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Bivariant K-theory of groupoids and the noncommutative geometry of limit sets

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Summary

We present a categorical setting for noncommutative geometry in the sense of Connes. This is done by introducing a notion of morphism for spectral triples. Spectral triples are the unbounded cycles for K-homology ([**11**]), and their bivariant generalization are the cycles for Kasparov's KK-theory ([**32**]). The central feature of KK-theory is the Kasparov product

$$KK_i(A,B)\otimes KK_j(B,C)\rightarrow KK_{i+j}(A,C).$$

Here A,B and C are C^*-algebras, and the product allows one to view KK as a category. The unbounded picture of this theory was introduced by Baaj and Julg ([**4**]). In this picture the external product

$$KK_i(A,B)\otimes KK_j(A',B')\rightarrow KK_{i+j}(A\overline{\otimes}B, A'\overline{\otimes}B'),$$

is given by an algebraic formula, as opposed to Kasparov's original approach, which is more analytic in nature, and highly technical.

In order to describe the internal Kasparov product of unbounded KK-cycles, we introduce a notion of connection for unbounded cycles $(\mathcal{E},D)$. This is a universal connection

$$\nabla:\mathcal{E}\rightarrow\mathcal{E}\tilde{\otimes}_B\Omega^1(B),$$

in the sense of Cuntz and Quillen ([**20**]), such that $[\nabla,D]$ extends to a completely bounded operator. The topological tensor product used here is the Haagerup tensor product for operator spaces. Blecher ([**7**]) showed this tensor product coincides with the C^*-module tensor product, in case both operator spaces are C^*-modules. His work plays a crucial role in our construction. The product of two cycles with connection is given by an algebraic formula and the product of connections can also be defined. Thus, cycles with connection form a category, and the bounded transform

$$(\mathcal{E},D,\nabla)\mapsto(\mathcal{E},D(1+D^2)^{-\frac{1}{2}}),$$

defines a functor from this category to the category KK.

We also describe a general construction for obtaining KK-cycles from real-valued groupoid cocycles. If $\mathcal{G}$ is a locally compact Hausdorff groupoid with Haar system and $c:\mathcal{G}\rightarrow\mathbb{R}$ a continuous closed cocycle, we show that pointwise multiplication by c in the convolution algebra $C_c(\mathcal{G})$, extends to an unbounded regular operator on the completion of $C_c(\mathcal{G})$ as a C^*-module over $C^*(\mathcal{H})$, where $\mathcal{H}$ is the kernel of c. It gives a KK-cycle for $(C^*(\mathcal{G}),C^*(\mathcal{H}))$. In case the groupoid $\mathcal{H}$ is unimodular with respect to a quasi-invariant measure, or more general, if $C^*(\mathcal{H})$ carries a trace, this KK-cycle gives rise to an index map $K_1(C^*(\mathcal{G}))\rightarrow\mathbb{C}$.

This result is general enough to be applied in a wide variety of examples. We use it to obtain the noncommutative torus as a smooth quotient (in the above categorical sense) of the irrational rotation action on the circle. In the last chapter we sketch the promising range of applications the above categorical setting and cocycle construction may have in the noncommutative geometry of limit sets.

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Bob, Annemieke, Lot en Joost, zonder jullie was ik hier nooit gekomen.

Contents

"Nobody is truly original. We are all just uniquely derivative."

Chris DeGarmo

Introduction

Noncommutative geometry can be described as the study of operator algebras using methods originating in geometry, topology and homotopy theory. Connes both pioneered and established the basic tools for noncommutative geometry in [**11**]. In that paper, cyclic cohomology, a cohomology theory for algebras generalizing classical DeRham theory on manifolds is developed, and a Chern character map

$$\mathrm{Ch} : K^*(\mathscr{A}) \to HP^*(\mathscr{A}),$$

from K-homology to periodic cyclic cohomology HP of a suitable topological algebra $\mathscr{A}$ is constructed. $\mathscr{A}$ is usually taken to be a dense Fréchet subalgebra of some enveloping C^*-algebra A. This should be viewed in analogy to the inclusion of the smooth functions $C^\infty(M) \subset C(M)$ in the continuous functions on some smooth manifold M. In K-theory (and K-homology, for that matter), this inclusion is "invisible": the K-groups of $C^\infty(M)$ and $C(M)$ are isomorphic, and the isomorphism is induced by the inclusion of the algebras. It should be mentioned here that this invariance fails in cyclic cohomology, but we will not be concerned with this issue presently.

From the Gel'fand-Naimark theorem we know that commutative C^*-algebras are dual to locally compact Hausdorff spaces, and thus, arbitrary C^*-algebras can be viewed as quantized spaces. However, quantizing manifolds is a different issue than quantizing spaces. In [**12**] Connes argues that a noncommutative metric space should be given by a spectral triple $(A, \mathscr{H}, D)$. The triple consists of a $\mathbb{Z}/2\mathbb{Z}$-graded C^*-algebra A, represented on a likewise graded Hilbert space $\mathscr{H}$, together with an odd unbounded operator D in $\mathscr{H}$, with compact resolvent, such that the graded commutators $[D, a]$ extend to bounded operators in $\mathscr{H}$, for all a in some dense subalgebra $\mathcal{A}$ of A. Such triples are also the cycles for the K-homology groups of A. The motivation for this definition of noncommtuative manifold stems from the fact that Connes was able to recover the Riemannian distance function on a compact spin manifold from the Dirac operator on this manifold.

In recent years, many noncommutative spectral triples have been constructed, at first in settings related to physics and geometry, and later also in the realm of analytic number theory and arithmetic. The latter examples were the starting point for the work in this thesis. The papers by Manin-Marcolli ([**40**], [**41**]) consider the action of finite index subgroups Γ, of the modular group $PSL(2, \mathbb{Z})$ on the boundary of the upper half plane. This boundary is isomorphic to $\mathbb{P}^1(\mathbb{R})$, and Γ acts on it with dense orbits. The K-theory of the crossed product algebra $C(\mathbb{P}^1(\mathbb{R})) \rtimes \Gamma$ turns out to be closely related the homology of the modular curve X_Γ uniformized by Γ. In [**14**] and [**17**], unformization of curves by Schottky groups is considered, both for classical Kleinian Schottky groups, and in the p-adic setting. Spectral triples for the action of such groups on the limit set of its Bruhat-Tits tree are constructed, and shown to contain information about the original curve. The Patterson-Sullivan measure, a special measure on the limit set, is a vital ingredient in these constructions.

The common denominator of the above situations is that they are both examples of a group acting on a space H that is hyperbolic in the sense of Gromov ([**24**]). Such spaces admit a natural compactification by adding the Gromov boundary ∂H. The action of the group on H extends to ∂H and the set of accumulation points of orbits is the limit set Λ_Γ, on which Γ acts with dense orbits. The analogue of a Patterson-Sullivan measure always exists for these groups [**15**], and the properties of this measure on $\Lambda_\Gamma \subset \partial H$ reflect the geometry of the interior space H. The construction of spectral triples in [**17**] depends heavily on the fact that Schottky groups are free. The original purpose of the research presented in this thesis was to construct spectral triples for general groups acting on limit sets of hyperbolic spaces.

Relations given by inclusions of groups and orbit equivalence should give rise to relations between the corresponding noncommutative geometries. An inclusion of groups $\Gamma \subset \Gamma'$ gives rise to an inclusion of limit sets $\Lambda_\Gamma \subset \Lambda_{\Gamma'}$. An orbit equivalence is given by a (partial) endomorphism $\sigma : \Lambda \to \Lambda$, that generates the same orbits as the Γ-action. The appropriate setting to consider these relations is that of groupoids. The crossed product algebra $C(\Lambda) \rtimes \Gamma$ can be obtained from the transformation groupoid $\Lambda \rtimes \Gamma$ of the action, and an inclusion of groups gives an inclusion of transformation groupoids, but not a homomorphism of algebras. An orbit equivalence also gives a homomorphism of groupoids, but not a homomorphism of algebras.

Instead of algebra homomorphisms, groupoid homomorphisms give rise to bimodules over the respective groupoid algebras. The Patterson-Sullivan measure gives rise to a homomorphism from the transformation groupoid $\Lambda \rtimes \Gamma$ to the real numbers $\mathbb{R}$. The kernel of such a homomorphism is again a groupoid, and thus gives rise to a bimodule again. This bimodule comes equipped with extra structure similar to that of a spectral triple but more general. It will be a cycle of Kasparov's bivariant K-theory [**32**].

Kasparov's theory associates to a pair of C^*-algebras (A, B) a $\mathbb{Z}/2\mathbb{Z}$-graded abelian group $KK_*(A, B)$. It comes equipped with the structure of a category by the intricate Kasparov product

$$KK_*(A, B) \otimes KK_*(B, C) \to KK_*(A, C),$$

where A, B and C are C^*-algebras. It unifies K-theory and K-homology in the sense that there are natural isomorphisms

$$KK_*(\mathbb{C}, A) \cong K_*(A), \quad KK_*(A, \mathbb{C}) \cong K^*(A).$$

Viewed as such, elements of $KK_*(A, B)$ in particular induce homomorphisms

$$K^*(B) \to K^*(A) \quad \text{and} \quad K_*(A) \to K_*(B).$$

The cycles for the KK-theory of (A, B) are given by bivariant spectral triples. Here the notion of a C^*-module over B is important, which is a right B-module with a B-vlaued inner product. These objects behave very much like Hilbert spaces. In particular, the notion of unbounded operators with compact resolvent makes sense in this setting. A C^*-B-module $\mathcal{E}$ carrying a representation of A, together with an unbounded operator D in $\mathcal{E}$, with compact resolvent and such that $[D, a]$ is bounded for all a in some dense subalgebra $\mathcal{A}$ of A, defines an element in $KK_*(A,B)$. These objects are called unbounded bimodules.

It turns out that by equipping the cycles $(\mathcal{E}, D)$ for KK-theory with the extra structure of a universal connection $\nabla : \mathcal{E} \to \mathcal{E}\tilde{\otimes}\Omega^1(A)$ in the sense of Cuntz-Quillen [**20**], satisfying certain analytic properties, is enough to create a category of spectral triples, in which morphisms are given by such cycles. Composition of morphisms, after KK-equivalence, corresponds to the Kasparov product. The construction of this functor is the topic of chapter 1. It gives a new way of thinking about morphisms of spectral triples, and probably even of Riemannian manifolds. It also allows for considering things in a relative setting, where the objects are not spectral triples, but unbounded bimodules (without connection).

In chapter 2 we explore groupoids $\mathcal{G}$ equipped with a homomorphism $c : \mathcal{G} \to \mathbb{R}$, as in the case of the Patterson-Sullivan measure. We show that under certain analytic conditions, this yields an unbounded bimodule for $(C^*(\mathcal{G}), C^*(\mathcal{H}_c))$. Here $\mathcal{H}_c$ is the kernel of the homomorphism c, a subgroupoid of $\mathcal{G}$. In many cases of interest, the groupoid algebra $C^*(\mathcal{H}_c)$ comes equipped with a canonical trace, giving a homomorphism $K_0(C^*(\mathcal{H}_c)) \to \mathbb{C}$. Combined with the Kasparov product this gives a homomorphism $K_1(C^*(\mathcal{G})) \to \mathbb{C}$.

In the last chapter, we return to the subject of limit sets. Most of the results discussed there are work in progress at the moment this thesis is written. The bimodules given by orbit equivalences and inclusions of groups can be used to relate the different index homomorphisms obtained in the above way. For hyperbolic manifolds, the index maps associated to different Patterson-Sullivan measures can be globalized to give a map from a slghtly more sophisticated K-group to the functions on the manifold M uniformized by the group Γ. Finally, we explore the group $SL(2,\mathbb{Z})$ and its action on both its tree and the projective line $\mathbb{P}^1(\mathbb{R})$ as morphisms in noncommutative geometry.

CHAPTER 1

Unbounded bivariant K-theory

In order to obtain a more transparant description of the external product structure in Kasparov's bivariant K-theory, Baaj and Julg [**4**] gave a description of the cycles of this theory in terms of unbounded operators on C^*-modules. Later on, Kucerovsky [**35**] gave necessary conditions for an unbounded cycle to represent the *internal* Kasporov product of two given cycles. In this chapter we construct a category of unbounded cycles, on which the bounded transform induces a functor to the category KK. In particular, this allows for computing the Kasparov product of two KK-elements as the composition of their unbounded representatives in the newly constructed category. Furthermore, we present a way to view this category as a category of spectral triples.

1. C^*-modules

From the Gelfand-Naimark theorem we know that C^*-algebras are a natural generalization of locally compact Hausdorff topological spaces. In the same vein, the Serre-Swan theorem tells us that finite projective modules are analogues of locally trivial finite-dimensional complex vector bundles over a topological space.The subsequent theory of C^*-modules, pioneered by Paschke and Rieffel, should be viewed in the light of these theorems. They are like Hermitian vector bundles over a space.

1.1. C^*-modules and their endomorphism algebras. In the subsequent review of the established theory, we will assume all C^*-algebras and Hilbert spaces to be separable, and all modules to be countably generated. This last assumption means that there exists a countable set of generators whose algebraic span is dense in the module.

Definition 1.1.1.1. Let B be a C^*-algebra. A *right C^*-B-module* is a complex vector space $\mathcal{E}$ which is also a right B-module, equipped with a bilinear pairing

$$\mathcal{E} \times \mathcal{E} \to B$$
$$(e_1, e_2) \mapsto \langle e_1, e_2 \rangle,$$

such that

- $\langle e_1, e_2 \rangle = \langle e_2, e_1 \rangle^*$,
- $\langle e_1, e_2 b \rangle = \langle e_1, e_2 \rangle b$,
- $\langle e, e \rangle \geq 0$ and $\langle e, e \rangle = 0 \Leftrightarrow e = 0$,
- $\mathcal{E}$ is complete in the norm $\|e\|^2 := \|\langle e, e \rangle\|$.

We use Landsman's notation ([**37**]) $\mathcal{E} \leftrightharpoons B$ to indicate this structure.

For two such modules, $\mathcal{E}$ and $\mathcal{F}$, one can consider operators $T : \mathcal{E} \to \mathcal{F}$. As opposed to the case of a Hilbert space ($B = \mathbb{C}$), such operators need not always have an adjoint with respect to the inner product. As a consequence, we consider two kinds of operator between C^*-modules.

DEFINITION 1.1.1.2. Let $\mathcal{E}, \mathcal{F}$ be C^*-B-modules. The Banach algebra of continuous B-module homomorphims from $\mathcal{E}$ to $\mathcal{F}$ is denoted by $\mathrm{Hom}_B(\mathcal{E}, \mathcal{F})$. Furthermore let

$$\mathrm{Hom}^*_B(\mathcal{E}, \mathcal{F}) := \{T : \mathcal{E} \to \mathcal{E} : \quad \exists T^* : \mathcal{E} \to \mathcal{E}, \quad \langle Te_1, e_2\rangle = \langle e_1, T^*e_2\rangle\}.$$

Elements of $\mathrm{Hom}^*_B(\mathcal{E}, \mathcal{F})$ are called *adjointable operators.*

Similarly we let $\mathrm{End}_B(\mathcal{E})$ and $\mathrm{End}^*_B(\mathcal{E})$ denote the continuous, respectively adjointable endomorphisms of the C^*-module $\mathcal{E}$.

PROPOSITION 1.1.1.3. *Let* $T \in \mathrm{Hom}^*_B(\mathcal{E}, \mathcal{F})$*. Then* $\mathrm{End}^*_B(\mathcal{E})$ *is a closed subalgebra of* $\mathrm{End}_B(\mathcal{E})$*, and it is a* C^**-algebra in the operator norm and the involution* $T \mapsto T^*$*.*

The concept of unitary isomorphism of C^*-modules is the obvious one: Two C^*-modules $\mathcal{E}$ and $\mathcal{F}$ over B are *unitarily isomorphic* if there exists a unitary $u \in \mathrm{Hom}^*_B(\mathcal{E}, \mathcal{F})$. $\mathcal{E}$ and $\mathcal{F}$ are said to be merely *topologically isomorphic* if there exists an invertible element $S \in \mathrm{Hom}_B(\mathcal{E}, \mathcal{F})$. An *isometric isomorphism* is a topological isomorphism that is isometric. The following remarkable result is due to M.Frank.

THEOREM 1.1.1.4 ([**22**]). *Two countably generated* C^**-modules are unitarily ismorphic if and only if they are isometrically isomorphic if and only if they are topologically isomorphic.*

$\mathrm{End}^*_B(\mathcal{E})$ contains another canonical C^*-subalgebra. Note that the involution on B allows for considering $\mathcal{E}$ as a left B-module via $be := eb^*$. The inner product can be used to turn the algebraic tensor product $\mathcal{E} \otimes_B \mathcal{E}$ into a $*$-algebra:

$$e_1 \otimes e_2 \circ f_1 \otimes f_2 := e_1\langle e_2, f_1\rangle \otimes f_2, \quad (e_1 \otimes e_2)^* := e_2 \otimes e_1.$$

This algebra is denoted by $\mathrm{Fin}_B(\mathcal{E})$. There is an injective *-homomorphism

$$\mathrm{Fin}_B(\mathcal{E}) \to \mathrm{End}^*_B(\mathcal{E}),$$

given by $e_1 \otimes e_2(e) := e_1\langle e_2, e\rangle$. The closure of $\mathrm{Fin}_B(\mathcal{E})$ in the operator norm is the C^*-algebra of *B-compact operators* on $\mathcal{E}$. It is denoted by $\mathbb{K}_B(\mathcal{E})$.

EXAMPLE 1.1.1.5 (Free modules). For each $n \in \mathbb{N}$, the module B^{n+1} becomes a C^*-module in the inner product

$$\langle(a_0, \cdots, a_n), (b_0, \cdots, b_n)\rangle := \sum_{i=0}^{n} a_i^* b_i.$$

There is a natural isomorphism $\mathbb{K}_B(A^n) \cong M_n(A)$.

A *grading* on a C^*-algebra B is a self-adjoint unitary $\gamma \in \mathrm{Aut}B$. If such a grading is present, B decomposes as $B^0 \oplus B^1$, where B^0 is the C^*-subalgebra of *even* elements, and B^1 the closed subspace of *odd* elements. We have $B^iB^j \subset B^{i+j}$ for $i, j \in \mathbb{Z}/2\mathbb{Z}$. For $b \in B^i$, we denote the *degree* of b by $\partial b \in \mathbb{Z}/2\mathbb{Z}$. From now on, we assume all C^*-algebras to be graded, possibly trivially, i.e. $\gamma = 1$.

DEFINITION 1.1.1.6. A C^*-module $\mathcal{E} \leftrightharpoons B$ is *graded* if it comes equipped with a selfadjoint unitary $\gamma \in \mathrm{Aut}^*_B(\mathcal{E})$ such that

- $\gamma(eb) = \gamma(e)\gamma(b)$,
- $\langle\gamma(e_1), \gamma(e_2)\rangle = \gamma\langle e_1, e_2\rangle$.

In this case $\mathcal{E}$ also decomposes as $\mathcal{E}^0 \oplus \mathcal{E}^1$, and we have $\mathcal{E}^i B^j \subset \mathcal{E}^{i+j}$ for $i, j \in \mathbb{Z}/2\mathbb{Z}$. The algebras $\mathrm{End}_B(\mathcal{E}), \mathrm{End}^*_B(\mathcal{E})$ and $\mathbb{K}_B(\mathcal{E})$ inherit a natural grading from $\mathcal{E}$ by setting $\gamma\phi(e) := \phi(\gamma(e))$. For $e \in \mathcal{E}^i$, we denote the *degree* of e by $\partial e \in \mathbb{Z}/2\mathbb{Z}$.From now on we assume all C^*-modules to be graded, possibly trivially.

1.2. Tensor products. For a pair of C^*-modules $\mathcal{E} \leftrightharpoons A$ and $\mathcal{F} \leftrightharpoons B$, the vector space tensor product $\mathcal{E} \otimes \mathcal{F}$ (over $\mathbb{C}$, which will be always supressed in the notation) can be made into a C^*-module over the minimal C^*-tensor product $A \bar{\otimes} B$. The minimal or *spatial* C^*-tensor product is obtained as the closure of $A \otimes B$ in $\mathbb{B}(\mathcal{H} \otimes \mathcal{K})$, where $\mathcal{H}$ and $\mathcal{K}$ are graded Hilbert spaces that carry faithful graded representations of A and B respectively. In order to make $A \bar{\otimes} B$ into a graded algebra, the multiplication law is defined as

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{\partial b_1 \partial a_2} a_1 a_2 \otimes b_1 b_2. \tag{1.1}$$

The completion of $\mathcal{E} \otimes \mathcal{F}$ in the inner product

$$\langle e_1 \otimes f_1, e_2 \otimes f_2\rangle := \langle e_1, e_2\rangle \otimes \langle f_1, f_2\rangle,$$

is a C^*-module denoted by $\mathcal{E} \bar{\otimes} \mathcal{F}$. It inherits a grading by setting $\gamma := \gamma_{\mathcal{E}} \otimes \gamma_{\mathcal{F}}$.

The graded module so obtained is the *exterior tensor product* of $\mathcal{E}$ and $\mathcal{F}$. The *graded tensor product* of maps $\phi \in \mathrm{End}^*_A(\mathcal{E})$ and $\psi \in \mathrm{End}^*_B(\mathcal{F})$ is defined by

$$\phi \otimes \psi(e \otimes f) := (-1)^{\partial(e)\partial(\psi)} \phi(e) \otimes \psi(f),$$

gives a graded inclusion

$$\mathrm{End}^*_A(\mathcal{E}) \bar{\otimes} \mathrm{End}^*_B(\mathcal{F}) \to \mathrm{End}^*_{A \bar{\otimes} B}(\mathcal{E} \bar{\otimes} \mathcal{F}),$$

which restricts to an isomorphism

$$\mathbb{K}_A(\mathcal{E}) \bar{\otimes} \mathbb{K}_B(\mathcal{F}) \to \mathbb{K}_{A \bar{\otimes} B}(\mathcal{E} \bar{\otimes} \mathcal{F}).$$

A *-homomorphism $A \to \mathrm{End}^*_B(\mathcal{E})$ is said to be *essential* if

$$A\mathcal{E} := \{\sum_{i=0}^{n} a_i e_i : a_i \in A, e_i \in \mathcal{E}, n \in \mathbb{N}\},$$

is dense in $\mathcal{E}$. If a graded essential *-homomorphism $A \to \mathrm{End}^*_B(\mathcal{F})$ is given, one can complete the algebraic tensor product $\mathcal{E} \otimes_A \mathcal{F}$ to a C^*-module $\mathcal{E} \tilde{\otimes}_A \mathcal{F}$ over B. The norm in which to complete comes from the B-valued inner product

$$\langle e_1 \otimes f_1, e_2 \otimes f_2\rangle := \langle e_1, \langle f_1, f_2\rangle e_2\rangle. \tag{1.2}$$

There is a *-homomorphism

$$\begin{array}{rcl} \mathrm{End}^*_A(\mathcal{E}) & \to & \mathrm{End}^*_B(\mathcal{E} \tilde{\otimes}_A \mathcal{F}) \\ T & \mapsto & T \otimes 1, \end{array}$$

which restricts to a homomorphism $\mathbb{K}_A(\mathcal{E}) \to \mathbb{K}_B(\mathcal{E} \tilde{\otimes}_A \mathcal{F})$.

EXAMPLE 1.1.2.1 (The standard module). The above constructions coincide when $A = \mathbb{C}$ and the homomorphism $\mathbb{C} \to \mathrm{End}^*_B(\mathcal{F})$ is given by multiples of the identity. If $\mathcal{E} = \mathcal{H}$, a graded separable Hilbert space, and $\mathcal{F} = B$, then $\mathcal{H}_B := \mathcal{H} \bar{\otimes} B \cong \mathcal{H} \tilde{\otimes} B$ is the *standard* C^*-module over B. We have $\mathbb{K}_B(\mathcal{H}_B) = \mathbb{K} \bar{\otimes} B$, where $\mathbb{K}$ is the algebra of compact operators on $\mathcal{H}$.

The standard module $\mathcal{H}_B$ absorbs any countably generated C^*-module. The direct sum $\mathcal{E} \oplus \mathcal{F}$ of C^*-B-modules becomes a C^*-module in the inner product

$$\langle (e_1, f_1), (e_2, f_2) \rangle := \langle e_1, e_2 \rangle + \langle f_1, f_2 \rangle.$$

THEOREM 1.1.2.2 (Kasparov [**32**]). *Let $\mathcal{E} \leftrightharpoons B$ be a countably generated graded C^*-module. Then there exists a graded unitary isomorphism $\mathcal{E} \oplus \mathcal{H}_B \xrightarrow{\sim} \mathcal{H}_B$.*

1.3. Correspondences. Noncommutative rings behave very differently from commutative rings in many ways. In particular, a given noncommutative ring can have very few ideals, or none at all. $M_n(\mathbb{C})$ for instance, is a simple algebra, and it is a not at all pathological object.The ordinary notion of homomorphism does not give an adequate categorical setting for noncommutative rings, because of the above mentioned lack of ideals. In pure algebra, a more flexible notion of morphism is given by bimodules, whose composition is the module tensor product. We now describe a category of such *correspondences* for C^*-algebras, taking into account the topology of these objects. The resulting category is slightly different from the usual category $\mathfrak{C}^*$, in which morphisms are essential $*$-homomorphisms. This structure is well-suited for functoriality properties of groupoid algebras, which will be explored in the next chapter.

DEFINITION 1.1.3.1. Let A, B be C^*-algebras. A *C^*-correspondence from A to B* consists of a C^*-B-module $\mathcal{E}$ together with an essential $*$-homomorphism $\pi : A \to \mathrm{End}^*_B(\mathcal{E})$, written $A \rightarrowtail \mathcal{E} \leftrightharpoons B$.
Two such correspondences are called isomorphic when there exists a unitary in $\mathrm{Hom}^*_B(\mathcal{E}, \mathcal{F})$ intertwining the A-representations.

We can compose correspondences $A \rightarrowtail \mathcal{E} \leftrightharpoons B$ and $B \rightarrowtail \mathcal{F} \leftrightharpoons C$ via the internal tensor product. Denote by $\mathfrak{Cor}_{C^*}(A, B)$ the set of isomorphism classes of correspondences from A to B. It is straightforward to check that the correpondences $A \rightarrowtail A \leftrightharpoons A$ are units for the composition operation modulo unitary equivalence.

PROPOSITION 1.1.3.2. *Composition of correspondences as described above is associative on isomorphism classes of correspondences. Therefore the sets $\mathfrak{Cor}_{C^*}(A, B)$ are the morphism sets of a category $\mathfrak{Cor}_{C^*}$, whose objects are all C^*-algebras.*

The proof of this result is straightforward, as unitary equivalence provides enough freedom for associativity and identity to hold. There is a functor $\mathfrak{C}^* \to \mathfrak{Cor}_{C^*}$, which is the identity on objects. To a *-homomorphism $\pi \in \mathfrak{C}^*(A, B)$ it associates the correspondence $A \rightarrowtail B \leftrightharpoons B \in \mathfrak{Cor}_{C^*}(A, B)$.

DEFINITION 1.1.3.3. Let A, B be C^*-algebras. A and B are said to be *strongly Morita equivalent* if there exists a correspondence $A \rightarrowtail \mathcal{E} \leftrightharpoons B$ such that $\pi : A \to \mathrm{End}^*_B(\mathcal{E})$ is an isomorphism onto $\mathbb{K}_B(\mathcal{E})$.

Strong Morita equivalence is amongst the most important equivalence relations for C^*-algebras. Two commutative C^*-algebras are strongly Morita equivalent if and only if they are isomorphic. As such the relation can be viewed as an extension (via the Gelfand-Naimark theorem) of the notion of homeomorphism for locally compact Hausdorff spaces. The following result supports that view.

THEOREM 1.1.3.4. *Two C^*-algebras A, B are isomorphic in $\mathfrak{Cor}_{C^*}$ if and only if they are strongly Morita equivalent.*

The reader can consult [**38**] for a proof.

1.4. Unbounded operators. Similar to the Hilbert space setting, there is a notion of unbounded operator on a C^*-module. Many of the already subtle issues in the theory of unbounded operators should be handled with even more care. This is mostly due to the fact that closed submodules of a C^*-module need not be orthogonally complemented. We refer to [**3**], [**36**] and [**52**] for detailed expositions of this theory.

DEFINITION 1.1.4.1 ([**4**]). Let $\mathcal{E},\mathcal{F}$ be C^*-B-modules. A densely defined closed operator $D:\mathfrak{Dom}D\to\mathcal{F}$ is called *regular* if

- D^* is densely defined in $\mathcal{F}$
- $1+D^*D$ has dense range.

Such an operator is automatically B-linear, and $\mathfrak{Dom}D$ is a B-submodule of $\mathcal{E}$. There are two operators, $\mathfrak{r}(D),\mathfrak{b}(D)\in\mathrm{Hom}^*_B(\mathcal{E},\mathcal{F})$ canonically associated with a regular operator D. They are the *resolvent* of D

$$\mathfrak{r}(D):=(1+D^*D)^{-\frac{1}{2}}, \tag{1.3}$$

and the *bounded transform*

$$\mathfrak{b}(D):=D(1+D^*D)^{-\frac{1}{2}}. \tag{1.4}$$

PROPOSITION 1.1.4.2. *If $D:\mathfrak{Dom}D\to\mathcal{F}$ is regular, then D^*D is selfadjoint and regular. Moreover, $\mathfrak{Dom}D^*D$ is a core for D and $\mathfrak{Im}\mathfrak{r}(D)=\mathfrak{Dom}D$.*

It follows that D is completely determined by $\mathfrak{b}(D)$, as $\mathfrak{r}(D)^2=1-\mathfrak{b}(D)^*\mathfrak{b}(D)$. Due to this fact, selfadjoint regular regular operators share many properties with selfdajoint closed operators on Hilbert space. In particular, they admit a functional calculus.

THEOREM 1.1.4.3 ([**3**],[**36**]). *Let $\mathcal{E}\leftrightharpoons B$ be a C^*-module, and D a selfadjoint regular operator in $\mathcal{E}$. There is a *-homomorphism $f\mapsto f(D)$, from $C(\mathbb{R})$ into the regular operators on $\mathcal{E}$, such that $(x\mapsto x)\mapsto D$ and $(x\mapsto x(1+x^2)^{-\frac{1}{2}})\mapsto\mathfrak{b}(D)$. Moreover, it restricts to a *-homomorphism $C_0(\mathbb{R})\to\mathrm{End}^*_B(\mathcal{E})$.*

This theorem allows us to derive a useful formula for the resolvent of D. We include it here for later reference.

COROLLARY 1.1.4.4. *Let D be a selfadjoint regular operator on a C^*-module $\mathcal{E}$. Then the equality*

$$\mathfrak{r}(D)^2=(1+D^2)^{-1}=\int_0^\infty e^{-x(1+D^2)}dx,$$

holds in $\mathrm{End}^*_B(\mathcal{E})$.

PROOF. We have to check convergence of the integral at $x=0$ and for $x\to\infty$. To this end, let $s\leq t$ and compute:

$$\begin{aligned}\|\int_s^t e^{-x(1+D^2)}dx\| &\leq \int_s^t \|e^{-x(1+D^2)}\|dx\\ &\leq \int_s^t \sup_{y\in\mathbb{R}}|e^{-x(1+y^2)}|dx\\ &=\leq \int_s^t e^{-x}dx\\ &= e^{-t}-e^{-s}.\end{aligned}$$

Hence the integral converges for both $t \to 0$ and $s \to \infty$. $\square$

Recall that a submodule $\mathcal{F} \subset \mathcal{E}$ is *complemented* if $\mathcal{E} \cong \mathcal{F} \oplus \mathcal{F}^{\perp}$, where

$$\mathcal{F}^{\perp} := \{e \in \mathcal{E} : \forall f \in \mathcal{F} \quad \langle e, f \rangle = 0\}.$$

Contrary to the Hilbert space case, closed submodules of a C^*-module need not be complemented.

The *graph of* D is the closed submodule

$$\mathfrak{G}(D) := \{(e, De) : e \in \mathfrak{Dom}(D)\} \subset \mathcal{E} \oplus \mathcal{F}.$$

There is a canonical unitary $v \in \mathrm{Hom}_B(\mathcal{E} \oplus \mathcal{F}, \mathcal{F} \oplus \mathcal{E})$, defined by $v(e, f) := (-f, e)$.Note that $\mathfrak{G}(D)$ and $v\mathfrak{G}(D^*)$ are orthogonal submodules of $\mathcal{E} \oplus \mathcal{F}$. The following algebraic characterization of regularity is due to Woronowicz .

Theorem 1.1.4.5 ([**52**]). *A densely defined operator $D : \mathcal{E} \to \mathcal{F}$ is regular if and only if $\mathfrak{G}(D) \oplus v\mathfrak{G}(D^*) \cong \mathcal{E} \oplus \mathcal{F}$.*

The isomorphism is given by coordinatewise addition. Moreover, the operator

$$p_D := \begin{pmatrix} \mathfrak{r}(D)^2 & D\mathfrak{r}(D)^2 \\ D\mathfrak{r}(D)^2 & D^2\mathfrak{r}(D)^2 \end{pmatrix} \tag{1.5}$$

satisfies $p_D^2 = p_D^* = p_D$, i.e. it is a projection, and $p_D(\mathcal{E} \oplus \mathcal{F}) = \mathfrak{G}(D)$. $\mathfrak{G}(D)$, which is naturally in bijection with $\mathfrak{Dom}(D)$, inherits the structure C^*-module from $\mathcal{E} \oplus \mathcal{F}$, and hence so does $\mathfrak{Dom}D$. We denote its inner product by $\langle \cdot, \cdot \rangle_1$. Since D commutes with $\mathfrak{r}(D)$, D maps $\mathfrak{r}(D)\mathfrak{G}(D)$ into $\mathfrak{G}(D)$. We denote this operator by D_1.

Proposition 1.1.4.6. *Let $D : \mathfrak{Dom}D \to \mathcal{E}$ be a selfdajoint regular operator. Then $D_1 : \mathfrak{r}(D)\mathfrak{G}(D) \to \mathfrak{G}(D)$ is a selfadjoint regular operator.*

Proof. From proposition 1.1.4.2 it follows that

$$\mathfrak{r}(D)\mathfrak{G}(D) = \mathfrak{r}(D)^2\mathcal{E} = \mathfrak{Dom}D^2.$$

D_1 is closed as an operator in $\mathfrak{G}(D)$ for if $\mathfrak{r}(D)^2 e_n \to \mathfrak{r}(D)^2 e$ and $D\mathfrak{r}(D)^2 e_n \to e'$ in the topology of $\mathfrak{G}(D)$, then it follows immediatley that

$$e' = D(D\mathfrak{r}(D)^2 e) = D^2\mathfrak{r}(D)^2 e.$$

It is straightforward to check that D_1 is symmetric for the inner product of $\mathfrak{G}(D)$. Hence it is regular, because $(1 + D^2)\mathfrak{r}(D)^4\mathcal{E} = \mathfrak{r}(D)^2\mathcal{E}$. To prove selfadjointness, suppose $y \in \mathfrak{Dom}D$ is such that there exists $z \in \mathfrak{Dom}D$ such that for all $x \in \mathfrak{r}(D)^2\mathcal{E}$ $\langle D_1 x, y \rangle_1 = \langle x, z \rangle_1$. Then $z = Dy$, because

$$\begin{aligned} \langle Dx, y \rangle_1 &= \langle Dx, y \rangle + \langle D^2 x, Dy \rangle \\ &= \langle D\mathfrak{r}(D)^2 e, y \rangle + \langle D^2\mathfrak{r}(D)^2 e, Dy \rangle \\ &= \langle \mathfrak{r}(D)^2 e, Dy \rangle + \langle D^2\mathfrak{r}(D)^2 e, Dy \rangle \\ &= \langle e, Dy \rangle. \end{aligned}$$

A similar computation shows that $\langle x, z \rangle_1 = \langle e, z \rangle$. Since $\mathfrak{r}(D)^2$ is injective this holds for all $e \in \mathcal{E}$, and hence $z = Dy$. Therefore

$$\mathfrak{Dom}D_1^* = \{y \in \mathfrak{Dom}D : Dy \in \mathfrak{Dom}D\} = \mathfrak{Dom}D^2 = \mathfrak{r}(D)^2\mathcal{E} = \mathfrak{Dom}D_1,$$

so D_1 is selfadjoint. $\square$

COROLLARY 1.1.4.7. *A selfadjoint regular operator $D : \mathfrak{Dom}D \to \mathcal{E}$ induces a morphism of inverse systems of C^*-modules:*

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[dr, "D_{i+1}"] & \mathcal{E}_{i+1} \arrow[r] \arrow[dr, "D_i"] & \mathcal{E}_i \arrow[r] \arrow[dr, "D_{i-1}"] & \mathcal{E}_{i-1} \arrow[r] \arrow[dr, "D_{i-2}"] & \cdots \arrow[r] \arrow[dr, "D_1"] & \mathcal{E}_1 \arrow[r] \arrow[dr, "D"] & \mathcal{E} \\
\cdots \arrow[r] & \mathcal{E}_{i+1} \arrow[r] & \mathcal{E}_i \arrow[r] & \mathcal{E}_{i-1} \arrow[r] & \cdots \arrow[r] & \mathcal{E}_1 \arrow[r] & \mathcal{E}
\end{tikzcd}$$

PROOF. Set $\mathcal{E}_i = \mathfrak{G}(D_{i-1})$. Then the maps $\mathcal{E}_i \to \mathcal{E}_{i-1}$ are just projection on the first coordinate, whereas the maps $D_i : \mathcal{E}_{i+1} \to \mathcal{E}_i$ are the projections on the second coordinates. These maps are adjointable,and we have

$$D_i^*(e_i) = (D_i\mathfrak{r}(D_i)^2 e_i, D_i^2\mathfrak{r}(D_i)^2 e_i), \quad \phi_i^*(e_i) = (\mathfrak{r}(D_i)^2, D_i\mathfrak{r}(D_i)^2).$$

These are exactly the components of the Woronowicz projection 1.5. □

We will refer to this inverse system as the *Sobolev chain* of D.

To construct selfadjoint regular operators in practice, we include some remarks and results on the extension of symmetric regular operators. This material is to be found in [**36**]. A densely defined operator D in a C^*-module $\mathcal{E}$ is *symmetric* if, for $e, e' \in \mathfrak{Dom}D$ we have $\langle De, e'\rangle = \langle e, De'\rangle$. Symmetric operators are closable, and their closure is again symmetric. Hence we will tacitly assume all symmetric operators to be closed.

