsilon^2} F'(u^\varepsilon(x,t)) + \xi_t^\varepsilon$$

with an additional stochastic term ξ_t^ε is studied for small values of ε. This equation is a reaction diffusion equation with a particular shape of the reaction term $-F'$ which is the negative derivative of a double-well potential with two wells of equal depth.

In the first part the invariant measure for this equation is studied in the case, where $x \in [-1,1]$ takes values in a compact one-dimensional domain, and where ξ_t^ε denotes a space-time white noise. This measure is absolutely continuous with respect to a Brownian bridge with appropriate boundary conditions. A scaled version of this measure is transformed to a Gibbs-type measure on a growing interval. Then it is shown that these transformed measures concentrate around a one-dimensional curve of minimizers in the infinite-dimensional space of possible configurations. This implies that in the original scaling the measures concentrate on the set of step functions with precisely one jump.

In the second part the dynamical system is studied in higher dimensions. Here the noise term ξ_t^ε is constant in space, and smoothened in time, with a correlation length that goes to zero at a precise rate as $\varepsilon \downarrow 0$. In the limit one obtains almost surely configurations that are concentrated on $\{\pm 1\}$. The development of the phase boundaries is driven by its mean curvature with an additional stochastic forcing term.

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Chapter 1

Introduction

The Allen-Cahn equation was introduced in [AC79] to model the growth of grains in crystalline materials near their melting point. It is one of the simplest models conceivable for phase separation and the evolution of interfaces between these phases without preservation of mass. Therefore, it is taken as a simplified model in various contexts. In this work the influence of noise on the Allen-Cahn equation is studied. Some rigorous new results on the invariant measure in the one-dimensional case and on the dynamics of the higher-dimensional equation are given.

1. Derivation of the model

Let us briefly recall the main ingredients of Allen and Cahn's model. Imagine a physical system, which can be described locally by an order parameter $u \in [-1,1]$, which depends on time and space. The system might, for example, consist of a crystalline material with two possible lattice structures, which are described by the order parameter taking the values ± 1. It might also consist of a sample of ferromagnetic material with two possible spins ± 1. The order parameter taking values in between 1 and -1 then corresponds to mixtures of the two lattice structures in the model of the crystal or a local mean magnetization, which may be obtained by a coarse graining in the case of the ferromagnet. A possible first step in modelling such a situation is the derivation of an energy associated to a configuration u. In both situations it is more favorable for the sample to attain values close to ± 1 instead of an intermediate value in $(-1,1)$. In the case of the coarse grained ferromagnet this is true, because it is energetically favorable for an atom to have the same spin as its neighbors, in the case of the crystal because the pure lattice structures are the most favorable. This effect is taken into account by a *potential energy*

$$\int \frac{1}{\varepsilon} F(u(x)) \mathrm{dx}.$$

Here F takes the shape of a double-well potential (see Figure 11). A standard choice for such a potential is the function $F(u) = \frac{1}{4}(u^2-1)^2$ but most properties are independent of the particular choice of F. We will work under more general assumptions later, but for simplicity of exposition let us assume for the moment that F is this particular function. Now this description is purely local and one needs another term to avoid too rough transitions between the phases. This effect is covered by a second term which mathematically takes the form of a *kinetic energy*:

$$\int \varepsilon |\nabla u(x)|^2 \mathrm{dx},$$

such that one obtains the energy functional

$$\mathcal{H}^\varepsilon(u) = \int \varepsilon |\nabla u(x)|^2 + \frac{1}{\varepsilon} F(u(x)) \mathrm{dx}.$$

The parameter ε is used to calibrate the differing strength of the individual terms. We will see later that it corresponds to the width of the interface between the different phases.

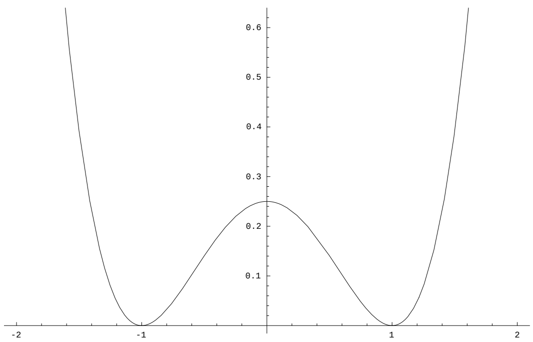

Figure 1.1: The potential F.

Now in order to give an equation of motion for the phase field u one postulates that the system tends towards configurations in which it looses energy quickly. In physicists notation this reads

$$\frac{\mathrm{d}}{\mathrm{dt}}u = \frac{\partial \mathcal{H}^{\varepsilon}(u)}{\partial u}.$$

In mathematical formulation this corresponds to the L^2-gradient flow of the energy functional $\mathcal{H}^{\varepsilon}$. We will see later that in order to obtain an interesting phenomenon in the *sharp interface limit* $\varepsilon \downarrow 0$ one has to accelerate the dynamics by a factor $\frac{1}{\varepsilon}$. The evolution equation one obtains in this way is the *Allen-Cahn equation*

$$\frac{\mathrm{d}}{\mathrm{dt}}u(x,t) = \Delta u(x,t) - \frac{1}{\varepsilon^2}F'(u(x,t)). \tag{1.0.1}$$

Mathematically there is no reason to restrict this equation to functions u that attain values only in $[-1,1]$ although other configurations have no physical meaning. Actually it can be shown easily with a comparison principle that solutions of the Allen-Cahn equation which initially only attain values in $[-1,1]$ will preserve this property for all times, whereas solutions with general initial conditions (say in L^∞) will quickly be drawn into the interval $[-1-\delta, 1+\delta]$ for every $\delta > 0$. This behavior seems reasonable, and for convenience, when introducing the noise we will not enforce that solutions attain values only in $[-1,1]$.

Furthermore, one should note that the choice of L^2-structure for the gradient flow is deliberate - one might also choose another metric and obtain a different equation. If for example one takes the H^{-1}-structure one obtains the Cahn-Hilliard equation, which is another well studied model for phase separation and evolution of boundaries. The most striking difference between these two models is that in the Allen-Cahn equation the *total mass* $\int u(x,t)\mathrm{dx}$ (corresponding to the total magnetization of the material or the total fraction of atoms within a given lattice structure) is not preserved, while it is preserved in the Cahn-Hilliard model. Which equation is more suitable, therefore, depends on the situation one wants to describe. A mathematical difference is that the Cahn-Hilliard equation is a fourth order equation, and that comparison principles, that are very useful in the study of the Allen-Cahn equation, are not available for the Cahn-Hilliard equation.

2. Heuristic analysis of the equation

Let us now give an non-rigorous description of the behavior of configurations which are favorable for the energy functional $\mathcal{H}^\varepsilon$ and for the dynamics given by the Allen-Cahn equation. We are particularly interested in the case where ε is small. Let us take a look at the one-dimensional case for $\varepsilon = 1$ first. Therefore, we look for minimizers of the functional

$$\mathcal{H}(u) = \int_{-\infty}^{\infty} (u'(x))^2 + F(u(x))\mathrm{dx}.$$

Obviously the minimizers are given by the constant functions $u(x) = 1$ and $u(x) = -1$. Nontrivial results can be obtained if one enforces the function to pass from -1 to 1 at least once by demanding $\lim_{x\to\pm\infty} u(x) = \pm 1$. Under this assumption one can write

$$\begin{aligned}\mathcal{H}(u) &= \int_{-\infty}^{\infty} \frac{1}{2}(u'(x))^2 + F(u(x))\mathrm{dx} \\ &= \int_{-\infty}^{\infty} \frac{1}{2}\Big(u'(x) - \sqrt{2F(u(x))}\Big)^2 + \sqrt{2F(u(x))}u'(x)\,\mathrm{dx} \\ &\geq G(+1) - G(-1) =: c_0, \end{aligned} \tag{1.0.2}$$

where G is an antiderivative of $\sqrt{2F(u)}$. In the case of the standard quartic double-well potential G is given by $G(u) = \frac{\sqrt{2}u}{2} - \frac{\sqrt{2}u^3}{6}$, such that one obtains $c_0 = \frac{2\sqrt{2}}{3}$. This energy is attained if and only if u solves

$$u'(x) - \sqrt{2F(u(x))} = 0, \tag{1.0.3}$$

which reads $u'(x) = \frac{\sqrt{2}}{2}(1-u^2)$ in the quartic case. The only solutions of this equation verifying the right boundary conditions are the functions $(m_\xi, \xi \in \mathbb{R})$ given as

$$m_\xi(x) = \tanh\Big(\frac{x-\xi}{\sqrt{2}}\Big). \tag{1.0.4}$$

In Chapter 2 an essential idea is, to consider this one-parameter family of minimizers as a one-dimensional submanifold of the infinite-dimensional space of possible configurations and then to analyze the behavior of $\mathcal{H}(u)$ in terms of the geometry of this submanifold.

In the ε dependent case the minimizers of $\mathcal{H}^\varepsilon$ under the above boundary conditions are given by a rescaled version of the m_ξ i.e.

$$m_\xi^\varepsilon = \tanh\Big(\frac{x-\xi}{\sqrt{2}\varepsilon}\Big).$$

One should note that the minimal energy $c_0 = \mathcal{H}(m_\xi) = \mathcal{H}^\varepsilon(m_\xi^\varepsilon)$ does not rescale in ε. It should be interpreted as the minimal energy required for a transition from -1 to $+1$. Another interesting observation is that the functions m_ξ decay exponentially towards ± 1 away from ξ. Therefore, almost all of the energy is concentrated near ξ. For the rescaled functions m_ξ^ε this implies that almost all the energy is concentrated on a region of the order ε. This explains why above ε was introduced as the width of a typical interface. Another observation is that by rewriting (1.0.3) for the rescaled functions m_ξ^ε one sees

$$\frac{1}{2}\varepsilon(m_\xi^\varepsilon)'(x)^2 = \frac{1}{\varepsilon}F(m_\xi^\varepsilon(x)),$$

a relation known as *equipartition of energy*, which has become crucial for studying the Allen-Cahn equation in the framework of geometric measure theory.

So far we have concluded that the minimizers of $\mathcal{H}^\varepsilon$ under the condition, that there is at least one jump, are given by the m_ξ^ε. These profiles are very close to -1 starting from $-\infty$, then perform a quick transition in a precisely known shape in a ε neighborhood of the separation point ξ, and then remain very close to $+1$ going to $+\infty$. One can also consider functions that perform several jumps at positions $\xi_1, \dots, \xi_n$. Around the transition points these functions will look similar to the m_ξ^ε for a transition from -1 to 1 or like $-m_\xi^\varepsilon$ for a transition from 1 to -1, and their energy is approximately given as nc_0. It is a fruitful idea to again think in geometrical terms, and to interpret this set of configurations with precisely n kinks indexed by $\xi_1, \dots, \xi_n$ as an n-dimensional submanifold of the space of possible configurations.

Having this picture of the energy landscape in mind one can quickly get a heuristic explanation of the behavior of the gradient flow. Solutions will very quickly be drawn to one of the *metastable* multikink configurations. Then as the energy is almost constant nc_0 on these manifolds the configuration remains almost static on this manifold. On the level of functions this means that functions will quickly be pushed into a configuration that is almost everywhere close to ± 1 with transitions of width of order ε around points $\xi_1, \dots, \xi_n$. The motion of these points ξ_i is extremely slow until two of them come close. When two such ξ_i meet the phase boundaries annihilate. Then the slow dynamics continue on the manifold with two kinks less. By such an annihilation the configuration quickly looses an energy of $2c_0$.

During this procedure the configuration will eventually have less and less kinks, which means on the other hand that the domains on which u attains only constant values will grow. This growth of domains is called coarsening, and much research is devoted to understand how quickly the growth of a typical domain proceeds in the Allen-Cahn equation but also in other models of surface dynamics.

In the higher-dimensional case for small ε energetically favorable configurations should also be characterized by regions in which they are almost constant close to either $+1$ or -1. The optimal configurations will again be constant, but interesting configurations should exhibit some kind of transition between these two phases. A natural candidate for a favorable configuration can be constructed as follows: Assume the space to be divided into two regions U^+ and U^-, corresponding to the sign of the phase field. Let d be the signed distance function to the boundary ∂U^+ i.e.

$$d(x) = \begin{cases} +\mathrm{dist}(x, \partial U^+) & \text{if } x \in U^+ \\ -\mathrm{dist}(x, \partial U^+) & \text{if } x \in U^-. \end{cases} \tag{1.0.5}$$

One would guess that the shape of the profile in normal direction of the boundary should be given by the shape of the one-dimensional transition which yields

$$u(x) = m_0\Big(\frac{d(x)}{\varepsilon}\Big). \tag{1.0.6}$$

Here m_0 denotes the energy minimizer defined by (1.0.4) for $\xi = 0$. Indeed if the radius of curvature of ∂U^+ is much bigger than ε such that one can ignore the curvature effects in the integration by part in tubular coordinates, for such a function the energy is given roughly by

$$\mathcal{H}^\varepsilon(u) = c_0 \mathcal{H}^{n-1}(\partial U^+),$$

where $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional area of the surface.

On the level of gradient flows this suggests that solutions should behave like the profile u, defined in (1.0.6) with evolving sets $U = U(t)$. The evolution of the sets $U(t)$ can be described in terms of the *normal velocity* of the separating surface $\partial U^+(t)$. For $x \in \partial U^+(t)$ this normal velocity is given by the time derivative of the signed distance function $d(x,t)$ of the surface at point x. Noting that by derivating (1.0.3) once, one directly obtains that the wave profile m_0 is a stationary solution of the one-dimensional Allen-Cahn equation

$$m_0''(x) - F'(m(x)) = 0,$$

and plugging the ansatz (1.0.6) into the Allen-Cahn equation (1.0.1) one obtains:

$$\frac{1}{\varepsilon} m_0'\Big(\frac{d(x)}{\varepsilon}\Big)\partial_t d(x,t) = \frac{1}{\varepsilon} m_0'\Big(\frac{d(x)}{\varepsilon}\Big)\Delta d(x,t).$$

In this calculation one uses the fact that the gradient of the signed distance function has norm $|\nabla d(x,t)| = 1$. Now for $x \in \partial U^+(t)$ the term $\Delta d(x,t)$ coincides with the sum of the principal curvatures of $\partial U^+(t)$ at point x. Thus we can conclude that the normal velocity is given by this sum or equivalently $(d-1)$ times the *mean curvature*. Actually this behavior was already conjectured in Allen and Cahn's paper.

This formal argument is reasonable also on the level of gradient flows: For small values of ε the energy of a configuration is given by the surface of the separation layer. Therefore, the dynamics should tend to decrease the surface. This is indeed true for the *mean curvature flow* which can be interpreted as a gradient flow of the surface area functional (with respect to the L^2-metric given by the surface measure).

The topic of this work is to study the influence of an additional noise term on these dynamics. An extra noise term may account for inaccuracies in the oversimplified model or for thermal effects. One observes that even a small noise influence drastically changes the behavior of the system. In general it is conjectured that the noise significantly accelerates the coarsening procedure.

Let us illustrate this in the one-dimensional case. Without a noise one observes a quick phase separation which is followed by a very slow dynamic on the metastable manifolds described above. Under the influence of a small noise term the phase separation which is driven by a strong decay of energy will take place in the same way. Then on the metastable manifold there is almost no influence of the energy. Therefore, the dynamics will be governed by the noise. As in the deterministic case one will quickly see a separation into phases on which the profile only attains values close to ± 1 with phase separation points $\xi_1, \ldots, \xi_n$. Then these ξ_i will evolve randomly. In the case of an additive space-time white noise one expects that this evolution is given by n independent Brownian motions which annihilate once they meet.

3. Survey of the mathematical literature

In the deterministic case treated above, the Allen-Cahn equation and the related energy functional, as well as the geometrical properties of motion by mean curvature, have been subject to extensive research and most of the ideas sketched above have been turned into rigorous mathematics. The stochastic case has also been treated but many questions remain open.

The convergence of the energy functional $\mathcal{H}^\varepsilon$ towards the perimeter functional of the boundary times the constant c_0 have been proven in [MM77, M87] on the level of Γ convergence. In fact the convergence of these functionals was one of the first examples for this framework, which is particular useful if one studies convergence of minimizers.

The one-dimensional dynamics have been considered by many authors, and the dynamics have been fully described. The idea of n-dimensional manifolds of multi-kink configurations was introduced in the pioneering work of Carr and Pego [CP89] and has been developed further by many authors. The most complete description available seems to be in a more recent paper of Chen [Ch04]. In [OR07] an approach based only on energy methods is discussed. There a detailed description of the energy landscape is given. Some ideas from this analysis will be used in Chapter 2.

The description of the evolution of surfaces according to its mean curvature is a rich mathematical topic. First approaches towards a study of the geometric evolution were obtained by

Brakke [B78] in the framework of geometric measure theory. In particular he introduced a concept of weak solutions to the mean curvature problem showed existence but no uniqueness of such solutions and proved a remarkable regularity theorem. (See [Ec04] for a more detailed account of Brakke's work.) In fact the non-uniqueness in Brakke's setup is not a technical inconvenience but an intrinsic feature of this evolution. There are configurations which are known to admit several weak Brakke solutions. There has been, and there continues to be today, a large interest in the study of the mean curvature flow, and we cannot review all of the approaches. Therefore, let us only discuss those which are most important for this work: In the 90's, starting with [ES91, ES92] and [CGG91], motion by mean curvature was studied as the level-set of a function Φ. The evolution of Φ is supposed to be such that *each* of its level sets evolves according to mean curvature. This means that Φ solves the following equation:

$$\partial_t \Phi = \left(\delta_{i,j} - \frac{\partial_i \Phi \partial_j \Phi}{|D\Phi|^2} \right) \partial_i \partial_j \Phi.$$

Such nonlinear degenerate parabolic equations were then studied in the context of *viscosity solutions* ([CIL92]). In particular, existence and uniqueness of solutions Φ was shown for a large class of initial data. The intrinsic non-uniqueness, observed in Brakke's approach, corresponds to the *fattening* phenomenon in the viscosity solutions approach: Level sets of Φ need not remain smooth surfaces - even if they are smooth initially. In fact, when Brakke's solutions fail to be unique, the level set flow of Φ contains all the possible Brakke solutions.

There are several other approaches to study motion by mean curvature. Let us discuss in more detail the approach developed in [ES92], which yields existence and uniqueness of smooth evolutions for short times. This idea will be used in Chapter 3 in order to construct a stochastically perturbed motion by mean curvature. Instead of taking a function Φ in which each level set evolves according to mean curvature, we study the evolution of the *signed distance function* d to the evolving surface.

To derive the right equation, suppose that $(\Gamma_t)_t = (\partial U_t)_t$ is a smooth family of surfaces evolving according to motion by mean curvature, and let $d(t, \cdot)$ be the signed distance function to Γ_t defined as above in (1.0.5). Then in a neighborhood of Γ_t the function $d(t, \cdot)$ is smooth, and for a point x in this neighborhood there exists a unique $y \in \Gamma_t$ such that $|x - y| = \mathrm{dist}(x, \Gamma_t)$. As seen above the time derivative of d corresponds to the normal velocity of Γ_t in y. Therefore, one has

$$\frac{\mathrm{d}}{\mathrm{dt}} d(x,t) = -\mathrm{div}\, \nu_t(y),$$

where $\nu_t = \mathrm{grad}\, d(t, \cdot)$ is the unit normal vector field pointing in the outside direction of Γ_t. On the other hand, the eigenvalues of the matrix $D^2 d(x,t)$ are

$$\lambda_i = -\frac{\kappa_i}{1 - \kappa_i d} \qquad 1 \le i \le (d-1) \qquad \lambda_d = 0,$$

where the λ_i denote the principal curvatures of $\Gamma(t)$ in y. Therefore, one can compute

$$\kappa_i = \frac{\lambda_i}{\lambda_i d - 1}$$

and with

$$\mathrm{div}\, \nu = -(\kappa_1 + \dots \kappa_{d-1})$$

one obtains

$$\frac{\mathrm{d}}{\mathrm{dt}} d(x,t) = \sum_{i=1}^{d} \frac{\lambda_i}{1 - \lambda_i} = \mathrm{trace}\big(D^2 d(x,t) (1 - d(x,t) D^2 d(x,t)) \big). \tag{1.0.7}$$

This equation is fully nonlinear, uniformly parabolic, and admits classical solutions for short times. Therefore, in this approach the concept of viscosity solutions can be avoided, but solutions can not be constructed globally in time.

The convergence of solutions of the Allen-Cahn equation towards phase indicator functions evolving according to motion by mean curvature was established in these several approaches. In [MS95] this was shown for classical solutions of the mean curvature evolution. Their proof relies on a Taylor expansion - very much in the spirit of Allen and Cahn's reasoning - and a spectral estimate to bound the remainder. See [C94] for a simplified exposition of such estimates. In the context of viscosity solutions a similar result was shown globally in time in [ESS92]. In [Il93] Ilmanen showed that such a result can also be obtained in the framework of geometric measure theory. His analysis relies on a beautiful intuition for the individual terms appearing in the Allen-Cahn equation as *diffuse mean curvature* and *diffuse surface* etc. On a technical level, the most difficult part in his analysis is to prove regularity of the limit of the diffuse energy measure, in order to control the discrepancy - a term introduced to describe how far the mean field u is away from the right shape.

The stochastic case has been treated in the physics literature. See for example [KO82, FV03] for a more detailed exposition of some ideas. But there are only partial rigorous results. In [Fu95, BMP95] the one-dimensional system is studied in the case that the system starts in a configuration m_ξ^ε defined above. They show that in the right scale the shape is preserved and the phase separation point performs a dynamic governed by the noise. The multi-kink case has not yet been treated rigorously.

There have also been approaches to study the evolutions of surfaces, which are driven by its mean curvature with an additional noise influence. The construction of such an object is a highly nontrivial endeavor. Yip [Y98] proposed a construction via a perturbation of a time-discrete approximation scheme of motion by mean curvature from [ATW93, LS95] via a stochastic flow. He proved energy bounds for this scheme and derived tightness. A characterization of the limiting dynamics could not be given. In [LS98II, LS98I] a theory for stochastic viscosity solutions was proposed, which should cover the case of the level set equation for motion by mean curvature with a noise that is white in time but constant in space. In this way motion by mean curvature with an additional stochastic forcing might be constructed. In the late nineties some questions concerning this approach remained open, and there has been progress in this direction only recently ([CFO09, BM02]). Even another approach for the construction was proposed in [DLN01]. Here the authors perturb equation (1.0.7) with an additive stochastic noise, which is white in time but constant in space. They rewrite it as a deterministic equation with Hölder continuous data and solve it pathwisely. In this way they were able to prove existence and uniqueness of the evolution for short times. This approach will be used below in Chapter 3. This approach is also limited to the case where the noise is constant in space.

The higher-dimensional Allen-Cahn equation with a stochastic forcing has been studied first by Funaki [Fu99]. In fact he studied the two-dimensional case and derived a description for the sharp interface limit for a noise which is constant in space and smoothened in time. A similar result in the stochastic viscosity setting was also announced in [LS98II]. In [KORV07] the limit is studied on a level of large deviation. In fact the authors study the action functional which arises when studying the large deviation behavior of a multi-dimensional Allen-Cahn equation with an additive space time white noise. Note that this functional is well defined although the stochastic evolution is not. Then they study the sharp interface limit on the level of this functional and propose a possible Γ-limit. The Γ-convergence has been proven rigorously in space-dimensions $d = 2, 3$ in [MR08]. Finally in [MR09] a sharp interface limit is studied on the level of geometric measure theory in the case where the perturbation is a L^2 function, that concentrates on the surface.

4. Contribution of this work

This work consists of two independent parts, which address the one-dimensional and the higher-dimensional situation, respectively. In the first part we study the one-dimensional system in equilibrium. The invariant measure of the stochastically perturbed evolution is studied. Due to the gradient structure of the Allen-Cahn equation this measure can be given very explicitly: It is absolutely continuous with respect to the distribution of a Brownian bridge. The Radon-Nikodym density corresponds to the potential energy of the path. The sharp interface limit is then studied on the level of these measures. To this end the random functions are stretched on an interval that grows as ε goes to zero, such that the potential energy, corresponding to the reaction term F, and the covariance of the Brownian Motion, corresponding to the diffusion term, scale with the same powers of ε. A discretization argument is then used to write the resulting measure as a Gibbs-type measure on a finite-dimensional space that is embedded in the infinite-dimensional space of configurations. Then analytical techniques can be applied to study the energy functional. In the end it is shown that if the intervals do not grow too fast, the rescaled measures concentrate around the set of minimizers of the energy functional. On the original fixed interval this corresponds to configurations that attain only the values $\{\pm 1\}$ with precisely one jump. This jump is distributed uniformly.

On a fixed interval this can be interpreted as an equilibrium version of the dynamical results obtained in [Fu95, BMP95]. The results from these works suggest that on a fixed interval with Dirichlet boundary data that enforces at least one jump in the right scaling one should have the following behavior for small ε. Solutions should concentrate around configurations just as above. The boundary point should perform a Browian motion with reflecting boundary condition. The result mentioned above says that the concentration also holds true on the level of invariant measures. As the uniform distribution is the unique reversible measure for reflected Brownian motion on would expect that the phases boundary is distributed uniformly. Then it should be possible to obtain dynamical results from our result using the theory of Mosco convergence of Dirichlet forms. This is subject of future research.

The result concerning the case of a growing interval is closely related to the results from [RV05]. Here the authors study the same kind of measure from a different point of view: They do not use the Gibbs-type measure interpretation but using Girsanov theorem rewrite the invariant measure as the distribution of a Markov process conditionned on the right boundary values. Using Freidlin-Wentzel theory they predict a phase transition in the behavior depending on the growth rate of the interval: For intervals growing slowly enough they predict configurations with one jump, but if the intervals grow exponentially quickly at the right exponential rate they expect a Poisson number of jumps. The growth rate we consider is only polynomially fast and much slower than the exponential growth rate. Nonetheless, it gives a partial proof of their conjectures (by different means). Furthermore, it gives the precise shape of the minimizers on a scale of order ε.

In a second part we study the geometric problem in higher dimensions. To be more precise we consider an equation that is perturbed by an additive noise that converges to a white noise as the interface width goes to zero. The result is that in the limit one sees a phase separation and the phase boundaries evolve according to a stochastically perturbed motion by mean curvature. Here the idea is to use the approach from [DLN01] to describe the limiting behavior of the boundaries. Then sub- and supersolutions can be constructed in a similar way as in [Fu99]. In order to pass to the limit the crucial ingredient is a stability result for the limiting dynamics obtained in [DLN01]. This result significantly simplifies the reasoning from [Fu99] and allows to include arbitrary dimensions and a more general class of initial surfaces.

A more detailed discussion and a precise statement of the individual results is given at the beginning of the two parts.

Chapter 2

Sharp interface limit for invariant measures

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The invariant measure of a one-dimensional Allen-Cahn equation with an additive space-time white noise is studied. This measure is absolutely continuous with respect to a Brownian bridge with a density which can be interpreted as a potential energy term. We consider the sharp interface limit in this setup. In the right scaling this corresponds to a Gibbs type measure on a growing interval with decreasing temperature. Our main result is that in the limit if the interval does not grow too fast we still see exponential convergence towards a curve of minimizers of the energy. In the original scaling the limit measure is concentrated on configurations with precisely one jump.

2.1 Introduction

Reaction-diffusion equations can be used to model phase separation and boundary evolutions in various physical contexts. Typically behavior of boundaries or geometric evolution laws are studied with the help of such equations. Often in such models one includes an extra noise term. This may happen for various reasons – the noise may be a simplified model for the effect of additional degrees of freedom that are not reflected in the reaction-diffusion equation. From a numerical point of view noise may improve stability in the simulations. In some systems there is even a justification for an extra noise term from a scaling limit of microscopic particle systems.

1. Setup and main result

The system considered here is the case of a symmetric bistable potential with two wells of equal depth. To be more precise, for a small parameter $\varepsilon > 0$ we are interested in the equation

$$\begin{aligned} &\partial_t u(x,t) = \Delta u(x,t) - \varepsilon^{-2\gamma} F'(u(x,t)) + \varepsilon^{(1-\gamma)/2}\sqrt{2}\,\partial_x\partial_t W(x,t) \\ &u(-1,t) = -1 \qquad u(1,t) = 1, \end{aligned} \tag{2.1.1}$$

for $(x,t) \in (-1,1) \times \mathbb{R}_+$. Here F is supposed to be a smooth (at least C^3) symmetric double-well potential i.e. we assume that F satisfies the following properties:

$$\begin{cases} (a) \quad F(u) \geq 0 \qquad \text{and} \quad F(u) = 0 \quad \text{iff } u = \pm 1, \\ (b) \quad F' \text{ admits exactly three zeros } \{\pm 1, 0\} \text{ and } F''(0) < 0,\ F''(\pm 1) > 0, \\ (c) \quad F \text{ is symmetric, } \forall u \geq 0 \quad F(u) = F(-u). \end{cases} \tag{2.1.2}$$

A typical example is $F(u) = \frac{1}{2}(u^2-1)^2$. The expression $\partial_x\partial_t W(x,t)$ is a formal expression denoting space-time white noise. Such equation can be given rigorous sense in various ways, for example in the sense of mild solutions ([Iw87, dPZ92]) or using Dirichlet forms [AR90]. We are interested in the behavior of the system in the sharp interface limit $\varepsilon \downarrow 0$. The parameter $\gamma > 0$ is a scaling factor. Our result will be valid for $\gamma < \frac{2}{3}$.

