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Three-position Dimensional Synthesis of Four-Bar Mechanism for Function Generation Using Fuzzy Logic Mathematics

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Abstract

Dimensional synthesis of the four-bar mechanism could not be determined precisely due to many constraints such as manufacturing tolerance, joint clearance, thermal deformation, the deflection and so on. All of these constraints are included in the uncertainty of the dimensions of the four-bar mechanism. In this research, this uncertainty will be modeled based on the fuzziness one of the precision points of Freudenstein's equation that builds intervals of link’s dimensions with membership functions. They represents the probability of the dimensions value depending on the uncertainty of the positions of the precision point itself rather than uncertainty of the external information about the mechanism dimensions. The results of the fuzzy synthesis will be defuzzified using the centroid defuzzification method to get the dimensions of the mechanism. Then, the resultant function from the fuzzy synthesis is comparing with the crisp one to study the range and limits of the fuzziness operation in the generated function.

Chapter 1: Introduction

Linkage mechanisms have a lot of applications in engineering field such as steering system that uses Ackerman steering linkage, robotics that use the actuation mechanisms to transform the electric energy to mechanical movement, and hydraulic actuators for heavy equipment that use hydraulic power to transform the mechanical movement.

The synthesis of these types of mechanism is deemed an essential factor for the performance efficiency. The links' lengths of the linkage mechanism that be determined precisely at the synthesis stage will be changed by many constraints in manufacturing such as joint clearance and manufacturing tolerance. Effects of these manufacturing process on the output of the linkage mechanism was investigated by Garrett and Hall in [1]. Joint clearance is a gap between parts in the joint to make them move relative to each other while the manufacturing tolerance is relating to incapable of making a perfect dimension links by the cutting machines due to wearing and temperature variation of internal parts. Other manufacturing errors for the linkage mechanisms was illustrated by Sutherland and Roth in [2] , then they studies in [3] the increasing of the structural error between desired output and the actual output due to the for a specific type of mechanism synthesis known as function generating mechanism due to these errors. In addition, the joint clearances and the tolerance of the link lengths were taken into consideration by Dhande and Chakraborty in [4] as they introduced a stochastic model for a specific kind of the function generating mechanisms called a four-bar mechanism to analyze and allocate the mechanical errors and to compare them with structural errors. Since the stochastic modeling may not predicted precisely as the input should be random variables, Faik and Erdman in [5] proposed the relation to reveal the contributions of every design parameter to the output of the mechanism. The relation between the change in the design parameter and the output of the mechanism is the sensitivity of the mechanism in which the work of Faik and Erdman introduced the global mechanism sensitivity with Nondimensional coefficients. A theory relating the tolerance sensitivity and the mechanism performance using a Jacobian matrix as well as determining the more sensitive design parameters to the dimensional tolerance were the main topics of Ting and Long in [6]. Similarly, the Jacobian matrix was used by Zhu and Ting in [7] to show the relationship between the performance of the mechanism and the tolerance of the whole mechanism rather than individual links. In addition, they introduced a bounded variation space of the performance sensitivity of the mechanism that manifests in an ellipsoid by a set of eigenvalues and eigenvectors of the Jacobian matrix. Because of that manifestation, the sensitivity of the design parameters are estimated by the comparing of the geometric characteristic of the performance space with the tolerance bounded space.

Three main methods are being used to model the uncertainty of dimensions of the links: a probabilistic method, an interval method and a fuzzy method. The interval method starts the dimensions of links within a range between two numbers, however, the membership function for the range of the numbers are the main feature of the fuzzy method. On the other hand, the mean value of the range and its standard deviation are the tools in the probabilistic method. In [1] and [4] mentioned in the second paragraph, interval and probabilistic methods were used to model the manufacturing tolerance and joints clearance: furthermore, the interval method was used to investigate their effects on the positions and orientations by Weidong and Rao in [8]. They also employed the fuzzy method to analyze the error of the four bar mechanism as they modeled its mechanical tolerance and joints clearance in [9]. In the same way, Diab and Smaili in [10] also used the interval number to model the same subject but they employed the fuzzy membership function to display the results before and after an optimization process. This optimization was concerned with minimization of the sensitivity of the design parameters to the mechanical tolerance and joints clearance.

Besides the manufacturing tolerance and the joints clearance, there are many constraints on the dimensions the mechanism's links such as thermal deformation and deflection. All of these constraints involve in the concept of uncertainty that affects the output of the mechanism. This concept adds different sources to the constraints on the dimensions without determine the probability contributions for everyone. Rao and Berke in [11] analyze the uncertainty of the engineering system using the interval method as they considered the given input parameters as interval numbers. For the same objective, Dua, Venigellaa, and Liu used both interval and probabilistic methods to assess the impact of uncertainty on the mechanism performance.

In this thesis, the uncertainty of the dimensional synthesis of the four bar mechanism will be modeled based on the uncertain precision point using the fuzzy method. This work is different from past papers, as the fuzzy input for the dimensional synthesis process will produce fuzzy lengths of the mechanism's links instead of fuzzifying the results. Then, the resultant lengths are estimated form the centroid diffuzification process and compared with the results from the traditional synthesis. The second difference is the case of the synthesis that be used in past works which is the path generation. In contract, the function generation synthesis with Freudenstein's equation that has three precision points will be fuzzified at one point to get the fuzzy linear system of equations, which they are solved by the interval analysis at different α-cut levels. The fuzziness of the Freudenstein's equation will be at middle point while the starting and ending points will not fuzzified; however, the resultant matrix from this process will be a fuzzy square matrix that be divided into several interval matrices using α-cut method. In the first chapter, brief statements about the four bar mechanism and the fuzzy logic followed by the mathematical model in the second chapter. Several examples are given in the third chapter and the forth one is the closure of the thesis.

1.1 Four bar Mechanism

The four-bar linkage is one of the most important mechanism that be used in many applications around the world. This linkage is used to transform the motion from one position to another. One of the most famous applications of that mechanism is the engine mechanism of a car which it has a four-bar linkage to transform the reciprocating movement of the piston to a rotating movement of the crank-shaft. Another application that has been popular recent years is the robot for achieving translational and rotational motions. They can be designed using different mechanisms, one of them can be a four bar linkage.

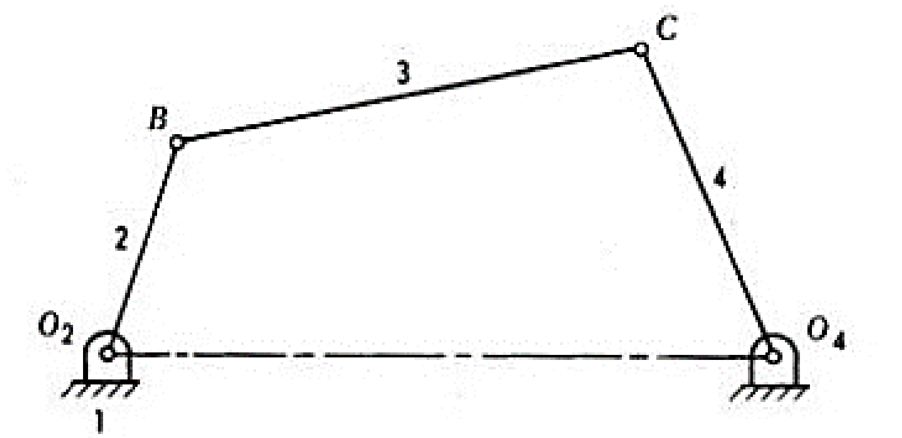

Figure 1-1. Four-Bar Linkage

Transformer mechanisms consist of rigid bars that from open or closed loop, and the four bar linkage is one of the simplest close loop mechanism that has three moving links and a fixed one with four pin joints. Determining of the lengths of the links required to achieve the task as well as a starting point of the mechanism is called a dimensional synthesis [12]. Norton [12] who divided the synthesis into two types: qualitative synthesis which means finding the solutions without depending on an algorithm to estimate these solutions. Dimensional synthesis is often one form of such a types [12]. The second type is called quantitative synthesis necessitate the generation of many solutions with an algorithm and using an optimization process to obtain the best solution from among them [12]. According to Wandling [13], dimensional synthesis is the second step of the synthesis process after determining the type of mechanism including how many links the mechanism is consisted of.

There are different methods to make dimensional synthesis: graphical method that could be considered the easiest one with limited design positions. According to Norton, three positions are the most number that works with this type of synthesis [12]. When more than three is needed, the design is much more involved with the graphical method. It is because of such complexity that a different approach to dimensional synthesis is called for. Analytical synthesis uses algebraic procedures to facilitate the design of the mechanism with more design positions. The kinematic synthesis that deals with velocity, acceleration and positions of the links can also be introduced by differentiating of the equation of motion rather than design positions. Norton states that finding the velocity and acceleration of the element of the mechanism using graphical method needed to require a lot of independent graphical solutions for every design position such that every one of them is completely different from the other. However, the analytical method readily incorporates them for all these positions [12].

There are three different forms of dimensional synthesis. First case is called motion generation that tracks of guides through different positions and orientations of an object. The maximum number of positions that could work for motion generation synthesis are 5 positions [14].

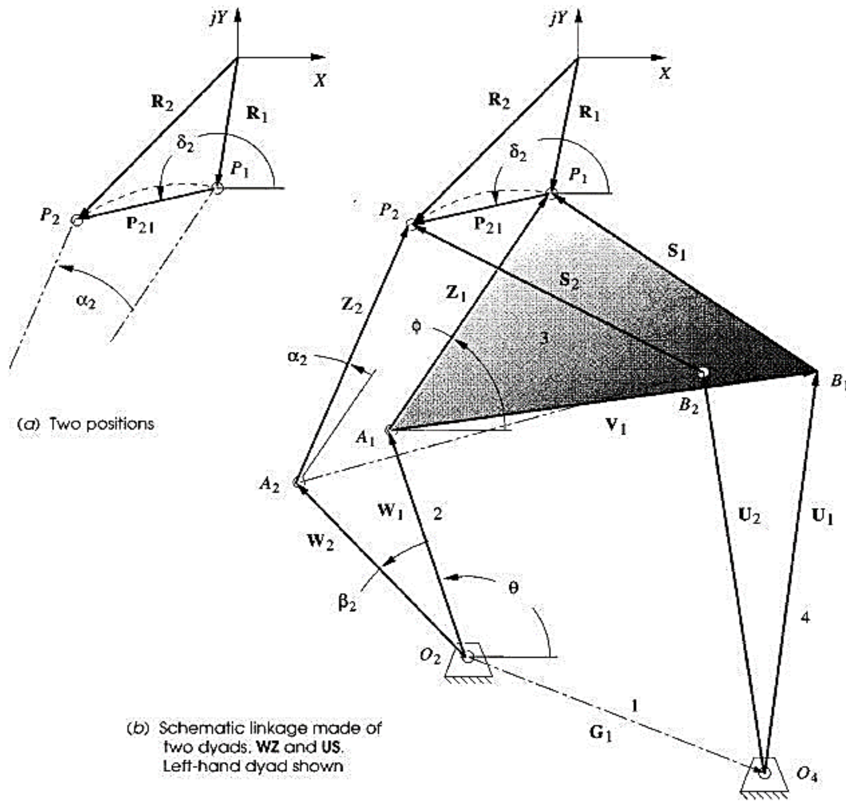

Figure 1-2. Two-position of the motion generation synthesis for Four-bar Linkage [1].

The second form is the path generation, which requires the coupler link "link 3" to pass through several precision points in space. According to Sandor & Erdman [14], the maximum number of positions for qualitative path generation synthesis is five precision points. However, more than five precision points can be achieved using a quantitative approach.

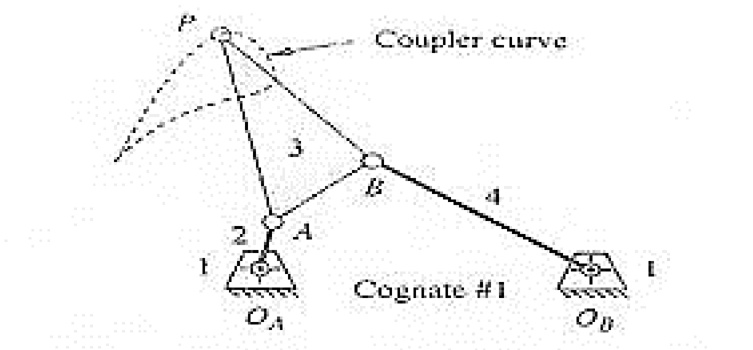

Figure 1-3. Path generation synthesis for Four-bar Linkage [15].

The last form is the function generation. The maximum number of precision points is seven. The form of the equations of motion will be a function between the input link "link 2" and the output link "link 4" such that the function will pass through the precision points [12]. However, if precision points are more than seven, the quantitative solutions will be the good choice to form the function between the input and the output links.

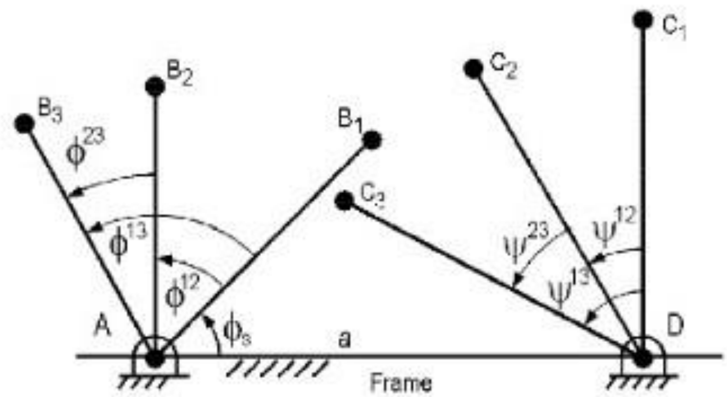

Figure 1-4. Funtion generation synthesis for Four-bar Linkage [16].

Hundreds of years ago, people use the graphical method to synthesize all types of mechanisms, especially four-bar mechanisms, and [17] was one of the last references in this method that published in the mid twentieth century which the same time when the analytical method was introduced. The first analytical method from all three cases was the case of function generation and it came on the mid-fifties when Freudenstein published his Phd thesis [18]. According to Ghosal, Freudenstein in [18] proposed an algebraic relation between the input angle and the output angle to the link lengths [16]. After that, the analytical method have developed by Freudenstein's student: Sandor, then Sandor's student: Eridman, Kaufman, and Loerch et al [12]. Sandor started presented the formulation for motion generation synthesis by solving the loop equations of the linkage mechanism using complex numbers[19]. His student Eridman used Sandor's results to synthesize the mechanism for the same case with three and four precision points [20].

While Loerch et al was interested in synthesizing the fixed pivot for the four-bar mechanism [21]and total design of the mechanism using a computer was the contribution by Kaufman [22]. For path generation, Freudenstein and Sandor applied the complex numbers with loop equations to the four bar mechanism for the generating a path passed five precision points with prescribed the input-link rotation [23]. Then, Blechschmidt and Uicker introduced a new way for the path generation that changed the approach to mechanism synthesis from the loop equation and complex number to the coupler curve. This technique increased the precision points to ten without prescribed the input link rotation and it was applied by Ananthasuresh and Kota to synthesis the four bar mechanism [24], [25].

The precision points have the same meaning in all three forms of kinematic synthesis. In motion generation, the concept of precision points consisted of the positions and orientations of the right-side dyad and the left-side dyad as shown in Figure 1-2. For the path generation, the precision points are on the path of the coupler curve as shown in Figure 1-3. On the subject of the function generation, the precision points made the function between the input link and the output link as shown in Figure 1-4 and the difference between the function the mechanism made and the desired function called a structural error [13]. To reduce the structural error, Wanldling proposed an analytical method to determine the spacing between the precision points in a way to minimize the structural error and it is called Chebyshev spacing [13].

1.2 Fuzzy Logic

A paradigm is a pattern of thinking that makes guidelines to approach an understanding of and to solve problems. That concept will be extended to many field such as scientific field and mathematical field. Kuhn defined the scientific one by “a universally recognized scientific achievements that for a time provide model problems and solutions to a community of practitioners” [26]. Changing a paradigm is called a paradigm shift which leads to a revolution in the field and that usually happens as a response to some challenges faced in the field due to growing of the given information in that field. The paradigm works through evolution while paradigm shift, such as in the case of genetic mutation, can change the structure of the gens to alter the product and the function of the gens. There are a lot of examples for the paradigm shifts in the scientific field such as the special relativity theory which was developed by Albert Einstein and in response to challenges posed by the limitation of Newtonian mechanics [26].

Regarding the mathematical paradigms, the most popular one is the classical logic or what is called Aerostatic logic that be formalizing by Aristotle thousands of years ago to assess the truth of the propositions or statements as true or false without consider the degrees of truth between the two extremes. This paradigm was dominant in many branches of knowledge such as philosophy, mathematics and science. The paradigm was challenged by schools that believe that knowledge must come from experience. A physician Sextus Empiricus who was a philosopher in the second century AD, and a supporter for this school, had a considerable criticism for this paradigm [27]. He explained that a syllogism (a logical argument consisted of major premise, minor premise and the conclusion), is valid if the two premises that form the sentences are also valid. Otherwise the validity will be uncertain [27]. Another significant criticism came from the legal Islamic scholar Ibn Taymiyya in the fourteenth century who explained that uncertainty in syllogism comes from incomplete induction using the method of analogy although Aristotelians deemed it uncertain evidence [27]. Sowa and Majumdar [27] have stated that “Ibn Taymiyya admitted that logical deduction is certain when it is applied to mental constructs in mathematics. However, reasoning about the real world can only be derived by induction guided by analogy which also leads to uncertain conclusions”.

The uncertainty in the scientific fields as it was explained above lead to paradigm shift for the classical logic and the probability theory was one of the alternative concept to build a new paradigm for the scientific fields [28]. A Paper was written by Bishop of Wells in 1685 that discussed the truth of statements given by independent sources whose reliability could invlove probabilities x and y, respectively [28], [29]. The theory of probability got noticeable progress during the eighteen century. A subjective probability theory proposes that the probability should acquired from an individual’s judgment about the chance of something happening and his judgment should be guided by past experience without any data or mathematical calculation. This theory studies the range of uncertainty in the human beliefs and how it changes with personal experience and bias. In the beginning of the twentieth century, other alternatives to classical logic and probability theory were developed to deal with uncertainty, such as multivalued and discrete logic by Jan Lukasiewicz in 1930, theory of evidence by Arthur Dempster in 1960, and a continues-value logic or what is called fuzzy logic by Lotfi Zadeh in 1965 [28].

From epistemological perspective, information could be understood as an opposite meaning of uncertainty. There is a variety of forms revealing the uncertainty in the engineering or scientific applications such as vagueness, fuzziness and imprecision[28]. Zadeh in forward of [30] defined the vagueness as shapeless, formless, or not specific information, as in “I will be back soon,” . However, he said that fuzziness associates with sharpness boundaries as in “I will be back in a few minutes”. Furthermore, the statement as “I will be back in a 5 minutes of 8 p.m.” is an example of the imprecision that could treated by the probability theory [28].

Chapter 2: Mathematical Model

2.1 Freudenstein's Equation

Based on the complex number theory, the loop equation of the four bar mechanism, showing in Figure 2-1, is given by

$$Z_1\, e^{i\theta_1} - Z_2\, e^{i\theta_2} - Z_3\, e^{i\theta_3} + Z_4\, e^{i\theta_4} = 0 \tag{1}$$

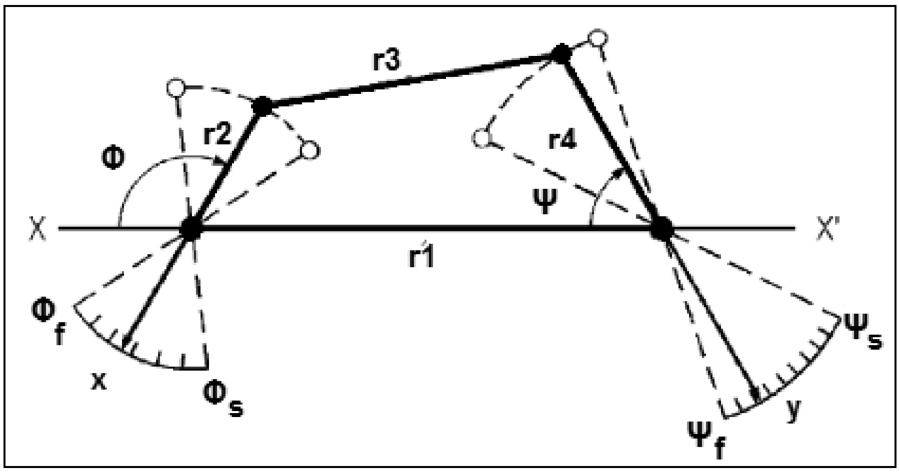

Figure 2-1. Coordinate system for function generation using Freudenstein's Equation

Using Euler’s formula, the loop equation spilt up into two equations as following:

$$Z_1 \cos(\theta_1) - Z_2 \cos(\theta_2) - Z_3 \cos(\theta_3) + Z_4 \cos(\theta_4) = 0 \tag{2}$$

$$Z_1 \sin(\theta_1) - Z_2 \sin(\theta_2) - Z_3 \sin(\theta_3) + Z_4 \sin(\theta_4) = 0 \tag{3}$$

As the ground link Z_1 has fixed position (Figure 2-1) and its angle $\theta_1 = 0$, thus $\cos(\theta_1) = 1$, and $\sin(\theta_1) = 0$. Substitutions yields (2) and (3) as:

$$Z_1 - Z_2 \cos(\theta_2) - Z_3 \cos(\theta_3) + Z_4 \cos(\theta_4) = 0 \tag{4}$$

$$-Z_2 \sin(\theta_2) - Z_3 \sin(\theta_3) + Z_4 \sin(\theta_4) = 0 \tag{5}$$

Since the function generation variables are associated with the input link “link 2” and the output link “link 4”, the coupler link “link 3” should be defined by them as following:

$$Z_1 - Z_2\cos(\theta_2) + Z_4\cos(\theta_4) = Z_3\cos(\theta_3) \tag{6}$$

$$-Z_2\sin(\theta_2) + Z_4\sin(\theta_4) = Z_3\sin(\theta_3) \tag{7}$$

Then, both (6) and (7) are squaring and adding to each other as following:

$$(Z_1 - Z_2\cos(\theta_2) + Z_4\cos(\theta_4))^2 + (-Z_2\sin(\theta_2) + Z_4\sin(\theta_4))^2$$
$$= {Z_3}^2(\cos(\theta_3) + \sin(\theta_3))^2 \tag{8}$$

$${Z_1}^2 + (Z_2\cos(\theta_2))^2 + (Z_4\cos(\theta_4))^2 + 2\,Z_1Z_4\cos(\theta_4) -$$
$$2\,Z_1Z_2\cos(\theta_2) + 2\,Z_2\,Z_4\cos(\theta_2)\,\cos(\theta_4) + (Z_2\sin(\theta_2))^2$$
$$-2\,Z_2\sin(\theta_2)\,Z_4\sin(\theta_4) + (Z_4\sin(\theta_4))^2 = {Z_3}^2(\cos(\theta_3) + \sin(\theta_3))^2 \tag{9}$$

$${Z_1}^2 + {Z_2}^2\,(\cos(\theta_2)^2 + \sin(\theta_2)^2) + {Z_4}^2(\cos(\theta_4)^2 + \sin(\theta_4)^2)$$
$$+2\,Z_1Z_4\cos(\theta_4) - 2\,Z_1Z_2\cos(\theta_2)$$
$$+2\,Z_2\,Z_4\,(\cos(\theta_2)\,\cos(\theta_4) - \sin(\theta_2)\,\sin(\theta_4)) = {Z_3}^2\,(\cos(\theta_3)^2 + \sin(\theta_3)^2) \tag{10}$$

Since $\cos(\theta)^2 + \sin(\theta)^2 = 1$, thus (9) should be written as:

$${Z_1}^2 + {Z_2}^2 + {Z_4}^2 + 2\,Z_1Z_4\cos(\theta_4) - 2\,Z_1Z_2\cos(\theta_2)$$
$$+2\,Z_2\,Z_4(\cos(\theta_2)\,\cos(\theta_4) - \sin(\theta_2)\,\sin(\theta_4)) = {Z_3}^2 \tag{11}$$

Since $[\cos(\theta_2)\,\cos(\theta_4) - \sin(\theta_2)\,\sin(\theta_4)] = \cos(\theta_2 - \theta_4)$, (11) will be as:

$${Z_1}^2 + {Z_2}^2 + {Z_4}^2 + 2\,Z_1Z_4\cos(\theta_4) - 2\,Z_1Z_2\cos(\theta_2)$$
$$+2\,Z_2\,Z_4\,\cos(\theta_2 - \theta_4) = {Z_3}^2 \tag{12}$$

Then, (11) could be arranged as:

$$\frac{Z_1}{Z_4}\cos(\theta_2) - \frac{Z_1}{Z_2}\cos(\theta_4) + \frac{{Z_1}^2 + {Z_2}^2 + {Z_4}^2 - {Z_3}^2}{2\, Z_2\, Z_4} = -\cos(\theta_2 - \theta_4) \qquad (13)$$

In a compact form, the final form of Freudenstein's Equation will be written as:

$$K_1 \cos(\theta_2) + K_2 \cos(\theta_4) + K_3 = -\cos(\theta_2 - \theta_4) \qquad (14)$$

$$K_1 \cos(\theta_2) + K_3 + \cos(\theta_4)\,(\cos(\theta_2) + K_2) + \sin(\,\theta_2)\sin(\theta_4) = 0$$

$$a + \cos(\theta_4)\, b + \mathrm{c}\sin(\theta_4) = 0$$

Where

$$K_1 = \frac{Z_1}{Z_4}$$

$$K_2 = -\frac{Z_1}{Z_2} \qquad (15)$$

$$K_3 = \frac{{Z_1}^2 + {Z_2}^2 + {Z_4}^2 - {Z_3}^2}{2\, Z_2\, Z_4}$$

In case of three prescribed positions with Freudenstein's Equation, as well as $\emptyset$ and ψ represent input and output angles, respectively, (14) expands to three equation that be written as:

$$K_1 \cos(\emptyset_1) + K_2 \cos(\psi_1) + K_3 = -\cos(\emptyset_1 - \psi_1)$$

$$K_1 \cos(\emptyset_2) + K_2 \cos(\psi_2) + K_3 = -\cos(\emptyset_2 - \psi_2) \qquad (16)$$

$$K_1 \cos(\emptyset_3) + K_2 \cos(\psi_3) + K_3 = -\cos(\emptyset_3 - \psi_3)$$

Where the link lengths be expressed in terms of K’s and the given the length of the ground link “link 1” as following:

$$Z_4 = \frac{Z_1}{K_1}$$

$$Z_2 = -\frac{Z_1}{K_2} \tag{17}$$

$$Z_3 = \sqrt{{Z_1}^2 + {Z_2}^2 + {Z_4}^2 + K_3 2\, Z_2\, Z_4}$$

By fuzzifying the second position from these prescribed positions, the Chebyshev spacing will not be used to determine the best location of the second point to reduce the structural error, and the second equation in (17) will be a fuzzy equation whose input and output angles are fuzzified. From the range of input and output angles beside the fuzzy region, a fuzzy linear system of equation will be formed. That system is solved in this project by decomposing the matrix into two matrices: the first part involves matrices of interval numbers at every level of α −cut method, and the second one is the crisp matrix at $\alpha = 1$. The solution will be acquired by the fuzzy mathematical operations to calculate the fuzzy constants K's. They are used to obtain the fuzzy lengths of the links for the four bar mechanism as defined in (17). These lengths will have been defuzzified by one of the defuzzification methods to get approximating lengths of the links for the mechanism. All of these steps and operation will explained in details at the next sections in this chapter.