LEMMA 1.1.4.8 ([**36**]). *Let D be a densely defined symmetric operator. Then the operators $D + i$ and $D - i$ are injective and have closed range.*

We can now define two isometries

$$\mathfrak{u}_+(D) := (D + i)\mathfrak{r}(D), \quad \mathfrak{u}_-(D) := (D - i)\mathfrak{r}(D),$$

and the *Cayley transform* of D is

$$\mathfrak{c}(D) := \mathfrak{u}_-(D)\mathfrak{u}_+^*(D). \tag{1.6}$$

In general, $\mathfrak{c}(D)$ is a partial isometry, with closed range. D can be recoverd from $\mathfrak{c}(D)$ by the formulas

$$\mathfrak{Dom}(D) = \mathfrak{Im}(1 - \mathfrak{c}(D))\mathfrak{c}^*(D)$$
$$D(1 - \mathfrak{c}(D))\mathfrak{c}^*(D)e = i(1 + \mathfrak{c}(D))\mathfrak{c}^*(D)e.$$

THEOREM 1.1.4.9 ([**36**]). *The Cayley transform $\mathfrak{c}$ furnishes a bijection between the set of symmetric regular operators in $\mathcal{E}$ and and the set of partial isometries $c \in \mathrm{End}_B^*(\mathcal{E})$ with the property that $(1 - c)c^*$ has dense range. Moreover, D' is an extension of D if and and only if $\mathfrak{c}(D')$ is an extension of $\mathfrak{c}(D)$.*

For a selfadjoint regular operator D, $1+D^2$ has dense range.Therefore by lemma 1.1.4.8, the operators $D + i$ and $D - i$ are bijective.

COROLLARY 1.1.4.10. *A symmetric regular operator D is selfadjoint if and only if $\mathfrak{c}(D)$ is unitary.*

1.5. Approximate projectivity. The work of Blecher [**7**] provides a metric description of C^*-modules which is useful in extending the theory to non C^*-algebras. We will discuss some of his work on these extensions in section 3, whereas in the present section we will discuss the characterization of C^*-modules as "approximately projective" modules.

For a countably generated C^*-A-module $\mathcal{E}$, the algebra $\mathbb{K}_A(\mathcal{E})$ has a countable approximate unit $\{u_\alpha\}_{\alpha\in\mathbb{N}}$ consisting of elements in $\mathrm{Fin}_A(\mathcal{E})$. Replacing u_α by $u_\alpha^* u_\alpha$ if necessary, we may assume

$$u_\alpha = \sum_{i=1}^{n_\alpha} x_i^\alpha \otimes x_i^\alpha.$$

For each n_α we get operators $\phi_\alpha \in \mathbb{K}_A(\mathcal{E}, A^{n_\alpha})$, defined by

$$\phi_\alpha : e \mapsto \sum_{i=1}^{n_\alpha} e_i \langle x_i^\alpha, e\rangle, \tag{1.7}$$

where e_i denotes the standard basis of A^{n_α}.We have

$$\phi_\alpha^* : x \mapsto \sum_{i=1}^{n_\alpha} x_i^\alpha \langle e_i, x\rangle, \tag{1.8}$$

and hence $\phi_\alpha^* \circ \phi_\alpha \to \mathrm{id}_\mathcal{E}$ pointwise. This structure determines the $\mathcal{E}$ completely as a C^*-module.

Theorem 1.1.5.1 ([**7**]). *Let A be a C^*-algebra and $\mathcal{E}$ be a Banach space which is also a right A-module. $\mathcal{E}$ is a countably generated C^*-module if and only if there exists a sequence $\{n_\alpha\}$ of positive integers and contractive module maps*

$$\phi_\alpha : \mathcal{E} \to A^{n_\alpha}, \quad \psi_\alpha : A^{n_\alpha} \to \mathcal{E},$$

such that $\psi_\alpha \circ \phi_\alpha$ converges pointwise to the identity on $\mathcal{E}$. In this case the inner product on $\mathcal{E}$ is given by

$$\langle e, f\rangle = \lim_{\alpha\to\infty} \langle \phi_\alpha(e), \phi_\alpha(f)\rangle.$$

For this reason we can think of C^*-modules as approximately finitely generated projective modules. Also note that the maps ϕ_α, ψ_α are by no means unique, and that different maps can thus give rise to the same inner product on $\mathcal{E}$.

2. KK-theory

Kasparov's bivariant K-theory KK [**32**] has become a central tool in noncommutative geometry since its creation. It is a bifunctor on pairs of C^*-algebras, associating to (A, B) a $\mathbb{Z}/2\mathbb{Z}$-graded group $KK_*(A, B)$. It unifies K-theory and K-homology in the sense that

$$KK_*(\mathbb{C}, B) \cong K_*(B) \text{ and } KK_*(A, \mathbb{C}) \cong K^*(A).$$

Much of its usefulness comes from the existence of internal and external product structures, by which KK-elements induce homomorphisms between K-theory and K-homology groups. In Kasparov's original approach, the definition and computation of the products is very complicated. In order to simplify the external product, Baaj and Julg [**4**] introduced another model for KK, in which the external product is given by a simple algebraic formula. The price one has to pay is working with unbounded operators. We will describe both models, and their relation, together with some results on the structure of KK as a category.

2.1. The bounded picture. The main idea behind Kasparov's approach to K-homology and KK-theory is that of a family of abstract elliptic operators. This was an idea pioneered by Atiyah, in his construction of K-homology for spaces and the family index theorem. We will consider bimodules $A \to \mathcal{E} \leftrightharpoons B$, without assuming the action of A to be essential, nor the inner product being full.

DEFINITION 1.2.1.1. For $p \in \mathbb{N}$, denote by $\mathbb{C}_p$ the complex unital graded algebra generated by symbols ε_j, $j = 1, \ldots, n$, of degree 1, satisfying the following relations:

$$\varepsilon_j^* = -\varepsilon_j, \quad \varepsilon_j^2 = -1, \quad [\varepsilon_i, \varepsilon_j] = 0.$$

Here we assume $i \neq j$, and the commutator is graded.

The algebra $\mathbb{C}_p$ is generated by the 2^n monomials $\varepsilon_{j_1} \ldots \varepsilon_{j_k}$, $0 \leq k \leq n$ and $j_1 < \cdots < j_k$. Considering these monomials as an orthonormal basis, the left regular representation of $\mathbb{C}_p$ on itself equips it with a C^*-norm. It is a well known fact that $\mathbb{C}_{p+2} \cong M_2(\mathbb{C}_p)$. This is sometimes referred to as *formal Bott periodicity.*

DEFINITION 1.2.1.2. Let $A \to \mathcal{E} \leftrightharpoons B$ be a graded bimodule and $F \in \mathrm{End}_B^*(\mathcal{E})$ an odd operator. $(\mathcal{E}, F)$ is a *Kasparov* (A, B)*-bimodule* if, for all $a \in A$,

- $[F, a], a(F^2 - 1), a(F - F^*) \in \mathbb{K}_B(\mathcal{E})$.

We denote by $\mathbb{E}_j(A, B)$ the set of Kasparov modules for $(A, B\tilde{\otimes}\mathbb{C}_j)$ modulo unitary equivalence. Unitary equivalence is defined by the existence of a unitary intertwining the action of the algebras and the operators. An *ungraded* C^*-module $\mathcal{E} \leftrightharpoons B$ equipped with a left action of A and an operator F satisfying the relations from definition 1.2.1.2 defines an element $[(\mathcal{E}', F)] \in \mathbb{E}_1(A, B)$. This is done by setting

$$\mathcal{E}' := \mathcal{E} \oplus \mathcal{E}, \quad \gamma := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad F' := \begin{pmatrix} 0 & F \\ F & 0 \end{pmatrix}, \quad \varepsilon_1 \mapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{1.9}$$

Here ε_1 is the generator of the Clifford algebra $\mathbb{C}_1$. Ungraded modules of this kind are therefore referred to as *odd* Kasparov modules.

The set of *degenerate* elements consists of bimodules for which

$$\forall a \in A : [F, a] = a(F^2 - 1) = a(F - F^*) = 0.$$

Denote by $e_i : C[0,1]\overline{\otimes}B \to B$ the evalution map at $i \in [0,1]$. Two Kasparov (A, B)-bimodules $(\mathcal{E}_i, F_i) \in \mathbb{E}_j(A, B)$, $i = 0, 1$ are *homotopic* if there exists a Kasparov $(A, C[0,1]\overline{\otimes}B)$-module $(\mathcal{E}, F) \in \mathbb{E}_j(A, C[0,1] \otimes B)$ for which $(\mathcal{E} \otimes_{e_i} B, F \otimes 1)$ is unitarily equivalent to $(\mathcal{E}_i, F_i)$, $i = 0, 1$. It is an equivalence relation, denoted $\sim$. Define

$$KK_j(A, B) := \mathbb{E}_j(A, B)/\sim .$$

KK_j is a bifunctor, contravariant in A, covariant in B, taking values in abelian groups. It is not hard to show that $KK_*(\mathbb{C}, A)$ and $KK_*(A, \mathbb{C})$ are naturally isomorphic to the K-theory and K-homology of A, respectively. Moreover, Kasparov proved the following deep theorem.

THEOREM 1.2.1.3 ([**32**]). *For any C^*-algebras A, B, C there exists an associative bilinear pairing*

$$KK_i(A,B) \otimes_{\mathbb{Z}} KK_j(B,C) \xrightarrow{\otimes_B} KK_{i+j}(A,C).$$

Therefore, the groups $KK_*(A,B)$ are the morphism sets of a category KK whose objects are all C^*-algebras.

The standard module A^n, viewed as an $(M_n(A), A)$-bimodule, defines an invertible element for the Kasparov product. Hence, in both variables, the KK-groups of A and $M_n(A)$ are isomorphic. Combining this with formal Bott periodicity yields a natural isomorphism $KK_i(A,B) \xrightarrow{\sim} KK_{i+2}(A,B)$. It follows that KK-theory can be defined using just $\mathbb{E}_0$ and $\mathbb{E}_1$. Moreover KK_1 can be defined using just odd (that is, ungraded) Kasparov modules. Because of this result we will refer to elements of $\mathbb{E}_0(A,B)$ as *even* Kasparov modules. There also is a notion of external product in KK-theory.

THEOREM 1.2.1.4 ([**32**]). *For any C^*-algebras A, B, C, D there exists an associative bilinear pairing*

$$KK_i(A,C) \otimes_{\mathbb{Z}} KK_j(B,D) \xrightarrow{\overline{\otimes}} KK_{i+j}(A\overline{\otimes}B, C\overline{\otimes}D).$$

The external product makes KK into a symmetric monoidal category

The main result of this chapter is the construction of a category that lifts the structure of the the above theorem to the level of cycles. This only works if one works with unbounded cycles, which are to be introduced shortly.

2.2. The unbounded picture. Developing KK-theory using unbounded operators has the advantage that the theory becomes more algebraic in nature. This happens, of course, at the expense of the difficulties introduced by working with regular operators. These problems can mostly be solved at a general level, and one need not worry about them when dealing with concrete examples.

DEFINITION 1.2.2.1. Let $A \to \mathcal{E} \leftrightarrows B$ be a graded bimodule and $D : \mathfrak{Dom} D \to \mathcal{E}$ an odd regular operator. $(\mathcal{E}, D)$ is an *unbounded (A,B)-bimodule* if, for all $a \in \mathcal{A}$, a dense subalgebra of A

- $[D, a]$, extends to an adjointable operator in $\mathrm{End}^*_B(\mathcal{E})$
- $a\mathfrak{r}(D) \in \mathbb{K}_B(\mathcal{E})$.

Denote the set of unbounded bimodules for $(A, B\tilde{\otimes}\mathbb{C}_i)$ modulo unitary equivalence by $\Psi_i(A,B)$. An ungraded module equipped with an operator satisfying the relations from definition 1.2.2.1 is called an *odd* unbounded (A,B)-bimodule. As in the bounded case, they define elements in $\Psi_1(A,B)$, by replacing F with D in 1.9. As in the bounded case, we will refer to elements of Ψ_0 as *even* unbounded bimodules. In [**4**] it is shown that $(\mathcal{E}, \mathfrak{b}(D))$ is a Kasparov bimodule, and that every element in $KK_*(A,B)$ can be represented by an unbounded bimodule. The motivation for introducing unbounded modules is the following result.

THEOREM 1.2.2.2 ([**4**]). *Let $(\mathcal{E}_i, D_i)$ be unbounded bimodules for (A_i, B_i), $i = 1,2$. The operator*

$$D_1 \otimes 1 + 1 \otimes D_2 : \mathfrak{Dom} D_1 \otimes \mathfrak{Dom} D_2 \to \mathcal{E} \otimes \mathcal{F},$$

extends to a selfadjoint regular operator with compact resolvent. Moreover, the diagram

$$\begin{array}{ccc} \Psi_i(A_1,B_1)\times\Psi_j(A_2,B_2) & \longrightarrow & \Psi_{i+j}(A_1\overline{\otimes}A_2, B_1\overline{\otimes}B_2) \\ \Big\downarrow \mathfrak{b} & & \Big\downarrow \mathfrak{b} \\ KK_i(A_1,B_1)\times KK_j(A_2,B_2) & \xrightarrow{\overline{\otimes}} & KK_{i+j}(A_1\overline{\otimes}A_2, B_1\overline{\otimes}B_2) \end{array}$$

commutes.

Consequently, we can *define* the external product in this way, using unbounded modules, and this is what we will do. Note that lemma 1.1.4.4 can be used to show that the resolvent of the operator $D_1\otimes 1 + 1\otimes D_2$ is compact. Indeed, writing $s = D_1\tilde{\otimes}1$ and $t = 1\tilde{\otimes}D_2$, we have $[s,t]=0$, i.e. s and t anticommute, and hence

$$\mathfrak{Dom}(s+t) = \mathfrak{Dom}s\cap\mathfrak{Dom}t, \quad 1+(s+t)^2 = 1+s^2+t^2, \quad [s^2,t^2]=0.$$

Now

$$(2+s^2+t^2)^{-1} = \int_0^\infty e^{-x(2+s^2+t^2)}dx = \int_0^\infty e^{-x(1+s^2)}e^{-x(1+t^2)}dx,$$

and $e^{-x(1+s^2)}e^{-x(1+t^2)} = e^{-x(1+D_1^2)}\otimes e^{-x(1+D_2^2)}$ is compact because both the $e^{-x(1+D_i^2)}$ are. Hence by lemma 1.1.4.4, $(2+s^2+t^2)^{-1}$ is a limit of compact operators, which is compact.

In [**35**], Kucerovsky gives sufficient conditions for an unbounded module $(\mathcal{E}\tilde{\otimes}_A\mathcal{F}, D)$ to be the internal product of $(\mathcal{E},S)$ and $(\mathcal{F},T)$. For each $e\in\mathcal{E}$, we have an operator

$$\begin{aligned} T_e : \mathcal{F} &\to \mathcal{E}\tilde{\otimes}_A\mathcal{F} \\ f &\mapsto e\otimes f. \end{aligned}$$

It's adjoint is given by $T_e^*(e'\otimes f) = \langle e,e'\rangle f$. We also need the concept of *semi-boundedness* which carries over from the Hilbert space setting.

Definition 1.2.2.3 ([**35**]). Let D be a symmetric operator in a C^*-module $\mathcal{E}\leftrightarrows B$. D is *semi-bounded below* if there exists a real number κ such that $\langle De,e\rangle \geq \kappa\langle e,e\rangle$. If $\kappa\geq 0$, D is *form-positive.*

It is immediate that D is semibounded below if and only if it is the sum of an operator in $\mathrm{End}_B^*(\mathcal{E})$ and a form positive operator. Kucerovsky's result now reads as follows.

Theorem 1.2.2.4 ([**35**]). *Let* $(\mathcal{E}\tilde{\otimes}_A\mathcal{F}, D)\in\Psi_0(A,C)$. *Supppose that* $(\mathcal{E},S)\in\Psi_0(A,B)$ *and* $(\mathcal{F},T)\in\Psi_0(B,C)$ *are such that*

- *For e in some dense subset of* $A\mathcal{E}$, *the operator*
$$\left[\begin{pmatrix} D & 0 \\ 0 & T\end{pmatrix}, \begin{pmatrix} 0 & T_e \\ T_e^* & 0\end{pmatrix}\right]$$
is bounded on $\mathfrak{Dom}(D\oplus T)$;
- $\mathfrak{Dom}D\subset\mathfrak{Dom}S\tilde{\otimes}1$;
- $\langle Sx,Dx\rangle + \langle Dx,Sx\rangle \geq \kappa\langle x,x\rangle$ *for all x in the domain.*

Then $(\mathcal{E}\tilde{\otimes}_A\mathcal{F}, D)\in\Psi_0(A,C)$ *represents the internal Kasparov product of* $(\mathcal{E},S)\in\Psi_0(A,B)$ *and* $(\mathcal{F},T)\in\Psi_0(B,C)$.

2.3. Categorical properties of KK. As we have seen, the Kasparov product makes KK into a category, and it is immediate that this category is additive. By an *extension* of C^*-algebras we shall mean an exact sequence of *-homomorphisms

$$0 \to A \to B \to C \to 0,$$

between C^*-algebras, such that the map $B \to C$ has a completely positive continuous linear splitting $C \to B$. The extension is said to be *split* if their exists a *-homomorphism $C \to B$ splitting it.

THEOREM 1.2.3.1 ([**32**]). *Let A, B, C, D be C^*-algebras and*

$$0 \to A \to B \to C \to 0,$$

an extension. There are natural exact sequences

$$\cdots \to KK_{i+1}(D, C) \to KK_i(D, A) \to KK_i(D, B) \to KK_i(D, C) \to KK_{i-1}(D, A) \to \cdots,$$

and

$$\cdots \to KK_{i-1}(A, D) \to KK_i(C, D) \to KK_i(B, D) \to KK_i(A, D) \to KK_{i+1}(C, D) \to \cdots.$$

If the extension is split, both sequences

$$KK_i(D, A) \to KK_i(D, B) \to KK_i(D, C),$$

and

$$KK_i(C, D) \to KK_i(B, D) \to KK_i(A, D)$$

are split exact for all i.

This result is usually referred to as *split-exactness* of the functor KK. As with Kasparov modules, we can define two C^*-algebras $A$and B to be *homotopic* if there exit *-homomorphisms $f : A \to B$ and $g : B \to A$ such that $f \circ g$ and $g \circ f$ are homotopic to the identity. Two *-homomorphisms $f, g : A \to B$are homotopic if there is a *-homomorphism $F : A \to C([0,1], B)$ such that $F(0) = f$ and $F(1) = g$. By definition, the functor KK is homotopy invariant. Recall that two separable C^*-algebras A, B are strongly Morita equivalent if and only if $\mathbb{K}\overline{\otimes}A$ and $\mathbb{K}\overline{\otimes}B$ are isomorphic. Moreover Morita equivalent C^*-algebras have isomorphic KK-groups, the isomorphism being implemented by the operation of tensoring with the equivalence bimodule. The is turn corresponds to taking the Kasparov product with the KK-element defined by the equivalence bimodule. Let $p \in \mathbb{K}$ be a rank one projection. We say a functor is *stable*, if the inclusion

$$\begin{aligned} A &\to \mathbb{K}\overline{\otimes}A \\ a &\mapsto p \otimes a, \end{aligned}$$

induces an isomorphism in K-theory. Higson proved the following deep and striking result.

THEOREM 1.2.3.2 ([**28**]). *Let $F : \mathfrak{C}^* \to \mathfrak{Ab}$ be a split exact and stable functor from C^*-algebras to abelian groups. Then F is homotopy invariant.*

In this theorem, one could replace $\mathfrak{Ab}$ with an arbitrary exact category. It turns out that KK takes in a special place amongst the split exact stable functors. The next result was anticipated by Cuntz in [**18**], and later proved by Higson.

THEOREM 1.2.3.3 ([**29**]). *KK is the universal split exact stable functor. That is, if $F : \mathfrak{C}^* \to \mathfrak{Ab}$ is any such functor, then it factors uniquely through the category KK. this means it determines a unique functor $\tilde{F} : KK \to \mathfrak{Ab}$ making the diagram*

$$\begin{tikzcd} \mathfrak{C}^* \arrow[r] \arrow[d] & \mathfrak{Ab} \\ KK \arrow[ur] & \end{tikzcd}$$

commutative.

The most striking feature of the category KK is probably that it is triangulated. This allwos for the development of homological algebra in KK, and sheds new light on the assembly maps that occur in the study of the Baum-Connes conjecture. This line of thought is pursued by Nest and Meyer [**43**].

3. Operator modules

When dealing with unbounded operators, it becomes necessary to deal with dense subalgebras of C^*-algebras and modules over these. The theory of C^*-modules, which is the basis of Kasparov's approach to bivariant K-theory for C^*-algebras, needs to be extended in an appropriate way. The framework of operator spaces and the Haagerup tensor product provides with a category of modules over operator algebras which is sufficiently rich to accomodate for the phenomana occurring in the Baaj-Julg picture of KK-theory.

3.1. Operator spaces. We will frequently deal with algebras and modules that are not C^*, and with operators that are not adjointable. In this section we discuss the basic notions of the theory of operator spaces, in which all of our examples will fit. The intrinsic approach presented here was taken from [**26**]. In the classic literature, operator spaces are described using matrix norms. These are globalized to yield the approach involving compact operators given here.

DEFINITION 1.3.1.1. An *operator space* is a linear space X together with a norm $\|\cdot\|$ on the algebraic tensor product $\mathbb{K} \otimes X$ such that

- For all $b \in \mathbb{B}(\mathcal{H})$ and $v \in \mathbb{K} \otimes X$, $\max\{\|bv\|, \|vb\|\} \leq \|b\|\|v\|$,
- For all orthogonal projections $p, q \in \mathbb{K}$ and $v, w \in \mathbb{K} \otimes X$, $\|pxp + qyq\| = \max\{\|pxp\|, \|qyq\|\}$,
- For each rank one projection $p \in \mathbb{K}$, X is complete in the norm $\|x\| := \|p \otimes x\|$.

A linear map $\phi : X \to Y$ between operator spaces is called *completely bounded*, resp. *completely contractive*, resp. *completely isometric* if the induced map

$$1 \otimes \phi : \mathbb{K} \otimes X \to \mathbb{K} \otimes Y,$$

is bounded, resp. contractive, resp isometric.

The following theorem is very important in identifying operator spaces in practice.

THEOREM 1.3.1.2 ([**48**]). *For every operator space X there exists a Hilbert space $\mathcal{H}$ and and a complete isometry $\phi : X \to \mathbb{B}(\mathcal{H})$.*

Hence an alternative definition of an operator space is that of a complete normed space X that is isometrically isomorphic to a closed subspace of a C^*-algebra. The (unique) C^*-tensor norm on $\mathbb{K} \otimes X$ would then equip X with the structure of an operator space in the sense of definition 1.3.1.1.

EXAMPLE 1.3.1.3. Any C^*-module $\mathcal{E}$ over a C^*-algebra B is an operator space, as it is isometric to $\mathbb{K}(\mathcal{E}, B)$, which is a closed subspace of $\mathbb{K}(B \oplus \mathcal{E})$.

EXAMPLE 1.3.1.4. Let $(\mathcal{E}, D)$ be an unbounded cycle for (A, B) and $\delta : \mathcal{A} \to \mathrm{End}^*_B(\mathcal{E})$ the closed densely defined derivation $a \mapsto [D, a]$. Then $\mathcal{A}$ can be made into an operator space via

$$\pi : \mathcal{A} \to M_2(\mathrm{End}^*_B(\mathcal{E}))$$
$$a \mapsto \begin{pmatrix} a & 0 \\ \delta(a) & a \end{pmatrix}.$$

Note that, actually $\mathcal{A} \subset \mathrm{End}_B(\mathfrak{G}(D))$, but that π is not *-homomorphism. This example is tantamount in our discussion of the Kasparov product, and it is also the main example of a non-selfadjoint operator algebra.

DEFINITION 1.3.1.5. For operator spaces X, Y, Z, a bilinear map $\phi : X \times Y \to Z$ is called completely bounded, resp. completely contractive, resp. completely isometric if the operator

$$\begin{array}{rcl} \mathbb{K} \otimes X \times \mathbb{K} \otimes Y & \to & \mathbb{K} \otimes Z \\ (m \otimes x, n \otimes y) & \mapsto & (mn \otimes \phi(x, y)), \end{array} \tag{1.10}$$

is bounded, resp. contractive, resp. isometric.
An *operator algebra* is an operator space $\mathcal{A}$ with a completely contractive multiplication $m : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$. An *operator module* over an operator algebra $\mathcal{A}$ is an operator space X with a completely contractive $\mathcal{A}$-module structure $X \times \mathcal{A} \to X$.

Of course, C^*-algebras and -modules are examples that fit this definition. The module $\mathfrak{G}(D) \subset \mathcal{E} \oplus \mathcal{E}$ from example 2.3.2.1 is a (left)-operator module over $\mathcal{A}$. The natural choice of morphisms between operator modules are the completely bounded module maps. If E and F are operator modules over an operator algebra $\mathcal{A}$, we denote the set of these maps by $\mathrm{Hom}^c_{\mathcal{A}}(E, F)$.

3.2. The Haagerup tensor product. For operator spaces X and Y, one can define their spatial tensor product $X \otimes Y$ as the norm closure of the algebraic tensor product in the spatial tensor product of some containing C^*-algebras. This gives rise to an exterior tensor product of operator modules.

The internal tensor product of C^*-modules is an example of the Haagerup tensor product for operator spaces. This tensor product will be extremely important in what follows.

DEFINITION 1.3.2.1. Let X, Y be operator spaces. The *Haagerup norm* on $\mathbb{K} \otimes X \otimes Y$ is defined by

$$\|u\|_h := \inf\{\sum_{i=0}^n \|x_i\|\|y_i\| : u = m(\sum x_i \otimes y_i), x_i \in \mathbb{K} \otimes X, y_i \in \mathbb{K} \otimes Y\}.$$

Here $m : \mathbb{K} \otimes X \otimes \mathbb{K} \otimes Y \to \mathbb{K} \otimes X \otimes Y$ is the linearization of the map 1.10.

THEOREM 1.3.2.2. *The norm on $X \otimes Y$ induced by the Haagerup norm is given by*

$$\|u\|_h = \inf\{\|x\|\|y\| : x \in X^{n+1}, y \in Y^{n+1}, u = \sum_{i=0}^{n} x_i \otimes y_i\},$$

and the completion of $X \otimes Y$ in this norm is an operator space.

This completion is denoted $X\tilde{\otimes}Y$ and is called the *Haagerup tensor product* of X and Y. By construction, the multiplication in operator algebra $\mathcal{A}$ induces a continuous map $\mathcal{A}\tilde{\otimes}\mathcal{A} \to \mathcal{A}$. A similar statement holds for operator modules.

Now suppose M is a right operator $\mathcal{A}$-module, and N a left operator $\mathcal{A}$-module. Denote by $I_{\mathcal{A}} \subset M\tilde{\otimes}N$ the closure of the linear span of the expressions $(ma \otimes n - m \otimes an)$. The *module Haagerup tensor product* of M and N over $\mathcal{A}$ is

$$M\tilde{\otimes}_{\mathcal{A}}N := M\tilde{\otimes}N/I_{\mathcal{A}},$$

equipped with the quotient norm, in which it is obviously complete. Moreover, if M also carries a left $\mathcal{B}$ operator module structure, and N a right $\mathcal{C}$ operator module structure, then $M\tilde{\otimes}_{\mathcal{A}}N$ is an operator $\mathcal{B}, \mathcal{C}$-bimodule. Graded operator algebras and -modules can be defined by the same conventions as in definition 1.1.1.6 and the discussion preceding it.. If the modules and operator algebras are graded, so are the Haagerup tensor products, again in the same way as in the C^*-case, as in the discussion around equation 1.1. The following theorem resolves the ambiguity in the notation for the interior tensor product of C^*-modules and the Haagerup tensor product of operator spaces.

THEOREM 1.3.2.3 ([**7**]). *Let $\mathcal{E}, \mathcal{F}$ be C^*-modules over the C^*-algebras A and B respectively, and $\pi : A \to \mathrm{End}^*_B(\mathcal{F})$ a nondegenrate *-homomorphism. Then the interior tensor product and the Haagerup tensor product of $\mathcal{E}$ and $\mathcal{F}$ are completely isometrically isomorphic.*

This result provides us with a convenient description of algebras of compact operators on C^*-modules. The *dual module* of a C^*-module $\mathcal{E}$ is equal to $\mathcal{E}$ as a linear space, but we equip it with a left C^*-A-module structure using the involution:

$$ae := ea^*, \quad (e_1, e_2) \mapsto \langle e_1, e_2\rangle^*.$$

THEOREM 1.3.2.4 ([**7**]). *There is a complete isometric isomorphism*

$$\mathbb{K}_A(\mathcal{E}\tilde{\otimes}\mathcal{F}) \xrightarrow{\sim} \mathcal{E}\tilde{\otimes}_A\mathbb{K}_B(\mathcal{F})\tilde{\otimes}_A\mathcal{E}^*.$$

In particular $\mathbb{K}_A(\mathcal{E}) \cong \mathcal{E}\tilde{\otimes}_A\mathcal{E}^*$.

The notion of direct sum of operator modules turns out to be a problematic issue. In the C^*-module case, the existence of a canonical inner product on direct sums prevents us from running into problems. This is one of the reasons to work with a more restricted class of modules, resembling C^*-modules in many ways.

3.3. Rigged modules. Blecher's characterization of C^*-modules as approximately finitely generated projective modules (theorem1.1.5.1) allows for a generalization of C^*-modules to non-selfadjoint operator algebras. The resulting theory is only slightly more involved than that for the C^*-case, and is exposed in [**6**]. The following definition is modelled on theorem1.1.5.1.

DEFINITION 1.3.3.1. Let $\mathcal{A}$ be an operator algebra and E a right $\mathcal{A}$-operator module. E is an $\mathcal{A}$-*rigged module* if there exists a sequence of positive integers $\{n_\alpha\}$ and completely contractive $\mathcal{A}$-module maps

$$\psi_\alpha : E \to \mathcal{A}^{n_\alpha}, \quad \phi_\alpha : \mathcal{A}^{n_\alpha} \to E,$$

such that

- ψ_α and ψ_α are compeltely contractive ;
- $\psi_\alpha \circ \phi_\alpha \to \mathrm{id}_E$ strongly on E ;
- ψ_α is $\mathcal{A}$-essential ;
- $\forall \beta : \phi_\beta \circ \psi_\alpha \circ \phi_\alpha \to \phi_\beta$ uniformly.

Subsequently define the *dual rigged module* of E by

$$E^* := \{e^* \in \mathrm{Hom}^c_{\mathcal{A}}(E, \mathcal{A}) : e^* \circ \psi_\alpha \circ \phi_\alpha \to e^*\},$$

and the algebra of $\mathcal{A}$-*compact operators* as $\mathbb{K}_{\mathcal{A}}(E) := E \tilde{\otimes}_{\mathcal{A}} E^*$.

It is immediate from this definition that $E^* = \mathbb{K}_{\mathcal{A}}(E, , \mathcal{A})$. A rigged module can be viewed as the direct limit of the spaces $\mathcal{A}^{n_\alpha}$, by letting the transition maps $t_{\alpha\beta} : A^{n_\beta} \to A^{n_\alpha}$ be defined as $t_{\alpha\beta} := \psi_\alpha \circ \phi_\beta$. As such it has the following universal property: If completely contractive module maps $g_\alpha : \mathcal{A}^{n_\alpha} \to W$ into some operator space are given, satisfying $g_\alpha t_{\alpha\beta} \to g_\beta$, then there is a unique completely contractive morphism $g : E \to W$.

Emphasizing both the absence of a genuine inner product and the similarites with C^*-modules, Blecher choose to revive Rieffel's terminology of rigged modules. Instead of an inner product, we do have at our disposal the duality pairing $E \times E^* \to \mathcal{A}$. By abuse of notation, we will denote this pairing by $(e, e^*) \mapsto \langle e, e^* \rangle$. Rigged modules can be characterized using this pairing, yielding a description that is closer to the direct definition of a C^*-module.

THEOREM 1.3.3.2. *Suppose $\mathcal{A}, \mathcal{B}$ are operator algebras, E a $(\mathcal{B}, \mathcal{A})$-operator bimodule and $\tilde{E}$ an $(\mathcal{A}, \mathcal{B})$-operator bimodule. Suppose there exist completely contractive pairings $E \times \tilde{E} \to \mathcal{B}$ and $\tilde{E} \times E \to \mathcal{A}$, such that $\langle e, \tilde{e} \rangle f = e \langle \tilde{e}, f \rangle$ and $\langle \tilde{e}, e \rangle \tilde{f} = \tilde{e} \langle e, \tilde{f} \rangle$. If $\mathcal{B}$ has an approximate identity of the form*

$$u_\beta = \sum_{i=0}^{n_\beta} \langle x_i^\beta, \tilde{x}_i^\beta \rangle, \quad \|(x_i^\beta)\| \leq 1, \quad \|(\tilde{x}_i^\beta)\| \leq 1.$$

Then E is a right $\mathcal{A}$-rigged module, $\mathcal{B} \cong \mathbb{K}_{\mathcal{A}}(E)$, and $\tilde{E} \cong E^*$. Moreover every right $\mathcal{A}$-rigged module arises in this way.

This description will be the one useful for us in dealing with unbounded bivariant K-theory. There is an analogue of adjointable operators on rigged modules. Their defintion is straightforward.

DEFINITION 1.3.3.3. A completely bounded operator $T : E \to F$ between rigged modules is called *adjointable* if there exists an operator $T^* : F^* \to E^*$ such that

$$\forall e \in E, f^* \in F^* : \quad \langle Te, f^* \rangle = \langle e, T^* f^* \rangle.$$

The space of adjointable operators from E to F is denoted $\mathrm{End}^*_{\mathcal{A}}(E, F)$.

The compact and adjointable operators satisfy the usual relation $\mathrm{End}^*_{\mathcal{A}}(E) = \mathscr{M}(\mathbb{K}_{\mathcal{A}}(E))$, where $\mathscr{M}$ denotes the multiplier algebra. The direct sum of rigged modules is canonically defined. If $(E, \psi^E_\alpha, \phi^E_\alpha)$ and $(F, \psi^F_\alpha, \phi^F_\alpha)$ are rigged modules, $(E \oplus F, \psi^E_\alpha \oplus \psi^F_\alpha, \phi^E_\alpha \oplus \phi^E_\alpha)$ equips $E \oplus F$ with the structure of a rigged module. For the constrcution of general infinite direct sums, see [**6**]. As can be expected from theorem 1.3.2.3, the Haagerup tensor product of rigged modules behaves like the interior tensor product of C^*-modules.

THEOREM 1.3.3.4. *Let E be a right $\mathcal{A}$-rigged module and F an $(\mathcal{A}, \mathcal{B})$ rigged bimodule. Then $E\tilde{\otimes}_{\mathcal{A}}F$ is a $\mathcal{B}$-rigged module and $\mathbb{K}_{\mathcal{B}}(E\tilde{\otimes}_{\mathcal{A}}F) \cong E\tilde{\otimes}_{\mathcal{A}}\mathbb{K}_{\mathcal{B}}(F)\tilde{\otimes}E^*$.*

If $\mathcal{B} = B$ happens to be a C^*-algebra, then $E\tilde{\otimes}_{\mathcal{A}}\mathscr{F}$ is a C^*-module. The rigged structure on $E\tilde{\otimes}_{\mathcal{A}}\mathscr{F}$ can be implemented by the approximate unit

$$\sum_{i,j=1}^{n_\alpha, n_\beta} e^\alpha_i \otimes f^\beta_j \otimes f^\beta_j \otimes \tilde{e}^\alpha_i,$$

where

$$\sum_{i=1}^{n_\alpha} e^\alpha_i \otimes \tilde{e}^\alpha_i \quad \text{and} \quad \sum_{j=1}^{n_\beta} f^\beta_j \otimes f^\beta_j,$$

are approximate units for $\mathbb{K}_{\mathcal{A}}(E)$ and $\mathbb{K}_B(\mathscr{F})$, respectively. The inner product on $E\tilde{\otimes}_{\mathcal{A}}\mathscr{F}$ is then given by

$$\begin{aligned} \langle e \otimes f, e' \otimes f' \rangle : &= \lim_{\alpha,\beta} \sum_{i,j=1}^{n_\alpha, n_\beta} \langle \langle \tilde{e}^\alpha_i, e\rangle f, f^\beta_j \rangle \langle f^\beta_j, \langle \tilde{e}^\alpha_i, e\rangle f\rangle \\ &= \lim_{\alpha} \sum_{i=1}^{n_\alpha} \langle \langle \tilde{e}^\alpha_i, e\rangle f, \langle \tilde{e}^\alpha_i, e\rangle f\rangle. \end{aligned} \tag{1.11}$$

In this way one constructs C^*-modules from noninvolutive representations $\mathcal{A} \to \mathrm{End}^*_B(\mathscr{F})$.

EXAMPLE 1.3.3.5 (The standard module). Let $\mathscr{H}$ be an infinite dimensional separable Hilbert space and $\mathcal{A}$ an operator algebra. Then the $\mathscr{H}_{\mathcal{A}} := \mathscr{H}\tilde{\otimes}\mathcal{A}$ is the *standard rigged module* over $\mathcal{A}$.

The Haagerup tensor product can be used to define a notion of projective rigged module, which in the finitely generated case coincides with the usual algebraic notion of projectivity. This notion is different from Connes topological projective modules [**11**], but the definition is completely analoguous.

DEFINITION 1.3.3.6. Let E be a rigged module over an operator algebra $\mathcal{A}$. E is a *projective* rigged module if there exists a Hilbert space space $\mathscr{H}$ such that E is completely isometrically isomorphic to a direct summand in $\mathscr{H}\tilde{\otimes}\mathcal{A}$.

Such an E has the usual properties of a projective object in a category. We will state one of them.

Proposition 1.3.3.7. *An $\mathcal{A}$-rigged module P is projective if and only if any diagram of completely bounded $\mathcal{A}$-module maps*

$$\begin{tikzcd} & P \arrow[d, "\phi"'] \\ M \arrow[r, "\psi"] & N \end{tikzcd}$$

such that ψ admits a completely bounded linear splitting, can be completed to a diagram

$$\begin{tikzcd} & P \arrow[d, "\phi"] \arrow[dl, "\chi"'] \\ M \arrow[r, "\psi"] & N. \end{tikzcd}$$

Proof. $\Rightarrow$ Let Q be such that $P \oplus Q \cong \mathcal{H}\tilde{\otimes}\mathcal{A}$ and replace ϕ, ψ by $\phi \oplus \mathrm{id} : P \oplus Q \to M \oplus Q$ and $\psi \oplus \mathrm{id} : M \oplus Q \to N \oplus Q$. Then the hypotheses on these maps are still valid, and we can define

$$\begin{aligned} \mathcal{H}_{\mathcal{A}} &\to M \oplus Q \\ e_\alpha &\mapsto \psi^{-1} \circ \phi(e_\alpha), \end{aligned}$$

where e_α is a basis for $\mathcal{H}$. This fills in the diagram.
$\Leftarrow$ If any such diagram can be filled in, we chose $N = P$ and $M = \mathcal{H}\tilde{\otimes}\mathcal{A}$, where $\mathcal{H} = \ell^2(X)$, where X is a generating set for P. $\square$

4. Smoothness

There are several definitions of smoothness to be found in the literature. We adopt the philosophy that a smooth structure on a C^*-algebra should come from a spectral triple (or, equivalently, from an unbounded bimodule). The most important features of a smooth subalgebra are stability under holomorphic functional calculus, implying K-equivalence, and smooth functional calculus for selfadjoint elements. We will show our smooth algebras satisfy these properties. Moreover, we give sufficient conditions for an unbounded module to define a smooth structure. Subsequently, we turn to the notion of a smooth C^*-module over a C^*-algebra equipped with a smooth structure.