We study the behavior of the invariant measure of (2.1.1). This measure can be described quite explicitly as follows ([dPZ96, RV05]): Let $\tilde{\nu}^\varepsilon$ be the law of a rescaled Brownian bridge on $[-1,1]$ with boundary points ± 1. More precisely $\tilde{\nu}^\varepsilon$ is the law of a Gaussian process $(\tilde{u}(s), s\in[-1,1])$ with expectations $\mathbb{E}[\tilde{u}(s)] = s \quad \forall s \in [-1,1]$ and covariance $\mathrm{Cov}(\tilde{u}(s),\tilde{u}(s')) = \varepsilon^{1-\gamma}\big(s\wedge s' + 1 - \frac{(s+1)(s'+1)}{2}\big)$. Another equivalent way to characterize $\tilde{\nu}^\varepsilon$ is to say that it is a Gaussian measure on $L^2[-1,1]$ with expectation function $s\mapsto s$ and covariance operator $\varepsilon^{1-\gamma}(-\Delta)^{-1}$ where Δ denotes the one-dimensional Dirichlet Laplacian. Even another equivalent way is to say that $\tilde{u}(s)$ is the solution to the stochastic differential equation (SDE)

$$\mathrm{d}\tilde{u}(s) = \varepsilon^{\frac{1-\gamma}{2}}\mathrm{d}B(s) \qquad \tilde{u}(-1) = -1$$

with some Brownian motion $B(s)$ conditionned on $\tilde{u}(1) = 1$. Then the invariant measure $\tilde{\mu}^\varepsilon$ of (2.1.1) is absolutely continuous with respect to $\tilde{\nu}^\varepsilon$ and is given as

$$\tilde{\mu}^\varepsilon(\mathrm{d}\tilde{u}) = \frac{1}{Z^\varepsilon}\exp\Big(-\frac{1}{\varepsilon^{1+\gamma}}\int_{-1}^{1} F(\tilde{u}(s))\,\mathrm{d}s\Big)\tilde{\nu}^\varepsilon(\mathrm{d}\tilde{u}). \tag{2.1.3}$$

Here $Z^\varepsilon = \int \exp\Big(-\frac{1}{\varepsilon^{1+\gamma}}\int_{-1}^{1} F(\tilde{u}(s))\,\mathrm{d}s\Big)\tilde{\nu}^\varepsilon(\mathrm{d}\tilde{u})$ is the appropriate normalization constant.

Often important intuition on a measure on path space can be gained from considering Feynman's heuristic interpretation. In our context this heuristic interpretation states that $\tilde{\nu}^\varepsilon(\mathrm{d}\tilde{u})$ is proportional to a measure

$$\exp\Big(-\frac{1}{\varepsilon^{1-\gamma}}\int_{-1}^{1}\frac{|\tilde{u}'(s)|^2}{2}\,\mathrm{d}s\Big)\mathrm{d}\tilde{u}$$

where $\mathrm{d}\tilde{u}$ is a flat reference measure on path space. Of course this picture is non-rigorous: Such a measure $\mathrm{d}\tilde{u}$ does not exist and the quantity $\int_{-1}^{1}\frac{\tilde{u}'(s)^2}{2}\,\mathrm{d}s$ is almost surely not finite under $\tilde{\nu}^\varepsilon(\mathrm{d}\tilde{u})$. Nonetheless, it is rigorous on the level of finite-dimensional distributions and various classical statements about Brownian motion such as Schilder's theorem or Girsanov's theorem have an interpretation in terms of this heuristic picture. The measure $\tilde{\mu}^\varepsilon(\mathrm{d}\tilde{u})$ can then be interpreted as proportional to

$$\exp\Big(-\frac{1}{\varepsilon^{1+\gamma}}\int_{-1}^{1} F(\tilde{u}(s))\,\mathrm{d}s - \frac{1}{\varepsilon^{1-\gamma}}\int_{-1}^{1}\frac{|\tilde{u}'(s)|^2}{2}\,\mathrm{d}s\Big)\mathrm{d}\tilde{u}.$$

As one wants to observe an effect which results from the interaction of the *potential term* $\frac{1}{\varepsilon^{1+\gamma}}\int F\big(\tilde{u}(s)\big)\,\mathrm{d}s$ and the *kinetic energy term* $\frac{1}{\varepsilon^{1-\gamma}}\int\frac{\tilde{u}'(s)^2}{2}\mathrm{d}s$ it seems reasonable to transform the system in a way that guarantees that these terms scale with the same power of ε. This transformation is given by stretching the random functions onto a growing interval $[-\varepsilon^{-\gamma},\varepsilon^{-\gamma}]$. More precisely consider the operators

$$T^\varepsilon\colon L^2[-1,1]\to L^2[-\varepsilon^{-\gamma},\varepsilon^{-\gamma}] \qquad T^\varepsilon\tilde{u}(s) = \tilde{u}(\varepsilon^\gamma s).$$

Then consider the pushforward measures $\mu^\varepsilon = T^\varepsilon_{\#}\tilde{\mu}^\varepsilon$. These measures are again absolutely continuous with respect to Gaussian measures: ν^ε is the Gaussian measure on $L^2[-\varepsilon^{-\gamma},\varepsilon^{-\gamma}]$ with expectation function $s\mapsto\varepsilon^\gamma s$ and covariance operator $\varepsilon(-\Delta)^{-1}$. The other equivalent characterizations for $\tilde{\nu}^\varepsilon$ can be adapted with the right powers of ε. The measure μ^ε is then given as

$$\mu^\varepsilon(\mathrm{d}u) = \frac{1}{Z^\varepsilon}\exp\Big(-\varepsilon^{-1}\int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} F(u(s))\,\mathrm{d}s\Big)\nu^\varepsilon(\mathrm{d}u).$$

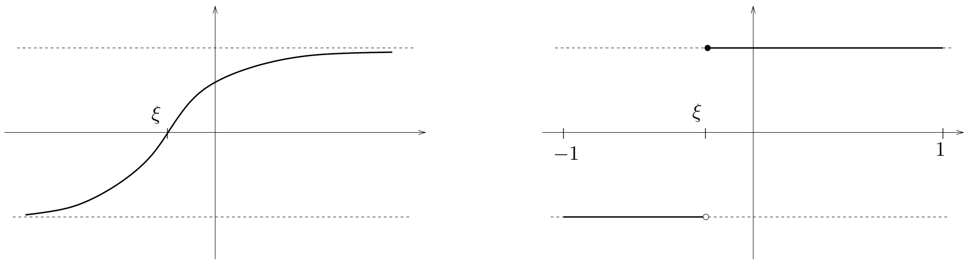

Figure 2.1: The instanton shape m_ξ and the function $-\mathbf{1}_{[-1,\xi[} + \mathbf{1}_{[\xi,1]}$.

Note that the normalization constant Z^ε is the same as above. In the Feynman picture this suggests that $\mu^\varepsilon(\mathrm{d}u)$ is proportional to

$$\exp\Big(-\frac{1}{\varepsilon}\int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} \Big[\frac{|u'(s)|^2}{2} + F\big(u(s)\big)\Big]\,\mathrm{d}s\Big)\mathrm{d}u.$$

This motivates to study the energy functional appearing in the exponent: For functions $u\colon \mathbb{R} \to \mathbb{R}$ defined on the whole line with boundary conditions $u(\pm\infty) = \pm 1$ consider the energy functional

$$\mathcal{E}(u) = \int_{-\infty}^{\infty} \Big[\frac{|u'(s)|^2}{2} + F(u(s))\Big]\,\mathrm{d}s - C_*.$$

Here C_* is a constant chosen in a way to guarantee that the minimizers of $\mathcal{E}$ with the right boundary conditions verify $\mathcal{E}(u) = 0$. This is the one-dimensional version of the well known real Ginzburg-Landau energy functional. There is a unique minimizer m of $\mathcal{E}$ subject to the condition $m(0) = 0$ and all the other minimizers are obtained via translation of m. More details on the energy functional and the minimizers can be found in Section 2.2. Denote by M the set of all these minimizers and by $m + L^2(\mathbb{R}) := \{u\colon \mathbb{R} \to \mathbb{R},\ u - m \in L^2(\mathbb{R})\}$ and $m + H^1(\mathbb{R}) := \{u\colon \mathbb{R} \to \mathbb{R}, u - m \in H^1(\mathbb{R})\}$ the spaces of functions with the right boundary values. Note that every random function distributed according to $\mu^\varepsilon(\mathrm{d}u)$ can be considered as function in $m + L^2(\mathbb{R})$ by trivial extension with ± 1 outside of $[-\varepsilon^{-\gamma}, \varepsilon^{-\gamma}]$. In this way $\mu^\varepsilon(\mathrm{d}u)$ can be interpreted as measure on $m + L^2(\mathbb{R})$. We can now state the main result of this work:

Theorem 2.1.1. *Assume $0 < \gamma < \frac{2}{3}$. Assume $p = 2$ or $p = \infty$. Then there exist positive constants c_0 and δ_0 such that for every $0 < \delta \le \delta_0$ one has*

$$\limsup_{\varepsilon \downarrow 0} \varepsilon \log \mu^\varepsilon \Big\{ \mathrm{dist}_{L^p}(u, M) \ge \delta \Big\} \le -c_0 \delta^2. \tag{2.1.4}$$

In particular the measures μ^ε concentrate around the set of minimizers exponentially fast.

The crucial step in the proof is to find a bound on the exponential decay of the normalization constant Z^ε. This lower bound can be found in Section 2.4. The asymptotic behavior of Z^ε is given in Corollary 2.4.15.

On the fixed interval $[-1, 1]$ this bound implies the following:

Corollary 2.1.2. *Assume $0 < \gamma < \frac{2}{3}$. Then the measures $\tilde{\mu}^\varepsilon(\mathrm{d}u)$ are tight for $\varepsilon \downarrow 0$ as measures on $L^2[-1, 1]$. Every limitting measure $\tilde{\mu}$ is concentrated on random functions of the type*

$$\tilde{u}(s) = -\mathbf{1}_{[-1,\xi[} + \mathbf{1}_{[\xi,1]},$$

where $\xi \in [-1, 1]$.

Remark 2.1.3. It is expected that the measures $\tilde{\mu}^\varepsilon(\mathrm{d}u)$ converge toward a measure $\tilde{\mu}$. Comparison with the dynamical results (e.g. [Fu95]) suggests that the phase separation point ξ should be distributed uniformly on $[-1, 1]$ in the limit.

Remark 2.1.4. Note that by Schilder's theorem together with an exponential tilting argument (such as [dH00] Theorem III.17 on page 34), in the case where $\gamma = 0$ the measures $\tilde{\mu}^\varepsilon$ concentrate exponentially fast around the unique minimizer of

$$u \mapsto \int_{-1}^{1} \left[\frac{|u'(s)|^2}{2} + F\big(u(s)\big) \right] \mathrm{d}s,$$

under the appropriate boundary conditions. In particular the weak limit is a Dirac measure on this minimizer, which is not a step function.

Remark 2.1.5. One can remark that by an application of Girsanov's theorem the measure $\tilde{\mu}^\varepsilon$ can be considered as the distribution of the solution of an SDE which is conditioned on the right boundary values (see [RY99] Chapter VIII §3 and also [HSV07, RV05]). It could be possible to obtain concentration results such as Theorem 2.1.2 by studying this SDE with the help of large deviation theory (see for example [S95]). We do not follow such an approach but conclude from Theorem 2.1.1 which is obtained by a discretization argument.

Remark 2.1.6. The reader might consider it unusual to work with $\tilde{\mu}^\varepsilon$ as measure on $L^2[-1, 1]$ instead of $C[-1, 1]$ or the space of càdlàg functions $D[-1, 1]$. The class of *continuous* processes is closed under weak convergence of measures on $D[-1, 1]$ such that tightness on this space cannot hold. But in fact the tightness holds in every topology τ in which the rescaled profiles $m_\xi(\varepsilon^\gamma x)$ converge to step functions and in which convergence in $L^\infty[-1, 1]$ implies convergence in τ.

2. Motivation and related works

The Allen-Cahn equation without noise was introduced in [AC79] to model the dynamics of interfaces between different domains of different lattice structure in crystals and has been studied since in various contexts. In the one-dimensional case the dynamics of the deterministic equation are well understood [Ch04, CP89, OR07] and can be described as follows: If one starts with arbitrary initial data, solutions will quickly tend to configurations which are locally constant close to ± 1 possibly with many transition layers that roughly look like the instanton shapes m introduced above. Then these interfaces move extremely slowly until eventually some two transition layers meet and annihilate. After that the dynamics continue very slowly with less interfaces.

In the two or more dimensions no such metastable behavior occurs. Solutions tend very quickly towards configurations which are locally constant with interfaces of width ε. Then on a slower scale these interfaces evolve according to motion by mean curvature (see [Il93] and the references therein).

Stochastic systems which are very similar to (2.1.1) have been studied in the classic paper by Farris and Jona-Lasinio [FJL82]. In the ninetees Funaki [Fu95] and Brasecco, de Masi, Presutti [BMP95] studied the one-dimensional equation in the case where the initial data is close to the instanton shape and showed that in an appropriate scaling the solution will stay close to such a shape. Then due to the random perturbation a dynamic along the one-parameter family of such shapes can be observed on a much faster time scale than in the deterministic case. Our result Theorem 2.1.2 corresponds to this on the level of invariant measures.

If the process does not start in a configuration with a single interface, it is believed that these different interfaces also follow a random induced dynamic which is much quicker than in the deterministic case. Different interfaces should annihilate when they meet [FV03]. More recently there were also investigations of the same system on a much bigger space interval where due to entropic effects noise induced nucleation should occur. This phenomenon has been studied on the level of invariant measures [RV05]. The limiting process should be related to the Brownian web which has recently been investigated e.g. in [FINR06].

From the point of view of statistical physics Theorem 2.1.1 can be interpreted as quite natural. In fact the Feynman picture suggests to view μ^ε as a Gibbs measure with energy $\mathcal{E}$ and decreasing temperature ε. On a fixed interval the result of Theorem 2.1.1 would therefore simply state that with decreasing temperature the Gibbs measure concentrates around the energy minimizers exponentially fast. On a rigorous level such results follow from standard large deviation theory (see e.g. [dH00, DS89]). Our result states that the entropic effects which originate from considering growing intervals do not change this picture. In fact also this is not very suprising - analysis of similar spin systems suggests that even on intervals that grow exponentially in ε^{-1} one should not observe more than one jump. But it is not clear if one can say anything about the shape of the interface in this settings.

3. Structure of the chapter

In Section 2.2 results about the energy landscape of the Ginzburg-Landau energy functional are summarized. In particular we discuss in some detail the minimizers of $\mathcal{E}$ and introduce tubular coordinates close to the curve of minimizers. The energy landscape is studied in terms of these tubular coordinates. In Section 2.3 some Gaussian concentration inequalities are discussed. In particular the discretization of the measure ν^ε is given and some error bounds are proven. The proof of Theorem 2.1.1 can then be found in Section 2.4. We will follow the convention that C denotes a generic constant which may change from line to line. Constants that appear several times will be numbered $c_1, c_2, \ldots$. Auxiliary scaling factors γ_1 to γ_3 that are supposed to satisfy a number of conditions will be introduced. These conditions are satisfied if γ_1 and γ_3 are very small and $\gamma_2 < 1$ is very close to 1.

2.2 The Energy Functional

In this section we discuss properties of the Ginzburg-Landau energy functional. We introduce the one parameter family of minimizers which we think of as a one-dimensional submanifold of the infinite-dimensional space of possible configurations. Then we discuss tubular coordinates of a neighborhood of this curve as well as a Taylor expansion of the energy landscape in these tubular coordinates. These ideas are mostly classical and go back to [CP89, Fu95, OR07]. Finally we give a discretized version of the minimizers and prove some error bounds.

For a function u defined on the whole real line consider the following energy functional:

$$\mathcal{E}(u) = \int_{\mathbb{R}} \left[\frac{1}{2} |u'(s)|^2 + F\big(u(s)\big) \right] \mathrm{d}s - C_*,$$

where the constant C_* is chosen in a way to guarantee that the minimum of $\mathcal{E}$ on the set of functions with the right boundary conditions is 0. In fact let m be the standing wave solution of the Allen-Cahn equation:

$$m''(s) - F'(m(s)) = 0 \qquad \forall s \in \mathbb{R}, \qquad m(\pm s) \to \pm 1 \quad \text{for} \quad s \to \infty. \tag{2.2.1}$$

As (2.2.1) is invariant under translations one can assume $m(0) = 0$. Then the solution can be found by solving the system

$$m'(s) - \sqrt{2F(m(s))} = 0 \qquad \forall s \in \mathbb{R}, \qquad m(0) = 0 \qquad m(\pm\infty) = \pm 1. \tag{2.2.2}$$

Note that the assumptions (2.1.2) on F imply that $\sqrt{F}$ is C^1 such that the solution to (2.2.2) is unique. The translations of m will be denoted by $m_\xi(s) = m(s - \xi)$. Note that the m_ξ are not the only solutions to (2.2.1) but that all the other solutions are either periodic or diverge such that the m_ξ are the only nonconstant critical points of $\mathcal{E}$ with finite energy. In fact the m_ξ are global minimizers of $\mathcal{E}$ subject to its boundary conditions. Completing the squares yields:

$$\begin{aligned}\int_{\mathbb{R}} \left[\frac{1}{2}|u'(s)|^2 + F\big(u(s)\big)\right] \mathrm{ds} \\ &= \int_{-\infty}^{\infty} \frac{1}{2}\Big(u'(s) - \sqrt{2F(u(s))}\Big)^2 + \sqrt{2F\big(u(s)\big)}u'(s)\,\mathrm{ds} \\ &\geq \int_{u(-\infty)}^{u(\infty)} \sqrt{2F(u)}\,\mathrm{du}.\end{aligned} \tag{2.2.3}$$

The term in the bracket is nonnegative and it vanishes if and only if u solves (2.2.2). In the sequel we will write

$$M = \{m_\xi, \xi \in \mathbb{R}\} \qquad \text{and} \qquad C_* = \int_{\mathbb{R}} \frac{1}{2}\left[|m'(s)|^2 + F\big(m(s)\big)\right] \mathrm{ds}.$$

For notational convenience we introduce the function $G(u) = \int_0^u \sqrt{2F(\tilde{u})}\mathrm{d}\tilde{\mathrm{u}}$. Then equation (2.2.3) states that $\int_{\mathbb{R}} \frac{1}{2}|u'(s)|^2 + F(u(s))\,\mathrm{ds} \geq G\big(u(\infty)\big) - G\big(u(-\infty)\big)$. Note that the assumptions (2.1.2) on F imply that G is a strictly increasing C^4 function with $G(0) = 0$. In the case of the standard double-well potential $F(u) = \frac{1}{2}(u^2-1)^2$ a calculation yields

$$m(s) = \tanh(s) \qquad \text{and} \qquad C_* = \frac{4}{3}.$$

Equation (2.2.2) shows that in general m can be given implicitly as

$$s = \int_0^m \frac{1}{\sqrt{2F(\tilde{m})}}\,\mathrm{d}\tilde{m}. \tag{2.2.4}$$

By expanding F around 1 one obtains exponential convergence to ± 1 for $s \to \pm\infty$. To be more precise there exist positive constants c_1 and c_2 such that

$$\begin{cases} |1 \mp m(\pm s)| \leq c_1 \exp(-c_2 s) & s \geq 0 \\ |m'(\pm s)| \leq c_1 c_2 \exp(-c_2 s) & s \geq 0 \\ |m''(\pm s)| \leq c_1 c_2^2 \exp(-c_2 s) & s \geq 0. \end{cases} \tag{2.2.5}$$

Recall that $m + L^2(\mathbb{R}) = \{u \colon u - m \in L^2(\mathbb{R})\}$. Due to (2.2.5) for all ξ one has $m - m_\xi \in L^2(\mathbb{R})$ such that the definition of the space $m + L^2$ is independent of the choice of minimizer.

We now introduce the concept of Fermi coordinates which was first used in this context in [CP89, Fu95]: Recall that for a function $u \in m + L^2(\mathbb{R})$ we write $\mathrm{dist}_{L^2}(u, M) := \inf_{\xi\in\mathbb{R}} \|u - m_\xi\|_{L^2(\mathbb{R})}$. If $\mathrm{dist}_{L^2}(u, M)$ is small enough there exists a unique $\xi \in \mathbb{R}$ such that $\mathrm{dist}(u, M) = \|u - m_\xi\|_{L^2(\mathbb{R})}$ and one has

$$\langle u - m_\xi, m_\xi' \rangle_{L^2(\mathbb{R})} = 0. \tag{2.2.6}$$

In fact the last equality (2.2.6) can easily be seen by differentiating $\xi \mapsto \|u - m_\xi\|^2_{L^2(\mathbb{R})}$. This has a simple geometric interpretation. The function m_ξ' can be seen as tangent vector to the curve M in m_ξ and the relation (2.2.6) can be interpreted as $v := u - m_\xi$ being normal to the tangent space in m_ξ. We will denote the space

$$N_\xi := \{v \in L^2(\mathbb{R}) : \langle v, m_\xi' \rangle_{L^2(\mathbb{R})} = 0\}$$

and interpret it as the normal space to M in m_ξ. For $u = m_\xi + v$ with $v \in N_\xi$ we will call the pair (ξ, v) Fermi or tubular coordinates of u.

One obtains information about the behavior of the energy functional close to M by considering the linearized Schrödinger type operators

$$\mathcal{A}_\xi = -\Delta + F''(m_\xi)$$

with domain of definition $H^2(\mathbb{R}) \subset L^2(\mathbb{R})$. The operator $\mathcal{A}_\xi$ is selfadjoint and nonnegative (see e.g. [Fu95]) and the eigenspace corresponding to the eigenvalue 0 is spanned by the function m'_ξ. This can be understood quite easily: The fact that the operator is nonnegative corresponds to the functional $\mathcal{E}$ attaining its minimum at m_ξ and the fact that m'_ξ is an eigenfunction to the eigenvalue 0 corresponds to the translational invariance of $\mathcal{E}$. The following more detailed description of the spectral behavior of $\mathcal{A}_\xi$ is taken from [OR07] Proposition 3.2 on page 391:

Lemma 2.2.1. *There exists a constant $c_3 > 0$ such that if $u \in H^1(\mathbb{R})$ satisfies*

$$(i) \quad u(\xi) = 0 \qquad or \qquad (ii) \quad \int_{\mathbb{R}} u(s)\, m'_\xi(s)\, \mathrm{ds} = 0,$$

then

$$c_3 \|u\|^2_{L^2(\mathbb{R})} \leq \int_{\mathbb{R}} \left[u'(s)^2 + F''\big(m_\xi(s)\big)\, u(s)^2\right] \mathrm{ds}. \tag{2.2.7}$$

This can be used to obtain the following description of the energy landscape. Similar results were already obtained in [Fu95] and [OR07]:

Proposition 2.2.2. *(i) There exist positive constants c_0, c_4, δ_1 such that for u with Fermi coordinates $u = m_\xi + v$ and $\|v\|_{H^1(\mathbb{R})} \leq \delta_1$ one has:*

$$c_0 \|v\|^2_{H^1(\mathbb{R})} \leq \mathcal{E}(u) \leq c_4 \|v\|^2_{H^1(\mathbb{R})}. \tag{2.2.8}$$

(ii) There exists a $\delta_0 > 0$ such that for $\delta \leq \delta_0$ the relation $\mathrm{dist}_{H^1}(u, M) \geq \delta$ implies

$$\mathcal{E}(u) \geq c_0 \delta^2. \tag{2.2.9}$$

Here $\mathrm{dist}_{H^1}(u, M) = \inf_{\xi \in \mathbb{R}} \|u - m_\xi\|_{H^1(\mathbb{R})}$. Statement (i) will be used as a local description of the energy landscape close to the curve of minimizers whereas the statement (ii) will be useful as a rough lower bound for the energy away from the curve. For the proof of Proposition 2.2.2 one needs the following lemma:

Lemma 2.2.3. *For every $\varepsilon > 0$ there exists $\delta > 0$ such that if $u \in m + L^2$ with $\mathcal{E}(u) \leq \delta$ then there exists $\xi \in \mathbb{R}$ such that*

$$\|u - m_\xi\|_{L^\infty(\mathbb{R})} \leq \varepsilon.$$

Furthermore, ξ can be chosen in a such a way that $u(\xi) = 0$.

Proof. For a small $\delta > 0$ assume $\mathcal{E}(u) \leq \delta$. We want to find a $\xi \in \mathbb{R}$ such that by choosing δ sufficiently small we can deduce that $\|u - m_\xi\|_{L^\infty(\mathbb{R})}$ becomes arbitrarily small. As $\mathcal{E}(u) < \infty$ we have $u \in m + H^1$ and in particular $u \in C^0(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Note that a similar calculation as (2.2.3) implies that $\mathcal{E}(u) \geq \big(G\big(\sup_{s\in\mathbb{R}} u(s)\big) - G\big(\inf_{s\in\mathbb{R}} u(s)\big)\big) - (G(1) - G(-1))$. Therefore by the properties of G by choosing δ sufficiently small, one can assume that $\|u\|_{L^\infty(\mathbb{R})} \leq 2$. By the assumptions (2.1.2) on F there exists a C such that for $u \in [-2, 2]$ one has

$$F(u) \geq C \min\left(|u-1|, |u+1|\right)^2,$$

and in particular we know that for every interval I the H^1-norm of $\min\left(|u-1|, |u+1|\right)$ can be controlled by the energy. As u is continuous and converges to ± 1 as s goes to $\pm\infty$, there exists a ξ with $u(\xi) = 0$. Without loss of generality one can assume that $\xi = 0$. We will show that in this case $\|u - m\|_{L^\infty(\mathbb{R})}$ can be made arbitrarily small.

According to (2.2.5) for every $\varepsilon > 0$ there exists T such that for $s \geq T$ one has $|m(s) - 1| \leq \varepsilon$ and for $s \leq -T$ it holds that $|m(s) + 1| \leq \varepsilon$. We will first give a bound on $u - m$ in $[-T, T]$. We consider only the case $s \geq 0$ the other one being similar. Note that as according to (2.2.3)

$$\mathcal{E}(u) = \int_{\mathbb{R}} \frac{1}{2} \left(u'(s) - \sqrt{2F(u)}\right)^2 \mathrm{ds},$$

one can write

$$\begin{aligned} u'(s) &= \sqrt{F\big(u(s)\big)} + r(s) \\ u(0) &= 0 \end{aligned} \tag{2.2.10}$$

where $\int_0^T r(s)^2 \mathrm{ds} \le 2\delta$ and using Cauchy-Schwarz inequality

$$\int_0^T |r(s)|\mathrm{ds} \le \sqrt{2T\delta}.$$

Thus using (2.2.2) one obtains for $v = |u - m|$

$$v(t) \le \int_0^t \left| \sqrt{F\big(u(s)\big)} - \sqrt{F\big(m(s)\big)} \right| + |r(s)|\mathrm{ds} \le C \int_0^t v(s)\,\mathrm{ds} + \int_0^t |r(s)|\mathrm{ds}, \tag{2.2.11}$$

where the constant C is given by $C = \sup_{u\in[-2,2]} \frac{\mathrm{d}}{\mathrm{du}} \left(\sqrt{F(u)}\right)$. Thus Gronwall's Lemma implies

$$|v(s)| \le \int_0^s |r(t)|\, e^{C(s-t)}\,\mathrm{dt},$$

and so $\sup_{s\in[0,T]} |v(s)| \le \sqrt{2T\delta} e^{CT}$. Thus by choosing δ small enough one can assure that $\sup_{s\in[0,T]} |v(s)| \le \frac{\varepsilon}{2}$.

Now let us focus on the case $s \in [-T, T]^c$. We will again only focus on $s \ge T$. Note that by the above calculations and the choice of T one has $u(-T) \le -(1-\varepsilon)$ and $u(T) \ge 1-\varepsilon$. Therefore, using

$$\begin{aligned} \int_{-\infty}^{-T} \frac{u'(s)^2}{2} + F(u(s))\mathrm{ds} + \int_{-T}^{T} \frac{u'(s)^2}{2} + F(u(s))\mathrm{ds} + \int_{T}^{\infty} \frac{u'(s)^2}{2} + F(u(s))\mathrm{ds} \\ \le G(1) - G(-1) + \delta, \end{aligned}$$

as well as

$$\int_{-T}^{T} \frac{u'(s)^2}{2} + F(u(s))\mathrm{ds} \ge G(u(T)) - G(u(-T)),$$

we get

$$\int_{T}^{\infty} \frac{u'(s)^2}{2} + F(u(s))\mathrm{ds} \le (G(1) - G(u(T))) - (G(-1) - G(u(-T))) + \delta \le C\varepsilon + \delta,$$

where $C = 2\sup_{u\in[-2,2]} F(u)$. Therefore, by using the fact that $\int_T^\infty \frac{u'(s)^2}{2} + F(u(s))$ controls the H^1-norm and thus also the L^∞ -norm of $\min(|u-1|, |u+1|)$ on $[T, \infty)$, one can conclude that possibly by choosing a smaller δ one obtains $\sup_{s\in[T,\infty)} v(s) \le C\varepsilon$. Thus by redefining ε one obtains the desired result. □

Proof. (Of Proposition 2.2.2): (i) First of all remark that for $v \in N_\xi$ one has

$$\tilde{c_0}\|v\|^2_{H^1(\mathbb{R})} \le \langle v, \mathcal{A}_\xi v\rangle_{L^2(\mathbb{R})} \le \tilde{c_4}\|v\|^2_{H^1(\mathbb{R})}. \tag{2.2.12}$$

In fact Lemma 2.2.1 (ii) implies that

$$c_3\|v\|^2_{L^2(\mathbb{R})} \le \langle v, \mathcal{A}_\xi v\rangle_{L^2(\mathbb{R})}. \tag{2.2.13}$$

To get the lower bound in (2.2.12) write

$$\begin{aligned} \langle \mathcal{A}v, v\rangle_{L^2(\mathbb{R})} &= \|\nabla v\|^2_{L^2(\mathbb{R})} + \int_{\mathbb{R}} F''(m(s))v^2(s)\,\mathrm{ds} \\ &\ge \|v\|^2_{H^1(\mathbb{R})} - (c_5+1)\|v\|^2_{L^2(\mathbb{R})}, \end{aligned} \tag{2.2.14}$$

where $c_5 = \max_{|v|\le 1} F''(v)$. Then (2.2.12) follows with $\tilde{c_0} = \frac{c_3}{c_3+c_5+1}$. In fact if $\|v\|_{L^2} \le \frac{1}{c_3+\tilde{c}_0+1}\|v\|_{H^1}$ one can use (2.2.14) and one can use (2.2.13) else. The upper bound in (2.2.12) is immediate noting that $\sup_{u\in[-1,+1]} |F''(u)| < \infty$.