2.2 Fuzzy Mathematical Operations

2.2.1 Fuzzy Number

In the fuzzy logic, a fuzzy number is used to represent the uncertainty with a membership function such as a triangular membership function, a trapezoidal membership function, Gaussian membership function, and generalized bell membership function is used.

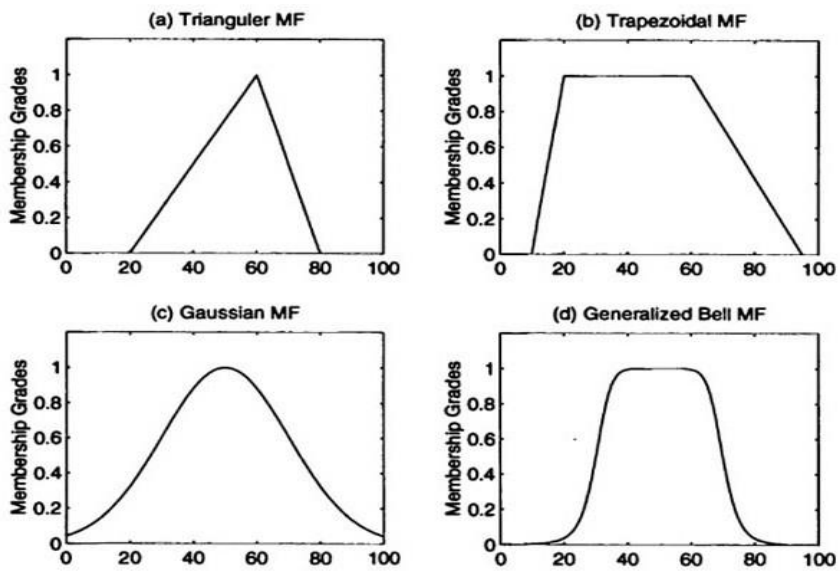

Figure 2-2. Examples of four classes of membership function:

(a) Triangle (x; 20, 60, 80); (b) Trapezoid (x; 10, 20, 60, 95);

(c) Gaussian (x; 50, 20); (d) generalized bell (x; 20, 4, 50) [42].

The fuzzy number is a convex, single point normal fuzzy set [28]. According to [28], "the convexity of the fuzzy set means that its membership values are strictly monotonically increasing, then they strictly monotonically decreasing with increasing values for elements in the universe, and the single point fuzzy set is the one whose membership function has an element belong to the set whose membership value is unity".

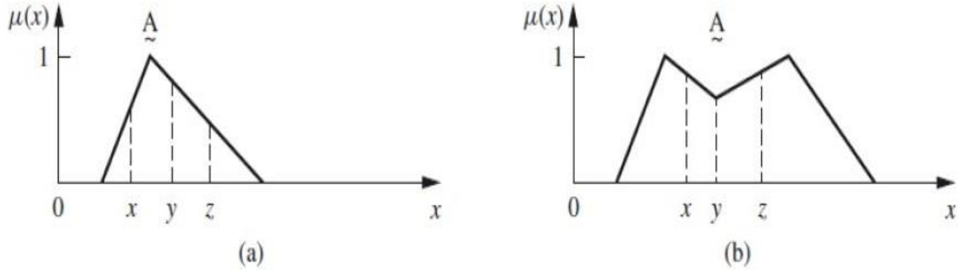

Figure 2-2. (a) Convex, single point normal fuzzy set. (b) Nonconvex, not single point normal fuzzy set [16].

The fuzzy number whose membership function is triangular will be the selected to represent the uncertainty in the input angle and the output angle for the function generation synthesis.

Definition 2-1 [31]. A triangular fuzzy number $\tilde{u} = (a, b, c)$ and its membership function is defined by:

$$\mu_{\tilde{u}}(x) = \begin{cases} \dfrac{x-a}{b-a} & a \leq x \leq b, \\ \dfrac{x-c}{b-c} & b \leq x \leq c, \\ 0 & otherwise \end{cases}$$

Definition 2-2 [31]. A triangular fuzzy number $\tilde{u} = (a, b, c)$ is non-negative if $a \geq 0$

Definition 2-3 [31]. Let $\tilde{u} = (a, b, c)$ and $\tilde{v} = (d, f, g)$ be two triangular fuzzy numbers, then:

1. $\tilde{u} = \tilde{v}$ $if\ and\ only\ if\ a = a, b = f, and\ c = g$
2. $\tilde{u} + \tilde{v} = (a + d, b + f, c + g)$
3. $\tilde{u} - \tilde{v} = (a - g, b - f, c - d)$

Definition 2-4 [31]**.** Let ũ be any triangular fuzzy number and ṽ be a non-negative one, then the multiplication is defined by:

1. $$\tilde{u} * \tilde{v} = \begin{cases} (ag, bf, cg) & a \geq 0, \\ (ag, bf, cg) & a < 0, c \geq 0, \\ (ag, bf, ce) & a < 0, c < 0, \end{cases}$$

Representing the membership function of the triangular fuzzy number could be infinite crisp sets that be identified by a variable α.

Definition 2-5 [32]**.** Let ũ be any triangular fuzzy number, so that, its membership function is a function that satisfies the following conditions:

1. $\tilde{u}_\alpha = [\tilde{u}_\alpha^L, \tilde{u}_\alpha^R]$
2. $\tilde{u}_\alpha^L$ *is a bounded monotonic increasing continuous* function $\forall\, \alpha \in [0;\ 1]$
3. $\tilde{u}_\alpha^R$ *is a bounded monotonic decreasing continuous* function $\forall\, \alpha \in [0;\ 1]$
4. $\tilde{u}_1^L = \tilde{u}_1^R$

2.2.2 Fuzzy α −cut method

An α −Cut set u_α is a crisp set derived from a fuzzy set ũ that be defined by

$$u_\alpha = \{x \mid \mu_{\tilde{u}}(x) \geq \alpha\}, \qquad \alpha \in [0,1] \tag{18}$$

There are infinite number of α −cut sets derived from a fuzzy set which every element belong to u_α have to belong to ũ with a level of membership greater of equal the value of α [28]. The arithmetic operations that be applied to α −cut set came from arithmetic interval operations such as addition, multiplication, subtraction, and division. These basic arithmetic interval operations are applied on the interval number determined by α −cut method.

Definition 2-6 [33]**.** An interval number is the bounded, closed subset of real number defined by

$$u_\alpha = [\underline{u}_\alpha, \overline{u}_\alpha] \qquad where\ \overline{u}_\alpha \geq \underline{u}_\alpha$$

Where $\underline{u}_\alpha$ the lower limit of the interval number and $\overline{u}_\alpha$ is the upper limit of the interval number.

Definition 2-7 [28], [33]**.** Let u_α and v_α be two interval numbers, then:

1. $u_\alpha + u_\alpha = [\underline{u}_\alpha + \underline{v}_\alpha , \overline{u}_\alpha + \overline{v}_\alpha]$
2. $u_\alpha - v_\alpha = [\underline{u}_\alpha + \overline{v}_\alpha , \overline{u}_\alpha + \underline{v}_\alpha]$
3. $u_\alpha * v_\alpha = [\min(\underline{u}_\alpha * \underline{v}_\alpha ,\ \underline{u}_\alpha * \overline{v}_\alpha , \overline{u}_\alpha * \underline{v}_\alpha , \overline{u}_\alpha * \overline{v}_\alpha),\ \max(\underline{u}_\alpha * \underline{v}_\alpha ,\ \underline{u}_\alpha * \overline{v}_\alpha , \overline{u}_\alpha * \underline{v}_\alpha , \overline{u}_\alpha * \overline{v}_\alpha)]$
4. $\frac{u_\alpha}{v_\alpha} = u_\alpha * \left(\frac{1}{v_\alpha}\right) = u_\alpha * \left[\frac{1}{\overline{v}_\alpha} , \frac{1}{\underline{v}_\alpha}\right]\ \ where\ 0 \notin v_\alpha$
5. $k * u_\alpha = k * [\underline{u}_\alpha , \overline{u}_\alpha] = \begin{cases} [k * \underline{u}_\alpha , k * \overline{u}_\alpha], & k > 0 \\ [k * \overline{u}_\alpha , k * \underline{u}_\alpha] & k < 0 \end{cases}$

Note that subtraction and division operations of the interval number are not inverses of addition and multiplication operation. So that, the authors in[33] present two extra interval operations that are defined as inverse of interval numbers for addition and multiplication. These operations are denoted by $\ominus$ for the inverse of addition operation and $\oslash$ for the inverse of multiplication operation.

Definition 2-8. Let $u_\alpha = [\underline{u}_\alpha , \overline{u}_\alpha]$ and $v_\alpha = [\underline{v}_\alpha , \overline{v}_\alpha]$ be two interval numbers, then:

1. $u_\alpha \ominus v_\alpha == [\underline{u}_\alpha - \underline{v}_\alpha , \overline{u}_\alpha - \overline{v}_\alpha]$

2. $u_\alpha \oslash v_\alpha =$

$$\begin{cases} [-\infty, \infty] & \underline{u}_\alpha = \overline{u}_\alpha = 0, \quad \underline{v}_\alpha = \overline{v}_\alpha = 0 \\ [0, 0] & \underline{u}_\alpha = \overline{u}_\alpha = 0, \quad (\underline{v}_\alpha \neq 0 \ or \ \overline{v}_\alpha \neq 0) \\ undefined & (\exists x \neq 0,\ x \in \underline{u}_\alpha), \quad \underline{v}_\alpha = \overline{v}_\alpha = 0 \\ undefined & 0 \notin \overline{u}_\alpha, \quad 0 \in \overline{v}_\alpha \\ \left[\frac{\underline{u}_\alpha}{\underline{v}_\alpha}, \frac{\overline{u}_\alpha}{\overline{v}_\alpha}\right] & \underline{u}_\alpha \geq 0, \quad \underline{v}_\alpha \geq 0 \\ \left[0, \frac{\overline{u}_\alpha}{\overline{v}_\alpha}\right] & \underline{u}_\alpha = 0, \quad \underline{v}_\alpha = 0 \\ \left[\frac{\underline{u}_\alpha}{\overline{v}_\alpha}, \frac{\overline{u}_\alpha}{\overline{v}_\alpha}\right] & \underline{u}_\alpha \leq 0 \leq \overline{u}_\alpha, \quad \underline{v}_\alpha \geq 0 \\ \left[\frac{\underline{u}_\alpha}{\overline{v}_\alpha}, \frac{\overline{u}_\alpha}{\underline{v}_\alpha}\right] & \overline{u}_\alpha \leq 0, \quad \underline{v}_\alpha \geq 0 \\ \left[\frac{\underline{u}_\alpha}{\overline{v}_\alpha}, 0\right] & \overline{u}_\alpha = 0, \quad \underline{v}_\alpha = 0 \\ u_\alpha \oslash (-v_\alpha) & \overline{v}_\alpha \leq 0 \\ \left[\max\left(\frac{\underline{u}_\alpha}{\overline{v}_\alpha}, \frac{\overline{u}_\alpha}{\underline{v}_\alpha}\right), \min\left(\frac{\overline{u}_\alpha}{\overline{v}_\alpha}, \frac{\underline{u}_\alpha}{\underline{v}_\alpha}\right)\right] & \underline{u}_\alpha \leq 0 \leq \overline{u}_\alpha \quad, \underline{v}_\alpha < 0 < \overline{v}_\alpha \\ \left[\frac{\overline{u}_\alpha}{\underline{v}_\alpha}, \frac{\underline{u}_\alpha}{\underline{v}_\alpha}\right] & 0 \geq \overline{u}_\alpha \quad, \underline{v}_\alpha < 0 < \overline{v}_\alpha \end{cases}$$

Proofs for these new arithmetic interval operations as inverse of addition and multiplication are explained in detail in [33].

2.2.3 Fuzzy Function

There several types of fuzzy function with respect to which parts of function is fuzzy. A function has three parts: the domain, the range, and the relation between them that transforms to a function as every element in the domain has a unique image in the range. The fuzziness of first two parts gives the first type of the fuzzy function called a crisp function with fuzzy constraints, which the domain and the range are fuzzy [34].

Definition 2-9 [34]**.** Suppose X and Y are crisp sets which a fuzzy set $\widetilde{\mathrm{A}} \in \mathrm{X}$ and a fuzzy set $\widetilde{\mathrm{B}} \in \mathrm{Y}$. Let f is a crisp function, then f will be a crisp function with fuzzy constraints on fuzzy domain A and fuzzy range B $f: X \rightarrow Y$ if it satisfies the condition:

$$\mu_{\widetilde{\mathrm{A}}}(x) \leq \mu_{\widetilde{\mathrm{B}}}(f(x))$$

The second type of the fuzzy function is the crisp function that transforms the fuzziness of the independent variable to the dependent variable and it is called fuzzy extension function. This type of the fuzzy function is applying in this project to solve the cosine of the fuzzy input angles in the function generation synthesis to produce a fuzziness in the output angles.

Definition 2-10 [34]**.** Suppose X and Y are crisp sets which a fuzzy set $\widetilde{\mathrm{A}} \in \mathrm{X}$. Let f is a crisp function $f: X \rightarrow Y$ and f will be a fuzzy extension function that define the image $\mathrm{f}(\widetilde{\mathrm{A}})$ on Y set. The extension principle is the base of the fuzzy extension function that satisfy the conditions:

$$\mu_{\mathrm{f}(\tilde{\mathrm{x}})}(y) \begin{cases} \max\limits_{x \in f^{-1}(y)} \left(\mu_{\tilde{x}}\left(f(x)\right)\right), & if\ \ f^{-1}(y) \neq \emptyset \\ 0\,, & if\ \ f^{-1}(y) = \emptyset \end{cases}$$

Where $f^{-1}(y)$ is the inverse image for y

Third type of fuzzy function is the fuzziness of the function itself while the domain is crisp set which the fuzzy function produce fuzzy image of that domain.

Definition 2-11 [34]**.** Suppose X a crisp sets. Let f is a fuzzy function that mapping of X in fuzzy set $\tilde{P}(Y)$:

$$\tilde{f}: X \rightarrow \tilde{P}(Y)$$

Where

$$\mu_{\tilde{\mathrm{f}}(\mathrm{x})}(y) = \mu_R(x, y)\,,\ \forall (x, y) \in X \times Y$$

2.3 Fuzzy and Interval Linear System of Equations

Consider an $n \times n$ fully fuzzy linear system “FFLS”:

$$\begin{aligned} a_{11}\, x_1 + a_{12}\, x_2 + \cdots + a_{1n}\, x_n &= b_1 \\ a_{21}\, x_1 + a_{22}\, x_2 + \cdots + a_{2n}\, x_n &= b_2 \\ &\vdots \\ a_{n1}\, x_1 + a_{n2}\, x_2 + \cdots + a_{nn}\, x_n &= b_n \end{aligned} \tag{19}$$

That could be written as:

$$\hat{A}\, \hat{x} = \hat{b} \tag{20}$$

Which $A = (a_{ij})_{n \times n}$ is a triangular fuzzy number that represents a coefficient matrix, however, $x = (x_i)_{n \times 1}$ and $b = (b_i)_{n \times 1}$ are column vectors of triangular fuzzy numbers. To solve this type of matrix, the authors in [33] suggest a practical method using the interval number which divided the matrix to two matrices: one is an interval matrix

$$A^0 x^0 = b^0 \quad (21)$$

The other is a crisp matrix

$$A^1 x^1 = b^1 \quad (22)$$

Where the interval matrix involves the lower and upper number of fuzzy numbers at level zero of α −cut method. This interval matrix according to LU decomposition method [35] represented by the following formula:

$$\hat{A} = \hat{L} \times \hat{U} \quad (23)$$

While the other matrix is a crisp matrix at level of $\alpha = 1$ and its solution consists of crisp numbers.

This selection of this method was depending on two factors: the singularity of the coefficient matrix $\hat{A}$ and the efficiency of the method. Determining the singularity of the coefficient matrix or getting closer to the singularity gives rise to the flouting points in the results. They show themselves clearly in the membership functions of the constants K's, in which the resultant fuzzy set has not been a fuzzy number as it mentioned above in the definition 2-5. The third example in the chapter 3 will demonstrate this situation clearly. Avoiding the flouting points comes from the sign of the determinant of the coefficient matrix $\hat{A}$ which the determinant must not contain zero. The advantage of the LU decomposition is the calculating of the determinant could simply done by multiplications of the diagonal elements in the $\hat{U}$ matrix while the determinant of the $\hat{L}$ matrix always be one because of the unity diagonal [36]. The second feature of the LU decomposition is the efficiency of the calculation which means the steps to get the solution are less than other methods as well as getting narrower solution comparing to the output of others [37]. The LU decomposition is classified as indirect method versus the direct methods that produce the exact results such as Gaussian elimination. Since the solutions have much difficulty with direct methods for multi-dimensional matrices, the indirect methods such as Krawczyk Method, Gauss–Seidel method and LU decomposition method was proposing to solve the interval system of equations with approximate solutions that converge to the exact one [38]. According to [36], [37], LU decomposition deem to be one of the most efficient method comparing to others especially in the computational time.

Generalized interval numbers are mentioned in[37], in which it is defined as the intervals whose limits are not conditioned by the ascending order and they have divided into three kinds:

1- *Proper interval set* that identifies as the classical interval definition as the following:

$$u_\alpha = [\underline{u}_\alpha , \overline{u}_\alpha] \qquad where\ \underline{u}_\alpha \leq \overline{u}_\alpha$$

2- *Improper interval set* that defined as following:

$$u_\alpha = [\overline{u}_\alpha, \underline{u}_\alpha] \qquad where\ \overline{u}_\alpha \geq \underline{u}_\alpha \tag{24}$$

3- Degenerated interval number that identified as

$$u_\alpha = [\overline{u}_\alpha, \underline{u}_\alpha] \qquad where\ \overline{u}_\alpha = \underline{u}_\alpha$$

Which it is identical with the crisp number.

Since the proper interval matrices have gotten from the fuzzy α-cut method agree with the proper interval matrices or the classical interval matrices, the proper solution set called a united solution set is defined in [39] as the following:

$$\hat{x} \subseteq \sum(\hat{A}, \hat{b}) , \qquad \sum(\hat{A}, \hat{b}) = \{x \in R^n | \exists\ A \in \hat{A}, \exists\ b \in \hat{b} , Ax = b\} \tag{25}$$

The authors in [33] have proposed procedures for LU decomposition method as the following:

$$\hat{A} = \hat{L} \times \hat{U} \implies \hat{L} \times \hat{y} = \hat{b} \implies \hat{U} \times \hat{x} = \hat{y} \tag{26}$$

Where $\hat{L}$ is the lower triangular interval matrix and $\hat{U}$ is the upper triangular interval matrix. To get these matrices according to [33], [37], firstly the diagonal elements of $\hat{L}$ are deemed to be unity , then the rest of elements of $\widehat{U}$ and $\hat{L}$ could be computed by these interval equations that derived from the Gaussian elimination direct method :

$For\ i \leq j$

$$\hat{a}_{ij} = \sum_{k=1}^{i-1}\left(\hat{l}_{ik} \times \hat{u}_{kj}\right) + \hat{u}_{ij} \quad \Rightarrow \quad \hat{u}_{ij} = \hat{a}_{ij} \ominus \sum_{k=1}^{i-1}\left(\hat{L}_{ik} \times \hat{u}_{kj}\right) \qquad (27)$$

$For\ i > j$

$$\hat{a}_{ij} = \sum_{k=1}^{j-1}\left(\hat{l}_{ik} \times \hat{u}_{kj}\right) + \hat{l}_{ij} \times \hat{u}_{jj} \quad \Rightarrow \quad \hat{l}_{ij} = \frac{\left(\hat{a}_{ij} \ominus \sum_{k=1}^{j-1}\left(\hat{L}_{ik} \times \hat{u}_{kj}\right)\right)}{\hat{u}_{jj}} \qquad (28)$$

Where $\hat{u} \in \widehat{U}\ \&\ \hat{l} \in \hat{L}$ and $\hat{u}_{jj} \in R^{+}$ or $\hat{u}_{jj} \in R^{-}$.

Note that $\hat{u}_{jj}$ must not contain zero as interval number as it mention above in Definition 2-6. This condition make a limitation on fuzziness of the input angle as the examples show in the next chapter. The Algorithms had been proved in details at both references mentioned above.

2.4 Defuzzification

2.4.1 Defuzzification Method

The fuzzy input always produce a fuzzy output, however there are many applications that need to get a crisp output instead of fuzzy output. These applications lead to a concept called a defuzzification that transforms the fuzzy output membership function to get a number that represents a crisp output as an opposite of the fuzzification process. Ross in [28] presents six methods to defuzzify the fuzzy output membership function as following:

1- Maximum membership principle: The method that determines the output crisp number that has a highest grade of membership function. The algebraic expression of the method as following:

$$\mu_{\tilde{u}}(z^*) \geq \mu_{\tilde{u}}(z) \quad \forall\, z \in Z \tag{29}$$

Where z^* is the output crisp value.

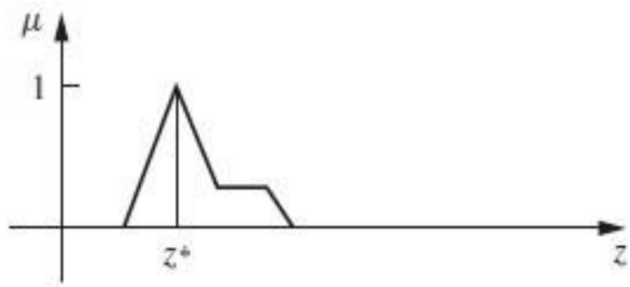

Figure 2-3: Max membership method

2- Centroid method: It gives the output crisp value has a grade of membership function from the center area of this function. It has an algebraic expression as following:

$$z^* = \frac{\int \mu_{\tilde{u}}(z) \cdot z \, dz}{\int \mu_{\tilde{u}}(z) \, dz} \tag{30}$$

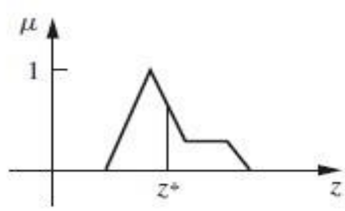

Figure 2-4: Centroid method

3- Weighted average method: Author in [28] describes it as the most appearing in the fuzzy applications, and it is conditional for the symmetric membership function. It has the algebraic expression as following:

$$z^* = \frac{\sum \mu_{\tilde{u}}(\bar{z}) \cdot \bar{z}}{\sum \mu_{\tilde{u}}(\bar{z})} \tag{31}$$

Where $\bar{z}$ is the centroid of the symmetric membership function.

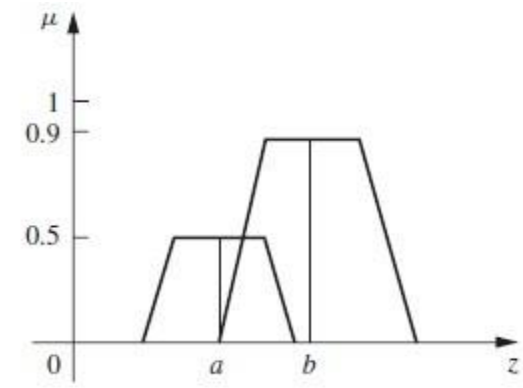

Figure 2-5: Weighted average method

4- Mean maximum membership: It is the mostly like the first method except the location of the maximum grade of the membership, in which the location be given by the algebraic expression as following:

$$z^* = \frac{a+b}{2} \tag{32}$$

Where locations of a and b are shown in Figure 2-6

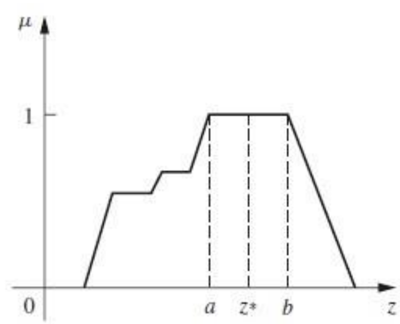

Figure 2-6: Mean maximum method

5- Center of sums: It does not conditional for the symmetric membership function; however, it requires the location of centroid for every membership function. The disadvantage of this method is duplication the intersecting area. Its algebraic expression is given as following:

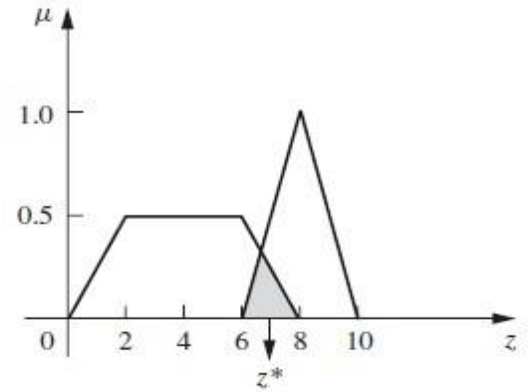

Figure 2-6: Center of sums method

$$z^* = \frac{\sum_{k=1}^{n} \mu_{\tilde{u}}(z) \cdot \int \bar{z}\, dz}{\sum_{k=1}^{n} \mu_{\tilde{u}}(z) \int dz} \tag{33}$$

Where $\bar{z}$ is the distance from z^* to the centroid of every membership function.

