THE MAXIMAL SUBGROUPS OF THE CLASSICAL GROUPS IN DIMENSION 13, 14 AND 15

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Abstract

One might easily argue that the Classification of Finite Simple Groups is one of the most important theorems of group theory. Given that any finite group can be deconstructed into its simple composition factors, it is of great importance to have a detailed knowledge of the structure of finite simple groups.

One of the classes of finite groups that appear in the classification theorem are the simple classical groups, which are matrix groups preserving some form. This thesis will shed some new light on almost simple classical groups in dimension 13, 14 and 15. In particular we will determine their maximal subgroups.

We will build on the results by Bray, Holt, and Roney-Dougal [8] who calculated the maximal subgroups of all almost simple finite classical groups in dimension $\leqslant 12$. Furthermore, Aschbacher [2] proved that the maximal subgroups of almost simple classical groups lie in nine classes. The maximal subgroups in the first eight classes, i.e. the subgroups of geometric type, were determined by Kleidman and Liebeck and all maximal subgroups in dimension $\geqslant 13$ that are of geometric type can be found in [26].

Therefore this thesis concentrates on the ninth class of Aschbacher's Theorem. This class roughly consists of subgroups which are almost simple modulo scalars and do not preserve a geometric structure. As our final result we will give tables containing all maximal subgroups of almost simple classical groups in dimension 13, 14 and 15.

Declarations

Candidate's declarations

I, Anna Katharina Schröder, hereby certify that this thesis, which is approximately 39 500 words in length, has been written by me, and that it is the record of work carried out by me, or principally by myself in collaboration with others as acknowledged, and that it has not been submitted in any previous application for a higher degree.

I was admitted as a research student in September 2010 and as a candidate for the degree of PhD in September 2010; the higher study for which this is a record was carried out in the University of St Andrews between 2010 and 2015.

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I hereby certify that the candidate has fulfilled the conditions of the Resolution and Regulations appropriate for the degree of PhD in the University of St Andrews and that the candidate is qualified to submit this thesis in application for that degree.

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1 Introduction

Finding the maximal subgroups of almost simple classical groups is useful for several reasons. Ever since the completion of the Classification of Finite Simple Groups it has become even more important to understand the structure of the finite simple groups since they are in some way the building blocks of all finite groups by Jordan-Hölder. Furthermore, Cannon and Holt [10] found a method to computationally construct the maximal subgroups of any finite group G, using the maximal subgroups of the almost simple extensions of the composition factors of G.

A lot of research has been done in order to understand the structure of the maximal subgroups of classical groups. Probably the most important result is Aschbacher's Theorem [2]. Aschbacher managed to show that every maximal subgroup of a classical group lies in one of nine classes, denoted by $\mathcal{C}_1$ to $\mathcal{C}_9$. The subgroups lying in the classes $\mathcal{C}_1$ to $\mathcal{C}_8$ preserve some geometric structure and are said to be of geometric type. The first mathematician to systematically determine the maximal subgroups of specific classical groups was Peter Kleidman. In his PhD thesis (1987) [25] he found the maximal subgroups of the simple classical groups in dimension up to 12. A few years later, Kleidman and Liebeck published a book ([26]) with tables containing the maximal subgroups of geometric type of the almost simple classical groups in dimension greater than 12. Furthermore, Bray, Holt, and Roney-Dougal [8] achieved a complete classification of the maximal subgroups of all almost simple classical groups in dimension $\leqslant 12$. For a more thorough literature review see [8, Preface, p.x].

This thesis will build on the results by Bray, Holt and Roney-Dougal to determine the maximal subgroups of all almost simple classical groups in dimension 13, 14 and 15. As the maximal subgroups of geometric type are given in [26], we will concentrate on subgroups lying in class $\mathcal{C}_9$, which we will denote by $\mathscr{S}$. Roughly speaking these are the subgroups that do not preserve some geometric structure and are almost simple modulo scalars.

Let $\Omega \in \{\mathrm{SL}_n(q), \mathrm{SU}_n(q), \mathrm{Sp}_n(q), \Omega_n^\epsilon(q)\}$, where $\epsilon \in \{\circ, +, -\}$, $q = p^e$ for some prime p, $e \in \mathbb{N}$, and $n \in \{13, 14, 15\}$. Let T be a group such that $\Omega \leqslant T \leqslant A$, where $A/\mathrm{Z}(A) = \mathrm{Aut}(\Omega/\mathrm{Z}(\Omega))$. Even though our main aim is to find the maximal subgroups of the almost simple groups $T/\mathrm{Z}(T)$ we will determine the maximal subgroups of T. The reason for this is that it is a lot easier to work with matrix than with permutation representations.

Furthermore, we can show that once we know the maximal subgroups of T then we can easily deduce the maximal subgroups of $T/\mathrm{Z}(T)$.