4.1. Smooth algebras. The following notion of smoothness will be used. It is slightly more general than Connes' notion of smoothness for spectral triples.

Definition 1.4.1.1. Let $\mathcal{E}$ be an unbounded (A, B)-bimodule and view the Sobolev modules $\mathcal{E}_i$ as submodules of $\mathcal{E}$. $\mathcal{E}$ is said to be *(left) smooth* if the subalgebra

$$\mathscr{A} := \bigcap_{i=0}^{\infty} \{a \in A : [D_i, a] \in \mathrm{End}^*_B(\mathcal{E}_i)\},$$

is dense in A.

If $\mathscr{A} \subset \mathfrak{Dom}^{\infty}(\mathrm{ad}D)$, the bimodule will be referred to as being *naively smooth.* Recall that Connes [**12**] calls a spectral triple $(\mathscr{A}, \mathcal{H}, D)$ smooth if both $\mathscr{A}$ and $[D, \mathscr{A}]$ are in $\mathfrak{Dom}^{\infty}\mathrm{ad}|D|$.

LEMMA 1.4.1.2. *Let $(\mathcal{E}, D)$ be an unbounded (A, B)-bimodule. Then for $(\mathcal{E}, D)$ to be smooth it suffices that it be naively smooth or smooth in the sense of Connes.*

PROOF. Since naive smoothness implies smoothness in the sense of Connes, we show that the latter implies smoothness in the sense of definition 1.4.1.1. Note that for any unbounded regular operator S, $\mathfrak{Dom}S = \mathfrak{Dom}|S|$ as C^*-modules, since $\mathfrak{r}(S) = \mathfrak{r}(|S|)$. Connes conditions assure that $\mathscr{A} \to \mathrm{End}^*_B(\mathfrak{Dom}|D|^i)$, and hence the module is smooth in our sense. □

Denote by $\pi_i : \mathscr{A} \to \mathrm{End}^*_B(\mathcal{E}_i)$ the representations

$$a \mapsto \begin{pmatrix} a & 0 \\ [D_i, a] & a \end{pmatrix}.$$

Let $\mathcal{A}_i$ be the closure of $\mathscr{A}$ in the norm inherited via π_i. It is clear that the $\mathcal{A}_i$ are operator algebras and that $\mathscr{A}$ equals their inverse limit. Hence it carries a Fréchet topology.

PROPOSITION 1.4.1.3. *The inclusions $\mathcal{A}_{i+1} \to \mathcal{A}_i$ are completely contractive.*

PROOF. We have to show that for all $a \in \mathcal{A}_i$, for all n, $\|\pi^n_i(a)\| \leq \|\pi^n_{i+1}(a)\|$, where $\pi^n_i : M_n(\mathcal{A}_i) \to \mathrm{End}^*_B(\bigoplus_{j=1}^n \mathcal{E}_i)$ and

$$\begin{aligned} \pi^n_{i+1} : M_n(\mathcal{A}_{i+1}) &\to \mathrm{End}^*_B(\bigoplus_{j=1}^{2n} \mathcal{E}_i) \\ (a_{kn}) &\mapsto (\pi_i(a_{km})). \end{aligned}$$

Denote by $\iota : \mathcal{E}_i \to \mathcal{E}_i \oplus \mathcal{E}_i$ the inclusion in the first coordinate, and by $p : \mathcal{E}_i \oplus \mathcal{E}_i \to \mathcal{E}_i$ the projection on the first coordinate. Set

$$\iota_n := \oplus^n_{j=1}\iota : \bigoplus_{j=1}^{n} \mathcal{E}_i \to \bigoplus_{j=1}^{2n} \mathcal{E}_i, \quad p_n := \oplus^n_{j=1}p : \bigoplus_{j=1}^{2n} \mathcal{E}_i \to \bigoplus_{j=1}^{n} \mathcal{E}_i.$$

Then we have

$$\|\pi^n_i(a)\| = \|p_n\pi^n_{i+1}(a)\iota_n\| \leq \|p\|\|\pi^n_{i+1}(a)\|\|\iota_n\| = \|\pi^n_{i+1}(a)\|,$$

as desired. □

Now we turn to spectral invariance of the $\mathcal{A}_i$. The following definition is a modification of [**5**], definition 3.11:

DEFINITION 1.4.1.4. Let $\mathscr{A}$ be an algebra with Banach norm $\|\cdot\|$. A norm $\|\cdot\|_\alpha$ on $\mathscr{A}$ is said to be *analytic* with respect to $\|\cdot\|$ if for each $x \in \mathcal{A}$, with $\|x\| < 1$ we have

$$\limsup_{n\to\infty} \frac{\ln \|x^n\|_\alpha}{n} \leq 0.$$

The reason for introducing the concept of analyticity is that analytic inclusions are spectral invariant.

PROPOSITION 1.4.1.5 ([**5**]). *Let* $\mathcal{A}_\beta \to \mathcal{A}_\alpha$ *be a continuous dense inclusion of unital Banach algebras. If* $\|\cdot\|_\beta$ *is analytic with respect to* $\|\cdot\|_\alpha$*, then for all* $a \in \mathcal{A}_\beta$ *we have* $\mathrm{Sp}_\beta(a) = \mathrm{Sp}_\alpha(a)$*.*

PROOF. It suffices to show that if $x \in \mathcal{A}_\beta$ is invertible in $\mathcal{A}_\alpha$, then $x^{-1} \in \mathcal{A}_\beta$. To this end choose $y \in \mathcal{A}_\beta$ with $\|x^{-1} - y\| < \frac{1}{2\|x\|_\alpha}$. Then $\|2 - 2xy\|_\alpha < 1$. By analyticity, there exists n such that $\|(2-2xy)^n\|_\beta < 1$, and hence $2 \notin \mathrm{Sp}_\beta(2-2xy)$. But then $0 \notin \mathrm{Sp}_\beta(2xy)$, hence $2xy$ has an inverse $u \in \mathcal{A}_\beta$. Therefore $x^{-1} = 2yu$. $\square$

In order to prove spectral invariance of the inclusions $\mathcal{A}_{i+1} \to \mathcal{A}_i$ we need the following straightforward result, whose proof we include for the sake of completeness.

LEMMA 1.4.1.6. *Let* $\mathcal{A}$ *be a Banach algebra and* $\delta : \mathcal{A}_\alpha \to M$ *a densely defined closed derivation into a Banach* $\mathcal{A}$*-module* M*. Then* $\|a\|_\alpha := \|a\| + \|\delta(a)\|$ *is analytic with respect to* $\|\cdot\|$*.*

PROOF. Let $\|x\| < 1$. We have $\|\delta(x^n)\| \leq n\|\delta(x)\|$, by an obvious induction. Then

$$\begin{aligned}
\limsup_{n\to\infty} \frac{\ln \|x^n\|_\alpha}{n} &= \limsup_{n\to\infty} \frac{\ln(\|x^n\| + \|\delta(x^n)\|)}{n} \\
&\leq \limsup_{n\to\infty} \frac{\ln(1 + n\|\delta(x)\|)}{n} \\
&\leq \limsup_{n\to\infty} \left(\frac{\ln n}{n} + \frac{\ln(1 + \|\delta(x)\|)}{n}\right) \\
&= 0.
\end{aligned}$$

$\square$

THEOREM 1.4.1.7. *Let* $(\mathcal{E}, D)$ *be a smooth unbounded* (A, B) *bimodule. Then all inclusions* $\mathcal{A}_{i+1} \to \mathcal{A}_i$ *are spectral invariant, and hence* $\mathscr{A}$ *and all the* $\mathcal{A}_i$ *are stable under holomorphic functional calculus in* A*.*

PROOF. Observe that

$$\|\pi_{i+1}(a)\| \leq \|\pi_i(a)\| + \|\pi_i(a) - \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\| \leq 2\|\pi_i(a)\| + \|\pi_{i+1}(a)\| \leq 3\|\pi_{i+1}(a)\|;$$

and $\|\pi_{i+1}(a) - \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\| = \|[D_i, a]\|$. Thus, by lemma 1.4.1.6, $\|\cdot\|_{i+1}$ is equivalent to a norm analytic with respect to $\|\cdot\|_i$. $\square$

In the sequel, by a *smooth structure* on a C^*-algebra A we shall mean an inverse system of operator algebras

$$\cdots \to \mathcal{A}_{i+1} \to \mathcal{A}_i \to \cdots \to A$$

where the maps are spectral invariant complete contractions with dense range. In that case , denote $\mathscr{A} = \lim_{\leftarrow} \mathcal{A}_i$. A *smooth* C^**-algebra* shall be a C^*-algebra with a smooth structure coming from a smooth unbounded bimodule.

4.2. Smooth C^*-modules. In differential geometry, a finite dimensional topological vector bundle over a smooth manifold M can always be smoothened, i.e. it can be equipped with a smooth structure. From the algebro-analytic perspective this can be understood in terms of the spectral invariance of the algebra of smooth functions $C^\infty(M) \subset C(M)$. This spectral invariance passes to the matrix algebras, and any projection $p \in M_k(C(M))$ is close to a smooth projection q. By a now standard result, the bundles defined by p and q are isomorphic, giving the smooth structure. For infinite dimensional bundles the situation is more complicated, and I am not aware of any results of this kind in this setting. Thefore we will demand our modules to be smooth.

DEFINITION 1.4.2.1. Let B be a smooth C^*-algebra, with smooth structure $\{\mathcal{B}_i\}$. A C^*-B-module $\mathcal{E}$ is a C^k-B-module, if there is an approximate unit

$$u_\alpha := \sum_{i=0}^{n_\alpha} x_i^\alpha \otimes x_i^\alpha \in \mathrm{Fin}_B(\mathcal{E}),$$

such that for each α and $0 \leq i, j \leq n_\alpha$, $\langle x_i^\alpha, x_j^\alpha \rangle \in \mathcal{B}_k$, and $\|\langle x_i^\alpha, x_j^\alpha \rangle\|_k \leq 1$. It is a *smooth* C^*-module if there is such an approximate unit that makes it a C^k-module for all k.

PROPOSITION 1.4.2.2. *Let B be a smooth C^*-algebra and $\mathcal{E}$ a smooth C^*-B-module, with corresponding approximate unit $u_\alpha := \sum_{i=0}^{n_\alpha} x_i^\alpha \otimes x_i^\alpha$. Then*

$$E_k := \{e \in \mathcal{E} : \langle x_i^\alpha, e \rangle \in \mathcal{B}_k, \quad \sup_\alpha \| \sum_{i=1}^{n_\alpha} e_i \langle x_i^\alpha, e \rangle \|_k < \infty\},$$

is a rigged $\mathcal{B}_k$-module. Moreover, the inclusions $E_{k+1} \to E_k$ are completely contractive with dense range, and $E_{k+1} \tilde{\otimes}_{\mathcal{B}_{k+1}} \mathcal{B}_k \cong E_k$.

PROOF. Recall the discussion before theorem 1.1.5.1. The maps $\phi_\alpha, \phi_\alpha^*$ of 1.7,1.8 restrict to maps

$$\phi_\alpha^k : \mathcal{A}_i^{n_\alpha} \to E_k, \quad \psi_\alpha^k : E_k \to \mathcal{B}_k^{n_\alpha}.$$

These are completely contractive for the matrix norms on E_k given by

$$\|(e_{ij})\| := \sup \|(\psi_k^\alpha(e_{ij}))\|,$$

and E_k is (by definition) complete in these matrix norms. It is straightforward to check that E_k is a rigged-$\mathcal{B}_k$-module in this way. For the last statement, the isomorphism will be implemented by the multiplication map

$$\begin{aligned} m : E_{k+1} \tilde{\otimes}_{\mathcal{B}_{k+1}} \mathcal{B}_k &\to E_k \\ e \otimes b &\mapsto eb. \end{aligned}$$

The inverse to this map is constructed via the direct limit property of E_k. Via the identification $\mathcal{A}_k^{n_\alpha} \cong \mathcal{A}_{k+1}^{n_\alpha} \tilde{\otimes}_{\mathcal{A}_{k+1}} \mathcal{A}_k$ define maps

$$\begin{aligned} m_\alpha^{-1} : \mathcal{A}_k^{n_\alpha} &\to E_{k+1} \tilde{\otimes}_{\mathcal{B}_{k+1}} \mathcal{B}_k \\ e_i &\mapsto \phi_\alpha^{k+1}(e_i) \otimes 1. \end{aligned}$$

They obviously satisfy the compatibility condition mentioned after definition 1.3.3.1 and induce a map $m^{-1} : E_k \to E_{k+1} \tilde{\otimes}_{\mathcal{B}_{k+1}} \mathcal{B}_k$, inverting m. □

If B is a smooth C^*-algebra and $(\mathcal{E}, D)$ a left smooth unbounded (A, B)-bimodule $(\mathcal{E}, D)$, the appropriate notion of smoothness is the following. For each k, the Sobolev module $\mathcal{E}_i$ is smooth over B, and, denoting the associated inverse system by $\{E_i^k\}$, the adjointable operator $D_i : \mathcal{E}_{i+1} \to \mathcal{E}_i$ restricts to an adjointable operator $D_{i,k} : E_{i+1}^k \to E_i^k$. We then require the algebra

$$\mathscr{A}_i := \bigcap_{i=0}^{\infty} \{a \in \mathcal{A}_i : [D_{i,j}, a] \in \mathrm{End}^*_{\mathcal{B}_i}(E_i^j)\},$$

to be dense in $\mathcal{A}_i$, for each i. We will call an unbounded bimodule *smooth* if it is smooth in this sense. This can be visualized by a diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] \\
\cdots \arrow[r] & E_{i+1}^{j+1} \arrow[r] \arrow[d] & E_i^{j+1} \arrow[r] \arrow[d] & E_{i-1}^{j+1} \arrow[r] \arrow[d] & \cdots \arrow[r] & E^{j+1} \arrow[d] \\
\cdots \arrow[r] & E_{i+1}^{j} \arrow[r] \arrow[d] & E_i^{j} \arrow[r] \arrow[d] & E_{i-1}^{j} \arrow[r] \arrow[d] & \cdots \arrow[r] & E^{j} \arrow[d] \\
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] \\
\cdots \arrow[r] & \mathcal{E}_{i+1} \arrow[r] & \mathcal{E}_i \arrow[r] & \mathcal{E}_{i-1} \arrow[r] & \cdots \arrow[r] & \mathcal{E}.
\end{tikzcd}$$

Here each E_i^j is a rigged $(\mathcal{A}_i^j, \mathcal{B}_j)$-bimodule. The bottom row is just the Sobolev chain of D.

4.3. Inner products and stabilization. For any operator algebra $\mathcal{A}$, a dual algebra $\mathcal{A}^*$ is defined, obtained via its realization as a non selfadjoint subalgebra of some C^*-algebra. $\mathcal{A}^*$ is the algebra of adjoints in this C^*-algebra. In general, $\mathcal{A}$ and $\mathcal{A}^*$ are not completely isometrically isomorphic. C^k-algebras do have this property, which makes working with rigged modules over them very similar to working with C^*-modules.

PROPOSITION 1.4.3.1. *Let A be a smooth C^*-algebra with smooth structure $\{\mathcal{A}_i\}$. For i, $\mathcal{A}_i \cong \mathcal{A}_i^*$ completely isometrically. In particular, the involution on $\mathcal{A}_i$ induces an anti-isomorphism of $\mathcal{A}_i$ with itself.*

PROOF. The operator algebra structure of $\mathcal{A}_i^*$ is given by

$$a \mapsto \begin{pmatrix} a & -[D, a] \\ 0 & a \end{pmatrix}.$$

This means that $\mathcal{A}^* = v\mathcal{A}v^* \subset \mathrm{End}^*_B(\mathcal{E})$, where v is the unitary $(x, y) \mapsto (-y, x)$. This isomorphism clearly extends to matrix algebras over $\mathcal{A}^*$. $\square$

Since the $\mathcal{A}_i$ are anti isomorphic to themselves, any right rigged $\mathcal{A}_i$-module has a canonically associated left rigged $\mathcal{A}_i$-module $\overline{E}$. As a linear space, this is just E with the left module structure $ae := ea^*$. The rigged structure comes from considering the modules $\mathcal{A}^{n_\alpha}$ as left modules via the same trick. The structural maps ϕ_α, ψ_α then become left-module maps having the desired properties.

Corollary 1.4.3.2. *Let $\mathcal{E}$ be a smooth C^*-module over a smooth C^*-algebra B with smooth structure $\{\mathcal{B}_i\}$. There is an isomorphism of rigged modules $\overline{E} \cong E^*$ given by restrcition of the inner product pairing on $\mathcal{E}$.*

Proof. The inner product on $\mathcal{E}$ induces an injection $\overline{E_i} \to E_i^*$. Conversely, for $f^* \in E_i^*$ we have

$$f^*(e) = \lim_\alpha \sum_{i=0}^{n_\alpha} f^*(x_i^\alpha \langle x_i^\alpha, e\rangle).$$

Thus, if we define $f := \lim_\alpha \sum_{i=0}^{n_\alpha} x_i^\alpha f^*(x_i^\alpha)$, it satifies $f^*(e) = \langle f, e\rangle$. $\square$

As a consequence, C^k-modules over a C^k-algebra can be constructed similarly to C^*-modules, by defining a nondegenerate innerproduct pairing satisfying all the properties of definition 1.1.1.1 and then completing. Stability under holomorphic functional calculus assures us that many properties of C^*-modules carry over to the smooth setting. In particular we can think of adjointable operators in the same way as we do in C^*-modules, and also the notion of unbounded regular operator makes perfect sense. Kasparov's stabilization theorem is a key tool in C^*-modules and KK-theory. There is no such result for general rigged modules over operator algebras, see [**6**], but in the case of smooth C^*-algebras the result does hold.

Theorem 1.4.3.3. *Let B be a smooth graded C^*-algebra, and $\mathcal{E}$ a countably generated smooth graded C^*-module. Then $\mathcal{E} \oplus \mathcal{H}_B$ is smoothly isomorphic to $\mathcal{H}_B$. That is, there is an isomorphism of graded inverse systems*

$$\begin{tikzcd}
\cdots \arrow[r] & E_{i+1} \oplus \mathcal{H}_{\mathcal{B}_{i+1}} \arrow[r] \arrow[d] & E_i \oplus \mathcal{H}_{\mathcal{B}_i} \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathcal{E} \oplus \mathcal{H}_B \arrow[d] \\
\cdots \arrow[r] & \mathcal{H}_{\mathcal{B}_{i+1}} \arrow[r] & \mathcal{H}_{\mathcal{B}_i} \arrow[r] & \cdots \arrow[r] & \mathcal{H}_B
\end{tikzcd}$$

Proof. The proof is based on the method of almost orthogonalization as described in [**21**]. We incorporate it in the proof. For simplicity we ignore the gradings, but note that the proof can be adapted as to respect all gradings involved. Let $u_\alpha := \sum_{i=0}^{n_\alpha} x_i^\alpha \otimes x_i^\alpha$ be an approximate unit for $\mathbb{K}_B(\mathcal{E})$ implementing the smooth structure. The x_i^α form a generating set for $\mathcal{E}$. Denote by $\{e_i\}$ the standard basis of $\mathcal{H}_B$. Let $\{x_n\} \subset \{e_n\} \cup \{x_i^\alpha\}$ be a sequence which meets all the e_n, and all the x_i^α infinitely many times. We proceed by induction. Suppose that orthonormal ellements $h_1, ..., h_n$ and the number $m(n)$ have been constructed in such a way that

- $\{h_1, ..., h_n\} \subset \mathrm{span}_{\mathscr{A}}\{x_1, ..., x_n, e_1, ..., e_{m(n)}\}$
- $d(x_k, \mathrm{span}_{\mathscr{A}}\{h_1, ..., h_n\}) \leq \frac{1}{k}, k = 1, ..., n.$

There exists $m' > m(n)$ such that $e_{m'} \perp \{x_{n+1}, h_1, ..., h_n\}$. Let

$$x' := x_{n+1} - \sum_{i=1}^{n} h_i \langle h_i, x_{n+1} \rangle, \quad x'' = x' + \frac{1}{n+1} e_{m'}.$$

Then $\langle x'', x'' \rangle = \langle x', x' \rangle + \frac{1}{(n+1)^2} > 0$, and hence this element is invertible in $\mathscr{A}$, and $\langle x'', x'' \rangle^{-\frac{1}{2}} \in \mathscr{A}$ by 1.4.1.7. Set $h_{n+1} := x'' \langle x'', x'' \rangle^{-\frac{1}{2}}$. Then

$$h_{n+1} \in \mathrm{span}_{\mathscr{A}} \{x', e_{m'}\} \perp \{h_1, ..., h_n\}.$$

Thus $\{h_1, ..., h_{n+1}\}$ is an orthonormal set. Moreover,

$$x + \frac{1}{n+1} e_{m'} \in \mathrm{span}_{\mathscr{A}} \{h_1, ..., h_{n+1}\},$$

so

$$d(x_{n+1}, \mathrm{span}_{\mathscr{A}} \{h_1, ..., h_{n+1}\}) \leq \frac{1}{n+1}.$$

Thus, by setting $m' = m(n+1)$ we complete the induction step. The sequence $\{h_i\}$ thus constructed is orthonormal and its $\mathscr{A}$ span is dense in each of the modules $E_i \oplus \mathscr{H}_{\mathcal{B}_i}$. □

5. Universal connections

In differential geometry, connections on Riemannian manifolds are a vital tool for differentiating functions and vector fields over the manifold. Cuntz and Quillen [**20**] developed a purely algebraic theory of connections on modules, which is gives a beautiful characterization of projective modules. They are exactly those modules that admit a universal connection. We review their results, but will recast everything in the setting of operator modules. This is only straightforward, because the Haagerup tensor product linearizes the multiplication in an operator algebra in a continuous way. We then proceed to construct a category of modules with connection, and finally pass to inverse systems of modules.

5.1. Universal forms. The notion of universal differential form is widely used in noncommutative geometry, especially in connection with cyclic homology [**11**]. For topological algebras, their exact definition depends on a choice of topological tensor product. The default choice is the Grothendieck projective tensor product, because it linearizes the multiplication in a topological algebra continuously. However, when dealing with operator algebras, the natural choice is the Haagerup tensor product.

Definition 1.5.1.1. Let $\mathcal{A}$ be an operator algebra. The module of *universal 1-forms* over $\mathcal{A}$ is defined as

$$\Omega^1(\mathcal{A}) := \ker(m : \mathcal{A} \tilde{\otimes} \mathcal{A} \to \mathcal{A}).$$

By definition, there is an exact sequence of operator bimodules

$$0 \to \Omega^1(\mathcal{A}) \to \mathcal{A} \tilde{\otimes} \mathcal{A} \xrightarrow{m} \mathcal{A} \to 0.$$

When $\mathcal{A}$ is graded, $\Omega^1(\mathcal{A})$ inherits a grading from $\mathcal{A} \tilde{\otimes} \mathcal{A}$. The map

$$\begin{array}{rcl} d : \mathcal{A} & \to & \Omega^1(\mathcal{A}) \\ a & \mapsto & 1 \otimes a - (-1)^{\partial a} a \otimes 1 \end{array}$$

is a graded bimodule derivation. $\Omega^1(\mathcal{A})$ carries a natural involution, defined by

$$(adb)^* := -(-1)^{\partial b} db^* a^*. \tag{1.12}$$

LEMMA 1.5.1.2. *The derivation d is universal. For any completely bounded graded derivation $\delta : \mathcal{A} \to M$ into an $\mathcal{A}$ operator bimodule, there is a unique completely bounded bimodule homomorphism $j_\delta : \Omega^1(\mathcal{A}) \to M$ such that the diagram*

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "\delta"] \arrow[dr, "d"'] & & M \\ & \Omega^1(\mathcal{A}) \arrow[ur, "j_\delta"'] & \end{tikzcd}$$

commutes.

PROOF. Set $j_\delta(da) = \delta(a)$. This determines j_δ because da generates $\Omega^1(\mathcal{A})$ as a bimodule. □

Any derivation $\delta : \mathcal{A} \to M$ has its associated module of forms

$$\Omega^1_\delta := j_\delta(\Omega^1(\mathcal{A})) \subset M.$$

The inner product on $\mathcal{E}$ induces a pairing

$$\begin{aligned} \mathcal{E} \times \mathcal{E}\tilde{\otimes}_\mathcal{A}\Omega^1(\mathcal{A}) &\to \Omega^1(\mathcal{A}) \\ (e_1, e_2 \otimes \omega) &\mapsto \langle e_1, e_2 \rangle \otimes \omega. \end{aligned}$$

By abuse of notation we write $\langle e_1, e_2 \otimes \omega \rangle$ for this pairing. A pairing

$$\mathcal{E}\tilde{\otimes}_\mathcal{A}\Omega^1(\mathcal{A}) \times \mathcal{E} \to \Omega^1(\mathcal{A}),$$

is obtained by setting $\langle e_1 \otimes \omega, e_2 \rangle := \langle e_2, e_1 \otimes \omega \rangle^*$.

DEFINITION 1.5.1.3. Let $\delta : \mathcal{A} \to M$ be a derivation as above, and E a right operator $\mathcal{A}$-module. A δ-*connection* on E is a completely bounded linear map

$$\nabla_\delta : E \to E\tilde{\otimes}_\mathcal{A}\Omega^1_\delta,$$

satifying the Leibniz rule

$$\nabla(ea) = \nabla(e)a + e \otimes \delta(a).$$

If $\delta = d$, the connection will be denoted ∇, and referred to as a *universal connection.* If moreover $\mathcal{E}$ is a C^*-module, a connection is *Hermitian* if

$$\langle e_1, \nabla(e_2) \rangle - \langle \nabla(e_1), e_2 \rangle = d\langle e_1, e_2 \rangle.$$

Note that a universal connection ∇ on a module E gives rise to δ-connections for any completely bounded derivation δ, simply by setting $\nabla_\delta := 1 \otimes j_\delta \circ \nabla$. If δ is of the form $\delta(a) = [S, a]$, for $S \in \mathrm{End}_\mathbb{C}(X, Y)$, where X and Y are left $\mathcal{A}$-operator modules, we write simply ∇_S for ∇_δ.

Not all modules admit a universal connection. Cuntz and Quillen showed that universal connections characterize algebraic projectivity. Their proof shows that projective rigged modules admit universal connections, but the class of modules admitting a connection might be larger. for our purposes however, this is sufficient.

PROPOSITION 1.5.1.4 ([**20**]). *A right $\mathcal{A}$ operator module E admits a universal connection if and only if the multiplication map $m : E\tilde{\otimes}\mathcal{A} \to E$ is $\mathcal{A}$-split.*

PROOF. Consider the exact sequence

$$0 \longrightarrow E\tilde{\otimes}_{\mathcal{A}}\Omega^1(\mathcal{A}) \xrightarrow{j} E\tilde{\otimes}\mathcal{A} \xrightarrow{m} E \longrightarrow 0,$$

where m is the multiplication map and $j(s \otimes da) = sa \otimes 1 - s \otimes a$. A linear map

$$s : E \to E\tilde{\otimes}\mathcal{A}$$

determines a linear map

$$\nabla : E \to E\tilde{\otimes}_{\mathcal{A}}\Omega^1(\mathcal{A})$$

by the formula $s(e) = e \otimes 1 - j(\nabla(e))$, since j is injective. Now

$$s(ea) - s(e)a = j(\nabla(e)a + e \otimes da - \nabla(ea)),$$

whence s being an $\mathcal{A}$-module map is equivalent to ∇ being a connection. $\square$

COROLLARY 1.5.1.5. *A C^*-module $\mathcal{E} \leftrightharpoons A$ admits a Hermitian connection.*

PROOF. By the stabilization theorem 1.1.2.2 $\mathcal{E}$ is an orthogonal direct summand in $\mathcal{H}_A = \mathcal{H}\tilde{\otimes}A$, i.e. $\mathcal{E} = p\mathcal{H}_a$, with $p^2 = p^* = p \in \mathrm{End}^*(\mathcal{H}_A)$. Observe that $\mathcal{H}_A\tilde{\otimes}_A\Omega^1 A \cong \mathcal{H}\tilde{\otimes}\Omega^1(A)$.The Levi-Cevita connection

$$\nabla : \mathcal{H}_A \to \mathcal{H}\tilde{\otimes}\Omega^1(A)$$
$$h \otimes a \mapsto h \otimes da,$$

is clearly Hermitian, and since p is a projection, so is $p\nabla p : \mathcal{E} \to \mathcal{E}\tilde{\otimes}_A\Omega^1(A)$. $\square$

5.2. Inverse systems and smoothness. As we have seen in corollary 1.1.4.7, an unbounded operator can be viewed as a morphism of inverse systems of C^*-modules, namely its Sobolev chain.

DEFINITION 1.5.2.1. Let $\{E_i, \phi_i\}$ be an inverse system of $\mathcal{A}$ rigged modules. A *connection* on $\{E_i, \phi_i\}$ is a family of connections $\nabla_i : E_i \to E_i\tilde{\otimes}\Omega^1(\mathcal{A})$ such that $\phi_{i+1} \otimes 1 \circ \nabla_{i+1} = \nabla_i \circ \phi_{i+1}$.

DEFINITION 1.5.2.2. Let $(\mathcal{E}, D)$ be an unbounded bimodule and $\nabla : \mathcal{E} \to \mathcal{E}\tilde{\otimes}_B\Omega^1(B)$ a Hermitian connection. ∇ is said to be a *D-connection* if $[\nabla, D]$ extends to a completely bounded operator $\mathcal{E} \to \mathcal{E}\tilde{\otimes}_B\Omega^1(B)$. ∇ is said to be a smooth D -connection if it is a D_i-connection, for all i, where we view the Sobolev modules $\mathcal{E}_i$ as dense submodules of $\mathcal{E}$.

PROPOSITION 1.5.2.3. *Let $(\mathcal{E}, D)$ be an unbounded (A, B)-bimodule and $\nabla : \mathcal{E} \to \mathcal{E}\tilde{\otimes}_B\Omega^1(B)$ a smooth D-connection. Then ∇ induces a connection $\{\nabla_i\}$ on the Sobolev chain of D.*

PROOF. We apply the usual trick. Define $\{\nabla_i\}$ inductively by

$$\nabla_{i+1} := \begin{pmatrix} \nabla_i & 0 \\ [D_i, \nabla_i] & \nabla_i \end{pmatrix}.$$

By definition of smoothness this defines a connection on the Sobolev chain of D. $\square$

Using the smooth stabilization theorem 1.4.3.3, we get:

COROLLARY 1.5.2.4. *A smooth C^*-module $\mathcal{E} \leftrightharpoons A$ admits a Hermitian connection.*

A smooth D-connection is said to be simply a smooth connection if it restricts to a $D_{i-1,k}$-connection $\nabla_{i,k} : E_i^k \rightarrow E_i^k \tilde{\otimes}_{\mathcal{B}_k} \Omega^1(\mathcal{B}_k)$.

5.3. Product connections. We now proceed to connections on tensor products of projective modules. Anticipating the use of connections on unbounded bimodules, a category of modules with connection is constructed.

PROPOSITION 1.5.3.1. *Let P be a right projective rigged $\mathcal{A}$-module with a universal connection ∇, P' a right projective rigged $(\mathcal{A}-\mathcal{B})$-bimodule with universal connection ∇'. Then $P \otimes_{\mathcal{A}} P'$ is $\mathcal{B}$-projective, and ∇ and ∇' determine a universal $\mathcal{B}$-connection on $P \otimes_{\mathcal{A}} P'$. If both connections are Hermitian, then so is the induced connection.*

PROOF. Let, Q, Q' be such that $P \oplus Q \cong \mathcal{H}\tilde{\otimes}\mathcal{A}$, $P' \oplus Q' \cong \mathcal{H}'\tilde{\otimes}\mathcal{B}$. Then:

$$P\tilde{\otimes}_{\mathcal{A}}P' \oplus Q\tilde{\otimes}_{\mathcal{A}}P' \oplus \mathcal{H}\tilde{\otimes}Q' \cong \mathcal{H}\tilde{\otimes}\mathcal{H}'\tilde{\otimes}\mathcal{B}.$$

Thus $P\tilde{\otimes}_{\mathcal{A}}P'$ is projective. Consider the derivation

$$\begin{aligned}\delta : \mathcal{A} &\rightarrow \mathrm{End}_{\mathcal{B}}(P', P'\tilde{\otimes}_{\mathcal{B}}\Omega^1(\mathcal{B}))\\ a &\mapsto [\nabla', a].\end{aligned}$$

By universality there is a unique map

$$j_\delta : \Omega^1(\mathcal{A}) \rightarrow \Omega^1_\delta,$$

intertwining d and δ. Thus, ∇ induces a connection

$$\nabla_\delta : P \rightarrow P\tilde{\otimes}_{\mathcal{A}}\Omega^1_\delta,$$

by composing with j_δ. Subsequently define

$$\begin{aligned}\nabla\tilde{\otimes}_{\mathcal{A}}\nabla' : P\tilde{\otimes}_{\mathcal{A}}P' &\rightarrow P\tilde{\otimes}_{\mathcal{A}}P'\tilde{\otimes}_{\mathcal{B}}\Omega^1(\mathcal{B})\\ p \otimes p' &\mapsto \nabla'(p') + \nabla_\delta(p)p',\end{aligned}$$

which is a connection. It is a straightforward calculation to check that this connection is Hermitian if ∇ and ∇' are. $\square$

We will refer to the connection of proposition 1.5.3.1 as the *product connection.* Taking product connections is associative up to isomorphism.

THEOREM 1.5.3.2. *Let P, P', P'' be right projective rigged $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$-modules respectively, with universal connections $\nabla, \nabla', \nabla''$. Suppose P', P'' are left $\mathcal{A}$ and $\mathcal{B}$ modules, respectively. The natural isomorphism*

$$P\tilde{\otimes}_{\mathcal{A}}(P'\tilde{\otimes}_{\mathcal{B}}P'') \xrightarrow{\sim} (P\tilde{\otimes}_{\mathcal{A}}P')\tilde{\otimes}_{\mathcal{B}}P''$$

intertwines the product connections $\nabla\tilde{\otimes}_{\mathcal{A}}(\nabla'\tilde{\otimes}_{\mathcal{B}}\nabla'')$ and $(\nabla\tilde{\otimes}_{\mathcal{A}}\nabla')\tilde{\otimes}_{\mathcal{B}}\nabla''$

Proof. The two product connections on $M = P\tilde{\otimes}_{\mathcal{A}}P'\tilde{\otimes}_{\mathcal{B}}P''$ correspond to splittings of the universal exact sequence given by the follwing diagram:

$$\begin{tikzcd}
0 \arrow[r] & \ker(1\otimes_{\mathcal{B}} m) \arrow[r] \arrow[d] & M_{\mathcal{A}}\tilde{\otimes}P''\otimes\mathcal{C} \arrow[r, shift left, "1\otimes_{\mathcal{B}} m"] \arrow[d] & M_{\mathcal{A}}\tilde{\otimes}_{\mathcal{B}}P'' \arrow[r] \arrow[l, shift left] & 0 \\
0 \arrow[r] & M\tilde{\otimes}_{\mathcal{C}}\Omega^1(\mathcal{C}) \arrow[r, "j"] & M\tilde{\otimes}\mathcal{C} \arrow[r, "m"] & M \arrow[r] \arrow[u, "\sim"] \arrow[d, "\sim"'] & 0 \\
0 \arrow[r] & \ker(m\otimes_{\mathcal{A}} 1) \arrow[r] \arrow[u] & P\tilde{\otimes}M_{\mathcal{B}}\tilde{\otimes}\mathcal{C} \arrow[r, shift left, "1\otimes_{\mathcal{A}} m"] \arrow[u] & P\tilde{\otimes}_{\mathcal{A}}M_{\mathcal{B}} \arrow[r] \arrow[l, shift left] & 0.
\end{tikzcd}$$

Here $M_{\mathcal{A}} = P\tilde{\otimes}_{\mathcal{A}}P'$ and $M_{\mathcal{B}} = P'\tilde{\otimes}_{\mathcal{B}}P''$. To show that this diagram commutes, observe that the given connections induce natural splittings for the maps

$$P\tilde{\otimes}P'\tilde{\otimes}P'' \to P\tilde{\otimes}_{\mathcal{A}}M_{\mathcal{B}} \quad \text{and} \quad P\tilde{\otimes}P'\tilde{\otimes}P'' \to M_{\mathcal{A}}\tilde{\otimes}_{\mathcal{B}}P''.$$

They correspond to the decompositions

$$P\tilde{\otimes}P'\tilde{\otimes}P'' \cong P\tilde{\otimes}_{\mathcal{A}}M_{\mathcal{B}} \oplus Q\tilde{\otimes}_{\mathcal{A}}M_{\mathcal{B}} \oplus P\tilde{\otimes}Q'\tilde{\otimes}_{\mathcal{B}}P'',$$

and

$$P\tilde{\otimes}P'\tilde{\otimes}P'' \cong M_{\mathcal{A}}\tilde{\otimes}_{\mathcal{B}}P'' \oplus Q_{\mathcal{A}}\tilde{\otimes}_{\mathcal{B}}P \oplus Q\tilde{\otimes}_{\mathcal{A}}P'\tilde{\otimes}P'',$$

where Q, Q' and $Q_{\mathcal{A}}$ are such that

$$P\oplus Q \cong P\tilde{\otimes}\mathcal{A}, \quad P'\oplus Q' \cong P'\tilde{\otimes}\mathcal{B}, \quad M_{\mathcal{A}}\oplus Q_{\mathcal{A}} \cong M_{\mathcal{A}}\tilde{\otimes}\mathcal{B}.$$

That is, Q and Q' come from ∇ and ∇' respectively, and $Q_{\mathcal{A}}$ from $\nabla\tilde{\otimes}_{\mathcal{A}}\nabla'$.

Therefore, the given connections induce natural splittings for the maps

$$P\tilde{\otimes}P'\tilde{\otimes}P'' \to P\tilde{\otimes}_{\mathcal{A}}M_{\mathcal{B}} \quad \text{and} \quad P\tilde{\otimes}P'\tilde{\otimes}P'' \to M_{\mathcal{A}}\otimes_{\mathcal{B}}P''.$$

These splittings correspond to the factorizations

$$\begin{tikzcd}
 & P\tilde{\otimes}P'\tilde{\otimes}P'' \arrow[dl] \arrow[dr] & \\
M_{\mathcal{A}}\tilde{\otimes}P'' \arrow[dr] & & P\tilde{\otimes}M_{\mathcal{B}} \arrow[dl] \\
 & P\tilde{\otimes}_{\mathcal{A}}P'\tilde{\otimes}_{\mathcal{B}}P'' &
\end{tikzcd}$$

of the map $P\tilde{\otimes}P'\tilde{\otimes}P'' \to P\tilde{\otimes}_{\mathcal{A}}P'\tilde{\otimes}_{\mathcal{B}}P''$. These factorizations are exactly the ones that give rise to the product connections $\nabla\tilde{\otimes}_{\mathcal{A}}(\nabla'\tilde{\otimes}_{\mathcal{B}}\nabla'')$ and $(\nabla\tilde{\otimes}_{\mathcal{A}}\nabla')\tilde{\otimes}_{\mathcal{B}}\nabla''$. Therefore the different splittings in the first diagram coincide under the intertwining isomorphisms. $\square$

The upshot of theorems 1.5.3.1 and 1.5.3.2 is that there is a category whose objects are operator algebras, and whose morphisms $\mathrm{Mor}(\mathcal{A},\mathcal{B})$ are given by pairs $(P,\nabla^{\mathcal{B}})$ consisting of an $(\mathcal{A},\mathcal{B})$-bimodule with a universal $\mathcal{B}$ connection (whence by proposition 1.5.1.4 P is $\mathcal{B}$-projective). The identiy morphisms are the pairs $(1_{\mathcal{A}},d)$ consisiting of the trivial bimodule $1_{\mathcal{A}}$ and the universal derivation $d:\mathcal{A}\to\Omega^1(\mathcal{A})$. Of course this category is described equivalently as the category of pairs (P,s) of bimodules together with a splitting s of the universal exact sequence.