6- Center of largest area: It works when we have two convex subregions, in which the defuzzified value is calculated by the centroid method of the convex fuzzy subregion with the largest area. This method is appropriate, as the completely fuzzy membership function is non-convex. Its algebraic expression is given as following:

$$z^* = \frac{\int \mu_{\tilde{u}_m}(z) \cdot z\, dz}{\int \mu_{\tilde{u}_m}(z)\, dz} \tag{34}$$

Where $\mu_{\tilde{u}_m}$ has the largest area form all convex subregions.

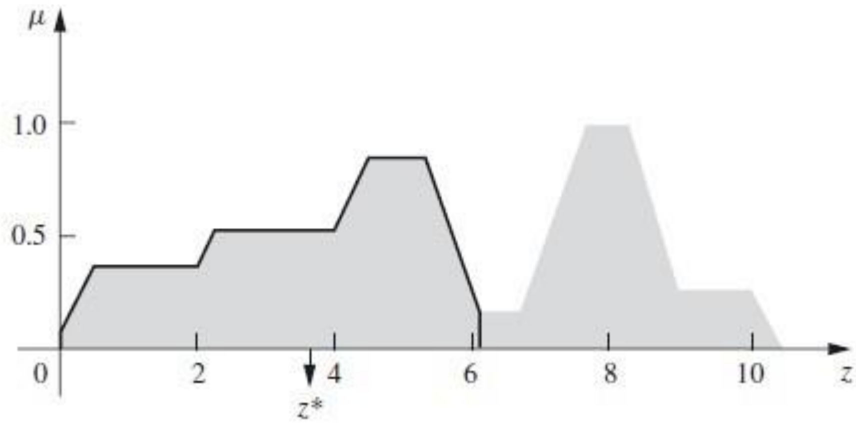

Figure 2-7: Center of largest area method

2.4.2 Centroid Defuzzificatio of Non-Linear membership function

The nonlinear membership function has a challenge to defuzzify directly because it is hard to get the explicit function that represent this function. Consequently, this type of function needs to be defuzzified approximately using the piecewise functions at α-cuts sets as they are defined in (18) and definition 2-6 as well as demonstrated clearly by Wang in [40] as the following:

Definition 2-12: Let ũ is a fuzzy number, then the α-cuts sets are defined as

$$u_\alpha = \{x \mid \mu_{\tilde{u}}(x) \geq \alpha\} = [min\{x \mid \mu_{\tilde{u}}(x) \geq \alpha\}, \max\{x \mid \mu_{\tilde{u}}(x) \geq \alpha\},$$
$$= [\underline{u}_\alpha, \overline{u}_\alpha], \qquad \alpha \in [0,1]$$

These linear piecewise functions are assumed as linear functions and they are be displayed as

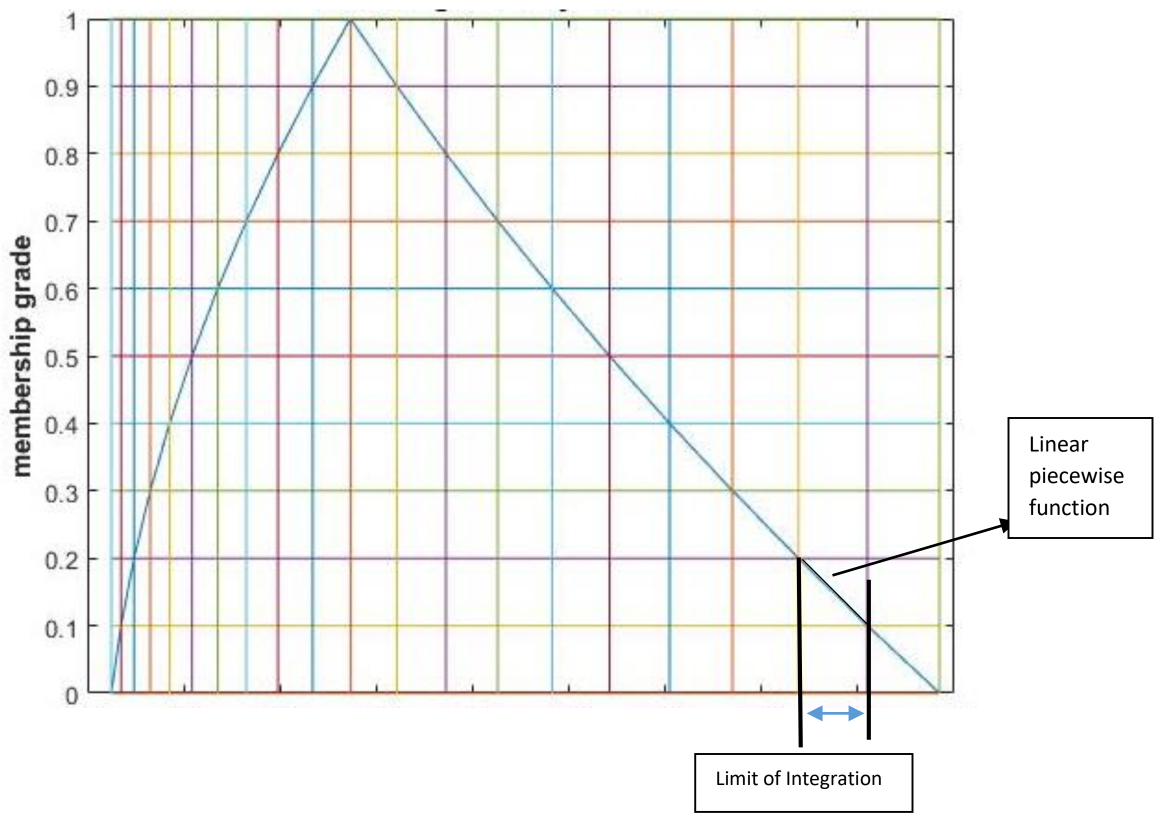

Figure 2-8: Linear piecewise function with the limits of integration at every α-cut set

The centroid defuzzification of the linear piecewise functions as demonstrated in (30) will be the summation of the limited integration of every linear piecewise function at numerator and denominator as are shown in Fig. 2-8

Chapter 3: Examples

3.1 Example

The example of the four-bar synthesis using Freudenstein's equation In [14] that mechaniclly generats the function of $y = \sin(x)$, $0° \leq x \leq 90°$ and the range of the function is $0 \leq y \leq 1$. The selected range of the input and output angles are $120°$ and $60°$ degrees, respectively, and the calculated chebyshef precesion points accourding to [14] will be:

$$x_1 = 3.75$$

$$x_2 = 42.75$$

$$x_3 = 81.75$$

Then, the initial input angle $\emptyset_i = 100$ and the initial output angle ψ_i =60. Using the linear mapping to get $\emptyset's$ and $\psi's$ from the function $y = \mathrm{f}(x)$ by scaling parameters a, b, c and d that be calculated from the relation in [41] as

$$\emptyset = a\,x + \emptyset_i \rightarrow \emptyset_f - \emptyset_i = a\,x_f \rightarrow a = \frac{120}{90} = 1.3333\,, \qquad b = \emptyset_i = 100$$

$$\psi = c\,y + \psi_i \rightarrow \psi_f - \psi_i = a\,y_f \rightarrow c = \frac{60}{1} = 60\,, \qquad d = \psi_i = 100$$

From these parameters, precision points of the input and output angles are calculted by the scaling parameters as

$$\emptyset = [105\ 157\ 209]$$

$$\psi = [66.27\ 102.42\ 119.68]$$

Then, the middle angle will be fuzzified at different degrees: 12, 6 and 4 degrees to see the differences between the centroid diffuzzified synthsis and the traditional crisp synthesis. The fuzzifying process of the middle angles will be the same at every case, so the steps will be demonstrated at the first case with the results; however, only results will be shown for other cases.

Regarding the first case with 12 degrees of fuzziness, the fuzzy membership functions of the middle input and output angles are shown in the following figures:

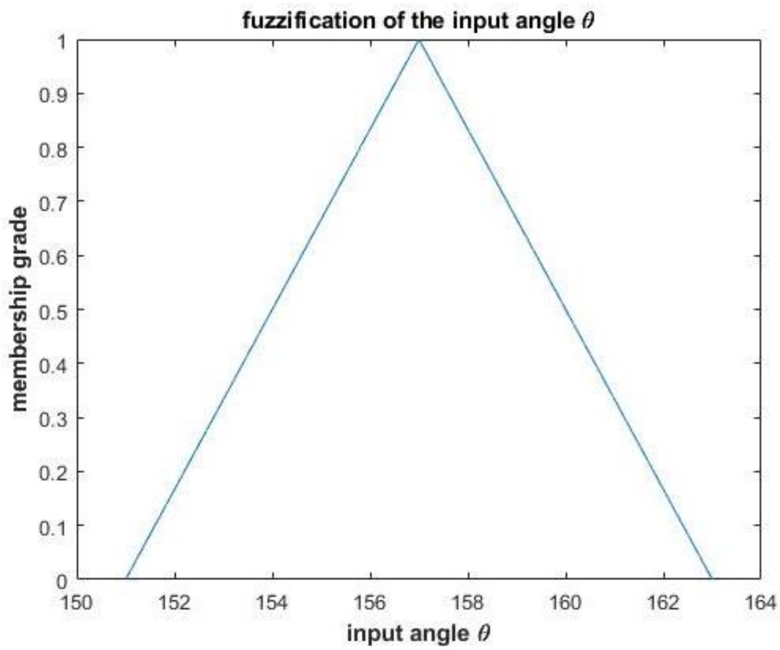

Figure 3-1. Fuzzification the second input angle $\emptyset_2$

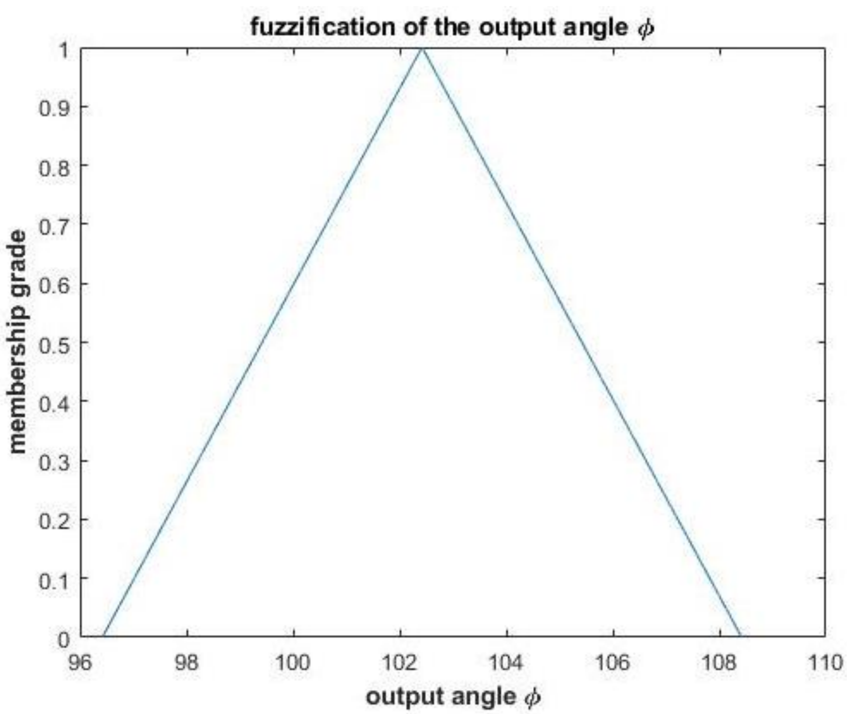

Figure 3-2. Fuzzification the second output angle ψ_2

Then, the resultant membership functions of crisp cosine functions of the fuzzy input $\emptyset_2$ and the fuzzy output ψ_2 as they were defined in Definition 3-3 are shown as

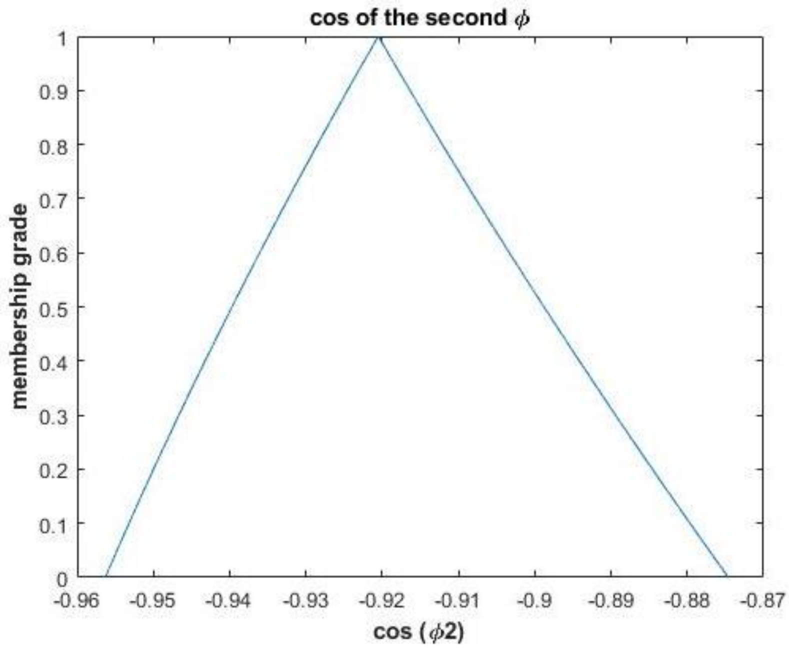

Figure 3-3. Crisp Cos function of fuzzy $\emptyset_2$

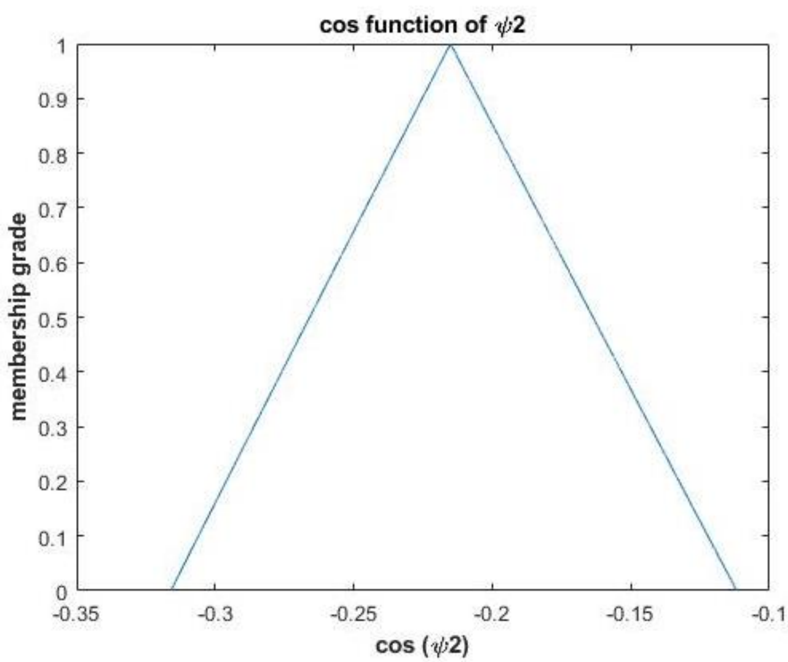

Figure 3-4. Crisp cos function of fuzzy ψ_2

The resultant fuzzy number can be expressed as an interval numbers with lower and upper limits using the α -cut mehtod.

$$\cos(\widetilde{\emptyset}_2) = [\,0.0358\,\alpha - 0.9563\,, -0.0459\,\alpha - 0.8746\,]$$
$$\cos(\tilde{\psi}_2) = [0.1009\,\alpha - 0.3160\,, -0.1033\,\alpha - 0.1118\,]$$

In addition, the right hand side in (13) that involves the cosine of the difference between the fuzzy input and the fuzzy output angles can be expressed by the α-cut mehtod as

$$\cos(\widetilde{\emptyset}_2 - \tilde{\psi}_2) = [0.1821\,\alpha + 0.3975\,, 0.7363 - 0.1568\,\alpha\,]$$

Then, the fully fuzzy linear system of equation will be as

$$\tilde{A} = \begin{bmatrix} -0.2588 & 0.4024 & 1 \\ [0.0358\alpha - 0.9563\,, -0.0459\,\alpha - 0.8746] & [0.1009\,\alpha - 0.3160\,, -0.1033\,\alpha - 0.1118] & 1 \\ -0.8746 & -0.4952 & 1 \end{bmatrix}$$

$$\tilde{\mathrm{b}} = \begin{bmatrix} -0.7801 \\ 0.1821\,\alpha + 0.3975\,, 0.7363 - 0.1568\,\alpha \\ -0.0119 \end{bmatrix}, \quad \tilde{x} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix}$$

The fuzzy linear system of equation will be solved at every level of α-cut sets as they defined in (18). There are 10 interval matrices besides a crisp matrix at $\alpha = 1$ that be solved to get fuzzy constants K's as they defined in (14) and (15). The solutions were defined in (25), (27) and (28) that lead to the resultant fuzzy numbers K's as shown in the following figures:

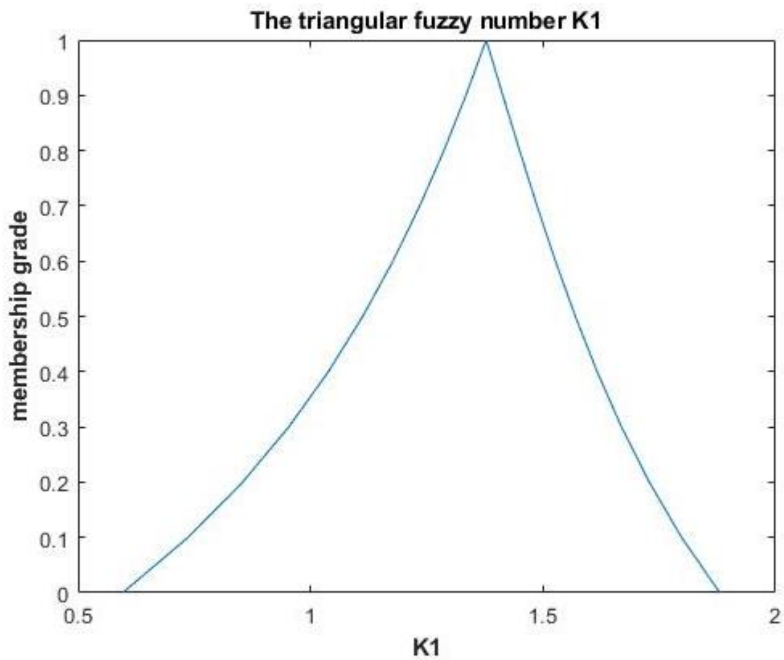

Figure 3-5. The fuzzy number K1 of the first case

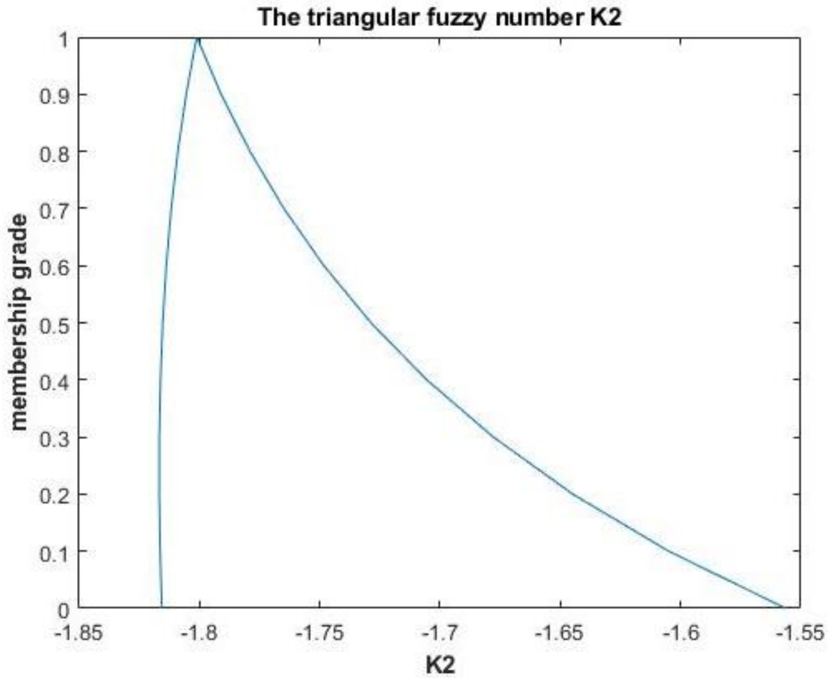

Figure 3-6. The fuzzy number K2 of the first case

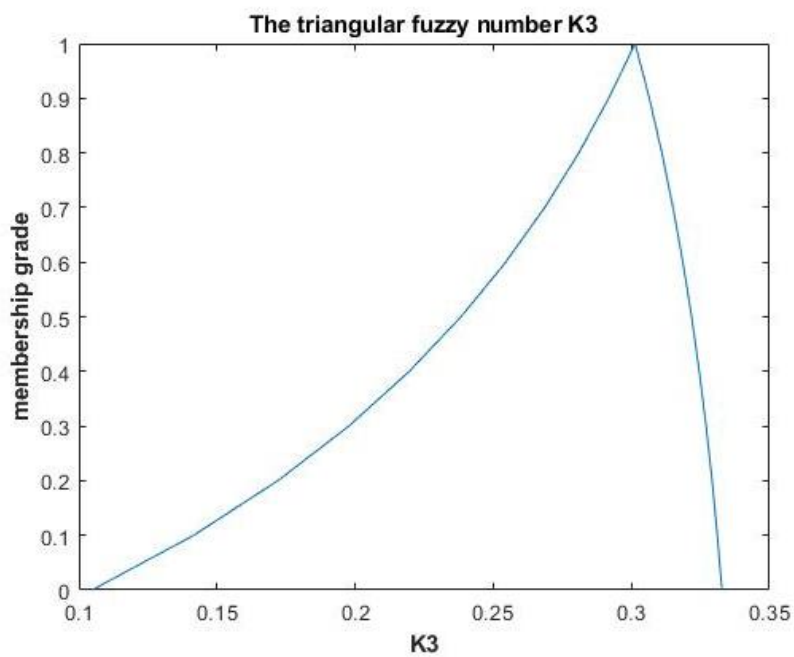

Figure 3-7. The fuzzy number K3 of the first case

Then, we get the fuzzy lengths of links as the defined in the definition 2-7 and (17) and they are as shown in the following figures:

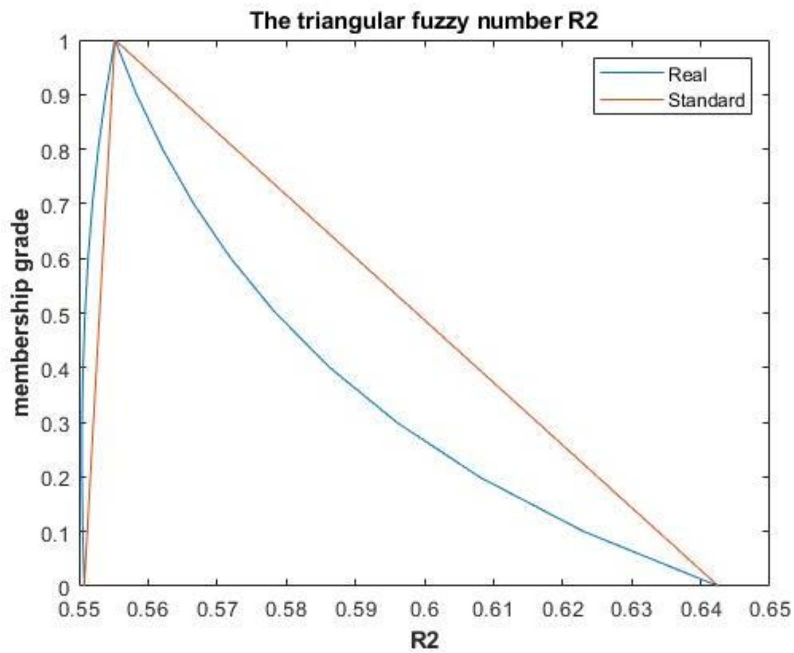

Figure 3-8. The fuzzy length of the input link "link 2" of the first case

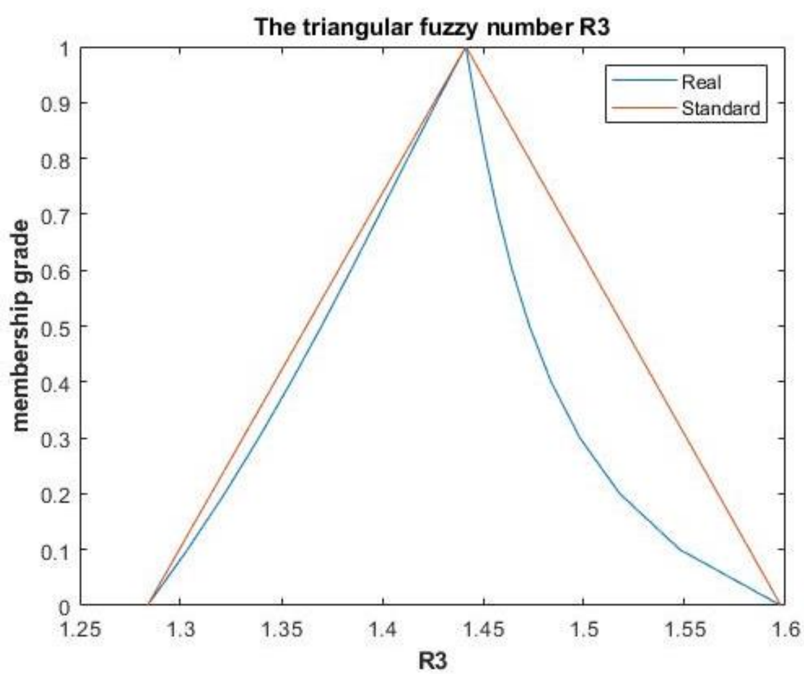

Figure 3-9. The fuzzy length of the coupler link "link 3" of the first case

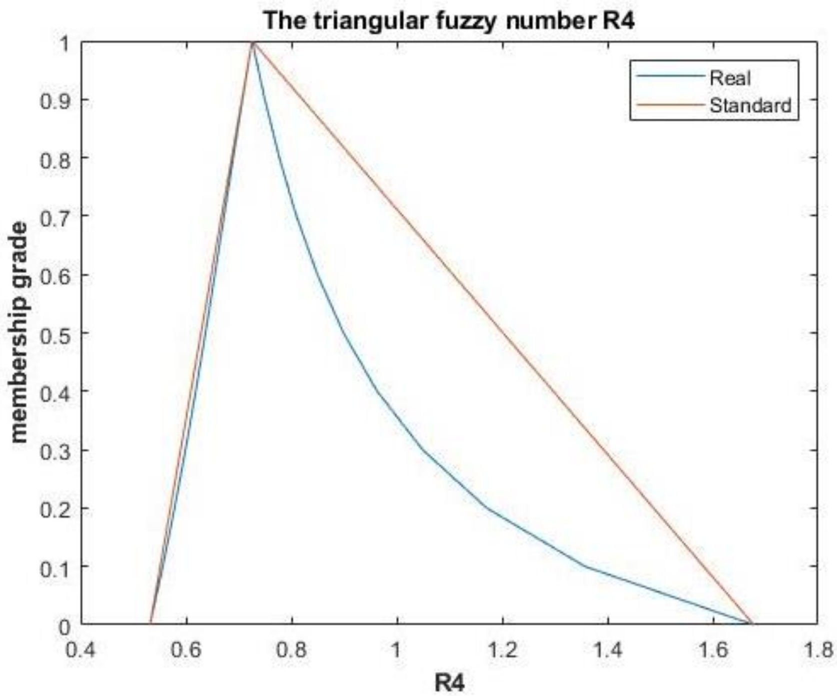

Figure 3-10. The fuzzy length of the output link “link 4” of the first case

Note that triangular membership functions of the fuzzy lengths are not linear the same as figure 2-2, in which the blue line represents the nonlinear results from the fuzzy inputs as they were defined in the definition 2-7 and the definition 2-8, while the red line represents the assumed linear result from these operations.