We will first give a brief discussion of some useful results in group and representation theory before introducing classical groups in Chapter 3. This will be followed by the theory that is needed to determine the maximal subgroups that lie in Class $\mathscr{S}$.

We will first consider the $\mathscr{S}$-subgroups $G \leqslant \Omega$ that are of cross characteristic, i.e. the $\mathscr{S}_1$-subgroups, in Chapter 4. In particular we will look at the form preserved by G and we will find a way to determine which of the automorphisms of Ω stabilise G.

In Chapters 5, 6 and 7 we will calculate the maximal $\mathscr{S}_1$-subgroups in dimension 13, 14 and 15 respectively. We will use the tables by Hiss and Malle [18] to get our potential maximal subgroups G. In particular in these three chapters we will often use Magma to determine the behaviour of our groups G. In these cases we will use the phrase by 'computer calculations' which is followed by the name of the file containing the Magma commands. These files can be found on a separate CD attached to this thesis.

The other type of $\mathscr{S}$-subgroups, the $\mathscr{S}_2$-subgroups, consisting of $\mathscr{S}$-subgroups in defining characteristic, will then be considered in the following two chapters. In Chapter 8 we will look at the theory behind $\mathscr{S}_2$-subgroups. In particular we will define the heighest weight of a representation and give an introduction to exterior and symmetric power modules and adjoint modules. In Chapter 9 we will show which of the potential $\mathscr{S}_2$-subgroups given by Luebeck in [29] are indeed $\mathscr{S}_2$-maximal in dimension 13, 14 and 15.

Containments between $\mathscr{S}_1$- and $\mathscr{S}_2$-subgroups will then be determined in Chapter 10. In this chapter we will also give a brief introduction to the tables in [26] containing the maximal subgroups of all almost simple classical groups in dimension $\geqslant 13$ that are of geometric type. Finally, we will show which of our $\mathscr{S}$-maximal subgroups also preserve a geometric structure, i.e are subgroups of one of the geometric maximal subgroups and hence are not maximal.

Our main results, the tables containing the maximal subgroups of classical groups in dimension 13, 14 and 15, can be found in Chapter 11.

2 Groups, Fields and Representations

In this chapter we will set up notation and give some definitions and lemmas which will be helpful throughout this thesis. We will begin with group theory, talk briefly about fields and finish this chapter with some useful results in representation theory.

2.1 Groups and Fields

We will begin with a few group-theoretic definitions - mainly to set up some notation.

Definition 2.1.1. A finite group G is **almost simple** if there exists a non-abelian simple group S such that $S \trianglelefteq G \leqslant \mathrm{Aut}(S)$. Here we identify S with $\mathrm{Inn}(S)$, as $S \cong \mathrm{Inn}(S)$. A finite group G is **quasisimple** if G is perfect and $G \cong \mathrm{Z}(G).S$, where $\mathrm{Z}(G) = \{z \in G \mid zg = gz \ \forall g \in G\}$ is the centre of G.

Definition 2.1.2. The **derived series** of a finite group G is a series of subgroups of G,

$$G = G^{(0)} \trianglerighteq G^{(1)} \trianglerighteq G^{(2)} \trianglerighteq \dots,$$

where $G^{(i)} = \langle [g,h] \mid g,h \in G^{(i-1)} \rangle$. Let $\boldsymbol{G}^{\infty}$ be the intersection of all the $G^{(i)}$.

Definition 2.1.3. Let G be a finite simple group and assume that there exists a group H such that $H/\mathrm{Z}(H) \cong G$. If H is perfect then H is a **covering group** of G. There exists a unique maximal covering group of G which is finite ([33]) and is called the **full covering group** of G. The **Schur multiplier** of G is the centre of its full covering group.

Definition 2.1.4. Let $(x_1, \dots, x_n)$ be a sequence of elements of a group G together with a set of conditions. These conditions can be e.g. conjugacy class membership or orders of specific elements of G. Then $(x_1, \dots, x_n)$ is a sequence of **standard generators** of G if $\langle x_1, \dots, x_n \rangle = G$ and for any other generating sequence $(y_1, \dots, y_n)$ of G satisfying the same conditions as the x_i there exists an $\alpha \in \mathrm{Aut}(G)$ such that $y_i = x_i^{\alpha}$ for all i.

Note that in general we use K to denote an arbitrary field and $\mathbb{F}_q$ to denote the unique (up to isomorphism) finite field with $q = p^e$ elements for some prime p and some $e \geqslant 1$. Hence $\mathbb{F}_q$ will always have characteristic p. In general we will use ω to denote a primitive element of $\mathbb{F}_q^{\times}$.

Definition 2.1.5. Let K be a field and let $K[x]$ be the ring of polynomials in the variable x with coefficients in K. Then K is **algebraically closed** if it contains a root of every non-constant polynomial in $K[x]$.