In order to obtain (2.2.8) one writes:

$$\mathcal{E}(u) = \frac{1}{2}\langle \mathcal{A}_\xi v, v\rangle + \int_{\mathbb{R}} U(s,\xi,v)\mathrm{ds}, \tag{2.2.15}$$

where

$$U(s,\xi,v) = F(m_\xi(s)+v(s)) - F(m_\xi(s)) - F'(m_\xi(s))v(s) - \frac{1}{2}F''(m_\xi(s))v(s)^2.$$

Here equation (2.2.1) is used. Using the Sobolev embedding $\|v\|_{L^\infty(\mathbb{R})} \le C\|v\|_{H^1(\mathbb{R})}$ one obtains by Taylor formula

$$\Big|\int_{\mathbb{R}} U\Big| \le \frac{1}{6}\sup_{|v|\le C\delta_1+1} |F'''(v)| \|v\|^3_{L^3(\mathbb{R})} \le C\|v\|_{L^\infty(\mathbb{R})}\|v\|^2_{L^2(\mathbb{R})} \le C\|v\|^3_{H^1(\mathbb{R})}. \tag{2.2.16}$$

This implies the inequality (2.2.8).

(ii) To show the second statement, first note that there exists a $\tilde{\delta}_0 > 0$ such that if $\mathcal{E}(u) \le \tilde{\delta}_0$ there exists a ξ such that

$$c_0\|u - m_\xi\|^2_{H^1(\mathbb{R})} \le \mathcal{E}(u). \tag{2.2.17}$$

In fact choosing ξ as in Lemma 2.2.3 and noting that if one uses the case (i) of Lemma 2.2.1 instead (ii) one sees that inequalities (2.2.12) and (2.2.16) remain valid for $v = u - m_\xi$. Then by using the L^∞ bound on v from Lemma 2.2.3 instead of Sobolev embedding in the last step of (2.2.16) one obtains the above statement. In order to obtain (2.2.9) choose $\delta_0 = \sqrt{\frac{\tilde{\delta}_0}{c_0}}$ and assume $\mathrm{dist}_{H^1}(u, M) \ge \delta$ for a $\delta \le \delta_0$. If $\mathcal{E}(u) \ge \tilde{\delta}_0$ the bound (2.2.9) holds automatically. Otherwise (2.2.17) holds and gives the desired estimate. □

We now pass to some bounds on approximated wave shapes. To this end fix $\gamma_1 < \gamma$. This parameter will be fixed throughout the paper. Denote by m^ε the profile m cut off outside of $[-\varepsilon^{-\gamma_1}, \varepsilon^{-\gamma_1}]$. More precisely assume that m^ε is a smooth monotone function that coincides with m on $[-\varepsilon^{-\gamma_1}, \varepsilon^{-\gamma_1}]$ and that verifies $m^\varepsilon(s) = \pm 1$ for $\pm s \ge \varepsilon^{-\gamma_1}+1$. Furthermore, assume that on the intervals $[\varepsilon^{-\gamma_1}, \varepsilon^{-\gamma_1}+1]$ (respectively $[-\varepsilon^{-\gamma_1}-1, -\varepsilon^{-\gamma_1}]$) one has $m(s) \le m^\varepsilon(s) \le 1$ (resp. $m(s) \ge m^\varepsilon(s) \ge -1$). Due to (2.2.5) one can also assume that $|(m^\varepsilon)'(s)| \le 2c_1c_2e^{-c_2\varepsilon^{-\gamma_1}}$ on both of these intermediate intervals. Then define $m^\varepsilon_\xi(s) = m^\varepsilon(s-\xi)$.

Furthermore, for $N \in \mathbb{N}$ and $k \in \{-N, -(N-1), \dots, (N-1), N\}$ set $s_k^{N,\varepsilon} = \frac{k\varepsilon^{-\gamma}}{N}$ and define

$$m_\xi^{N,\varepsilon}(s) = \begin{cases} m^\varepsilon_\xi(s) & \text{if } s = s_k^{N,\varepsilon} \text{ for } k = -(N-1),\dots,(N-1) \\ & \text{the linear interpolation between these points.} \end{cases} \tag{2.2.18}$$

One then gets the following bound:

Lemma 2.2.4. *For ε small enough and $\xi \in [-\varepsilon^{-\gamma}+\varepsilon^{-\gamma_1}+1, \varepsilon^{-\gamma}-\varepsilon^{-\gamma_1}-1]$ one has*

(i) $\|m_\xi - m^\varepsilon_\xi\|_{L^2(\mathbb{R})} \le C\exp(-c_2\varepsilon^{-\gamma_1})$
$\|(m_\xi)' - (m^\varepsilon_\xi)'\|_{L^2(\mathbb{R})} \le C\exp(-c_2\varepsilon^{-\gamma_1})$.

(ii) $\|m_\xi - m_\xi^{N,\varepsilon}\|_{L^2(\mathbb{R})} \le C\varepsilon^{-\gamma_1/2}\frac{\varepsilon^{-2\gamma}}{N^2}$
$\|(m_\xi)' - (m_\xi^{N,\varepsilon})'\|_{L^2(\mathbb{R})} \le C\varepsilon^{-\gamma_1/2}\frac{\varepsilon^{-\gamma}}{N}$.

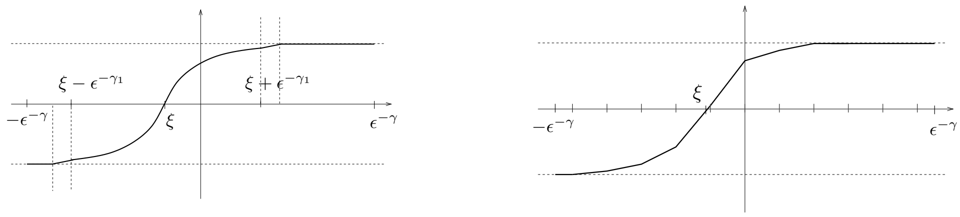

Figure 2.2: The approximated waveshapes m_ξ^ε and $m_\xi^{N,\varepsilon}$.

Proof. To see (i) write

$$\begin{aligned}\|m_\xi - m_\xi^\varepsilon\|^2_{L^2(\mathbb{R})} &\le \int_{\varepsilon^{-\gamma_1}}^{\infty} \Big(m(s) - m^\varepsilon(s)\Big)^2 \mathrm{ds} + \int_{-\infty}^{-\varepsilon^{-\gamma_1}} \Big(m(s) - m^\varepsilon(s)\Big)^2 \mathrm{ds} \\ &\le 2 \int_{\varepsilon^{-\gamma_1}}^{\infty} c_1^2 \exp(-2c_2 s) \mathrm{ds} \le C \exp(-2c_2 \varepsilon^{-\gamma_1})\end{aligned}$$

and

$$\begin{aligned}\|m'_\xi - (m_\xi^\varepsilon)'\|^2_{L^2(\mathbb{R})} &\le \int_{\varepsilon^{-\gamma_1}}^{\infty} \Big(m'(s) - (m^\varepsilon)'(s)\Big)^2 \mathrm{ds} + \int_{-\infty}^{\varepsilon^{-\gamma_1}} \Big(m'(s) - (m^\varepsilon)'(s)\Big)^2 \mathrm{ds} \\ &\le C \exp(-2c_2 \varepsilon^{-\gamma_1}).\end{aligned}$$

Here one uses the inequalities (2.2.5) as well as the properties of m^ε.

To see (ii) write

$$\|m'_\xi - (m_\xi^{N,\varepsilon})'\|_{L^2(\mathbb{R})} \le \|m'_\xi - (m_\xi^\varepsilon)'\|_{L^2(\mathbb{R})} + \|(m_\xi^\varepsilon)' - (m_\xi^{N,\varepsilon})'\|_{L^2(\mathbb{R})}. \tag{2.2.19}$$

To bound the second term assume without loss of generality that $\xi = 0$ and write

$$\begin{aligned}\|(m^\varepsilon)' - (m^{N,\varepsilon})'\|^2_{L^2(\mathbb{R})} &= \sum_{k=-N}^{N-1} \int_{s_k^{N,\varepsilon}}^{s_{k+1}^{N,\varepsilon}} \Big((m^\varepsilon)'(s) - (m^{N,\varepsilon})'(s)\Big)^2 \mathrm{ds} \\ &= \sum_{k=-N^\varepsilon}^{N^\varepsilon-1} \int_{s_k^{N,\varepsilon}}^{s_{k+1}^{N,\varepsilon}} \Big((m^\varepsilon)'(s) - (m^{N,\varepsilon})'(s)\Big)^2 \mathrm{ds}.\end{aligned} \tag{2.2.20}$$

In the second equality $N^\varepsilon = \lceil \varepsilon^{-\gamma_1} \frac{N}{\varepsilon^{-\gamma}} \rceil$. Here we use the fact that u^ε is constant outside of $[-\varepsilon^{-\gamma_1}, \varepsilon^{-\gamma_1}]$ and therefore coincides with its piecewise linearization. The integrals can be bounded using the Poincaré inequality:

$$\int_{s_k^{N,\varepsilon}}^{s_{k+1}^{N,\varepsilon}} \Big((m^\varepsilon)'(s) - (m_\xi^{N,\varepsilon})'(s)\Big)^2 \mathrm{ds} \le \frac{\varepsilon^{-2\gamma}}{N^2\pi^2} \int_{s_k^{N,\varepsilon}}^{s_{k+1}^{N,\varepsilon}} (m^\varepsilon)''(s) \mathrm{ds} \le \frac{\varepsilon^{-3\gamma}}{N^3\pi^2} \sup_{s\in\mathbb{R}} |(m^\varepsilon)''(s)|^2. \tag{2.2.21}$$

Plugging this into (2.2.19) one gets:

$$\|(m^\varepsilon)' - (m^{N,\varepsilon})'\|^2_{L^2} \le \varepsilon^{-\gamma_1} \frac{\varepsilon^{-2\gamma}}{N^2\pi^2} \sup_{s\in\mathbb{R}} |(m^\varepsilon)''(s)|^2.$$

Due to (i) the term involving $|m'_\xi - (m_\xi^\varepsilon)'|$ can be absorbed in the constant for ε small enough. This yields the second estimate in (ii). For the bound on $\|m'_\xi - (m_\xi^\varepsilon)'\|_{L^2(\mathbb{R})}$ one proceeds in the same manner with another use of Poincaré inequality. The details are left to the reader. □

2.3 Gaussian estimates

In this section concentration properties of some discretized Gaussian measures are discussed and the bounds which are needed in Section 2.4 are provided. To this end we recall a classical Gaussian concentration inequality. Then we introduce the discretized version of the Gaussian reference measure ν^ε and give an error bound. We also study another discretized measure which can be viewed as a discretized massive Gaussian free field.

Let E be a separable Banach space equipped with its Borel-σ-field $\mathcal{F}$ and norm $\|\cdot\|$. Recall that a probability measure μ on $(E, \mathcal{F})$ is called Gaussian if for every η in the dual space X^* the pushforward measure $\eta_{\#}\mu$ is Gaussian. For the moment all Gaussian measures are assumed to be centered i.e. for all $\eta \in X^*$ it holds $\int \eta(x)\mu(\mathrm{d}x) = 0$. Denote by

$$\sigma = \sup_{\eta \in X^*, \|\eta\|_{X^*} \leq 1} \left(\int \eta(x)^2 \mu(\mathrm{d}x) \right)^{1/2}.$$

Note that σ is finite [Le96]. Then one has the following classical concentration inequality (see [Le96] page 203):

$$\mu\Big(y;\ \|y\| \ \geq \int \|x\|\, \mu(\mathrm{d}x) + r\Big) \ \leq\ e^{-r^2/2\sigma^2}.$$

There are several ways to prove this inequality. It can, for example, be derived as a consequence of the Gaussian isoperimetric inequality.

The difficulty in applying this inequality to specific examples is to evaluate the quantities σ and $\int \|x\|\mu(\mathrm{d}x)$. This is easier in the case where E is a Hilbert space. Then a centered Gaussian measure μ is uniquely characterized by the covariance operator Σ which satisfies

$$\int \langle \eta_1, x\rangle\langle \eta_2, x\rangle\, \mu(\mathrm{d}x) \ =\ \langle \eta_1, \Sigma\eta_2\rangle \qquad \forall \eta_1, \eta_2 \in E.$$

It is known [dPZ92] that Σ must be a nonnegative symmetric trace class operator. Then σ^2 is the spectral radius of Σ and using Jensen's inequality one obtains

$$\int \|x\|\, \mu(\mathrm{d}x) \leq \Big(\int \|x\|^2\, \mu(\mathrm{d}x)\Big)^{1/2} = \big(\mathrm{Tr}\,\Sigma\big)^{1/2}.$$

Therefore, one can write

Lemma 2.3.1. *Let μ be a centered Gaussian measure on a Hilbert space E with covariance operator Σ. Then one has*

$$\mu\big(x; \|x\| \geq \big(\mathrm{Tr}\,\Sigma\big)^{1/2} + r\big) \leq e^{-r^2/2\sigma^2}. \tag{2.3.1}$$

We now want to use this inequality to study the behavior of the measure ν^ε under discretization. To this end fix an integer N and consider piecewise affine functions $u \in L^2[-\varepsilon^{-\gamma}, -\varepsilon^{-\gamma}]$ of the following type

$$u(x) = \begin{cases} \pm 1 & \text{for} \quad x = \pm\varepsilon^{-\gamma} \\ \text{arbitrary} & \text{for} \quad x = s_k^{N,\varepsilon} \quad k = -(N-1), \ldots, (N-1) \\ & \text{the linear interpolation between those points,} \end{cases} \tag{2.3.2}$$

and denote by $H^{N,\varepsilon}$ the affine space of all such functions. Recall that $s_k^{N,\varepsilon} = \frac{k\varepsilon^{-\gamma}}{N}$. The space $H^{N,\varepsilon}$ can canonically be identified with $\mathbb{R}^{2N-1}$. In particular typical finite-dimensional objects such as Lebesgue- and codimension one Hausdorff measures make sense on $H^{N,\varepsilon}$. Denote these measures by $\mathcal{L}^{N,\varepsilon}$ and $\mathcal{H}^{N,\varepsilon}$. There are several bilinear forms on (the tangential space of) $H^{N,\varepsilon}$ which will be important in the sequel: The H^1- and L^2-scalar product correspond to the fact that $H^{N,\varepsilon}$ is a subset of $m + H^1$. But there is also the Euclidean scalar product $\langle u, v\rangle = \sum_{k=-(N-1)}^{(N-1)} u(s_k^{N,\varepsilon})v(s_k^{N,\varepsilon})$ which determines the behavior of the measures $\mathcal{L}^{N,\varepsilon}$ and $\mathcal{H}^{N,\varepsilon}$.

Recall that ν^ε is the distribution of a Gaussian process $(u(s), s \in [-\varepsilon^{-\gamma}, \varepsilon^{-\gamma}])$ with $\mathbb{E}[u(s)] = \varepsilon^\gamma s$ and $\mathrm{Cov}(u(s), u(s')) = \varepsilon\Big(s \wedge s' + \varepsilon^{-\gamma} - \frac{(s+\varepsilon^{-\gamma})(s'+\varepsilon^{-\gamma})}{2\varepsilon^{-\gamma}}\Big)$. According to the Kolmogorov-Chentsov Theorem we can assume that u has continuous paths. Consider now the piecewise linearization of u^N of u:

$$u^N(s) = \begin{cases} \pm 1 & \text{for} \quad s = \pm\varepsilon^{-\gamma} \\ u(s) & \text{for} \quad x = s_k^{N,\varepsilon} \quad k = -(N-1), \dots, (N-1) \\ & \text{the linear interpolation between those points.} \end{cases}$$

Lemma 2.3.2. *(i) The distribution of u^N is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^{N,\varepsilon}$ on $H^{N,\varepsilon}$. The density is given by*

$$\frac{1}{\sqrt{(\varepsilon 2\pi)^{2N-1}}} \Big(\frac{N}{\varepsilon^{-\gamma}}\Big)^N \left(2\varepsilon^{-\gamma}\right)^{1/2} \exp\big(-\varepsilon^{\gamma-1}\big) \exp\Big(-\frac{1}{2\varepsilon}\int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} |u'(s)|^2 \mathrm{ds}\Big). \tag{2.3.3}$$

(ii) The random function $u - u^N$ consists of $2N$ independent rescaled Brownian bridges. To be more precise for each $k \in \{-N, \dots (N-1)\}$ the process $(u(s) - u^N(s) \colon s \in [s_k^{N,\varepsilon}, s_{k+1}^{N,\varepsilon}])$ is a centered Gaussian process with covariance

$$\mathrm{Cov}(u(s) - u^N(s), u(s') - u^N(s')) = \varepsilon\Big(s \wedge s' - s_k^{N,\varepsilon} - \frac{N}{\varepsilon^{-\gamma}}(s - s_k^{N,\varepsilon})(s' - s_k^{N,\varepsilon})\Big). \tag{2.3.4}$$

These processes are mutually independent and independent of u^N.

Proof. (i) The measure ν^ε can be considered as the distribution of a rescaled Brownian motion u on $[-\varepsilon^{-\gamma}, \varepsilon^{-\gamma}]$ starting at $u(-\varepsilon^{-\gamma}) = -1$ and conditioned on $u(\varepsilon^{-\gamma}) = 1$. Therefore, the finite-dimensional distributions can be obtained by finite-dimensional conditioning:

$$\begin{aligned} &\nu^\varepsilon\Big(u(s_{-(N-1)}^{N,\varepsilon}) \in \mathrm{d}x_{-(N-1)}, \dots, u(s_{N-1}^{N,\varepsilon}) \in \mathrm{d}x_{(N-1)}\Big) \\ &= \left(\prod_{i=-N}^{(N-1)} \frac{1}{\sqrt{(\varepsilon 2\pi)\delta}} \exp\Big(-\frac{(x_{i+1}-x_i)^2}{2\varepsilon\delta}\Big)\right) \left(\frac{1}{\sqrt{(\varepsilon 2\pi) 2\varepsilon^{-\gamma}}} \exp\Big(-\frac{(1-(-1))^2}{4\varepsilon^{-\gamma}\varepsilon}\Big)\right)^{-1} \\ &= \frac{1}{\sqrt{(\varepsilon 2\pi)^{2N-1}}} \delta^{-N} \sqrt{2\varepsilon^{-\gamma}} \exp(\varepsilon^{\gamma-1}) \exp\Big(-\frac{1}{2\varepsilon} \sum_{i=-N}^{N-1} \delta \frac{(x_{i+1}-x_i)^2}{\delta^2}\Big) \end{aligned}$$

Here $\delta = \frac{\varepsilon^{-\gamma}}{N}$ and $x_{\pm N} = \pm 1$. By noting that the Riemann sum appearing in the last line is equal to the integral of the squared derivative of the piecewise linearization one obtains the result.

(ii) Denote for $k = -N, \dots, (N-1)$ and $s \in [0, \delta]$ by $\tilde{u}_k(s) = u(s_k^{N,\varepsilon} + s) - u^N(s_k^{N,\varepsilon} + s) = u(s_k^{N,\varepsilon} + s) - \Big(1 - \frac{s}{\delta}\Big)u(s_k^{N,\varepsilon}) - \frac{s}{\delta}u(s_{k+1}^{N,\varepsilon})$. We want to show that the processes $(\tilde{u}_k(s), s \in [0, \delta])$ posses the right covariances and are mutually independent and independent of u^N. To this end calculate for $s, s' \in [0, \delta]$ and $i = -N, \dots, (N-1)$:

$$\begin{aligned} \mathrm{Cov}(\tilde{u}_k(s), \tilde{u}_k(s')) = \mathrm{Cov}\Big[&u(s_k^{N,\varepsilon} + s) - \Big(1 - \frac{s}{\delta}\Big)u(s_k^{N,\varepsilon}) - \frac{s}{\delta}u(s_{k+1}^{N,\varepsilon}), \\ &u(s_k^{N,\varepsilon} + s') - \Big(1 - \frac{s'}{\delta}\Big)u(s_k^{N,\varepsilon}) - \frac{s'}{\delta}u(s_{k+1}^{N,\varepsilon})\Big]. \end{aligned}$$

By plugging in the explicit expression for the covariances of the $u(s)$ and some tedious but elementary calculations one obtains the desired expression. In a similar way one can see that for $i \neq j$ one has

$$\mathrm{Cov}(\tilde{u}_j(s), \tilde{u}_i(s')) = 0 \qquad \text{and} \qquad \mathrm{Cov}(\tilde{u}_j(s), u^N(t)) = 0$$

for all $s, s' \in [0, \delta]$ and $t \in [-\varepsilon^{-\gamma}, \varepsilon^{-\gamma}]$. ☐

Denote the Gaussian normalization constant by

$$Z_1^{N,\varepsilon} := \left(\frac{1}{\sqrt{(2\pi\varepsilon)^{2N-1}}} \left(\frac{N}{\varepsilon^{-\gamma}}\right)^N \sqrt{2\varepsilon^{-\gamma}} \exp\bigl(-\varepsilon^{\gamma-1}\bigr) \right)^{-1}.$$

We define the discrete Dirichlet-Laplace operator $\Delta_{N,\varepsilon}$ as

$$\Delta_{N,\varepsilon}^{k,j} = \frac{N}{\varepsilon^{-\gamma}} \begin{cases} -2 & \text{for } k = j \\ 1 & \text{for } |k-j| = 1 \\ 0 & \text{else.} \end{cases} \tag{2.3.5}$$

Then a direct computation shows

$$Z_1^{N,\varepsilon} = \sqrt{(2\pi\varepsilon)^{2N-1}} \exp\bigl(\varepsilon^{\gamma-1}\bigr)\bigl(\det(-\Delta_{N,\varepsilon})\bigr)^{-1/2}. \tag{2.3.6}$$

We now want to apply the Gaussian concentration inequality to obtain a bound on the probability of large $u - u^N$:

Lemma 2.3.3. *The following bounds hold:*

1. *L^2-bound on the whole line:*

$$\nu^\varepsilon\Bigl(u : \|u-u^N\|_{L^2[-\varepsilon^{-\gamma},\varepsilon^{-\gamma}]} \geq \sqrt{\varepsilon\frac{\varepsilon^{-2\gamma}}{3N}} + r\Bigr) \leq \exp\left(-\frac{r^2\pi^2N^2}{\varepsilon^{1-2\gamma}}\right) \tag{2.3.7}$$

2. *L^2-bound on the short intervals:*

$$\nu^\varepsilon\left(\|u-u^N\|_{L^2[s_k^{N,\varepsilon},s_{k+1}^{N,\varepsilon}]} \geq \sqrt{\varepsilon\frac{\varepsilon^{-2\gamma}}{6N^2}} + r\right) \leq \exp\left(-\frac{r^2\pi^2N^2}{\varepsilon^{1-2\gamma}}\right). \tag{2.3.8}$$

3. *L^∞-bound on the whole line:*

$$\nu^\varepsilon\left(\|u-u^N\|_{L^\infty[-\varepsilon^{-\gamma},\varepsilon^{-\gamma}]} \geq r\right) \leq 4N\exp\left(-\frac{r^2N}{8\varepsilon^{1-\gamma}}\right). \tag{2.3.9}$$

Proof. Let us consider (2.3.7) first. Note that $u-u^N$ is a centered Gaussian process such that Lemma 2.3.1 can be applied. The expected L^2-norm can be calculated as follows:

$$\begin{aligned}\nu^\varepsilon\Bigl[\|u-u^N\|^2_{L^2[-\varepsilon^{-\gamma},\varepsilon^{-\gamma}]}\Bigr] &= \sum_{k=-N}^{N-1} \nu^\varepsilon\|\tilde{u}_k\|^2_{L^2} = \sum_{k=-N}^{N-1}\int_{s_k^{N,\varepsilon}}^{s_{k+1}^{N,\varepsilon}} \nu^\varepsilon\bigl(\tilde{u}(s)^2\bigr)\mathrm{ds} \\ = \sum_{k=-N}^{N-1}\int_{s_k^{N,\varepsilon}}^{s_{k+1}^{N,\varepsilon}} \varepsilon &\left(s - s_k^{N,\varepsilon} - \frac{\left(s-s_k^{N,\varepsilon}\right)^2}{\frac{\varepsilon^{-\gamma}}{N}}\right)\mathrm{ds} = 2N\varepsilon\frac{1}{6}\left(\frac{\varepsilon^{-\gamma}}{N}\right)^2.\end{aligned}$$

Here for the third equality equation (2.3.4) is used.

To get information about the spectral radius of the covariance operator Σ calculate for $f, g \in L^2[-\varepsilon^{-\gamma},\varepsilon^{-\gamma}]$:

$$\begin{aligned}\langle f, \Sigma g\rangle &= \nu^\varepsilon\left[\langle f, u-u^N\rangle\langle g, u-u^N\rangle\right] \\ &= \sum_{k=-N}^{N-1}\int_{s_k^{N,\varepsilon}}^{s_{k+1}^{N,\varepsilon}} \varepsilon\Bigl(s\wedge s' - \frac{(s-s_k^{N,\varepsilon})(s'-s_k^{N,\varepsilon})}{\frac{\varepsilon^{-\gamma}}{N}}\Bigr) f(s)g(s')\,\mathrm{ds}.\end{aligned}$$

Here in the last step the independence of the different bridges is used as well as formula (2.3.3). Note that the integral kernel in the last line is the Green function of the negative Dirichlet-Laplace operator on the interval $[s_k^{N,\varepsilon}, s_{k+1}^{N,\varepsilon}]$. Denoting this operator by $\varepsilon(-\Delta_{T_k})^{-1}$ one can write

$$\langle f, \Sigma g\rangle = \sum_{k=-N}^{N-1} \langle f, \varepsilon(-\Delta_{T_k})^{-1} g\rangle_{L^2(T_k)}.$$

The spectral decomposition of the inverse Dirichlet-Laplace operator on intervals of length T is well known. In fact on $L^2[0,T]$ the smallest eigenvalue λ_0 and the corresponding eigenfunction $e_0(x)$ are given as:

$$e_0(s) = \sin\Big(\frac{\pi s}{T}\Big) \quad \text{and} \quad \lambda_0 = \frac{\varepsilon T^2}{\pi^2}.$$

The spectral radius of $\varepsilon\big(-\Delta_{T_k}\big)^{-1}$ is thus given as

$$\sigma_k^2 = \varepsilon \frac{\varepsilon^{-2\gamma}}{(N\pi)^2}.$$

Therefore, one can write

$$\begin{aligned}
\sigma^2 &= \sup_{f,\|f\|=1} \langle f, \Sigma f\rangle = \sup_{f,\|f\|=1} \sum_{k=-N}^{N-1} \langle f, \varepsilon(\Delta_{T_k})^{-1} g\rangle_{L^2(T_k)} \\
&\leq \sup_{f,\|f\|=1} \sum_{k=-N}^{N-1} \sigma_k^2 \langle f, f\rangle_{L^2(T_k)} = \varepsilon \left(\frac{\varepsilon^{-\gamma}}{\pi N}\right)^2 \sup_{f,\|f\|=1} \langle f, f\rangle.
\end{aligned}$$

On the other hand by taking f as a linear combination of the eigenfunctions on the shorter intervals one obtains

$$\sigma^2 = \varepsilon \left(\frac{\varepsilon^{-\gamma}}{\pi N}\right)^2.$$

Thus equation (2.3.1) gives the desired result. The proof of (2.3.8) proceeds in the same manner.