5.4. Induced operators and their graphs. One can proceed to enrich this category by considering triples (P,∇,T) consisting of rightprojective bimodules with connection and a distinguished endomorphism $T\in\mathrm{End}_{\mathcal{B}}(P)$. Denote by $1\tilde{\otimes}_\nabla T$ the operator

$$1\otimes_\nabla T(p\otimes p'):=(-1)^{\partial T\partial p}(p\otimes T(p')+\nabla_T(p)p'),$$

which is well defined on $P\tilde{\otimes}_{\mathcal{A}}P'$. The composition law then becomes

$$(P,\nabla,S)\circ(P',\nabla',T):=(P\tilde{\otimes}_{\mathcal{B}}P',\nabla\tilde{\otimes}_{\mathcal{B}}\nabla',S\tilde{\otimes}1+1\tilde{\otimes}_\nabla T).$$

Associativity of this composition is implied by the following proposition.

Proposition 1.5.4.1. *Let P be a right projective rigged $\mathcal{A}$-module, P' a right projective rigged $(\mathcal{A},\mathcal{B})$-bimodule and ∇,∇' universal connections. Furthermore let E,F be $(\mathcal{B},\mathcal{C})$-bimodules, and $D\in\mathrm{End}_{\mathcal{C}}(E,F)$. Then*

$$1\tilde{\otimes}_\nabla 1\tilde{\otimes}_{\nabla'}D=1\tilde{\otimes}_{\nabla\tilde{\otimes}_{\mathcal{A}}\nabla'}D,$$

under the intertwining isomorphism.

Proof. Recall the formula for the product connection

$$\nabla\tilde{\otimes}_{\mathcal{A}}\nabla'(p\otimes p'):=p\otimes\nabla'(p')+\nabla_\delta(p)p'.$$

Morevoer, write ∇_D for $\nabla_{\nabla'_D}$. It is straightforward to check that

$$(\nabla\tilde{\otimes}_{\mathcal{A}}\nabla')_D(p\otimes p')=p\otimes\nabla'_D(p')+\nabla_D(p)p'.$$

Therefore we have

$$\begin{aligned}1\otimes_{\nabla\tilde{\otimes}_{\mathcal{A}}\nabla'}D(p\otimes p'\otimes e)&=p\otimes p'\otimes De+\nabla\otimes\nabla'(p\otimes p')e\\&=p\otimes p'\otimes De+p\otimes\nabla'_D(p')e+\nabla_D(p)(p'\otimes e).\end{aligned}$$

On the other hand

$$\begin{aligned}1\tilde{\otimes}_\nabla 1\tilde{\otimes}_{\nabla'}D(p\otimes p'\otimes e)&=p\otimes(1\tilde{\otimes}_{\nabla'}D)(p'\otimes e)+\nabla_{1\tilde{\otimes}_{\nabla'}D}(p)(p'\otimes e)\\&=p\otimes p'\otimes De+p\otimes\nabla'_D(p')e+\nabla_{1\tilde{\otimes}_{\nabla'}}D(p)(p'\otimes e),\end{aligned}$$

thus, it suffices to show that $\nabla_D=\nabla_{1\tilde{\otimes}_{\nabla'}D}$. To this end, observe that

$$[1\tilde{\otimes}_{\nabla'}D,a]=[\nabla'_D,a]:P\otimes_{\mathcal{A}}P'\to P\otimes_{\mathcal{A}}P',$$

which gives a natural isomorphism $\Omega^1_{\nabla'_D}\xrightarrow{\sim}\Omega^1_{1\tilde{\otimes}_{\nabla'}D}$ intertwining the derivations. By universality this gives a commutative diagram

$$\begin{tikzcd} & \Omega^1(\mathcal{A}) \arrow[dl] \arrow[dr] & \\ \Omega^1_{1\tilde{\otimes}_{\nabla'}D} \arrow[rr, "\sim"] & & \Omega^1_{\nabla'_D}, \end{tikzcd}$$

which shows that $\nabla_D = \nabla_{1\tilde{\otimes}_{\nabla'}D}$. □

As we have seen, a connection $\nabla : \mathcal{E} \to \mathcal{E}\tilde{\otimes}\Omega^1(B)$ can be used to transfer operators on $\mathcal{F}$ to $\mathcal{E} \otimes_B \mathcal{F}$. We now show that this algebraic procedure is well behaved for selfadjoint regular operators T in $\mathcal{F}$, and describe the graph $\mathfrak{G}(1\tilde{\otimes}_\nabla T) \subset \mathcal{E}\tilde{\otimes}_B\mathcal{F} \oplus \mathcal{E}\tilde{\otimes}_B\mathcal{F}$ as a topological C^*-module, in terms of the graph of T.

Lemma 1.5.4.2. *Let $\mathcal{E}, \mathcal{F}$ be C^*-modules over B and C respectively, and $\nabla : \mathcal{E} \to \mathcal{E}\tilde{\otimes}\Omega^1(B)$ a Hermitian connection. Suppose $\mathcal{F}$ is a left B-module and $T : \mathfrak{Dom}(T) \to \mathcal{F}$ a selfadjoint regular operator such that $[T, b] \in \mathrm{End}^*_B(\mathcal{F})$ for all $b \in \mathcal{B}_1 \subset B$, a dense subalgebra of B. If ∇ and $\mathcal{E}$ are C^1 with respect to $\mathcal{B}_1$, then the operator $1\tilde{\otimes}_\nabla T$ is selfadjoint and regular. The map*

$$\begin{aligned} E^1\tilde{\otimes}_{\mathcal{B}_1}\mathfrak{G}(T) &\to \mathfrak{G}(1\tilde{\otimes}_\nabla T)\\ e \otimes (f, Tf) &\mapsto (e \otimes f, 1\tilde{\otimes}_\nabla T(e \otimes f)) \end{aligned}$$

is a topological isomorphism of C^*-modules.

Proof. Observe that $E^1\tilde{\otimes}_{\mathcal{B}_1}\mathcal{F} \cong \mathcal{E}\tilde{\otimes}_B\mathcal{F}$, since $\mathcal{E} = E^1\tilde{\otimes}_{\mathcal{B}_1}B$. To see that $t := 1 \otimes_\nabla T$ is selfadjoint regular, stabilize $\mathcal{E}$, and denote by $\tilde{\nabla}$ the Levi-Civita connection on $\mathcal{H}_B$. Then, via the stabilization isomorphism $\nabla' := \nabla \oplus \tilde{\nabla}$ defines a Hermitian connection on $\mathcal{H}_B \cong \mathcal{E} \oplus \mathcal{H}_B$. Since the difference $\nabla' - \tilde{\nabla}$ is a completely bounded module map, it suffices to prove regularity of t when ∇ is the Levi-Civita connection on $\mathcal{H}_B$. But in that case, for $e = \sum_{i=1}^\infty e_i \otimes b_i$,

$$\begin{aligned} t : \mathcal{H}_{\mathcal{B}_1}\tilde{\otimes}_{\mathcal{B}_1}\mathcal{F} &\to \mathcal{H}_{\mathcal{B}_1}\tilde{\otimes}_{\mathcal{B}_1}\mathcal{F}\\ e \otimes f &\mapsto \sum_{i=1}^\infty e_i \otimes T(b_i f), \end{aligned}$$

which is clearly selfadjoint regular. For the statement on the topological type of $\mathfrak{G}(t)$, it again suffices to consider the Levi-Civita connection $\mathcal{H}_\mathcal{B}$. We have $R := \nabla'_T - \tilde{\nabla}_T \in \mathrm{End}^*_C(\mathcal{E}\tilde{\otimes}_B\mathcal{F})$ and hence

$$\begin{aligned} \mathfrak{G}(t) &\xrightarrow{\sim} \mathfrak{G}(t + R)\\ (x, tx) &\mapsto (x, (t + R)x) \end{aligned}$$

topologically, due to the fact that $(i + t + R)(i + t)^{-1} \in \mathrm{Aut}^*_C(\mathcal{E}\tilde{\otimes}_B\mathcal{F})$. Note that the standard orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$ of $\mathcal{H}_B$ defines a C^1-approximate unit for $\mathbb{K}_B(\mathcal{H}_B)$. The inner product on $\mathcal{H}_{\mathcal{B}_1}\tilde{\otimes}_{\mathcal{B}_1}\mathfrak{G}(T)$ is thus given by

$$\begin{aligned} \langle e \otimes (f, Tf), e' \otimes (f', Tf')\rangle :&= \lim_{n\to\infty}\sum_{i=1}^n \langle\langle e_i, e\rangle(f, Tf), \langle e_i, e'\rangle(f', Tf)\rangle\\ &= \sum_{i=1}^\infty \langle(b_i f, T b_i f), (b'_i f', T b_i f)\rangle\\ &= \sum_{i=1}^\infty \langle b_i f, b'_i f'\rangle + \langle T b_i f, T b'_i f'\rangle. \end{aligned}$$

Therefore the map

$$\begin{aligned} \mathcal{H}_{\mathcal{B}_1}\tilde{\otimes}_{\mathcal{B}_1}\mathfrak{G}(T) &\to \mathfrak{G}(t)\\ e \otimes (f, Tf) &\mapsto (e \otimes f, t(e \otimes f)), \end{aligned}$$

is unitary. □

If the module $\mathcal{E}$ comes equipped with a regular operator S, the operators $S\tilde{\otimes}1$ and $1\tilde{\otimes}_\nabla T$ almost anticommute. That is, they anticommute up to a bounded operator. This implies that their sum is well defined as a regular operator.

PROPOSITION 1.5.4.3. *Let A, B, C be C^*-algebras, $(\mathcal{E}, S)$ and $(\mathcal{F}, T)$ be (A, B)- and (B, C)-bimodules equipped with selfadjoint regular operators S and T, respectively, such that $[T, b] \in \mathrm{End}^*_B(\mathcal{F})$ for all $b \in \mathcal{B}_1 \subset B$, a dense subalgebra of B. If $\nabla : \mathcal{E} \to \mathcal{E}\tilde{\otimes}_B\Omega^1(B)$ is a C^1- connection, then the operator*

$$S \otimes 1 + 1 \otimes_\nabla T$$

is selfadjoint and regular.

PROOF. It is a well known fact that $s := S \otimes 1$ is a regular operator on $\mathcal{E} \otimes_B \mathcal{F}$ and we saw that $t := 1 \otimes_\nabla T$ is regular. Thus, s and t are selfadjoint regular operators whose domains intersect densely, and the graded commutator $[s, t]$ is an adjointable operator. To show that $s + t$ is regular we have to show that $\mathfrak{G}(s + t) \oplus v\mathfrak{G}(s + t) \cong E\tilde{\otimes}_B\mathcal{F} \oplus \mathcal{E}\tilde{\otimes}_B\mathcal{F}$. Write $\mathfrak{r}_i(s) := (s + i)^{-1}$ and consider the endomorphism

$$g := \begin{pmatrix} \mathfrak{r}_i(s)\mathfrak{r}_i(t) & -(s+t)\mathfrak{r}_i(t)^*\mathfrak{r}_i(s)^* \\ (s+t)\mathfrak{r}_i(s)\mathfrak{r}_i(t) & \mathfrak{r}_i(t)^*\mathfrak{r}_i(s)^* \end{pmatrix} \in M_2(\mathrm{End}_B(\mathcal{E} \otimes_B \mathcal{F})).$$

The maps $(s+t)\mathfrak{r}_i(t)^*\mathfrak{r}_i(s)^*$ and $(s+t)\mathfrak{r}_i(s)\mathfrak{r}_i(t)$ are well defined because $[s, \mathfrak{r}_i(t)^*]$ and $[t, \mathfrak{r}_i(s)]$ are bounded. This follows from the fact that $[s, t]$ is bounded:

$$\begin{aligned} 0 &= [s, 1] \\ &= [s, \mathfrak{r}_i(t)^*(t - i)] \\ &= \mathfrak{r}_i(t)^*[s, (t - i)] + [s, \mathfrak{r}_i(t)^*](t - i) \\ &= \mathfrak{r}_i(t)^*[s, t] + [s, \mathfrak{r}_i(t)^*](t - i), \end{aligned}$$

hence $[s, \mathfrak{r}_i(t)^*](t - i)$ is bounded and so is $[s, \mathfrak{r}_i(t)^*]$. Computing g^*g gives

$$\begin{aligned} &\begin{pmatrix} \mathfrak{r}_i(t)^*\mathfrak{r}(s)^2\mathfrak{r}_i(t) & 0 \\ 0 & \mathfrak{r}_i(s)\mathfrak{r}(t)^2\mathfrak{r}_i(s)^* \end{pmatrix} \\ &+ \begin{pmatrix} \mathfrak{r}_i(t)^*\mathfrak{r}_i(s)^*(s+t)^2\mathfrak{r}_i(s)\mathfrak{r}_i(t) & 0 \\ 0 & \mathfrak{r}_i(s)\mathfrak{r}_i(t)(s+t)^2\mathfrak{r}_i(t)^*\mathfrak{r}_i(s)^* \end{pmatrix}. \end{aligned}$$

The operator

$$r = \begin{pmatrix} \mathfrak{r}_i(t)^*\mathfrak{r}(s)^2\mathfrak{r}_i(t) & 0 \\ 0 & \mathfrak{r}_i(s)\mathfrak{r}(t)^2\mathfrak{r}_i(s)^* \end{pmatrix}$$

is positive and has dense range, and $r \leq g^*g$, so g^*g has dense dense range, and hence g does so too. Since g maps $\mathcal{E} \otimes_B \mathcal{F}^2$ into $\mathfrak{G}(s + t) \oplus v\mathfrak{G}(s + t)$, this is a dense and closed submodule, hence all of $\mathcal{E} \otimes_B \mathcal{F} \oplus \mathcal{E} \otimes_B \mathcal{F}$. Selfadjointness follows from the fact that s and t anticommute up to an adjointable operator. Hence $\mathfrak{Dom}(s + t) = \mathfrak{Dom}s \cap \mathfrak{Dom}(t)$, and s and t are selfadjoint. □

Of course the case of unbounded bimodules is contained in this theorem. It will be the case we focus on the next section.

COROLLARY 1.5.4.4. *Let A, B, C be C^*-algebras, $(\mathcal{E}, S)$ an unbounded (A, B) bimodule and $(\mathcal{F}, T)$ an unbounded (B, C)-bimodule. Let $\nabla : \mathcal{E} \to \mathcal{E}\tilde{\otimes}_B \Omega^1(B)$ be a C^1-connection on $\mathcal{E}$. Then the operator*

$$S \otimes 1 + 1 \otimes_\nabla T$$

is selfadjoint and regular.

The product construction preserves selfadjointness and regularity. On the level of the graphs of the operators, we now show it can be viewed as a pull-back construction of topological C^*-modules. By Frank's theorem 1.1.1.4, this yields a unitary isomorphism of the modules involved. This suggests the product construction might be defined intrinsically, without reference to the connection. However, the pull back need not be the graph of an operator.

THEOREM 1.5.4.5. *Let A, B, C be C^*-algebras, $(\mathcal{E}, S)$ and $(\mathcal{F}, T)$ be (A, B)- and (B, C)-bimodules equipped with selfadjoint regular operators S and T, respectively. Suppose that $[T, b] \in \mathrm{End}^*_B(\mathcal{F})$ for all $b \in \mathcal{B}_1 \subset B$, a dense subalgebra of B. Let $\nabla : \mathcal{E} \to \mathcal{E}\tilde{\otimes}_B \Omega^1(B)$ be a C^1- connection, and $\mathcal{G}$ the universal solution to the diagram*

$$\begin{array}{ccc} \mathcal{G} & \longrightarrow & \mathfrak{G}(S)\tilde{\otimes}_B \mathcal{F} \\ \downarrow & & \downarrow \\ E^1\tilde{\otimes}_{\mathcal{B}_1}\mathfrak{G}(T) & \longrightarrow & \mathcal{E}\tilde{\otimes}_B \mathcal{F}. \end{array}$$

Then the natural map

$$\mathfrak{G}(S\tilde{\otimes}1 + 1\tilde{\otimes}_\nabla T) \to \mathcal{G},$$

is a topological isomorphism of C^*-modules.

PROOF. As usual, write $s = S\tilde{\otimes}1$ and $t = 1\tilde{\otimes}_\nabla T$. Since $\mathfrak{G}(S)\tilde{\otimes}_B \mathcal{F} \cong \mathfrak{G}(\sqrt{2}s)$ and $E^1\tilde{\otimes}_{\mathcal{B}_1}\mathfrak{G}(T) \cong \mathfrak{G}(\sqrt{2}t)$, we may replace $\mathcal{G}$ by the pull back of the diagram

$$\begin{array}{ccc} \mathcal{G} & \longrightarrow & \mathfrak{G}(\sqrt{2}s) \\ \downarrow & & \downarrow \\ \mathfrak{G}(\sqrt{2}t) & \longrightarrow & \mathcal{E}\tilde{\otimes}_B \mathcal{F}. \end{array}$$

Thus

$$\mathcal{G} = \{((x, \sqrt{2}sx), (x, \sqrt{2}tx)) : x \in \mathfrak{Dom}(s + t)\},$$

and

$$\langle((x, \sqrt{2}sx), (x, \sqrt{2}tx)), ((x', \sqrt{2}sx'), (x', \sqrt{2}tx'))\rangle = 2(\langle x, x'\rangle + \langle sx, sx'\rangle + \langle tx, tx'\rangle).$$

Using

$$g := \begin{pmatrix} \mathfrak{r}(s+t)^2 & -(s^2+t^2)\mathfrak{r}(s+t)^2 \\ (s^2+t^2)\mathfrak{r}(s+t)^2 & \mathfrak{r}(s+t)^2 \end{pmatrix} \in M_2(\mathrm{End}_B(\mathcal{E} \otimes_B \mathcal{F})),$$

it follows that $s^2 + t^2$ is selfadjoint regular by the same reasoning as in proposition 1.5.4.3. Since the operator $(i + |s^2 + t^2|^{\frac{1}{2}})(i + s + t)^{-1} \in \mathrm{Aut}^*(\mathcal{E}\tilde{\otimes}_B \mathcal{F})$, the result follows. $\square$

If the module $\mathcal{F}$ is smooth, i.e. induces a smooth structure $\{\mathcal{B}_i\}$ on B, and $\mathcal{E}$ is a smooth C^*-module for this smooth structure, the Sobolev chain of $D = S\tilde{\otimes}1+1\tilde{\otimes}_\nabla T$ can be computed from the Sobolev chains of S and T by the following diagram:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
\cdots \arrow[r] & \mathfrak{G}(D_2) \arrow[r] \arrow[d] & \mathfrak{G}(t)_{D_1} \arrow[r] \arrow[d] & \mathfrak{G}(t_1)_D \arrow[r] \arrow[d] & E^3\tilde{\otimes}\mathcal{F}_3 \arrow[d] \\
\cdots \arrow[r] & \mathfrak{G}(s)_{D_1} \arrow[r] \arrow[d] & \mathfrak{G}(D_1) \arrow[r] \arrow[d] & \mathfrak{G}(t)_D \arrow[r] \arrow[d] & E^2\tilde{\otimes}\mathcal{F}_2 \arrow[d] \\
\cdots \arrow[r] & \mathfrak{G}(s_1)_D \arrow[r] \arrow[d] & \mathfrak{G}(s)_D \arrow[r] \arrow[d] & \mathfrak{G}(D) \arrow[r] \arrow[d] & E^1\tilde{\otimes}\mathcal{F}_1 \arrow[d] \\
\cdots \arrow[r] & \mathcal{E}_3\tilde{\otimes}\mathcal{F} \arrow[r] & \mathcal{E}_2\tilde{\otimes}\mathcal{F} \arrow[r] & \mathcal{E}_1\tilde{\otimes}\mathcal{F} \arrow[r] & \mathcal{E}\tilde{\otimes}\mathcal{F}
\end{tikzcd}$$

The Sobolev chain of D is on the diagonal, that of $s = S\tilde{\otimes}1$ is the bottom row, and that of $t = 1\tilde{\otimes}_\nabla T$ is the right vertical row. The upper triangular part consists of the Sobolev chains of t viewed as an operator in $\mathfrak{G}(D_i)$, and the lower triangular part of the Sobolev chains of s viewed as an operator in $\mathfrak{G}(D_i)$. Moreover, all squares all pull back squares.

6. Correspondences

Universal connections can be employed to give a transparent construction of the Kasparov product, on the level of unbounded bimodules. This observation leads to the construction of a category of spectral triples and even of unbounded bimodules themselves. They give a notion of morphism of noncommutative geometries, in such a way that the bounded transform induces a functor from correspondences to KK-groups. By considering several levels of differentiability and smoothness on correspondences, one gets subcategories of correspondences of C^k- and smooth C^*-algebras.

6.1. The Trotter-Kato formula. When dealing with addition of noncommuting unbounded operators s and t, on a Hilbert space $\mathcal{H}$, several subtleties arise. First of all one needs to check closability and density of the domain of $s+t$. If these things are in order, one would like to obtain information about $e^{-x(s+t)}$, the one parameter group generated by $s+t$, and its resolvent. These questions can be tantalizingly difficult and involve some deep analysis. Under favourable conditions, though, a satisfactory description of $e^{-x(s+t)}$ can be given in terms of e^{-xs} and e^{-xt}. It is quite striking that one might as well use other functions of s and

t instead of exponentials. For our purposes it is enough to consider the function $f(s) = (1+s)^{-1}$.

THEOREM 1.6.1.1 ([**31**]). *Let f be either one of the functions $s \mapsto e^{-s}$ or $s \mapsto (1+s)^{-1}$. Suppose s and t are nonnegative selfadjoint operators on a Hilbert space $\mathcal{H}$, such that their sum $s+t$ is selfadjoint on $\mathfrak{Dom}(s) \cap \mathfrak{Dom}(t)$. Then*

$$\lim_{n\to\infty}(f(\frac{xs}{2n})f(\frac{xt}{n})f(\frac{xs}{2n}))^n = e^{-x(s+t)},$$

in norm for x in compact intervals in $(0,\infty)$. If $s+t$ is strictly positive, the convergence holds for $x \in [\epsilon, \infty)$ for any $\epsilon > 0$.

We now argue that a similar result holds for unbounded operators in C^*-modules. Let s and t be nonnegative regular operators in a C^*-B-module $\mathcal{E}$, such that their sum $s+t$ is densely defined and regular. By representing B faithfully and nondegenerate on a Hilbert space $\mathcal{H}'$, one obtains a second Hilbert space $\mathcal{H} := \mathcal{E}\tilde{\otimes}_B\mathcal{H}'$ and operators $s\otimes 1, t\otimes 1$ and $(s+t)\otimes 1 = s\otimes 1 + t\otimes 1$. Moreover, $\text{End}^*_B(\mathcal{E})$ is faithfully represented on $\mathcal{H}$, and $f(s)\otimes 1 = f(s\otimes 1)$ for any $f \in C_0(\mathbb{R})$. Also, $s\otimes 1, t\otimes 1$ and $(s+t)\otimes 1$ are positive whenever $s, t, s+t$ are. Therefore we have

COROLLARY 1.6.1.2. *Let f be either one of the functions $s \mapsto e^{-s}$ or $s \mapsto (1+s)^{-1}$. Suppose s and t are nonnegative selfadjoint regular operators on a C^*-module $\mathcal{E}$, such that their sum $s+t$ is selfadjoint and regular on $\mathfrak{Dom}(s)\cap\mathfrak{Dom}(t)$. Then*

$$\lim_{n\to\infty}(f(\frac{xs}{2n})f(\frac{xt}{n})f(\frac{xs}{2n}))^n = e^{-x(s+t)},$$

in norm for x in compact intervals in $(0,\infty)$. If $s+t$ is strictly positive, the convergence holds for $x \in [\epsilon, \infty)$ for any $\epsilon > 0$.

The Trotter-Kato formula in C^*-modules will be a crucial tool in what follows.

6.2. The KK-product. Everything is in place now to establish that compact resolvents are preserved under taking products. Then we will see that the product operator satisfies Kucerovsky's conditions for an unbounded Kasparov product. Thus, if two unbounded bimodules are compatible in the sense that there exists a C^1-connection for them, the KK-product of these modules is given by an explicit algebraic formula. Let us put the pieces together.

LEMMA 1.6.2.1. *Let s, t be selfadjoint regular operators on a C^*-module $\mathcal{E}$, and $R \in \text{End}^*_B(\mathcal{E})$ be a selfadjoint element. If $(1+s^2)^{-1}(t+i)^{-1} \in \mathbb{K}_B(\mathcal{E})$, then $(1+s^2)^{-1}(t+R+i)^{-1} \in \mathbb{K}_B(\mathcal{E})$.*

PROOF. One has the identity

$$(1+s^2)^{-1}(i+t+R)^{-1} = (1+s^2)^{-1}(i+t)^{-1}(1-R(t+i)^{-1}),$$

which is a compact operator. □

We now employ the Trotter-Kato formula from the previous section to show that the product of cycles is a cycle. Note that this result is a generalization of the stability property of spectral triples proved in [**10**]. There it was shown that tensoring a given spectral triple by a finitely generated projective module yields again a spectral triple.

PROPOSITION 1.6.2.2. *Let A, B, C be C^*-algebras, $(\mathcal{E}, S)$ an unbounded (A, B) bimodule and $(\mathcal{F}, S)$ an unbounded (B, C)-bimodule. Let $\nabla : \mathcal{E} \to \mathcal{E}\tilde{\otimes}_B\Omega^1(B)$ be a C^1-connection on $\mathcal{E}$. Then the operator*

$$S \otimes 1 + 1 \otimes_\nabla T$$

has compact resolvent.

PROOF. The operator $s^2 + t^2$ is selfadjoint and regular, as we saw in the proof of theorem 1.5.4.5. Moreover, since $s^2 + t^2$ is positive we have

$$s^2 + t^2 = |s^2 + t^2|.$$

Since $(s+t)^2$ is a bounded perturbation of s^2+t^2, for $s+t$ to have compact resolvent it is sufficient that $(1 + s^2 + t^2)^{-1}$ be compact. By applying lemma 1.1.4.4 to the operator $|s^2 + t^2|^{\frac{1}{2}}$, we get the identity

$$(2 + s^2 + t^2)^{-1} = \int_0^\infty e^{-x(2+s^2+t^2)}dx.$$

By this same lemma it suffices to show that the integrand $e^{-x(2+s^2+t^2)}$ is compact for $x > 0$. The Trotter-Kato formula 1.6.1.2 gives the equality

$$e^{-x(2+s^2+t^2)} = \lim_{n\to\infty}((1 + \frac{x}{2n}s^2)^{-1}(1 + \frac{x}{n}t^2)^{-1}(1 + \frac{x}{2n}s^2)^{-1})^n.$$

Therefore it suffices to show that $(1 + \frac{1}{2}s^2)^{-1}(i + t)^{-1}$ is compact. By the previous lemma, we only have to check this in case ∇ is the Levi-Civita connection on $\mathcal{H}_B$. In that case

$$(1 + \frac{1}{2}s^2)^{-1}(i + t)^{-1} = \sum_{i=0}^{\infty}(1 + \frac{1}{2}S^2)^{-1}e_i \otimes (i + T)^{-1} \otimes e_i,$$

which is a norm convergent series in $\mathbb{K}_C(\mathcal{H}_B\tilde{\otimes}\mathcal{F}) = \mathcal{H}_B\tilde{\otimes}\mathbb{K}_C(\mathcal{F})\tilde{\otimes}\mathcal{H}_B$. □

At this point, we would like to note that for a given pair of cycles $(\mathcal{E}, S)$ and $(\mathcal{F}, T)$, the existence of a C^1-connection is not guaranteed. In the presence of such a connection, we have the follwoing result.

THEOREM 1.6.2.3. *The diagram*

$$\begin{tikzcd}
\Psi_0(A,B) \times \Psi_0(B,C) \arrow[r, "(S,T) \mapsto S\otimes 1 + 1\otimes_\nabla T"] \arrow[d, "\mathfrak{b}"'] & \Psi_0(A,C) \arrow[d, "\mathfrak{b}"'] \\
KK_0(A,B) \otimes KK_0(B,C) \arrow[r, "\otimes_B"] & KK_0(A,C)
\end{tikzcd}$$

commutes, whenever the composition in the top row is defined.

PROOF. We just need to check that the unbounded bimodules $(\mathcal{E}, S)$, $(\mathcal{F}, T)$ and $(\mathcal{E}\tilde{\otimes}_B\mathcal{F}, S\tilde{\otimes}1 + 1 \otimes_\nabla T)$ satisfy the conditions of theorem 1.2.2.4. If we write D for $S \otimes +1 \otimes_\nabla T$, we have to check that

$$J := \left[\begin{pmatrix} D & 0 \\ 0 & T \end{pmatrix}, \begin{pmatrix} 0 & T_e \\ T_e^* & 0 \end{pmatrix}\right]$$

is bounded on $\mathfrak{Dom}(D \oplus T)$. This is a straightforward calculation:

$$J\begin{pmatrix} e' \otimes f' \\ f \end{pmatrix} = \begin{pmatrix} Se \otimes f + (-1)^{\partial e}\nabla_T(e)f \\ \langle e, Se'\rangle f + [T, \langle e, e'\rangle]f + (-1)^{-\partial e'}\langle e, \nabla_T(e')\rangle f \end{pmatrix}$$
$$= \begin{pmatrix} Se \otimes f + (-1)^{\partial e}\nabla_T(e)f \\ \langle Se, e'\rangle f + \langle \nabla_T(e), e'\rangle f \end{pmatrix}.$$

This is valid whenever $e \in E_1^1$, which is dense in $\mathcal{E}$.

The second condition $\mathfrak{Dom}(D) \subset \mathfrak{Dom}(S\tilde{\otimes}1)$ is obvious, so we turn the semiboundedness condition

$$\langle S\tilde{\otimes}1x, Dx\rangle + \langle Dx, S\tilde{\otimes}1x\rangle \geq \kappa\langle x, x\rangle, \tag{1.13}$$

must hold for all x in the domain. The expression 1.13 is equal to

$$\langle [D, S\tilde{\otimes}1]x, x\rangle = \langle [s,t]x, x\rangle \geq -\|[s,t]\|\langle x, x\rangle,$$

and the last estimate is valid since $[s,t]$ is in $\mathrm{End}^*_C(\mathcal{E}\tilde{\otimes}_B\mathcal{F})$. □

6.3. Formal Bott periodicity. We obtained a description of the KK-product of even unbounded bimodules, in the presence of a connection. This construction can be lifted to $\Psi_i(A,B)$ for each i. The way to go is indicated by the following result of Kasparov.

THEOREM 1.6.3.1 ([**32**]). *For all j, the map*

$$\Psi_i(A,B) \to \Psi_{i+j}(A\tilde{\otimes}\mathbb{C}_j, B)$$
$$(\mathcal{E}, D) \mapsto (\mathcal{E}\tilde{\otimes}\mathbb{C}_j, D\tilde{\otimes}1),$$

induces an isomorphism $KK_j(A,B) \to KK_{i+j}(A\tilde{\otimes}\mathbb{C}_j, B)$.

Using this, we can define the composition of two unbounded bimodules with connection as the composition

$$\Psi_i(A,B) \times \Psi_j(B,C) \to \Psi_i(A,B) \times \Psi_{i+j}(B\tilde{\otimes}\mathbb{C}_i, C) \to \Psi_{i+j}(A,C).$$

From theorem 1.6.2.3 we directly obtain the analoguous result in all degrees, whenever a connection for two given cycles exists.

THEOREM 1.6.3.2. *The diagram*

$$\begin{tikzcd} \Psi_i(A,B) \times \Psi_j(B,C) \arrow[r, "(S,T) \mapsto S\otimes 1 + 1\otimes_\nabla T"] \arrow[d, "\mathfrak{b}"] & \Psi_{i+j}(A,C) \arrow[d, "\mathfrak{b}"] \\ KK_i(A,B) \otimes KK_j(B,C) \arrow[r, "\otimes_B"] & KK_{i+j}(A,C) \end{tikzcd}$$

commutes, whenever the composition in the top row is defined.

In order to obtain a useful formula in the case of odd modules, we only have to delve a little deeper into formal Bott periodicity. Recall that elements of $\Psi_i(A,B)$ by definition equals $\Psi_0(A, B\tilde{\otimes}\mathbb{C}_i)$. Hence its elements are given by unbounded $(A, B\tilde{\otimes}\mathbb{C}_i)$ bimodules $(\mathcal{E}, D)$. Thus, D is an operator that commutes with the action of B and the action of $\mathbb{C}_i$. From $(\mathcal{E}, D)$ we can construct $(\mathcal{E}', D')$ in the following way (cf.[**30**], appendix A.3):

$$\mathcal{E}' := \mathcal{E} \oplus \mathcal{E}, \quad D' := \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix},$$

as in 1.9. The action of $\mathbb{C}_{i+2}$ is given by

$$\varepsilon_j := \begin{pmatrix} \varepsilon_j & 0 \\ 0 & \varepsilon_j \end{pmatrix}, \quad \varepsilon_{i+1} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \varepsilon_{i+1} := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Here $j = 1, \ldots, i$. In view of 1.9 we could denote the set of odd Kasparov modules by $\Psi_{-1}(A, B)$. The map $\Psi_i(A, B) \to \Psi_{i+2}(A, B)$ so defined is the *formal periodicity map*, and induces an isomorphism $KK_i(A, B) \to KK_{i+2}(A, B)$. It's inverse, on the level of unbounded bimodules $(\mathcal{E}, D) \in \Psi_{i+2}(A, B)$, with $i \geq 1$, is given by compressing the operator D to the $+1$ eigenspace of the involution $-i\varepsilon_{i+1}\varepsilon_{i+2}$. For $i = -1$, one compresses to the $+1$ eigenspace of ϵ_1. Applying this procedure to the case of two composable odd modules $(\mathcal{E}, S, \nabla)$ and $(\mathcal{F}, T)$ yields that the product operator is

$$\begin{pmatrix} 0 & S\tilde{\otimes}1 - i\tilde{\otimes}_\nabla T \\ S\tilde{\otimes}1 + i\tilde{\otimes}_\nabla T & 0 \end{pmatrix}, \tag{1.14}$$

on the module $\mathcal{E}\tilde{\otimes}_B\mathcal{F} \oplus \mathcal{E}\tilde{\otimes}_B\mathcal{F}$.

6.4. The nonunital case. So far, we have only been working with unital C^*-algebras. In this section we show that this restriction, imposed for the sake of clarity, is harmless. In [**42**] it is shown that any operator algebra is contained in a unital operator algebra, and that the operator norms on the unization are uniquely defined.

Definition 1.6.4.1. Let $\mathcal{A}$ be an operator algebra and $A \to B(\mathcal{H})$ a complete isometry. Its *unitization* $\mathcal{A}^+$ is the algebraic unitization $\mathcal{A} \oplus \mathbb{C}$ with product $(a, z)(b, w) = (ab + aw + zb, zw)$. Identifying $\mathcal{A}^+$ with

$$\{a + \lambda \cdot 1 : a \in \mathcal{A}, \lambda \in \mathbb{C}\},$$

$\mathcal{A}^+$ becomes an operator algebra.

This definition is independent of the choice of complete isometry [**42**].The standard C^*-unitization is a special case of this. Now note that a rigged module over $\mathcal{A}$ is automatically a rigged module over $\mathcal{A}^+$. Hence for a smooth C^*-algebra A, with smooth structure $\{\mathcal{A}_i\}$, any smooth C^*-module $\mathcal{E}$ is a direct summand in $\mathcal{H}_{A^+}$. Hence a smooth Hermitian connection $\nabla : \mathcal{E} \to \mathcal{E}\tilde{\otimes}_A\Omega^1(A^+)$ always exists. Since

$$E^i\tilde{\otimes}_{\mathcal{A}^+_{i+1}}\mathcal{A}^+_i = E^i,$$

and the modules $E^i \subset \mathcal{E}$ are $\mathcal{A}_i$-essential, the property $E^{i+1}\tilde{\otimes}_{\mathcal{A}^+_{i+1}}\mathcal{A}^+_i = E^i$ of proposition 1.4.2.2 remains valid:

$$E^i = E^i\mathcal{A}_i \cong E^{i+1}\tilde{\otimes}_{\mathcal{A}^+_{i+1}}\mathcal{A}^+_i\mathcal{A}_i = E^{i+1}\tilde{\otimes}_{\mathcal{A}_{i+1}}\mathcal{A}_i.$$

A smooth connection on a smooth KK-cycle $(\mathcal{E}, D)$ for (A, B) is now a connection $\nabla : \mathcal{E} \to \mathcal{E}\tilde{\otimes}_{B^+}\Omega^1(B^+)$, such that $[\nabla, D_i]$ is completely bounded for all i. If B was already unital, then B^+ decomposes as $B \oplus \mathbb{C}$, so $\Omega^1(B)$ is a direct summand in $\Omega^1(B^+)$ in this case. A connection $\nabla^+ : \mathcal{E} \to \mathcal{E}\tilde{\otimes}^+_B\Omega^1(B^+)$ therefore induces a connection $\nabla : \mathcal{E} \to \mathcal{E}\tilde{\otimes}_B\Omega^1(B)$ and vice versa. Although this is not a bijective correspondence, the ambiguity is irrelevant, as the subsequent discussion shows.

Let $(\mathcal{E}, S, \nabla)$ and $(\mathcal{F}, T, \nabla')$ be nonunital unbounded bimodules, with connections. The tensor product $\mathcal{E}\tilde{\otimes}_A \mathcal{F}$ equals $\mathcal{E}\tilde{\otimes}_{A^+}\mathcal{F}$, by definition. To show that the product of nonunital cycles is again a cycle, only the compact resolvent property needs some care. Thus we have to show that $a(S\tilde{\otimes}1 + 1\tilde{\otimes}_\nabla T)$ is compact, using compactness of $a(1+S^2)^{-1}$ and $b(1+T^2)^{-1}$, for $a \in \mathcal{A}$, $b \in \mathcal{B}$. The argument in proposition 1.6.2.2 carries through up to the reduction of the necessity to show that $a(1+\frac{1}{2}s^2)^{-1}(i+t)^{-1}$ is compact for $a \in \mathcal{A}$. To achieve this, one employs an approximate unit e_α for $\mathcal{B}$. The operators $a(1+\frac{1}{2}s^2)^{-1}e_\alpha(i+t)^{-1}$ are shown to be compact in the the same way as before. Then, $a(1+\frac{1}{2}s^2)^{-1}(i+t)^{-1}$ is their norm limit, hence compact. The validity of theorem 1.6.2.3 follows readily from this.