Definition 2.1.6. Let K be a field and let $K \leqslant L$ be such that for each $a \in L$ there exists $f \in K[x]$ such that $f(a) = 0$. Then L is an **algebraic extension** of K. Let $h \in K[x]$ be such that $h(a) = 0$ and let $g \in K[x]$ be the greatest common divisor of all such h. Then the roots $a, a_1, \ldots, a_n$ of g in L are the **algebraic conjugates** of a. Furthermore, the group of automorphisms of L permutes the algebraic conjugates ([39, Lemma 2.7.12, p.27]).

The following lemmas will be useful for this thesis.

Lemma 2.1.7. *The element -1 is a square in $\mathbb{F}_q^\times$ if and only if either q is even or $q \equiv 1 \pmod 4$.*

Proof. If q is even then all elements of $\mathbb{F}_q^\times$ are square.

If q is odd then -1 is a square if and only if there exists a primitive element ω of $\mathbb{F}_q^\times$ such that $\omega^{2j} = -1$ for some j. This is the case if and only if $2j = \frac{q-1}{2} \pmod{q-1}$ which has a solution if and only if $q \equiv 1 \pmod 4$. □

Lemma 2.1.8. *Let $\mu \in \mathbb{F}_{q^2}^\times$. If μ^{q+1} is square in $\mathbb{F}_q^\times$ then μ is square in $\mathbb{F}_{q^2}^\times$.*

Proof. If q is even then any element of $\mathbb{F}_q^\times$ is square. Hence suppose that q is odd. Let $\mu^{q+1} = a^2 \in \mathbb{F}_q^\times$, with $a \in \mathbb{F}_q^\times$. Then $a = \omega^{i(q+1)}$ for some primitive element ω of $\mathbb{F}_{q^2}^\times$ and some i. Hence $\mu^{q+1} = \omega^{2i(q+1)}$ and $\mu = z(\omega^i)^2$ for some $(q+1)^{\text{th}}$ root of unity z. Then $z = \omega^{(q-1)k} = \omega^{2k\frac{(q-1)}{2}}$ for some k and μ is indeed a square. □

2.2 Representation Theory

We will first state a few basic definitions and lemmas. Then there will be a short introduction to splitting and fusion of representations, algebraic irrationalities and Brauer characters.

We will denote a representation of a group G by $\rho : G \to \mathrm{GL}(V)$, where $\mathrm{GL}(V)$ is the group of linear transformations of a vector space V over a field K. If $\dim(V) = n$ is finite, then we use $\mathrm{GL}_n(K)$ instead of $\mathrm{GL}(V)$ and say that n is the **dimension** of ρ. If we also have $K = \mathbb{F}_q$ then we write $\mathrm{GL}_n(q)$ instead of $\mathrm{GL}_n(\mathbb{F}_q)$.

Definition 2.2.1. Let ρ be a representation of a finite group G with $\rho : G \to \mathrm{GL}_n(K)$. The corresponding **character** of G is the function $\chi : G \to K$ defined by $\chi(g) = \mathrm{Trace}(g\rho)$ for all $g \in G$.

Definition 2.2.2. Let G be finite and let $\rho : G \to \mathrm{GL}_n(K)$ be a representation of G. The **character ring** of ρ is the ring generated by the character values of ρ.

Definition 2.2.3. Let $\alpha \in \mathrm{Aut}(G)$ and let $\rho : G \to \mathrm{GL}_n(K)$ be a representation. Then $\boldsymbol{^{\alpha}\rho} : G \to \mathrm{GL}_n(K)$ is defined by $g(^{\alpha}\rho) = (g^{\alpha})\rho$ for all $g \in G$. Let $G\rho \leqslant H$. If $\beta \in \mathrm{Aut}(H)$ then $\boldsymbol{\rho^{\beta}}$ is defined by $g(\rho^{\beta}) = (g\rho)^{\beta}$ for all $g \in G$.

Definition 2.2.4. Two representations $\rho_1, \rho_2 : G \to \mathrm{GL}_n(K)$ are **equivalent** if there exists $h \in \mathrm{GL}_n(K)$ such that $h^{-1}(g\rho_1)h = g\rho_2$ for all $g \in G$. Furthermore, ρ_1 and ρ_2 are **quasiequivalent** if there exists $\alpha \in \mathrm{Aut}(G)$ such that $^{\alpha}\rho_1$ and ρ_2 are equivalent. We say that α **stabilises** ρ_1 if $^{\alpha}\rho_1$ is equivalent to ρ_1.

Lemma 2.2.5 ([8, Lemma 1.8.6, p.39])**.** *Let $\rho, \rho' : G \to \mathrm{GL}_n(K)$ be two faithful representations. Then ρ and ρ' are quasiequivalent if and only if their images are conjugate subgroups of $\mathrm{GL}_n(K)$.*

We will now consider the character values of representations over algebraically closed fields.