To prove the third statement (2.3.9) note that by Lemma 2.3.2, the deviations of a the random function u from the piecewise linearizations u^N between the points $s_k^{N,\varepsilon}$ are independent Brownian bridges. Therefore, such a process $\left(u(s_k^{N,\varepsilon}+s) - u^N(s_k^{N,\varepsilon}+s), 0 \leq s \leq \frac{\varepsilon^{-\gamma}}{N}\right)$ has the same distribution as $\varepsilon^{\frac{1}{2}}\left(B_s - \frac{sN}{\varepsilon^{-\gamma}} B_{\frac{\varepsilon^{-\gamma}}{N}}\right)$ for a Brownian motion B defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. So one can write

$$\begin{aligned}
&\nu^\varepsilon\Big(\|u(s) - u^N(s)\|_{L^\infty[-\varepsilon^{-\gamma}, \varepsilon^{-\gamma}]} \geq r\Big) \\
&\qquad \leq \sum_{k=-N}^{N-1} \nu^\varepsilon\left(\max_{s_k^{N,\varepsilon} \leq s \leq s_{k+1}^{N,\varepsilon}} |u(s) - u^N(s)| \geq r\right) \\
&\qquad \leq 2N \quad \mathbb{P}\left(\max_{0 \leq s \leq \frac{\varepsilon^{-\gamma}}{N}} \left|\varepsilon^{1/2}\left(B_s - \frac{sN}{\varepsilon^{-\gamma}} B_{\frac{\varepsilon^{-\gamma}}{N}}\right)\right| \geq r\right) \\
&\qquad \leq 2N \quad \mathbb{P}\left(\max_{0 \leq s \leq \frac{\varepsilon^{-\gamma}}{N}} |B_s| \geq \frac{r}{2\varepsilon^{1/2}}\right).
\end{aligned}$$

Using the exponential version of the maximal inequality for martingales (see Proposition 1.8 in Chapter II in [RY99]) one can see that

$$\nu^\varepsilon\left(\|u(s) - u^N(s)\|_{L^\infty[-\varepsilon^{-\gamma}, \varepsilon^{-\gamma}]} \geq r\right) \leq 4N \exp\left(-\frac{r^2 N}{8\varepsilon^{1-\gamma}}\right).$$

□

We now want to study the properties of another discrete Gaussian measure. In fact denote by $H_0^{N,\varepsilon}$ the space of affine functions defined as in 2.3.2 with the only change that they are assumed to possess zero boundary conditions. The Lebesgue measure on this space is defined in the same manner. For a fixed constant κ consider the centered probability measure ϱ whose density with respect to $\mathcal{L}^{N,\varepsilon}$ is proportional to

$$\exp\left(-\kappa\frac{\int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}}|u(s)|^2+|\nabla u(s)|^2\,\mathrm{ds}}{2\varepsilon}\right).$$

In fact this measure is a version of what is known in the literature as discrete massive free field, discrete Ornstein-Uhlenbeck bridge or pinned $\nabla\phi$ surface model [S07, HSV05]. The H^1-norm in the exponent can be rewritten in terms of the finite-dimensional coordinates. In fact for $u\in H_0^{N,\varepsilon}$ with $u(s_k^{N,\varepsilon})=u_k$ for $k=-N,\ldots,N$ one has

$$\begin{aligned}\|u\|^2_{H^1(\mathbb{R})}&=\|u\|^2_{L^2(\mathbb{R})}+\|u'\|^2_{L^2(\mathbb{R})}\\&=\frac{1}{3}\frac{\varepsilon^{-\gamma}}{N}\sum_{k=-N}^{N}u_k^2+u_{k+1}^2+u_ku_{k+1}+\frac{N}{\varepsilon^{-\gamma}}\sum_{k=-N}^{N}u_k^2+u_{k+1}^2-2u_ku_{k+1}\\&=\sum_{k,j=-(N-1)}^{N-1}u_k\left(I_{N,\varepsilon}^{k,j}-\Delta_{N,\varepsilon}^{k,j}\right)u_j,\end{aligned}\tag{2.3.10}$$

where the $(2N-1)\times(2N-1)$ matrix $(I_{N,\varepsilon}^{k,j})$ is given as

$$I_{N,\varepsilon}^{k,j}=\frac{1}{3}\frac{\varepsilon^{-\gamma}}{N}\begin{cases}2&\text{for } k=j\\\frac{1}{2}&\text{for } |k-j|=1\\0&\text{else,}\end{cases}\tag{2.3.11}$$

and the discrete Laplace operator $\Delta_{N,\varepsilon}$ is defined as in (2.3.5). Denote the normalization constant

$$Z_2^{N,\varepsilon}=\int\exp\left(-\kappa\frac{\int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}}|u(s)|^2+|\nabla u(s)|^2\quad\mathrm{ds}}{2\varepsilon}\right)\mathcal{L}^{N,\varepsilon}(\mathrm{d}u).$$

Lemma 2.3.4. *(i) $Z_2^{N,\varepsilon}$ is given as*

$$\sqrt{\left(\frac{2\varepsilon\pi}{\kappa}\right)^{2N-1}}\det(-\Delta_{N,\varepsilon}+I_{N,\varepsilon})^{-\frac{1}{2}}.\tag{2.3.12}$$

(ii) In the sense of symmetric matrices we have the following Poincaré inequality

$$I_{N,\varepsilon}\le C\varepsilon^{-\gamma}(-\Delta_{N,\varepsilon}).\tag{2.3.13}$$

(iii) We have the following bound: For $r\ge 0$

$$\varrho^{N,\varepsilon}\left\{u\colon\|u\|_{H^1}\ge\sqrt{\frac{(2N-1)\varepsilon}{\kappa}}+r\right\}\le\exp\left(-\kappa r^2/2\varepsilon\right).\tag{2.3.14}$$

Proof. (i) To see this one only has to note that $\kappa\varepsilon^{-1}\left(-\Delta_{N,\varepsilon}+I_{N,\varepsilon}\right)$ is the inverse covariance matrix of this finite-dimensional Gaussian measure.

(ii) Using the usual Poincaré inequality one can write for $u \in H_0^{N,\varepsilon}$

$$
\begin{aligned}
-\sum_{k,j=-(N-1)}^{N-1} u\left(s_k^{N,\varepsilon}\right) \Delta_{N,\varepsilon}^{k,j} u\left(s_j^{N,\varepsilon}\right) &= \|u'\|^2_{L^2[-\varepsilon^{-\gamma},\varepsilon^{-\gamma}]} \\
&\geq C\varepsilon^{\gamma} \|u\|^2_{L^2[-\varepsilon^{-\gamma},\varepsilon^{-\gamma}]} = C\varepsilon^{\gamma} \sum_{k,j=-(N-1)}^{N-1} u\left(s_k^{N,\varepsilon}\right) I_{N,\varepsilon}^{k,j} u\left(s_j^{N,\varepsilon}\right).
\end{aligned}
$$

(iii) To see (2.3.14) write with a finite-dimensional change of variables:

$$
\begin{aligned}
\varrho^{N,\varepsilon}\left\{u \colon \|u\|_{H^1} \geq r\right\} &= \frac{1}{Z_2^{N,\varepsilon}} \int_{\{u\colon \|u\|_{H^1} \geq r\}} \exp\left(-\kappa \frac{\|u\|^2_{H^1}}{2\varepsilon}\right) \mathcal{L}^{N,\varepsilon}(\mathrm{d}u) \\
= \sqrt{\left(\frac{\kappa}{2\varepsilon\pi}\right)^{2N-1}} &\int_{\{\sum_{k=-(N-1)}^{N-1} x_k^2 \geq r\}} \exp\left(-\kappa \frac{\sum_{k=-(N-1)}^{N-1} x_k^2}{2\varepsilon}\right) \mathrm{d}x_{-(N-1)} \dots \mathrm{d}x_{N-1}.
\end{aligned}
$$

In fact here one uses the linear transformation that transforms a Gaussian random variable on a finite-dimensional space to a Gaussian random variable with Id covariance matrix. Therefore, the problem reduces to considering a vector of $2N-1$ independent centered Gaussian random variables X_k with variance $\frac{\varepsilon}{\kappa}$. The expectation

$$
\mathbb{E}\left[\sum_{k=-(N-1)}^{N-1} X_i^2\right] = \frac{(2N-1)\varepsilon}{\kappa}
$$

and the spectral radius

$$
\sigma^2 = \frac{\varepsilon}{\kappa}
$$

are calculated easily such that (2.3.1) gives the desired result. □

2.4 Concentration around a curve in infinite-dimensional space

In this section we give the proof of Theorem 2.1.1. To this end we consider the finite-dimensional measure

$$
\mu^{N,\varepsilon}(\mathrm{d}u) = \frac{1}{Z^{N,\varepsilon}} \exp\left(-\frac{1}{\varepsilon}\int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} F(u(s))ds\right) \nu^{N,\varepsilon}(\mathrm{d}u),
$$

with the normalization constant $Z^{N,\varepsilon} = \int \exp\left(-\frac{1}{\varepsilon}\int F(u(s))\mathrm{d}s\right) \nu^{N,\varepsilon}(\mathrm{d}u)$. Note that although $\nu^{N,\varepsilon}$ is given by the finite-dimensional marginals of ν^{ε}, the measure $\mu^{N,\varepsilon}$ does not coincide with the finite-dimensional distribution of μ^{ε}. The strategy is now as follows: In Proposition 2.4.4 a lower bound on the discrete normalization constant $Z^{N,\varepsilon}$ is given. This is achieved by calculating the integral in a tubular neighborhood of the set of minimizers M. Then in Proposition 2.4.8 the rough energy bound given in Proposition 2.2.2 is used to conclude concentration of the discretized measure $\mu^{N,\varepsilon}$ around the curve of minimizers. Finally Lemma 2.4.13 gives a bound on the discretization error which allows to finish the proof of concentration around the curve of minimizers in the continuous case.

Recall the following version of the coarea formula:

Lemma 2.4.1. *Let f be a Lipschitz function $f : A \subseteq E \to I \subseteq \mathbb{R}$, where E is a n-dimensional Euclidean space and A is an open subset and I some interval. Denote by $\mathcal{L}^n, \mathcal{L}^1$ and $\mathcal{H}^{n-1}$ the Lebesgue measure on E, on $\mathbb{R}$ and the $(n-1)$-dimensional Hausdorff measure on E respectively. Suppose that the gradient (which exists $\mathcal{L}^n$-a.e.) Df does not vanish $\mathcal{L}^n$ a.e. in A. Then for every nonnegative measurable test function $\varphi : A \to \mathbb{R}$ one has the following formula:*

$$
\int_A \varphi(x)\mathcal{L}^n(\mathrm{d}x) = \int_I \mathcal{L}^1(\mathrm{d}\xi) \int_{f^{-1}(\xi)} \mathcal{H}^{n-1}(\mathrm{d}x) \frac{1}{|Df(x)|_E} \varphi(x). \tag{2.4.1}
$$

In order to apply formula (2.4.1) to $\mu^{N,\varepsilon}$ one needs the following:

Lemma 2.4.2. *Consider the function $f : A \to \mathbb{R}$, where $A := \{x \in m + L^2 \colon \mathrm{dist}_{L^2}(x, M) < \beta\}$ is the open set in which the Fermi coordinates are defined, given by*

$$f(x) = f(m_\xi + s) = \xi,$$

where $x = m_\xi + s$ are the Fermi coordinates of x. Then f is Fréchet differentiable and one has

$$Df(x)[h] = Df(m_\xi + s)[h] = \frac{\langle m'_\xi, h\rangle}{|m'_\xi|^2 - \langle s, m''_\xi\rangle}. \tag{2.4.2}$$

Proof. The differentiability follows from the implicit function theorem. To calculate the derivative at $x = m_\xi + s$ in direction h consider the function

$$\Phi(v, w) = \langle m_\xi - m_w + s + vh, m'_w\rangle,$$

defined in an environment of $(0, \xi) \in \mathbb{R}^2$. Noting that one has $\Phi(v, f(m_\xi + s + vh)) = 0$ one can write

$$0 = \partial_v \Phi(v, f(m_\xi + s + vh))|_{v=0} + \partial_w \Phi(v, f(m_\xi + s + vh))|_{v=0} Df(m_\xi + s)[h].$$

Observing that

$$\partial_v \Phi(v, f(m_\xi + s + vh))|_{v=0} = \langle h, m'_\xi\rangle$$

and

$$\partial_w \Phi(v, f(m_\xi + s + vh))|_{v=0} = -\langle m'_\xi, m'_\xi\rangle + \langle s, m''_\xi\rangle$$

concludes the proof. □

We want to apply the coarea formula to the function f just defined, restricted to $H^{N,\varepsilon}$. There is a slight inconvenience which originates from the fact that the norm of the gradient which appears in 2.4.1 is the norm in the finite-dimensional space E whereas the gradient of the function f is a function in $L^2(\mathbb{R})$. To resolve this is the content of the next lemma:

Lemma 2.4.3. *Let $g : m + L^2(\mathbb{R}) \to \mathbb{R}$ be a Fréchet differentiable function and denote by $\nabla g(x)$ its L^2-gradient at point x. Consider then the function $\tilde{g}$ defined on $\mathbb{R}^{2N-1}$ obtained by composition of the embedding $\mathbb{R}^{2N-1} \to H^{N,\varepsilon}$ and g. Denote by $\tilde{\nabla}\tilde{g}$ its gradient. Then one has the following inequality:*

$$\|\tilde{\nabla}\tilde{g}\|_{\mathbb{R}^{2N-1}} \le 2\sqrt{\frac{\varepsilon^{-\gamma}}{N}}\|\nabla g\|_{L^2}.$$

Proof. We calculate the derivative of $\tilde{g}$ in direction $\tilde{e}_k = (0, \ldots, 0, 1, 0, \ldots 0)$ with the 1 on k-th position. Embedding $\tilde{e}_k$ into $H^{N,\varepsilon}$ gives the hat function

$$e_k(s) = \begin{cases} 0 & \text{for } s \notin [s^{N,\varepsilon}_{k-1}, s^{N,\varepsilon}_{k+1}] \\ \frac{N}{\varepsilon^{-\gamma}}\left(s - s^{N,\varepsilon}_{k-1}\right) & \text{for } s \in]s^{N,\varepsilon}_{k-1}, s^{N,\varepsilon}_{k}] \\ \frac{N}{\varepsilon^{-\gamma}}\left(s^{N,\varepsilon}_{k+1} - s\right) & \text{for } s \in]s^{N,\varepsilon}_{k}, s^{N,\varepsilon}_{k+1}]. \end{cases} \tag{2.4.3}$$

Therefore, one obtains

$$(\tilde{\nabla}\tilde{g})_k = \int_{\mathbb{R}} e_k(s)\nabla g(s)\mathrm{ds} = \int_{s^{N,\varepsilon}_{k-1}}^{s^{N,\varepsilon}_{k+1}} e_k(s)\nabla g(s)\mathrm{ds}.$$

Applying Cauchy-Schwarz inequality and using $\|e_k\|_\infty \le 1$ one gets:

$$\begin{aligned} \|\tilde{\nabla}\tilde{g}\|^2_{\mathbb{R}^{2N-1}} &= \sum_{k=-(N-1)}^{N-1}\left(\int_{s^{N,\varepsilon}_{k-1}}^{s^{N,\varepsilon}_{k+1}} e_k(s)\nabla g(s)\mathrm{ds}\right)^2 \le 2\frac{\varepsilon^{-\gamma}}{N}\sum_{k=-(N-1)}^{N-1}\int_{s^{N,\varepsilon}_{k-1}}^{s^{N,\varepsilon}_{k+1}} (\nabla g(s))^2\,\mathrm{ds} \\ &\le 2\frac{\varepsilon^{-\gamma}}{N}2\|\nabla g\|^2_{L^2(\mathbb{R})}. \end{aligned} \tag{2.4.4}$$

□

Now we are ready to derive a lower bound on the normalization constant $Z^{N,\varepsilon}$ of the finite-dimensional approximation of μ^ε. Recall that $\mu^{N,\varepsilon}(\mathrm{d}u) = \frac{1}{Z^{N,\varepsilon}} \exp\left(-\frac{1}{\varepsilon}\int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} F(u(s))\mathrm{d}s\right)\nu^{N,\varepsilon}(\mathrm{d}u)$ where $\nu^{N,\varepsilon}$ is a discretized Brownian bridge. One gets the following bound:

Proposition 2.4.4. *Assume $N = N(\varepsilon)$ grows like $\varepsilon^{-\gamma_2}$ for ε decreasing to 0. Assume that*

$$0 < \gamma_1 < \gamma < \gamma_2 < 1. \tag{2.4.5}$$

Then the following bound holds for ε small enough:

$$\begin{aligned} Z^{N,\varepsilon} \geq \exp\Big(-\frac{C_*}{\varepsilon}\Big)\exp\left(-2C\left(\frac{\varepsilon^{-2\gamma-\gamma_1}}{\varepsilon N^2}\right)\right)\sqrt{\frac{N}{\varepsilon^{-\gamma}}}\left(\varepsilon^{-\gamma}-\varepsilon^{-\gamma_1}-1\right) \\ \sqrt{\varepsilon}\sqrt{\frac{\epsilon^{-1-\gamma}}{N}}\exp\left(-C\frac{\varepsilon^{-4\gamma-1-\gamma_1}}{N^4}\right)\exp\left(-\varepsilon^{\gamma-1}\right)c_4^{-\frac{2N-1}{2}}\left(1+C\varepsilon^{-\gamma}\right)^{-\frac{2N-1}{2}}. \end{aligned} \tag{2.4.6}$$

In particular, if one assumes that

$$-2\gamma - \gamma_1 + 2\gamma_2 > 0 \tag{2.4.7}$$

one obtains

$$\liminf_{\varepsilon\downarrow 0} \varepsilon \log Z^{N,\varepsilon} \geq -C_*. \tag{2.4.8}$$

Proof. Using the definition of $\nu^{N,\varepsilon}$ one can write

$$\begin{aligned} Z^{N,\varepsilon} &= \int_{H^{N,\varepsilon}} \exp\Big(-\frac{1}{\varepsilon}\int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} F(u(s))\mathrm{d}s\Big)\nu^{N,\varepsilon}(\mathrm{d}u) \\ &= \frac{1}{Z_1^{N,\varepsilon}}\exp\Big(-\frac{C_*}{\varepsilon}\Big)\int_{H^{N,\varepsilon}}\exp\Big(-\frac{1}{\varepsilon}\int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} F(u(s))\mathrm{d}s \\ &\qquad -\frac{1}{\varepsilon}\int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}}\frac{1}{2}|u'(s)|^2\mathrm{d}s + \frac{C_*}{\varepsilon}\Big)\mathcal{L}^{N,\varepsilon}(\mathrm{d}u) \\ &= \frac{1}{Z_1^{N,\varepsilon}}\exp\Big(-\frac{C_*}{\varepsilon}\Big)\int_{H^{N,\varepsilon}}\exp\Big(-\frac{1}{\varepsilon}\mathcal{E}(u)\Big)\mathcal{L}^{N,\varepsilon}(\mathrm{d}u). \end{aligned} \tag{2.4.9}$$

Recall that $Z_1^{N,\varepsilon} = \int \exp\Big(-\frac{1}{\varepsilon}\int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}}\frac{1}{2}|u'(s)|^2\mathrm{d}s\Big)\mathcal{L}^{N,\varepsilon}(\mathrm{d}u)$ is the normalization constant of the discretized Brownian bridge and $\mathcal{L}^{N,\varepsilon}$ is the Lebesgue measure on the finite-dimensional space $H^{N,\varepsilon}$. In order to find a lower bound on $Z^{N,\varepsilon}$ we can restrict the integration to a tubular neighborhood of M. More precisely set $I_\varepsilon := [-\varepsilon^{-\gamma}+\varepsilon^{-\gamma_1}, \varepsilon^{-\gamma}-\varepsilon^{-\gamma_1}]$ and

$$A := \left\{u \in H^{N,\varepsilon}\colon u = m_\xi + v\colon \langle v, m_\xi'\rangle_{L^2(\mathbb{R})} = 0 \text{ for some } \xi \in I_\varepsilon \text{ and } \|v\|_{H^1(\mathbb{R})} \leq \delta\right\},$$

for some δ to be determined later. For the moment we will only assume δ to be small enough to be able to apply Funaki's estimate (2.2.8) on the energy landscape. Furthermore, denote by

$$A_\xi := \left\{u \in H^{N,\varepsilon}\colon u = m_\xi + v\colon \langle v, m_\xi'\rangle = 0 \text{ and } \|v\|_{H^1(\mathbb{R})} \leq \delta\right\}.$$

Using Funaki's estimate (2.2.8) for $u = m_\xi + v \in A$ one can write

$$\exp\Big(-\frac{1}{\varepsilon}\mathcal{E}(u)\Big) \geq \exp\left(-\frac{c_4}{\varepsilon}\|v\|_{H^1(\mathbb{R})}^2\right).$$

Note the v is not an element of the discretized space $H^{N,\varepsilon}$ but a general function in $L^2(\mathbb{R})$ that needs not vanish outside of $[-\varepsilon^{-\gamma}, \varepsilon^{-\gamma}]$. But v can be well approximated by a function $v^{N,\varepsilon} = u - m_\xi^{N,\varepsilon} \in H_0^{N,\varepsilon}$. In fact using Lemma 2.2.4 one gets

$$\|v^{N,\varepsilon} - v\|_{H^1(\mathbb{R})} = \|m_\xi^{N,\varepsilon} - m_\xi\|_{H^1(\mathbb{R})} \leq C\frac{\varepsilon^{-\gamma}}{N}\varepsilon^{\frac{-\gamma_1}{2}}.$$

Putting this together one gets:

$$\begin{aligned} Z^{N,\varepsilon} Z_1^{N,\varepsilon} \exp\Big(\frac{C_*}{\varepsilon}\Big) &\geq \int_A \exp\left(-\frac{c_4}{\varepsilon}\|v\|^2_{H^1(\mathbb{R})}\right) \mathcal{L}^{N,\varepsilon}(\mathrm{d}u) \\ &\geq \exp\left(-2C\left(\frac{\varepsilon^{-2\gamma-\gamma_1}}{\varepsilon N^2}\right)\right) \int_A \exp\left(-\frac{2c_4}{\varepsilon}\|v^{N,\varepsilon}\|^2_{H^1(\mathbb{R})}\right) \mathcal{L}^{N,\varepsilon}(\mathrm{d}u). \end{aligned} \tag{2.4.10}$$

Let us concentrate on the integral term in equation (2.4.10). Using the coarea formula (2.4.1) one gets:

$$\int_A \exp\left(-\frac{2c_4}{\varepsilon}\|v^{N,\varepsilon}\|^2_{H^1}\right) \mathcal{L}^{N,\varepsilon}(\mathrm{d}u) = \int_{I_\varepsilon} \mathrm{d}\xi \int_{A_\xi} \frac{1}{|\tilde{\nabla} \tilde{f}|} \exp\Big(-\frac{2c_4}{\varepsilon}\|v^{N,\varepsilon}\|^2_{H^1}\Big) \mathcal{H}^{N,\varepsilon}(\mathrm{d}u). \tag{2.4.11}$$

where $\mathcal{H}^{N,\varepsilon}$ is the codimension one Hausdorff measure on $H^{N,\varepsilon}$. Using Lemma 2.4.2 and the observation from Lemma 2.4.3 one knows:

$$\frac{1}{|\tilde{\nabla} \tilde{f}|} \geq \frac{1}{2}\sqrt{\frac{N}{\varepsilon^{-\gamma}}} \frac{\|m'_\xi\|^2_{L^2(\mathbb{R})} - \langle v, m''_\xi\rangle_{L^2(\mathbb{R})}}{\|m'_\xi\|_{L^2(\mathbb{R})}}.$$

By choosing a smaller δ if necessary this can be bounded uniformly from below on A by $C\sqrt{\frac{N}{\varepsilon^{-\gamma}}}$ such that one gets:

$$\begin{aligned} &\int_A \exp\left(-\frac{2c_4}{\varepsilon}\|v^{N,\varepsilon}\|^2_{H^1(\mathbb{R})}\right) \mathcal{L}^{N,\varepsilon}(\mathrm{d}u) \\ &\quad\geq C\sqrt{\frac{N}{\varepsilon^{-\gamma}}} \int_{I_\varepsilon} \mathrm{d}\xi \int_{A_\xi} \exp\left(-\frac{2c_4}{\varepsilon}\|v^{N,\varepsilon}\|^2_{H^1(\mathbb{R})}\right) \mathcal{H}^{N,\varepsilon}(\mathrm{d}u). \end{aligned} \tag{2.4.12}$$

Let us focus on the last integral. By a linear change of coordinates one can write

$$\int_{A_\xi} \exp\left(-\frac{2c_4}{\varepsilon}\|v^{N,\varepsilon}\|^2_{H^1(\mathbb{R})}\right) \mathcal{H}^{N,\varepsilon}(\mathrm{d}u) = \int_{B_\xi} \exp\left(-\frac{2c_4}{\varepsilon}\|v\|^2_{H^1(\mathbb{R})}\right) \mathcal{H}^{N,\varepsilon}(\mathrm{d}v), \tag{2.4.13}$$

where $B_\xi = \Big\{v \in H_0^{N,\varepsilon} \colon \langle v, m'_\xi\rangle_{L^2(\mathbb{R})} = \langle m_\xi - m_\xi^{N,\varepsilon}, m'_\xi\rangle_{L^2(\mathbb{R})} \text{ and } \|v\|_{H^1(\mathbb{R})} \leq \delta\Big\}$. In order to conclude, we need the following lemma:

Lemma 2.4.5. *Let E be a finite-dimensional Euclidean space with Lebesgue measure $\mathcal{L}$ and codimension 1 Hausdorff measure $\mathcal{H}$. Let $a^* = \langle a, \cdot\rangle \in E^*$ be a linear form and $x \mapsto \langle x, \Sigma x\rangle$ be a symmetric, positive bilinear form. Furthermore, write for $b \in \mathbb{R}$ and $\delta > 0$*

$$\tilde{B}^{b,\delta^2} = \left\{x \in E \colon a^* x = b \text{ and } \langle x, \Sigma x\rangle \leq \delta^2\right\}.$$

Furthermore, set $d^2 = \inf_{x\in\tilde{B}^{b,\infty}} \langle x, \Sigma x\rangle$ and let n be a Σ-unit normal vector on $\tilde{B}^{0,\infty}$, i.e. $\langle n, \Sigma x\rangle = 0$ for all $x \in \tilde{B}^{0,\infty}$ and $\langle n, \Sigma n\rangle = 1$. Then one has for every b

$$\int_{\langle x,\Sigma x\rangle \leq \delta^2 - d^2} \exp\left(-\langle x, \Sigma x\rangle\right) \mathcal{L}(\mathrm{d}x) \leq 2\delta\sqrt{\frac{1}{\langle \Sigma n, \Sigma n\rangle}} \exp\left(d^2\right) \int_{\tilde{B}^{b,\delta^2}} \exp\left(-\langle x, \Sigma x\rangle\right) \mathcal{H}(\mathrm{d}x). \tag{2.4.14}$$

Furthermore, one has the following expressions for d^2:

$$d^2 = \frac{b^2}{\langle a, \Sigma^{-1} a\rangle}, \qquad \text{and} \quad \langle a, \Sigma^{-1} a\rangle = \Big(\sup_{\eta\colon \langle \eta, \Sigma\eta\rangle = 1} a^*(\eta)\Big)^2. \tag{2.4.15}$$

The vector n satisfies $n = \pm\frac{\Sigma^{-1} a}{\sqrt{\langle a, \Sigma^{-1} a\rangle}}$ such that

$$\langle \Sigma n, \Sigma n\rangle = \frac{1}{\langle a, \Sigma^{-1} a\rangle}\left(\sup_{\eta\colon \langle \eta, \eta\rangle = 1} a^*(\eta)\right)^2. \tag{2.4.16}$$

Proof. (Of Lemma 2.4.5): Using the coarea formula one can write:

$$
\begin{aligned}
\int_{\langle x,\Sigma x\rangle\leq\delta^2-d^2} & \exp\left(-\langle x,\Sigma x\rangle\right)\mathcal{L}(\mathrm{d}x) \\
&\leq \int_{-\delta}^{\delta}\int_{\tilde{B}^{0,\delta^2-d^2}} \exp\left(-\langle (y+\lambda n),\Sigma(y+\lambda n)\rangle\right)\sqrt{\frac{1}{\langle\Sigma n,\Sigma n\rangle}}\mathcal{H}(\mathrm{d}y)\mathrm{d}\lambda \\
&\leq \sqrt{\frac{1}{\langle\Sigma n,\Sigma n\rangle}}\int_{-\delta}^{\delta}\int_{\tilde{B}^{0,\delta^2-d^2}} \exp\left(-\langle y,\Sigma y\rangle\right)\mathcal{H}(\mathrm{d}y)\mathrm{d}\lambda \\
&= 2\delta\sqrt{\frac{1}{\langle\Sigma n,\Sigma n\rangle}}\int_{\tilde{B}^{0,\delta^2-d^2}} \exp\left(-\langle y,\Sigma y\rangle\right)\mathcal{H}(\mathrm{d}y) \\
&= 2\delta\sqrt{\frac{1}{\langle\Sigma n,\Sigma n\rangle}}\exp\left(d^2\right)\int_{\tilde{B}^{0,\delta^2-d^2}} \exp\left(-\langle (y+dn),\Sigma(y+dn)\rangle\right)\mathcal{H}(\mathrm{d}y) \\
&= 2\delta\sqrt{\frac{1}{\langle\Sigma n,\Sigma n\rangle}}\exp\left(d^2\right)\int_{\tilde{B}^{b,\delta^2}} \exp\left(-\langle y,\Sigma y\rangle\right)\mathcal{H}(\mathrm{d}y).
\end{aligned}
\tag{2.4.17}
$$

The other assertions are elementary. □

In order to apply this lemma to the case $E = H_0^{N,\varepsilon}$, $a^*(v) = \langle v, m'_\xi\rangle_{L^2(\mathbb{R})}$ $b = \langle m_\xi - m_\xi^{N,\varepsilon}, m'_\xi\rangle_{L^2(\mathbb{R})}$ and $\langle v,\Sigma v\rangle = \frac{2c_4}{\varepsilon}\|v\|^2_{H^1(\mathbb{R})}$ one needs to evaluate the constants d and $\langle\Sigma n,\Sigma n\rangle$ in this context. This is the subject of the next lemma. Note that the scalar product $\langle\cdot,\cdot\rangle$ from Lemma 2.4.5 corresponds to the Euclidean scalar product on $H_0^{N,\varepsilon}$.