The resultant defuzzified lengths from the centroid defuzzification process will be used to simulate the four bar linkage by ADAMS View as it shown in Figure 3-11.

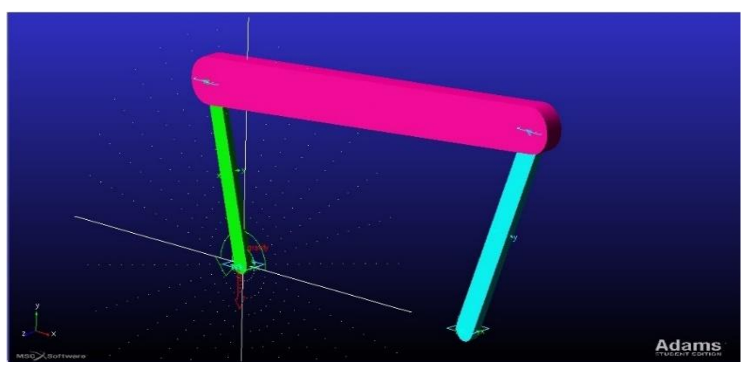

Figure 3-11. ADAMS model for the Four bar Mechanism within diffuzzified lengths using centroid method

All of steps mentioned above are repeated at every case, so the only resultant results of the simulation will be shown for other cases. These results are manifested at the relations between the crisp linkage approximation and the fuzzy linkage approximation passing through the selected points of $\emptyset's$ and $\psi's$ as well as the structural error over the exact function for both them. The results of the first case where the fuzziness area is $144\ cm^2$ is shown in the following Figures:

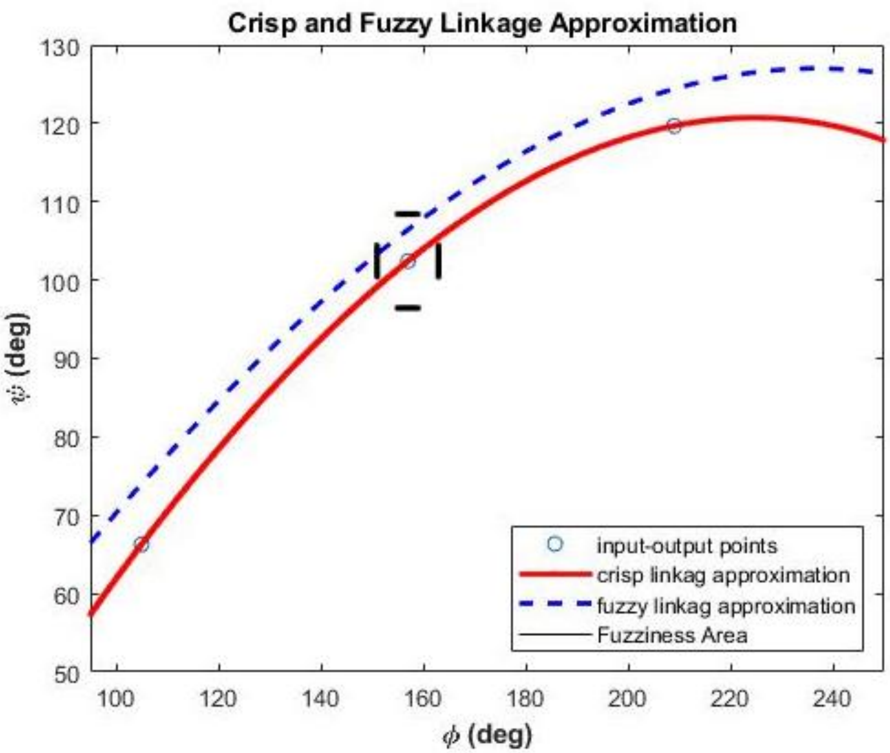

Figure 3-12. The actual crisp and fuzzy linkage function of the first case

Where the area surrounded by black lines are the fuzziness area resulting from the fuzzification process at x and y axises. This area is the base of rectangular pyramid representing the membership function of the continuous fuzzy function of the two variables, in which the fuzzy linkage approximation must pass through it to without determining the exact location as the crisp linkage approximation do.

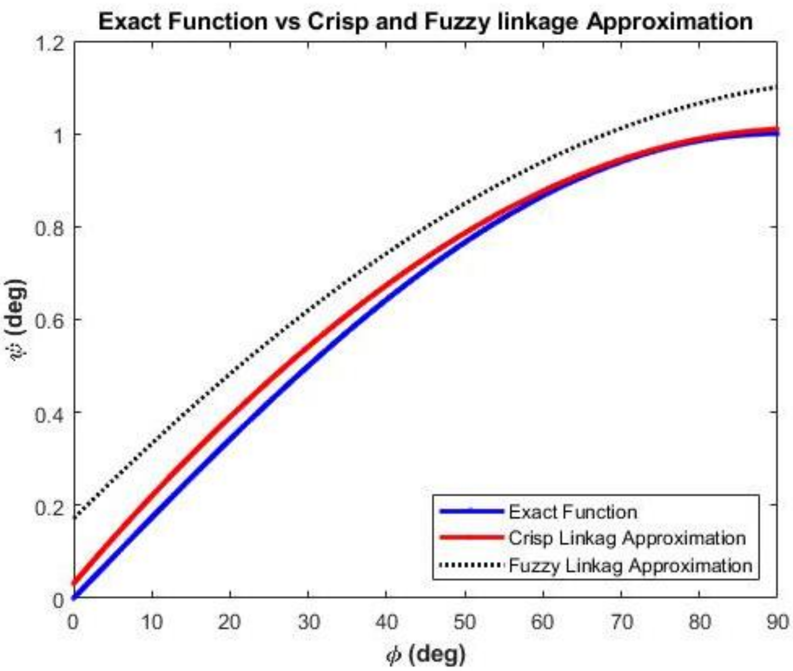

Figure 3-13. The exact function and actual linkage approximations of the of the first case

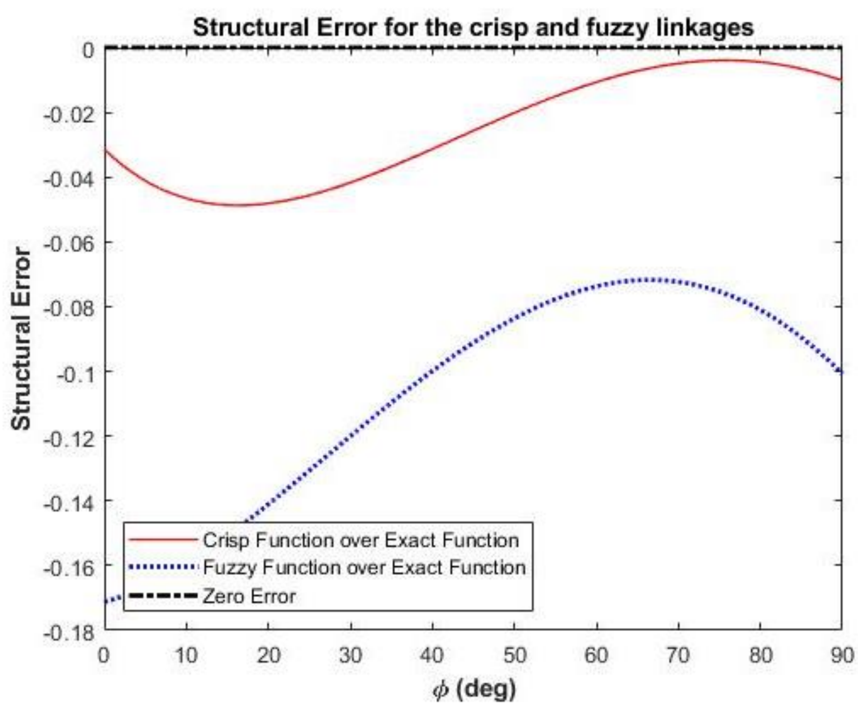

Figure 3-14. Structural Error of the 12 degrees fuzziness of the first case

Table 3-1. Results of the first case

The link	R1	R2	R3	R4
Crisp length	1.0	0.5551894	1.4414696	0.7257699
Fuzzified length	1.0	0.5769539	1.4249247	0.8872722
Structural Error of crisp approximating	7.378			
Structural Error of Fuzzy approximating	27.035			

The same case with exachnging the first row of the fuzzy matrix with the last one by changing the order of the input and output angles as

$$\emptyset = [209\ 157\ 105],\ \psi = [119.68\ 102.42\ 66.27]$$

The purpose is to invistegate the effect of the changing the order of the fuzzy linear system since the definition of the solution set $\hat{x}$ as mentioned in (25) deemed as subset of the united soution set $\sum(\hat{A}, \hat{b})$. The results of the modified first case will be as

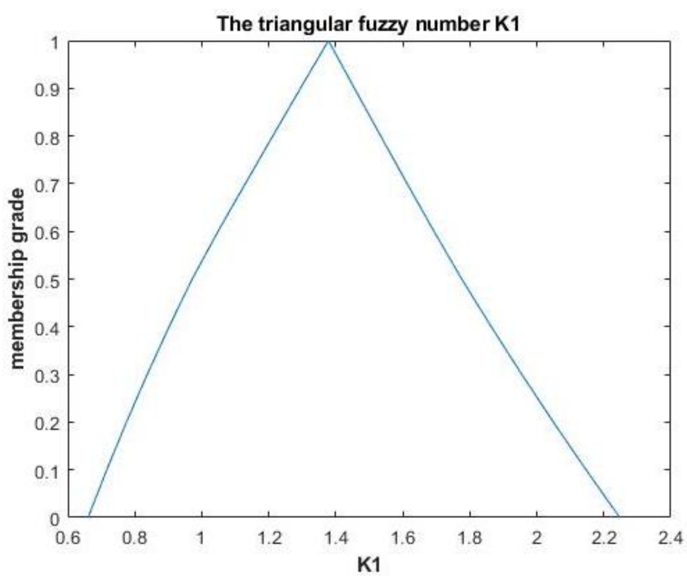

Figure 3-15. The fuzzy number K1 of of the modified first case

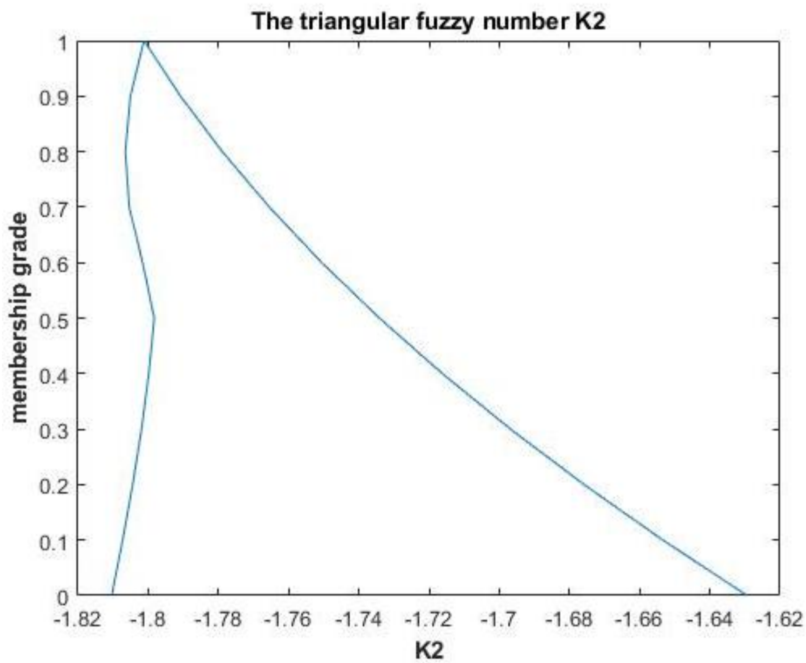

Figure 3-16. The fuzzy number K2 the modified first case

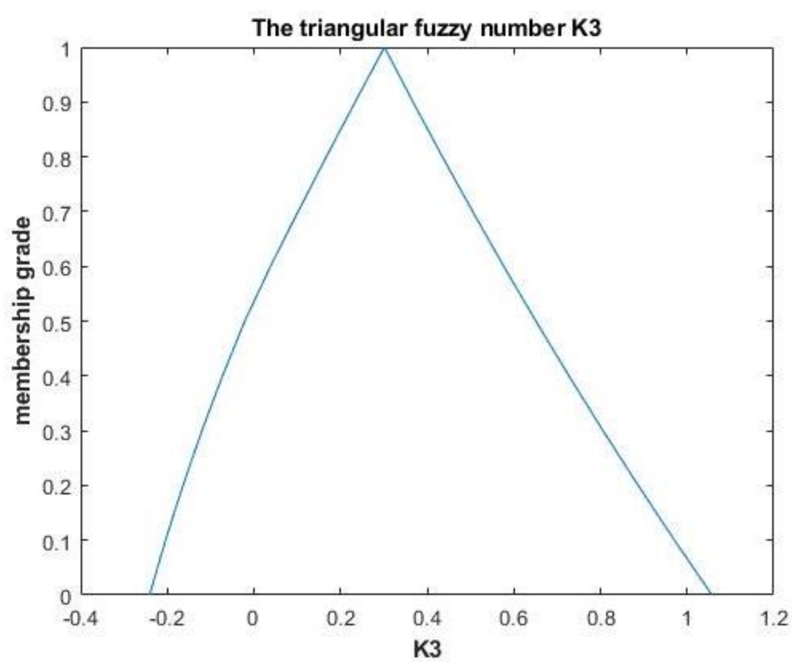

Figure 3-17. The fuzzy number K3 the modified first case

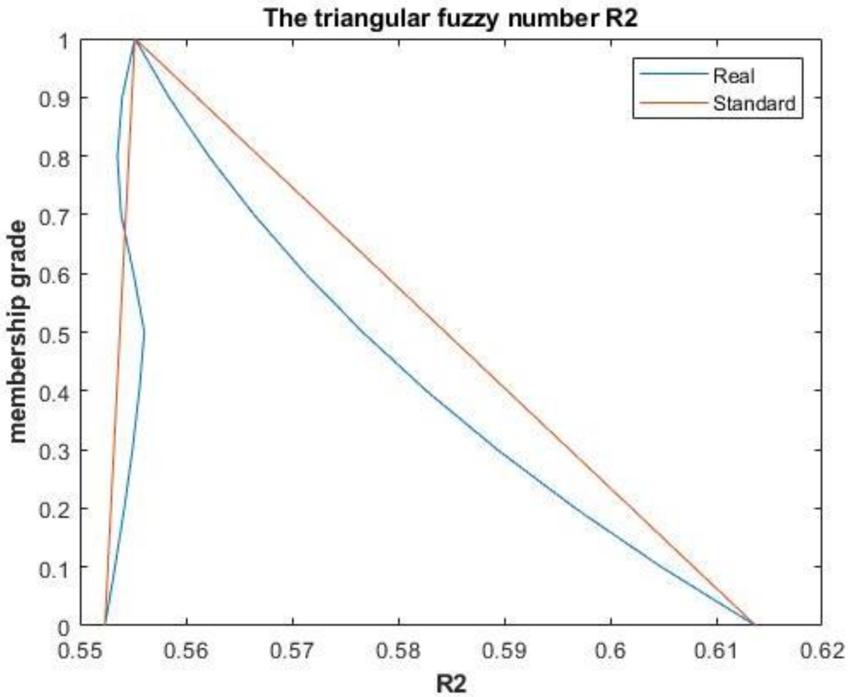

Figure 3-18. The fuzzy length of the input link "link 2" the modified first case

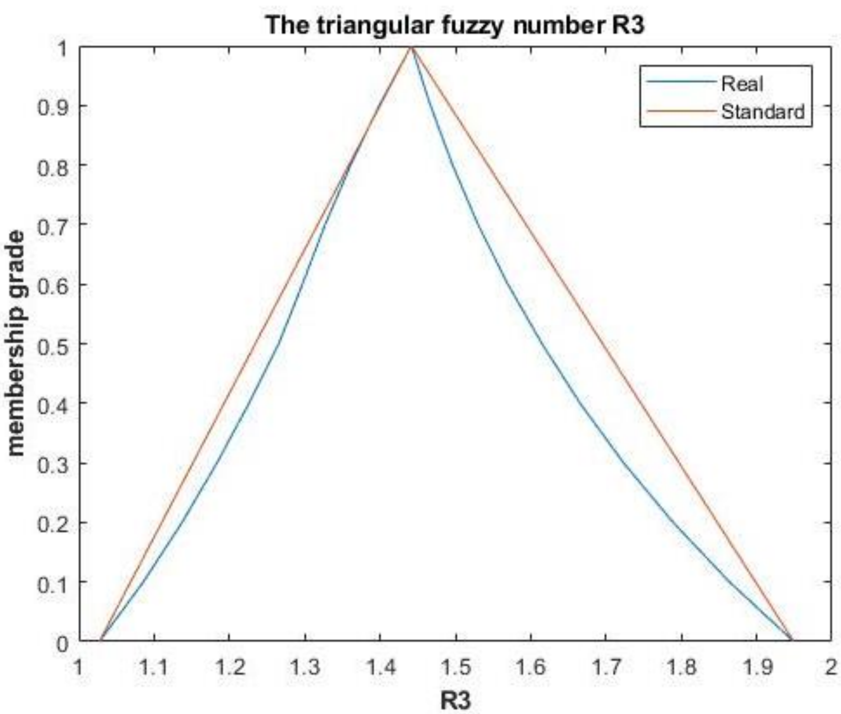

Figure 3-19. The fuzzy length of the coupler link "link 3" the modified first case

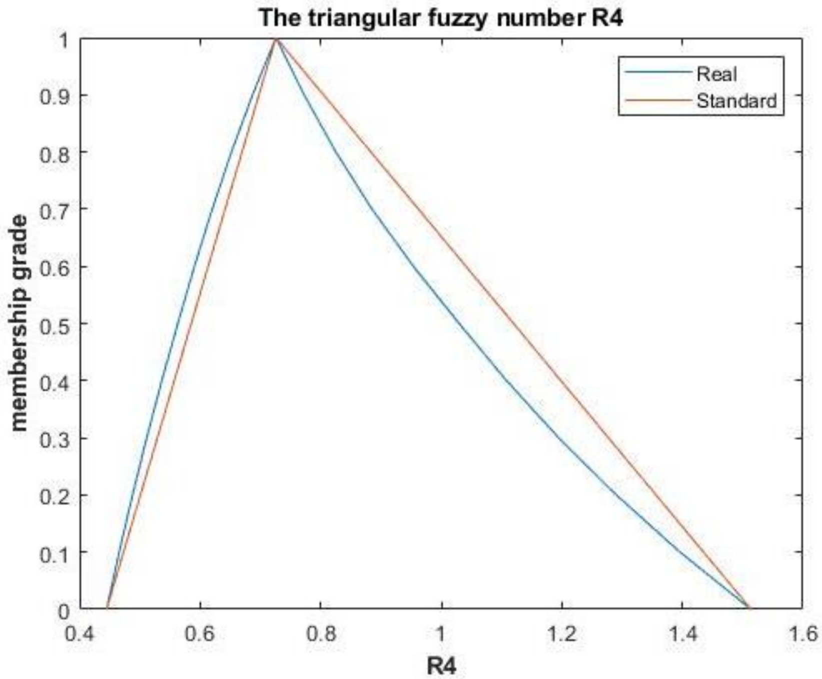

Figure 3-20. The fuzzy length of the output link “link 4” the modified first case

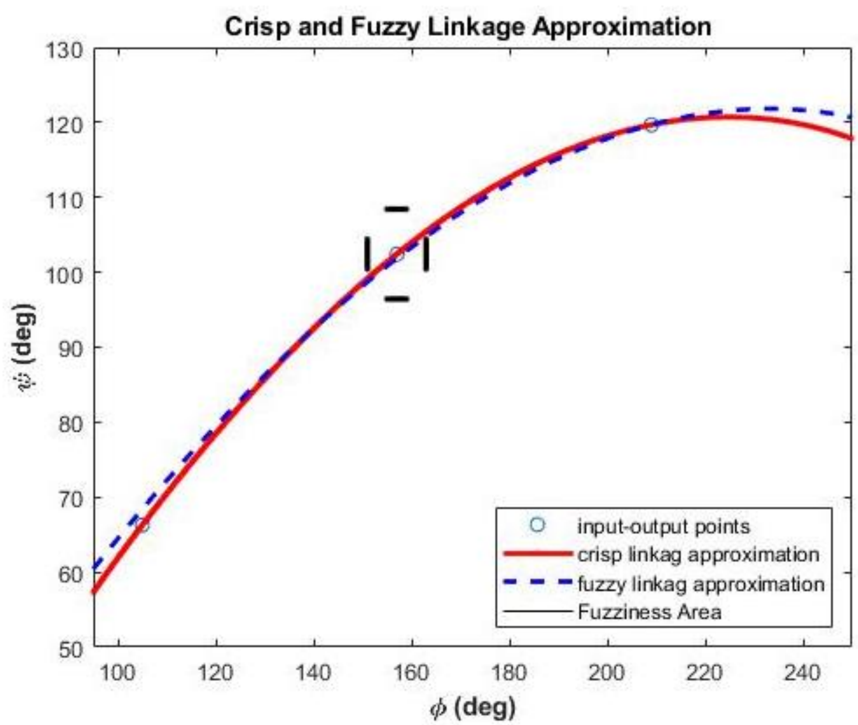

Figure 3-21. The actual crisp and fuzzy linkage function of the modified first case

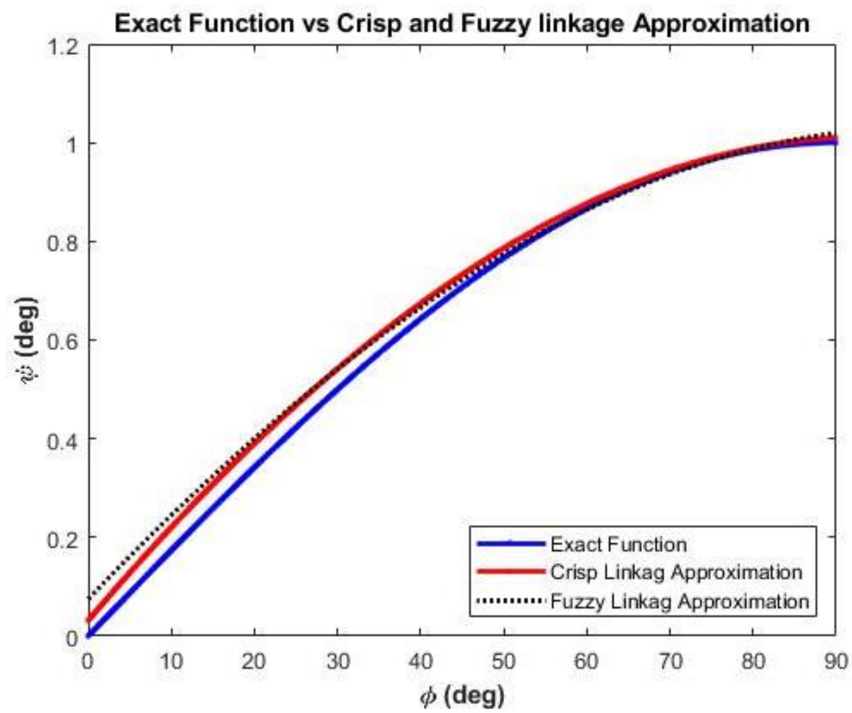

Figure 3-22. The exact function and actual linkage approximations of the of the modified first case

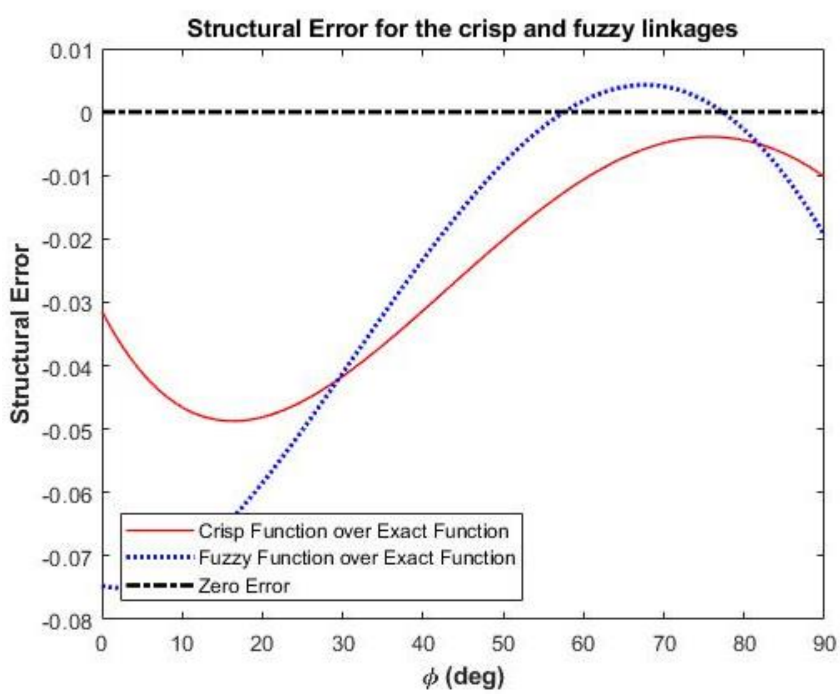

Figure 3-23. Structural Error of the of the modified first case

Table 3-2. Results of the modified first case

The link	R1	R2	R3	R4
Crisp length	1.0	0.5551894	1.4414696	0.7257699
Fuzzified length	1.0	0.5725990	1.4572117	0.8599609
Structural Error of the crisp approximating		7.3780		
Structural Error of the fuzzy approximating		9.437		

It is obvious that exchanging of rows' order minimizes the degree of nonlinearity for the fuzzy K's constants comparing to the first case, which leads to better behavior for the membership functions of the fuzzy lengths of the links. In addition, it is clear that membership function of K2 and the input link in Fig. 3-16 and Fig. 3-18, respectively, have appeared as multivalued functions and the crisp function have not intersect with the exact sin function for both situations of the first case although the structure error of the crisp function less than the fuzzy one.