6.5. A category of spectral triples. Let A and B be smooth C^*-algebras. The results from 5.4 suggest that triples $(\mathcal{E}, D, \nabla)$ consisting of a smooth (A, B)-bimodule equipped with a smooth regular operator D and a smooth connection ∇ form a category, in which the composition law is

$$(\mathcal{E}, D, \nabla) \circ (\mathcal{E}', D', \nabla') := (\mathcal{E}\tilde{\otimes}\mathcal{F}, D\tilde{\otimes}1 + 1\tilde{\otimes}_\nabla D', \nabla\tilde{\otimes}_B\nabla').$$

An essential piece for this statement to hold is missing, and we will prove it now.

PROPOSITION 1.6.5.1. *Let A, B, C be C^*-algebras, $(\mathcal{E}, S, \nabla)$ and $(\mathcal{F}, T, \nabla')$ be (A, B)- and (B, C)-bimodules equipped with selfadjoint regular operators S and T, and C^1-connections ∇ and ∇', respectively. Suppose that $[T, b] \in \mathrm{End}^*_B(\mathcal{E})$ for all $b \in \mathcal{B}_1 \subset B$, a dense subalgebra of B. Then the product connection $\nabla\tilde{\otimes}_B\nabla'$ is an $S\tilde{\otimes}1 + 1\tilde{\otimes}_\nabla T$-connection.*

PROOF. Since

$$[\nabla\tilde{\otimes}_B\nabla', S\tilde{\otimes}1 + 1\tilde{\otimes}_\nabla T] = [\nabla\tilde{\otimes}_B\nabla', S\tilde{\otimes}1] + [\nabla\tilde{\otimes}_B\nabla', 1\tilde{\otimes}_\nabla T],$$

and $[\nabla\tilde{\otimes}_B\nabla', S\tilde{\otimes}1] = [\nabla, S]\tilde{\otimes}1$, which is completely bounded, we compute

$$(-1)^{\partial e}[\nabla\tilde{\otimes}_B\nabla', 1\tilde{\otimes}_\nabla T](e\otimes f)$$

to find

$$e\otimes[\nabla', T]f + \nabla_{\nabla'}(e)Tf + \nabla\tilde{\otimes}_B\nabla'(\nabla_T(e)f) - \nabla_T(e)\nabla'(f) - 1\tilde{\otimes}_\nabla T(\nabla_{\nabla'}(e)f).$$

The first term is completely bounded, and in working out the last four terms write $\nabla(e) = \sum e_i \otimes db_i$. Then

$$\begin{aligned}
\nabla_{\nabla'}(e)Tf &= \sum e_i \otimes [\nabla', b_i]Tf, &(1.15)\\
\nabla_T(e)\nabla'(f) &= \sum e_i \otimes [T, b_i]\nabla'(f), &(1.16)\\
\nabla\tilde{\otimes}_B\nabla'(\nabla_T(e)f) &= \sum e_i \otimes \nabla'[T, b_i]f + \nabla_{\nabla'}(e_i)[T, b_i]f, &(1.17)\\
1\tilde{\otimes}_\nabla T(\nabla_{\nabla'}(e)f) &= \sum_i e_i \otimes T[\nabla', b_i]f + \nabla_T(e_i)[\nabla', b_i]f. &(1.18)
\end{aligned}$$

Combining 1.15,1.16 and the first terms on the right hand sides of 1.17 and 1.18 give a term

$$\sum_i e_i \otimes [[\nabla', T], b_i]f = \nabla_{[\nabla', T]}(e)f,$$

and the terms remaining from 1.17 and 1.18 give a term

$$(\nabla_{\nabla'}\nabla_T - \nabla_T\nabla_{\nabla'})(e\otimes f).$$

Thus, we have shown that

$$[\nabla\tilde{\otimes}_B\nabla', 1\tilde{\otimes}_\nabla T] = 1\tilde{\otimes}_\nabla[\nabla', T] + [\nabla_{\nabla'}, \nabla_T],$$

which is a completely bounded map $\mathcal{E}\tilde{\otimes}_B\mathcal{F} \to \mathcal{E}\tilde{\otimes}_B\mathcal{F}\tilde{\otimes}_C\Omega^1(C)$. □

DEFINITION 1.6.5.2. Let A and B be C^*-algebras, and $(\mathcal{H}, D)$ and $(\mathcal{H}', D')$ be smooth spectral triples for A and B respectively. A C^k-*correspondence* $(\mathcal{E}, S, \nabla)$ between $(\mathcal{H}, D)$ and $(\mathcal{H}', D')$ is an unbounded C^k-(A, B)-bimodule with C^k-connection , such that $[S, \mathcal{A}_{i+1}] \subset \mathrm{End}^*_{\mathcal{B}_i}(E^i)$ and $\mathcal{H} \cong \mathcal{E}\tilde{\otimes}_B\mathcal{H}'$ and $D_i = (S\tilde{\otimes}1 + 1\tilde{\otimes}_\nabla D')_i$ for $i = 0, ..., k$ under this isomorphism. The correspondence is *smooth* if it is C^k for all k. Two correspondences are said to be equivalent if they are C^k- or smoothly unitarily isomorphic such that the unitary intertwines the operators. The set of isomorphism classes of such correspondences is denoted by $\mathfrak{Cor}_k(D, D')$ or $\mathfrak{Cor}(D, D')$ in the smooth case.

The requirement $[S, \mathcal{A}_{i+1}] \subset \mathrm{End}^*_{\mathcal{B}_i}(E^i)$ can be viewed as a transversality condition.

THEOREM 1.6.5.3. *There is a category whose objects are C^k-spectral triples and whose morphisms are the sets $\mathfrak{Cor}_k(D, D')$. The bounded transform $\mathfrak{b}(\mathcal{E}, D, \nabla) = (\mathcal{E}, \mathfrak{b}(D))$ defines a functor $\mathfrak{Cor}_k \to KK$.*

PROOF. Composition of correspondences $(\mathcal{E}, S, \nabla)$ and $(\mathcal{F}, T, \nabla')$ is defined by

$$(\mathcal{E}\tilde{\otimes}_B\mathcal{F}, S\tilde{\otimes}1 + 1\tilde{\otimes}_\nabla T, \nabla\tilde{\otimes}_B\nabla').$$

This is associative by theorem 1.5.3.2 and proposition 1.5.4.1, and defines a correspondence again by by propositions 1.5.4.3 and 1.6.2.2. That the composite of two C^k-correspondences is again a C^k-correspondence, follows by examining the diagram the diagram after theorem 1.5.4.5 and using the transversality condition. □

As mentioned in the introduction, a category with unbounded cycles as objects can be constructed in a similar way. A morphism of unbounded cycles $A \to (\mathcal{E}, D) \leftrightharpoons B$ and $A' \to (\mathcal{E}', D') \leftrightharpoons B'$ is given by a correspondence $A \to (\mathcal{F}, S, \nabla) \leftrightharpoons A'$ and a bimodule $B \to \mathcal{F}' \leftrightharpoons B'$, where B is represnted by compact operators. The bounded transform functor then takes values in the *morphim category* KK^2.

Furthermore, we would like to note that the category of spectral triples constructed is a 2-category. A morphism of morphisms $f : (\mathcal{E}, D, \nabla) \to (\mathcal{E}', D'\nabla')$ is given by an element $F \in \mathrm{Hom}^*_B(\mathcal{E}, \mathcal{F})$, commuting with the left A-module structures and making the diagrams

$$\begin{tikzcd} \mathcal{E} \arrow[r, "F"] \arrow[d, "D"'] & \mathcal{E}' \arrow[d, "D'"] \\ \mathcal{E} \arrow[r, "F"] & \mathcal{E}' \end{tikzcd} \qquad \begin{tikzcd} \mathcal{E} \arrow[r, "F"] \arrow[d, "\nabla"'] & \mathcal{E}' \arrow[d, "\nabla'"] \\ \mathcal{E}\tilde{\otimes}_B\Omega^1(B) \arrow[r, "F"] & \mathcal{E}'\tilde{\otimes}_B\Omega^1(B), \end{tikzcd}$$

commutative.

The external product of correspondences is defined in the expected way:

$$(\mathcal{E}, D, \nabla) \otimes (\mathcal{E}', D'\nabla') := (\mathcal{E}\overline{\otimes}\mathcal{E}', D\overline{\otimes}1 + 1\overline{\otimes}D', \nabla\overline{\otimes}1 + 1\overline{\otimes}\nabla).$$

In this way, $\mathfrak{Cor}$ becomes a symmetric monoidal category.

Definition 1.6.5.4. Let $(\mathcal{E}, S)$ and $(\mathcal{F}, T)$ be unbounded (A, B)-bimodules. They are said to be *weakly equivalent* if there exists a unitary in $u \in \mathrm{Hom}^*_B(\mathcal{E}, \mathcal{F})$ with the property that $u^*Tu - S$ is densely defined and extends to a bounded operator in $\mathrm{End}^*_B(\mathcal{E})$.

Weak equivalence is an equivalence relation. On the level of correspondences, all compatible connections (if they exist) become equivalent. The resulting category $\mathfrak{WCor}$, is the category of *weak correspondences* of spectral triples (or unbounded bimodules). The functor $\mathfrak{Cor} \to KK$ factors through the quotient map $\mathfrak{Cor} \to \mathfrak{WCor}$. When one is merely interested in obtaining K-theoretic information of some sort, working in $\mathfrak{WCor}$ can be much easier than working in $\mathfrak{Cor}$.

CHAPTER 2

Groupoids

Since its inception, one of the most important areas of appliciation for noncommutative geometry has been the realm of group actions. If a group Γ acts on a topological space X, it is natural to look at the orbit space X/Γ. The topology of this space depends heavily on properties of the action of Γ on X. If the action is free and proper, many properties of X, such as local compactness, or being a manifold, carry over to X/Γ. However, if this hypothesis is not satisfied, X/Γ can be far from a nice space. An extreme case is when the orbits are dense, and the topology of X/Γ is trivial. A now well known result due to Rieffel states that for a free and proper action, the C^*-algebras $C_0(X/\Gamma)$ and $C_0(X) \rtimes \Gamma$ are strongly Morita equivalent. The latter C^*-algebra continues to have good properties in case of a bad action. Another issue is that of orbit equivalence. In favourable cases, the orbits of the action of a discrete countable group Γ are generated by a single (partial) endomorphism $\sigma : X \to X$. Since σ need not be invertible, we can view this only as an action of the semigroup $\mathbb{N}$ of natural numbers. There is however a notion of semigroup crossed product, and the algebras $C(X) \rtimes_\sigma \mathbb{N}$ can be viewed as generalizations of Cuntz-Krieger algebras. Both types of crossed products can be obtained as groupoid C^*-algebras. It is expected that, from the groupoid point of view, the relation of orbit equivalence can, in favourable cases, be realized as a correspondence of groupoids. These groupoid correspondences yield C^*-bimodules over the groupoid C^*-algebras, thus making contact with the material discussed in chapter 1. Moreover, we show that a closed 1-cocycle $c : \mathcal{G} \to \mathbb{R}$ induces an odd unbounded $(C^*(\mathcal{G}), C^*(\mathcal{H}))$-bimodule, where $\mathcal{H} = \ker c$. Again, in favourable cases, the kernel algebra $C^*(\mathcal{H})$ carries a trace, inducing an index map $K_1(C^*(\mathcal{G})) \to \mathbb{C}$.

1. A category of groupoids

To account for the notion of Morita equivalence of groupoids, as well as the nonfunctoriality of groupoid algebras with respect to the ordinary notion of groupoid homomorphism, it is useful to broaden ones horizon and allow for a notion of correspondence of groupoids. A correspondence between groupoids $\mathcal{G}$ and $\mathcal{H}$ will be a space Z carrying suitable commuting left $\mathcal{G}$- and right $\mathcal{H}$-actions. This notion of morphism was pioneered by Hilsum and Skandalis [**27**] in the case of smooth étale groupoids. In later papers, other authors modified it to adapt to the general smooth case ([**37**]) and to the locally compact case ([**49**]). Our applications consider locally compact étale groupoids with Haar system. We do however obtain some results that hold in the non-étale case.

1.1. Groupoid bundles. In general, topological groupoids can be viewed as generalizations of both groups and topological spaces. Both of these occur as extreme cases of the following definition.

Definition 2.1.1.1. A *groupoid* $\mathcal{G}$ is a small category in which every morphism is invertible. The set of morphisms of $\mathcal{G}$ is denoted $\mathcal{G}^{(1)}$, and the objects $\mathcal{G}^{(0)}$. We identify $\mathcal{G}^{(0)}$ with a subset of $\mathcal{G}^{(1)}$ as identity morphisms. $\mathcal{G}$ is said to be a *locally compact Hausdorff groupoid* if $\mathcal{G}^{(1)}$ carries such a topology, and the domain and range maps

$$d, r : \mathcal{G}^{(1)} \to \mathcal{G}^{(0)} \subset \mathcal{G}^{(1)},$$

are continuous for this topology. $\mathcal{G}$ is said to be *étale* if r and d are local homeomorphisms.

Thus, a group can be regarded as a groupoid with just one object, and a topological space as groupoid with only identity morphisms. We will tacitly assume all groupoids to be locally compact and Hausdorff.

Example 2.1.1.2 (Crossed product groupoid). Let Γ be a locally compact group acting on a space X. We associate to it the *crossed product* $X \rtimes \Gamma$ of X by Γ. As a set this is just the cartesian product $X \times \Gamma$, and we identify X with $X \times \{e\}$ as the unit space of $X \rtimes \Gamma$. The structural maps are

$$d(x, \gamma) := x\gamma, \quad r(x, \gamma) := x, \quad (x, \gamma) \circ (x\gamma, \delta) := (x, \gamma\delta).$$

As with groups, there is a notion of a groupoid action on a space. This in turn can be used to define new groupoids from old. If $\phi_i : X_i \to Y$, $i = 1, 2$, are continuous maps between topological spaces X_i and Y, we denote the pull back, or *fibered product*, of the X_i over Y by

$$X_1 *_Y X_2 := \{(x_1, x_2) : \phi_1(x_1) = \phi_2(x_2)\}.$$

$X_1 *_Y X_2$ is the universal solution for commutative diagrams

$$\begin{tikzcd} X \arrow[r, "\psi_1"] \arrow[d, "\psi_2"'] & X_1 \arrow[d, "\phi_1"] \\ X_2 \arrow[r, "\phi_2"] & Y. \end{tikzcd}$$

In case one of the X_i is a groupoid $\mathcal{G}$ and a map $\rho : Z \to \mathcal{G}^{(0)}$ is given, it is convenient to write $\mathcal{G} \ltimes_\rho Z$ for the pull back with respect to d and ρ, and $Z \rtimes_\rho \mathcal{G}$ for the pull back with respect to r and ρ.

Definition 2.1.1.3. Let Z be a topological space and $\mathcal{G}$ a groupoid. A *left action* of $\mathcal{G}$ on Z consists of a continuous map $\rho : Z \to \mathcal{G}^{(0)}$, called the *moment map*, and a continuous map

$$\begin{aligned} \mathcal{G} \ltimes_\rho Z &\to Z \\ (\xi, z) &\mapsto \xi z, \end{aligned}$$

(the pull back is with respect to $d : \mathcal{G} \to \mathcal{G}^{(0)}$) with following properties:

- $\rho(\xi z) = r(\xi)$,
- $\rho(z)z = z$,
- If $(\xi_1, \xi_2) \in \mathcal{G}^2$ and $(\xi_2, z) \in \mathcal{G} \ltimes_\rho Z \to Z$ then $(\xi_1\xi_2)z = \xi_1(\xi_2 z)$.

The space Z is said to be a *left $\mathcal{G}$-bundle*.

A groupoid action gives a commutative diagram

$$\begin{tikzcd} Z & \mathcal{G} \ltimes_\rho Z \arrow[l] \arrow[r] \arrow[d] & Z \arrow[d, "\rho"] \\ \mathcal{G}^{(0)} \arrow[from=u, "\rho"'] & \mathcal{G} \arrow[l, "r"'] \arrow[r, "d"] & \mathcal{G}^{(0)}. \end{tikzcd}$$

The notion of right action is obtained by switching r and d and considering $Z \rtimes_\rho \mathcal{G}$. The spaces $Z \rtimes_\rho \mathcal{G}$ and $\mathcal{G} \ltimes_\rho Z$ are groupoids over Z. We will describe the structure for $Z \rtimes_\rho \mathcal{G}$. The structure for $\mathcal{G} \ltimes_\rho Z$ is similar but transposed. We have

$$Z \rtimes_\rho \mathcal{G} = \{(z,\xi) \in Z \times \mathcal{G} : \rho(z) = r(\xi)\},$$

and define

$$d(z,\xi) := z\xi, \quad r(z,\xi) = z, \quad (z,\xi)^{-1} = (z\xi, \xi^{-1}), \quad (z,\xi)(z\xi,\eta) = (z,\xi\eta).$$

This is well defined because Z is a $\mathcal{G}$-bundle.If Z carries both a left $\mathcal{G}$- and a right $\mathcal{H}$-action the actions are said to *commute* if

- $\forall(\xi,z) \in \mathcal{G} \ltimes_\rho Z, (z,\chi) \in Z \rtimes_\sigma \mathcal{H}, \quad (\xi z)\chi = \xi(z\chi),$
- $\forall(z,\chi) \in Z \rtimes_\sigma \mathcal{H}, \quad \rho(z\chi) = \rho(z),$
- $\forall(\xi,z) \in \mathcal{G} \ltimes_\rho Z, \quad \sigma(\xi z) = \sigma(z).$

Such a Z is called a *$\mathcal{G}$-$\mathcal{H}$-bibundle.* Moreover, the action is said to be (left- resp. right-) *proper* if the map

$$\begin{aligned} \mathcal{G} \ltimes_\rho Z &\to Z \times Z \\ (\xi, z) &\mapsto (\xi z, z), \end{aligned}$$

is proper, that is, inverse images of compact sets are compact.

Definition 2.1.1.4. Let Z be a $\mathcal{G}$-$\mathcal{H}$ -bibundle with moment maps $\rho : Z \to \mathcal{G}^{(0)}$ and $\sigma : Z \to \mathcal{H}^{(0)}$. The $\mathcal{G}$ action is said to be *left principal* if the map

$$\begin{aligned} \mathcal{G} \ltimes_\rho Z &\to Z *_{\mathcal{H}^{(0)}} Z \\ (\xi, z) &\mapsto (\xi z, z), \end{aligned}$$

is a homeomorphism. This is equivalent to saying that the $\mathcal{G}$-action is free, σ is an open surjection and induces a bijection $\mathcal{G}\backslash Z \to \mathcal{H}^{(0)}$. If the bibundle is both left- and right-principal, it is said to be an equivalence bibundle. Two groupoids $\mathcal{G}, \mathcal{H}$ are *Morita equivalent* if there exists an equivalence $\mathcal{G}$-$\mathcal{H}$-bibundle.

Example 2.1.1.5. If Γ is a locally compact group acting freely and properly on the space X, it becomes an equivalence bibundle for the crossed product groupoid $X \rtimes \Gamma$ (example 2.1.1.2) and the trivial groupoid X/Γ. The moment map ρ is the identity, and the action given by the action of Γ. For the right action, σ is the quotient map, while the action itself is trivial.

1.2. Correspondences. There is an obvious notion of groupoid homomorphism, it being a proper map $\phi : \mathcal{G}_1 \to \mathcal{G}_2$ that respects all the structure. These morphisms form a category $\mathfrak{G}$, with groupoids as objects. However, this notion of morphism is not appropriate when one considers groupoid algebras or Morita equivalence. As in the case of C^*-algebras, we can define correspondences which accomodate for the analogue of Morita equivalence of groupoids. We will see later that this extension is also well behaved with respect to taking groupoid C^*-algebras and C^*-correspondences.

DEFINITION 2.1.2.1. Let $\mathcal{G}, \mathcal{H}$ be locally compact Hausdorff groupoids. A *correspondence* from $\mathcal{G}$ to $\mathcal{H}$ is a proper left principal $\mathcal{G}$-$\mathcal{H}$ -bibundle Z. We tentatively denote this data by $\mathcal{G} \rightarrowtail Z \leftrightarrows \mathcal{H}$. Two correspondences $\mathcal{G} \rightarrowtail Z \leftrightarrows \mathcal{H}$ and $\mathcal{G} \rightarrowtail Z' \leftrightarrows \mathcal{H}$ are called isomorphic if there exists a homeomorphism $Z \to Z'$ intertwining all the relevant structures.

Groupoid correspondences can be composed, in the following way. Let $\mathcal{G}_1 \rightarrowtail Z_1 \leftrightarrows \mathcal{H}$ and $\mathcal{H} \rightarrowtail Z_2 \leftrightarrows \mathcal{G}_2$ with moment maps ρ_i, σ_i, $i = 1, 2$ be given. Form the fiber product $Z_1 *_{\mathcal{H}^{(0)}} Z_2$ with respect to the moment maps σ_i. There is an $\mathcal{H}$-action with respect to the moment map

$$\begin{aligned} \sigma : Z_1 *_{\mathcal{H}^{(0)}} Z_2 &\to \mathcal{H}^{(0)} \\ (z_1, z_2) &\mapsto \sigma_i(z_i). \end{aligned}$$

$\mathcal{H}$ acts (from the left) by $\chi(z_1, z_2) := (z_1\chi^{-1}, \chi z_2)$. Define

$$Z = Z_1 *_{\mathcal{H}} Z_2 := \mathcal{H} \backslash Z_1 *_{\mathcal{H}^{(0)}} Z_2.$$

Z carries a left $\mathcal{G}_1$- and a right $\mathcal{G}_2$-action, which can be checked to be a correspondence. The correspondence $\mathcal{G} \rightarrowtail \mathcal{G} \leftrightarrows \mathcal{G}$ is a unit for this composition up to isomorphism. Denote by $\mathfrak{Cor}_G(\mathcal{G}, \mathcal{H})$ the set of isomorphism classes of correspondences from $\mathcal{G}$ to $\mathcal{H}$. We have the following straightforward result.

PROPOSITION 2.1.2.2 ([**38**]). *Composition of correspondences is associative and unital up to isomorphism, and therefore the sets $\mathfrak{Cor}_G(\mathcal{G}, \mathcal{H})$ are the morphism sets of a category $\mathfrak{Cor}_G$ whose objects are all locally compact Hausdorff groupoids.*

There is a canonical (contravariant) functor $\mathfrak{G} \to \mathfrak{Cor}_G$ which is the identity on objects. To a proper groupoid homomorphism $\phi : \mathcal{G} \to \mathcal{H}$ it associates the correspondence

$$Z_\phi := \mathcal{H} *_{\mathcal{H}^{(0)}} \mathcal{G}^{(0)} = \{(\chi, u) \in \mathcal{H} \times \mathcal{G}^{(0)} : d(\chi) = \phi(u)\}. \tag{2.1}$$

The moment maps

$$\rho : Z_\phi \to \mathcal{G}^{(0)}, \quad \sigma : Z_\phi \to \mathcal{H}^{(0)},$$

are given by the coordinate projections and the actions by

$$((\chi, u), \xi) \mapsto (\chi\phi(\xi), d(\xi)), \quad (\eta(\chi, u)) \mapsto (\eta\chi, u).$$

Note that for the $\mathcal{G}$-action to be proper it is crucial that ϕ be proper.

THEOREM 2.1.2.3 ([**38**]). *Let $\mathfrak{Cor}_G$ be the category with objects all locally compact Hausdorff groupoids and morphisms given by correspondences. Two groupoids are Morita equivalent if and only if they are isomorphic as objects in $\mathfrak{Cor}_G$.*

In fact, in [**38**] proposition 2.1.2.2 and theorem 2.1.2.3 are proved in the setting of smooth groupoids, but these proofs carry over verbatim to the locally compact case.

1.3. Equivariant KK-theory. In his second paper on the subject, [**33**], Kasparov developed a version of KK-theory for C^*-algebras acted upon by a locally compact group G, and a version he called *representable* KK-theory, and denoted RKK. This last version is defined for $C_0(X)$-algebras, with X some locally compact space. The notion will be defined below, and the intuition one should have with it is that of a bundle of C^*-algebras over X. Both theories are extreme cases of the equivariant theory for groupoids, developed by LeGall in [**23**]. We will describe this theory here, even though we will only deal with the above mentioned extreme cases. The action of a groupoid on a space was has been described in some detail above. In this section, we are concerned with the action of a groupoid on a C^*-algebra. Recall that he center of an algebra A is denoted by ZA.

Definition 2.1.3.1. Let A be a C^*-algebra and X a locally compact space. A is said to be a $C_0(X)$-algebra if there is a *-homomorphism $C_0(X) \to Z\mathscr{M}(A)$ such that, under this map $\overline{C_0(X)A} = A$.

If A is a $C_0(X)$-algebra, we set $A_x := A/C_xA$. Here C_x is the maximal ideal of $C_0(X)$ consisting of functions vanishing at x. A_x is called the *fiber* of A at x. We denote by $\pi_x : A \to A_x$ the quotient map. There is an injection $A \to \prod_{x\in X} A_x$.

Definition 2.1.3.2. A $C_0(X)$-algebra A is a *continuous field of C^*-algebras* if for all $a \in A$ the function $x \mapsto \|a_x\|$ is continuous.

There is another way of defining a continuous field of C^*-algebras, as a continuous field of Banach spaces whose fibers are C^*-algebras. A *continuous field of Banach spaces* over X is a set $\mathscr{E}$ together with a map $\pi : \mathscr{E} \to X$, such that every fiber $E_x := \pi^{-1}(x)$ is a Banach space, and a subset $\mathscr{S} \subset \prod_{x\in X} E_x$ of sections with the following properties:

- For all $x \in X$, $E_x = \{s(x) : s \in \mathscr{S}\}$;
- for each $s \in \mathscr{S}$ the function $x \mapsto \|s(x)\|$ is continuous;
- $\mathscr{S}$ is *locally uniformly closed*: If $t \in \prod_{x\in X} E_x$ and for each $\epsilon > 0$ and $x_0 \in X$ there exists $s \in \mathscr{S}$ such that for all x in a neighbourhood of x_0 we have $\|t(x) - s(x)\| < \epsilon$, then $t \in \mathscr{S}$.

One straightforwardly checks that the algebra of C_0-sections of a continuous field of Banach spaces whose fibers are C^*-algebras is a $C_0(X)$-algebra with the norm continuity property. We will tacitly identify a continuous field of C^*-algebras with its C^*-algebra of C_0-sections. Similarly, one defines a continuous field of C^*-modules. If $\mathcal{E}$ is a C^*-module over a $C_0(X)$-algebra B, it decomposes into fibers. The fiber at x is

$$\mathcal{E}_x := \mathcal{E}\tilde{\otimes}_B B_x,$$

where we tensor over the fiber homomorphism $B \to B_x$. Clearly, $\mathcal{E}_x$ is a C^*-module over B_x. By identifying $\mathcal{E}$ with $\mathcal{E}B$, we obtain a homomorphism of $C_0(X)$ into $Z(\mathrm{End}^*_B(\mathcal{E}))$, which makes $\mathbb{K}_B(\mathcal{E})$ into a $C_0(X)$-algebra. If B is a continuous field of C^*-algebras, $\mathcal{E}$ is a continuous field of C^*-modules, and $\mathbb{K}_B(\mathcal{E})$ is a continuous field of C^*-algebras as well. $\mathrm{End}^*_B(\mathcal{E})$ is not a $C_0(X)$-algebra in general, as mentioned before. However, the function $x \mapsto \|T_x\|$ is still continuous. This follows from the description of the norm:

$$\|T_x\| := \sup_{e\in\mathcal{E}_x\setminus\{0\}} \frac{\|T_x e\|}{\|e\|},$$

and the fact that $\mathcal{E}$ is a continuous field.

Lemma 2.1.3.3. *Let B be a continuous field of C^*-algebras, and $\mathcal{E} \leftrightharpoons B$ a C^*-module. Then $T \in \mathbb{K}_B(\mathcal{E})$ if and only if $T_x \in \mathbb{K}_{B_x}(\mathcal{E}_x)$ for all $x \in X$.*

Proof. The only if direction is trivial. So suppose $T \in \mathrm{End}^*_B(\mathcal{E})$ and $T_x \in \mathbb{K}_{B_x}(\mathcal{E}_x)$. We will use local uniform closure of the set of C_0-sections of the continuous field $\mathbb{K}_B(\mathcal{E})$. Let $x_0 \in X$ and $\epsilon > 0$. We can find $e_{i,0}, f_{i,0} \in \mathcal{E}_x$, $i = 0, ..., n$,

such that
$$\|T_{x_0} - \sum_{i=0}^{n} e_{i,0} \otimes f_{i,0}\|_{x_0} < \epsilon.$$
Since the map $\mathcal{E} \to \mathcal{E}_x$ is surjective, we can lift the $e_{i,0}, f_{i,0}$ to elements $e_i, f_i \in \mathcal{E}$. The set
$$V := \{x \in X : \|T_x - \sum_{i=0}^{n} e_{i,x} \otimes f_{i,x}\|_x < \epsilon\},$$
is an open neighbourhood of x_0, which by definition has the desired property. Hence $T \in \mathbb{K}_B(\mathcal{E})$. □

A *morphism* of $C_0(X)$-algebras is a *-homomorphism that is also a morphism of $C_0(X)$-modules. Thus a morphism $\phi : A \to B$ of $C_0(X)$-algebras induces a map $\prod_{x\in X} A_x \to \prod_{x\in X} B_x$, whose components are denoted ϕ_x. For locally compact X, ϕ is injective, resp. surjective if and only if all its fiber maps ϕ_x are [**23**].

The notion of *restriction* of a $C_0(X)$-algebra to a closed subset $Y \subset X$ is defined to be the $C_0(Y)$-algebra $A_Y := A/C_Y A$, where C_Y is the ideal of functions vansihing on Y. Given a continuous map $p : Z \to X$ the $C_0(Z)$-algebra p^*A is the restriction of the $C_0(Z \times X)$-algebra $A\overline{\otimes}C_0(Z)$ to the graph of p, a closed subset of $Z \times X$. If $d, r : \mathcal{G} \to \mathcal{G}^{(0)}$ are the domain and range maps of a locally compact groupoid and A a $C_0(X)$-algebra, there are natural identifications $(d^*A)_\xi = A_{d(\xi)}$ and $(r^*A_\xi) = A_{r(\xi)}$. These constructions allow for the definition of $C_0(X)$ tensor products. Given two $C_0(X)$-algebras A and B, their tensor product $A\tilde{\otimes}_{C_0(X)}B$ is the restriction of the *maximal* C^*-tensor product $A\tilde{\otimes}B$, which is a $C_0(X \times X)$-algebra, to the diagonal
$$\{(x, x) \in x \times X : x \in X\}.$$
If $\mathcal{E}$ is a C^*-module over a $C_0(X)$-algebra and $p : Y \to X$ a continuous map, $p^*\mathcal{E} \coloneqq p^*B$ denotes the C^*-module $\mathcal{E}\tilde{\otimes}_B p^*B$.

Definition 2.1.3.4. Let $\mathcal{G}$ be a locally compact groupoid and A a $C_0(\mathcal{G}^{(0)})$-algebra. An action of $\mathcal{G}$ on A is an isomorphism $\phi : d^*A \to r^*A$ of $C_0(\mathcal{G})$-algebras, such that for any pair of composable elements (ξ, η) we have $\phi_{\xi\eta} = \phi_\xi\phi_\eta$. A is said to be a $\mathcal{G}$-algebra. Given a C^*-module $\mathcal{E}$ over a $\mathcal{G}$-algebra A, an action of $\mathcal{G}$ on $\mathcal{E}$ is given by a unitary $u \in \mathrm{End}^*_B(d^*\mathcal{E}, r^*\mathcal{E})$, such that $u_\xi u_\eta = u_{\xi\eta}$ for composable pairs (ξ, η).

We will always consider $C_0(\mathcal{G})$ with the trivial grading. As such, an action of $\mathcal{G}$ on a graded C^*-algebra is given by an isomorphism preserving the degree. A representation $\pi : A \to \mathrm{End}^*_B(\mathcal{E})$ of a $\mathcal{G}$-algebra A on a C^*-$\mathcal{G}$-module $\mathcal{E}$ is said to be *equivariant* if
$$\forall\xi \in \mathcal{G}, \forall a_{d(\xi)} \in A_{d(\xi)} : u_\xi \pi_{d(\xi)}(a_{d(\xi)})u^*_\xi = \pi_{r(\xi)}(\alpha_\xi a_{d(\xi)}).$$

Definition 2.1.3.5 ([**23**]). Let A, B be two $\mathcal{G}$-algebras, $\mathcal{E}$ an equivariant (A, B)-bimodule, and $F \in \mathrm{End}^*_B(\mathcal{E})$. $(\mathcal{E}, F)$ is said to be an *equivariant Kasparov module* if it is a Kasparov module in the ordinary sense and
$$\forall a \in r^*A : \pi(a)(u(s^*F)u^* - r^*F) \in r^*\mathbb{K}_B(\mathcal{E}).$$
Two such modules are said to be *unitarily equivalent* if there exists a degree 0 equivariant unitary intertwining the representations and the operators. We denote the set of unitary equivalence classes of such bimodules by $\mathbb{E}^{\mathcal{G}}(A, B)$. A *homotopy*

of such modules is an element of $\mathbb{E}^{\mathcal{G}}(A, B\tilde{\otimes}C([0,1]))$. The set of homotopy classes of $\mathcal{G}$-equivariant Kasparov modules is denoted $KK_0^{\mathcal{G}}(A,B)$.

As in the ordinary case, one defines higher order Kasparov modules using the Clifford algebras $\mathbb{C}_i$ (definition 1.2.1.1), i.e. $KK_i^{\mathcal{G}}(A,B) := KK_0^{\mathcal{G}}(A, B\tilde{\otimes}\mathbb{C}_i)$. $KK^{\mathcal{G}}(A,B)$ has the usual properties of a KK-bifunctor.

THEOREM 2.1.3.6 ([**23**]). *$KK_i^{\mathcal{G}}(A,B)$ is a group, and there exists an equivariant Kasparov product*

$$KK_i^{\mathcal{G}}(A,B)\otimes KK_j^{\mathcal{G}}(B,C)\to KK_{i+j}^{\mathcal{G}}(A,C),$$

for $\mathcal{G}$-algebras A, B and C, making $KK^{\mathcal{G}}$ into a category.

The Kasparov product is well behaved with respect to groupoid homomorphisms and even with respect to correspondences of groupoids. Since we will only make use of the first case, we state Le Gall's result in that setting.

THEOREM 2.1.3.7 ([**23**]). *Let $\mathcal{G}$,$\mathcal{H}$ and $\mathcal{K}$ be topological groupoids, A, B and C $\mathcal{H}$-algebras, and $\phi:\mathcal{K}\to\mathcal{H}$ and $\psi:\mathcal{G}\to\mathcal{K}$ continuous groupoid homomorphisms.*
- $\forall x\in KK_i^{\mathcal{H}}(A,B) : (\phi\circ\psi)^*(x)=\phi^*\circ\psi^*(x)\in KK_i^{\mathcal{G}}(\psi^*\phi^*A,\psi^*\phi^*B)$;
- $\phi^*(1_A)=1_{\phi^*A}\in KK_0^{\mathcal{G}}(\phi^*A,\phi^*A)$;
- $\forall x\in KK_i^{\mathcal{H}}(A,B), y\in KK_j^{\mathcal{H}}(B,C)$ *we have*

$$\phi^*(x)\otimes_{\phi^*B}\phi^*(y)=\phi^*(x\otimes_B y)\in KK^{\mathcal{G}}(\phi^*A,\phi^*C).$$

The case we will be most interested in is that of an ordinary space, viewed as a groupoid. Elements of $\mathbb{E}^X(A,B)$ are ordinary Kasparov modules subject to the extra condition that in the C^*-module $\mathcal{E}$ there is an equality $(af)eb = ae(fb)$ for all $a\in A$, $b\in B$ and $f\in C_0(X)$ [**33**]. The same adaptation is used for unbounded bimodules.

2. C^*-algebras and -modules

In order to obtain C^*-algebras from general groupoids, one needs the datum of a Haar system. This is a system of measures supported on the fibers of the range map r. Inversion in the groupoid yields a system of measures supported on the fibers of d. In the presence of a Haar system, a natural convolution product on the algebra $C_c(\mathcal{G})$ is defined, making it into a topological *-algebra. The reduced groupoid C^*-algebra $C_r^*(\mathcal{G})$ can then be obtianed as an algebra of endomorphisms of a canonical C^*-module over $C_0(\mathcal{G}^{(0)})$, as the completion of $C_c(\mathcal{G})$ in a canonical representation. If the groupoid happens to be étale, a canonical Haar system always exists.

2.1. Haar systems and the convolution algebra. The space $C_c(\mathcal{G})$ of continuous complex valued functions with compact support is a right module over $C_c(\mathcal{G}^{(0)})$ if we define

$$f*g(\xi):=f(\xi)g(d(\xi)),\quad f\in C_c(\mathcal{G}),\quad f\in C_c(\mathcal{G}^{(0)}).$$

$C_c(\mathcal{G})$ is an algebra under pointwise multiplication, but this algebra does not capture the structure of $\mathcal{G}$ as a groupoid. If $\mathcal{G}$ carries some extra structure, $C_c(\mathcal{G})$ carries both a noncommutative multiplication, encoding the groupoid structure, and a $C_c(\mathcal{G}^{(0)})$-valued inner product. This in turn allows for the definition of two canonical C^*-algebras, that need not be isomorphic.

DEFINITION 2.2.1.1 ([**46**]). Let $\mathcal{G}$ be a locally compact Hausdorff groupoid. A *Haar system* on $\mathcal{G}$ is a system of measures $\{\nu^x : x \in \mathcal{G}^{(0)}\}$ on $\mathcal{G}^{(1)}$ such that

- $\operatorname{supp} \nu^x = r^{-1}(x)$
- $\forall f \in C_c(\mathcal{G}), \quad \int_{\mathcal{G}} f(\xi) d\nu^{r(\eta)}(\xi) = \int_{\mathcal{G}} f(\eta\xi) d\nu^{d(\eta)}(\xi)$
- $\forall f \in C_c(\mathcal{G}), \quad g(x) := \int_{\mathcal{G}} f(\xi) d\nu^x(\xi) \in C(\mathcal{G}^{(0)})$.

Étale groupoids always admit a Haar system, consisting of counting measures on the fibers. There is a natural involution on $C_c(\mathcal{G})$ given by $f^*(\xi) := \overline{f(\xi^{-1})}$. The Haar system also allows us to define the *convolution product* on $C_c(\mathcal{G})$:

$$f * g(\eta) := \int_{\mathcal{G}} f(\xi) g(\xi^{-1}\eta) d\nu^{r(\eta)}.$$

This is an associative, distributive product that makes $C_c(\mathcal{G})$ into a topological *-algebra for the topology given by uniform convergence on compact subsets.