Lemma 2.4.6. *One has for ε small enough:*

(i) $\langle m_\xi - m_\xi^{N,\varepsilon}, m'_\xi\rangle_{L^2(\mathbb{R})} \leq C\varepsilon^{-\gamma_1/2}\frac{\varepsilon^{-2\gamma}}{N^2}$,

(ii) $d^2 \leq C\frac{\varepsilon^{-4\gamma-1-\gamma_1}}{N^4}$,

(iii) $\langle\Sigma n,\Sigma n\rangle \geq C\frac{\varepsilon^{-1-\gamma}}{N}$.

Proof. (Of Lemma 2.4.6) (i) Applying Cauchy-Schwarz inequality one gets

$$
\langle m_\xi - m_\xi^{N,\varepsilon}, m'_\xi\rangle_{L^2(\mathbb{R})} \leq \|m_\xi - m_\xi^{N,\varepsilon}\|_{L^2(\mathbb{R})}\|m'_\xi\|_{L^2(\mathbb{R})} \leq C\varepsilon^{-\gamma_1/2}\frac{\varepsilon^{-2\gamma}}{N^2}. \tag{2.4.18}
$$

Here Lemma 2.2.4 was used.

(ii) In order to get a lower bound on $\langle a,\Sigma^{-1}a\rangle$ we use the variational principle given in (2.4.15). Without loss of generality we may assume that $\xi = 0$. In the present context (2.4.15) reads:

$$
d^2 = \frac{b^2}{\langle a,\Sigma^{-1}a\rangle}, \qquad \text{and} \quad \langle a,\Sigma^{-1}a\rangle = \Big(\sup_{\eta\in H_0^{N,\varepsilon}:\ \frac{2c_4}{\varepsilon}\|\eta\|_{H^1}=1}\langle m',\eta\rangle_{L^2(\mathbb{R})}\Big)^2. \tag{2.4.19}
$$

Thus in order to find a lower bound on $\langle a,\Sigma^{-1}a\rangle$ one needs to choose an appropriate test function $\eta \in H_0^{N,\varepsilon}$. To this end set $\bar{k} = \inf\left\{k\colon s_k^{N,\varepsilon} > 1\right\}$. Recall that the $s_k^{N,\varepsilon} = \frac{k\varepsilon^{-\gamma}}{N}$ define the discretization of functions in $H_0^{N,\varepsilon}$. For ε small enough one has $1 \leq s_{\bar{k}}^{N,\varepsilon} \leq 2$. Set

$$
\bar{\eta}(s) = \begin{cases} 0 & \text{for } s \leq -s_{\bar{k}}^{N,\varepsilon} \\ \frac{s+s_{\bar{k}}^{N,\varepsilon}}{s_{\bar{k}}^{N,\varepsilon}} & \text{for } -s_{\bar{k}}^{N,\varepsilon} < s \leq 0 \\ \frac{s_{\bar{k}}^{N,\varepsilon}-s}{s_{\bar{k}}^{N,\varepsilon}} & \text{for } 0 < s \leq s_{\bar{k}}^{N,\varepsilon} \\ 0 & \text{for } s_{\bar{k}}^{N,\varepsilon} < s. \end{cases} \tag{2.4.20}
$$

Then $\bar{\eta} \in H_0^{N,\varepsilon}$ and

$$\frac{2c_4}{\varepsilon}\|\bar{\eta}\|_{H^1}^2 = \frac{2c_4}{\varepsilon}\left(\frac{2}{s_{\bar{k}}^{N,\varepsilon}} + \frac{2}{3}s_{\bar{k}}^{N,\varepsilon}\right) \leq \frac{20\,c_4}{3\varepsilon}. \tag{2.4.21}$$

On the other hand m' is strictly positive and we can write

$$\langle \bar{\eta}, m' \rangle_{L^2(\mathbb{R})} = \int_{-2}^{2} \bar{\eta}(s) m'(s) \mathrm{d}s \geq \inf_{s \in [-2,2]} m'(s) \int_{-2}^{2} \bar{\eta}(s) \mathrm{d}s \geq \inf_{s \in [-2,2]} m'(s). \tag{2.4.22}$$

Thus we get $\langle a, \Sigma^{-1} a \rangle \geq C\varepsilon$. Using (2.4.18) one obtains:

$$d^2 \leq C \frac{\varepsilon^{-4\gamma-1-\gamma_1}}{N^4}$$

(iii) To derive a lower bound on $\langle \Sigma n, \Sigma n \rangle$ first note that

$$\begin{aligned}\langle a, \Sigma^{-1} a \rangle &= \left(\sup_{\langle \eta, \Sigma\eta \rangle = 1} \langle a, \eta \rangle \right)^2 = \left(\sup_{\eta \in H_0^{N,\varepsilon}:\ \frac{2c_4}{\varepsilon}\|\eta\|_{H^1}^2 = 1} \langle \eta, m' \rangle_{L^2(\mathbb{R})} \right)^2 \\ &\leq \|m'\|_{L^2(\mathbb{R})}^2 \frac{\varepsilon}{2c_4}.\end{aligned} \tag{2.4.23}$$

To bound the second factor choose the test function $\bar{\eta}$ as above in (2.4.20). Then one gets noting $\|\bar{\eta}\|_\infty \leq 1$

$$\langle \bar{\eta}, \bar{\eta} \rangle = \sum_{k=-\bar{k}}^{\bar{k}} \left(\bar{\eta}(s_k^{N,\varepsilon}) \right)^2 \leq C \frac{N}{\varepsilon^{-\gamma}}.$$

We obtain the desired estimate from (2.4.16) together with (2.4.23) and (2.4.22). □

End of proof of Proposition 2.4.4: Applying Lemma 2.4.5 and 2.4.6 to equation (2.4.13) one gets:

$$\begin{aligned}&\int_{B_\xi} \exp\left(-\frac{2c_4}{\varepsilon} \|v\|_{H^1(\mathbb{R})}^2 \right) \mathcal{H}^{N,\varepsilon}(\mathrm{d}v) \\ &\geq C \frac{\sqrt{\varepsilon}}{\delta} \sqrt{\frac{\epsilon^{-1-\gamma}}{N}} \exp\left(-C \frac{\varepsilon^{-4\gamma-1-\gamma_1}}{N^4} \right) \int_B \exp\left(-\frac{2c_4}{\varepsilon} \|v\|_{H^1(\mathbb{R})}^2 \right) \mathcal{L}^{N,\varepsilon}(\mathrm{d}v),\end{aligned} \tag{2.4.24}$$

where $B = \left\{ v \in H_0^{N,\varepsilon} \colon \frac{2c_4}{\varepsilon} \|v\|_{H^1}^2 \leq \frac{2c_4}{\varepsilon} \delta^2 - d^2 \right\}$. Due to Lemma 2.4.6 (ii) and (2.4.5) $\frac{\varepsilon d^2}{2c_4} \downarrow 0$ as $\varepsilon \downarrow 0$ such that the last integral in (2.4.24) can be bounded from below by

$$\int_{\|v\|_{H^1}^2 \leq \delta^2} \exp\left(-\frac{2c_4}{\varepsilon} \|v\|_{H^1(\mathbb{R})}^2 \right) \mathcal{L}^{N,\varepsilon}(\mathrm{d}v) = Z_2^{N,\varepsilon} \varrho^{N,\varepsilon} \left(\|v\|_{H^1}^2 \leq \frac{\delta^2}{2} \right). \tag{2.4.25}$$

Here ϱ is the massive Gaussian free field discussed in Lemma 2.3.4 for $\kappa = 2c_4$. Lemma 2.3.4 together with (2.4.5) yields that $\varrho^{N,\varepsilon} \left(\|v\|_{H^1}^2 \leq \frac{\delta^2}{2} \right\} \to 1$ for $\varepsilon \downarrow 0$. This probability can thus be bounded from below by $\frac{1}{2}$ for ε small enough. Thus the following lemma finishes the proof. □

Lemma 2.4.7. *The Gaussian normalization constants $Z_1^{N,\varepsilon}$ and $Z_2^{N,\varepsilon}$ satisfy the following:*

$$\exp\left(-\varepsilon^{\gamma-1} \right) c_4^{-\frac{2N-1}{2}} \left(1 + C\varepsilon^{-\gamma} \right)^{-\frac{2N-1}{2}} \leq \frac{Z_2^{N,\varepsilon}}{Z_1^{N,\varepsilon}} \leq \exp\left(-\varepsilon^{\gamma-1} \right) c_4^{-\frac{2N-1}{2}}. \tag{2.4.26}$$

Proof. By Lemma 2.3.4 and equation (2.3.6)

$$\frac{Z_2^{N,\varepsilon}}{Z_1^{N,\varepsilon}} = (2c_4)^{-\frac{2N-1}{2}} \exp\left(-\varepsilon^{\gamma-1}\right) \left(\frac{\det\left(-\Delta_{N,\varepsilon}\right)}{\det\left(I_{N,\varepsilon} - \Delta_{N,\varepsilon}\right)} \right)^{\frac{1}{2}}.$$

By the Poincaré inequality (2.3.13) one has

$$-\Delta_{N,\varepsilon} \le \left(I_{N,\varepsilon} - \Delta_{N,\varepsilon}\right) \le \left(1 + C\varepsilon^{-\gamma}\right)\left(-\Delta_{N,\varepsilon}\right)$$

in the sense of symmetric matrices. This implies

$$\det(-\Delta_{N,\varepsilon}) \le \det(I_{N,\varepsilon} - \Delta_{N,\varepsilon}) \le \left(1 + C\varepsilon^{-\gamma}\right)^{2N-1} \det(-\Delta_{N,\varepsilon}).$$

This finishes the proof. □

As a next step an upper bound on $\mu^{N,\varepsilon}(A^c)$ is derived:

Proposition 2.4.8. *Choosing γ_1 and γ_2 according to (2.4.5) one has for $\delta \le \delta_0$:*

$$\limsup_{\varepsilon \downarrow 0} \varepsilon \log\Big(Z^{N,\varepsilon} \mu^{N,\varepsilon}\left(\operatorname{dist}_{H^1}(u, M) \ge \delta\right)\Big) \le -(C_* + c_0 \delta^2). \tag{2.4.27}$$

In particular, setting $\delta = 0$ one obtains

$$\limsup_{\varepsilon \downarrow 0} \varepsilon \log \; Z^{N,\varepsilon} \le -C_*. \tag{2.4.28}$$

Proof. Denote by $A^\delta := \{u \colon \operatorname{dist}_{H^1}(u, M) \ge \delta\}$. Then one has

$$\begin{aligned} Z^{N,\varepsilon} \mu^{N,\varepsilon}(A^\delta) &= \exp\Big(-\frac{C_*}{\varepsilon}\Big) \frac{1}{Z_1^{N,\varepsilon}} \int_{A^\delta} \exp\Big(-\frac{1}{\varepsilon}\mathcal{E}(u)\Big) \mathcal{L}^{N,\varepsilon}(\mathrm{d}u) \\ &\le \exp\Big(-\frac{C_* + c_0\delta^2}{\varepsilon}\Big) \frac{1}{Z_1^{N,\varepsilon}} \int_{A^\delta} \exp\Big(-\frac{1}{\varepsilon}\left(\mathcal{E}(u) - c_0\delta^2\right)\Big) \mathcal{L}^{N,\varepsilon}(\mathrm{d}u). \end{aligned} \tag{2.4.29}$$

Note that by (2.2.9) $\mathcal{E}(u) - c_0\delta^2 \ge 0$ on A^δ. So on this set one gets

$$\exp\Big(-\frac{1}{\varepsilon}\left(\mathcal{E}(u) - c_0\delta^2\right)\Big) \le \exp\Big(-\left(\mathcal{E}(u) - c_0\delta^2\right)\Big).$$

Therefore, one gets

$$\begin{aligned} \int_{A^\delta} \exp\Big(-\frac{1}{\varepsilon}\left(\mathcal{E}(u) - c_0\delta^2\right)\Big) \mathcal{L}^{N,\varepsilon}(\mathrm{d}u) &\le \int_{A^\delta} \exp\Big(-\left(\mathcal{E}(u) - c_0\delta^2\right)\Big) \mathcal{L}^{N,\varepsilon}(\mathrm{d}u) \\ &\le \int_{A^\delta} Z_3^{N,\varepsilon} \exp\left(\int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} -F\big(u(s)\big)\,\mathrm{d}s + c_0\delta^2 \right) \nu^{1,N}(\mathrm{d}u), \end{aligned} \tag{2.4.30}$$

where $\nu^{1,N}$ is the discretized Brownian bridge without rescaling and

$$Z_3^{N,\varepsilon} = \int \exp\Big(-\int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} \frac{1}{2}|u'(s)|^2\,\mathrm{d}s\Big)\, \mathcal{L}^{N,\varepsilon}(\mathrm{d}u)$$

is the appropriate normalization constant. Using the positivity of F the last term in (2.4.30) can thus be bounded by

$$Z_3^{N,\varepsilon} \exp\Big(c_0\delta^2\Big).$$

Plugging this into (2.4.29) yields

$$Z^{N,\varepsilon}\mu^{N,\varepsilon}(A^\delta) \le \exp\Big(-\frac{C_* + c_0\delta^2}{\varepsilon}\Big) \frac{1}{Z_1^{N,\varepsilon}} Z_3^{N,\varepsilon} \exp\Big(c_0\delta^2\Big).$$

This finishes the proof together with the following bound on the normalization constants $Z_1^{N,\varepsilon}$ and $Z_3^{N,\varepsilon}$. □

Lemma 2.4.9. *One has*

$$\frac{Z_3^{N,\varepsilon}}{Z_1^{N,\varepsilon}} = \varepsilon^{-\frac{2N-1}{2}}.$$

Proof. This is a direct consequence of the fact that for matrices $A \in \mathbb{R}^{n\times n}$ and $\xi \in \mathbb{R}$

$$\det(\xi A) = \xi^n \det(A),$$

as well as the explicit formula for the Gaussian normalization constants. □

One can now summarize the finite-dimensional calculation in the following:

Corollary 2.4.10. *Choosing the constants γ_1 and γ_2 as in (2.4.5),(2.4.7) one obtains for $\delta \le \delta_0$:*

$$\limsup_{\varepsilon\downarrow 0} \varepsilon \log\left(\mu^{N,\varepsilon}(\mathrm{dist}_{H^1}(u,M) \ge \delta)\right) \le -c_0\delta^2.$$

Note that such a choice is possible for all $\gamma < 1$.

Proof. Dividing and using the estimates from above yields the result. □

Using the continuous embedding of H^1 into L^∞ one gets:

Corollary 2.4.11. *Choosing the constants γ_1 and γ_2 as in (2.4.5), (2.4.7) one obtains for $\delta \le \delta_0$:*

$$\limsup_{\varepsilon\downarrow 0} \varepsilon \log\left(\mu^{N,\varepsilon}(\mathrm{dist}_{L^\infty}(u,M) \ge \delta)\right) \le -\tilde{c}_0\delta^2.$$

Such a choice is possible for all $\gamma < 1$.

As a last step in this section we need to control the deviations from the discretized measure with the help of the Gaussian estimates derived in Section 2.3. To this end one has to estimate the deviations of the normalization constant Z^ε from $Z^{N,\varepsilon}$. In order to proof the following lemma we will need an additional assumption on the double well potential F.

Assumption 2.4.12.

$$|F'(u)| \text{ is bounded for } u \in \mathbb{R}\ . \tag{2.4.31}$$

In fact one can simply modify the potential F by cutting it off outside of some compact set, such that it satisfies (2.4.31). We will proceed now by proving Theorem 2.1.1 under the additional assumption (2.4.31). The general case will then follow as a corollary.

Proposition 2.4.13. *Assume that F satisfies (2.4.31). Then the following holds:*

(i) For every $\gamma_3 > 0$ we have for ε small enough:

$$\begin{aligned}&\int_{H^\varepsilon} \exp\Big(-\frac{1}{\varepsilon}\|F'\|_\infty \int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} |u(s)-u^N(s)|\mathrm{d}s\Big)\nu^\varepsilon(\mathrm{d}u) \\ &\qquad \ge \left(\frac{1}{2}\right)^{2N} \exp\left(-\frac{\sqrt{6}\|F'\|_\infty}{3}\frac{\varepsilon^{-\frac{1}{2}-\frac{3}{2}\gamma-\gamma_3}}{N^{\frac{1}{2}}}\right)\end{aligned} \tag{2.4.32}$$

(ii) For ε small enough we have:

$$\begin{aligned}&\int_{H^\varepsilon} \exp\Big(\frac{1}{\varepsilon}\|F'\|_\infty \int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} |u(s)-u^N(s)|\mathrm{d}s\Big)\nu^\varepsilon(\mathrm{d}u) \\ &\le \left(\frac{4\|F'\|_\infty^2\varepsilon^{1-3\gamma}}{\pi N^3}\right)^N \exp\left(\frac{\sqrt{6}\|F'\|_\infty}{3}\frac{\varepsilon^{-\frac{1}{2}-\frac{3}{2}\gamma-\gamma_3}}{N^{\frac{1}{2}}} + \frac{1}{2\pi^2}\|F'\|_\infty^2\frac{\varepsilon^{-1-3\gamma}}{N^2}\right).\end{aligned} \tag{2.4.33}$$

(iii) For $\gamma < \frac{2}{3}$ the the normalization constant satisfies:

$$\liminf_{\varepsilon \downarrow 0} \varepsilon \log Z^\varepsilon = -C_*. \tag{2.4.34}$$

Recall that u^N denotes the discretization of the Brownian bridge as introduced in Section 2.3.

Proof. Applying Cauchy-Schwarz inequality and the independence of the Brownian bridges on the short intervals derived in Lemma 2.3.2 one gets:

$$\begin{aligned} &\int_{H^\varepsilon} \exp\Big(-\frac{1}{\varepsilon}\|F'\|_\infty \int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} |u(s)-u^N(s)|\mathrm{d}s\Big)\nu^\varepsilon(\mathrm{d}u) \\ &\geq \int_{H^\varepsilon} \exp\Big(-\frac{1}{\varepsilon}\left(\frac{\varepsilon^{-\gamma}}{N}\right)^{\frac{1}{2}} \|F'\|_\infty \sum_{k=-N}^{N-1} \|u-u^N\|_{L^2[s_k^{N,\varepsilon}, s_{k+1}^{N,\varepsilon}]}\Big)\nu^\varepsilon(\mathrm{d}u) \\ &= \prod_{k=-N}^{N-1} \int_{H^\varepsilon} \exp\Big(-\frac{1}{\varepsilon}\left(\frac{\varepsilon^{-\gamma}}{N}\right)^{\frac{1}{2}} \|F'\|_\infty \|u-u^N\|_{L^2[s_k^{N,\varepsilon}, s_{k+1}^{N,\varepsilon}]}\Big)\nu^\varepsilon(\mathrm{d}u). \end{aligned} \tag{2.4.35}$$

In the same way one can see that

$$\begin{aligned} &\int_{H^\varepsilon} \exp\Big(\frac{1}{\varepsilon}\|F'\|_\infty \int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} |u(s)-u^N(s)|\mathrm{d}s\Big)\nu^\varepsilon(\mathrm{d}u) \\ &\leq \prod_{k=-N}^{N-1} \int_{H^\varepsilon} \exp\Big(\frac{1}{\varepsilon}\left(\frac{\varepsilon^{-\gamma}}{N}\right)^{\frac{1}{2}} \|F'\|_\infty \|u-u^N\|_{L^2[s_k^{N,\varepsilon}, s_{k+1}^{N,\varepsilon}]}\Big)\nu^\varepsilon(\mathrm{d}u). \end{aligned} \tag{2.4.36}$$

Thus in order to prove (2.4.32) and (2.4.33) we have to bound the integrals over the Brownian bridges on the short intervals. To simplify the equations we introduce the following notation:

$$\begin{aligned} \alpha_\varepsilon &= \sqrt{\varepsilon \frac{\varepsilon^{-2\gamma}}{6N^2}} \\ \beta_\varepsilon &= \frac{1}{\varepsilon}\left(\frac{\varepsilon^{-\gamma}}{N}\right)^{\frac{1}{2}} \|F'\|_\infty \\ \delta_\varepsilon &= \frac{\pi^2 N^2}{\varepsilon^{1-2\gamma}}. \end{aligned}$$

In this notation the concentration inequality (2.3.8) reads

$$\nu^\varepsilon\left(\|u-u^N\|_{L^2[s_k^{N,\varepsilon}, s_{k+1}^{N,\varepsilon}]} \geq \alpha_\varepsilon + r\right) \leq \exp\left(-r^2\delta_\varepsilon\right). \tag{2.4.37}$$

Now let us proceed to prove (2.4.32): Using the formula

$$\mathbb{E}[e^{\beta X}] = 1 + \beta \int_0^\infty e^{\beta x}\mathbb{P}\left[X \geq x\right]\mathrm{d}x, \tag{2.4.38}$$

which holds for every non-negative random variable X and every $\beta \in \mathbb{R}$ one obtains:

$$\begin{aligned} &\int_{H^\varepsilon} \exp\Big(-\beta_\varepsilon\|u-u^N\|_{L^2[s_k^{N,\varepsilon}, s_{k+1}^{N,\varepsilon}]}\Big)\nu^\varepsilon(\mathrm{d}u) \\ &= 1 - \beta_\varepsilon \int_0^\infty e^{-\beta_\varepsilon x}\, \mathbb{P}\left[\|u-u^N\|_{L^2[s_k^{N,\varepsilon}, s_{k+1}^{N,\varepsilon}]} \geq x\right]\mathrm{d}x \\ &\geq 1 - \beta_\varepsilon \int_0^{\varepsilon^{-\gamma_3}\alpha_\varepsilon} e^{-\beta_\varepsilon x}\,\mathrm{d}x - \beta_\varepsilon \int_{\varepsilon^{-\gamma_3}\alpha_\varepsilon}^\infty e^{-\beta_\varepsilon x} \exp\Big(-(x-\alpha_\varepsilon)^2\delta_\varepsilon\Big)\mathrm{d}x \\ &= e^{-\beta_\varepsilon \varepsilon^{-\gamma_3}\alpha_\varepsilon} - e^{-\beta_\varepsilon\varepsilon^{-\gamma_3}\alpha_\varepsilon}\beta_\varepsilon \int_0^\infty e^{-\beta_\varepsilon x} \exp\Big(-\big(x+\alpha_\varepsilon(\varepsilon^{-\gamma_3}-1)\big)^2\delta_\varepsilon\Big)\mathrm{d}x. \end{aligned} \tag{2.4.39}$$

Using the elementary inequality

$$\int_0^\infty \exp(-\beta x - \delta x^2)\mathrm{dx} \leq \int_0^\infty \exp(-\beta x)\mathrm{dx} \leq \frac{1}{\beta},$$

which holds for $\beta, \delta > 0$ the last integral in (2.4.39) can be bounded by

$$\begin{aligned}&\int_0^\infty e^{-\beta_\varepsilon x} \exp\Big(- \big(x + \alpha_\varepsilon(\varepsilon^{-\gamma_3} - 1)\big)^2 \delta_\varepsilon\Big)\mathrm{dx} \\ &\qquad \leq \exp\Big(- \delta_\varepsilon \alpha_\varepsilon^2 (\varepsilon^{-\gamma_3} - 1)^2\Big) \frac{1}{\beta_\varepsilon + 2\delta_\varepsilon \alpha_\varepsilon (\varepsilon^{-\gamma_3} - 1)}\end{aligned}$$

Noting that $\delta_\varepsilon \alpha_\varepsilon^2$ is a quantity of order $O(1)$ one sees that this term decays to zero exponentially in ε. In particular for ε small enough

$$\beta_\varepsilon \int_0^\infty e^{-\beta_\varepsilon x} \exp\Big(- \big(x + \alpha_\varepsilon(\varepsilon^{-\gamma_3} - 1)\big)^2 \delta_\varepsilon\Big)\mathrm{dx} \leq \frac{1}{2}.$$

Plugging this into (2.4.39) and using (2.4.35) finishes the proof of (2.4.32).

To derive the lower bound in (2.4.34) using (2.4.38) one gets

$$\begin{aligned}&\int_{H^\varepsilon} \exp\Big(\beta_\varepsilon \|u - u^N\|_{L^2[s_k^{N,\varepsilon}, s_{k+1}^{N,\varepsilon}]}\Big) \\ &\quad \leq 1 + \beta_\varepsilon \int_0^{\alpha_\varepsilon} e^{\beta_\varepsilon x}\,\mathrm{dx} + \beta_\varepsilon \int_{\alpha_\varepsilon}^\infty e^{\beta_\varepsilon x} \exp\Big(- \delta_\varepsilon \left(x - \alpha_\varepsilon\right)^2\Big)\mathrm{dx} \qquad (2.4.40)\\ &\quad = e^{\beta_\varepsilon \alpha_\varepsilon} + e^{\beta_\varepsilon \alpha_\varepsilon} \beta_\varepsilon \int_0^\infty \exp\Big(\beta_\varepsilon x - \delta_\varepsilon x^2\Big)\,\mathrm{dx}.\end{aligned}$$

Completing the squares the last integral can be bounded by

$$\begin{aligned}&\int_0^\infty \exp\Big(\beta_\varepsilon x - \delta_\varepsilon x^2\Big)\,\mathrm{dx} = \exp\Big(\frac{\beta_\varepsilon^2}{4\delta_\varepsilon}\Big) \int_0^\infty \exp\Big(-\delta_\varepsilon \Big(x - \frac{\beta_\varepsilon}{2\delta_\varepsilon}\Big)^2\Big) \qquad (2.4.41)\\ &\quad \leq \exp\Big(\frac{\beta_\varepsilon^2}{4\delta_\varepsilon}\Big)\sqrt{\frac{\pi}{\delta_\varepsilon}}.\end{aligned}$$

Plugging this into (2.4.40) and using (2.4.36) yields the desired result.

To see (2.4.33) write:

$$\begin{aligned}Z^\varepsilon &= \int_{H^\varepsilon} \exp\left(-\frac{1}{\varepsilon}\int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} F(u(s))\mathrm{ds}\right) \nu^\varepsilon(\mathrm{d}u) \qquad (2.4.42)\\ &= \int_{H^\varepsilon} \exp\left(-\frac{1}{\varepsilon}\int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} F(u^N(s))\mathrm{ds}\right) \exp\left(-\frac{1}{\varepsilon}\int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} \big(F(u(s)) - F(u^N(s))\big)\,\mathrm{ds}\right) \nu^\varepsilon(\mathrm{d}u)\\ &\geq \int_{H^\varepsilon} \exp\left(-\frac{1}{\varepsilon}\int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} F(u^N(s))\mathrm{ds}\right) \exp\Big(- \frac{1}{\varepsilon}\|F'\|_\infty \int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} |u(s) - u^N(s)|\mathrm{ds}\Big)\nu^\varepsilon(\mathrm{d}u).\end{aligned}$$

Using the independence of the discretized Brownian bridge and the bridges on the small intervals derived in Lemma 2.3.2 the last term can be rewritten as

$$Z^{N,\varepsilon} \int_{H^\varepsilon} \exp\Big(- \frac{1}{\varepsilon}\|F'\|_\infty \int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} |u(s) - u^N(s)|\mathrm{ds}\Big)\nu^\varepsilon(\mathrm{d}u). \qquad (2.4.43)$$

If one chooses $N = N(\varepsilon)$ growing like $\varepsilon^{-\gamma_2}$ according to (2.4.32) the exponent in the error terms scales like

$$\varepsilon^{-\frac{1}{2} - \frac{3}{2}\gamma - \gamma_3 + \frac{\gamma_2}{2}},$$

so that if one chooses the γ_i such that (2.4.5),(2.4.7) hold and in addition

$$-\frac{1}{2}-\frac{3\gamma}{2}-\gamma_3+\frac{\gamma_2}{2} > -1 \tag{2.4.44}$$

$$\gamma_3 > 0 \tag{2.4.45}$$

the result follows from the bound (2.4.8) on the discretized normalization constant. Note that such a choice is possible for $\gamma < \frac{2}{3}$.