The fuzziness of second case will be 6 degrees of the middle angles. The input and output angles will keep the descending order of the modified first case and the results will be in the following figures.

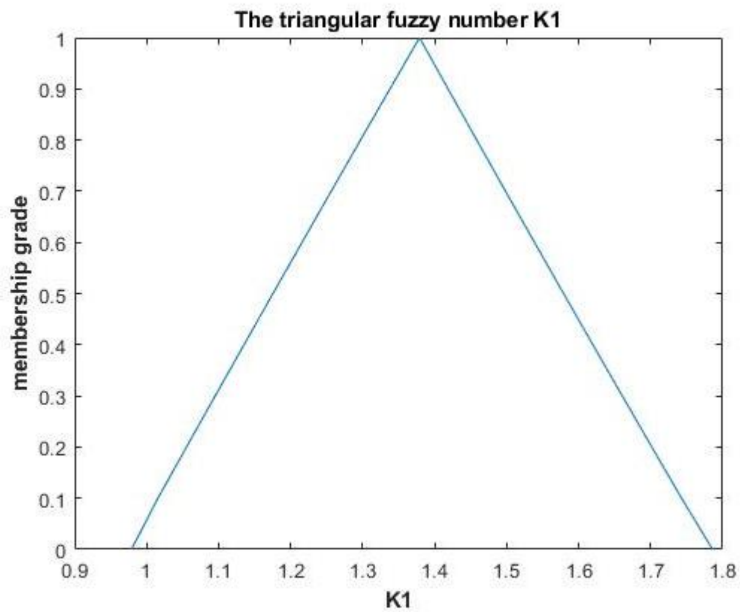

Figure 3-24. The fuzzy number K1 of the second case

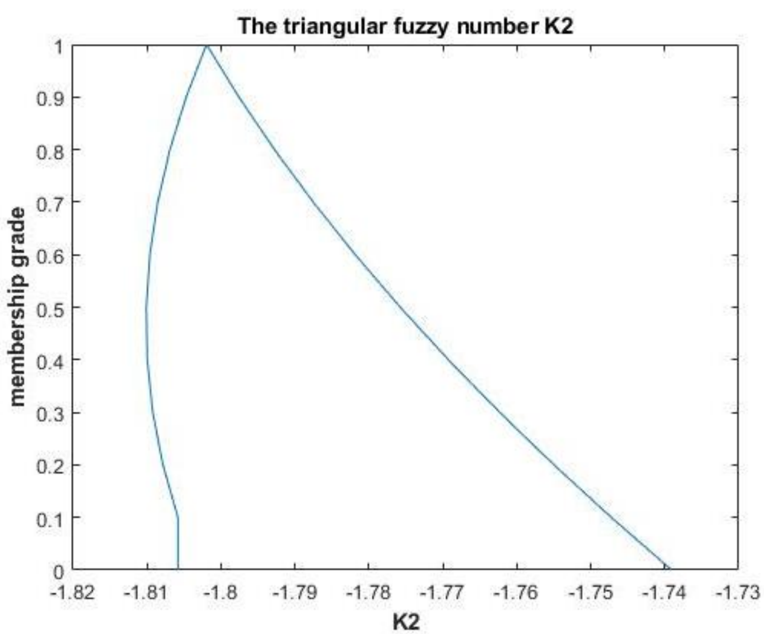

Figure 3-25. The fuzzy number K2 of the second case

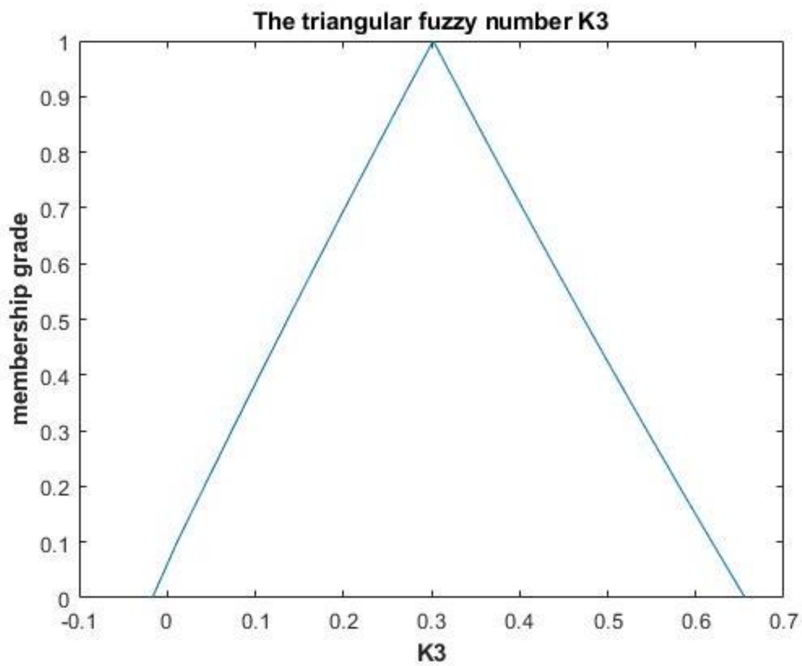

Figure 3-26. The fuzzy number K3 of the second case

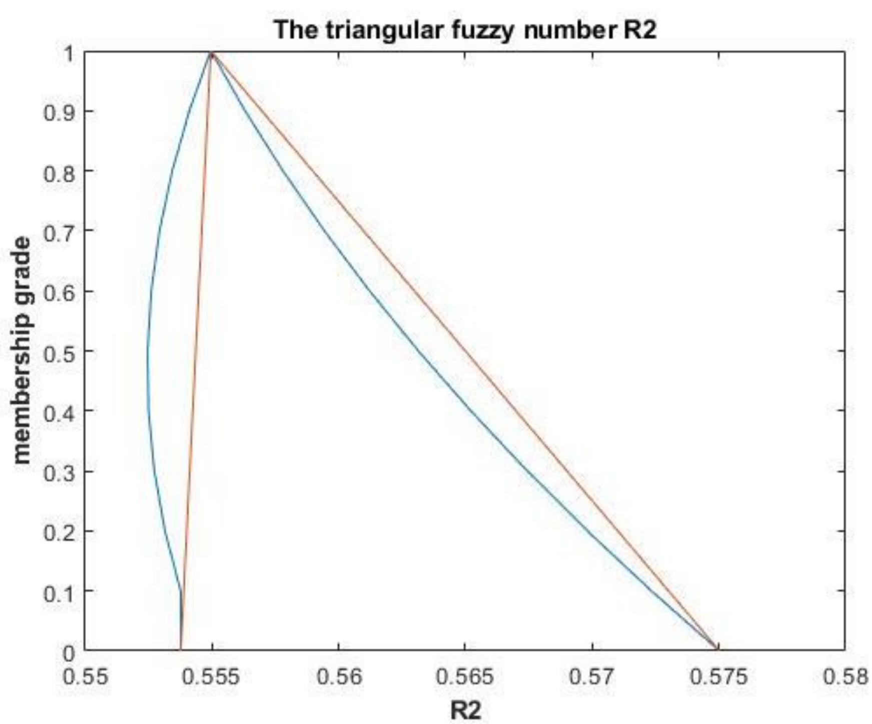

Figure 3-27. The fuzzy length of the input link "link 2" of the second case

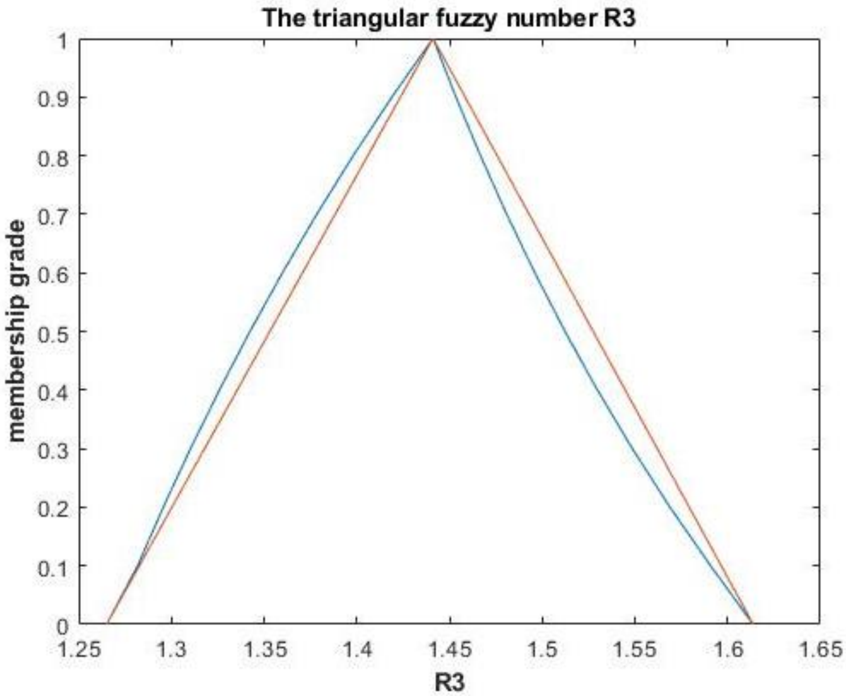

Figure 3-28. The fuzzy length of the coupler link "link 3" of the second case

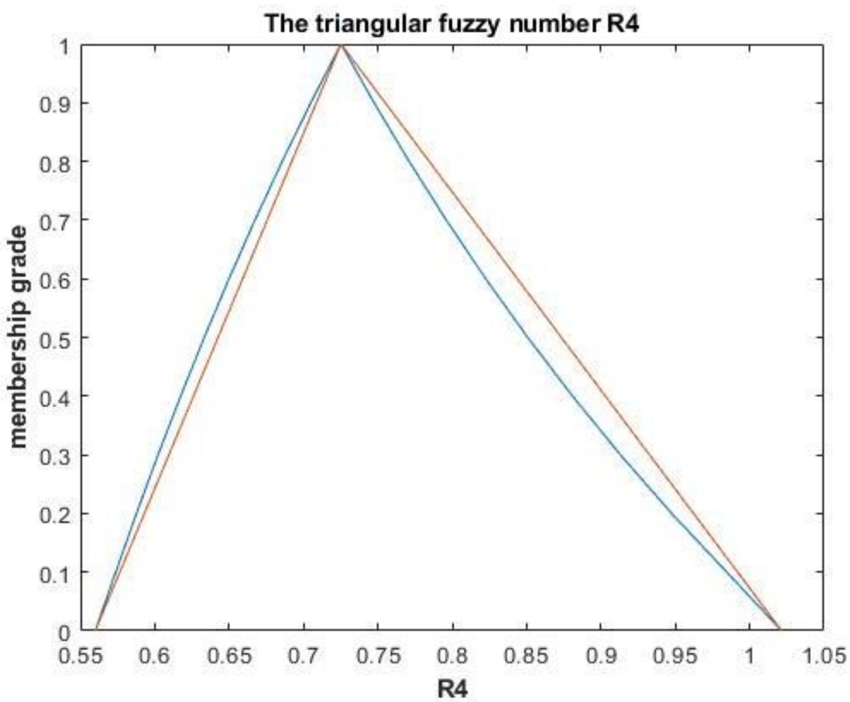

Figure 3-29. The fuzzy length of the output link "link 4" of the second case

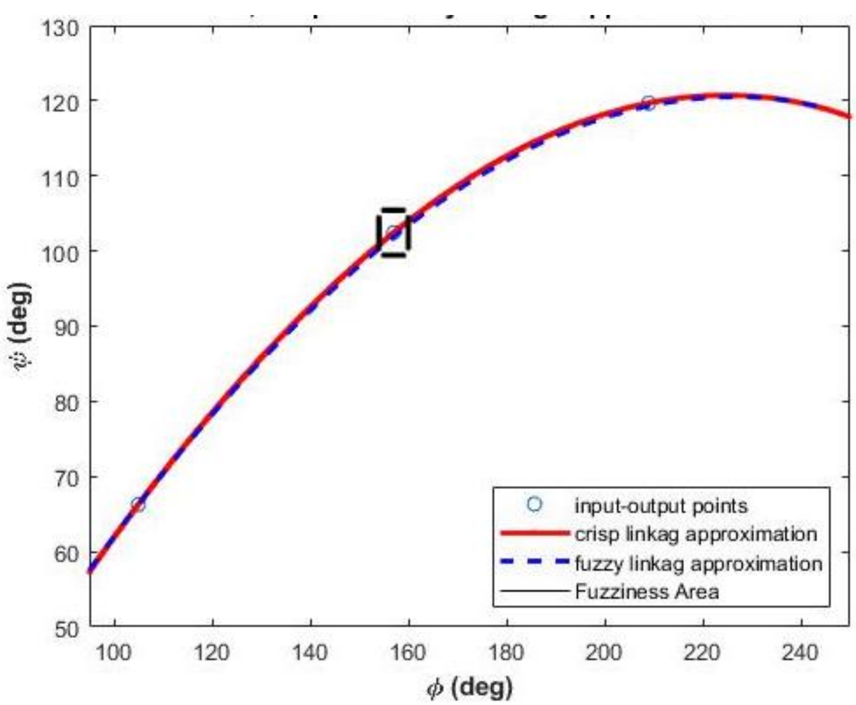

Figure 3-30. The actual crisp and fuzzy linkage function of the second case

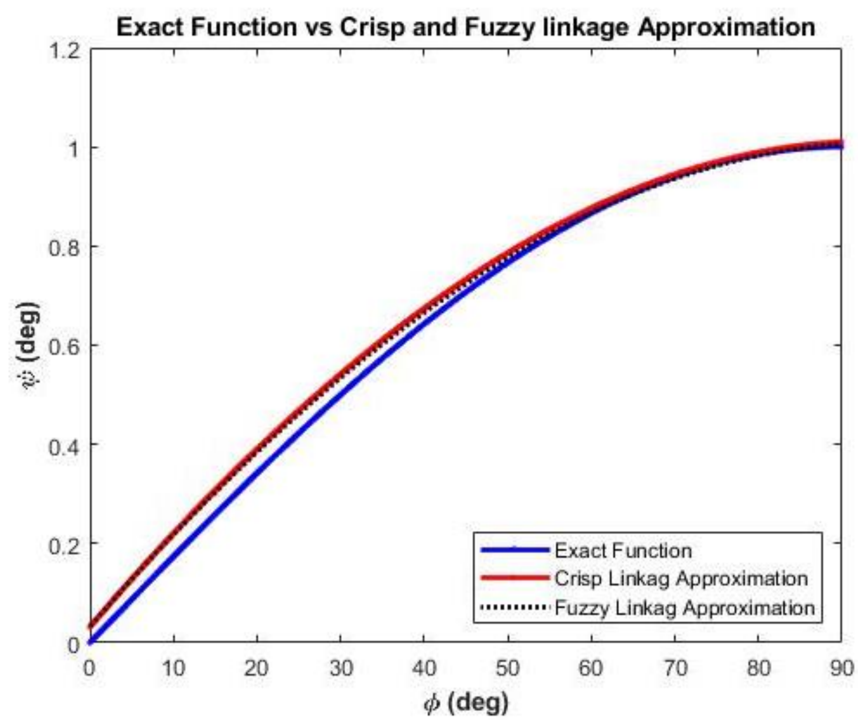

Figure 3-31. The exact function and actual linkage approximations of the of the second case

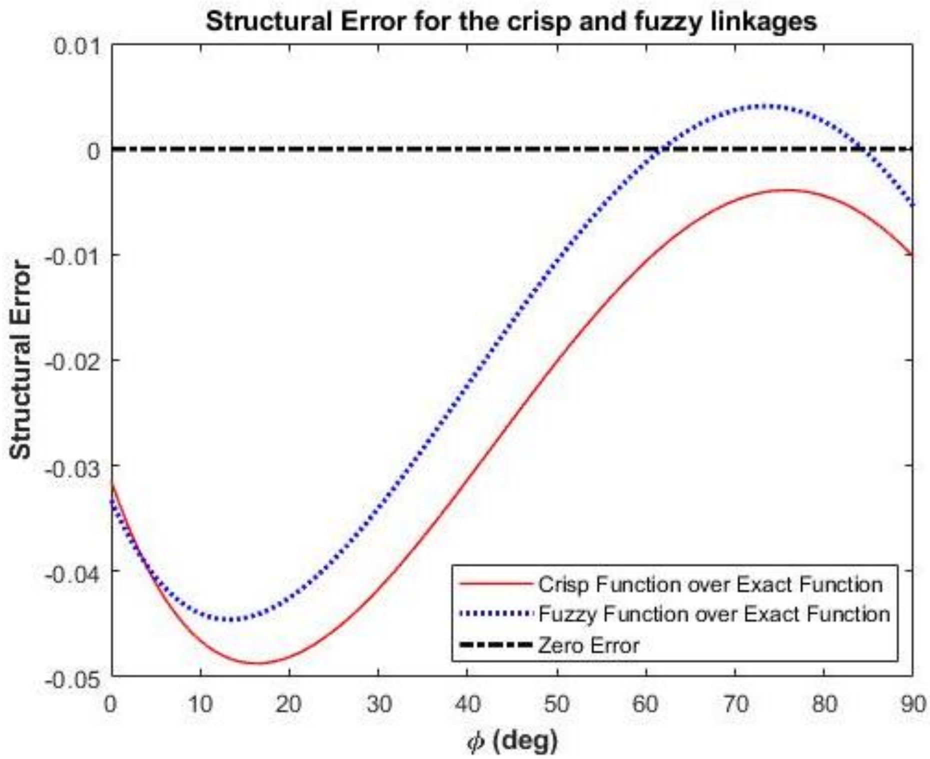

Figure 3-32. Structural Error of the of the second case

Table 3-3. Results of the second case

The link	R1	R2	R3	R4
Crisp length	1.0	0.5551894	1.4414696	0.7257699
Fuzzified length	1.0	0.5600600	1.4319182	0.7590060
Structural Error of the crisp approximating		7.3780		
Structural Error of the fuzzy approximating		6.329		

It is clear that fuzzy approximating has less amount of structural error comparing to the crisp approximating as well as intersecting with the exact function at two points, the first point at $\emptyset = 61.79°$ and the second one at $\emptyset = 84.28°$ which is close to the precision point x_3 .

In order to change the crisp approximating to get more accurate crisp approximating, the new chebyshef precesion points will be recalculated accouring to the formula

$$x_k = \frac{x_f - x_i}{2} - \frac{x_f - x_i}{2} * \cos\big(30 * (2 * k - 1)\big), \qquad k = 1,2,3 \tag{35}$$

The new chebyshef precesion points are as

$$x_1 = 6.028\,, \qquad x_2 = 45\,, \qquad x_3 = 83.971 \tag{36}$$

So that, the new precision points of the input and output angles keeping the descending order are as

$$\emptyset = [211.962\ 160\ 108.038]$$
$$\psi = [119.668\ 102.426\ 66.302]$$

The results for the first case with 12 degrees fuzziness for the new precision points will be as