Lemma 2.2.6. *Let G be a finite group and let K be an algebraically closed field. Let $\rho : G \to \mathrm{GL}_n(K)$ be a representation of G with character χ. Then for all $g \in G$, $\chi(g\rho)$ equals the sum of the eigenvalues of $g\rho$.*

Proof. Over algebraically closed fields any $g\rho$ is conjugate to a matrix in Jordan normal form. Furthermore, $\mathrm{Trace}(g\rho) = \mathrm{Trace}(A^{-1}(g\rho)A)$ for any $A \in \mathrm{GL}_n(K)$. □

Lemma 2.2.7. *Let $A \in \mathrm{GL}_n(K)$, where K is an algebraically closed field. Suppose that A has order $t < \infty$. Then the eigenvalues of A are t^{th} roots of unity.*

Proof. Since A is finite, $A^t = I$ for some t. Let λ be an eigenvalue of A. Then there exists a vector $v \neq 0$ such that $vA = \lambda v$. Therefore, $vA^i = vAA^{i-1} = \lambda vA^{i-1} = v\lambda^i$, implying that λ^i is an eigenvalue of A^i. Hence, $vA^t = vI = \lambda^t v$. This proves that $\lambda^t = 1$. □

The following will be used throughout this thesis.

Definition 2.2.8. Let $G \leqslant \mathrm{GL}_n(q)$. Then G acts on $(\mathbb{F}_{q^r})^n$ for every $r \in \mathbb{N}\backslash\{0\}$. If G stabilises no proper non-zero subspace of $\mathbb{F}_q^n$ then G is **irreducible**. Else G is **reducible**. Furthermore, G is **absolutely irreducible** if for all r, G stabilises no proper nonzero subspace of $(\mathbb{F}_{q^r})^n$. Let $\rho : G \to \mathrm{GL}_n(q)$ be a representation of a finite group G. Then ρ is **absolutely irreducible** if $G\rho$ is absolutely irreducible.

Lemma 2.2.9. *Schur's Lemma [23, Thm 9.2, p.145]*
The centraliser of an absolutely irreducible group $G \leqslant \mathrm{GL}_n(q)$ consists of all the scalar matrices of $\mathrm{GL}_n(q)$.

Definition 2.2.10. Let $G \leqslant H \leqslant \mathrm{GL}_n(q)$. Then G is **scalar-normalising** in H if $\mathrm{N}_H(G) \leqslant G\mathrm{Z}(\mathrm{GL}_n(q))$.

2.2.1 Splitting and Fusion

Let ρ be an absolutely irreducible representation of a quasisimple group G, let n be the dimension of ρ and let a be prime. Let ρ' be defined on $G.a$, using ATLAS ([12]) notation for cyclic groups and extensions. Then we can either find a ρ' such that for all $g \in G$, $g\rho' = g\rho$ in which case we say that ρ splits. Otherwise there does not exist any ρ' such that for all $g \in G$, $g\rho' = g\rho$ in which case ρ is fused.

The description of the two cases was taken from [12, p.xxxviii].

Splitting Case

The first possibility is that ρ extends to ρ' in such a way that $g\rho' = g\rho$ for all $g \in G$. Then ρ' is absolutely irreducible and has dimension n.

Definition 2.2.11. Let V be a vector space. Suppose that $\rho : G \to \mathrm{GL}_n(V)$ is an absolutely irreducible representation and that there exists $\rho' : G.a \to \mathrm{GL}_n(V)$ such that $g\rho' = g\rho$ for all $g \in G$. Then ρ **splits**.

If ρ splits then we get a non-equivalent absolutely irreducible representations ρ_{ω^t} on $G.a$, where ω is a primitive element of $\mathbb{F}_a^\times$, $0 \leqslant t < a$, and $g\rho_{\omega^t} = g\rho$ for all $g \in G$.

Let χ_{ω^t} denote the character values of ρ_{ω^t} and let χ be the character value of ρ. Then

$$\chi_{\omega^t}(g) = \begin{cases} \chi(g) & \text{if } g \in G \\ \omega^t \chi_{\omega^0}(g) & \text{otherwise.} \end{cases}$$

Lemma 2.2.12. *Let $\rho : G \to \mathrm{GL}_n(V)$ be an absolutely irreducible representation and suppose that there exists $\rho' : G.a \to \mathrm{GL}_n(V)$ such that $g\rho' = g\rho$ for all $g \in G$. Let α generate the automorphism of G of order a such that $G\langle\alpha\rangle = G.a$. Then ${}^{\alpha}\rho$ is equivalent to ρ, i.e. α stabilises ρ.*