For the upper bound in (2.4.34) similar to (2.4.42) and (2.4.43) one can write

$$Z^\varepsilon \leq Z^{N,\varepsilon} \int_{H^\varepsilon} \exp\Big(\frac{1}{\varepsilon}\|F'\|_\infty \int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} |u(s)-u^N(s)| \mathrm{ds}\Big)\nu^\varepsilon(\mathrm{d}u). \tag{2.4.46}$$

For $\gamma < \frac{2}{3}$ one can chose the parameters γ_i such that (2.4.7)-(2.4.5) and (2.4.44)-(2.4.45) hold and in addition

$$-3\gamma + 2\gamma_2 > -1. \tag{2.4.47}$$

Then the bound (2.4.33) implies

$$\limsup_{\varepsilon\downarrow 0} \varepsilon \log \int_{H^\varepsilon} \exp\Big(\frac{1}{\varepsilon}\|F'\|_\infty \int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} |u(s)-u^N(s)| \mathrm{ds}\Big)\nu^\varepsilon(\mathrm{d}u) = 0. \tag{2.4.48}$$

Thus (2.4.28) implies the upper bound in (2.4.34). □

In the sequel we will always assume that the γ_i satisfy (2.4.7)-(2.4.5), (2.4.44)-(2.4.45) and (2.4.47). Now one can conclude:

Proposition 2.4.14. *The statement of Theorem 2.1.1 holds under the additional assumption (2.4.31).*

Proof. One can estimate

$$\begin{aligned}\mu^\varepsilon\Big(\mathrm{dist}_{L^2}(u,M)\geq\delta\Big) &= \frac{1}{Z^\varepsilon}\int_{\{\mathrm{dist}_{L^2}(u,M)\geq\delta\}} \exp\Big(-\frac{1}{\varepsilon}\int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} F(u(s))\mathrm{ds}\Big)\nu^\varepsilon(\mathrm{du}) \\ &\leq \frac{1}{Z^\varepsilon}\int_{A_1} \exp\Big(-\frac{1}{\varepsilon}\int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} F(u(s))\mathrm{ds}\Big)\nu^\varepsilon(\mathrm{du}) \\ &\quad + \frac{1}{Z^\varepsilon}\int_{A_2} \exp\Big(-\frac{1}{\varepsilon}\int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} F(u(s))\mathrm{ds}\Big)\nu^\varepsilon(\mathrm{du}),\end{aligned} \tag{2.4.49}$$

where

$$\begin{aligned} A_1 &= \Big\{\|u-u^N\|_{L^2[-\varepsilon^{-\gamma},\varepsilon^{-\gamma}]} \geq \frac{\delta}{2}\Big\} \\ A_2 &= \Big\{\mathrm{dist}_{L^2}(u^N,M) \geq \frac{\delta}{2}\Big\}.\end{aligned}$$

The concentration inequality (2.3.7) implies together with the lower bound on the normalization constant (2.4.34) that the first integral decays to zero on a quicker exponential scale than ε^{-1}. For the integral over A_2 we can write using the independence of the discretized Brownian bridges and the bridges on the intermediate intervals again:

$$\begin{aligned}&\frac{1}{Z^\varepsilon}\int_{A_2} \exp\Big(-\frac{1}{\varepsilon}\int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} F(u(s))\mathrm{ds}\Big)\nu^\varepsilon(\mathrm{du}) \\ &\qquad\leq \frac{Z^{N,\varepsilon}}{Z^\varepsilon}\frac{1}{Z^{N,\varepsilon}}\int_{A_2} \exp\Big(-\frac{1}{\varepsilon}\int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} F(v(s))\mathrm{ds}\Big)\nu^\varepsilon(\mathrm{du})\times \\ &\qquad\qquad \int_{H^\varepsilon} \exp\Big(\frac{1}{\varepsilon}\|F'\|_\infty \int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} \big|u(s)-u^N(s)\big|\,\mathrm{ds}\Big)\nu^\varepsilon(\mathrm{du}).\end{aligned} \tag{2.4.50}$$

The fraction $\frac{Z^{N,\varepsilon}}{Z^{\varepsilon}}$ can be bounded using (2.4.32), the integral in the last line is bounded due to (2.4.48). Thus the statement follows from the result on the discretized measures in Corollary 2.4.10. This finishes the proof for the L^2-norm. To the see analogue result for the L^∞-norm repeat the same reasoning with Lemma 2.4.10 replaced by Lemma 2.4.11 and the L^2 bound (2.3.7) replaced by the L^∞-bound (2.3.9). □

Proof. (Of Theorem 2.1.1 in the general case): Denote by dist either dist_{L^2} or dist_{L^∞}. Assume that F only satisfies assumptions (2.1.2). By cutting F off outside of $[-2,2]$ one can choose a function $\bar{F}$ that coincides with F on $[-2,2]$ that satisfies (2.1.2) and (2.4.31) as well as

$$\bar{F}(u) \leq F(u) \qquad \text{for } u \in \mathbb{R}.$$

Then one can write

$$\begin{aligned}\mu^{\varepsilon}\left(\mathrm{dist}(u,M) \geq \delta\right) &= \frac{\int_{\{\mathrm{dist}(u,M)\geq\delta\}} \exp\Big(-\varepsilon^{-1}\int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} F(u(s))\mathrm{ds}\Big)\nu^{\varepsilon}(\mathrm{d}u)}{\int \exp\Big(-\varepsilon^{-1}\int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} F(u(s))\mathrm{ds}\Big)\nu^{\varepsilon}(\mathrm{d}u)} \\ &\leq \frac{\int_{\{\mathrm{dist}(u,M)\geq\delta\}} \exp\Big(-\varepsilon^{-1}\int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} F(u(s))\mathrm{ds}\Big)\nu^{\varepsilon}(\mathrm{d}u)}{\int_{\{\|u\|_{L^\infty}\leq 2\}} \exp\Big(-\varepsilon^{-1}\int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} F(u(s))\mathrm{ds}\Big)\nu^{\varepsilon}(\mathrm{d}u)}\end{aligned} \tag{2.4.51}$$

The denominator of this fraction coincides with

$$\int_{\{\|u\|_{L^\infty(\mathbb{R})}\leq 2\}} \exp\Big(-\varepsilon^{-1}\int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} \bar{F}(u(s))\mathrm{ds}\Big)\nu^{\varepsilon}(\mathrm{d}u)$$

and the numerator is bounded from above by

$$\int_{\{\mathrm{dist}(u,M)\geq\delta\}} \exp\Big(-\varepsilon^{-1}\int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} \bar{F}(u(s))\mathrm{ds}\Big)\nu^{\varepsilon}(\mathrm{d}u),$$

such that one can write

$$\begin{aligned}\mu^{\varepsilon}\left(\mathrm{dist}(u,M) \geq \delta\right) \leq& \frac{\int_{\{\mathrm{dist}(u,M)\geq\delta\}\}} \exp\Big(-\varepsilon^{-1}\int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} \bar{F}(u(s))\mathrm{ds}\Big)\nu^{\varepsilon}(\mathrm{d}u)}{\int \exp\Big(-\varepsilon^{-1}\int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} \bar{F}(u(s))\mathrm{ds}\Big)\nu^{\varepsilon}(\mathrm{d}u)}\times \\ &\times \frac{\int \exp\Big(-\varepsilon^{-1}\int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} \bar{F}(u(s))\mathrm{ds}\Big)\nu^{\varepsilon}(\mathrm{d}u)}{\int_{\{\|u\|_{L^\infty}\leq 2\}} \exp\Big(-\varepsilon^{-1}\int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} \bar{F}(u(s))\mathrm{ds}\Big)\nu^{\varepsilon}(\mathrm{d}u)}.\end{aligned} \tag{2.4.52}$$

Now applying Proposion 2.4.14 shows that the second factor can be bounded by 2 for ε small enough and thus applying Proposion 2.4.14 to the first factor yields the desired result. □

With a similar argument one can see that the bounds on the normalization constant in Proposition 2.4.13 also holds without assumption (2.4.31):

Corollary 2.4.15. *Suppose $\gamma < \frac{2}{3}$. Then one has the following bound:*

$$\liminf_{\varepsilon\downarrow 0} \varepsilon \log Z^{\varepsilon} = -C_*. \tag{2.4.53}$$

It remains to prove Corollary 2.1.2:

Proposition 2.4.16. *The family of measures $\tilde{\mu}^{\varepsilon}$ is tight. All points of accumulation are concentrated on functions of the type*

$$\tilde{m}_{\xi}(s) = -\mathbf{1}_{[-1,\xi]}(s) + \mathbf{1}_{[\xi,1]}(s). \tag{2.4.54}$$

Proof. Denote by $\tilde{M} = \{\tilde{m}_\xi : \xi \in [-1,1]\}$ and $\operatorname{dist}(\tilde{u}, \tilde{M}) = \inf_{\xi\in[-1,1]} \|\tilde{u} - \tilde{m}_\xi\|_{L^2[-1,1]}$. Furthermore, denote by $\tilde{m}_\xi^\varepsilon(s) = m\left(\frac{s-\xi}{\varepsilon}\right)$. Note that for all $\xi \in [-1,1]$ $\tilde{m}_\xi^\varepsilon$ converges to $\tilde{m}_\xi$ in L^2. Now choose $\delta > 0$ and ε_0 such that $\|\tilde{m}_\xi^\varepsilon - \tilde{m}_\xi\|_{L^2} \leq \frac{\delta}{2}$ for all $\varepsilon \leq \varepsilon_0$. Then Theorem 2.1.1 implies that

$$\begin{aligned}\tilde{\mu}^\varepsilon\left(\operatorname{dist}_{L^2}(\tilde{u}, \tilde{M}) \geq \delta\right) &\leq \tilde{\mu}^\varepsilon\left(\inf_\xi \|\tilde{u} - \tilde{m}_\xi^\varepsilon\|_{L^2[-1,1]} \geq \frac{\delta}{2}\right) \\ &\leq \mu^\varepsilon\left(\operatorname{dist}_{L^2}(T^\varepsilon(\tilde{u}), M) \geq \frac{\delta}{2\sqrt{\varepsilon^{-\gamma}}}\right) \downarrow 0.\end{aligned} \tag{2.4.55}$$

This is sufficient to show the tightness of the measures $\{\tilde{\mu}^\varepsilon\}$. In fact fix a small constant $\kappa > 0$. Let us construct a precompact set K such that $\tilde{\mu}^\varepsilon(K^C) \leq \kappa$. For a fixed $N \in \mathbb{N}$ due to (2.4.55) there exists ε^N such that for all $\varepsilon \leq \varepsilon^N$

$$\tilde{\mu}\left(\operatorname{dist}(\tilde{u}, \tilde{M}) \geq \frac{1}{2N}\right) \leq \frac{\kappa}{2^N}.$$

In particular, there exist finitely many $\xi_i^N \in [-1,1]$ $i = 1, \ldots, i_N$ such that for all $\varepsilon \leq \varepsilon^N$

$$\tilde{\mu}^\varepsilon\left(\cup_i B\left(\tilde{m}_{\xi_i^N}, \frac{1}{N}\right)\right) \geq 1 - \frac{\kappa}{2^N}.$$

Furthermore, due to tightness of the measures $(\tilde{\mu}^\varepsilon, \varepsilon \in [\varepsilon_N, 1])$ there exist finitely many balls $\tilde{B}_i^N$ of radius $\frac{1}{N}$ such that for all $\varepsilon \in [\varepsilon_N, 1]$ one has

$$\tilde{\mu}^\varepsilon\left(\cup_i B_i\right) \geq 1 - \frac{\kappa}{2^N}.$$

Set $K^N = \left(\bigcup_i B_i\right) \cup \left(\bigcup_i B\left(\tilde{m}_{\xi_i^N}, \frac{1}{N}\right)\right)$ and $K = \cap_N K^N$. Then K is precompact and for all ε has measure $\geq 1 - \kappa$. This shows tightness. The concentration follows from (2.4.55). $\square$

Chapter 3

The short time asymptotic of the stochastic Allen-Cahn equation

This chapter has been accepted for publication in Annales Scientifique de l'Institut Henri Poincaré [W08].

A description of the short time behavior of solutions of the Allen-Cahn equation with a smoothened additive noise is presented. The key result is that in the sharp interface limit solutions move according to motion by mean curvature with an additional stochastic forcing. This extends a similar result of Funaki [Fu99] in spatial dimension $n = 2$ to arbitrary dimensions.

3.1 Introduction and main result

1. Setting and main result: For a small parameter $\varepsilon > 0$ consider the following stochastic Allen-Cahn equation in an open domain D in $\mathbb{R}^n$ for some $n \geq 2$:

$$\begin{aligned} \frac{\partial}{\partial t} u^\varepsilon(x,t) &= \Delta u^\varepsilon(x,t) + \varepsilon^{-2} f(u(x,t)) + \varepsilon^{-1} \xi^\varepsilon(t) && (x,t) \in D \times [0,\infty) \\ \frac{\partial}{\partial \nu} u^\varepsilon(x,t) &= 0 && x \in \partial D \\ u^\varepsilon(x,0) &= u_0^\varepsilon(x) && x \in D. \end{aligned} \tag{3.1.1}$$

Here $f(u) = -F'(u)$ is the negative derivative of a symmetric double-well potential. For fixing ideas, assume that $F(u) = \frac{(u^2-1)^2}{4}$ and $f(u) = u - u^3$. In particular F has two global minima at ± 1 and solutions of the dynamical system $\dot{x} = f(x)$, that start outside of zero, converge to one of these minima. The expression $\xi^\varepsilon(t)$ denotes a noise term defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The noise $\xi^\varepsilon(t)$ is constant in space and smooth in time. For $\varepsilon \downarrow 0$ the correlation length goes to zero at a precise rate and $\int_0^t \xi^\varepsilon(s)ds$ converges to a Brownian motion pathwisely. The details of the construction and further properties can be found below.

We study the short time evolution of developed surfaces for (3.1.1). More precisely let Σ_0 the boundary of a set U_0 be compactly embedded in D of class $C^{2,\alpha}$ for some $\alpha > 0$. Assume that the initial configuration $u^\varepsilon(x,t)$ is close to -1 on U_0 and close to $+1$ on $D \setminus U_0$ with a transition layer of order $O(\varepsilon)$. We show that for short times there exist two phases and the evolution of the phase boundary follows two influences - the tendency to minimize the boundary and a stochastic effect. The main result is:

Theorem 3.1.1. *Consider the problem (3.1.1) with the noise term $\xi^\varepsilon(t)$ as constructed below. In particular suppose that the approximation rate γ verifies $\gamma < \frac{2}{3}$. Then for any compactly embedded* *hypersurface* $\Sigma_0 = \partial U_0$ *of class* $C^{2,\alpha}$ *there exist initial conditions* u^ε*, a positive stopping time* τ *and randomly evolving closed hypersurfaces* $(\Sigma(t))_{0\leq t\leq\tau}$ *such that the following hold:*

(i) The surfaces $(\Sigma_t)_{0\leq t\leq\tau}$ *evolve according to stochastically perturbed motion by mean curvature, e.g. the normal velocity* V *at each point is given by*

$$V = (n-1)\kappa - c_0\dot{W}(t).$$

(ii) $\sup_{0\leq t\leq\tau} \|u^\varepsilon(x,t) - \chi_{\Sigma_t}\|_{L^2(D)} \to 0$ *almost surely as* ε *goes to zero.*

Here κ denotes the mean curvature of the surface at a given point. The constant c_0 is given by

$$c_0 = \frac{\sqrt{2}}{\int_{-1}^{1}\sqrt{F(u)}\mathrm{du}}.$$

The function χ_{Σ_t} is a step function taking the value -1 in the interior and $+1$ on the exterior. The precise meaning of the geometric evolution will be given in the next section.

The noise scaling $\varepsilon^{-1}\xi^\varepsilon(t)$ can be interpreted as follows: Consider the stochastic equation

$$\frac{\partial v}{\partial t} = \Delta v + f(v) + \varepsilon\xi^\varepsilon(t). \tag{3.1.2}$$

Equation (3.1.1) can be obtained from this equation by diffusive scaling: $u(x,t) = v(\varepsilon^{-1}x, \varepsilon^{-2}t)$. The intuition is that in (3.1.2) surfaces should move with velocity $V = (n-1)\kappa + c(\varepsilon\xi^\varepsilon(t))$. Here c is the speed of a travelling wave solution corresponding to a perturbation of the potential through $\varepsilon\xi^\varepsilon(t)$. Then after rescaling one obtains as normal velocity $V = \kappa + \varepsilon^{-2}\times\varepsilon^1 c(\varepsilon\xi^\varepsilon(t))$ such that the random term becomes a quantity of order $O(1)$. The significant observation is that the noise term does not rescale. Actually this observation is characteristic for our result. Even in the limit the Brownian motion can be considered pathwise and there is nowhere any need to work with stochastic integrals.

2. The white noise approximation: Let $(W(t), t\geq 0)$ be a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For technical reasons extend the definition of $(W(t), t\geq 0)$ to negative times by considering an independent Brownian motion $(\widetilde{W}(t), t\geq 0)$ and setting $W(t) = \widetilde{W}(-t)$ for $t<0$. Then $(W(t), t\in\mathbb{R})$ is a gaussian process with independent stationary increments and a distinguished point $W(0)=0$ a.s. Let ρ be a mollifying kernel i.e. $\rho\colon \mathbb{R}\to\mathbb{R}_+$ is smooth and symmetric with $\rho(x)=0$ outside of $[-1,1]$ and $\int\rho(x)\mathrm{dx} = 1$. For $\gamma>0$ set $\rho^\varepsilon(x) = \varepsilon^{-\gamma}\rho(\frac{x}{\varepsilon^\gamma})$. Then the approximated Brownian motion $W^\varepsilon(t)$ is defined as usual as

$$W^\varepsilon(t) = W*\rho^\varepsilon(t) = \int_{-\infty}^{\infty}\rho^\varepsilon(t-s)W(s)\mathrm{ds}.$$

Note that it is only here that the Brownian motion at negative times is needed. So actually only negative times in $(-\varepsilon^\gamma, 0]$ will play a role. The parameter γ determines how quickly the approximations converge to the true integrated white noise. We will always assume

$$\gamma < \frac{2}{3}$$

in order to have the needed pathwise bounds on the white noise approximations.

Proposition 3.1.2. *Let* $\xi^\varepsilon(t) = \dot{W}^\varepsilon(t)$ *denote the derivative of* W^ε*. Then the following properties hold:*

(i) $\xi^\varepsilon(t)$ *is a stationary centered gaussian process with* $\mathbb{E}[\xi^\varepsilon(t)^2] = \varepsilon^{-\gamma}|\rho|^2_{L^2}$*.*

(ii) The correlation length of $\xi^\varepsilon(t)$ *is* $2\varepsilon^\gamma$ *i.e. if* $|s-t|\geq 2\varepsilon^\gamma$ *then* $\xi^\varepsilon(t)$ *and* $\xi^\varepsilon(s)$ *are independent.*

(iii) If $\gamma < \tilde{\gamma}$ for every positive time T there exists a non-random constant C such that

$$\mathbb{P}\Big[\exists \varepsilon_0 \text{ s.th. } \forall \varepsilon \le \varepsilon_0 \sup_{0\le t\le T} |\xi^\varepsilon(t)| \le C\varepsilon^{-\frac{\tilde{\gamma}}{2}}\Big] = 1.$$

In particular for $\gamma < \frac{2}{3}$ for ε small enough

$$\xi^\varepsilon(t) \le C\varepsilon^{-\frac{1}{3}}. \tag{3.1.3}$$

Proof. One can write

$$\xi^\varepsilon(t) = \int_{-\infty}^{\infty} \frac{\mathrm{d}}{\mathrm{dt}}\rho^\varepsilon(t-s)W(s)\mathrm{ds} = \int_{-\infty}^{\infty} \rho^\varepsilon(t-s)\mathrm{d}W(s) \quad \text{a.s.},$$

where the first equality follows from differentiating under the integral and the second from stochastic integration by parts. Then properties (i) and (ii) follow from standart properties of the stochastic integral. To see (iii) write

$$\begin{aligned}|\xi^\varepsilon(t)| &= \Big|\int_{t-\varepsilon^\gamma}^{t+\varepsilon^\gamma} \varepsilon^{-2\gamma}\rho'\left(\frac{t-s}{\varepsilon^\gamma}\right)W(s)\mathrm{ds}\Big| \\ &\le \Big|\varepsilon^{-2\gamma}\int_{t-\varepsilon^\gamma}^{t+\varepsilon^\gamma} \rho'\left(\frac{t-s}{\varepsilon^\gamma}\right)W(t)\mathrm{ds}\Big| + \Big|\varepsilon^{-2\gamma}\int_{t-\varepsilon^\gamma}^{t+\varepsilon^\gamma} \rho'\left(\frac{t-s}{\varepsilon^\gamma}\right)(W(t)-W(s))\mathrm{ds}\Big|.\end{aligned}$$

The first term vanishes due to $\int_{t-\varepsilon^\gamma}^{t+\varepsilon^\gamma} \rho'\left(\frac{t-s}{\varepsilon^\gamma}\right) ds = 0$. One obtains

$$\begin{aligned}|\xi^\varepsilon(t)| &\le \Big|\varepsilon^{-2\gamma}\int_{t-\varepsilon^\gamma}^{t+\varepsilon^\gamma} \rho'\left(\frac{t-s}{\varepsilon^\gamma}\right)(W(t)-W(s))\mathrm{ds}\Big| \\ &\le \Big|\varepsilon^{-2\gamma}2\varepsilon^\gamma\|\rho'\|_\infty \mathrm{osc}_{s\in[t-\varepsilon^\gamma,t+\varepsilon^\gamma]}W(s)\Big|.\end{aligned}$$

The oscillation is defined as $\mathrm{osc}_{s\in[t-\varepsilon^\gamma,t+\varepsilon^\gamma]}W(s) := \sup_{s\in[t-\varepsilon^\gamma,t+\varepsilon^\gamma]} W(s) - \inf_{s\in[t-\varepsilon^\gamma,t+\varepsilon^\gamma]} W(s)$.

Now one can apply Lévy's well known result on the modulus of continuity of Brownian paths (See e.g. [KS91] Theorem 9.25 on page 114):

$$\mathbb{P}\Big[\limsup_{\delta\to 0} \frac{1}{g(\delta)} \max_{\substack{0\le s<t\le T \\ t-s\le\delta}} |W(t)-W(s)| = 1\Big] = 1,$$

where the modulus of continuity is given by $g(\delta) = \sqrt{2\delta\log(\frac{1}{\delta})}$. In particular there exists almost surely a (random!) ε_0 such that for $\varepsilon \le \varepsilon_0$ we have $\sup_{t\in[0,T]} \mathrm{osc}_{s\in[t-\varepsilon^\gamma,t+\varepsilon^\gamma]}W(s) \le (2\varepsilon^\gamma)^{\frac{1}{2}-\frac{\tilde{\gamma}-\gamma}{2\gamma}}$. This gives the desired estimate

$$|\xi^\varepsilon(t)| \le \varepsilon^{-\gamma}2\|\rho'\|_\infty(2\varepsilon^\gamma)^{\frac{1}{2}-\frac{\tilde{\gamma}-\gamma}{2\gamma}} \le C\varepsilon^{-\tilde{\gamma}/2}.$$

□

We will need a similar bound on the derivatives of ξ^ε

Proposition 3.1.3. *Consider the process $\dot{\xi}^\varepsilon(t)$. Then if $\gamma < \tilde{\gamma}$ for every positive time T there exists a constant C such that*

$$\mathbb{P}\Big[\exists \varepsilon_0 \quad \forall \varepsilon \le \varepsilon_0 \sup_{0\le t\le T} |\dot{\xi}^\varepsilon(t)| \le C\varepsilon^{-\frac{3\tilde{\gamma}}{2}}\Big] = 1.$$

In particular for $\gamma < \frac{2}{3}$ and ε small enough

$$|\dot{\xi}^\varepsilon(t)| \le C\varepsilon^{-1}. \tag{3.1.4}$$

Proof. The proof is similar to the one above:

$$\begin{aligned}|\dot{\xi}^\varepsilon(t)| &\leq \Big|\int_{t-\varepsilon^\gamma}^{t+\varepsilon^\gamma} \varepsilon^{-3\gamma}\rho''\left(\frac{t-s}{\varepsilon^\gamma}\right)W(s)\mathrm{d}s\Big| \\ &\leq \Big|\int_{t-\varepsilon^\gamma}^{t+\varepsilon^\gamma} \varepsilon^{-3\gamma}\rho''\left(\frac{t-s}{\varepsilon^\gamma}\right)(W(t)-W(s))\mathrm{d}s\Big| \\ &\leq 2\varepsilon^{-2\gamma}\|\rho''\|_\infty \mathrm{osc}_{s\in[t-\varepsilon^\gamma,t+\varepsilon^\gamma]}W(s) :\end{aligned}$$

Then one applies Lévy's modulus of continuity again to see that almost surely for $\varepsilon \leq \varepsilon_0(\omega)$ one has $\mathrm{osc}_{s\in[a,b]}W(s) \leq (2\varepsilon^\gamma)^{\frac{1}{2}-\frac{3\tilde{\gamma}-3\gamma}{2\gamma}}$ and obtains the desired result:

$$|\dot{\xi}^\varepsilon(t)| \leq 2\varepsilon^{-2\gamma}\|\rho''\|_\infty(2\varepsilon^\gamma)^{\frac{1}{2}-\frac{3\tilde{\gamma}-3\gamma}{2\gamma}} = C\varepsilon^{\frac{3\tilde{\gamma}}{2}}.$$

□

3. Motivation and related works: Solutions of the Allen-Cahn equation

$$\frac{\partial u}{\partial t} = \Delta u + \frac{1}{\varepsilon^2}f(u)$$

evolve according to the L^2 gradient flow of the real Ginzburg-Landau energy functional:

$$\mathcal{H}^\varepsilon(u) = \int |\nabla u|^2 + \frac{1}{\varepsilon^2}F(u).$$

There are two different effects. The reaction term $\varepsilon^{-2}f(u)$ pushes solutions to the two minima ± 1 and the diffusion term Δu tends to smoothen the solution. For small ε there will be two phases, corresponding to regions where the solution is close to ± 1. The width of the transition layer between those two phases is of the order $O(\varepsilon)$. Then the evolution gradually shrinks the transition layer.

This behavior is the motivation to consider the Allen-Cahn equation as a simple model of a two phase system which is driven by the surface energy without conservation of mass. Allen and Cahn [AC79] introduced it to model the interface motion between different crystalline structures in alloys. In the deterministic setting there were major advances in connection with the improved understanding of the theory of geometric flows of surfaces as initiated for example by [ES91, CGG91] in the early nineties. In particular in [ESS92] it was shown that in the limit $\varepsilon \downarrow 0$ solutions only attain the values ± 1 and the phase boundary evolves according to motion by mean curvature. The key difficulty here is to find a description of the geometric evolution which is global in time. A similar result for short times was established in [MS95].

Stochastic perturbations of this effect have also been considered. From a modelling point of view an additional noise term can account for inaccuracies of the simplified model or as effects of thermal perturbations. From a mathematical point of view it is a very interesting and challenging question to study stochastically perturbed evolutions of surfaces and the Allen-Cahn setup is one possible point of view. In [Fu95] Funaki considered the case of the Allen-Cahn equation in one space dimension with a space-time white noise. He showed that in the limit $\varepsilon \downarrow 0$ on the right time-scale solutions only attain values ± 1 and the boundary point essentially performs a Brownian motion. In [Fu99] he studies the two dimensional case with a smoothened noise and shows that for short times solutions evolve according to a stochastically perturbed motion by mean curvature. His analysis relies on a comparison theorem which requires the noise to be smooth and a very subtle analysis of a quasi-linear stochastic PDE which describes the boundary evolution. On the level of stochastic surface evolution there were advances by Yip [Y98] and Dirr, Luckhaus and Novaga [DLN01] but a fully satisfactory description is not yet available. Some results based on a stochastic version of the concept of viscosity solutions were announced in [LS98II]. Recently the model has enjoyed an increasing interest in the numerical analysis community. For example in [KKL07] numerical approximations of the one dimensional equation are studied. Numerical analysis of this equation is challenging because all the interesting dynamics happen on a very thin layer which requires to develop adaptive methods which work in the stochastic setting.

Our result is a generalization of Funaki's result to arbitrary dimension. We use the same comparison technique to study the equation. Therefore, we also need to assume a smoothened noise with correlation length going to zero as ε goes to zero. The description of the surface and the convergence result is based on [DLN01] and fully avoids Funaki's result of weak convergence. In fact this is also a strictly pathwise result so that all results hold almost surely.

4. Structure of the paper: In Section 2 the technique of [DLN01] to describe motion by mean curvature is briefly reviewed and the main results are stated. In Section 3 the results about the geometric flow are used to proof the behavior of the Allen-Cahn equation.

3.2 Stochastic motion by mean curvature

This section reviews the description of a stochastically perturbed motion by mean curvature given in [DLN01]. A short time existence result for surfaces moving with normal velocity $\mathrm{d}V = (n-1)\kappa\mathrm{d}t + c\mathrm{d}W(t)$, where κ denotes the mean curvature, and a pathwise stability result under approximations of the integrated noise are given.

Motivated by [ES92] consider the following system

$$\begin{aligned} \mathrm{d}d(x,t) &= g(D^2 d(x,t), d(x,t))\mathrm{d}t + \mathrm{d}W(t) && (x,t)\in\mathcal{O}\times(0,T) \\ |\nabla d|^2 &= 1 && (x,t)\in\partial\mathcal{O}\times(0,T) \\ d(x,0) &= d_0(x) && x\in\mathcal{O}, \end{aligned} \tag{3.2.1}$$

on some open bounded domain $\mathcal{O}$. Here $D^2 d$ denotes the Hessian of d and $g(A,q) = \mathrm{tr}(A(I-qA)^{-1})$ for a symmetric matrix A and $q\in\mathbb{R}$. The initial condition d_0 is supposed to be of class $C^{2,\alpha}$ and to verify $|\nabla d| = 1$ in $\mathcal{O}$. Furthermore, it is assumed that ∇d is nowhere tangent to the boundary.