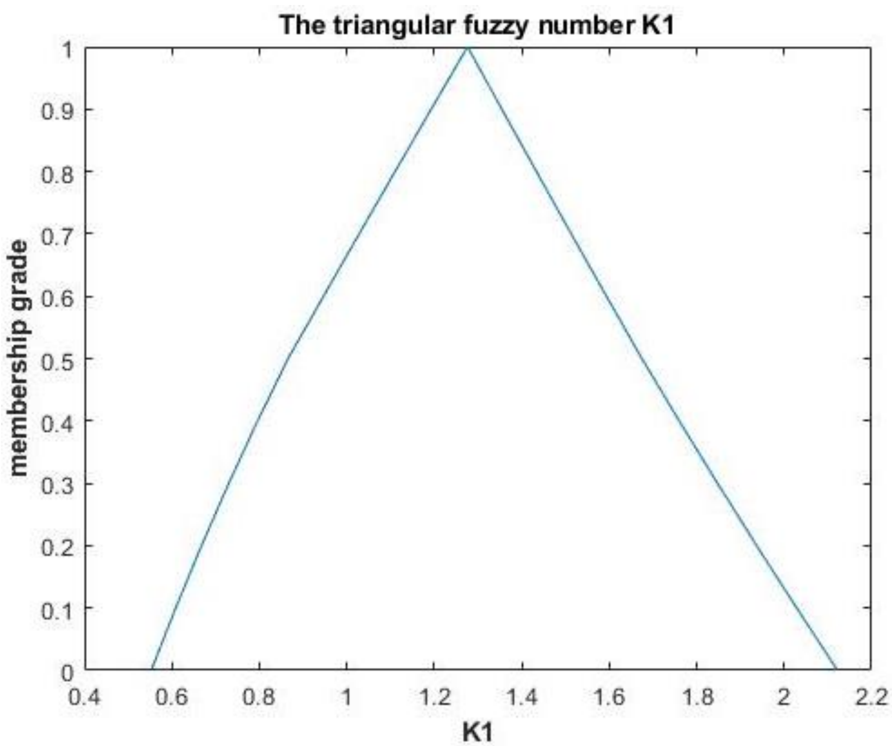

Figure 3-33. The fuzzy number K1 of the first case for the new precision points

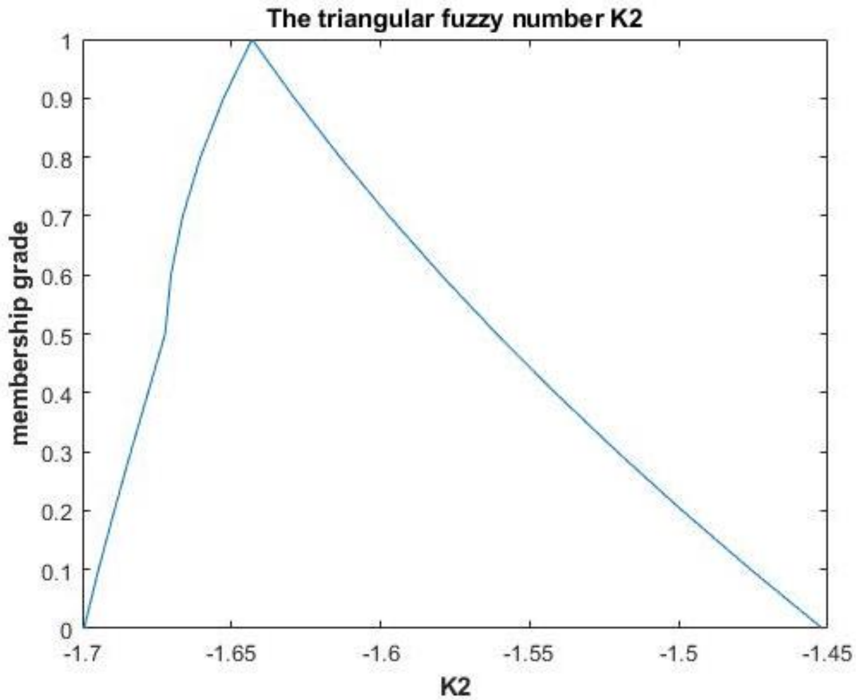

Figure 3-34. The fuzzy number K2 of the first case for the new precision points

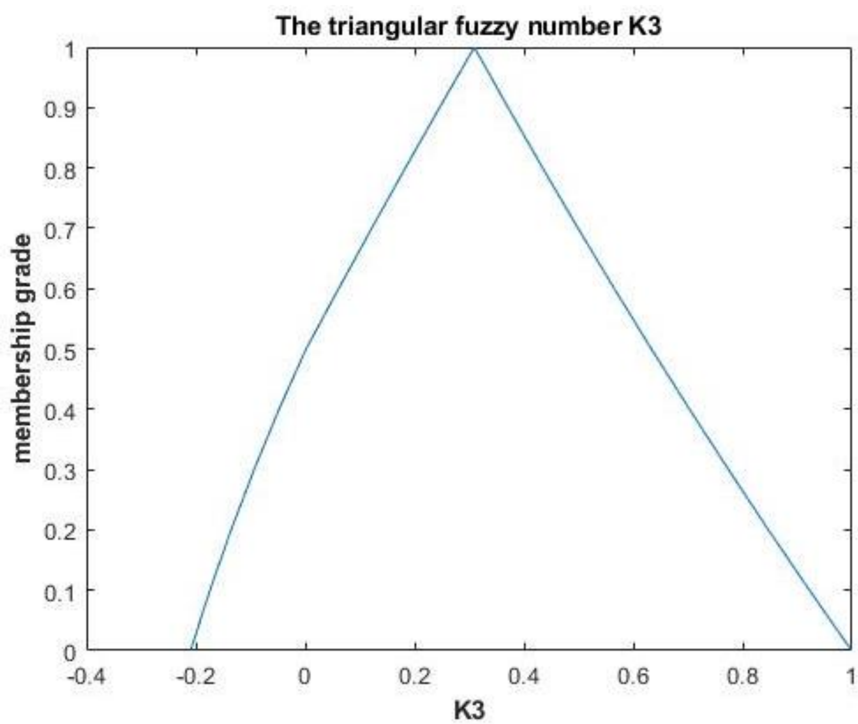

Figure 3-35. The fuzzy number K2 of the first case for the new precision points

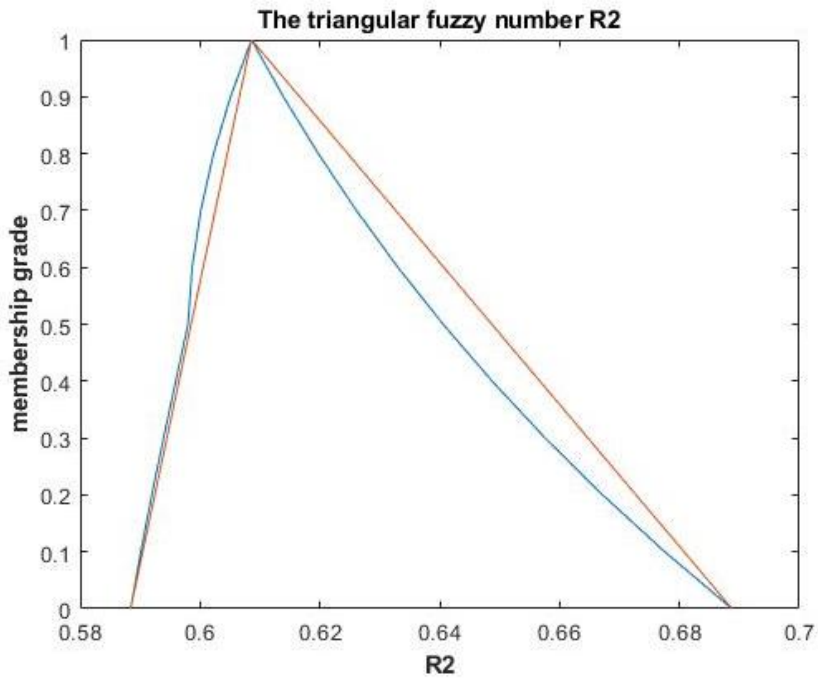

Figure 3-36. The fuzzy length of the input link "link 2" of the first case for the new precision points

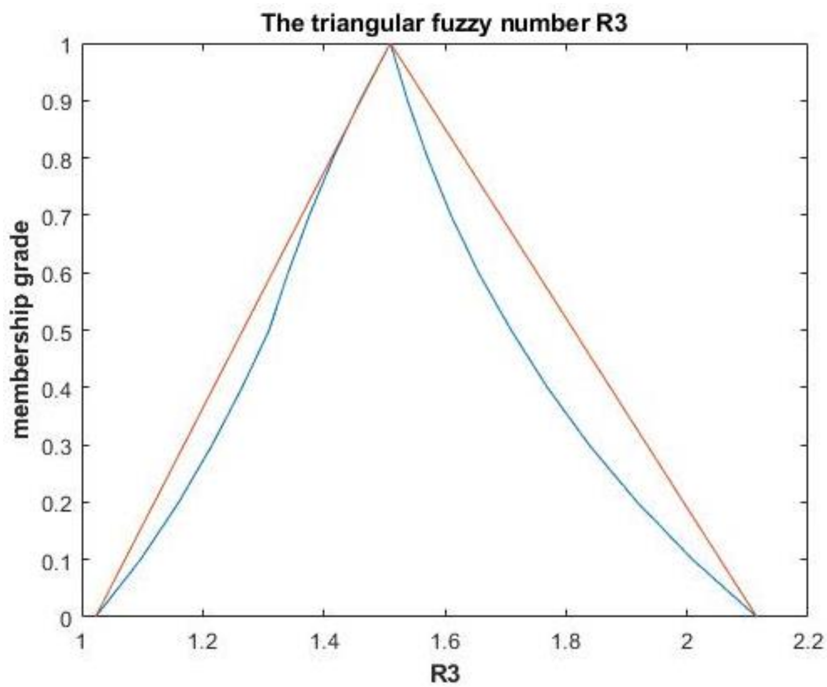

Figure 3-37. The fuzzy length of the coupler link "link 3" of the first case for the new precision points

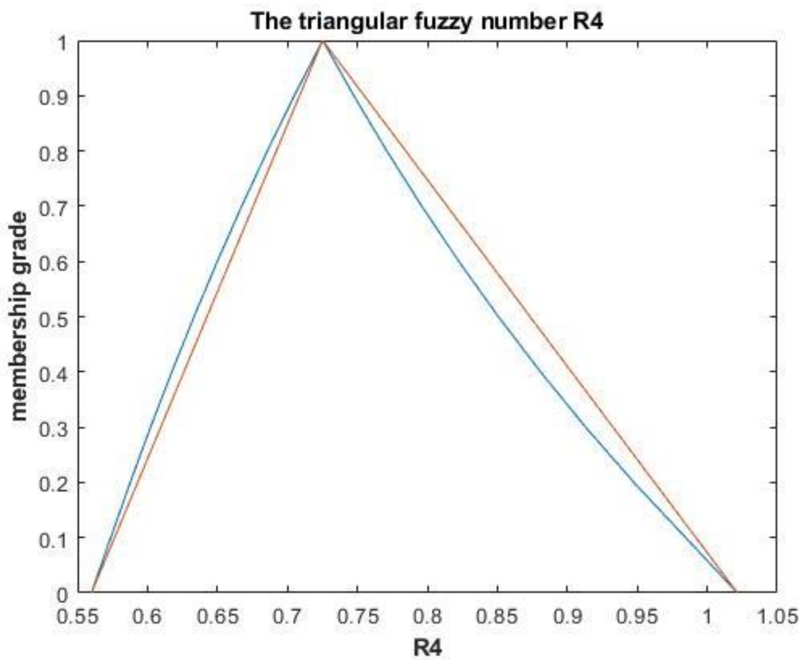

Figure 3-38. The fuzzy length of the output link "link 4" of the first case for the new precision points

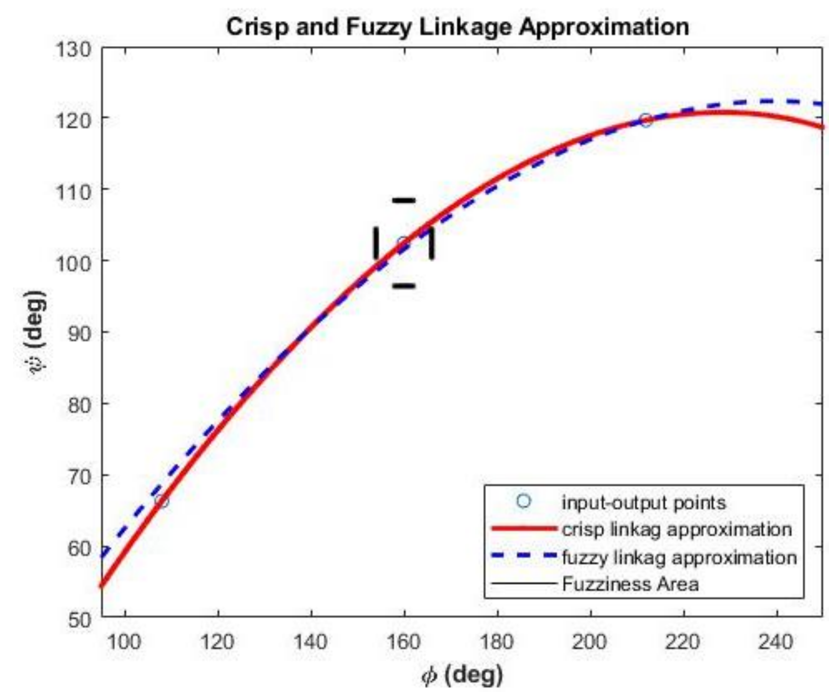

Figure 3-39. The actual crisp and fuzzy linkage function of the first case for the new precision points

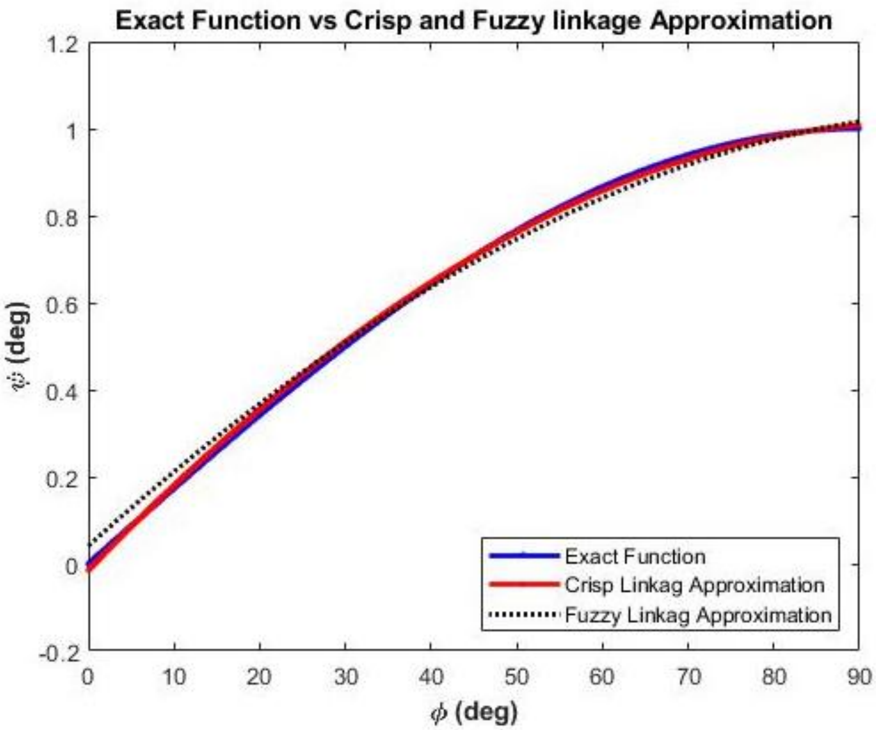

Figure 3-40. The exact function and actual linkage approximations of the of the first case for the new precision points

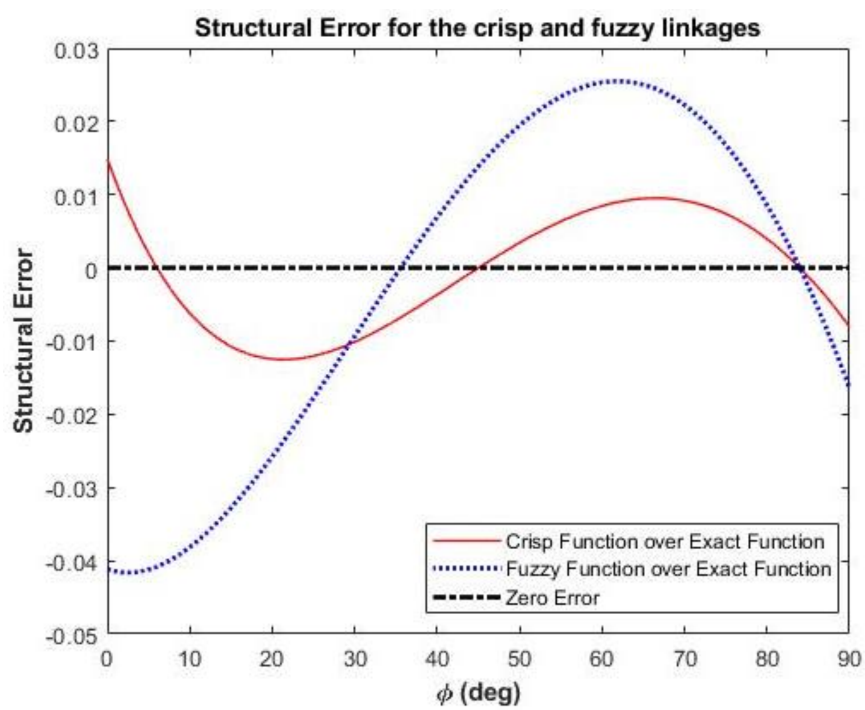

Figure 3-41. Structural Error of the of the first case for the new precision points

Table 3-4. Results of the first case for the new precision points

The link	R1	R2	R3	R4
Crisp length	1.0	0.6085875	1.5098086	0.7836283
Fuzzified length	1.0	0.6258844	1.5309920	0.9689904
Structural Error of the crisp approximating		1.875		
Structural Error of the fuzzy approximating		5.553		

It is obvious that amount of structural error of the crisp approximating is decreasing from the previous precision points with 7.378 by 5.503 to get 1.873 and intersecting with the sin function at all new precision points. Consequently, the fuzzy approximating that is got by the centroid defuzzification process has less amount of the structural error with 5.553 comparing to 9.437 for the first case with old precision points. In addition, the fuzzy approximating intersecting with the sin function at two points: the first one is at $\emptyset = 7.378°$ and the second one is at third precision point $x_3 = 83.971°$. additionally, the multivalued membership function in Fig 3-16 and Fig 3-18 have disappeared in the fuzziness of the first case for the new precision points which a positive indication to get more accurate centroid defuzzified length of the input link.

The results of the second case with 6 degrees fuzziness for the new precision points keeping the descending order will be as

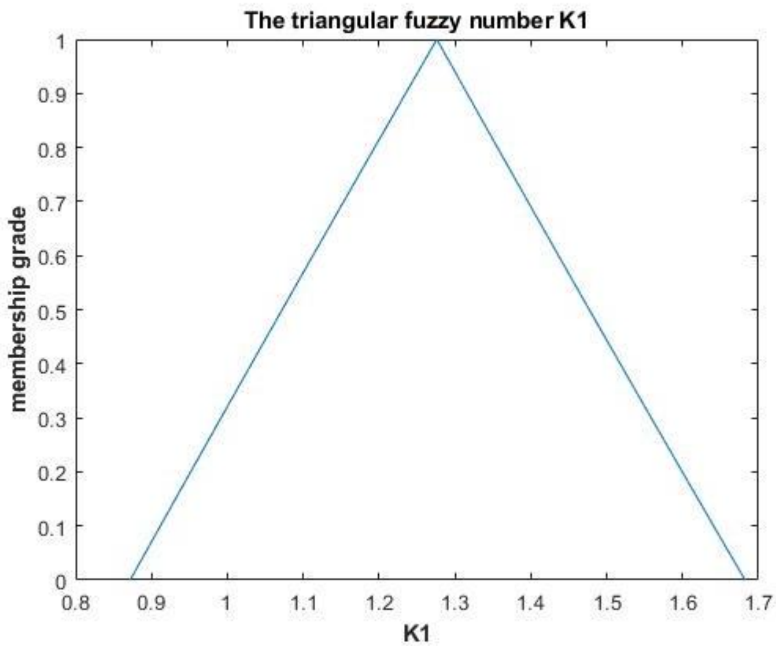

Figure 3-42. The fuzzy number K1 of the second case for the new precision points

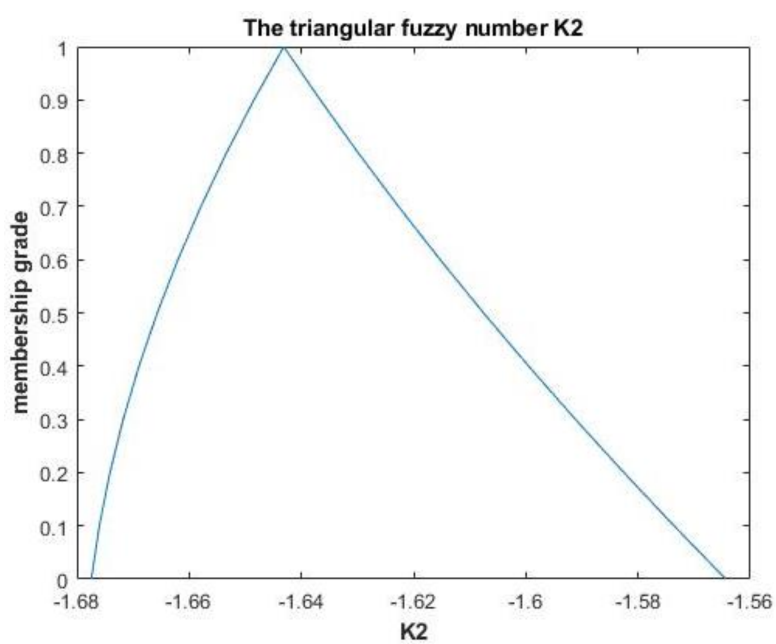

Figure 3-43. The fuzzy number K2 of the first case for the new precision points

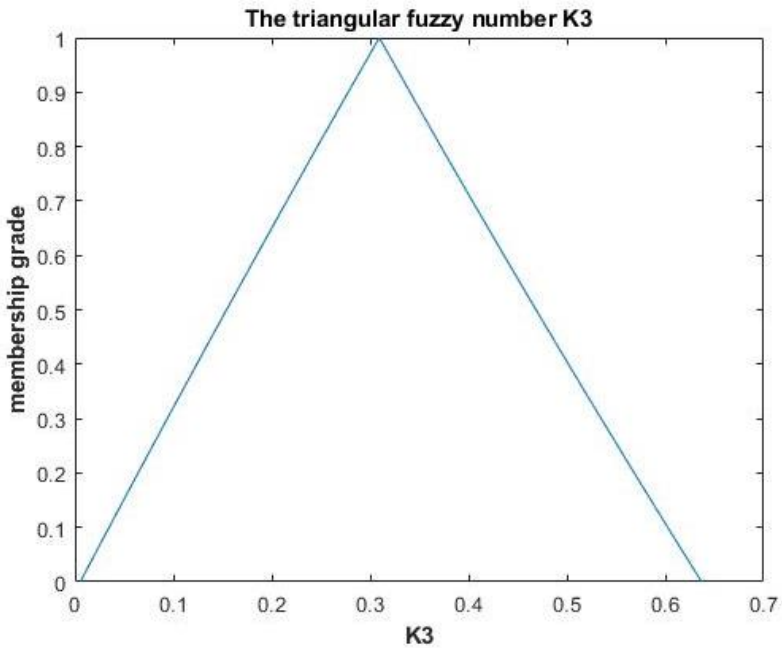

Figure 3-44. The fuzzy number K3 of the second case for the new precision points

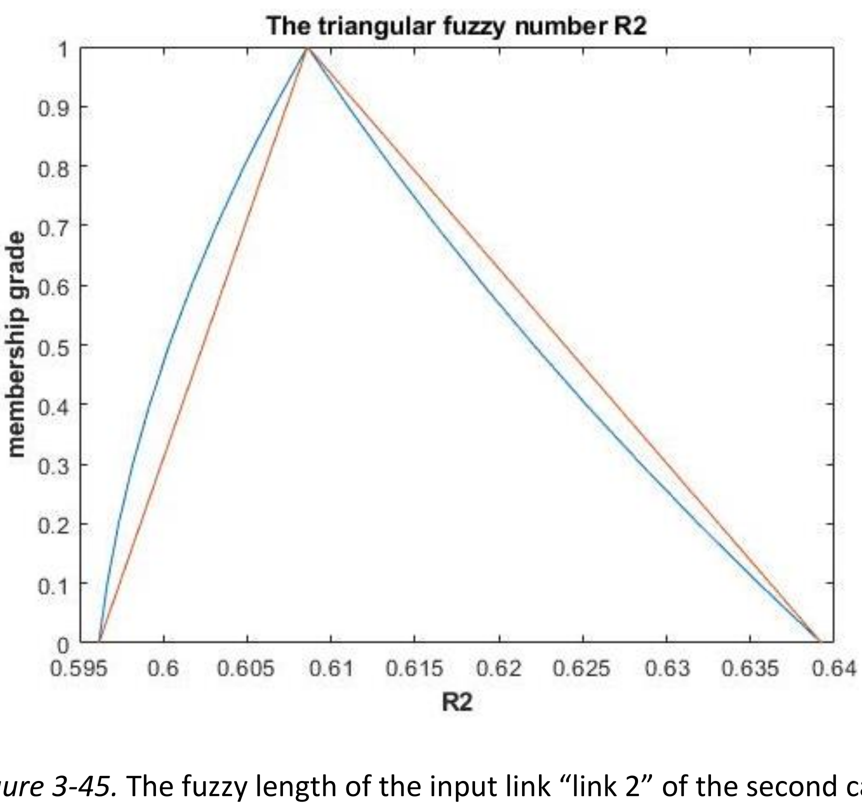

Figure 3-45. The fuzzy length of the input link “link 2” of the second case for the new precision points

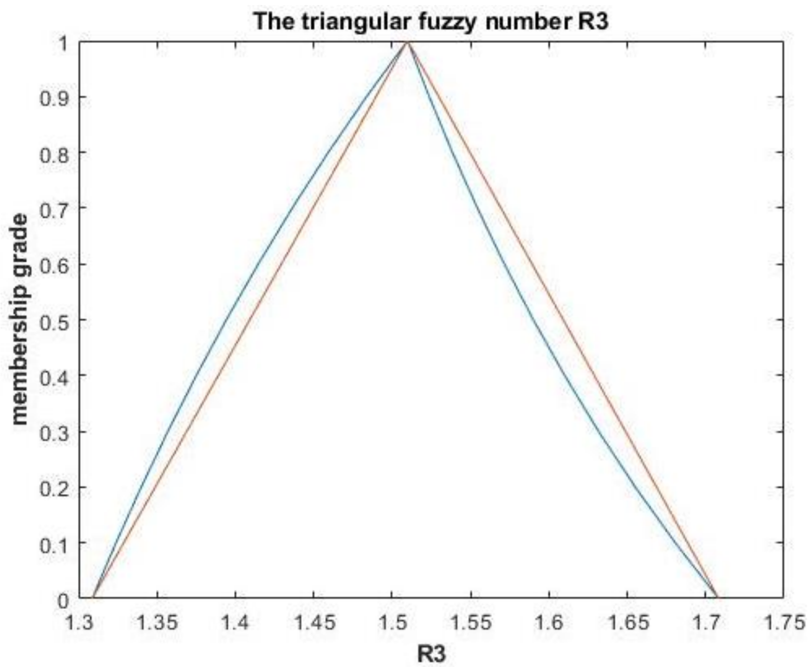

Figure 3-46. The fuzzy length of the coupler link "link 3" of the second case for the new precision points

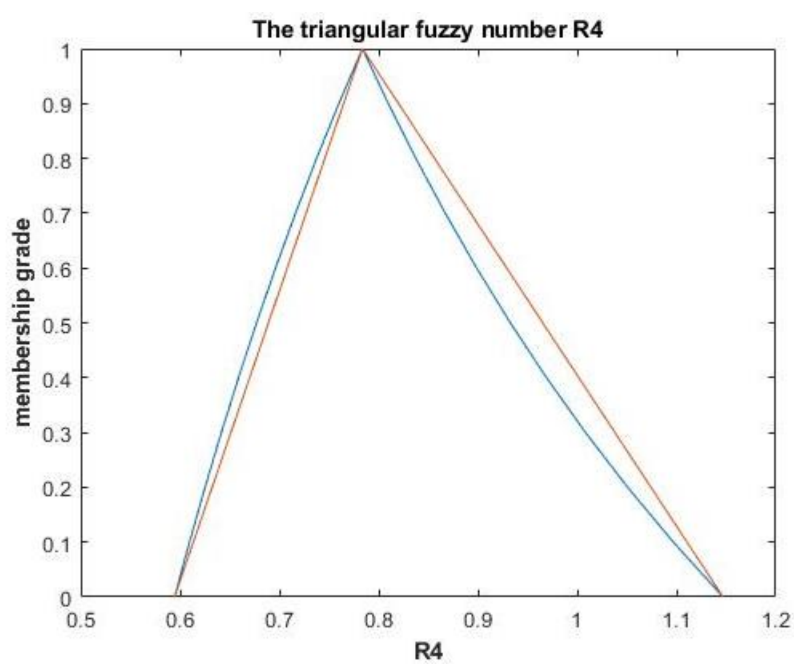

Figure 3-47. The fuzzy length of the output link "link 4" of the second case for the new precision points