DEFINITION 2.2.1.2 ([**25**]). Let $\mathcal{G}$ be a locally compact Hausdorff groupoid with Haar system. Define

$$\|f\|_\nu := \sup_{u \in \mathcal{G}^{(0)}} \int_{\mathcal{G}} |f(\xi)| d\nu^u, \quad \|f\|_{\nu^{-1}} := \sup_{u \in \mathcal{G}^{(0)}} \int_{\mathcal{G}} |f(\xi^{-1})| d\nu^u,$$

and

$$\|f\|_I := \max\{\|f\|_\nu, \|f\|_{\nu^{-1}}\}.$$

Let $\mathcal{H}$ be a Hilbert space. A representation $\pi : C_c(\mathcal{G}) \to B(\mathcal{H})$ is called *admissible* if it is continuous with respect to the inductive limit topology on $C_c(\mathcal{G})$ and the weak operator topology on $B(\mathcal{H})$, and $\|\pi(f)\| \leq \|f\|_I$.

DEFINITION 2.2.1.3. The *full C^*-norm* on $C_c(\mathcal{G})$ is defined by

$$\|f\| := \sup\{\|\pi(f)\| : \pi \text{ admissible}\}.$$

The *full C^*-algebra* $C^*(\mathcal{G})$ is the completion of $C_c(\mathcal{G})$ with respect to this norm.

We can associate a canonical C^*-$C_0(\mathcal{G}^{(0)})$-module to a groupoid with Haar system via the pairing

$$\begin{aligned} C_c(\mathcal{G}) \times C_c(\mathcal{G}) &\to C_c(\mathcal{G}^{(0)}) \\ \langle f, h\rangle(u) \quad &:= \int_{\mathcal{G}} \overline{f(\xi^{-1})} h(\xi^{-1}) d\nu^u \xi. \end{aligned}$$

As usual, $C_c(\mathcal{G})$ gets a norm

$$\|f\|^2 := \|\langle f, f\rangle\| := \sup_{u \in \mathcal{G}^{(0)}} \int_{\mathcal{G}} |f(\xi^{-1})|^2 d\nu^u \xi.$$

We denote the completion by $L^2(\mathcal{G}, \nu)$. Since $C_c(\mathcal{G})$ acts on itself by convolution we get an embedding

$$C_c(\mathcal{G}) \hookrightarrow \operatorname{End}^*_{C(\mathcal{G}^{(0)})}(L^2(\mathcal{G}, \nu)).$$

DEFINITION 2.2.1.4 ([**46**]). The *reduced C^*-algebra* $C^*_r(\mathcal{G})$ of $\mathcal{G}$, is the completion of $C_c(\mathcal{G})$ in the norm $\|.\|_r$ it gets as an algebra of operators on $L^2(\mathcal{G}, \nu)$.

The approach we've taken to defining $C^*_r(\mathcal{G})$ is different from that in [**46**] and was first considered in [**34**]. As mentioned before, the C^*-algebras $C^*(\mathcal{G})$ and $C^*_r(\mathcal{G})$ are not isomorphic in general.

EXAMPLE 2.2.1.5. In the literature the C^*-algebras $C^*(X \rtimes \Gamma)$ and $C_r^*(X \rtimes \Gamma)$, with $X \rtimes \Gamma$ as in example 2.1.1.2, are referred to as the full and reduced *crossed products of* $C(X)$ *by* Γ.They are commonly denoted $C(X) \rtimes \Gamma$ and $C(X) \rtimes_r \Gamma$, but we will stick with the first notation.

2.2. Amenability. A sufficient analytic condition for the full and reduced groupoid C^*-algebras to be isomorphic is that of amenability. All groupoids of interest to us will turn out to be amenable, but the condition is very technical in nature, and the results we will use are proved elsewhere. Therefore we will restrict ourselves here to an overview of the concept, for the sake of completeness.We consider the following weakening of the notion of Haar system, with which the notion of amenability is formulated.

DEFINITION 2.2.2.1 ([**2**]). Let $\mathcal{G}$ be a locally compact groupoid, and $\pi : X \to Y$ an equivariant surjection of $\mathcal{G}$-bundles. A *continuous π-system* of measures on X is a family of measures $\{\nu^x\}$ such that

- supp $\nu^x = r^{-1}(x)$
- $\forall f \in C_c(\mathcal{G}), \quad g(x) := \int_{\mathcal{G}} f(\xi) d\nu^x(\xi) \in C(\mathcal{G}^{(0)})$.

π is called *amenable* if there exists a net $\{\nu_i\}$ of π-systems of probability measures such that

$$\int_{\mathcal{G}} d|\xi\nu_i^x - \nu_i^{\xi x}| \to 0,$$

uniformly on compact subsets of $X \rtimes \mathcal{G}$.

The net $\{\nu_i\}$ is sometimes referred to as an *approximate invariant continuous mean.* A groupoid $\mathcal{G}$ is called *amenable* if the map $r : \mathcal{G} \to \mathcal{G}^{(0)}$ is an amenable surjection. The action of a group Γ on a space X is said to be amenable if the crossed product groupoid $X \rtimes \Gamma$ is. For amenable groupoids, the full and reduced C^*-algebras coincide.

THEOREM 2.2.2.2 ([**2**],[**46**]). *Let $\mathcal{G}$ be a groupoid with Haar system. If $\mathcal{G}$ is amenable, then $C_r^*(\mathcal{G}) \cong C^*(\mathcal{G})$. That is, the full and reduced norms on $C_c(\mathcal{G})$ coincide.*

This is indeed a very useful result, since the reduced norm is much easier to work with than the full-norm. The notion of amenability is compatible with that of equivalence of groupoids as defined in the previous section.

THEOREM 2.2.2.3 ([**2**]). *Amenabiltiy is invariant under equivalence of locally compact groupoids.*

2.3. C^*-modules from correspondences. The construction of $C^*(\mathcal{G})$ and $C_r^*(\mathcal{G})$ is compatible with both the notion of correspondence and that of Morita equivalence.

THEOREM 2.2.3.1 ([**45**]). *Let $\mathcal{G}$,$\mathcal{H}$ be second countable locally compact Hausdorff groupoids and Z a biprincipal $\mathcal{G}$-$\mathcal{H}$- bibundle. The space $C_c(Z)$ can be completed into Morita equivalence correspondences*

$$C^*(\mathcal{G}) \rightarrowtail \mathcal{E}^Z \leftrightharpoons C^*(\mathcal{H}) \quad \text{and} \quad C_r^*(\mathcal{G}) \rightarrowtail \mathcal{E}_r^Z \leftrightharpoons C_r^*(\mathcal{H}).$$

That is, the maps

$$\mathcal{G} \mapsto C^*(\mathcal{G}), \quad \mathcal{G} \mapsto C_r^*(\mathcal{G}),$$

preserve Morita equivalence classes. They also carry correspondences into correspondences.

THEOREM 2.2.3.2 ([**49**]). *Let $\mathcal{G} \rightarrowtail Z \leftrightharpoons \mathcal{H}$ be a groupoid correspondence. The space $C_c(Z)$ can be completed into C^*-correpondences*

$$C^*(\mathcal{G}) \rightarrowtail \mathcal{E}^Z \leftrightharpoons C^*(\mathcal{H}) \quad \text{and} \quad C^*_r(\mathcal{G}) \rightarrowtail \mathcal{E}^Z_r \leftrightharpoons C^*_r(\mathcal{H}).$$

On the dense subspaces $C_c(\mathcal{G})$, $C_c(\mathcal{H})$ and $C_c(Z)$, explicit formulae for both the inner product(s) and module structures can be given. For later reference and completeness we give them here. For $\Phi \in C_c(Z)$, the right module action of $h \in C_c(\mathcal{H})$ is given by

$$\Phi \cdot h(z) := \int_{\mathcal{H}} \Phi(z\chi)h(\chi^{-1})d\nu^{\sigma(z)}\chi. \tag{2.2}$$

Similarly, the left action of $g \in C_c(\mathcal{G})$ on Φ is

$$g \cdot \Phi(z) := \int_{\mathcal{G}} g(\xi)\Phi(\xi^{-1}z)d\nu^{\rho(z)}\xi. \tag{2.3}$$

There is a $C_c(\mathcal{H})$-valued inner product on $C_c(Z)$:

$$\langle \Phi, \Psi\rangle_{\mathcal{H}}(\chi) := \int_{\mathcal{G}} \overline{\Phi(\xi^{-1}z)}\Psi(\xi^{-1}z\chi)dv^{\rho(z)}\xi. \tag{2.4}$$

In this formula, $z \in Z$ is chosen such that $\sigma(z) = r(\chi)$, and it is independent of choice because $\mathcal{G} \setminus Z \cong \mathcal{H}^{(0)}$, and finite because the $\mathcal{G}$-action is proper. $\langle \Phi, \Psi\rangle_{\mathcal{H}} \in C_c(\mathcal{H})$ by virtue of the properness of the $\mathcal{H}$-action. In case the $\mathcal{H}$ action is transitive, one defines a $C_c(\mathcal{G})$-valued inner product by

$$\langle \Phi, \Psi\rangle_{\mathcal{G}}(\eta) := \int_{\mathcal{H}} \Phi(\eta^{-1}z\chi)\overline{\Psi(z\chi)}d\nu^{\sigma(z)}\chi, \tag{2.5}$$

where $z \in Z$ is chosen in such a way that $\rho(z) = r(\eta)$. Again, the integral is independent of this choice by transitivity of the $\mathcal{H}$-action.

2.4. Functoriality for étale groupoids. The next step is to establish functoriality of the maps $\mathcal{G} \mapsto C^*(\mathcal{G})$ and $\mathcal{G} \mapsto C^*_r(\mathcal{G})$, viewed as maps $\mathfrak{Cor}_G \to \mathfrak{Cor}_{C^*}$. Recall that $\mathfrak{Cor}_{C^*}$ is the category of C^*-correspondences described in proposition 1.1.3.2. Functoriality has been established for Lie groupoids in [**37**], but a proof in the locally compact case is still lacking. However, in [**44**] it is shown that for smooth étale groupoids, the bimodule construction is functorial on the level of the uncompleted $(C^\infty_c(\mathcal{G}), C^\infty_c(\mathcal{G}))$-bimodule $C^\infty_c(Z)$. Combined with the line of proof employed in [**37**], this yields the following result.

THEOREM 2.2.4.1. *Let $\mathfrak{Cor}^{et}_G \subset \mathfrak{Cor}_G$ be the subcategory of étale groupoids. The maps $\mathcal{G} \mapsto C^*(\mathcal{G})$ and $\mathcal{G} \mapsto C^*_r(\mathcal{G})$ define functors $\mathfrak{Cor}^{et}_G \to \mathfrak{Cor}_{C^*}$.*

Let $\mathcal{G}_1 \to Z \leftrightharpoons \mathcal{H}$ and $\mathcal{H} \to W \leftrightharpoons \mathcal{G}_2$ be groupoid correspondences with moment maps ρ, σ and ρ', σ' respectively. The important part of the argument is showing that the C^*-modules $\mathcal{E}^Z \tilde{\otimes}_{C^*(\mathcal{H})} \mathcal{E}^W$ and $\mathcal{E}^{Z *_{\mathcal{H}} W}$ are canonically ismorphic. To this end the map

$$i : C_c(Z) \otimes_{C_c(\mathcal{H})} C_c(W) \to C_c(Z *_{\mathcal{H}} W),$$

defined by

$$(z, w) \mapsto \int_{\mathcal{H}} \Phi(zh)\Psi(h^{-1}w)d\nu^{\sigma(z)},$$

is shown to extend to an isometry ([**37**],[**44**])

$$\mathcal{E}^Z \tilde{\otimes}_{C^*(\mathcal{H})} \mathcal{E}^W \hookrightarrow \mathcal{E}^{Z *_{\mathcal{H}} W}.$$

Thus it suffices to show that this map has dense range. But for this Mrčun's argument applies: For a function $\Xi \in C_c(Z *_{\mathcal{H}} W)$ with support in an open set of the form $U * V$, where $U \subset Z$ and $V \subset W$ are such that σ and σ' are homeomorphisms onto their images and $\rho'(V) \subset \sigma(U)$, the map

$$\begin{aligned} j : V &\to U * V \\ v &\mapsto [\sigma^{-1}(\rho'(v)) \cap U, v], \end{aligned}$$

is a homeomorphism. Define $\Psi(v) := \Xi(j(v))$. The set $S := \sigma^{-1}\rho(\text{supp } \Psi) \cap U$ is compact in U, and hence there exists Φ with supp $\Psi \subset U$ such that $\Psi(S) = 1$. Then we have $\Xi = i(\Phi \otimes \Psi)$. Since the functions Ξ span $C_c(Z *_{\mathcal{H}} W)$, this shows that U has dense range. We believe that the functoriality result should be valid in the general case, maybe with some extra condition on the correspondences.

3. Cocycles and K-theory

The continuous cohomology of a groupoid generalizes that of a group. In this section we develop a connection between the cocycles defining the cohomology group $H^1(\mathcal{G}, \mathbb{R})$ and $K_1(C^*(\mathcal{G}))$. This is done by constructing for each closed real-valued 1-cocycle $c : \mathcal{G} \to \mathbb{R}$ an odd unbounded $(C^*(\mathcal{G}), C^*(\mathcal{H}))$-bimodule, where $\mathcal{H} = \ker c$. This in turn induces maps $K_1(C^*(\mathcal{G})) \to K_0(C^*(\mathcal{H}))$ and $K_0(C^*(\mathcal{G})) \to K_1(C^*(\mathcal{H}))$. According to properties of c, the K-groups of $C^*(\mathcal{H})$ can be more accessible than those of $C^*(\mathcal{G})$, thus paving a way to the calculation of invariants of $C^*(\mathcal{G})$.

3.1. Groupoid cocycles. The cohomology of groupoids can be developed in complete generality, by adapting the theory for groups, in a similar way as the notion of action is adapted. We will only be interested in the first cohomology group $H^1(\mathcal{G}, \mathbb{R})$ of a groupoid $\mathcal{G}$ with coefficients in $\mathbb{R}$. This group has a fairly straightforward definition in terms of homomorphisms $\mathcal{G} \to \mathbb{R}$.

DEFINITION 2.3.1.1. Denote by $Z^1(\mathcal{G}, \mathbb{R})$ the set of continuous homomorphisms $\mathcal{G} \to \mathbb{R}$. We will refer to the elements of $Z^1(\mathcal{G}, \mathbb{R})$ as *cocycles* on $\mathcal{G}$. Denote by $B^1(\mathcal{G}, \mathbb{R})$ the subset of those $c \in Z^1(\mathcal{G}, \mathbb{R})$ such that there exists a continuous function $f : \mathcal{G}^{(0)} \to \mathbb{R}$ such that $c(\xi) = f(r(\xi)) - f(d(\xi))$. The elements of $B^1(\mathcal{G}, \mathbb{R})$ are referred to as *coboundaries*. Finally, define $H^1(\mathcal{G}, \mathbb{R}) := Z^1(\mathcal{G}, \mathbb{R})/B^1(\mathcal{G}, \mathbb{R})$.

The kernel

$$\ker c := \{\xi \in \mathcal{G} : c(\xi) = 0\}$$

is a closed subgroupoid of $\mathcal{G}$, which we will denote by $\mathcal{H}_c$. It is immediate that $\mathcal{H}^{(0)} = \mathcal{G}^{(0)}$. $\mathcal{H}$ acts on $\mathcal{G}$ by both left- and right multiplication, and these actions are proper. We will always consider the action by multiplication from the right. $\mathcal{G} \to \mathcal{G} \leftrightharpoons \mathcal{H}$ is a groupoid correspondence. Any closed subgroupoid with Haar system $\mathcal{H} \subset \mathcal{G}$ is Morita equivalent to the crossed product $\mathcal{G} \ltimes_r \mathcal{G}/\mathcal{H}$, where the moment map $\mathcal{G}/\mathcal{H} \to \mathcal{G}^{(0)}$ for the action of $\mathcal{G}$ on $\mathcal{G}/\mathcal{H}$ is given by $[\chi] \mapsto r(\chi)$, whence the notation. $\mathcal{G} \ltimes_r \mathcal{G}/\mathcal{H}$ inherits a Haar system from $\mathcal{G}$, since we have

$$r^{-1}([\eta]) = \{(\xi, [\eta]) \in \mathcal{G} \ltimes_r \mathcal{G}/\mathcal{H} : d(\xi) = r(\eta)\} \cong d^{-1}(r(\eta)).$$

The equivalence correspondence is given by $\mathcal{G}$ itself with moment map

$$\begin{aligned} \rho : \mathcal{G} &\to (\mathcal{G} \ltimes_r \mathcal{G}/\mathcal{H})^{(0)} = \mathcal{G}/\mathcal{H} \\ \eta &\mapsto [\eta] \end{aligned}$$

equal to the quotient map. The left action is given by $(\xi, [\eta_1])\eta_2 = \xi\eta_2$ whenever $[\eta_1] = [\eta_2]$, and hence the bundle is left principal. The map $\sigma : \mathcal{G} \to \mathcal{H}^{(0)}$ is just equal to d. The bundle is right principal by construction.

Definition 2.3.1.2. A cocycle $c : \mathcal{G} \to \mathbb{R}$ is *regular* if $\mathcal{H} = \ker c$ admits a Haar system, and *closed* if the canonical projection $\mathcal{G} \to \mathcal{G}/\mathcal{H}$ is closed.

From the above discussion, it follows that for a regular cocycle, the full and reduced C^*-agebras of $\mathcal{G}$ and $\mathcal{G} \ltimes_r \mathcal{G}/\mathcal{H}$ are Morita equivalent. If $\mathcal{G}$ is an étale groupoid, any closed subgroupoid admits a Haar system, as is the case when $\mathcal{G}$ is a Lie groupoid and c is smooth. For the map $\mathcal{G} \to \mathcal{G}/\mathcal{H}$ to be closed it is sufficient that the image of c be a discrete subgroup of $\mathbb{R}$. The sum of two closed cocycles is again a closed cocycle.

Renault [**46**] showed that a 1-cocycle $c \in Z^1(\mathcal{G}, \mathbb{R})$ defines a one-parameter group of automorphisms of $C^*(\mathcal{G})$ by

$$u_t f(\xi) = e^{itc(\xi)} f(\xi).$$

Furthermore he showed that if $c \in B^1(\mathcal{G}, \mathbb{R})$, the automorphism group is inner, i.e. implemented by a strongly continuous family of unitaries in the multiplier algebra of $C^*(\mathcal{G})$.

Proposition 2.3.1.3. *Let $c : \mathcal{G} \to \mathbb{R}$ be a regular cocycle. The operators*

$$\begin{aligned} U_t : C_c(\mathcal{G}) &\to C_c(\mathcal{G}) \\ U_t f(\xi) &= e^{itc(\xi)} f(\xi) \end{aligned}$$

extend to a one parameter group of unitaries in $\mathrm{End}_{C^*(\mathcal{H})}(\mathcal{E}^{\mathcal{G}})$ *(resp.* $\mathrm{End}_{C^*_r(\mathcal{H})}(\mathcal{E}^{\mathcal{G}}_r)$*), implementing the one parameter group of automorphisms u_t of (the image of) $C^*(\mathcal{G})$, resp. $C^*_r(\mathcal{G})$.*

Proof. The identity $\langle U_t f, U_t g\rangle_{\mathcal{H}} = \langle f, g\rangle_{\mathcal{H}}$ is proved by a straightforward computation. To see that U_t implements u_t, just compute:

$$\begin{aligned} U_t(f * U_t^* g)(\eta) &= e^{itc(\eta)} \int_{\mathcal{G}} f(\xi) e^{-itc(\xi^{-1}\eta)} g(\xi^{-1}\eta) d\nu^{r(\eta)} \\ &= \int_{\mathcal{G}} f(\xi) e^{itc(\xi)} g(\xi^{-1}\eta) d\nu^{r(\eta)} \\ &= (u_t f) * g(\eta). \end{aligned}$$

□

3.2. An odd bimodule. The generator of the one parameter group described in proposition 2.3.1.3 is closely related to the cocycle c. On the level of $C_c(\mathcal{G})$, pointwise multiplication by c induces a derivation, which we wil further investigate in this section.

PROPOSITION 2.3.2.1. *Let $\mathcal{G}$ be a locally compact Hausdorff groupoid with Haar system, $c : \mathcal{G} \to \mathbb{R}$ a regular cocyle, and $\mathcal{H} = \ker c$. The operator*

$$\begin{aligned} D : C_c(\mathcal{G}) &\to C_c(\mathcal{G}) \\ f(\xi) &\mapsto c(\xi)f(\xi), \end{aligned}$$

is a $C_c(\mathcal{H})$-linear derivation of $C_c(\mathcal{G})$ considered as a bimodule over itself. Moreover, it extends to a selfadjoint regular operator in the C^*-modules $\mathcal{E}^{\mathcal{G}} \leftrightharpoons C^*(\mathcal{H})$ and $\mathcal{E}^{\mathcal{G}}_r \leftrightharpoons C^*_r(\mathcal{H})$.

PROOF. It is clear that D is $C_c(\mathcal{H})$-linear and the following computation

$$\begin{aligned} f * Dg(\eta) &= \int_{\mathcal{G}} f(\xi) Dg(\xi^{-1}\eta) d\nu^{r(\eta)} \\ &= \int_{\mathcal{G}} f(\xi) c(\xi^{-1}\eta) g(\xi^{-1}\eta) d\nu^{r(\eta)} \\ &= c(\eta) \int_{\mathcal{G}} f(\xi) g(\xi^{-1}\eta) d\nu^{r(\eta)} - \int_{\mathcal{G}} c(\xi) f(\xi) g(\xi^{-1}\eta) d\nu^{r(\eta)} \\ &= \mathcal{D}(f * g)(\eta) - (Df) * g(\eta), \end{aligned}$$

shows it is a derivation. Furthermore, it is straightforward to check that

$$\langle Df, g\rangle_{\mathcal{H}} = \langle f, Dg\rangle_{\mathcal{H}},$$

using formula 2.4. Thus, D is closable, and we will denote its closure by D as well. It is regular because on $C_c(\mathcal{G})$ we have

$$(1 + D^*D)f(\xi) = (1 + c^2(\xi))f(\xi),$$

and this clearly has dense range. The same goes for $D + i$ and $D - i$, restricted to $C_c(\mathcal{G})$. Therefore, by lemma 1.1.4.8, these operators are bijective, and hence the Cayley transform $\mathfrak{c}(D)$ (1.6) is unitary. Then, by corollary 1.1.4.10, it follows that D is selfadjoint. $\square$

The operator D is of course the generator of the one-parameter group of proposition 2.3.1.3. If the cocycle c happens to be closed and $K \subset \mathcal{G}^{(0)}$ is compact, the induced map $c : (K \times \mathbb{R}) \cap \mathcal{G}/\mathcal{H} \to \mathbb{R}$ is proper. This can be seen using the identification

$$\mathcal{G}/\mathcal{H}_c = \{(r(\xi), c(\xi)) : \xi \in \mathcal{G}\}.$$

It is a key fact in the subsequent proof.

THEOREM 2.3.2.2. *Let $\mathcal{G}$ be a locally compact Hausdorff groupoid and $c : \mathcal{G} \to \mathbb{R}$ a closed regular cocycle. The operator D from proposition 2.3.2.1, makes the correspondences*

$$C^*(\mathcal{G}) \rightarrowtail \mathcal{E}^{\mathcal{G}} \leftrightharpoons C^*(\mathcal{H}), \quad C^*_r(\mathcal{G}) \rightarrowtail \mathcal{E}^{\mathcal{G}}_r \leftrightharpoons C^*_r(\mathcal{H}),$$

into unbounded bimodules.

PROOF. The derivation property implies that the commutators $[D, f]$ are bounded for $f \in C_c(\mathcal{G})$. They are given by convolution by Df. So it remains to show that D has $C^*(\mathcal{H})$-compact resolvent. To this end, let $f, \Phi \in C_c(\mathcal{G})$. The operator $f \circ (1 + D^2)^{-1}$ acts as

$$f \circ (1 + D^2)^{-1}\Phi(\eta) = \int_{\mathcal{G}} f(\xi)(1 + c^2(\xi^{-1}\eta))^{-1}\Phi(\xi^{-1}\eta) d\nu^{r(\eta)}\xi.$$

The action of

$$g \in C_c(\mathcal{G} \ltimes_r \mathcal{G}/\mathcal{H}) \subset C^*(\mathcal{G} \ltimes_r \mathcal{G}/\mathcal{H}) = \mathbb{K}_{C^*(\mathcal{H})}(\mathcal{E}^{\mathcal{G}}),$$

is given by

$$\begin{aligned} g\Psi(\eta) &= \int_{\mathcal{G} \ltimes_r \mathcal{G}/\mathcal{H}} g(\xi_1, [\xi_2])\Psi(\xi^{-1}\eta) d\nu^{[\eta]}(\xi_1, [\xi_2]) \\ &= \int_{\mathcal{G}} g(\xi, [\xi^{-1}\eta])\Psi(\xi^{-1}\eta) d\nu^{r(\eta)}\xi. \end{aligned}$$

Thus, if we show that for each $f \in C_c(\mathcal{G})$ the function

$$k_f(\xi, [\eta]) := (1 + c^2(\eta))^{-1} f(\xi)$$

is a norm limit of elements in $C_c(\mathcal{G} \ltimes_r \mathcal{G}/\mathcal{H})$, then we are done.

Define

$$K_n := (r(\text{supp } f) \times \mathbb{R}) \cap c^{-1}([-n, n]) \subset \mathcal{G}/\mathcal{H},$$

where we view c as a map $\mathcal{G}/\mathcal{H} \to \mathbb{R}$. Then

$$\cdots \subset \cdots \subset K_n \subset K_{n+1} \subset \cdots \subset \mathcal{G}/\mathcal{H},$$

is a filtration of $(r(\text{supp } f) \times \mathbb{R}) \cap \mathcal{G}/\mathcal{H}$ by compact sets. Moreover, we may assume that the image of c is not a bounded set in $\mathbb{R}$, and that $K_n \neq K_{n+1}$ (if not, just rescale). Thus, there exist cutoff functions

$$e_n : \mathcal{G}/\mathcal{H} \to [0, 1],$$

with

$$e_n = 1 \quad \text{on } K_n, \quad e_n = 0 \quad \text{on } \mathcal{G}/\mathcal{H} \setminus K_{n+1}.$$

Define

$$k_f^n(\xi, [\eta]) := e_n([\eta]) k_f(\xi, [\eta])$$

such that $c(K_n) \subset [-n, n]$. Recall from definitions 2.2.1.2 and 2.2.1.4 that $\| \cdot \|_r \leq \| \cdot \| \leq \| \cdot \|_I$, so it suffices to show that $\|k_f^n - k_f^m\|_I \to 0$ as $n > m \to \infty$. For $n > m$ we can estimate:

$$\begin{aligned} \|k_f^n - k_f^m\|_\nu &= \sup_{[\eta] \in \mathcal{G}/\mathcal{H}} \int_{\mathcal{G} \ltimes_r \mathcal{G}/\mathcal{H}} |k_f^n(\xi, [\eta]) - k_f^m(\xi, [\eta])| d\nu^{[\eta]} \\ &= \sup_{[\eta] \in \mathcal{G}/\mathcal{H}} \int_{\mathcal{G}} |k_f^n(\xi, [\eta]) - k_f^m(\xi, [\eta])| d\nu^{r(\eta)} \\ &= \sup_{[\eta] \in \mathcal{G}/\mathcal{H}} \int_{\mathcal{G}} |(e_n - e_m)(\eta)(1 + c^2(\eta))^{-1} f(\xi)| d\nu^{r(\eta)} \\ &\leq \frac{1}{1 + m^2} \sup_{[\eta] \in \mathcal{G}/\mathcal{H}} \int_{\mathcal{G}} |f(\xi)| d\nu^{r(\eta)} \\ &= \frac{1}{1 + m^2} \|f\|_\nu. \end{aligned}$$

For $\|k_f^n - k_f^m\|_{\nu^{-1}}$ a similar computation yields the estimate

$$\|k_f^n - k_f^m\|_I \leq \frac{1}{1 + m^2} \|f\|_I,$$

proving that the sequence k_f^n is Cauchy for $\| \cdot \|_I$ and hence for $\| \cdot \|$ and $\| \cdot \|_r$. Furthermore, it converges to $f(1 + D^2)^{-1}$. □

Several well known examples of spectral triples can be obtained using this theorem. In the sequel we encounter some truly bivariant examples.

EXAMPLE 2.3.2.3 (The circle). Consider $\mathbb{R}$ as a groupoid, and take $c = \mathrm{id} : \mathbb{R} \to \mathbb{R}$. The kernel of c is a point, so $C^*(\mathcal{H}_c) = \mathbb{C}$. The spectral triple so obtained is the Fourier transform of the canonical Dirac triple $(C(S^1), L^2(S^1), i\frac{\partial}{\partial x})$ on S^1. One obtains this triple directly from the embedding $\mathbb{Z} \to \mathbb{R}$.

EXAMPLE 2.3.2.4 (Crossed products by subgroups of $\mathbb{R}$). Let $(C(X), \mathcal{H}, \mathcal{D})$ be a commutative smooth spectral triple, $G \subset \mathbb{R}$ a subgroup acting smoothly (in the sense of the given spectral triple) on X. The projection homomorphism $c : X \rtimes G \to G \subset \mathbb{R}$ gives an unbounded $C^*(X \rtimes G) \to (\mathcal{E}, D_c) \leftrightharpoons C(X)$. $\mathcal{E}$ is a completion of $C_c(G) \otimes C(X)$, which is a free $C(X)$-module, and the Levi-Civita connection is a D_c connection in this case. Thus one can form the Kasparov product of this module with the given specral triple to obtain a spectral triple for $C(X) \rtimes G$. One in particular obtains the noncommutative tori in this way (see section 3.4).

3.3. Continuous quasi-invariant measures. An interesting class of cocyles $c : \mathcal{G} \to \mathbb{R}$ comes from certain well-behaved measures on the unit space $\mathcal{G}^{(0)}$. For this class of cocycles, the kernel algebra $C^*(\mathcal{H})$ carries a canonical trace.$\tau : C^*(\mathcal{H}) \to \mathbb{C}$. Composing the induced homomorphism $\tau_* : K_0(C^*(\mathcal{H})) \to \mathbb{C}$ with the homomorphism $K_1(C^*(\mathcal{G})) \to K_0(C^*(\mathcal{H}))$ induced by the bimodule coming from c, yields an index map $K_1(C^*(\mathcal{G})) \to \mathbb{C}$.

DEFINITION 2.3.3.1. Let $\mathcal{G}$ be a groupoid with Haar system $\{\nu^x\}$ and μ be a positive Radon measure on $\mathcal{G}^{(0)}$. ν^μ denotes a measure on $\mathcal{G}$, the measure *induced by* μ, and is defined by

$$\int_{\mathcal{G}} f(\xi) d\nu^\mu(\xi) := \int_{\mathcal{G}^{(0)}} \int_{\mathcal{G}} f(\xi) d\nu^x(\xi) d\mu(x).$$

The measure μ is said to be *quasi-invariant* in ν^μ is equivalent to its inverse ν_μ, induced by the corresponding right Haar system on $\mathcal{G}$. The function

$$\Delta := \frac{d\nu^\mu}{d\nu_\mu} : \mathcal{G} \to \mathbb{R}_{\geq 0},$$

is called the *modular function* of μ. If this function is continuous, then μ is said to be *continuous.*

The modular function is an almost everywhere homomorphism [**46**]. That is, it is a *measurable* cocycle on $\mathcal{G}$. We will only be interested in continuous measures, and it that case Renault's result is rephrased as follows.

PROPOSITION 2.3.3.2. *Let $\mathcal{G}$ be a groupoid with Haar system and μ a continuous quasi-invariant measure on $\mathcal{G}^{(0)}$. Then the modular function $\Delta : \mathcal{G} \to \mathbb{R}_{\geq 0}$ is a continuous cocycle.*

The measure μ defines a positive functional τ on the algebra $C_c(\mathcal{G})$.

$$\begin{aligned} \tau : C_c(\mathcal{G}) &\to \mathbb{C} \\ f &\mapsto \int_{\mathcal{G}^{(0)}} f d\mu. \end{aligned}$$

It extends to both $C^*(\mathcal{G})$ and $C^*_r(\mathcal{G})$, but in general does not yield a trace. However, if the measure μ is continuous, the Radon-Nikodym cocycle c induces a one-parameter group u_t of automorphisms of $C^*(\mathcal{G})$, as mentioned before proposition 2.3.1.3.

DEFINITION 2.3.3.3. Let A be a C^*-algebra and u_t a strongly continuous one parameter group of automorphisms of A. A KMS- β-*state* on A, relative to u_t, is a state $\sigma : A \to \mathbb{C}$, such that the for all a, b in some dense subalgebra of A the function

$$F : t \mapsto \sigma(a u_t b)$$

admits a continuous bounded continuation to the strip $\{z \in \mathbb{C} : 0 \leq \mathfrak{Im} z \leq \beta\}$ that is homolomorphic on the interior, such that

$$F(t + i\beta) = \sigma(u_t(b)a).$$

THEOREM 2.3.3.4 ([**46**]). *Let μ be a continuous quasi-invariant measure on $\mathcal{G}$. The functional τ is a KMS -1-state for the one parameter group of automorphisms associated to the Radon-Nikodym cocycle on $\mathcal{G}$.*

A measured groupoid is called *unimodular* if $\Delta = 1$ ν^μ-almost everywhere. For continuous measures, the following proposition is a corollary of theorem 2.3.3.4, but it holds for general measures.

PROPOSITION 2.3.3.5. *Let $\mathcal{G}$ be a unimodular measured groupoid. Then the functional $\tau : C_c(\mathcal{G}) \to \mathbb{C}$ is a trace.*

PROOF. Compute

$$\begin{aligned} \tau(f * g) &= \int_{\mathcal{G}^{(0)}} \int_{\mathcal{G}} f(\xi) g(\xi^{-1} x) d\nu^x \xi d\mu(x) \\ &= \int_{\mathcal{G}^{(0)}} \int_{\mathcal{G}} f(\xi) g(\xi^{-1}) d\nu^x \xi d\mu(x) \\ &= \int_{\mathcal{G}^{(0)}} \int_{\mathcal{G}} f(\xi^{-1}) g(\xi) d\nu^x \xi d\mu(x) \\ &= \tau(g * f), \end{aligned}$$

where we used unimodularity of $\mathcal{G}$ in the third line. ☐

COROLLARY 2.3.3.6. *Let $\mathcal{G}$ be a groupoid with Haar system, μ a continuous quasi-invariant measure such that the cocycle Δ is regular. Then $\tau : C^*(\mathcal{H}) \to \mathbb{C}$ is a trace, for $\mathcal{H} = \ker \Delta$.*

If μ is quasi-invariant, $\Delta(\xi) \neq 0$ for all ξ in $\mathcal{G}$. Hence we can compose it with the logarithm $\ln : \mathbb{R}_+ \xrightarrow{\sim} \mathbb{R}$, to obtain a real valued cocycle $c_\mu \in Z^1(\mathcal{G}, \mathbb{R})$. We will refer to this element as the *Radon-Nikodym cocycle* on $\mathcal{G}$.

COROLLARY 2.3.3.7. *Let $\mathcal{G}$ be a groupoid with Haar system, μ a continuous quasi-invariant measure such that the cocycle Δ is closed and regular. Then μ induces an index homomorphism $\mathrm{Ind}_\mu : K_1(C^*(\mathcal{G})) \to \mathbb{C}$.*

PROOF. By theorem 2.3.2.2, the Radon-Nikodym cocycle c_μ defines an element $[D] \in KK_1(C^*(\mathcal{G}), C^*(\mathcal{H}))$ and the Kasparov product with $[D]$ gives a group homomorphism $\otimes_{[D]} : K_1(C^*(\mathcal{G})) \to K_0(C^*(\mathcal{H}))$. The trace τ induces a homomorphism $\tau_* : K_0(C^*(\mathcal{H})) \to \mathbb{C}$. Hence we can define $\mathrm{Ind}_\mu := \tau_* \circ \otimes_{[D]} : K_1(C^*(\mathcal{G})) \to \mathbb{C}$. ☐

Note that, in fact, we get an index map $K_1(C^*(\mathcal{G})) \to \mathbb{C}$ for any closed regular cocycle whose kernel is unimodular with respect to some quasi invariant measure.

3.4. The noncommutative torus as a quotient. Recall that the noncommutative 2-torus, topologically is the C^*-algebra of an irrational rotation action on the circle S^1. More precisely, for $\theta \in (0,1)$ consider the action of $\mathbb{Z}$ on S^1, given by rotation over an angle $2\pi\theta$:

$$e^{2\pi it} \cdot n := e^{2\pi i(t+n\theta)}.$$

We denote the corresponding crossed product groupoid by $S^1 \rtimes_\theta \mathbb{Z}$. Lebesgue measure λ is quasi-invariant for this action, and we get a representation of $C^*(S^1 \rtimes_\theta \mathbb{Z})$ as bounded operators on the Hilbert space $\mathcal{H} := L^2(S^1 \rtimes_\theta \mathbb{Z}, \nu^\lambda)$, where the Haar system ν is given by counting measures on the fiber. The subalgebra $C_c^\infty(S^1 \times_\theta \mathbb{Z})$ comes equipped with two canonical derivations:

$$\partial_1 f(x,n) := inf(x,n), \quad \partial_2 f(x,n) := \partial f(x,n).$$

The operator

$$D := \begin{pmatrix} 0 & -i\partial_1 - \partial_2 \\ -i\partial_1 + \partial_2 & 0 \end{pmatrix},$$

is an odd, unbounded operator on $\mathcal{H} \oplus \mathcal{H}$, with compact resolvent. Moreover, $C^*(S^1 \rtimes_\theta \mathbb{Z})$ acts on this graded Hilbert space by the diagonal representation. The commutators $[D,f]$ are bounded, for $f \in C_c^\infty(S^1 \times_\theta \mathbb{Z})$, which is dense in $C^*(S^1 \rtimes_\theta \mathbb{Z})$. The above described structure is the *canonical spectral triple on* $C^*(S^1 \rtimes_\theta \mathbb{Z})$. It is smooth in the sense of Connes.

The trivial cocycle $c : S^1 \rtimes_\theta \mathbb{Z} \to \mathbb{Z}$, given by projection on the first factor, gives us an unbounded bimodule via theorem 2.3.2.2. As a C^*-module, this is just $\ell^2(\mathbb{Z})\tilde{\otimes}C(S^1)$, and the operator D_c acts as $e_n \mapsto ne_n$, where e_n are the canonical basis vectors for $\ell^2(\mathbb{Z})$. The Levi-Civita connection

$$\nabla : \ell^2(\mathbb{Z})\tilde{\otimes}C(S^1) \to \ell^2(\mathbb{Z})\tilde{\otimes}\Omega^1(C(S^1)),$$

satisfies $[\nabla, D_c] = 0$. Now consider the canonical spectral triple on the circle algebra $C(S^1)$. This triple is odd and its operator is given by ordinary differentiation. The C^*-module is smooth for the ordinary smooth structure on the circle. Moreover, D_c satisfies

$$[D_c, C^*(S^1 \rtimes_\theta \mathbb{Z})_{i+1}] \subset \mathrm{End}^*_{C^i(S^1)}(\ell^2(\mathbb{Z})\tilde{\otimes}C^i(S^1)),$$

the transversality condition for correspondences.