In order to solve the above system consider $q(x,t) = d(x,t) - W(t)$. Then q solves the system

$$\begin{aligned} \mathrm{d}q(x,t) &= g(D^2 q(x,t), q(x,t) + W(t))\mathrm{d}t && (x,t)\in\mathcal{O}\times(0,T) \\ |\nabla q|^2 &= 1 && (x,t)\in\partial\mathcal{O}\times(0,T) \\ q(x,0) &= d_0(x) && x\in\mathcal{O}. \end{aligned} \tag{3.2.2}$$

Due to maximal regularity of the linearized system ([L95]) and a fix point argument the following results are obtained:

Theorem 3.2.1. ([DLN01] Section 4) *Let $t\mapsto W(t)$ be α-Hölder continuous for some $\alpha\in(0,1)$. Then there exists a time T depending only on the $C^{\alpha/2}$-norm of W and the $C^{2,\alpha}$-norm of d_0 such that on $\mathcal{O}\times[0,T]$ system (3.2.2) and also (3.2.1) admit a unique solution of class $C^{1+\alpha/2,2+\alpha}$. Moreover if $t\mapsto\tilde{W}(t)$ is another function of class C^α and $\tilde{q}$ is the solution (3.2.2) with W replaced by $\tilde{W}$ with interval of existence $[0,\tilde{T}]$ on has*

$$\sup_{t\in[0,\min\{\tilde{T},T\}]} \|q(t,\cdot) - \tilde{q}(t,\cdot)\|_{C^{2,\alpha}} \le C\|W-\tilde{W}\|_{C^{\alpha/2}([0,\min\{\tilde{T},T\}])}. \tag{3.2.3}$$

Now let $\Sigma_0 = \partial U_0$ be as above. In particular Σ_0 is assumed to be of class $C^{2,\alpha}$. Define the *signed distance function* d_0 and the indicator χ_{Σ_0} as

$$d_0(x) = \begin{cases} -\mathrm{dist}(x,\Sigma_0) & \text{for } x\in U_0 \\ \mathrm{dist}(x,\Sigma_0) & \text{for } x\in D\setminus U_0 \end{cases}$$

and

$$\chi_{\Sigma_0}(x) = \begin{cases} -1 & \text{for } x \in U_0 \\ 1 & \text{for } x \in D \setminus U_0. \end{cases}$$

There exists an open environment $\mathcal{O}$ of Σ_0 such that on $\mathcal{O}$ the function $d_0(x)$ is of class $C^{2,\alpha}$ and ∇d is nowhere tangent to $\partial\mathcal{O}$. Furthermore, on $\mathcal{O}$ it holds $|\nabla d_0| = 1$. Then for a given stochastic noise $W(t)$ consider the pathwise solution $d(x,t)$ of (3.2.1) with initial condition d_0 on $[0, T(\omega)]$. Define the evolving surfaces $(\Sigma(t), 0 \le t \le T(\omega))$ as the zero level sets of $d(x,t)$. Then the following holds:

Theorem 3.2.2. ([DLN01] Section 4)

(i) For every t the function $x \mapsto d(x,t)$ is the signed distance function of $\Sigma(t)$ on $\mathcal{O}$.

(ii) If $X(0)$ in $\Sigma(0)$. Then up to a stopping time there exists a solution $X(t)$ to the stochastic differential equation

$$dX(t) = (n-1)\nu(X(t),t)\kappa(X(t),t)\mathrm{d}t + \nu(X(t),t)\mathrm{d}W(t),$$

with $X(t) \in \Sigma(t)$ almost surely.

Here $\nu(x,t)$ denotes the exterior normal vector to $\Sigma(t)$ for $x \in \Sigma(t)$. The last observation justifies to say that the surfaces $\Sigma(t)$ evolve according to stochastic motion by mean curvature. Note that we use the convention that $\kappa = \frac{1}{n-1}\sum_{i=1}^{n-1} \kappa_i$ with the principal curvatures κ_i such that the factor $(n-1)$ appears which is not present in [DLN01].

3.3 Construction of sub- and supersolutions

In this section the link between the boundary dynamic and the Allen-Cahn equation is established. For a related calculation see [Fu99, CHL97].

In order to construct sub- and supersolutions to (3.1.1) consider the following modification of the reaction term: $f(u,\delta) = f(u) + \delta$. The implicit function theorem implies that there exists an interval $[-\tilde{\delta}_0, \tilde{\delta}_0]$ such that for $\delta \in [-\tilde{\delta}_0, \tilde{\delta}_0]$ there exist two solutions $m_\pm(\delta)$ of the equation $f(u,\delta) = 0$ which are close to ± 1 and that the mappings $\delta \mapsto m_\pm(\delta)$ are smooth. Consider the following auxiliary one dimensional problem

$$\begin{aligned} \frac{\partial}{\partial t}u(x,t) &= \frac{\partial^2}{\partial x^2}u(x,t) + f(u(x,t)) + \delta \\ u(\pm\infty) &= m_\pm(\delta). \end{aligned} \tag{3.3.1}$$

A travelling wave solution to (3.3.1) is a solution $u(x,t) = m(x - ct)$ with a fixed wavespeed c. Finding such a solution is equivalent to finding an appropriate waveshape $m(x,\delta)$ and wavespeed $c(\delta)$ such that

$$\begin{aligned} m''(x) + c(\delta)m'(x) + \{f(m(x)) + \delta\} &= 0 \\ m(\pm\infty) &= m_\pm(\delta). \end{aligned} \tag{3.3.2}$$

The following properties hold:

Lemma 3.3.1. ([CHL97] Lemma 3.3) *There exists a constant δ_0 such that for $\delta \in [-\delta_0, \delta_0]$ problem (3.3.2) admits a solution $(m(x,\delta), c(\delta))$ where m is increasing in x and this solution is unique up to translation. Furthermore, m can be chosen smooth in δ. There exist constants A and β such that the following properties hold:*

(i) $0 < \partial_x m(x,\delta) \le A$ for all $(x,\delta) \in \mathbb{R} \times [-\delta_0, \delta_0]$.

(ii) $|\partial_x m(\pm x,\delta)| + |(\partial_x)^2 m(\pm x,\delta)| + |m(\pm x,\delta) - m_\pm(\delta)| \le Ae^{-\beta x}$ *for all* $(x,\delta) \in \mathbb{R}_+ \times [-\delta_0,\delta_0]$.

(iii) The traveling wave velocity $c(\delta)$ *is smooth in* $[-\delta_0,\delta_0]$ *and* $c(0) = 0$.

Actually as pointed out in [Fu99] $\partial_\delta c(0) = -c_0 = -\frac{\sqrt{2}}{\int_{-1}^{1}\sqrt{F(u)}\mathrm{d}u}$.

The idea of the construction is the following: We expect the surface to evolve according to two influences - the surface tension and the stochastic perturbation of the potential making one of the stable states more attractive. Close to the surface the solution should look like a travelling wave interface which is moving with velocity $c(\varepsilon\xi)$. This means that solution should behave like

$$u(x,t) \approx m\left(\frac{d(x,t)}{\varepsilon}, \varepsilon\xi^\varepsilon(t)\right),$$

where d is the signed distance function of a surface moving with normal velocity $V = (n-1)\kappa + \varepsilon^{-1}c(\varepsilon\xi^\varepsilon)$. The standard way of making this idea rigorous is to modify it in such a way that such an approximate solution is a true sub/supersolution and show that the difference between the two cases evolves on a slower time scale than the original dynamic.

Fix some initial surface Σ_0 as in Theorem 3.1.1. As Σ_0 is compactly embedded one can fix an N such that all the principle curvatures of Σ_0 are bounded by N. As in Section 3.2 one can define a random evolution $(\Sigma^{\pm,\varepsilon}(t), 0 \le t \le T^{\pm}_{\varepsilon,N})$ evolving with normal velocity

$$V = (n-1)\kappa + \varepsilon^{-1}c(\varepsilon\xi^\varepsilon(t) \pm \varepsilon^\beta).$$

Here the stopping time $T^{\pm}_{\varepsilon,N}$ is defined as the largest time such that the evolution is well defined and such that on $[0, T^{\pm}_{\varepsilon,N}]$ the principle curvatures remain bounded by N. The constant β can be chosen such that $1 < \beta < 2$. The condition $\beta > 1$ ensures that in the original time scale the extra term does not have an effect and the condition $\beta < 2$ ensures that the effect is strong enough for the solution to remain a sub/supersolution. Furthermore, assume (by shortening the time interval if necessary) that there exists an open set $\mathcal{O}$ such that for all $t \in [0, T^{\pm}_{\varepsilon,N}]$ the η-neighborhood of $\Sigma^\pm(t)$ is contained in $\mathcal{O}$ for some small η. Then one can extend the signed distance functions $d^\pm(x,t)$ to a smooth function $\tilde{d}^\pm$ on all of $[0, T(\omega)] \times D$ such that on $U(t) \setminus \mathcal{O}$ the function $\tilde{d}^\pm$ is smaller than $-\eta$ and on $D \setminus (U(t) \cup \mathcal{O})$ it is larger than η, such that $|\nabla \tilde{d}^\pm| \le 1$ and such that $\tilde{d}$ is constant close to ∂D.

Define

$$u^\pm(x,t) = m\left(\frac{\tilde{d}^\pm(x,t) \pm \varepsilon^a e^{c_1 t}}{\varepsilon}, \varepsilon\xi^\varepsilon(t) \pm \varepsilon^\beta\right),$$

where a and c_1 are constants that will be chosen below. One gets the following conclusion:

Lemma 3.3.2. *If one chooses a and c_1 properly, there exists a (random) $\varepsilon_0 > 0$ such that for all $\varepsilon \le \varepsilon_0$ and for $0 \le t \le T_N^{\pm,\varepsilon}$*

$$u^{\varepsilon,-}(x,t) \le u^\varepsilon(x,t) \le u^{\varepsilon,+}(x,t),$$

for every solution $u^\varepsilon(x,t)$ of (3.1.1) with initial data verifying $u^{\varepsilon,-}(x,0) \le u^\varepsilon(x,0) \le u^{\varepsilon,+}(x,0)$.

Proof. The conclusion will follow by a PDE-comparison principle. We only show the inequality involving u^+ the other one being similar. Let us calculate

$$\begin{aligned}
\partial_t u^{\varepsilon,+}(x,t) &= \frac{m_x}{\varepsilon}\left(\partial_t \tilde{d}(x,t) + \varepsilon^a c_1 e^{c_1 t}\right) + \varepsilon m_\delta \dot{\xi}(t) \\
\Delta u^\varepsilon(x,t) &= \frac{m_x}{\varepsilon}\Delta\tilde{d}(x,t) + \frac{m_{xx}}{\varepsilon^2}|\nabla\tilde{d}(x,t)|^2.
\end{aligned}$$

Here m_x denotes the partial derivative of $m(x,t)$ with respect to x. Then rewrite the reaction term using (3.3.2):

$$\varepsilon^{-2}(f(m)+\varepsilon\xi^\varepsilon) = \varepsilon^{-2}\Big(-m'' - m'c(\varepsilon\xi^\varepsilon+\varepsilon^\beta) - \varepsilon^\beta\Big).$$

By properly arranging the terms one gets

$$\mathcal{L}(u^+) := \partial_t u^{\varepsilon,+}(x,t) - \Delta u^\varepsilon(x,t) - \varepsilon^{-2}(f(m(x,t)) + \varepsilon\xi^\varepsilon(t)) = I_1 + I_2 + I_3 + \varepsilon^{\beta-2},$$

where

$$\begin{aligned} I_1 &= \frac{m_x}{\varepsilon}\Big(\partial_t\tilde{d}(x,t) + \varepsilon^a c_1 e^{c_1 t} - \Delta\tilde{d}(x,t) + \varepsilon^{-1}c(\varepsilon\xi^\varepsilon(t)+\varepsilon^\beta)\Big)\\ I_2 &= \varepsilon m_\delta\dot{\xi}(t)\\ I_3 &= \frac{m_{xx}}{\varepsilon^2}\Big(1-|\nabla\tilde{d}(x,t)|^2\Big). \end{aligned}$$

Here the first term accounts for the boundary motion. The statement that this term is small essentially means that the surface evolves with normal velocity $V = (n-1)\kappa + \varepsilon^{-1}c(\varepsilon\xi^\varepsilon + \varepsilon^\beta)$. The second term corresponds to the change of wave profile due to the change of noise. It is here that we need the pathwise bound (3.1.4) on the derivative of ξ^ε to control this term. The third term essentially vanishes because close to $\Sigma(t)$ the function $\tilde{d}$ coincides with d. Consequently, $|\nabla d|^2 = 1$. Off the boundary the derivative m_{xx} becomes exponentially small such that we also control this term. In the end this means that the correction term $\varepsilon^{\beta-2}$ dominates the dynamic. Let us make these considerations rigorous:

By (3.1.4) $I_2 \leq C$ for every ε smaller than $\varepsilon_0(\omega)$. For $d(x,t)\leq\eta$ $\quad\nabla d(x,t) = 1$. Consequently, I_3 vanishes for such x. For $d(x,t)\geq\eta$ Lemma 3.3.1 (ii) implies:

$$\frac{m_{xx}}{\varepsilon^2}\Big(1-|\nabla\tilde{d}|^2\Big) \leq \frac{2A}{\varepsilon^2}e^{-C/\varepsilon} \to 0.$$

To bound I_1 consider points x close to $\Sigma(t)$. For all other x the reasoning is as for I_3. For x with $\operatorname{dist}(x,\Sigma^{\pm,\varepsilon}(t)) \leq \frac{1}{2N}$ the functions $d(x,t)$ and $\tilde{d}(x,t)$ coincide and one obtains

$$\partial_t d(x,t) = \Delta d(y,t) + \varepsilon^{-1}c(\varepsilon\xi^\varepsilon(t)+\varepsilon^\beta),$$

where as before y is the unique point in $\Sigma(t)$ such that $d(x,t) = \operatorname{dist}(x,y)$. Plugging this into I_1 gives

$$I_1 = \frac{m_x}{\varepsilon}\Big\{\Delta d^\varepsilon(y,t) - \Delta d^\varepsilon(x,t) + \varepsilon^a c_1 e^{c_1 t}\Big\}.$$

Here one uses the fact that all the principle curvatures $\kappa_i(t,y)$ of the $\Sigma^{\pm,\varepsilon}(t)$ are bounded by N to obtain

$$\begin{aligned} |\Delta d^\varepsilon(y,t) - \Delta d^\varepsilon(x,t)| &= \left|\sum_{i=1}^{n-1}\kappa_i(y,t) - \sum_{i=1}^{n-1}\frac{\kappa_i(y,t)}{1-d(x,t)\kappa_i(y,t)}\right|\\ &= \sum_{i=1}^{n-1}|\kappa_i(y,t)|\frac{d(x,t)|\kappa_i(y,t)|}{1-d(x,t)|\kappa_i(y,t)|}\\ &\leq 4N^2 d(x,t), \end{aligned}$$

because $\sup_{x\in[0,\frac{1}{2}]}\partial_x\frac{x}{1-x} = 4$. Plugging this in yields

$$|I_1| \leq m_x\left(\frac{\tilde{d}^\pm(x,t)\pm\varepsilon^a e^{c_1 t}}{\varepsilon}, \varepsilon\xi^\varepsilon \pm \varepsilon^\beta\right)\frac{4N^2 d(x,t) + \varepsilon^a c_1 e^{c_1 t}}{\varepsilon}.$$

Choosing c_1 larger than N^2 and using $\sup_x xm_x < \infty$ one obtains $|I_1| \leq C$. Thus altogether if ε is small enough the term $\varepsilon^{\beta-2}$ will dominate everything else and one obtains

$$\mathcal{L}(u^+) \geq 0.$$

On the boundary $\frac{\partial u_+}{\partial \nu} = 0$ due to the definition of $\tilde{d}$. So a standart comparison principle gives the desired result. The inequality for u_- is shown in a similar manner. □

To finish the proof of the main theorem one needs the following Lemma:

Lemma 3.3.3. *Fix any time interval* $[0, T]$. *Denote by* $W^{\pm,\varepsilon}$ *the random functions* $[0, T] \ni t \mapsto \frac{1}{c_0} \int_0^t \varepsilon^{-1} c(\varepsilon \xi^\varepsilon(s) \pm \varepsilon^\beta) \mathrm{ds}$. *Then* $c_0 W^{\pm,\varepsilon}$ *converges almost surely to* $t \mapsto c_0 W(t)$ *in* $C^{0,\alpha}([0, T])$ *for every* $\alpha < \frac{1}{2}$.

Proof. Consider only $W^{+,\varepsilon}(t)$ the calculation for $W^{-,\varepsilon}(t)$ being the same. Fix $\alpha < \frac{1}{2}$ and a ϑ with $\alpha < \vartheta < \frac{1}{2}$. Then for $\mathbb{P}$-almost every ω there exists a random constant C such that

$$\sup_{-1 \leq s < t \leq T} \frac{|W(s) - W(t)|}{|s - t|^\vartheta} \leq C.$$

Assume that ε is small enough to ensure $\varepsilon \xi^\varepsilon(t) + \varepsilon^\beta \in [-\delta_0, \delta_0]$. (Recall that c is only defined on $[-\delta_0, \delta_0]$.) Using Taylor-formula and $c(0) = 0$ one can write for every t:

$$\varepsilon^{-1} c(\varepsilon \xi^\varepsilon(t) + \varepsilon^\beta) = c'(0)(\xi^\varepsilon(t) + \varepsilon^{\beta-1}) + \frac{1}{2} c''(a(t)) \varepsilon^{-1} \big(\varepsilon \xi^\varepsilon(t) + \varepsilon^\beta\big)^2,$$

for some $a(t)$ verifying $|a(t)| \leq |\varepsilon \xi^\varepsilon(t) + \varepsilon^\beta|$. Therefore, one can write

$$\begin{aligned}\|c_0 W - c_0 W^{+,\varepsilon}\|_\infty \leq & \sup_{s \in [0,T]} |c_0 W(s) - c_0 \int_0^s \xi^\varepsilon(t) dt| + c_0 T \varepsilon^{\beta-1} \\ & + T \sup_{\delta \in [-\delta_0, \delta_0]} |c''(\delta)| \Big(\sup_{s \in [0,T]} \varepsilon(\xi^\varepsilon(s))^2 + \varepsilon^{2\beta-1} \Big).\end{aligned}$$

Due to (3.1.3) the last terms converge to zero almost surely. Therefore, it remains to consider the first term. Due to $\dot{W}^\varepsilon(s) = \xi^\varepsilon(s)$ one obtains:

$$\begin{aligned}\sup_{s \in [0,T]} |c_0 W(s) - c_0 \int_0^s \xi^\varepsilon(t) dt| &\leq c_0 \sup_{s \in [0,T]} |W(s) - W(s)^\varepsilon| + c_0 |W(0)^\varepsilon| \\ &= c_0 \sup_{s \in [0,T]} \left| \int_{-\varepsilon^\gamma}^{\varepsilon^\gamma} \Big(W(s) - W(s-t)\Big) \rho^\varepsilon(t) dt \right| + c_0 \left| \int_{-\varepsilon^\gamma}^{\varepsilon^\gamma} \Big(W(0) - W(t)\Big) \rho^\varepsilon(t) dt \right| \\ &\leq 2 c_0 C \Big(\varepsilon^\gamma\Big)^\vartheta \to 0.\end{aligned}$$

Consider now the Hölder-seminorm

$$\begin{aligned}&\sup_{0 \leq s < t \leq T} \frac{1}{(t-s)^\alpha} \Big| c_0 W(t) - c_0 W^{+,\varepsilon}(t) - c_0 W(s) + c_0 W^{+,\varepsilon}(s) \Big| \\ &= \sup_{0 \leq s < t \leq T} \frac{1}{(t-s)^\alpha} \Big| c_0 W(t) - c_0 W(s) - \int_s^t \varepsilon^{-1} c(\varepsilon \xi^\varepsilon(s) + \varepsilon^\beta) \mathrm{ds} \Big| \\ &\leq \sup_{0 \leq s < t \leq T} \frac{1}{(t-s)^\alpha} |c_0 W(t) - c_0 W(s) - c_0 W^\varepsilon(t) + c_0 W^\varepsilon(s)| \\ &\quad + \sup_{0 \leq s < t \leq T} \frac{1}{(t-s)^\alpha} \left| \int_s^t c_0 \varepsilon^{\beta-1} + \sup_{\delta \in [-\delta_0, \delta_0]} |c''(\delta)| \Big(\varepsilon(\xi^\varepsilon(u))^2 + \varepsilon^{2\beta-1} \Big) du \right|.\end{aligned}$$

Again the second term converges to zero. For the first term one gets:

$$\begin{aligned}
&\sup_{0\leq s<t\leq T}\frac{1}{(t-s)^{\alpha}}|c_0W(t)-c_0W(s)-c_0W^{\varepsilon}(t)+c_0W^{\varepsilon}(s)| \\
&\leq c_0\sup_{0\leq s<t\leq T}\frac{1}{(t-s)^{\alpha}}\int_{-\varepsilon^{\gamma}}^{\varepsilon^{\gamma}}\big(W(t)-W(t-u)-W(s)+W(s-u)\big)\rho^{\varepsilon}(u)du \\
&\leq c_0\sup_{0\leq s<t\leq T}\frac{(2(W(t)-W(s)))^{\frac{\alpha}{\vartheta}}}{(t-s)^{\alpha}}\int_{-\varepsilon^{\gamma}}^{\varepsilon^{\gamma}}\big(W(t)-W(t-u)-W(s)+W(s-u)\big)^{1-\frac{\alpha}{\vartheta}}\rho^{\varepsilon}(u)du \\
&\leq c_0\,(2C)^{\frac{\alpha}{\gamma}}\left(2C(2\varepsilon^{\gamma})^{\vartheta}\right)^{1-\frac{\alpha}{\vartheta}}.
\end{aligned}$$

This shows the desired convergence. □

Proof. (of Theorem (3.1.1)) Chose the initial configurations u_0^{ε} such that $u^{\varepsilon}(x,0)\leq u_0^{\varepsilon}(x)\leq u^{\varepsilon,+}(x,0)$. Define the stopping time $\tau(\omega):=\inf_{\varepsilon}T^{\pm}_{\varepsilon,N}$. Remark that τ is almost surely positive due to the boundedness of the $\|W^{\pm,\varepsilon}\|_{C^{\alpha/2}}$ and the $C^{2,\alpha}$ convergence of $d^{\varepsilon,\pm}$ to d.

Then by Lemma 3.3.2 one has for all times $0\leq t\leq T(\omega)$ that $u^{\varepsilon}(x,t)\leq u^{\varepsilon}(x,t)\leq u^{\varepsilon,+}(x,t)$. So one gets:

$$\begin{aligned}
\|u^{\varepsilon}(\cdot,t)-\chi_{\Sigma(t)}(\cdot,t)\|_{L^2} &\leq \|u^{\varepsilon}(\cdot,t)-u^{\varepsilon,+}(\cdot,t)\|_{L^2(D)}+\|u^{\varepsilon,+}(\cdot,t)-\chi_{\Sigma^{\varepsilon,+}}(t)(\cdot,t)\|_{L^2(D)} \\
&\quad+\|\chi_{\Sigma^{\varepsilon,+}}(t)(\cdot,t)-\chi_{\Sigma(t)}(\cdot,t)\|_{L^2(D)} \\
&\leq \|u^{\varepsilon,-}(\cdot,t)-u^{\varepsilon,+}(\cdot,t)\|_{L^2(D)}+\|u^{\varepsilon,+}(\cdot,t)-\chi_{\Sigma^{\varepsilon,+}}(t)(\cdot,t)\|_{L^2(D)}+ \\
&\quad\|\chi_{\Sigma^{\varepsilon,+}(t)}(\cdot,t)-\chi_{\Sigma(t)}(\cdot,t)\|_{L^2(D)} \\
&\leq \|u^{\varepsilon,-}(\cdot,t)-\chi_{\Sigma^{\varepsilon,-}(t)}(\cdot,t)\|_{L^2(D)}+2\|u^{\varepsilon,+}(\cdot,t)-\chi_{\Sigma^{\varepsilon,+}(t)}(\cdot,t)\|_{L^2(D)}+ \\
&\quad 2\|\chi_{\Sigma^{\varepsilon,+}(t)}(\cdot,t)-\chi_{\Sigma(t)}(\cdot,t)\|_{L^2(D)}+\|\chi_{\Sigma^{\varepsilon,-}(t)}(\cdot,t)-\chi_{\Sigma(t)}(\cdot,t)\|_{L^2(D)}.
\end{aligned}$$

The supremum in time of the first two terms converges to zero due to the definition of $u^{\varepsilon,\pm}$. Consider $\|\chi_{\Sigma^{\varepsilon,-}(t)}(\cdot)-\chi_{\Sigma(t}(\cdot)\|_{L^2(D)}=\int_{\mathcal{O}}\Big(\chi_{\Sigma^{\varepsilon,-}(t)}(x)-\chi_{\Sigma(t)}(x)\Big)dx$. By Lemma 3.3.3 and by Theorem 3.2.1 the signed distance functions converge in $C^{2,\alpha}(\mathcal{O})$ uniformly in time and this term converges to zero. The convergence of the term involving $\chi_{\Sigma^{\varepsilon,-}(t)}$ can be seen in the same way. □

Chapter 4

Appendix

4.1 Outlook

In this section some ideas for possible extensions of the results, which are presented in this thesis, are given:

- To determine the distribution of the phase separation point in the study of the invariant measure is an open problem. It should be uniform. Once this is established the results on the invariant measure should allow to obtain dynamical results using the theory of Dirichlet forms. Formally the problem of convergence of solutions of the one dimensional Allen-Cahn equation is very similar to the problem of approximating a graph by tubular neighborhoods studied for example in the last chapter of [B08]. Here the theory of Mosco-Convergence of Dirichlet forms on changing Hilbert spaces is used. This theory has been developed in [KS03, K06]. The generalization of this technique to the infinite dimensional space should be possible.

- A very challenging problem remains the rigorous verification of the conjectured behavior of multi kink configurations in the one dimensional case. Here a detailed analysis of the energy landscape similar to [OR07] might make it possible to extend the techniques used in [Fu95] to the multi kink case.

- Much remains to be done in a higher dimensional setting. The question of a reasonable construction of stochastically perturbed boundary evolutions is still wide open. Recently, a possible approach using the Allen-Cahn approximation has been studied in a joint work with Matthias Röger [RW09]. In fact we use an Allen-Cahn approximation to the evolution studied by Yip in [Y98]. Here we do not consider an additive stochastic term but rather a multiplicative noise which depends on the gradient of the phase field u. So far we were able to prove tightness of the Allen-Cahn approximation, which is similar to the results obtained in [Y98]. We want to use techniques similar to [MR08, Il93] to extract more information about the limit process.

4.2 Some results for stochastic reaction diffusion equations

In this section we recall existence and uniqueness results for the Allen-Cahn equation with stochastic forcing considered in Chapter 2. Furthermore, we show that the invariant measure of the one dimensional equation is indeed the measure considered in Chapter 2. These results are all well known. Nonetheless, we give the main arguments for the readers convenience. The exposition is rather sketchy and roughly follows [Z89, dPZ92, dPZ96].

We treat the systems:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}}u(x,t) &= \Delta u(x,t) - \frac{1}{\varepsilon^2}F'(u(x,t)) + \sqrt{2}\dot{W}(x,t) \qquad (x,t) \in (-1,1)\times\mathbb{R}_+ \\ u(x,0) &= u_0(x) \qquad u(\pm 1,t) = \pm 1 \qquad x \in (-1,1),\, t\in\mathbb{R}_+, \end{aligned} \tag{4.2.1}$$

where F is as above in (2.1.2). The formal term $\dot{W}(x,t)$ denotes space-time white noise. In the physical literature it is defined as a centered space-time Gaussian process with correlation function

$$\mathbb{E}\big[\dot{W}(x,t)\dot{W}(y,s)\big] = \delta_0(x-y)\delta_0(t-s).$$

In the mathematical literature one defines the *cylindrical Wiener process* $W(t) = \sum_{k=1}^{\infty} e_k W_t^k$ for an orthonormal basis $(e_k)_{k\in\mathbb{N}}$ of $L^2[-1,1]$ and independent one dimensional Brownian motions W^k. Then equation (4.2.1) is interpreted as a stochastic differential equation in some function space. In this particular case one faces the problem that the cylindrical Wiener process does not attain values in $L^2[-1,1]$ but only in some distribution space, whereas the nonlinearity $\frac{1}{\varepsilon^2}F'(u(x,t))$ can best be dealt with on the space of continuous functions. This problem can be overcome by rewriting (4.2.1) in the mild form

$$u(\cdot,t) = S_0(t)u_0 + \sqrt{2}\int_0^t S_0(t-s)dW(s) - \frac{1}{\varepsilon^2}\int_0^t S_0(t-s)F'(u(s))\mathrm{ds}. \tag{4.2.2}$$

Here $S_0(t)$ denotes the solution semigroup to the heat equation with the same Dirichlet boundary conditions as in (4.2.1). This corresponds to the classical variation of constants formula for solving inhomogeneous equations. It turns out that although the noise takes values in a much rougher space, due to the smoothing properties of the semigroup $S_0(t)$ the stochastic convolution attains values in the space of continuous functions. The nonlinear equation can then be solved with a fixed point argument in this space. Note that this argument is strictly restricted to one space dimension. In higher dimensions the smoothing property of the semigroup does not suffice to ensure that the process $\int_0^t S_0(t-s)dW(s)$ takes values in a function space. Therefore, the solution of such nonlinear equation in higher space dimensions is in general not possible. We therefore restrict ourselves to the one dimensional case. The disadvantage of this approach is that solutions to (4.2.2) only formally correspond to equation (4.2.1). In particular in this setting one has to be very careful, when one wants to apply the Ito formula. We will now describe this procedure in some detail and then show that the process one obtains is Markovian, reversible, and that the unique invariant measure is given by

$$\mu^\varepsilon(\mathrm{d}u) = \frac{1}{Z^\varepsilon}\exp\Big(-\frac{1}{\varepsilon^2}\int_{-1}^{1} F(u(s))\,\mathrm{ds}\Big)\nu(\mathrm{d}u), \tag{4.2.3}$$

where ν is the distribution of the Brownian bridge with the same boundary condition as in (4.2.1). Z^ε denotes the appropriate normalization constant that ensures that μ^ε is a probability measure. One should note that also the opposite direction of reasoning is possible. With the measure (4.2.3) one can write down the infinite dimensional Dirichlet form corresponding to (4.2.1). Then the general theory of Dirichlet forms gives the existence of a reversible process, which can be identified as a solution to (4.2.1). See [AR90] and the references therein for more about this approach.