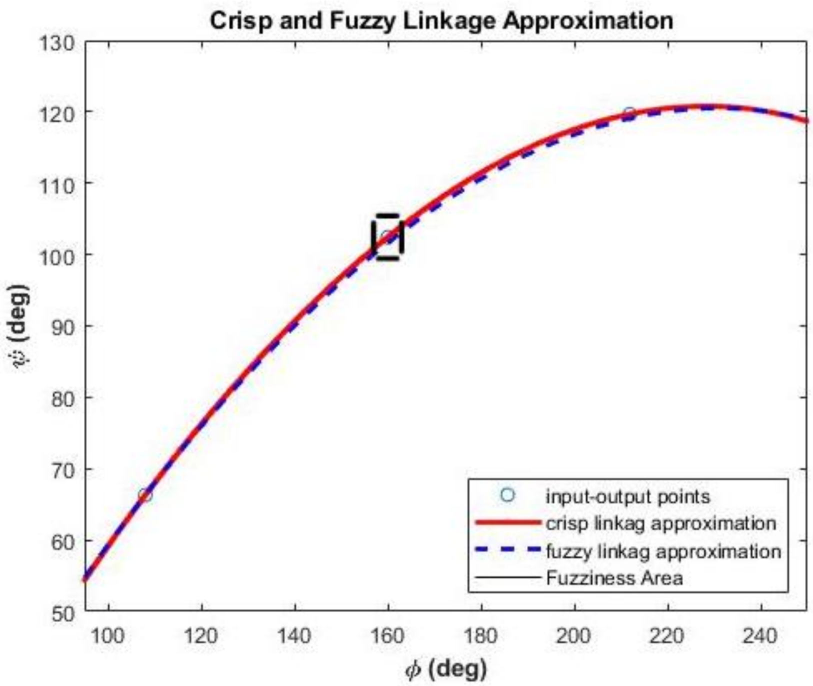

Figure 3-48. The actual crisp and fuzzy linkage function of the second case for the new precision points

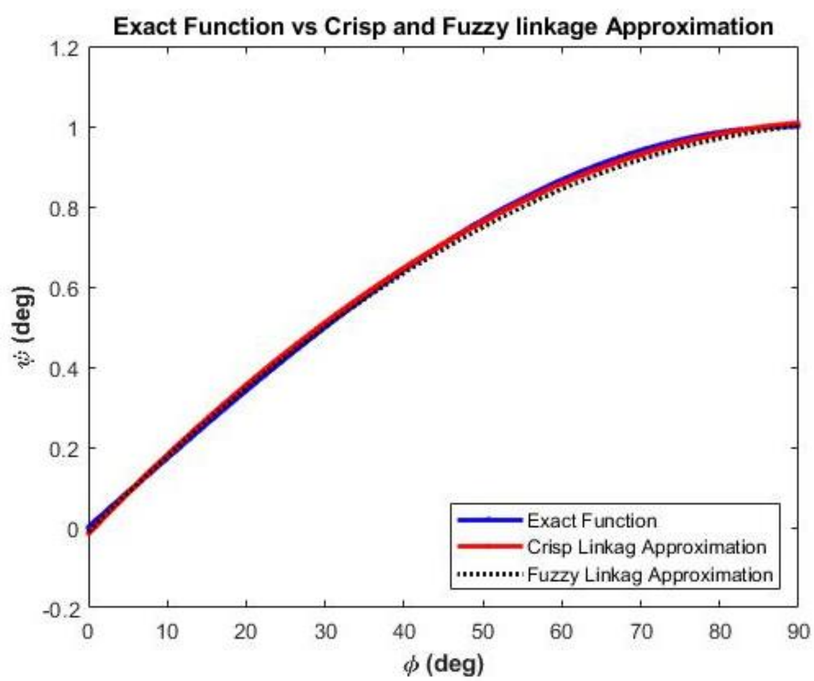

Figure 3-49. The exact function and actual linkage approximations of the of the second case for the new precision points

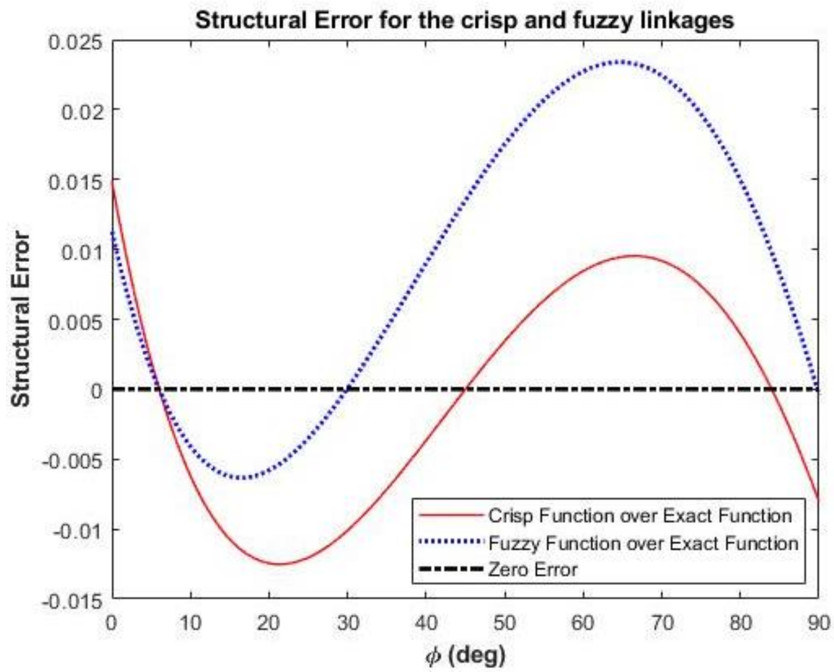

Figure 3-50. Structural Error of the of the second case for the new precision points

Table 3-5. Results of the second case for the new precision points

The link	R1	R2	R3	R4
Crisp length	1.0	0.6085875	1.5098086	0.7836283
Fuzzified length	1.0	0.6133654	1.4976248	0.8273258
Structural Error of the crisp approximating		1.875		
Structural Error of the fuzzy approximating		3.318		

The fuzziness of the second case for the new precision points has less amount of structural error by 2.2 points comparing to the same fuzziness in the second case for the old precision points and 2.4 points comparing to the decreasing with 6 degrees in the fuzziness from the first case for the new precision points. The precision point will be five to both crisp and fuzzy approximating for the first time. The fuzzy precision points will be coincided with the crisp precision points at the first point; however, the others are $30.17°$ and $89.27°$ versus $45°$ and $83.971°$ for the crisp approximating, respectively.

Lastly, the results of the third case with 4 degrees fuzziness for the new precision points keeping the descending order will be as

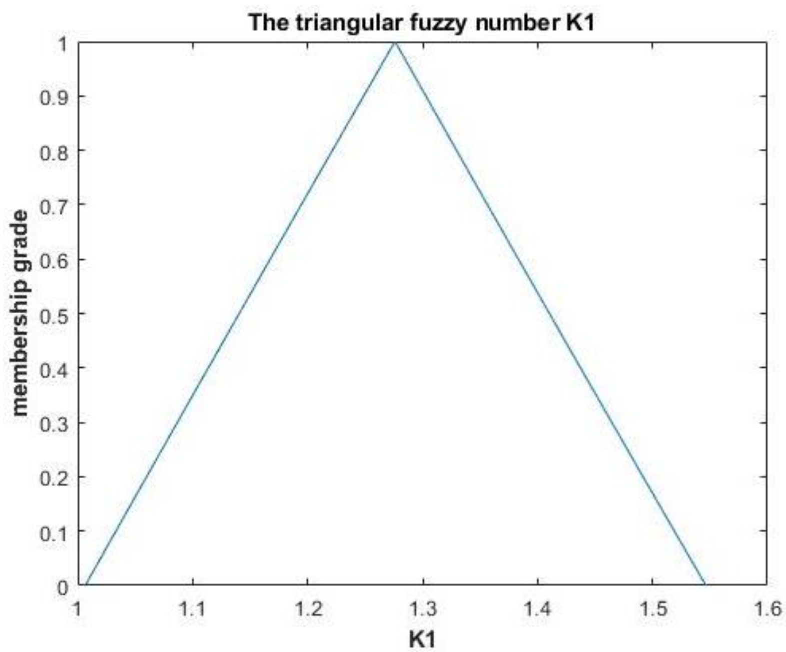

Figure 3-51. The fuzzy number K1 of the third case for the new precision points

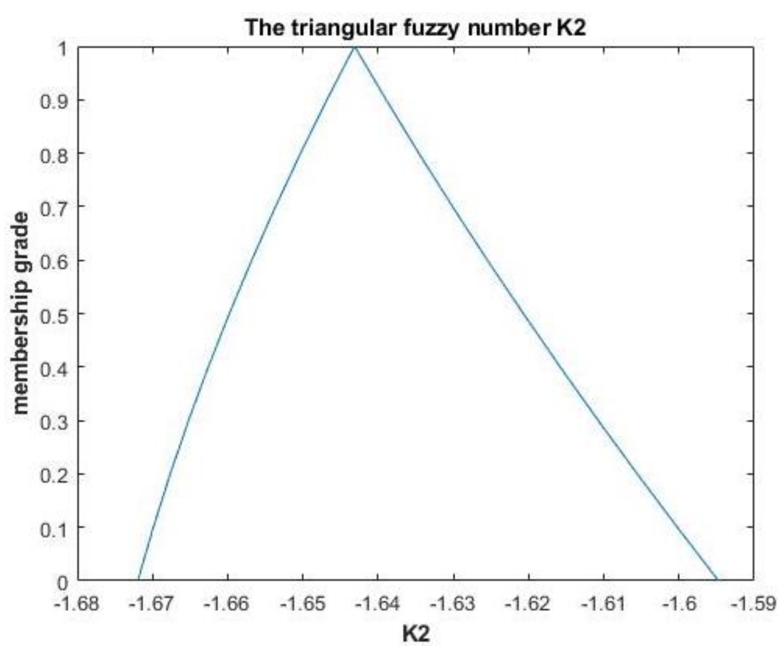

Figure 3-52. The fuzzy number K2 of the third case for the new precision points

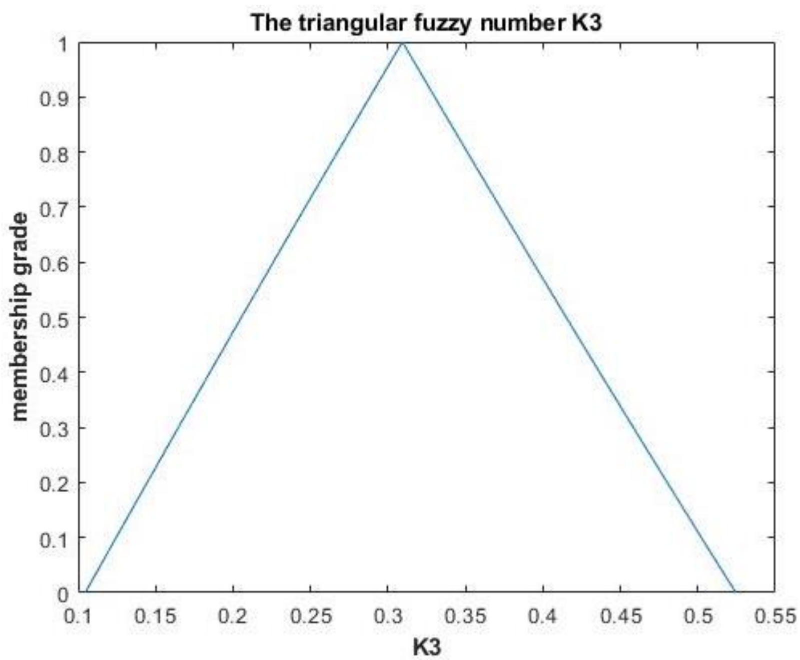

Figure 3-53. The fuzzy number K3 of the third case for the new precision points

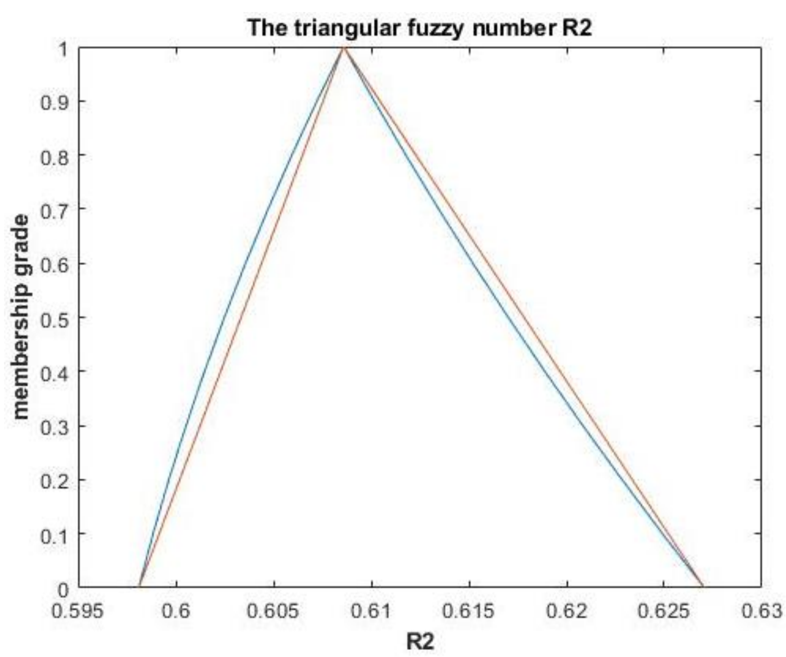

Figure 3-54. The fuzzy length of the input link "link 2" of the *third* case for the new precision points

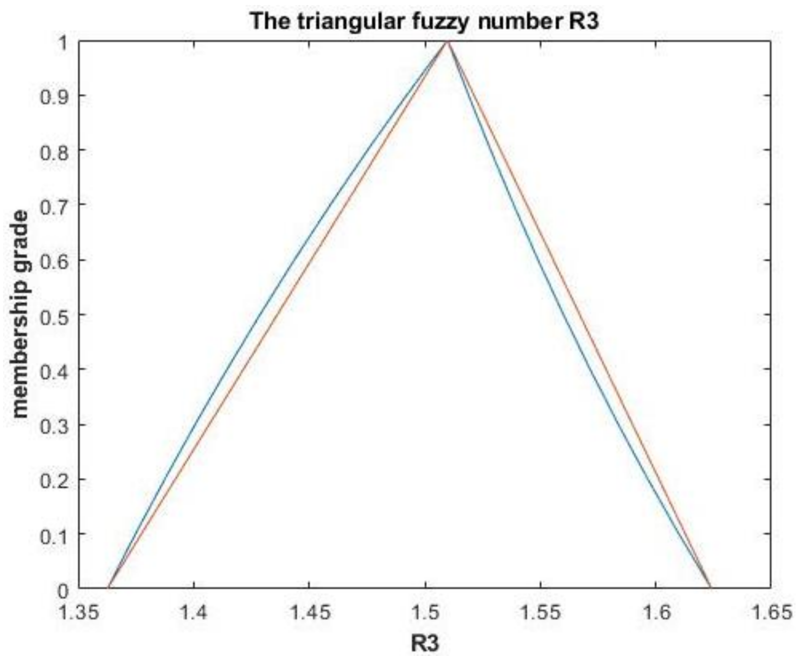

Figure 3-55. The fuzzy length of the coupler link “link 3” of the third case for the new precision points

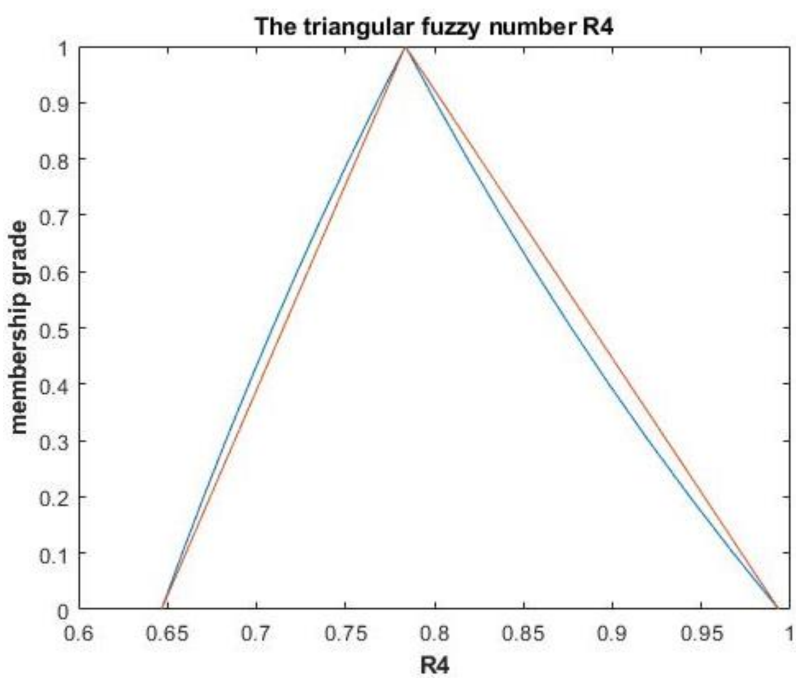

Figure 3-56. The fuzzy length of the output link “link 4” of the third case for the new precision points

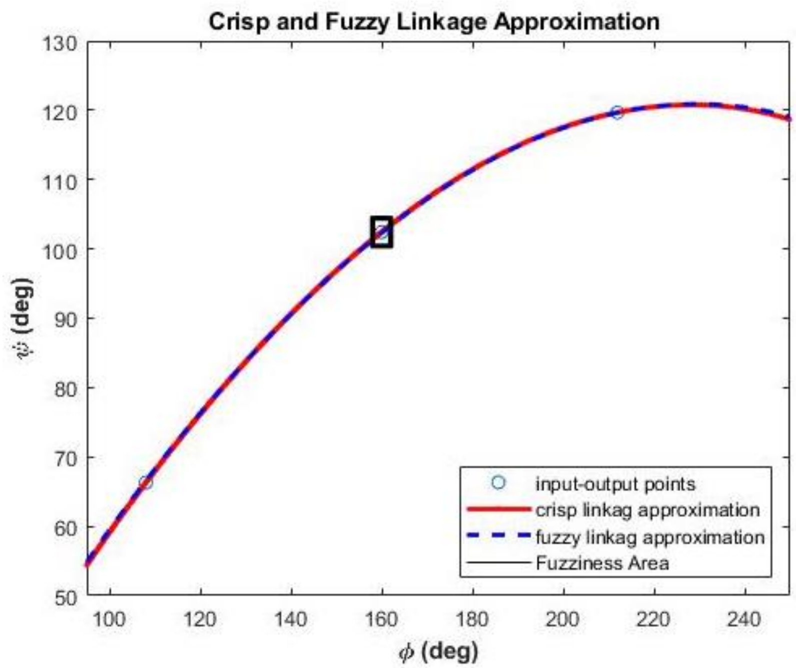

Figure 3-57. The actual crisp and fuzzy linkage function of the third case for the new precision points

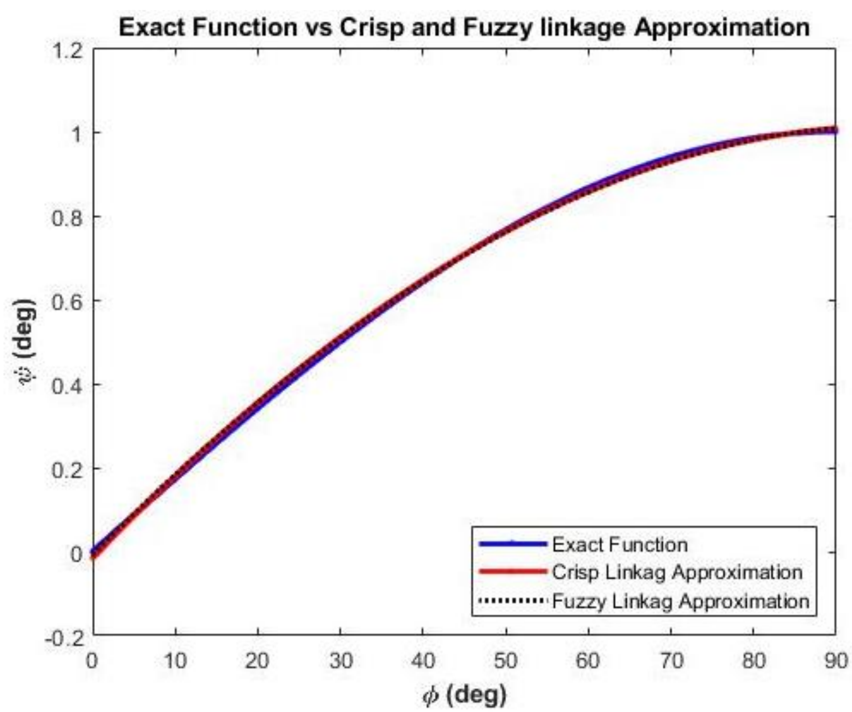

Figure 3-58. The exact function and actual linkage approximations of the of the third case for the new precision points

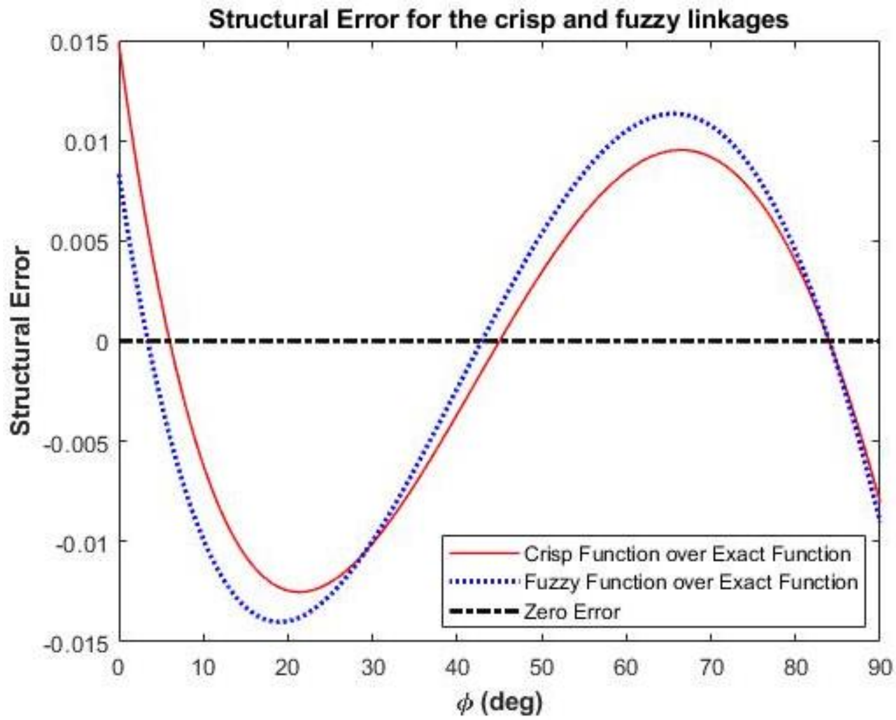

Figure 3-59. Structural Error of the of the third case for the new precision points

Table 3-6. Results of the third case for the new precision points

The link	R1	R2	R3	R4
Crisp length	1.0	0.6085875	1.5098086	0.7836283
Fuzzified length	1.0	0.6133654	1.4976248	0.8273258
Structural Error of the crisp approximating		1.875		
Structural Error of the fuzzy approximating		2.111		

Form the results of the third case with 4 degrees fuzziness, the precision points of the fuzzy approximating with the sin function will be coincided with the crisp approximating in the third precision point $x_3 = 89.971$;however, the other two points will precede the crisp approximating by 2 degrees approximately to get $3.43°$ for the first precision point and $42.9°$ for the last one. In addition, the structural error is reduced by 2.2 points comparing to the decreasing in the fuzziness with 2 degrees from the second case.