Proof. Let $G.a = \langle G, k\rangle$ such that $g^k = g^\alpha$ for all $g \in G$. Let $h = k\rho'$. Then

$$\begin{aligned}(g\rho)^h &= h^{-1}(g\rho)h \\ &= (k^{-1}gk)\rho' \\ &= g^\alpha\rho' = g^\alpha\rho\end{aligned}$$

from which it follows that ρ is equivalent to ${}^{\alpha}\rho$. □

Fusion Case

If there does not exist any $\rho' : G.a \to \mathrm{GL}(V)$ such that $g\rho' = g\rho$ for all $g \in G$ then there exist a non-equivalent representations $\rho_1, \ldots, \rho_a$ $(\rho = \rho_1)$ with dimension n such that $\rho_1, \ldots, \rho_a$ fuse to give a single representation $\rho' = \rho_1 + \ldots + \rho_a$ with dimension $n \cdot a$ defined on $G.a$. Let χ_i and χ' denote the character values of ρ_i and ρ' respectively, where $1 \leqslant i \leqslant a$. Then

$$\chi'(g) = \begin{cases} \sum_{i=1}^{a} \chi_i(g) & \text{if } g \in G \\ 0 & \text{otherwise.}\end{cases}$$

2.2.2 Algebraic Irrationalities

For the algebraic irrationalities that are needed in this thesis we will use the same notation as in the ATLAS [12]. In general, we let $\mathrm{i} \in \mathbb{C}$ be a fixed square root of -1 and we let

$$\mathrm{z}_n := e^{2\pi\mathrm{i}/n} = \cos(2\pi/n) + \mathrm{i}\sin(2\pi/n)$$

be a particular primitive n^{th} root of unity. Furthermore,

$$\mathrm{b}_n := \sum_{r=1}^{(n-1)/2} \mathrm{z}_n^{r^2} = \begin{cases} (-1+\sqrt{n})/2 & \text{if } n \equiv 1 \pmod 4 \\ (-1+\mathrm{i}\sqrt{n})/2 & \text{if } n \equiv 3 \pmod 4 \end{cases}$$

by Gauss. We will also need the following irrationalities:

$$\begin{aligned}\mathrm{c}_n &:= \frac{1}{3}\sum_{r=1}^{n-1} \mathrm{z}_n^{r^3}; \\ \mathrm{r}_n &:= \sqrt{n}; \\ \mathrm{i}_n &:= \sqrt{-n} = \mathrm{i}\cdot\mathrm{r}_n; \\ \mathrm{y}_n &:= \mathrm{z}_n + \mathrm{z}_n^{-1} = 2\cos(2\pi/n).\end{aligned}$$

Note that these algebraic irrationalities are defined as elements of $\mathbb{C}$. However, in most cases we will need to find the **p-modular reduction** of these irrationalities, i.e. the interpretation of these irrationalities as elements of the finite field we are interested in.

The following table, Table 2.2.1 (p.19), shows some properties of the algebraic irrationalities that are needed for this thesis.

The first column of the table gives the name of the irrationality, followed by its definition. The column 'Real' answers the question whether the irrationality is a real number or not, whilst the last column gives the smallest finite field over which these algebraic irrationalities can be realised as elements of this field. For example, if $p \equiv 2, 3 \pmod 5$ then z_5 can be realised in $\mathbb{F}_{p^4}$ but not in any subfield of $\mathbb{F}_{p^4}$.

The content of this table was calculated using a number of different methods of which we will demonstrate the most important ones.

We will start with the roots of unity z_n. To find the p-modular reductions of z_n, it suffices to find the smallest q such that $q-1$ is divisible by n.

The properties of the irrationality $\mathrm{b}_5 = \sum_{t=1}^{2} \mathrm{z}_5^{t^2} = \mathrm{z}_5 + \mathrm{z}_5^{-1}$ can be determined as follows. Since $\mathrm{b}_5 = \frac{-1+\sqrt{5}}{2}$ by definition it follows that $2\mathrm{b}_5 + 1 = \sqrt{5}$ and hence that $\mathrm{b}_5^2 + \mathrm{b}_5 - 1 = 0$ squaring both sides. Therefore b_5 has a minimal polynomial of degree 2 and $\mathrm{b}_5 \in \mathbb{F}_p$ if and only if $\sqrt{5}$ is. Equivalently, $\mathrm{b}_5 \in \mathbb{F}_p$ if and only if $\left(\frac{5}{p}\right) = 1$ using Legendre symbols ([34, p.70]). By the quadratic reciprocity law ([34, p.72])

$$\left(\frac{5}{p}\right) = \left(\frac{p}{5}\right)(-1)^{\frac{p-1}{2}\cdot\frac{5-1}{2}}.$$

Note that $\left(\frac{5}{p}\right) = \left(\frac{p}{5}\right)$ unless $p = 2$. Furthermore, p is square in $\mathbb{F}_5$ if and only if $p \equiv 1, 4 \pmod 5$. Hence

$$\left(\frac{5}{p}\right) = \begin{cases} (-1)\cdot(-1) = 1 & \text{if } p = 2 \\ 1 \cdot 1 = 1 & \text{if } p \equiv 1, 4 \pmod 5 \\ (-1)\cdot 1 = -1 & \text{if } p \equiv 2, 3 \pmod 5,\ p \neq 2. \end{cases}$$