Reversibility of finite dimensional gradient flow

To motivate the considerations below, let us consider the case of finite dimensional gradient flows first. Consider the equation

$$\mathrm{d}x_t = -\nabla\mathcal{H}(x_t)\mathrm{dt} + \sqrt{2}\mathrm{d}W_t,$$

where now x_t is supposed to be an $\mathbb{R}^d$-valued process, W_t is a d dimensional Brownian motion, and $\mathcal{H} : \mathbb{R}^d \to \mathbb{R}$ is a function. In order to obtain well-posedness of the equation assume that $\mathcal{H}$ is at least of class $C^{1,1}$ and satisfies a growth condition at ∞, say $\mathcal{H}(x) \geq C|x|$ for $|x|$ sufficiently large. Then the measure

$$\mu(\mathrm{dx}) = \frac{1}{Z}\exp(-\mathcal{H}(x))\mathrm{dx}$$

is the unique reversible measure for the above diffusion. In fact, in order to see that μ is indeed a reversible measure, denote by

$$\mathcal{L} = \Delta - \nabla\mathcal{H}\cdot\nabla$$

the generator of the diffusion and calculate for f, g sufficiently nice:

$$\begin{aligned}
\int f(x)\,(\mathcal{L}g(x))\,\mu(\mathrm{dx}) &= \int f(x)\,(\Delta g(x))\,\mu(\mathrm{dx}) - \int f(x)\,(\nabla\mathcal{H}(x)\cdot\nabla g(x))\,\mu(\mathrm{dx}) \\
&= -\int \nabla f(x)\cdot\nabla g(x)\mu(\mathrm{dx}) - \int f(x)\nabla g(x)\cdot(-\nabla\mathcal{H}(x))\,\mu(\mathrm{dx}) \\
&\qquad - \int f(x)\,(\nabla\mathcal{H}(x)\cdot\nabla g(x))\,\mu(\mathrm{dx}) \\
&= -\int \nabla f(x)\cdot\nabla g(x)\mu(\mathrm{dx}) \\
&= \int g(x)\,(\mathcal{L}f(x))\,\mu(\mathrm{dx}). \qquad (4.2.4)
\end{aligned}$$

In this setting the symmetry of the generator with respect to μ also implies the symmetry of the semigroup. The stochastic interpretation of this property is, that, if $\mathbb{P}^\mu$ denotes the distribution of the process with a random initial condition distributed according to μ, then for every T the processes $(x_t, t \in [0,T])$ and $(x_{T-t}, t \in [0,T])$ have the same law. In particular μ is an invariant measure for the process. Uniqueness of this invariant measure essentially follows from the non-degeneracy of the noise, which implies that the semigroup of the process has the strong Feller property.

The infinite dimensional equation has a similar structure. In fact the Allen-Cahn equation is given as the accelerated L^2-gradient flow of the real Ginzburg-Landau energy

$$\mathcal{H}^\varepsilon(u) = \int \varepsilon\frac{|\nabla u(x)|^2}{2} + \frac{1}{\varepsilon}F(u(x))\mathrm{dx},$$

i.e. the above equation can formally be written as

$$\frac{\mathrm{d}}{\mathrm{dt}}u = -\frac{1}{\varepsilon}\nabla_{L^2}\mathcal{H}^\varepsilon(u) + \sqrt{2}\dot{W}_t,$$

where W_t is a standard Wiener process on $L^2[-1,1]$. A formal application of the finite dimensional results would suggest that the invariant measure should be proportional to

$$\mu(\mathrm{d}u) \sim \exp\left(-\frac{1}{\varepsilon}\mathcal{H}^\varepsilon(u)\right)\mathrm{d}u \sim \exp\left(-\int \frac{|\nabla u(x)|^2}{2} + \frac{1}{\varepsilon^2}F(u(x))\mathrm{dx}\right)\mathrm{d}u,$$

and, if one accepts Feynman's heuristic that

$$\exp\left(-\int \frac{|\nabla u(x)|^2}{2}\right)\mathrm{d}u$$

is proportional to the distribution of a Brownian bridge, one obtains the desired form of the invariant measure.

Existence and uniqueness for the infinite dimensional system

We will now sketch how to prove global existence and Markov property for the solution of (4.2.1). In order to avoid the inhomogeneous boundary data consider $v(x) = u(x) - x$ instead. Then v solves

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}} v(x,t) &= \Delta v(x,t) + g(v(x),x) + \sqrt{2}\dot{W}(x,t) \qquad (x,t) \in (-1,1) \times \mathbb{R}_+ \\ v(x,0) &= v_0(x) \qquad v(\pm 1,t) = 0 \qquad x \in (-1,1), \quad t \in \mathbb{R}_+, \end{aligned} \tag{4.2.5}$$

where

$$g(v,x) = -\frac{1}{\varepsilon^2} F'(v+x),$$

and $v_0(x) = u_0(x) - x$. We will solve this equation on the space $C_0[-1,1] = \{u : [-1,1] \to \mathbb{R} \colon u \text{ continuous and } u(\pm 1) = 0\}$.

Before treating the nonlinear equation let us first recall some facts about the linear equation

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}} z(x,t) &= \Delta z(x,t) + \sqrt{2}\dot{W}(x,t) \qquad (x,t) \in (-1,1) \times \mathbb{R}_+ \\ z(x,0) &= v_0(x) \qquad z(\pm 1,t) = 0 \qquad x \in (-1,1),\, t \in \mathbb{R}_+. \end{aligned} \tag{4.2.6}$$

There are several approaches to study this equation. The easiest approach is probably the decomposition into a system of infinitely many independent stochastic ordinary differential equations. To this end for $k = 1, 2, \ldots$ define the functions e_k as

$$e_k(x) = \sin\left(\frac{k\pi}{2}(x+1)\right).$$

The $(e_k)_{k\in\mathbb{N}}$ form the basis of eigenfunctions of the Dirichlet-Laplace operator on $[-1,1]$ and the associated eigenvalues are given as $\lambda_k = -\left(\frac{k\pi}{2}\right)^2$. Then (4.2.6) can be written as $z(x,t) = \sum x_k(t) e_k(x)$ where the coefficients x_k solve

$$\mathrm{d}x_k(t) = -\lambda_k x_k(t)\mathrm{dt} + \sqrt{2}\mathrm{d}W^k(t), \tag{4.2.7}$$

for independent Brownian motions $W^k = \langle W(t), e_k \rangle$. Similar to the finite dimensional case solutions to (4.2.6) are referred to as *infinite dimensional Ornstein-Uhlenbeck process.* [1] The convergence of the formal sum $u(x,t) = \sum x_k(t)e_k(x)$ in $C([0,T], L^2[-1,1])$ follows directly with an application of the maximal inequality. In order to obtain convergence in $C([0,T], C_0[-1,1])$, one has to apply the Kolmogorov-Chentsov criterion. To be more precise this argument yields that solutions $z(x,t)$ are locally α-Hölder-continuous for every $\alpha < \frac{1}{2}$ as function of the space variable x for fixed t but only locally β-Hölder continuous for every $\beta < \frac{1}{4}$ as function of the time variable t for fixed x. In particular for every $x \in (-1,1)$ the process $(z(x,t), t \geq 0)$ is not a semimartingale[2]. As in the finite dimensional case the process is reversible and the reversible measure is given by a normal distribution. To be more precise (4.2.7) shows that the k-th mode $\langle u(t), e_k \rangle_{L^2}$ has as reversible measure $\mathcal{N}\left(0, \frac{1}{\lambda_k}\right)$, such that for the full process one obtains a normal distribution on

[1] Note that this Ornstein-Uhlenbeck process does not coincide with the Ornstein-Uhlenbeck process usually treated in Malliavin-Calculus. That process corresponds to the equation

$$\mathrm{d}z(x,t) = -z(x,t)\mathrm{dt} + \sqrt{2}(-\Delta)^{-1/2}\mathrm{d}W(x,t).$$

Both processes possess the same reversible measure.

[2] For a discussion of the regularity of the linear equation the reader is referred to the very motivating lecture notes [H09]. There the regularity is discussed on a formal level on page 5 and on a rigorous level in section 5.1

L^2 with covariance operator $(-\Delta)^{-1}$, which can be identified with the distribution of a Brownian bridge. In fact, for a Brownian bridge $\big(X_t,\, t \in [-1,1]\big)$ one has for test functions φ and χ:

$$\begin{aligned}
\mathbb{E}\Big[\int_{-1}^{1} \varphi(s) X_s \mathrm{d}s \int_{-1}^{1} \chi(s) X_s \mathrm{d}s\Big] &= \int_{-1}^{1}\int_{-1}^{1} \varphi(s)\chi(t)\, \mathbb{E}\big[X_s X_t\big] \mathrm{d}s\, \mathrm{d}t \\
&= \int_{-1}^{1}\int_{-1}^{1} \varphi(s)\chi(t)\Big(s \wedge t + 1 - \frac{(s+1)(t+1)}{2}\Big) \mathrm{d}s\, \mathrm{d}t \\
&= \int_{-1}^{1} \varphi(s)(-\Delta)^{-1}\chi(s) \mathrm{d}s. \qquad (4.2.8)
\end{aligned}$$

Here in the last line we have used the fact that the integral kernel given by the covariance structure of the Brownian bridge coincides with the Green function of the Dirichlet-Laplace operator. Note that the same solutions can also be written as *mild solutions*:

$$z(t) = S(t)z(0) + \sqrt{2}\int_0^t S(t-s)\mathrm{d}W_s.$$

Here $S(t)$ denotes the heat semigroup on $[-1,1]$ with Dirichlet boundary conditions. Rewriting the semigroup in the eigenbasis $(e_k)_{k\in\mathbb{N}}$ one can see immediately that this is equivalent to the approach just discussed.

Let us now deal with the nonlinear equation. The nonlinearity g in (4.2.5) is not continuous from $L^2[-1,1] \to L^2[-1,1]$ but it defines a nice continuous operator from C_0 into itself. In fact one has even more: If u and v satisfy $\|u\|_\infty \le r$ and $\|v\|_\infty \le r$ for some large r one has for all $x \in [-1,1]$

$$\begin{aligned}
|g(u(x),x) - g(v(x),x)| &= \frac{1}{\varepsilon^2}|F'(u(x)+x) - F'(v(x)+x)| \\
&\le \frac{1}{\varepsilon^2} \sup_{|w|\le r+1} |F''(w)|\|u-v\|_\infty, \qquad (4.2.9)
\end{aligned}$$

such that one can conclude, that the nonlinear operator $G : C_0[-1,1] \to C_0[-1,1]$, given by $G(u)(x) = g(u(x),x)$ is locally Lipschitz. In fact setting $C_r = \frac{1}{\varepsilon^2}\sup_{|w|\le r+1}|F''(w)|$, equation (4.2.9) can be restated as

$$\|G(u) - G(v)\|_\infty \le C_r \|u-v\|_\infty \qquad (4.2.10)$$

for $\|u\|_\infty, \|v\|_\infty \le r$. Furthermore, defining the operator $H\colon C_0[-1,1] \to \mathbb{R}$ as $H(u) = \int_{-1}^1 \frac{1}{\varepsilon^2}F(u(x)+x)dx$ the operator G can be interpreted as L^2-gradient of H in the following sense: H is Fréchet differentiable and for every $u,h \in C_0[-1,1]$ one has

$$\begin{aligned}
DH[u](h) &:= \lim_{\delta\downarrow 0}\frac{1}{\delta}\big(H(u+\delta h) - H(u)\big) \\
&= \frac{1}{\varepsilon^2}\int_{-1}^{1} F'(u(x)+x)h(x)\mathrm{d}x \\
&= -\langle G(u), h\rangle_{L^2}. \qquad (4.2.11)
\end{aligned}$$

Now we want to construct global solutions to (4.2.5). We look for mild solutions to the nonlinear equation, i.e. functions v, that satisfy

$$v(t) = S(t)v(0) + \sqrt{2}\int_0^t S(t-s)\mathrm{d}W_s + \int_0^t S(t-s)G(v(s))\,\mathrm{d}s. \qquad (4.2.12)$$

To get local existence we will fix the path $z(t) = S(t)v(0)+\sqrt{2}\int_0^t S(t-s)\mathrm{d}W_s \in C([0,T], C_0[-1,1])$ and look for solutions of the mild formulation of the equation pathwisely. To this end note that the mapping

$$\Theta v(t) = z(t) + \int_0^t S(t-s)G(v(s))\mathrm{ds}$$

maps $C([0,T], C_0[-1,1])$ into itself. As the maximum principle implies $\|S(t)\|_{C_0\to C_0} \le 1$, one gets

$$\|\Theta v(t) - z(t)\|_\infty \le \int_0^t \|G(v(s))\|_\infty \mathrm{ds}.$$

Therefore, with (4.2.10) one can conclude, that for fixed r and a $T = T(r)$ chosen sufficiently small Θ maps $\overline{B}_r(z) = \{u(t) \in C([0,T], C_0[-1,1])\colon \sup_{t\in[0,T]} \|u(t) - z(t)\|_\infty \le r\}$ into itself. In a similar way for $u, v \in \overline{B}_r(z)$ one can see with (4.2.10) that

$$\begin{aligned}\|\Theta u(t) - \Theta v(t)\|_\infty &\le \int_0^t \|G(u(s)) - G(v(s))\|_\infty \,\mathrm{ds} \\ &\le Ct \sup_{s\in[0,t]} \|u(s) - v(s)\|_\infty\end{aligned}$$

such that– possibly by choosing a smaller t– the mapping Θ is a contraction. The Banach fixed point theorem then yields the local existence and uniqueness of mild solutions. In order to get global existence, due to the local boundedness of G, it is sufficient to prove that on a finite time interval $[0,T]$ the norm $\|u(t)\|_\infty$ cannot explode. To this end for a fixed path $z(t)$ write $\tilde{v} = v - z$. Formally we would like to conclude that $\tilde{v}$ solves the (nonstochastic) PDE

$$\frac{d}{dt}\tilde{v}(x,t) = \Delta\tilde{v}(x,t) - F'(\tilde{v}(x,t) + x + z(x,t)). \tag{4.2.13}$$

Then, using the fact that $\mathrm{sign}(u) - F'(u)$ attains only negative values for $|u|$ sufficiently large, we could conclude using a comparison principle that $\|\tilde{v}\|_\infty$ cannot explode on a compact time interval $[0,T]$.

Unfortunately, equation (4.2.13) is not true, as we only know that the mild formulation of the nonlinear SPDE holds. In order to make the argument rigorous, one can take the mild solutions of (4.2.5) with the Laplace operator replaced by its Yoshida approximation. Then one can prove nonexplosion for these regularized solutions. In this argument the maximum principle is replaced by the fact that by the Hille-Yoshida Theorem the Yoshida approximation of the Laplacian generates a contraction semigroup on $C_0[-1,1]$. Then, using the fact that the solutions of the approximated equation converge to solutions of the original problem, one can conclude nonexplosion for the original process. See [dPZ92] page 201 for the detailed calculations. The Markov property follows from the pathwise uniqueness.

Reversible measure

In order to show that the reversible measure has the right form, we perform a finite dimensional approximation: For $u \in L^2[-1,1]$ denote by $\widetilde{\Pi_n}(u)$ the orthogonal projection on the linear space spanned by $e_1, \dots, e_n$. For fixed u in $C_0[-1,1]$ the sequence $\widetilde{\Pi_n}(u)$ converges to u in $L^2[-1,1]$ but in general it is not true that this convergence holds in $C_0[-1,1]$. Nonetheless it is possible to define symmetric projection operators $\Pi_n : L^2[-1,1] \to \mathrm{span}(e_1, \dots e_n)$ such that the convergence alsl holds in $C_0[-1,1]$ (see[Z89], Proposition 3 on page 244). [3] Now instead of the full system

[3] In the case of usual Fourier polynomials this approximation is given for example by the Césaro mean of the $\widetilde{\Pi_i}$:

$$\Pi_n u = \frac{1}{n}\sum_{i=1}^n \widetilde{\Pi_i}(u).$$

This result is well known as Fejer's theorem.

(4.2.5), we first treat the following finite dimensional system:

$$v_n(t) = \widetilde{\Pi_n} z(t) + \int_0^t S(t-s)\Pi_n(G(\Pi_n v(s)))\mathrm{ds}. \tag{4.2.14}$$

As this system is finite dimensional, the mild solution can rigorously be written as the solution to an equation of the type (4.2.4) and thus the same reasoning as above yields, that the measure

$$\mu^n = \frac{1}{Z^n}\exp(-H_n(u))\nu^n(\mathrm{d}u),$$

is reversible for (4.2.13). Here ν^n is the finite dimensional projection of the Brownian bridge and $H_n(u) = H(\Pi_n(u))$. In this last step the symmetry of the projections Π_n is used. One can thus conclude that for $\varphi, \chi \in C_b(C_0[-1,1])$ one has

$$\int_{C_0} \varphi(u)P_t^n\chi(u)\mu^n(\mathrm{d}u) = \int_{C_0} \chi(u)P_t^n\varphi(u)\mu^n(\mathrm{d}u),$$

where P_t^n denotes the semigroup associated to (4.2.14). Then, by passing to the limit $n \to \infty$, (which requires a little more work and the uniform approximation property of the operators Π_n) one obtains:

$$\int_{C_0} \varphi(u)P_t\chi(u)\mu(\mathrm{d}u) = \int_{C_0} \chi(u)P_t\varphi(u)\mu(\mathrm{d}u).$$

This yields the desired result if one inverts the transformation back to the nonhomogeneous boundary data. As in the finite dimensional case, the nondegeneracy of the noise can be used to show the strong Feller property and the uniqueness of the invariant measure.

Bibliography

[AR90] Albeverio, S.; Röckner, M. Stochastic differential equations in infinite dimensions: solutions via Dirichlet forms. *Probab. Theory Related Fields* 89 (1991), no. 3, 347–386.

[AC79] Allen, S.M., Cahn, J.W. a microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening, *Acta Metall.* 27 (1979), 1085 –1095.

[ATW93] Almgren, F.; Taylor, J.; Wang, L. Curvature-driven flows: a variational approach. *SIAM J. Control Optim.* 31 (1993), no. 2, 387–438.

[ASZ09] Ambrosio L.,Savaré G.,Zambotti L., Existence and Stability for Fokker-Planck equations with log-concave reference measure, *Prob. Theory and Related Fields*, 145 (2009), no. 3–4, 517–564.

[BM07] Bakhtin, Y.; Mattingly, J. Malliavin calculus for infinite-dimensional systems with additive noise, *Journal of Functional Analysis*, vol. 249 no. 2 (2007), pp. 307–353

[B08] Bonciocat, A. Curvature bounds and heat kernels, *Dissertation* (2008), Universität Bonn.

[B78] Brakke, K. The motion of a surface by its mean curvature. *Mathematical Notes, 20. Princeton University Press*, Princeton, N.J., 1978. i+252 pp.

[BMP95] Brassesco, S.; De Masi, A.; Presutti, E. Brownian fluctuations of the interface in the $D = 1$ Ginzburg-Landau equation with noise. *Ann. Inst. H. Poincaré Probab. Statist.* 31 (1995), no. 1, 81–118.

[BM02] Buckdahn R.; Ma J. Pathwise stochastic Taylor expansions and stochastic viscosity solutions for fully nonlinear stochastic PDEs. *Ann. Probab.*, 30 (2002), no. 3, 1131 – 1171.

[CP89] Carr, J.; Pego, R. L. Metastable patterns in solutions of $u_t = \varepsilon^2 u_{xx} - f(u)$. *Comm. Pure Appl. Math.* 42 (1989), no. 5, 523–576.

[CFO09] Caruana, M.; Friz, P.; Oberhauser, H. A (rough) pathwise approach to fully non-linear stochastic partial differential equations *preprint* (2009) arXiv: 0902.3352.

[Ch04] Chen, X. Generation, propagation, and annihilation of metastable patterns. *J. Differential Equations* 206 (2004), no. 2, 399–437.

[CGG91] Chen, Y; Giga, Y.; Goto, S. Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations. *J. Differential Geom.* 33 , no. 3, 749–786, 1991.

[C94] Chen, X. Spectrum for the Allen-Cahn, Cahn-Hilliard, and phase-field equations for generic interfaces. *Comm. Partial Differential Equations* 19 (1994), no. 7-8, 1371–1395.

[CHL97] Chen, X.; Hilhorst, D.; Logak, E. Asymptotic behavior of solutions of an Allen-Cahn equation with a nonlocal term. *Nonlinear Anal.* 28, no. 7, 1283–1298, 1997.

[CIL92] Crandall, M.; Ishii, H.; Lions, P.-L. User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc.* 27 (1992), no. 1, 1–67.

[dPZ92] Da Prato, G.; Zabczyk, J. Stochastic equations in infinite dimensions. *Encyclopedia of Mathematics and its Applications*, 44. Cambridge University Press, Cambridge, 1992.

[dPZ96] Da Prato, G.; Zabczyk, J. Ergodicity for infinite-dimensional systems. *London Mathematical Society Lecture Note Series*, 229. Cambridge University Press, Cambridge, 1996. xii+339 pp.

[DS89] Deuschel, J.-D.; Stroock, D. Large deviations. *Pure and Applied Mathematics*, 137. Academic Press, Inc., Boston, MA, 1989.

[DLN01] Dirr, N.; Luckhaus, S.; Novaga, M. A stochastic selection principle in case of fattening for curvature flow, *Calc. Var. Partial Differential Equations* 13 no. 4: 405–425, 2001.

[Ec04] Ecker, K. Regularity theory for mean curvature flow. *Progress in Nonlinear Differential Equations and their Applications*, 57. Birkhäuser Boston, Inc., Boston, MA, 2004. xiv+165 pp.

[ESS92] Evans, L.; Soner, H.; Souganidis, P. Phase transitions and generalized motion by mean curvature. *Comm. Pure Appl. Math.* 45, no. 9: 1097–1123, 1992.

[ES91] Evans, L.; Spruck, J. Motion of level sets by mean curvature. I. *J. Differential Geom.* 33 , no. 3, 635–681, 1991.

[ES92] Evans, L.; Spruck, J. Motion of level sets by mean curvature. II. *Trans. Amer. Math. Soc.* 330 , no. 1, 321–332, 1992.

[FJL82] Faris, W.; Jona-Lasinio, G. Large fluctuations for a nonlinear heat equation with noise. *J. Phys. A* 15 (1982) , no. 10, 3025–3055.

[FV03] Fatkullin, I.;Vanden-Eijnden, E. Coarsening by diffusion-annihilation in a bistable system driven by noise. *Preprint*, 2003.

[FINR06] Fontes, L. R. G.; Isopi, M.; Newman, C. M.; Ravishankar, K. Coarsening, nucleation, and the marked Brownian web. *Ann. Inst. H. Poincaré Probab. Statist.* 42 (2006), no. 1, 37–60.

[Fu95] Funaki, T. The scaling limit for a stochastic PDE and the separation of phases. *Probab. Theory Related Fields* 102 (1995), no. 2, 221–288.

[Fu99] Funaki, T. Singular limit for stochastic reaction-diffusion equation and generation of random interfaces, *Acta Math. Sin. (Engl. Ser.)* 15, no. 3: 407–438, 1999.

[H09] Hairer, M. An introduction to Stochastic PDEs. *Lecture notes* (2009), http://www.hairer.org/notes/SPDEs.pdf.

[HSV05] Hairer, M.; Stuart, A. M.; Voss, J.; Wiberg, P. Analysis of SPDEs arising in path sampling. I. The Gaussian case. *Commun. Math. Sci.* 3 (2005), no. 4, 587–603.

[HSV07] Hairer, M.; Stuart, A. M.; Voss, J. Analysis of SPDEs arising in path sampling. II. The nonlinear case. *Ann. Appl. Probab.* 17 (2007), no. 5-6, 1657–1706.

[dH00] den Hollander, F. Large deviations. *Fields Institute Monographs*, 14. American Mathematical Society, Providence, RI, 2000. x+143 pp.

[Il93] Ilmanen, T. Convergence of the Allen-Cahn equation to Brakke's motion by mean curvature. *J. Differential Geom.* 38 (1993), no. 2, 417–461.

[Iw87] Iwata, K. An infinite-dimensional stochastic differential equation with state space $C(R)$. *Probab. Theory Related Fields* 74 (1987), no. 1, 141–159.

[KS91] Karatzas, I.; Shreve, S. Brownian motion and stochastic calculus. Second edition. *Graduate Texts in Mathematics*, 113. Springer-Verlag, New York, 1991.

[KKL07] Katsoulakis, M; Kossioris, G.; Lakkis, O. Noise regularization and computations for the 1-dimensional stochastic Allen-Cahn problem. *Interfaces Free Bound.* 9 (2007) , no. 1, 1–30,.

[KO82] Kawasaki, K.; Ohta, T. Kinetic Drumhead Model of Interface. I *Progress of Theoretical Physics*, Vol. 67, No. 1, pp. 147-163

[KORV07] Kohn, R.; Otto, F.; Reznikoff, M.; Vanden-Eijnden, E. Action minimization and sharp-interface limits for the stochastic Allen-Cahn equation. *Comm. Pure Appl. Math.* 60 (2007), no. 3, 393–438.

[K06] Kolesnikov, A. Mosco convergence of Dirichlet forms in infinite dimensions with changing reference measures. *J. Funct. Anal.* 230 (2006), no. 2, 382–418.

[KS03] Kuwae, K.; Shioya, T. Convergence of spectral structures: a functional analytic theory and its applications to spectral geometry. *Comm. Anal. Geom.* 11 (2003), no. 4, 599–673.

[Le96] Ledoux, M. Isoperimetry and Gaussian analysis. *Lectures on probability theory and statistics (Saint-Flour, 1994)*, 165–294, Lecture Notes in Math., 1648, Springer, Berlin, 1996.

[LS98I] Lions, P.-L.; Souganidis, P. Fully nonlinear stochastic partial differential equations. *C. R. Acad. Sci. Paris Sér. I Math.* 326 (1998), no. 9, 1085–1092.

[LS98II] Lions, P.; Souganidis, P. Fully nonlinear stochastic partial differential equations: non-smooth equations and applications. *C. R. Acad. Sci. Paris Sér. I Math.* 327 (1998) , no. 8, 735–741.

[LS95] Luckhaus, S.; Sturzenhecker, T. Implicit time discretization for the mean curvature flow equation. *Calc. Var. Partial Differential Equations* 3 (1995), no. 2, 253–271.

[L95] Lunardi, A. Analytic Semigroups and Optimal regularity in Parabolic Problems, volume 16, Birkhäuser, 1995.

[MM77] Modica L.; Mortola, S. Il limite nella Γ -convergenza di una famiglia di funzionali ellittici. *Boll. Un. Mat. Ital.* A 14 (1977), 526 –529.

[M87] Modica, L. The gradient theory of phase transitions and the minimal interface criterion. *Arch. Rational Mech. Anal.* 98 (1987), no. 2, 123–142.

[MS95] de Mottoni, P.; Schatzman, M. Geometrical evolution of developed interfaces. *Trans. Amer. Math. Soc.* 347 (1995), no. 5: 1533–1589.

[MR08] Mugnai, L.; Röger, M. The Allen-Cahn action functional in higher dimensions. *Interfaces Free Bound.* 10 (2008), no. 1, 45–78.

[MR09] Mugnai, L.; Röger, M. Convergence of perturbed Allen-Cahn equations to forced mean curvature flow. *preprint* (2009) arXiv:0902.1816v1 [math.AP].

[OR07] Otto, F.; Reznikoff, M. G. Slow motion of gradient flows. *J. Differential Equations* 237 (2007), no. 2, 372–420.

[vRS09] von Renesse, M. ; Es-Sarhir, A. Well-Posednessof Stochastic Curve Shortening Flow in the Plane *preprint* (2009).

[RY99] Revuz, D.; Yor, M. Continuous martingales and Brownian motion. Third edition. *Grundlehren der Mathematischen Wissenschaften* 293. Springer-Verlag, Berlin, 1999. xiv+602 pp.

[RV05] REZNIKOFF, M. G.; VANDEN-EIJNDEN, E. Invariant measures of stochastic partial differential equations and conditioned diffusions. *C. R. Math. Acad. Sci. Paris* 340 (2005), no. 4, 305–308.

[RW09] RÖGER, M; WEBER, H Tightness for a stochastic Allen-Cahn equation. *preprint* arXiv:0911.5706v1 [math.AP].

[S07] SHEFFIELD, S. Gaussian free fields for mathematicians. *Probab. Theory Related Fields* 139 (2007), no. 3-4, 521–541.

[SY04] SOUGANIDIS, P. E.; YIP, N. K. Uniqueness of motion by mean curvature perturbed by stochastic noise. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 21 (2004), no. 1, 1–23.

[S95] SUGIURA, M. Metastable behaviors of diffusion processes with small parameter *J. Math. Soc. Japan* 47 (1995), no.4, 755–788.

[W08] WEBER, H. On the short time asymptotic of the stochastic Allen-Cahn equation. *to appear in Ann. Inst. H. Poincaré Probab. Statist.*

[W09] WEBER, H. Sharp interface limit for invariant measures of a stochastic Allen-Cahn equation. *to appear in Comm. Pure Appl. Math.*

[Y98] YIP, N. Stochastic motion by mean curvature. *Arch. Rational Mech. Anal.* 144 (1998) , no. 4: 313–355.

[Z89] ZABCZYK, J. Symmetric solutions of semilinear stochastic equations. *Stochastic partial differential equations and applications*, II (Trento, 1988), 237–256, Lecture Notes in Math., 1390, Springer, Berlin, 1989.