THEOREM 2.3.4.1. *$(\ell^2(\mathbb{Z})\tilde{\otimes}C(S^1), D_c, \nabla)$ is a correspondence of the spectral triples $(C^*(S^1 \rtimes \mathbb{Z}), \mathcal{H} \oplus \mathcal{H}, D)$ and $(C(S^1), L^2(S^1), i\partial)$.*

PROOF. By the above discussion, $(\ell^2(\mathbb{Z})\tilde{\otimes}C(S^1), D_c, \nabla)$ is a smooth bimodule with connnection. The product of these two odd cycles is given by

$$\begin{pmatrix} 0 & D_c\tilde{\otimes}1 - i\tilde{\otimes}_\nabla i\partial \\ D_c\tilde{\otimes}1 + i\tilde{\otimes}_\nabla i\partial & 0 \end{pmatrix},$$

cf.1.14. This exactly reduces to the operator D on $\mathcal{H} \oplus \mathcal{H}$. $\square$

In this way, we can regard the noncommutative torus as the quotient manifold for the rotation action of $\mathbb{Z}$ on the circle.

4. Crossed products

For any locally compact group Γ acting on a locally compact space X, one can form the crossed product groupoid $X \rtimes \Gamma$, cf. example 2.1.1.2. A Haar system on $X \rtimes \Gamma$ is obtained by setting $\nu^x = \delta_x \otimes \lambda$, with δ_x the Dirac measures at $x \in X$ and λ a left Haar measure on Γ. If the group Γ is discrete, λ is counting measure, and this groupoid is étale. In that case the algebra $C_c(X \rtimes \Gamma)$ can be identified with the algebraic tensor product $C(X) \otimes \mathbb{C}[\Gamma]$ of $C(X)$ by the complex group ring of Γ. We will give explicit descriptions of the modular function of crossed products by discrete groups, and canonical connections associated to equivariant maps.

4.1. The modular function. Suppose now that a Γ-quasi-invariant measure μ on X, is given. Define a map

$$\begin{aligned} c' : X \rtimes \Gamma &\to \mathbb{R}_{>0} \\ (x,\gamma) &\mapsto \frac{d\gamma\mu}{d\mu}(x\gamma). \end{aligned}$$

LEMMA 2.4.1.1. *Let Γ be a topological group acting on the space X, and μ a Γ-quasi invariant measure on X. The modular function of the measured groupoid $X \rtimes \Gamma$ is*

$$\Delta(x,\gamma) = \Delta_\Gamma(\gamma)\frac{d\gamma\mu}{d\mu}(x\gamma).$$

PROOF. We compute the relevant integral:

$$\begin{aligned} \int_{X\rtimes\Gamma} f(\xi^{-1})d\nu^\mu &= \int_X\int_\Gamma f(x\gamma,\gamma^{-1})d\lambda(\gamma)d\mu(x) \\ &= \int_\Gamma\int_X f(x,\gamma^{-1})d\gamma\mu(x)d\lambda(\gamma) \\ &= \int_\Gamma\int_X f(x,\gamma)d\gamma^{-1}\mu(x)d\lambda^{-1}(\gamma) \\ &= \int_\Gamma\int_X f(x,\gamma)\frac{d\gamma^{-1}\mu}{d\mu}d\mu(x)\frac{d\lambda^{-1}}{d\lambda}d\lambda(\gamma) \\ &= \int_X\int_\Gamma f(x,\gamma)\frac{d\lambda^{-1}}{d\lambda}\frac{d\gamma^{-1}\mu}{d\mu}d\lambda(\gamma)d\mu(x) \\ &= \int_X\int_\Gamma f(\xi)\frac{d\lambda^{-1}}{d\lambda}\frac{d\gamma^{-1}\mu}{d\mu}d\lambda(\gamma)d\mu(x). \end{aligned}$$

Therefore

$$\Delta(x,\gamma) = \frac{d\nu^\mu}{d\nu^{-\mu}}(x,\gamma) = \left(\frac{d\lambda^{-1}}{d\lambda}(\gamma)\frac{d\gamma^{-1}\mu}{d\mu}(x)\right)^{-1} = \Delta_\Gamma(\gamma)\frac{d\mu}{d\gamma^{-1}\mu(x)}(x)$$

Then using that $\frac{d\mu}{d\gamma^{-1}\mu(x)}(x) = \frac{d\gamma\mu}{d\mu(x)}(x\gamma)$ gives the desired expression ☐

In particalur we see that c' is a homomorphism, which for unimodular groups, and in particular for discrete groups, equals the modular fucntion of $X \rtimes \Gamma$. In general the maps $x \mapsto \frac{d\gamma^*\mu}{d\mu}(x)$ are only measurable, but they still define a measurable homomorphism $X \rtimes \Gamma \to \mathbb{R}$.

4.2. Equivariant maps. Let Γ and Γ' be countable discrete groups and $\phi : \Gamma \to \Gamma'$ a proper homomorphism. Correspondingly, let X and X' be locally compact spaces, acted upon by Γ and Γ', respectively. A proper map $\psi : X \to X'$ is called ϕ-*equivariant* if $\psi(x\gamma) = \psi(x)\phi(\gamma)$. If ψ is ϕ-equivariant, the pair (ψ, ϕ) will be referred to as an equivariant pair. Such an equivariant pair induces a proper groupoid homomorphism $\psi \rtimes \phi : X \rtimes \Gamma \to X' \rtimes \Gamma'$, and hence a correspondence

$$Z_{\psi\rtimes\phi} := \{((\psi(x), \phi(\gamma)), x) : (x, \gamma) \in X \rtimes \Gamma\}.$$

For fixed $\gamma \in \Gamma$, the subset

$$Z^{(\gamma)} := \{((y, \gamma), x) : \psi(x) = y\gamma\} = Z_{\psi\rtimes\phi} \cap Y \times \{\gamma\} \times X \subset Z_{\psi\rtimes\phi},$$

is open, closed and compact, since the crossed product groupoid $Y \rtimes \Gamma'$ is étale. The function $1_{Z^{(\gamma)}}$ is equal to $\pi^*1_{Y\times\{\gamma\}}$, where $\pi : Z_{\psi\rtimes\phi} \to Y \rtimes \Gamma'$ is projection on the first factor. We denote by $\mathbb{P}$ the coset space $\phi(\Gamma)/\Gamma'$ and set

$$Z^{[\gamma]} := \bigcup_{\delta\in[\gamma]} Z^{(\delta)},$$

where $[\gamma] \in \mathbb{P}$ denotes the class of γ. Now we identify $\mathbb{P}$ with a subset of $\phi(\Gamma)/\Gamma'$ by choosing a representative in each equivalence class. We choose e as the representative of $[e]$. For $\delta \in \Gamma'$, there is a map

$$\begin{aligned} \psi \rtimes_\delta \phi : X \times \Gamma &\to Z_{\psi\rtimes\phi} \\ (x, \gamma) &\mapsto ((\psi(x)\delta^{-1}, \delta\phi(\gamma)), x\gamma). \end{aligned}$$

Note that for $\delta = e$, this map is defined for any correspondence coming from a groupoid homomorphism.

LEMMA 2.4.2.1. *The maps $\psi \rtimes_\delta \phi$ are $C_c(X \rtimes \Gamma)$-module maps. Moreover, the module $C_c(Z_{\phi\rtimes\psi})$ is a free right module over $C_c(X \rtimes \Gamma)$, with basis*

$$\{\pi^*1_{Y\times\gamma} \in C_c(Z_{\phi\rtimes\psi}) : \gamma \in \mathbb{P}\}.$$

PROOF. Recall the right module structure given by a correspondence (equation 2.2), and compute

$$\begin{aligned} (\psi \rtimes_\delta \phi)^*(\Psi \cdot g)(\eta) &= \Psi \cdot g(\psi \rtimes_\delta \phi(\eta), d(\eta)) \\ &= \int_{X\rtimes\Gamma} \Psi(\psi \rtimes_\delta \phi(\eta)\psi \rtimes \phi(\xi), d(\xi))g(\xi^{-1})d\nu^{d(\eta)} \\ &= \int_{X\rtimes\Gamma} (\psi \rtimes_\delta \phi)^*\Psi(\eta\xi)g(\xi^{-1})d\nu^{d(\eta)} \\ &= \int_{X\rtimes\Gamma} (\psi \rtimes_\delta \phi)^*\Psi(\xi)g(\xi^{-1}\eta)d\nu^{r(\eta)} \\ &= (\psi \rtimes_\delta \phi)^*\Psi \cdot g(\eta). \end{aligned}$$

Next, we will prove that

$$\Psi = \frac{1}{|\ker\phi|} \sum_{\gamma\in\mathbb{P}} \pi^*1_{Y\times\gamma}(\psi \rtimes_\gamma \psi)^*\Psi,$$

which is always a finite sum since Ψ has compact support. This proves the second statement. We compute

$$\begin{aligned}\pi^*1_{Y\times\gamma}\cdot(\psi\rtimes\phi)^*\Psi(\eta,x) &= \int_{X\rtimes\Gamma}\pi^*1_{Y\times\gamma}(\eta\psi\rtimes\phi(\xi),d(\xi))(\psi\rtimes\phi)^*\Psi(\xi^{-1})d\nu^x\\ &= \int_{X\rtimes\Gamma}\pi^*1_{Y\times\gamma}(\eta\psi\rtimes\phi(\xi),d(\xi))\Psi(\psi\rtimes\phi(\xi^{-1}),r(\xi))d\nu^x\\ &= |\ker\phi|\Psi|_{Z^{[\gamma]}}(\eta,x).\end{aligned}$$

The last equality follows because $\pi^*1_{Y\times\gamma}(\eta\psi\rtimes\phi(\xi),d(\xi))$ is nonzero only if

$$\eta\psi\rtimes\phi(\xi)\in Y\times\gamma,$$

whereas $r(\xi)=x$. This determines ξ up to a choice of elements $\gamma\in\phi^{-1}(\gamma')$, with γ' such that $\eta=(x,\gamma')$. There are exactly $|\ker\phi|$ such γ, because $\psi\rtimes\phi(x,\gamma)=(\psi(x),\phi(\gamma))$. □

Lemma 2.4.2.2. *If ϕ is injective, the functions*

$$\{\pi^*1_{Y\times\gamma}\in C_c(Z_{\phi\rtimes\psi}):\gamma\in\mathbb{P}\},$$

form an orthonormal set for the $C_c(X\rtimes\Gamma)$-valued inner product.

Proof. By equation 2.4,

$$\begin{aligned}\langle\pi^*1_{Y\times\gamma},\pi^*1_{Y\times\gamma}\rangle(\chi) &= \int_{Y\rtimes\Gamma'}\pi^*1_{Y\times\gamma}(\xi^{-1}z)\pi^*1_{Y\times\gamma}(\xi^{-1}z\chi)d\nu^{\rho(z)}\xi\\ &= \int_{Y\rtimes\Gamma'}1_{Y\times\gamma}(\xi^{-1}\eta)1_{Y\times\gamma}(\xi^{-1}\eta(\psi\rtimes\phi)(\chi))d\nu^x\xi\\ &= 1_X(\chi).\end{aligned}$$

Here we have written $z=(\eta,x)$, and used the fact that for the penultimate expression to be nonzero ξ is completely determined by η, and that $(\psi\rtimes\phi)(\chi)=(\psi(x),\phi(\gamma))$ for $\chi=(x,\gamma)$. Injectivity of ϕ then yields the last equality. If γ and δ are such that $\gamma\neq\delta\in\mathbb{P}$, then $\langle\pi^*1_{Y\times\gamma},\pi^*1_{Y\times\gamma}\rangle(\chi)=0$ because $\psi\rtimes\phi(\chi)$ can only move the Γ'-part of $\xi^{-1}\eta$ around in the same coset. □

Combining the preceding lemmas, we get and explicit formula for the Levi-Civita connection associated to the basis of $C_c(Z_{\phi\rtimes\psi})$ coming from a choice of representatives of elements of $\mathbb{P}$.

Theorem 2.4.2.3. *Let X,Y be compact spaces acted upon by discrete countable groups Γ and Γ', respectively. For an equivariant pair (ψ,ϕ), the map*

$$\begin{aligned}\nabla: C_c(Z_{\psi\rtimes\phi}) &\to C_c(Z_{\psi\rtimes\phi})\otimes_{C_c(X\rtimes\Gamma)}\Omega^1(C_c(X\rtimes\Gamma))\\ \Phi &\mapsto \sum_{\gamma\in\mathbb{P}}\frac{1}{|\ker\phi|}\pi^*1_{Y\times\{\gamma\}}\otimes d(\psi\rtimes_\gamma\phi)^*(\Phi),\end{aligned}$$

defines a universal connection

$$\nabla:\mathcal{E}^{Z_{\psi\rtimes\phi}}\to\mathcal{E}^{Z_{\psi\rtimes\phi}}\tilde{\otimes}_{C^*(X\rtimes\Gamma)}\Omega^1(C^*(X\rtimes\Gamma)).$$

Moreover, if ϕ is injective, i.e. $|\ker\phi|=1$, then ∇ is Hermitian.

Proof. By lemma 2.4.2.1, the above expression is just the Levi-Civita connection associated to the basis from lemma 2.4.2.2. If ϕ is injective this same lemma says the basis orthonormal, hence ∇ is Hermitian. □

5. Semidirect products

We now discuss some generalities concerning semidirect product groupoids. These arise when one deals with a single, not necessarily invertible endomorphism of a topological space. Our examples are related to symbolic dynamics and topological Markov shifts. The C^*-algebras associated to groupoids of topological Markov shifts are the Cuntz-Krieger algebras and their generalizations. We show that an arbitrary map admitting a Markov partition can be covered by a topological Markov shift, and that this yields a correspondence of semidirect product groupoids.

5.1. Markov maps. In this section we recall the basic properties of Markov maps. This theory works most nicely in the case of a totally disconnected space. To get a fruitful theory for arbitrary spaces, it is convenient to consider partial endomorphisms. These are maps defined on an open subset $U \subset X$ into X.

Definition 2.5.1.1 ([**9**]). Let X be a compact Hausdorff space, and $\sigma : U \to X$ a continuous partial endomorphism. A finite partition $\mathscr{P} = \{U_i\}$ of U into open subsets U_i is called a *Markov partition with respect to* σ if $\sigma|_{U_i}$ is a homeomorphism onto its image and

$$U_i \cap U_j = \emptyset, \quad \bigcup_i U_i = U, \quad U_i \cap \sigma(U_j) \neq \emptyset \Rightarrow U_i \subset \sigma(U_j).$$

A partial endomorphism σ for which a Markov partition exists is called a *Markov map.*

As mentioned before, topological Markov shifts are the easiest and most frequently ocurring examples of Markov maps.

Example 2.5.1.2 (Shifts and subshifts). Let $\mathfrak{A} = \{a_0, a_1, \dots, a_n\}$ be a finite set of symbols, equipped with the discrete topology. Let

$$\mathcal{S}_{\mathfrak{A}} := \mathfrak{A}^{\mathbb{N}} = \prod_{i=0}^{\infty} \mathfrak{A},$$

the set of infinite words in the alphabet $\mathfrak{A}$, with the product topology. It can be metrized by

$$d((x_i),(y_i)) := e^{-\min\{i : x_i \neq y_i\}}.$$

The map

$$\begin{aligned} \sigma_{\mathfrak{A}} : \mathcal{S}_{\mathfrak{A}} &\to \mathcal{S}_{\mathfrak{A}} \\ (x_0 x_1 \dots) &\mapsto (x_1 x_2 \dots) \end{aligned}$$

is continuous and Markov for the partition given by the alphabet. The pair $(\mathcal{S}_{\mathfrak{A}}, \sigma_{\mathfrak{A}})$ is called the *full $\mathfrak{A}$-shift.* A closed $\sigma_{\mathfrak{A}}$ invariant subset W is called a *subshift.* It is called a *subshift of finite type* if there exists $n \in \mathbb{N}$ and a subset $w \subset \mathfrak{A}^n$ such that

$$W = \{(x_0 x_1 \dots) \in \mathcal{S}_{\mathfrak{A}} : \forall i \in \mathbb{N}, (x_i \dots x_{i+n}) \in w\}.$$

Proposition 2.5.1.3 ([**16**], 2.5.4). *Let W a subshift of finite type. Then W is isomorphic to a subshift of finite type such that $w \subset \mathfrak{A}^2$.*

Here, by isomorphism we mean the existence of an equivariant homeomorphism. Thus, the subset w can be defined by a $n+1 \times n+1$-matrix A with entries in $\{0,1\}$. Conversely, this matrix completely determines the subshift, which we will therefore denote by $\mathcal{S}_A$. For a Markov map $\sigma : U \to X$, with partition $\mathscr{P} = \{U_i\}_{i=0}^n$ one can consider the refinements

$$\mathscr{P}^n := \{\bigcap_{j=0}^n \sigma^{-j}(U_{i_j})\}.$$

LEMMA 2.5.1.4. *For all $k \leq n$, $\mathscr{P}^n$ is a Markov partition for σ^k.*

PROOF. Let U^k denote the domain of σ^k. Then we have

$$U^k = \sigma^{-1}(U^{k-1}).$$

Since $\bigcup \mathscr{P} = U$, it follows that $\bigcup \mathscr{P}^1 = U^1$ and inductively that $\bigcup \mathscr{P}^n = U^n$. To see that each $\mathscr{P}^n$ is Markov with respect to σ^k, $k \leq n$, denote

$$U_{i_0 \ldots i_n} = \bigcap_{j=0}^n \sigma^{-j}(U_{i_j}).$$

We have $\sigma(U_{i_0 \ldots i_n}) \subset U_{i_1 \ldots i_n}$. To see this, assume $U_{i_0 \ldots i_n} \neq \emptyset$ (otherwise the statement is vacuous). Then,

$$\sigma(\bigcap_{j=0}^n \sigma^{-j}(U_{i_j})) = \sigma(U_{i_0}) \cap U_{i_1} \cap \sigma^{-1}(U_{i_2}) \cap \cdots \cap \sigma^{-n+1}(U_{i_n}),$$

because σ is injective on U_{i_0}. Hence $\sigma(U_{i_0}) \cap U_{i_1} \neq \emptyset$ so $U_{i_1} \subset \sigma(U_{i_0})$ and

$$\sigma(U_{i_0}) \cap U_{i_1} \cap \sigma^{-1}(U_{i_2}) \cap \cdots \cap \sigma^{-n+1}(U_{i_n}) = U_{i_1} \cap \sigma^{-1}(U_{i_2}) \cap \cdots \cap \sigma^{-n+1}(U_{i_n}).$$

Using this, it follows by induction that if $U_{i_0 \ldots i_n} \cap U_{j_0 \ldots j_n} \neq \emptyset$ then $i_k = j_k$ for $k = 0, \ldots, n$. Also by induction, for $k \leq n$, one gets $\sigma^k(U_{i_0 \ldots i_n}) = U_{i_k \ldots i_n}$. Moreover, if

$$\sigma^k(U_{i_0 \ldots i_n}) \cap U_{j_0 \ldots j_n} \neq \emptyset,$$

then

$$U_{i_k \ldots i_n} \cap U_{j_0 \ldots j_n} \neq \emptyset,$$

and $j_0 = i_k, \ldots, i_n = j_{n-k}$, i.e. $U_{j_0 \ldots j_n} \subset U_{i_k \ldots i_n} = \sigma^k(U_{i_0 \ldots i_n})$ □

DEFINITION 2.5.1.5. Let $\sigma : U \to X$ be a Markov map such that $\overline{U} = X$. A Markov partition $\{U_i\}$ is called a *topological generator* if σ^k has dense domain for all k, and

$$\lim_{n \to \infty} \sup_{V \in \mathscr{P}^n} \text{diam}\,(V) = 0$$

Markov generators allow for a special type of equivariant coverings. The following theorem was proved in [**1**] for Markov automorphisms. We generalize it to arbitrary partial endomorphisms.

THEOREM 2.5.1.6. *Let X be a compact metric space and $\sigma : U \to X$ a Markov map, with partition $\mathscr{P} = \{U_i\}_{i=0}^m$. σ determines a subshift $\mathcal{S}_A$ of the full $m+1$-shift, with the matrix A determined by $\mathscr{P}$. Moreover, if $\mathscr{P}$ is a topological generator, there is a σ-equivariant map $\pi : \mathcal{S}_A \to X$.*

PROOF. Define $A = (a_{ij})$ by

$$a_{ij} = 1 \Leftrightarrow U_i \cap \sigma(U_j) \neq \emptyset, \quad a_{ij} = 0 \Leftrightarrow U_i \cap \sigma(U_j) = \emptyset.$$

Then we can identify

$$W := \{(i_0 i_1 \ldots) : \forall n \in \mathbb{N}, \bigcap_{j=0}^n \sigma^{-j}(U_{i_j}) \neq \emptyset\},$$

with $\mathcal{S}_A$. To show that W is shift invariant, we must show that for all n

$$\bigcap_{j=0}^{n} \sigma^{-j}(U_{i_j}) \neq \emptyset \Rightarrow \bigcap_{j=0}^{n-1} \sigma^{-j}(U_{i_{j+1}}) \neq \emptyset.$$

This follows from the proof of lemma 2.5.1.4. In case $\mathscr{P}$ is a topological generator, we have

$$\bigcap_{n}^{\infty} \overline{\bigcap_{j=0}^{n} \sigma^{-j}(U_{i_j})} \neq \emptyset \Rightarrow \bigcap_{n}^{\infty} \overline{\bigcap_{j=0}^{n} \sigma^{-j}(U_{i_j})} = \{p\}$$

for some unique $p \in X$, since diam $\bigcap_j^\infty \sigma^{-j}(A_{i_j}) = 0$, and the intersection is over a nested sequence of compact sets. This defines a map $\pi : W \to X$, which is equivariant by construction. It is continuous because for $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that $\mathrm{diam}(V) < \epsilon$ for all $V \in \mathscr{P}^n$. If $x, y \in W$ are such that $d(x, y) < e^{-n-1}$, then we have $x_i = y_i$ for $i = 0, ..., n$ and hence

$$\pi(x), \pi(y) \in \overline{\bigcap_{j=0}^{n} \sigma^{-i}(U_{i_j})},$$

whence $d(\pi(x), \pi(y)) < \epsilon$. □

For infinite Markov partitions, the situation is considerably more complicated. We do expect however, that a result similar to the above can be obtained.

5.2. Generalized Cuntz-Krieger algebras. In [**47**], Renault discusses transformation groupoids associated to continuous partial endomorphisms. The C^*-algebras of the groupoids coming from subshifts of finite type are the classical Cuntz-Krieger algebras.

Definition 2.5.2.1 ([**47**]). Let $\sigma : U \to X$ be a continuous partial endomorphism. The *semidirect product* groupoid of X by σ is defined as

$$X \rtimes \sigma := \{(x, n, y) \in X \times \mathbb{Z} \times X : \exists k, l \in \mathbb{N} \quad \sigma^k(x) = \sigma^l(y), k - l = n\},$$

$$r(x, n, y) = x, d(x, n, y) = y$$

and composition law

$$(x, n, y)(y, k, z) = (x, n + k, z).$$

Note that the unit space of $X \rtimes \sigma$ can be identified with X. This groupoid is étale, for the topology defined by the basis of open sets

$$(U, m, V, n) := \{(x, m - n, y) : (x, y) \in U \times V, \sigma^m(x) = \sigma^n(y)\},$$

where U and V are open subsets of the domain on which σ^m, respectively σ^n are injective. It comes equipped with a continuous homomorphism

$$c : X \rtimes \sigma \to \mathbb{Z} \subset \mathbb{R}$$
$$(x, n, y) \mapsto n.$$

This homomorhism is known as the *fundamental cocycle* of $X \rtimes \sigma$.

Proposition 2.5.2.2 ([**47**]). *The groupoid $X \rtimes \sigma$ is amenable, and the C^*-algebra $C^*(X \rtimes \sigma)$ is nuclear.*

Example 2.5.2.3 (Cuntz-Krieger algebras). $W \subset \mathfrak{A}^{\mathbb{N}}$ be a subshift of finite type of the full $m+1$-shift. W is given by a $(m+1)\times(m+1)$ matrix A with entries in $\{0,1\}$. The C^*-algebra $C^*(W \rtimes \sigma)$ is canonically isomorphic to the Cuntz-Krieger algebra $\mathcal{O}_A$, [**19**]. Recall that $\mathcal{O}_A$ is the universal C^*-algebra generated by partial isometries S_i, $i = 0, ..., m$, satisfying the relations

$$P_i P_j = 0, \quad Q_i = \sum_{j=0}^{m} A(i,j) P_j,$$

where $P_i = S_i S_i^*$ and $Q_i = S_i^* S_i$.

In case of an arbitrary partial endomorphism $\sigma : U \to X$ that admits a Markov generator, theorem 2.5.1.6 gives us an equivariant map $\pi : \mathcal{S}_A \to X$. Here A is the matrix given by the Markov partition. π defines a proper homomorphism $\mathcal{S}_A \rtimes \sigma_A \to X \rtimes \sigma$ of étale groupoids. This map fits into a commutative diagram of groupoids

$$\begin{tikzcd} \mathcal{S}_A \rtimes \sigma_A \arrow[r, "\pi"] \arrow[d, "c_A"'] & X \rtimes \sigma \arrow[d, "c"] \\ \mathbb{Z} \arrow[r, phantom, "="] & \mathbb{Z}. \end{tikzcd}$$

Here we denote by c_A and c the fundamental cocycles the respective groupoids. In particular, π maps $\ker c_A$ to $\ker c$. Denote by Z_π the $\mathcal{S}_A \rtimes \sigma_A$-$X \rtimes \sigma$ correspondence induced by π and by Z_c the $\ker c$-$\ker c_A$ induced by the restriction $\pi : \ker c_A \to \ker c$.

Proposition 2.5.2.4. *The $X \rtimes \sigma$-$\mathcal{H}_{c_A}$ bibundles Z_π and $X \rtimes \sigma *_{\mathcal{H}_c} Z_c$ are isomorphic.*

Proof. Recall that

$$Z_\pi = \{((x,n,y),y') \in X \rtimes \sigma \times \mathcal{S}_A : y = \pi(y')\},$$

and that

$$X \rtimes \sigma *_{\mathcal{H}_c} Z_c = X \rtimes \sigma *_X Z_c/\mathcal{H}_c.$$

Thus an element of $X \rtimes \sigma *_{\mathcal{H}_c} Z_c$ is represented by a pair $((x,n,y),((y,0,z),z'))$ with $\pi(z') = z$, modulo equivalence by the action of $\ker c$ on both factors. Using this it is straightforward to check that the map

$$Z_\pi \to X \rtimes \sigma *_{\mathcal{H}_c} Z_c$$
$$((x,n,y),y') \mapsto ((x,n,y),((y,0,y),y'))$$

is an equivariant bijection. In fact, its inverse is given by

$$((x,n,y),((y,0,z),z')) \mapsto ((x,n,z),z').$$

□

Corollary 2.5.2.5. *Let*

$$C^*(X \rtimes \sigma) \to (\mathcal{E}, D_c) \leftrightharpoons C^*(\mathcal{H}_c)$$

and

$$\mathcal{O}_A \to (\mathcal{E}_A, D_A) \leftrightharpoons C^*(\mathcal{H}_{c_A}),$$

be the unbounded bimodules associated to the cocycles c and c_A, cf. theorem 2.3.2.2. The bimodules $\mathcal{E}^{Z_\pi}$ and $\mathcal{E}^{Z_c}$ satisfy

$$\mathcal{E}\tilde{\otimes}_{C^*(\mathcal{H}_c)}\mathcal{E}^{Z_c} \xrightarrow{\sim} \mathcal{E}^{Z_\pi}\tilde{\otimes}_{\mathcal{O}_A}\mathcal{E}_A,$$

where the isomorphism is induced by the isomorphism of correspondences above.

PROOF. This follows directly from theorem 2.2.4.1. □

In order to make the above pair of bimodules into a correspondence of KK-cycles, we need to equip $\mathcal{E}^{Z_\pi}$ with a connection.In order to find an explicit connection, some further information on the structure of $\mathcal{E}^{Z_\pi}$ as a module over $\mathcal{O}_A$ is required. This is work in progress.

CHAPTER 3

Limit sets

Fundamental groups of Riemann surfaces or higher dimensional hyperbolic manifolds, those of Mumford curves, the modular group $SL(2,\mathbb{Z})$ and its finite index subgroups, free groups and automorphism groups of trees are all examples of a discrete countable group Γ acting on a space which is hyperbolic in the sense of Gromov. The *limit set* $\Lambda_\Gamma \subset \partial H$, on which Γ acts with dense orbits, has some remarkable properties. In important cases, such as that of modular curves, it recovers topological information of the curve $\mathbb{H}^2/\Gamma$. This can be viewed as a holography principle cf.([**40**],[**41**]). In [**14**] and [**17**], spectral triples for such boundary actions were constructed, in the case of Kleinian and p-adic Schottky groups. These constructions depend heavily on the fact that the groups under consideration are free. They employ the Patterson-Sullivan measure on the limit set, and the orbit equivalence of the boundary action of the free group with the full one-sided shift. In this chapter we pursue these ideas further, allowing for more general groups, in particular all the examples mentioned above. The Patterson-Sullivan measure remains the vital ingredient to obtain a noncommutative geometry for these limit sets, but the construction given does not yield spectral triples, but unbounded bimodules. They come from the cocycle construction described in theorem 2.3.2.2. In all cases of interest, the kernel groupoid is unimodular, and hence the bimodule defines an index map, as in 2.3.3.7. The ultimate goal is to show that orbit equivalences, inclusions of groups (coverings of curves), and the equivariant map from the tree of $SL(2,\mathbb{Z})$ to $\mathbb{P}^1(\mathbb{R})$ all give rise to correspondences of KK-cycles, relating the different index maps. Most of the material discussed in this chapter is work in progress at the moment this thesis is written.

1. Limit sets and Patterson-Sullivan measures

There is, in general, no preferred way to compactify a topological space. If the only purpose is to obtain a compact space $\overline{H}$ that contains a given space H as a dense subset, one might choose the one point compactification. This is the simplest choice possible, but not always the best. The largest reasonable compact space containing H as a dense subset is the Stone-Čech compactification $\check{H}$. From the operator algebraic point of view, these two spaces correspond to the unitization and the multiplier algebra of $C_0(H)$, respectively. We will be interested in spaces H that are hyperbolic in the sense of Gromov. These spaces admit a natural compactification, intermediate with respect to the extreme cases mentioned above. Much of the geometry of H is reflected in properties of the Gromov boundary ∂H. In the presence of the action of a discrete countable group Γ, ∂H decomposes into two parts, according to the behaviour of the action of Γ.

1.1. Hyperbolic spaces. We review some general definitions and results concerning Gromov hyperbolic spaces. The reason to present this general discussion is that it includes all the examples of interest to us at once. Recall that a metric space (H, d) is said to be *geodesic* if for any two points $x, y \in H$ there exists a continuous path of length $d(x, y)$ from x to y. We will assume all metric spaces to be geodesic.

DEFINITION 3.1.1.1. Let (H, d) be a metric space, and $p \in H$ a fixed basepoint. The *Gromov product* on H (with respect to p) is the map

$$\begin{aligned} &H \times H \to \mathbb{R} \\ &(x, y) \mapsto x \diamond_p y := \frac{1}{2}(d(x, p) + d(y, p) - d(x, y)). \end{aligned}$$

Let $\delta > 0$. H is said to be δ-*hyperbolic* if

$$\forall z \in H \quad x \diamond_p y \geq \min\{x \diamond_p z, y \diamond_p z)\} - \delta.$$

H is called *hyperbolic* if it is δ-hyperbolic for some $\delta > 0$. Furthermore a hyerbolic metric space H is called *proper* if all closed balls

$$B_\epsilon(x_0) := \{x \in H : d(x_0, x) \leq \epsilon\}$$

are compact.

Examples of hyperbolic metric spaces are the Poincaré half spaces $\mathbb{H}^n$ (or any equivalent Riemannian manifold), and trees. The latter have hyperbolicity constant $\delta = 0$. There is a natural way to compactify proper hyperbolic spaces. A sequence $(x_n)_{n\in\mathbb{N}}$ in H is said to *converge to infinity* if

$$\liminf_{i,j} x_i \diamond_p x_j = \infty.$$

Although the Gromov product does depend on the basepoint p, the notion of convergence to infinity does not. Let H_∞ be the set of sequences converging to infinity. We can define an equivalence relation on this set by

$$(x_n) \sim (y_n) \Leftrightarrow \liminf_{i,j} x_i \diamond_p y_j = \infty.$$

DEFINITION 3.1.1.2. Let H be a proper hyperbolic metric space. Define the *boundary of* H as

$$\partial H := H_\infty / \sim,$$

the quotient of H_∞ by the above equivalence relation.

We denote elements of the boundary by $[(x_n)]$, where (x_n) is some representative sequence. The boundary ∂H carries a natural topology, a basis of open sets for which is given by

$$U(x, r) := \{y \in \partial H : \exists (x_n), (y_n), [(x_n)] = x, [(y_n)] = y, \liminf_{i,j} x_i \diamond_p y_j \geq r > 0\}.$$

On $H \cup \partial H$ we can define a topology by extending the topology of H by the basis

$$V(x, r) := U(x, r) \cup \{y \in H : \exists (x_n), [(x_n)] = x, \liminf_{i} x_i \diamond_p y \geq r > 0\},$$

for $x \in \partial H$.

PROPOSITION 3.1.1.3. *Let H be a hyperbolic metric space. The spaces ∂H and $\overline{H} := H \cup \partial H$ are compact in the above topologies.*

Although the metric on H does not extend to the Gromov compactification $\overline{H}$, it is a metrizable space.

THEOREM 3.1.1.4 ([**24**]). *Let (H, d) be a proper hyperbolic geodesic metric space. There exists a real number $a_0 > 1$ such that for every $a \in [1, a_0]$ the boundary ∂H admits a metric d_a with the following property. Let $r : \mathbb{R} \to H$ be a geodesic connecting two distinct points $x, y \in \partial H$, then there exists $C \in \mathbb{R}$ such that*

$$C^{-1} a^{-d(p,r)} \leq d_a(x, y) \leq C a^{-d(p,r)},$$

where $d(p, r) = \inf\{d(p, r(t)) : t \in \mathbb{R}\}$.

A metric d_a with the above property is called a *visual metric with parameter a*. From now on we'll fix a and a visual metric d_a.

1.2. Quasiconformal measures. Let Γ denote a discrete countable group of isometries of some Gromov hyperbolic space H. The action of Γ on H, which we assume to be properly discontinuous, extends to an action on the boundary ∂H, by setting

$$[(x_n)]\gamma := [(x_n\gamma)].$$

Isometries of H can be classified according to their behaviour as homeomorphisms of ∂H. Isometries are named accordingly.

PROPOSITION 3.1.2.1 (Classification of isometries [**24**]). *Let γ be an isometry of a hyperbolic space H. Then exactly one of the following occurs:*

(1) *For all $p \in H$, the set $\{p\gamma^n : n \in \mathbb{Z}\}$ is bounded in H. γ is said to be* elliptic*.*
(2) *The homeomorphism $\gamma : \partial H \to \partial H$ has exactly two fixed points. γ is said to be* loxodromic*.*
(3) *The homeomorphism $\gamma : \partial H \to \partial H$ has exactly one fixed point. γ is said to be* parabolic*.*

EXAMPLE 3.1.2.2 (Kleinian and Fuchsian groups). Recall that a *Kleinian group* is a discrete subgroup of the group of orientation preserving isometries of hyperbolic n-space $\mathbb{H}^n$. If $n = 3$, the full isometry group of $\mathbb{H}^3$ is isomorphic to $PGL_2(\mathbb{C})$. A *Fuchsian group* is a Kleinian group of isometries of $\mathbb{H}^3$ that is conjugate to a subgroup of $PGL_2(\mathbb{R})$. These were the type of groups for which the subsequent theory was first considered [**50**],[**51**]. The elliptic, loxodromic and parabolic elements of proposition 3.1.2.1 coincide with the classical notions in this case.

EXAMPLE 3.1.2.3 (Hyperbolic groups). A discrete group Γ is said to be *word-hyperbolic* if its Cayley graph is Gromov-hyperbolic for the word metric.

DEFINITION 3.1.2.4. Let Γ be a discrete properly discontinuous group of isometries of a Gromov hyperbolic space H, with basepoint p. The *limit set* $\Lambda_\Gamma \subset \partial H$ is the set of accumulation points of $p\Gamma$. That is

$$\Lambda_\Gamma := \overline{p\Gamma} \setminus H \subset \partial H.$$

Γ is said to be *non-elementary* if Λ_Γ contains more than two points, and *elementary* otherwise.

This definition is independent of the choice of basepoint p. The complement of the limit set in ∂H is sometimes referred to as the *domain of discontinuity* of Γ. It is denoted by $\Omega_\Gamma := \partial H \setminus \Lambda_\Gamma$. This name is justified by the following theorem of Gromov.

THEOREM 3.1.2.5 ([**24**]). *Λ_Γ is the unique minimal domain for the action of Γ in ∂H. That is, any closed Γ-invariant subset of ∂H contains Λ_Γ.*

In particular, Γ has dense orbits in Λ_Γ. The limit set admits a natural measure with very specific properties. The construction of these measures was first developed by Patterson, for Fuchsian groups, and later extended by Sullivan to arbitrary Kleinian groups. Coornaert then extended their construction to the case of a general Gromov hyperbolic space.

DEFINITION 3.1.2.6. Let $x \in \partial H$ and $r_x : [0,\infty] \to H$ a geodesic ray such that $r(\infty) = x$. The *Busemann horofunction* $h_x : H \to \mathbb{R}$ associated to r_x is the function

$$h_x(p) := \lim_{t\to\infty} d(p, r_x(t)) - t.$$

The Busemann functions are continuous and satisfy

$$|h_x(p_1) - h_x(p_2)| \leq d(p_1, p_2).$$

In what follows we will be interested in the functions

$$h_\gamma(x) = h_x(p) - h_x(p\gamma),$$

viewed as functions of x. From the above inequality, we get an upper bound $|h_\gamma(x)| \leq d(p, p\gamma)$. We use them to define

$$\begin{aligned} j_\gamma : \partial H &\to \mathbb{R} \\ x &\mapsto a^{h_x(p) - h_x(p\gamma)}, \end{aligned}$$

where a is the visual parameter. Given a Borel measure μ on ∂H, we define $\gamma^*\mu(A) := \mu(A\gamma^{-1})$.

DEFINITION 3.1.2.7. A Borel measure μ on ∂H is called *quasiconformal of dimension* N, $N \in \mathbb{R}$, if the measures $\gamma^*\mu$ are absolutely continuous with respect to one another and there exists a constant $C \geq 1$ such that the Radon-Nikodym-derivatives $\frac{d\gamma^*\mu}{d\mu}$ satisfy

$$C^{-1} j_\gamma^N \leq \frac{d\gamma^*\mu}{d\mu} \leq C j_\gamma^N \qquad \text{mod } \mu. \tag{3.1}$$

The *Poincare series* of the group Γ is the series

$$g_s(x,y) := \sum_{\gamma\in\Gamma} a^{-sd(x,y\gamma)},$$

and the *critical exponent* $\delta_a(\Gamma)$ is the real number such that g_s converges for $s > \delta_a(\Gamma)$ and diverges for $s < \delta_a(\Gamma)$. Coornaert [**15**] proved the following:

THEOREM 3.1.2.8. *Suppose $\delta_a(\Gamma) < \infty$. For each $p \in H$, there exists a Γ-quasiconformal measure μ_p of dimension $\delta_a(\Gamma)$ on ∂H whose support is the limit set Λ_Γ.*

The measure μ_p is referred to as the *Patterson-Sullivan measure based at p*. Note that the dependence on the basepoint p actually gives us a field of measures, parametrized by H/Γ, as two points in the same orbit give the same measure. Sullivan calls this a *conformal density*. We will see in due course that, for ordinary hyperbolic space $H = \mathbb{H}^n$, this density can be interpreted as a generalized KMS-state on a continuous field of C^*-algebras over H/Γ. In the sequel we will always assume our groups to be of finite critical exponent, guaranteeing the existence of a Patterson-Sullivan meausre.