Finally, we will consider $\mathrm{y}_{24} = \mathrm{z}_{24} + \mathrm{z}_{24}^{-1} = 2\cos(\frac{2\pi}{24})$. If $p \neq 2, 3$ then $\mathrm{y}_{24} \in \mathbb{F}_q$ if and only if $q \equiv \pm 1 \pmod{24}$ by [8, Lemma 4.2.1, p.156]. Hence $\mathrm{y}_{24} \in \mathbb{F}_p$ if and only if $p \equiv 1, 23 \pmod{24}$. Furthermore, it is straightforward to check that $\mathrm{y}_{24} \in \mathbb{F}_{p^2} \backslash \mathbb{F}_p$ if $p \equiv 5, 7, 11, 13, 17, 19 \pmod{24}$.

The p-modular reductions of the other algebraic irrationalities can be determined using similar methods to the above. Note that we only find the p-modular reduction of c_{19} when $p = 11$ as this is sufficient for this thesis.

Table 2.2.1: Algebraic Irrationalities

Irr	**Definition**	**Real**	***p*-mod reduction**
z_3	$e^{\frac{2\pi \mathrm{i}}{3}} = \cos(\frac{2\pi}{3}) + \mathrm{i}\sin(\frac{2\pi}{3})$	No	Deg 1: $p \equiv 1 \pmod 3$ Deg 2: $p \equiv 2 \pmod 3$, $p \neq 2$
z_5	$e^{\frac{2\pi \mathrm{i}}{5}} = \cos(\frac{2\pi}{5}) + \mathrm{i}\sin(\frac{2\pi}{5})$	No	Deg 1: $p \equiv 1 \pmod 5$ Deg 2: $p \equiv 4 \pmod 5$ Deg 4: $p \equiv 2, 3 \pmod 5$
b_5	$\frac{1}{2}(-1+\sqrt{5}) = \mathrm{z}_5 + \mathrm{z}_5^4$	Yes	Deg 1: $p \equiv 1, 4 \pmod 5$ Deg 2: $p \equiv 2, 3 \pmod 5$, $p \neq 2$
b_{11}	$\frac{1}{2}(-1+\mathrm{i}\sqrt{11})$	No	Deg 1: $p \equiv 1, 3, 4, 5, 9 \pmod{11}$ Deg 2: $p \equiv 2, 6, 7, 8, 10 \pmod{11}$
b_{27}	$\frac{1}{2}(-1+\mathrm{i}\sqrt{27}) = 2\mathrm{z}_3 - \mathrm{z}_3^2$	No	Deg 1: $p \equiv 1 \pmod 3$ Deg 2: $p \equiv 2 \pmod 3$, $p \neq 2$
b_{29}	$\frac{1}{2}(-1+\sqrt{29}) = \sum_{r=1}^{14} \mathrm{z}_{29}^{r^2}$	Yes	Deg 1: $p \equiv 1, 4, 5, 6, 7, 9, 13, 16, 20, 22,$ $23, 24, 25, 28 \pmod{29}$ Deg 2: $p \equiv 2, 3, 8, 10, 11, 12, 14, 15, 17,$ $18, 19, 21, 26, 27 \pmod{29}$
b_{31}	$\frac{1}{2}(-1+\mathrm{i}\sqrt{31})$	No	Deg 1: $p \equiv 1, 2, 4, 5, 7, 8, 9, 10, 14, 16, 18,$ $19, 20, 25, 28 \pmod{31}$ Deg 2: $p \equiv 3, 6, 11, 12, 13, 15, 17, 21, 22, 23,$ $24, 26, 27, 29, 30 \pmod{31}$
c_{19}	$\frac{1}{3}\sum_{r=1}^{18} \mathrm{z}_{19}^{r^3}$	Yes	Deg 1: $p = 11$
i	$\sqrt{-1} = \mathrm{z}_4$	No	Deg 1: $p \equiv 1 \pmod 4$ Deg 2: $p \equiv 3 \pmod 4$
i_6	$\mathrm{i} \cdot \mathrm{r}_6$	No	Deg 1: $p \equiv 1, 5, 7, 11 \pmod{24}$ Deg 2: $p \equiv 13, 17, 19, 23 \pmod{24}$
r_2	$\sqrt{2} = \mathrm{z}_8^7 + \mathrm{z}_8$	Yes	Deg 1: $p \equiv 1, 7 \pmod 8$ Deg 2: $p \equiv 3, 5 \pmod 8$
r_3	$\sqrt{3} = -(\mathrm{z}_{12}^3 + 2\mathrm{z}_{12}^7)$	Yes	Deg 1: $p \equiv 1, 11 \pmod{12}$ Deg 2: $p \equiv 5, 7 \pmod{12}$
r_5	$\sqrt{5} = 2\mathrm{z}_5 + 2\mathrm{z}_5^4 + 1$	Yes	Deg 1: $p \equiv 1, 4 \pmod 5$ Deg 2: $p \equiv 2, 3 \pmod 5$, $p \neq 2$
r_6	$\sqrt{6} = \sqrt{2}\sqrt{3}$ $= -(\mathrm{z}_{24}^3 + \mathrm{z}_{24}^9 + 2\mathrm{z}_{24}^{11} + 2\mathrm{z}_{24}^{17})$	Yes	Deg 1: $p \equiv 1, 5, 19, 23 \pmod{24}$ Deg 2: $p \equiv 7, 11, 13, 17 \pmod{24}$
r_7	$\sqrt{7}$ $= 2\mathrm{z}_{28} + 2\mathrm{z}_{28}^9 + \mathrm{z}_{28}^{21} + 2\mathrm{z}_{28}^{25}$	Yes	Deg 1: $p \equiv 1, 3, 9, 19, 25, 27 \pmod{28}$ Deg 2: $p \equiv 5, 11, 13, 15, 17, 23 \pmod{28}$
y_{17}	$\mathrm{z}_{17} + \mathrm{z}_{17}^{16} = \cos(\frac{2\pi}{17})$	Yes	Deg 1: $p \equiv 1, 16 \pmod{17}$ Deg 2: $p \equiv 4, 13 \pmod{17}$ Deg 4: $p \equiv 2, 8, 9, 15 \pmod{17}$ Deg 8: $p \equiv 3, 5, 6, 7, 10, 11, 12, 14 \pmod{17}$
y_{24}	$\mathrm{z}_{24} + \mathrm{z}_{24}^{23} = \cos(\frac{2\pi}{24})$	Yes	Deg 1: $p \equiv 1, 23 \pmod{24}$ Deg 2: $p \equiv 5, 7, 11, 13, 17, 19 \pmod{24}$