3.2 Example

In the second exampe, the input and output angles are given without using Chebyshev spacing as the follwing:

$$\emptyset = [80\ \ 50\ \ 30]$$

$$\psi = [60\ \ 30\ \ 0]$$

This example should clearify the limitation of the fuzziness that be determined by the free zero interval for the constants K's as mintioned in definition 2-6 and (28). So that, the fuzziness of the second point in this case will be 0.2 degree which is very small comparing to the cases of the previous example. The steps of the solutions should be the same as the previous example and their figures are as the following:

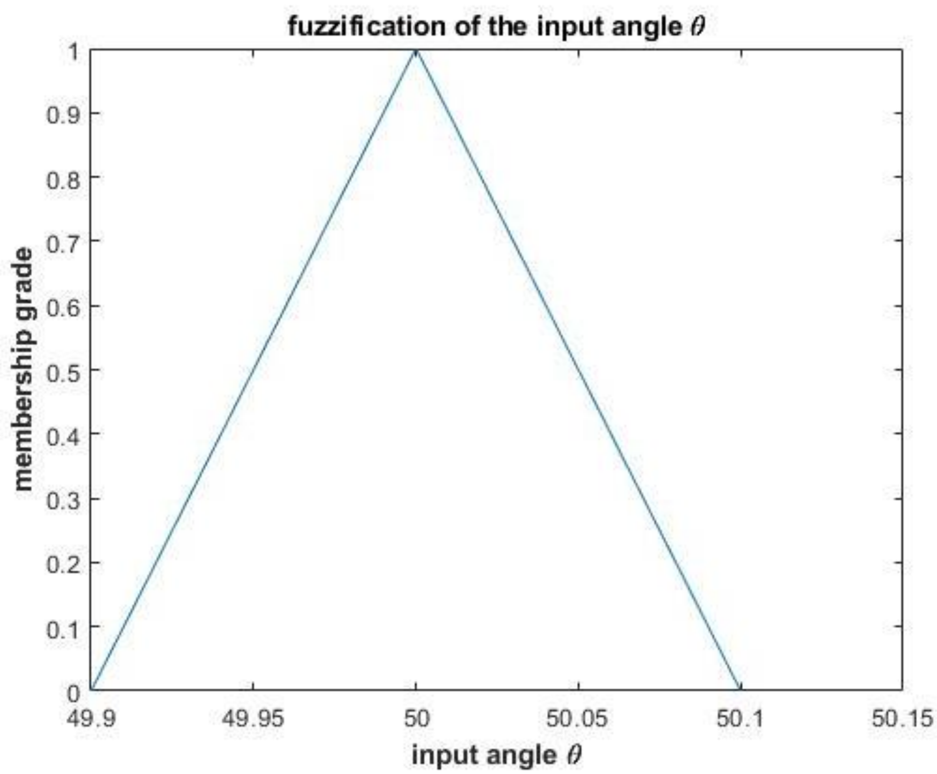

Figure 3-60. Fuzzification the second input angle $\emptyset_2$

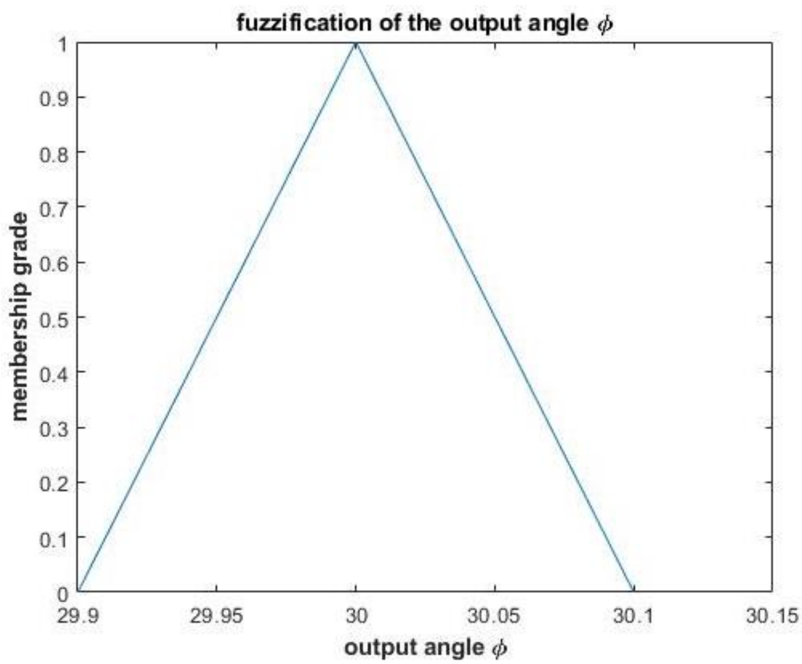

Figure 3-61. Fuzzification the second output angle ψ_2

Then, the constants K's have the following membership functions.

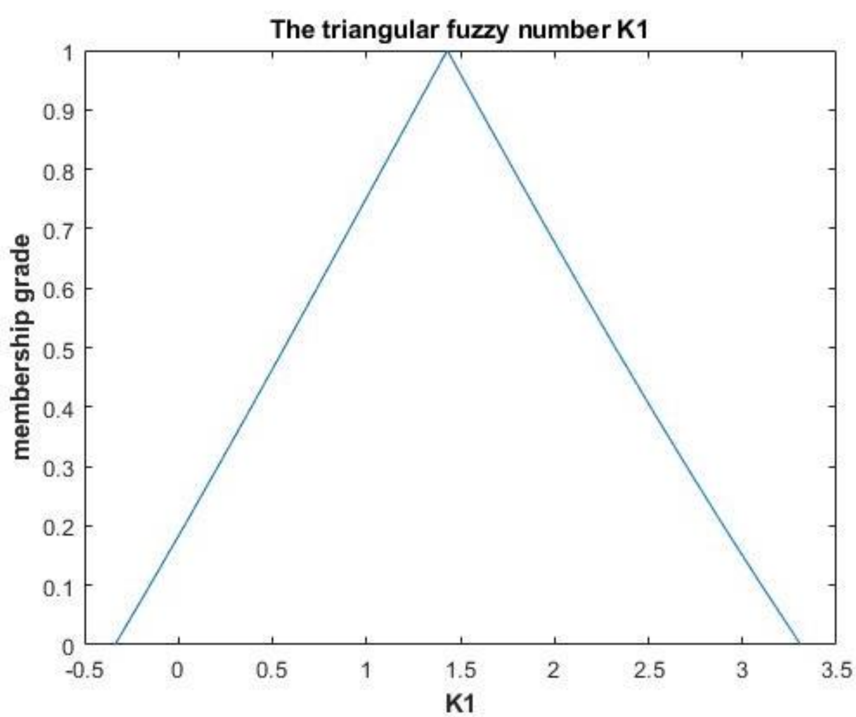

Figure 3-62. The fuzzy number K1

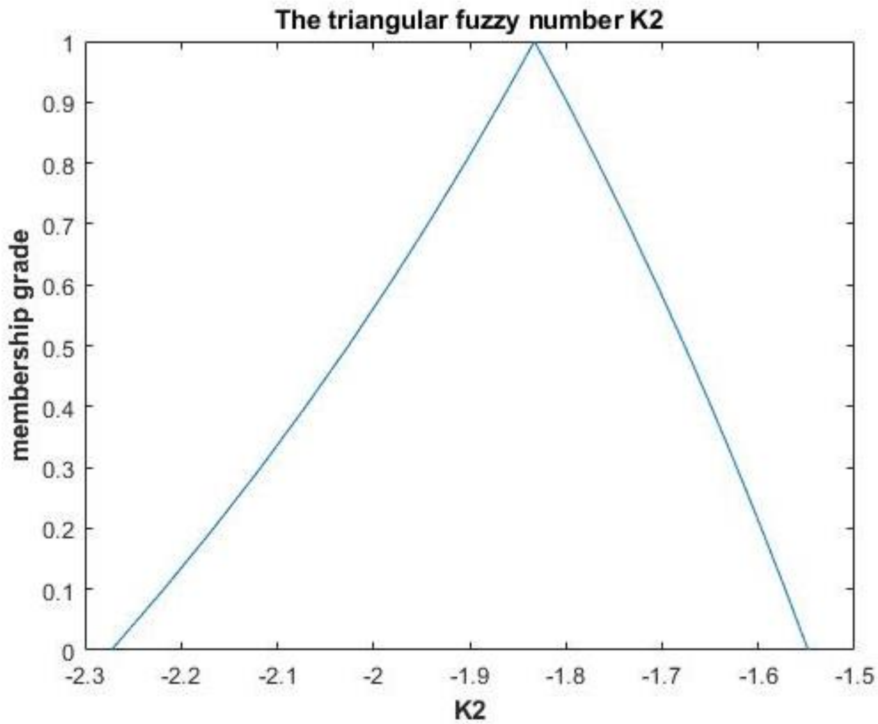

Figure 3-63. The fuzzy number K2

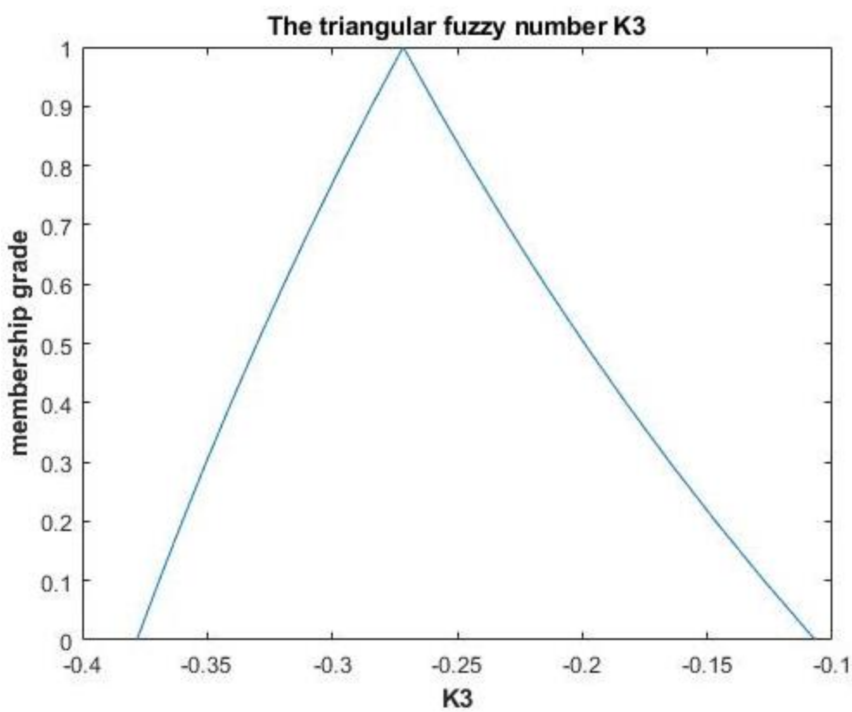

Figure 3-64. The fuzzy number K3

It is clear that the approximating triangular fuzzy number for constant K1 has contained a zero. That leads to make the resultants fuzzy lengths of the links not a fuzzy number, in which this constraint restricts the fuzziness operation. The link that will not be affected is the fuzzy lengths of the input link due to the free zero interval of the fuzzy constant K2 as shown in the following figure.

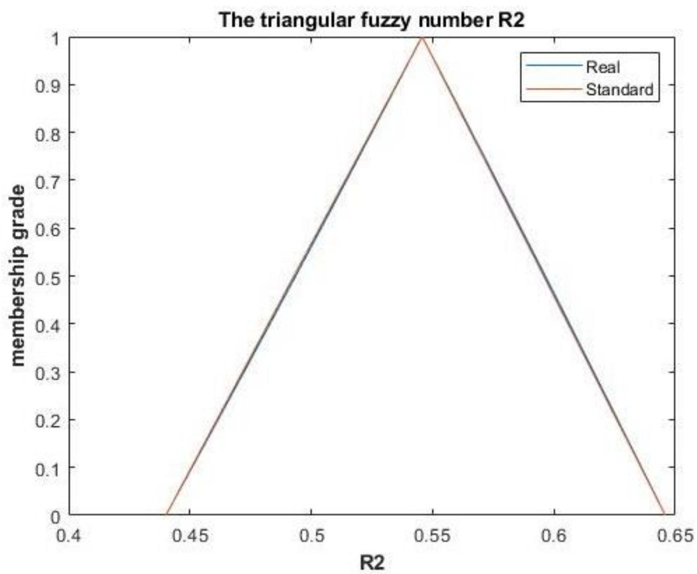

Figure 3-65. The fuzzy length of the input link "link 2"

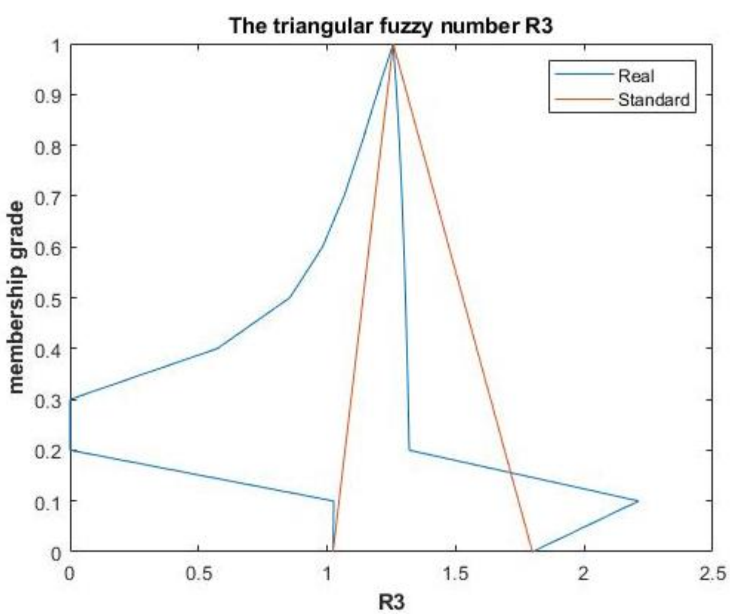

Figure 3-66. The fuzzy length of the coupler link "link 3"

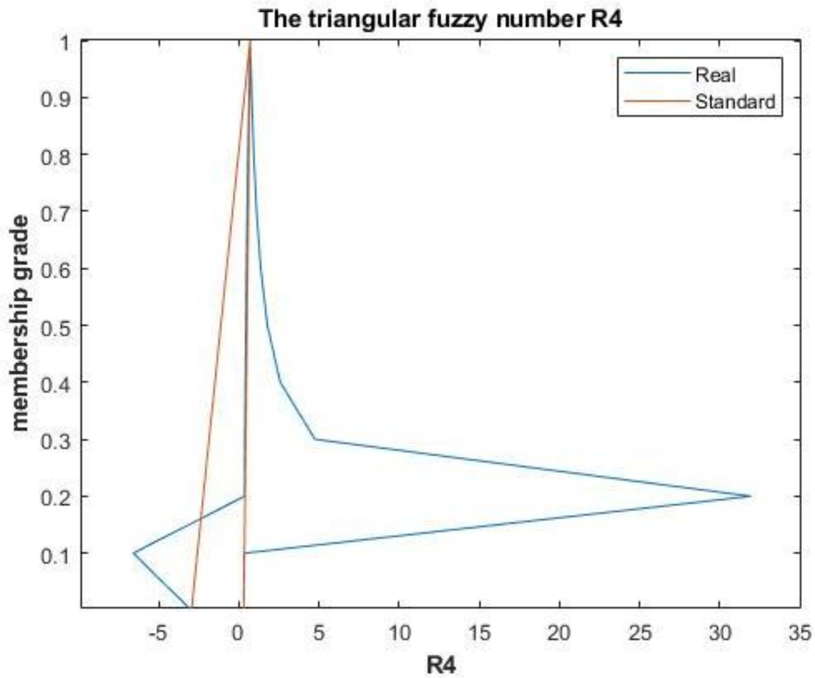

Figure 3-67. The fuzzy length of the output link "link 4"

It can be seen from Figure 3-29 and 3-30 that fuzzy lengths of the coupler and output links are not fuzzy number as were defined in the Definition 2-5. That shape will be excluded from the diffuzzification process due to no convexity for the membership function for the resultant fuzzy set. Consequently, a simple change is made on the selected input angles to go far away from the zero interval, in which the input angles will be away from 90° and 270° because the cosine function at these values equal zero. As the output angles stay the same, the new input angles will be:

Ø = [75 50 30]

The resultant lengths of the links after the modification are shown in the following figures:

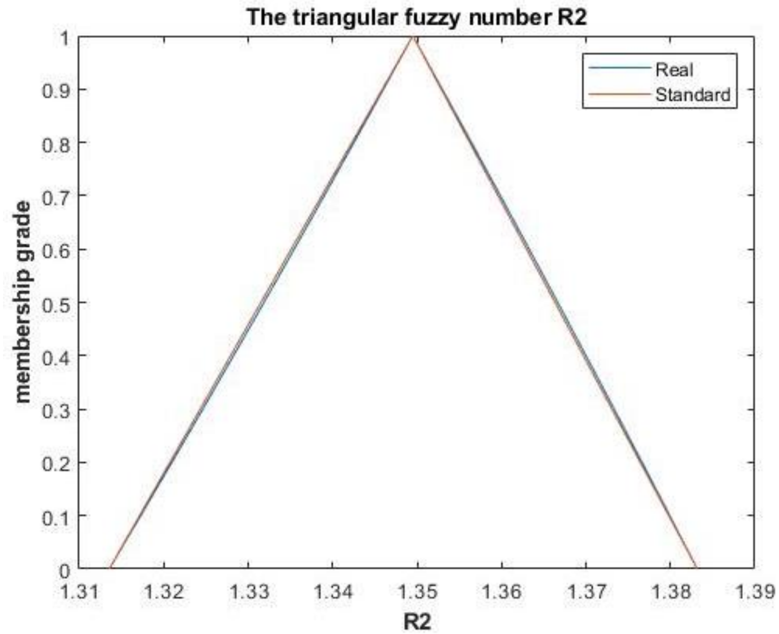

Figure 3-68. The fuzzy length of the input link “link 2”

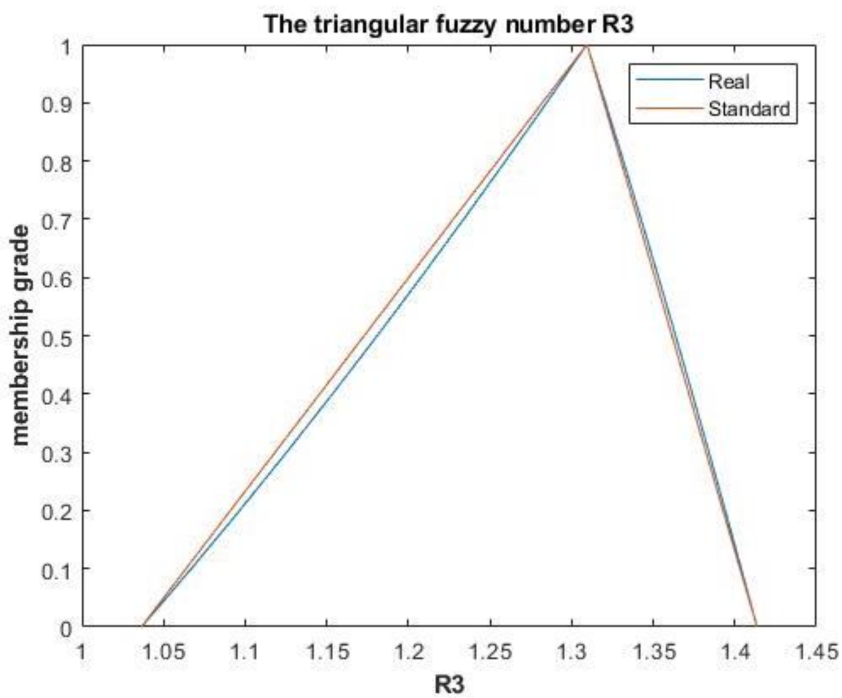

Figure 3-69. The fuzzy length of the coupler link “link 3”

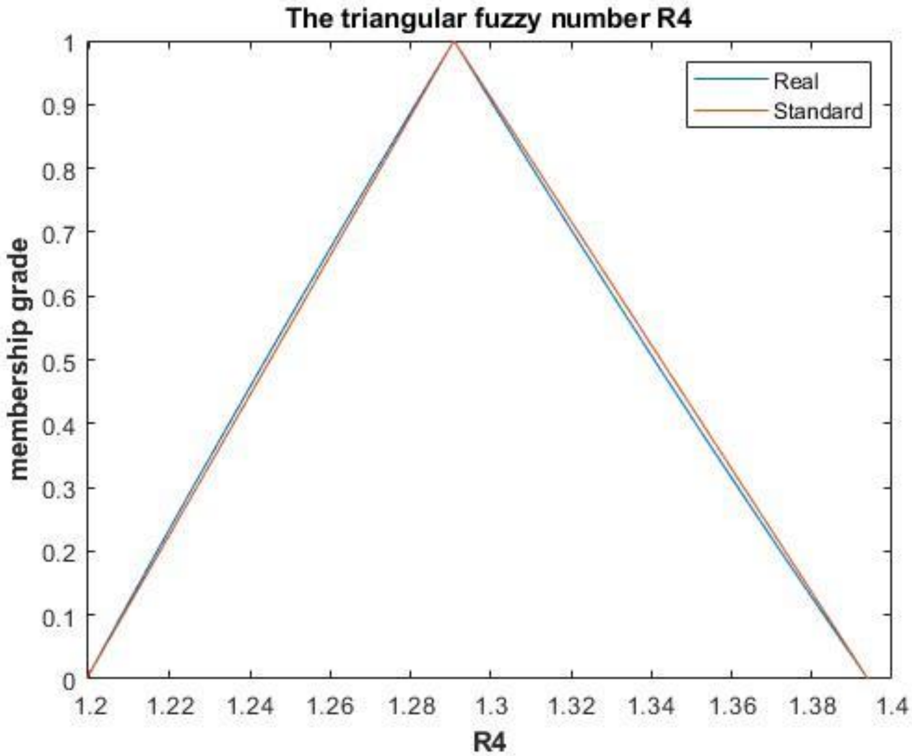

Figure 3-70. The fuzzy length of the output link "link 4"

It is clear from Figure 3-59, 3-60 and 3-61 above that linearity of the membership functions for the fuzzy lengths is achieved by changing the input angles away from 90°.

Chapter 4: Conclusion and Future Work

4.1 Conclusion

It is clear from all cases that have been investigated in the first example that fuzziness of Freudenstein's equation of synthesis of four-bar mechanism allows to the mechanism to work by the function passing through a region rather than an exact point. Although the fuzziness was only for the middle point, the other precision points at the resultant generated function have been fuzzified with different proportions as the lengths of the links were fuzzified. The estimated functions from the centroid defuzzification method have differential amount of structural error depending on the fuzziness area in the middle of the crisp function and the type of rows order of the fuzzy matrix as shown in the first example especially in Figure 3.14, Figure 3-23, Figure 3-32, Table 3-1, Table 3-2 and Table 3-3. Changing the type of rows order of the fuzzy matrix is affected the results because of the subset relation relating the solution of the interval linear system as shown in (25) rather than the equality relation of the solution in the crisp linear system.

In addition, the fuzzy approximating using the centroid defuzzification method has less amount of structural error comparing to the crisp approximating as shown in Figure 3-32 and Table 3-3. The reason behind is that fuzzification includes the new precision point as the fuzzy input was between $154°$ and $160°$; however, the crisp approximating with the old precision points has stuck to the wrong choice of these points as the reference did.

From the two situations mentioned in the first example, it is clear that fuzziness of the precision points to model the uncertainty of the links' lengths has an advantage of the interval solutions. In the fuzzy solutions, the membership function keeps the Chebyshev crisp approximating at specific level of α-cuts such as $\alpha = 1$ at the second situation and $\alpha = 0.5, 0$ at the first situation. Consequently, all three methods that be used to model the uncertainty of the link's lengths are considered in the results of the fuzzy approximating as the crisp approximating has preserved in the fuzzy lengths. In other words, the probabilistic method is included in the fuzzy synthesis as the membership functions of the fuzzy lengths have been estimated from the fuzzy precision points instead of the information that describe the uncertain variables. Besides the probabilistic method, the interval method are included as the results of the fuzzy synthesis produce the interval lengths at $\alpha = 0$.

On the subject the selecting of the input and output angles, the selecting in the fuzzy synthesis do not depend on the Chebyshev spacing because the middle point converts to a region with larger area for the function. That area should give a designer the flexibility to determine the input and output angles with guarantee of a limited extent of increasing for the structural error. The crisp approximating for three precision position for the function generation relies on the Chebyshev spacing theory to minimizing the structural error. On the other hand, the fuzzy approximating in this project excludes the Chebyshev spacing to looking for different factors with the flexibility has been gained such as the taking the input angles from one side of the cosine circle and the output angles from the other as shown in the Figure 4-1.

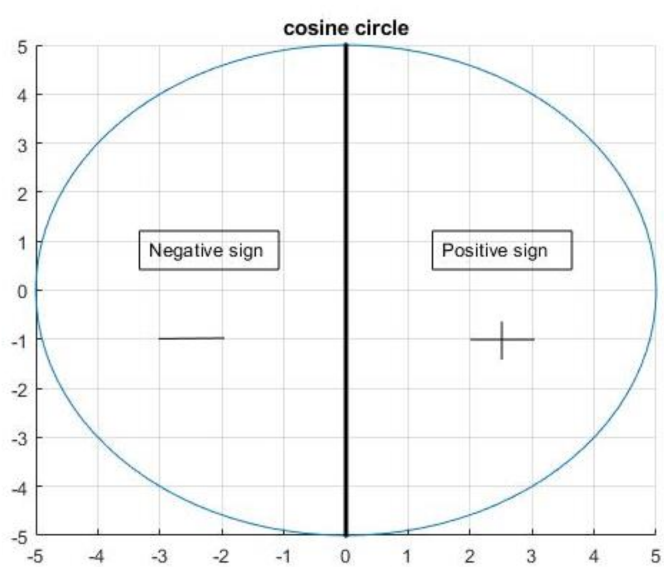

Figure 4-1. The sign cirlce for the cosine function

This factor gives the linearity of the membership for the fuzzy lengths that leads to a wide range of fuzziness as demonstrated by the second situation in the first example, in which the fuzziness that was taken around 12 degrees had been increased the structural error about 3.7 points. This amount of structural error is deemed a small structural error comparing to the fuzziness area. Another factor related to the linearity of the membership function for the fuzzy lengths is the avoid the zero values for the cosine function or near it as demonstrated in the second example that viewed the five degrees away from the 90° made the linear membership function for the fuzzy lengths.

Generally, the modeling of the uncertainty of the dimensional synthesis of the four-bar mechanism by fuzziness one of the precision point builds intervals of link's dimensions with membership functions that represent the probability of the dimensions value depending on the uncertainty of the positions of the precision point itself rather than uncertainty of the external information about the links' dimensions.

4.2 Future Work

It can be seen that project concerned with the four-bar mechanism; however, other types of mechanism such as Six-bar mechanism with three or more precision points that could be synthesized with the same fuzzy mathematical model with different loop equations to improve the synthesis operations as we did for the four-bar mechanism.

Another aspect that could be spotlighted is the effect of changing the type of fuzzy number from a triangular to a trapezoidal on the fuzziness area and the related structural error. That changing should convert the way to from the interval matrices because there are many at level $\alpha = 1$ and the crisp matrix that be defined at this level should be an interval matrix by which the solution has a subset relation $\hat{x} \subseteq \sum(\hat{A}, \hat{b})$ instead of equality relation in the crisp one.

Chapter 5: References

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