1.3. Quasi-convex-cocompact groups. General discrete group actions on Gromov hyperbolic space can be quite wicked. We now describe a class of actions, for which the Patterson Sullivan meausure is quite well-behaved. We denote by $Q(\Lambda_\Gamma) \subset H$ the *Gromov envelope* of the limit set Λ_Γ. It is defined as

$$Q(\Lambda_\Gamma) := \bigcup \{g(\mathbb{R}) : g : \mathbb{R} \to H, \quad \text{a geodesic for which} \quad g(\infty), g(-\infty) \in \Lambda_\Gamma\},$$

that is, the union of all geodesics in H with endpoints in Λ_Γ. Its closure is sometimes referred to as the *convex core* of the limit set.$Q(\Lambda_\Gamma) \subset H$ is closed and Γ-invariant.

DEFINITION 3.1.3.1. The group Γ is said to be *quasi-convex-cocompact* if the quotient space $Q(\Lambda_\Gamma)/\Gamma$ is compact.

In the theory of Kleinian groups, the above property is usually referred to as convex-cocompactness. Recall that a Kleinian group is said to be of the *first kind* if $\Lambda_\Gamma = S^{n-1}$. Such groups are *cofinite*, ie. the manifold $M := \mathbb{H}^n/\Gamma$ has finite volume. For Kleinian groups of the *second kind* (that is, those that are not of the first kind), convex cocompactness and cofiniteness are not he same concepts. Now let X be a metric space, $\delta \geq 0$ a real number. Recall that the expression

$$\mu_H^\delta(S) := \inf\{\sum_{i=0}^{\infty} \text{diam } (U_i)^\delta : S \subset \bigcap_{i=0}^{\infty} U_i\},$$

defines a Borel measure on X, called the δ-dimensional Hausdorff measure. It can be shown that if $0 < \mu_H^\delta(S) < \infty$ for some δ, then this δ is unique.

DEFINITION 3.1.3.2. Let X be a metric space. The *Hausdorff dimension* of X is the extended real number

$$\inf\{\delta : \mu_H^\delta(X) = 0\}.$$

For quasiconvex-cocompact groups, the Patterson-Sullivan and Hausdorff measures are closely related.

THEOREM 3.1.3.3 ([**15**]). *Let Γ be a quasiconvex-cocompact group and μ a quasi-conformal measure of dimension δ on Λ_Γ. There exists a constant $C \geq 1$ such that for all $S \subset \Lambda_\Gamma$*

$$C^{-1}\mu_H^\delta(S) \leq \mu(S) \leq C\mu_H^\delta(S).$$

In particular, $\delta(\Gamma)$-dimensional Hausdorff measure on Λ_Γ is nonzero and the Hausdorff dimension of Λ_Γ is $\delta(\Gamma)$.

2. Hyperbolic manifolds

Kleinian groups (example 3.1.2.2) occur as fundamental groups of hyperbolic manifolds. Their importance in this field stems from the Mostow rigidity theorem. This deep theorem losely reads as saying that two finite volume hyperbolic n-manifolds are isometric if and only if their fundamental groups are isomorphic.

2.1. A continuous field of C^*-algebras. A Kleinian group Γ uniformizes a an n-dimensional hyperbolic manifold $M := \mathbb{H}^n/\Gamma$. Recall that a *fundamental domain* for the action Γ is an open connected subset $D \subset \mathbb{H}^n$ with the following two properties:

- $\bigcup_{\gamma\in\Gamma} \overline{D}\gamma = \mathbb{H}^n$,
- $\forall \gamma \in \Gamma : \overline{D}\gamma \cap \overline{D} \subset \partial D$.

It is well known that a fundamental domain always exists, and it may or may not have vertices on the boundary $\partial\mathbb{H}^n = S^{n-1}$. Boundary vertices of a fundamental domain for Γ are called *cusps* of Γ. They correspond to the parabolic ends of the quotient manifold M, sometimes also referred to as cusps. If Γ has cusps, then M cannot possibly be compact, but can still have finite volume. We will review some constructions of Lott [**39**], relating the boundary operator algebra $C^*(\Lambda \rtimes \Gamma)$ to the quotient manifold M.

Lemma 3.2.1.1 ([**39**]). *The groupoid $\Lambda \rtimes \Gamma$ is amenable, and the C^*-algebra $C^*(\Lambda \rtimes \Gamma) = C^*_r(\Lambda \rtimes \Gamma)$ is nuclear, simple and purely infinite.*

It is convenient to choose the visual parameter equal to e. The Patterson-Sullivan measure μ_p is conformal (i.e. one may take $C = 1$ in 3.1), continuous and the Radon-Nikodym cocycle

$$\begin{aligned} c : \Lambda \rtimes \Gamma &\to \mathbb{R} \\ (x,\gamma) &\mapsto \ln \frac{d\gamma\mu_p}{d\mu_p}(x\gamma), \end{aligned}$$

can be expressed as

$$(x,\gamma) \mapsto \delta_\Gamma(\lim_{n\to\infty} d(p\gamma, p_n) - d(p,p_n)) = \delta_\Gamma h_\gamma(x),$$

where p_n is a sequence in $\mathbb{H}^n$ converging to $x \in \partial\mathbb{H}^n$. *In the sequel we will assume these cocycles to be closed.* This is definitely the case if the limit set is all of S^{n-1} and we expect it to be true for arbitrary limit sets. The measures μ_p and $\mu_{p'}$, for different basepoints $p, p' \in \mathbb{H}^n$, are related by

$$\frac{d\mu_p}{d\mu_{p'}}(x) = \lim_{n\to\infty} d(p,p_n) - d(p', p_n).$$

All this follows from the material in [**50**]. In [**39**], the *Busemann cocycle*

$$b : (x,\gamma) \mapsto h_\gamma(x), \tag{3.2}$$

is used to explicitly construct a one parameter group of automorphisms and a KMS δ_Γ-state on $C^*(\Lambda \rtimes \Gamma)$. This is in fact a special case of the constructions discussed in section 3.3. We denote the KMS-state associated with the measure μ_p by τ_p. Lott then proceeds to construct a continuous field of C^*-algebras over M, with fiber $C^*(\Lambda \rtimes \Gamma)$. We here give a description of this field that looks slightly different from the one in [**39**], but is essentially the same. Let

$$\mathscr{A} := \{f : \mathbb{H}^n \times \mathcal{G} \to \mathbb{C} : \forall m \in \mathbb{H}^n f(m,\cdot) \in C_c(\mathcal{G})\}, \tag{3.3}$$

where we require the functions f to be continuous. $\mathscr{A}$ becomes an algebra in the pointwise convolution product

$$f * g(m,\eta) := \int_{\mathcal{G}} f(m,\xi) g(m, \xi^{-1}\eta) d\nu^{r\eta}\xi.$$

The crossed product groupoid $\mathcal{G} = \Lambda \rtimes \Gamma$ carries an action of Γ:

$$(x,\gamma)\delta := (x\delta, \delta^{-1}\gamma\delta),$$

and hence the set $\mathbb{H}^n \times \Lambda \rtimes \Gamma$ and the algebra $\mathscr{A}$ do so too.

LEMMA 3.2.1.2 ([**39**]). *The expressions*

$$U_t F(m,x,\gamma) := e^{ith_\gamma(x)} F(m,x,\gamma) \tag{3.4}$$

$$\tau(F)(m) := \tau_m F(m), \tag{3.5}$$

define a one parameter group of automorphisms of $\mathscr{A}$, and a $C(\mathbb{H}^n)$-valued functional on $\mathscr{A}$, both of which are Γ-equivariant.

The subalgebras $\mathscr{A}_b$ and $\mathscr{A}_0$, of functions that are bounded, respectively vanish at infinity on $\mathbb{H}^n$ carry a natural norm:

$$\|f\| := \sup_{m\in\mathbb{H}^n} \|f(m,\cdot)\|,$$

where the norm on the righthand side is the C^*-norm on $C_c(\Lambda\rtimes\Gamma)$. The completion of $\mathscr{A}_0$ in this norm is the C^*-algebra of C_0-sections of the trivial field of C^*-algebras over $\mathbb{H}^n$ with fiber $C^*(\Lambda \rtimes \Gamma)$. The subspace of invariant functions $\mathscr{A}^\Gamma \subset \mathscr{A}$ gives rise to a continuous field of C^*-algebras over M, with fiber $C^*(\Lambda \rtimes \Gamma)$. We denote its C^*-algebra of C_0-sections by $C^*_M(\Lambda \rtimes \Gamma)$, and the $*$-algebra of all sections by $A^*_M(\Lambda \rtimes \Gamma)$. The center of $A^*_M(\Lambda \rtimes \Gamma)$ is $C(M)$. By lemma 3.2.1.2, we get a one parameter group of automorphisms U_t and a $C(M)$-valued functional τ of $A^*_M(\Lambda \rtimes \Gamma)$. Moreover, τ has the KMS-δ_Γ- property with respect ot U_t.

PROPOSITION 3.2.1.3 ([**39**]).
(1) *If Γ is convex cocompact then δ_Γ is the unique β for which $A^*_M(\Lambda \rtimes \Gamma)$ has a KMS-β-state.*
(2) *If Γ is not convex cocompact then for each $\beta \in [\delta_\Gamma, \infty)$, $A^*_M(\Lambda \rtimes \Gamma)$ has a KMS-β-state.*
(3) *If Γ is not convex cocompact and has no parabolic elements, then the set of β's for which $A^*_M(\Lambda \rtimes \Gamma)$ has a KMS-β-state is $[\delta_\Gamma, \infty)$.*

The spectral geometry of M is related to $A^*_M(\Lambda\rtimes\Gamma)$ as expressed in the following proposition. In the convex cocompact case, all of the above structure restricts to $C^*_M(\Lambda \rtimes \Gamma)$.

PROPOSITION 3.2.1.4 ([**39**]). *$\tau(1)$ is a positive eigenfunction of the Laplacian Δ_M on M, with eigenvalue $\delta_\Gamma(n-1-\delta_\Gamma)$. If Γ is convex-cocompact, then $\tau(1) \in C_0(M)$. In this case τ restricts to a functional $C^*_M(\Lambda \rtimes \Gamma) \to C_0(M)$, for which the KMS-property holds.*

2.2. A KK^M-cycle. In this section we globalize the cocycle construction 2.3.2.2 by incorporating the variation of the basepoint $p \in \mathbb{H}^n$ into the construction. We assume Γ is convex cocompact, such that proposition 3.2.1.4 is valid. Define

$$c : \mathbb{H}^n \times \Lambda \rtimes \Gamma \to \mathbb{C}$$
$$(p,x,\gamma) \mapsto \frac{d\gamma\mu_p}{\delta_\Gamma d\mu_p}(x\gamma),$$

which is a Γ-invariant function. The algebra $\mathscr{A}_0$ (3.3) carries a derivation

$$\mathscr{D} : \mathscr{A}_0 \to \mathscr{A}_0,$$

defined by $\mathscr{D}F(m,x,\gamma) := c(m,x,\gamma)F(m,x,\gamma)$. This is the generator of the one parameter group from proposition 3.2.1.4. Define

$$\mathscr{B}_0 := \{f \in \mathscr{A}_0 : f(p,\cdot) \in C_c(\mathcal{H}_p)\},$$

where $\mathcal{H}_p$ denotes the kernel of the Radon-Nikodym cocycle associated to μ_p. One straightforwardly checks that the action Γ on $\mathscr{A}_0$ restricts to an action on $\mathscr{B}_0$, and hence $\mathscr{B}_0^\Gamma \subset \mathscr{A}_0^\Gamma$. Note that the groupoids $\mathcal{H}_p$ and $\mathcal{H}_{p\delta}$ are isomorphic via

$$\begin{aligned}\mathcal{H}_p &\to \mathcal{H}_{p\delta}\\ (x,\gamma) &\mapsto (x\delta, \delta^{-1}\gamma\delta).\end{aligned}$$

This gives rise to a C^*-subalgebra of $C^*_M(\Lambda \rtimes \Gamma)$ corresponding to a continuous field of C^*-algebras over M with fiber $C^*(\mathcal{H}_m)$. Denote its C^*-algebra of C_0-sections by $C^*_M(\mu)$, and the *-algebra of all sections by $A^*_M(\mu)$..The algebra $\mathscr{A}_0^\Gamma \subset C^*_M(\Lambda \rtimes \Gamma)$ carries a $\mathscr{B}_0^\Gamma$-valued inner product:

$$\langle \Psi, \Phi\rangle(p,\chi) := \int_{\Lambda\rtimes\Gamma} \overline{\Psi(p,\xi^{-1})}\Phi(p,\xi^{-1}\chi)d\nu^{r(\chi)}\xi,$$

where $\chi \in \mathcal{H}_p$. Denote the completion of $\mathscr{A}_0^\Gamma$ in this inner product by $\mathcal{E}_M$. The operator $\mathscr{D} : \mathscr{A}_0^\Gamma \to \mathscr{A}_0^\Gamma$ is $\mathscr{B}_0^\Gamma$ linear and symmetric for this inner product.

THEOREM 3.2.2.1. *$\mathscr{D}$ extends to a selfadjoint regular operator with compact resolvent in $\mathcal{E}_M$ and as such defines an odd unbounded KK^M-cycle for the pair $(C^*_M(\Lambda \rtimes \Gamma), C^*_M(\mu))$.*

PROOF. Selfadjointness and regularity follow in the same way as for cocycles. The compact resolvent property follows from the fact that it holds fiberwise, which is sufficient by lemma 2.1.3.3. □

The restriction of the functional $\tau : A^*_M(\Lambda \rtimes \Gamma) \to C_0(M)$ to $A^*_M(\mu)$ is tracial, i.e. $\tau(f*g) = \tau(g*f)$, because it is defined by fiberwise integration and the fibers of the field defined by $C^*_M(\mu)$ are the C^*-algebras of unimodular groupoids. Moreover, an element of $KK_0^M(C_0(M), C^*_M(\mu))$ defines a formal difference of projections in $M_n(A^*_M(\mu))$. This is done by setting

$$\mathrm{Ind}(F)(m) := \mathrm{Ind}F_m,$$

that is, taking the fiberwise index of a KK_0^M-cycle $(\mathcal{E}, F)$. To a projection e in $M_n(A^*_M(\mu))$ we can apply τ:

$$\tau_*(e) := \tau(\sum_{i=1}^n e_{ii}),$$

which gives a well defined map $\tau_* : KK_0^M(C_0(M), C^*_M(\mu)) \to C(M)$ because τ is tracial. In the sequel $e_m : C_0(M) \to \mathbb{C}$ denotes the homomorphism $f \mapsto f(m)$, which is dual to the inclusion $m \hookrightarrow M$.

THEOREM 3.2.2.2. *The operator $\mathscr{D}$ and the functional τ induce an index map*

$$\mathrm{Ind}_\mu : KK_1^M(C_0(M), C^*_M(\Lambda \rtimes \Gamma)) \to C(M).$$

This map is compatible with the maps $\mathrm{Ind}_{\mu_p} : K_1(C^*(\Lambda \rtimes \Gamma)) \to \mathbb{C}$ *in the sense that the diagram*

$$\begin{tikzcd} KK_1^M(C_0(M), C^*_M(\Lambda \rtimes \Gamma)) \arrow[r, "\mathrm{Ind}_\mu"] \arrow[d, "e_m"'] & C(M) \arrow[d, "e_m"'] \\ K_1(C^*(\Lambda \rtimes \Gamma)) \arrow[r, "\mathrm{Ind}_{\mu_m}"] & \mathbb{C}, \end{tikzcd}$$

commutes.

Proof. Recall the definition of the restriction homomorphism

$$e_m : KK_1^M(C_0(M), C^*_M(\Lambda \rtimes \Gamma)) \to K_1(C^*(\Lambda \rtimes \Gamma)$$
$$(\mathcal{E}, F) \mapsto (\mathcal{E}\tilde{\otimes}_{C^*_M(\Lambda\rtimes\Gamma)} C^*(\Lambda \rtimes \Gamma), F\tilde{\otimes}1),$$

where we tensor over the fiber map $C^*_M(\Lambda \rtimes \Gamma) \to C^*(\Lambda \rtimes \Gamma)$. By LeGall's theorem 2.1.3.7, these restriction maps are compatible with the Kasparov product. That is, the diagram

$$\begin{tikzcd} KK_1^M(C_0(M), C^*_M(\Lambda \rtimes \Gamma)) \arrow[r, "\otimes_{\mathscr{D}}"] \arrow[d, "e_m"'] & KK_0^M(C_0(M), C^*_M(\mu)) \arrow[d, "e_m"'] \\ K_1(C^*(\Lambda \rtimes \Gamma)) \arrow[r, "\otimes_{D_m}"] & K_0(C^*(\mu_m)), \end{tikzcd}$$

commutes. Here D_m and $C^*(\mu_m)$ are to be understood as D_p and $C^*(\mu_p)$ where $p \in \mathbb{H}^n$ is any point in the fiber over $m \in M$. Furthermore, the diagram

$$\begin{tikzcd} KK_0^M(C_0(M), C^*_M(\mu)) \arrow[r, "\tau_*"] \arrow[d, "e_m"'] & C(M) \arrow[d, "e_m"'] \\ K_0(C^*(\mu_m)) \arrow[r, "\tau_m"] & \mathbb{C}, \end{tikzcd}$$

commutes by the very definition of τ. □

Although there is no solid evidence for this as of yet, we expect the image of the global index map to be related to automorphic forms on M. This will be the subject of further research.

2.3. Coverings. An inclusion of groups $\Gamma \subset \Gamma'$ gives rise to an inclusion of limit sets $\Lambda_\Gamma \subset \Lambda_{\Gamma'}$, since the orbits of Λ are a subset of the orbits of Γ'. We also get a possibly ramified covering $\mathbb{H}^n/\Gamma \to \mathbb{H}^n/\Gamma'$. In this section we assume that Γ is a finite index subgroup of Γ'. We believe it will be possible to obtain results similar to those discussed here for arbitrary inclusions, using the same ideas as employed here. Theorem 2.3.2.2, applied to the Busemann cocycles b and b' (3.2) gives us two odd unbounded bimodules

$$C^*(\Lambda \rtimes \Gamma) \to (\mathcal{E}, D) \leftrightharpoons C^*(\mu), \tag{3.6}$$

and

$$C^*(\Lambda' \rtimes \Gamma') \to (\mathcal{E}', D') \leftrightharpoons C^*(\mu'). \tag{3.7}$$

Here $C^*(\mu)$ denotes the groupoid C^*-algebra of the kernel of the Busemann cocycle, which equals the kernel of the Radon-Nikodym cocycle associated to μ. The inclusion of groups $\Gamma \subset \Gamma'$ gives rise to an inclusion of groupoids $\Lambda \rtimes \Gamma \subset \Lambda' \rtimes \Gamma'$. This map is obviously proper and gives a groupoid correpondence

$$\Lambda' \rtimes \Gamma' \to Z \leftrightharpoons \Lambda \rtimes \Gamma.$$

LEMMA 3.2.3.1. *Let $\Gamma \subset \Gamma'$ be an inclusion of groups, and b and b' the associated Busemann cocycles. The inclusion of groupoids $\Lambda \rtimes \Gamma \subset \Lambda' \rtimes \Gamma'$ restricts to an inclusion $\ker b \subset \ker b'$.*

PROOF. This is immediate from the definition of b and b':

$$b(x,\gamma) = \lim_{n\to\infty} d(p,p_n) - d(p\gamma, p_n),$$

with p_n a sequence converging to $x \in \Lambda$. Clearly we can choose the same sequence for $x \in \Lambda \subset \Lambda'$ in the definition of b, so $b(x,\gamma) = 0$ implies $b'(x,\gamma) = 0$. □

Consequently, there is a groupoid correspondence

$$\ker b' \to Z_{b,b'} \leftrightharpoons \ker b.$$

LEMMA 3.2.3.2. *The correspondences Z and $\Lambda' \rtimes \Gamma' *_{\ker b'} Z_{b,b'}$ are isomorphic.*

PROOF. Define a map $\Lambda' \rtimes \Gamma' *_{\ker b'} Z_{b,b'} \to Z$ by

$$((x',\gamma'),((x'\gamma',\delta'),x)) \mapsto ((x',\gamma'\delta'),x).$$

Its inverse is given by $((x',\gamma'),x) \mapsto ((x',\gamma'),(x'\gamma',e),x)$. □

Hence, by theorem 2.2.4.1, the associated bimodules

$$C^*(\Lambda' \rtimes \Gamma') \to \mathcal{E}^Z \leftrightharpoons C^*(\Lambda \rtimes \Gamma), \tag{3.8}$$

and

$$C^*(\mu') \to \mathcal{E}^{Z_{b,b'}} \leftrightharpoons C^*(\mu), \tag{3.9}$$

satisfy

$$\mathcal{E}^Z \tilde{\otimes}_{C^*(\Lambda\rtimes\Gamma)} \mathcal{E} \cong \mathcal{E}' \tilde{\otimes}_{C^*(\mu)} \mathcal{E}^{Z_{b,b'}}.$$

Moreover, from lemma 2.4.2.1 and 2.4.2.2 and theorem 2.4.2.3, we know that $\mathcal{E}^Z$ is a finitely generated free module with Hermitian connection

$$\nabla : C_c(Z) \to C_c(Z) \otimes_{C_c(\Lambda\rtimes\Gamma)} \Omega^1(C_c(\Lambda \rtimes \Gamma))$$
$$\Phi \mapsto \sum_{\gamma\in\mathbb{P}} \pi^* 1_{\Lambda'\times\{\gamma\}} \otimes d(i_\gamma)^*(\Phi).$$

Here $\mathbb{P}$ denotes the coset space Γ'/Γ, which we have identified with a choice of representatives in Γ'. The maps $i_\gamma^* : C_c(Z) \to C_c(\Lambda \rtimes \Gamma)$ are induced by

$$i_\gamma((x',\delta),x) := ((x'\gamma^{-1},\gamma\delta),x\delta).$$

THEOREM 3.2.3.3. *The pair $((\mathcal{E}^Z,\nabla),\mathcal{E}^{Z_{b,b'}})$ defines a weak correspondence of the KK-cycles 3.6 and 3.7.*

PROOF. The respective tensor products 3.8 and 3.9 are isomorphic, hence it remains to show that the operators on this tensor product coincide up to a bounded perturbation. This follwos from explicitly computing the induced operator on the tensor product $\mathcal{E}^Z \tilde{\otimes}_{C^*(\Lambda \rtimes \Gamma)} \mathcal{E}$. In doing so one sees that the action of $1\tilde{\otimes}_\nabla D_b$ on

$$C_c(Z) \cong \bigoplus_{\gamma \in \mathbb{P}} C_c(Z^{[\gamma]}),$$

on each component $C_c(Z^{[\gamma]})$ is given by multiplication by the Busemann cocycle for Γ based at $p\gamma$ instead of at p. On each component this is a bounded perturbation of the Busemann cocycle for Γ' based at p. Since there are only finitely many cosets, this suffices. □

3. Fuchsian groups

Lastly, we discuss some constructions with classical Fuchsian groups, that is, groups acting as isometries of the hyperbolic plane

$$\mathbb{H}^2 = \{z \in \mathbb{C} : \operatorname{Im} z > 0\},$$

by fractional linear transformations. These are subgroups of $PSL_2(\mathbb{R})$. The most prominent example is the modular group $PSL(2, \mathbb{Z})$ and its finite index subgroups, which are important in the theory of modular forms. Arbitrary Fuchsian groups uniformize Riemann surfaces by (branched) coverings. Explicit orbit equivalences exist for such groups, which seems to lead to a promising approach for computing index maps.

3.1. Orbit equivalence. In this section, Γ will be a Fuchsian group of the first kind. The quotient $X_\Gamma := \mathbb{H}^2/\Gamma$ is a Riemann surface, and for an inclusion $\Gamma \subset \Gamma'$ we obtain a (possibly ramified) covering $X_{\Gamma'} \to X_\Gamma$. In [**9**], Bowen and Series construct a partial endomorphism $\sigma : U \to X$, which is *orbit equivalent* to the action of Γ on $\mathbb{P}^1(\mathbb{R})$. This means that, except for a finite number of pairs of points, elements $x, y \in \mathbb{P}^1(\mathbb{R})$ are in the same Γ-orbit if and only if there exist $m, n \in \mathbb{N}$ such that $\sigma^n(x) = \sigma^m(y)$. Their maps have more special properties.

THEOREM 3.3.1.1. *Let Γ be a Fuchsian group of the first kind. There exists a partial endomorphism $\sigma : U \to X$, which is orbit equivalent to Γ. Moreover, if Γ has no cusps, σ is a Markov generator. The Markov partition $\{U_i\}$ has the property that the restriction of σ is to each U_i is equal to an element $\gamma_i \in \Gamma$.*

In view of theorem 2.5.1.6 there is an equivariant map $\mathcal{S}_A \to \mathbb{P}^1(\mathbb{R})$ from a subshift of finite type to the dynamical system given by σ. By 2.5.2.5 the gives rise to bimodules relating the KK-cycles coming from the fundamental cocycles of the semidirect product groupoids $X \rtimes \sigma$ and $\mathcal{S}_A \rtimes \sigma_A$.

PROPOSITION 3.3.1.2. *Let $\sigma : U \to X$ be a Markov generator with partition $\{U_i\}$, that is orbit equivalent to the action of some group Γ on X. Moreover assume that the restriction of σ to each U_i equals some $\gamma_i \in \Gamma$. Then there is a natural groupoid homomorphism $X \rtimes \sigma \to X \rtimes \Gamma$.*

PROOF. Define

$$\begin{aligned} X \rtimes \sigma &\to X \rtimes \Gamma \\ (x, n, y) &\mapsto (x, \delta_{i_1} \cdots \delta_{i_{n+k}} \gamma_{j_k}^{-1} \cdots \gamma_{j_1}^{-1}). \end{aligned}$$

Here k is such that $\sigma^{n+k}(x) = \sigma^k(y)$ and the δ_{i_m} and γ_{j_ℓ} are determined by this relation because

$$\sigma^{n+k}(x) = x\delta_{i_1}\cdots\delta_{i_{n+k}}, \quad \sigma^k(y) = y\gamma_{j_1}\cdots\gamma_{j_k},$$

according to whether $x \in U_{i_1}$, $x\delta_{i_1} \in U_{i_2}$, and so on. The map is independent of the choice of k because if k' is such that $\sigma^{k'+n}(x) = \sigma^{k'}(y)$ and $k < k'$, then the equality holds for all k'' with $k \leq k'' \leq k'$. Hence the corresponding group elements will be equal and cancel each other. □

Thus, for such an orbit equivalence, and in particular for Fuchsian groups, we have groupoid homomorphisms

$$\mathcal{A} \rtimes \sigma_A \to X \rtimes \sigma \to X \rtimes \Gamma.$$

For the last homomorphism, it is not clear under what conditions it is proper. This is a necessary condition to associate a C^*-bimodule to it. Also, the relation between the fundamental cocycle of $\Lambda \rtimes \sigma$ and the Radon-Nikodym cocyle on $\Lambda \rtimes \Gamma$ needs further investigation. Establishing a correspondence of KK-cycles for these orbit equivalences for Fuchsian groups might allow for computation of the index map $\mathrm{Ind}_\mu : K_1(C^*\Lambda \rtimes \Gamma)) \to \mathbb{C}$ via the index map $K_1(\mathcal{O}_A) \to \mathbb{C}$ coming from the fundamental cocycle. Since the K-theory of Cuntz-Krieger algebras is known, this might simplify the computations greatly.

3.2. The modular group. The group $SL(2,\mathbb{Z})$ acts as a group of isometries of the hyperbolic plane by fractional linear transformations. Effectively, this is an action of $PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\{1,-1\}$. This group can be written as the free product $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$, from which an action on a tree $\mathcal{T}$ is constructed. The noncommutative geometry of both boundary actions of this group is the subject of the papers [**40**],[**41**]. There it is shown that the homology of modular curves is recovered from the K-theory of the boundary crossed product algebra $C(\mathbb{P}^1(\mathbb{R})) \rtimes \Gamma$. More precisely, let $\Gamma \subset SL(2,\mathbb{Z})$ be a subgroup of finite index. Denote by $X_\Gamma := \mathbb{H}^2/\Gamma$ the corresponding modular curve, and by $\mathbb{P} = \mathbb{P}_\Gamma$ the coset space $\Gamma \backslash SL(2,\mathbb{Z})$. The set $\mathscr{C}$ of cusps of Γ can be identified with the orbits $\mathbb{P}^1(\mathbb{Q})/\Gamma$. X_Γ can be compactified, as a Riemann surface, by adding the cusps and a suitable choice of charts. The *cuspidal cohomology* of X_Γ is the relative cohomology group $H^1(X_\Gamma, \mathscr{C})$.

THEOREM 3.3.2.1 ([**40**]). *There is a natural isomorphism* $K_1(C^*(\mathbb{P}^1(\mathbb{R}) \rtimes \Gamma)) \xrightarrow{\sim} H^1(X_\Gamma, \mathscr{C}) \oplus \mathbb{Z}$.

This isomorphism comes from analyzing the maps in the Pimnser sequence for Γ given its action on the tree $\mathcal{T}$. We would like to describe the index map $K_1(C^*(\mathbb{P}^1(\mathbb{R}) \rtimes \Gamma)) \to \mathbb{C}$ coming from proposition 2.3.3.7. Again, without solid evidence as of yet, we expect it to pick out the $H^1(X_\Gamma, \mathscr{C})$ part of $K_1(C^*(\Lambda \rtimes \Gamma))$.

For $p \in \mathcal{T}$, let $\mu_{\mathcal{T}}$ denote the Patterson-Sullivan measure based at p and $c_{\mathcal{T}}$ the associated modular Radon-Nikodym cocycle on $\partial\mathcal{T} \rtimes \Gamma$. Similarly $\mu_{\mathbb{H}^2}$ and $c_{\mathbb{H}^2}$ denote the corresponding objects on $\mathbb{P}^1(\mathbb{R}) \rtimes \Gamma$. The cocycle construction theorem 2.3.2.2 applied to both cocycles yields two unbounded bimodules

$$C^*(\partial\mathcal{T} \rtimes \Gamma) \to (\mathcal{E}_{\mathcal{T}}, D_{\mathcal{T}}) \leftrightarrows C^*(\ker c_{\mathcal{T}}), \tag{3.10}$$

and

$$C^*(\mathbb{P}^1(\mathbb{R}) \times \Gamma) \to (\mathcal{E}_{\mathbb{P}^1(\mathbb{R})}, D_{\mathbb{P}^1(\mathbb{R})}) \leftrightarrows C^*(\ker c_{\mathbb{H}^2}). \tag{3.11}$$

We now proceed with the construction of a geometric correspondence between these KK-cycles in the sense of definition 1.6.5.2 and the subsequent modification to the bivariant case.

There is an equivariant embedding $\Upsilon : \mathcal{T} \to \mathbb{H}^2$. It is obtained by choosing a fundamental domain for the action of $SL(2,\mathbb{Z})$, and covering $\mathbb{H}^2$ by it translates, obtaining the well known tesselation of the plane. Then $\mathcal{T}$ can be identified with the union of the edges of this tesselation that have no vertex at infinity. This map extends to an equivariant proper surjection $\Upsilon : \partial\mathcal{T} \to \partial\mathbb{H}^2 = \mathbb{P}^1(\mathbb{R})$. Therefore, Υ induces a proper groupoid homomorphism

$$\begin{aligned} \partial\mathcal{T} \rtimes \Gamma &\to \mathbb{P}^1(\mathbb{R}) \rtimes \Gamma \\ (x,\gamma) &\mapsto (\Upsilon(x),\gamma), \end{aligned}$$

for $\Gamma \subset PSL(2,\mathbb{Z})$.

For $p, p', p'' \in \mathbb{H}^2$ denote by $[p,p']$ the geodesic segment connecting p to p' and by $[p,p',p'']$ the angle between the geodesic segments $[p,p']$ and $[p',p'']$. The following lemma gives a useful relation between the metrics on $\mathcal{T}$ and $\mathbb{H}^2$.

Lemma 3.3.2.2. *Let $p,q,r \in \mathcal{T} \subset \mathbb{H}^2$ and let q_1, r_1 be the predecessors of q,r on the paths connecting them to p. If $d_{\mathcal{T}}(p,q) = d_{\mathcal{T}}(p,r)$, then the triangles $\Delta(p,q,q_1)$ and $\Delta(p,r,r_1)$ are congruent. In particular $d_{\mathbb{H}^2}(p,q) = d_{\mathbb{H}^2}(p,r)$.*

Proof. We may assume $d_{\mathcal{T}}(p,p') \in \mathbb{N}$ for all $p,p' \in \mathcal{T}$ and proceed by induction. If $d_{\mathcal{T}}(p,q) = d_{\mathcal{T}}(p,r) = 1$ there is nothing to prove as $\mathcal{T} \subset \mathbb{H}^2$ has geodesic edges. For $n = 2$ the statement follows because $d_{\mathbb{H}^2}(p,q_1) = d_{\mathbb{H}^2}(p,r_1) = 1$ and the angles $[q,q_1,p]$ and $[r,r_1,p]$ are equal.

For the induction step, assume $d_{\mathcal{T}}(p,q) = d_{\mathcal{T}}(p,r) = n$. Denote by $q_2, r_2 \in \mathcal{T}$, the predecessors of q_1, r_1 in $[p,q], [p,r]$ respectively. Then $d_{\mathcal{T}}(p,q_i) = d_{\mathcal{T}}(p,r_i) = n - i$ and $d_{\mathcal{T}}(q,q_i) = d_{\mathcal{T}}(r,r_i) = i$ for $i = 1,2$. Hence $d_{\mathbb{H}^2}(p,q_i) = d_{\mathbb{H}^2}(p,r_i)$ and $\Delta(p,q,q_1)$, $\Delta(p,q_1,q_2)$, $\Delta(p,r,r_1)$ and $\Delta(p,r_1,r_2)$ are geodesic triangles in $\mathbb{H}^2$. The triangles $\Delta(p,q_1,q_2)$ and $\Delta(p,r_1,r_2)$ are congruent by the induction hypothesis. Since the angles $[r,r_1,r_2]$ and $[q,q_1,q_2]$ are equal, the angles $[p,q_1,q]$ and $[p,r_1,r]$ are equal. Hence $\Delta(p,q,q_1)$ and $\Delta(p,r,r_1)$ are congruent as well. □

Proposition 3.3.2.3. *The homomorphism $\Upsilon : \partial\mathcal{T} \rtimes \Gamma \to \mathbb{P}^1(\mathbb{R}) \rtimes \Gamma$ restricts to a homomorphism $\ker c_{\mathcal{T}} \to \ker c_{\mathbb{H}^2}$.*

Proof. Let $\delta_{\mathcal{T}}$ and $\delta_{\mathbb{H}^2}$ denote the critical exponents of the respective Poincaré series for $SL(2,\mathbb{Z})$. The Patterson-Sullivan measures satisfy

$$\frac{d\gamma\mu_{\mathcal{T}}}{d\mu_{\mathcal{T}}} = j_\gamma^{\delta_{\mathcal{T}}}, \quad \frac{d\gamma\mu_{\mathbb{H}^2}}{d\mu_{\mathbb{H}^2}} = j_\gamma^{\delta_{\mathbb{H}^2}}.$$

We have to show that $j_\gamma^{\delta_{\mathcal{T}}}(x\gamma) = 1$ implies $j_\gamma^{\delta_{\mathbb{H}^2}}(\Upsilon(x)\gamma) = 1$. Now

$$j_\gamma^{\delta_{\mathcal{T}}}(x\gamma) = \lim_{n\to\infty} e^{\delta_{\mathcal{T}}(d_{\mathcal{T}}(p,x_n\gamma) - d_{\mathcal{T}}(p,x_n))},$$

where $x_n \in \mathcal{T}$ is a sequence converging to $x \in \partial\mathcal{T}$. Hence $j_\gamma^{\delta_{\mathcal{T}}}(x\gamma) = 1$ is equivalent to

$$\lim_{n\to\infty} d_{\mathcal{T}}(p,x_n\gamma) - d_{\mathcal{T}}(p,x_n) = 0.$$

Since $\mathcal{T}$ is a tree, this means that there exists $m \in \mathbb{N}$ such that for all $n > m$ $d_{\mathcal{T}}(p, x_n\gamma) - d_{\mathcal{T}}(p, x_n) = 0$. But then $d_{\mathbb{H}^2}(p, x_n\gamma) - d_{\mathbb{H}^2}(p, x_n) = 0$ by lemma 3.3.2.2, and hence

$$j_\gamma^{\delta_{\mathbb{H}^2}}(\Upsilon(x)\gamma) = \lim_{n\to\infty} e^{\delta_{\mathbb{H}^2}(d_{\mathbb{H}^2}(p,x_n\gamma)-d_{\mathbb{H}^2}(p,x_n))} = 1.$$

$\square$

The map Υ induces a $C^*(\mathbb{P}^1(\mathbb{R}) \rtimes \Gamma)$-$C^*(\partial\mathcal{T} \rtimes \Gamma)$ bimodule $\mathcal{E}$, which is a completion of $C_c(Z_\Upsilon)$, with

$$Z_\Upsilon = \{((\Upsilon(x), \gamma), x\gamma) : x \in \partial\mathcal{T}, \gamma \in \Gamma\}.$$

This module comes from the equivariant pair (Υ, id), so by theorem 2.4.2.3 we get an explicit formula for a a Hermitian connection on it. It is given by

$$\begin{aligned} \nabla : C_c(Z_\Upsilon) &\to C_c(Z_\Upsilon) \otimes_{C_c(\partial\mathcal{T}\rtimes\Gamma)} \Omega^1(C_c(\partial\mathcal{T} \rtimes \Gamma)) \\ \Psi \quad &\mapsto \pi^* 1_{\mathbb{P}^1(\mathbb{R})} \otimes d\Upsilon^*\Psi. \end{aligned}$$

The restriction of Υ to the kernel groupoid induces a bimodule $C^*(\ker c_{\mathbb{H}^2}) \to \mathcal{E}' \leftrightarrows C^*(\ker c_{\mathcal{T}})$, by proposition 3.3.2.3. This bimodule comes from the correspondence

$$Z'_\Upsilon := \{((\Upsilon(x), \gamma), x\gamma) : (\Upsilon(x), \gamma) \in \ker c_{\mathbb{H}^2}\}.$$

This pair of modules does not constitute a correspondence of KK-cycles, as the operators do not coincide under the respective tensor products. They are most likely bounded perturbations of one another, and thus constiute a weak correpsondence. For K-theoretic purposes this is sufficient, but it is likely that by suitable perturbing the connection and the operator $D_{\mathcal{T}}$, a genuine correspondence can be obtained.

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