2.2.3 Brauer Characters

We will only give a short introduction to Brauer characters. For a more detailed version see [23, Chp 15, p.262].

Let R be the ring of algebraic integers in $\mathbb{C}$ and let M be a maximal ideal of R containing pR for some fixed prime p. Note that M is not necessarily uniquely determined. Then $K = R/M$ is a field of characteristic p. Let τ be the natural homomorphism from R into K.

Lemma 2.2.13 ([23, Lemma 15.1, p.263])**.**
Let $U = \{\kappa \in \mathbb{C} \mid \kappa^m = 1, m \in \mathbb{Z}$ with $p \nmid m\}$. Then

(i) $U \subseteq R$;

(ii) τ is an injection from U to $K^\times$ and there is an isomorphism between U and the roots of unity of $K^\times$;

(iii) K is algebraically closed and algebraic over its prime field.

Let K be an algebraically closed field of characteristic p and let $\rho : G \to \mathrm{GL}_n(K)$ be a representation of some finite group G. Then for all $g \in G$ the eigenvalues of $g\rho$ lie in $K^\times$ since K is algebraically closed. Let $S = \{g \in G \mid p \nmid |g|\}$ and let $x \in S$. Furthermore, let $\epsilon_1, \ldots, \epsilon_n \in K^\times$ denote the eigenvalues of $x\rho$. Then for all i there exists a unique $u_i \in U = \{\kappa \in \mathbb{C} \mid \kappa^m = 1, m \in \mathbb{Z}$ with $p \nmid m\}$ such that $u_i\tau = \epsilon_i$ by Lemma 2.2.13(ii).

Definition 2.2.14. We call the function $\upsilon(x) = \sum u_i$ a **Brauer character** of G.

Note that if $p \nmid |G|$ then the Brauer characters of G are in fact the ordinary characters of G as given in the ATLAS [12] by [23, Thm 15.6, p.265]. Equivalently, if we consider a representation ρ of G over a finite field $\mathbb{F}_{p^t}$ where $p \nmid |G|$ then the ATLAS contains the Brauer characters of G. If $p \mid |G|$ then the Brauer characters of some non-abelian simple groups G can be found in [24].

3 Classical Groups

There are 4 types of classical groups, namely linear, unitary, symplectic and orthogonal groups which will be defined in this chapter. As the theory of classical groups is quite complex we will only define concepts needed for this thesis. For a more detailed introduction see e.g. Taylor [36].

Note that some of the theory developed in this chapter is not valid for groups of smaller dimension. Therefore we will assume throughout that the dimension of the classical groups is $\geqslant 13$ unless otherwise stated.

We will first define classical groups and their automorphisms. Then we will briefly talk about the Schur indicator of a representation, quotient spaces and various types of maximal subgroups of classical groups. After that we will state Aschbacher's Theorem and give a discussion on why we will work with quasisimple classical groups even though we are interested in finding the maximal subgroups of the almost simple classical groups.

3.1 An Introduction to Classical Groups

Let V be a vector space of dimension $n > 0$ over a field K unless otherwise stated. We will define the forms preserved by the classical groups as maps. We will mention how to write these maps as matrices but will not go into any detail. For more information on the matrix formulation see [8, Section 1.5.1, p.17].

Definition 3.1.1. Let $\sigma \in \mathrm{Aut}(K)$. A map $\beta : V \times V \to K$ is a ***σ*-sesquilinear form** if
(i) $\beta(u + v, w) = \beta(u, w) + \beta(v, w)$,
(ii) $\beta(u, v + w) = \beta(u, v) + \beta(u, w)$, and
(iii) $\beta(\lambda u, \mu v) = \lambda\mu^\sigma\beta(u, v)$
for all $u, v, w \in V$ and all $\lambda, \mu \in K$. If $\beta(u, v) = \beta(v, u)$ then β is **symmetric**. If $\sigma = 1$ then β is **bilinear**.

A map $Q : V \to K$ is a **quadratic form** if
(i) $Q(\lambda v) = \lambda^2 Q(v)$ for all $v \in V$, $\lambda \in K$, and
(ii) $\beta(u, v) := Q(u + v) - Q(u) - Q(v)$ is a symmetric bilinear form for all $u, v \in V$.
We call the bilinear form corresponding to Q the **polar form** of Q.

Note that over a field that does not have even characteristic β and Q uniquely determine each other.

To write β and Q as matrices let $(e_1, \ldots, e_n)$ be a basis of V. Then the matrix of β is $B = (b_{ij})_{n\times n}$ where $b_{ij} = \beta(e_i, e_j)$ for all i, j. The matrix of Q is the upper triangular matrix $A = (a_{ij})$ where $a_{ij} = \beta(e_i, e_j)$ if $i < j$, $a_{ii} = Q(e_i)$ and $a_{ij} = 0$ for all $i > j$.

Definition 3.1.2. Let $v \in V$ be fixed. A σ-sesquilinear form β is **non-degenerate** if $\beta(v, u) = 0$ for all $u \in V$ implies that $v = 0$. A quadratic form Q is non-degenerate if its polar form is non-degenerate.

Note that if we write a σ-sesquilinear map β as a matrix B then B is non-degenerate if and only if $\det(B) \neq 0$.

Definition 3.1.3. Let β be a σ-sesquilinear form on V. A non-zero vector $v \in V$ is **singular** if $\beta(v, v) = 0$. Otherwise, v is **non-singular**. If $v, w \in V$ are singular and $\beta(v, w) = 1$ then $\langle v, w\rangle$ is a **hyperbolic line**.

Definition 3.1.4. Let $g \in \mathrm{GL}(V)$, let β be a σ-sesquilinear form and Q be a quadratic form on V. Then g is an **isometry** of β if $\beta(ug, vg) = \beta(u, v)$ for all $u, v \in V$ and an isometry of Q if $Q(vg) = Q(v)$ for all $v \in V$. If $\beta(ug, vg) = \lambda\beta(u, v)$ or $Q(vg) = \lambda Q(v)$ for some $\lambda \in K\backslash\{0\}$ then g is a **similarity** of β or Q.

Definition 3.1.5. Let V and W be vector spaces over K and let $\phi \in \mathrm{Aut}(K)$. If $f : V \to W$ is a map satisfying $(v + w)f = vf + wf$ and $(\lambda v)f = \lambda^{\phi}(vf)$ for all $v \in V$, $w \in W$ and $\lambda \in K$ then f is a **ϕ-semilinear map**. If f is a ϕ-semilinear map for some ϕ then f is **semilinear**.

Definition 3.1.6. Let f be a ϕ-semilinear map and let β be a σ-sesquilinear and Q be a quadratic form on V. Then f is a **semi-isometry** of β or Q if $\beta(vf, wf) = \beta(v, w)^{\phi}$ or $Q(vf) = Q(v)^{\phi}$ for all $v, w \in V$. If there exists $0 \neq \lambda \in K$ such that $\beta(vf, wf) = \lambda\beta(v, w)^{\phi}$ or $Q(vf) = \lambda Q(v)^{\phi}$ for all $v, w \in V$ then f is a **semi-similarity**.

Definition 3.1.7. Two σ-sesquilinear forms β and κ on V are **isometric** or **equivalent** if there exists $g \in \mathrm{GL}(V)$ such that $\beta(ug, vg) = \kappa(u, v)$ for all $u, v \in V$. If there exist $g \in \mathrm{GL}(V)$ and $\lambda \in K\backslash\{0\}$ such that $\beta(ug, vg) = \lambda\kappa(u, v)$ for all $u, v \in V$ then β and κ are said to be **similar**.

Lemma 3.1.8 ([8, Thm 1.5.13, p.16])**.** *Let β be a σ-sesquilinear form on V. Assume that there exist $\lambda \in K^{\times}$ and $\tau \in \mathrm{Aut}(K)$ such that $\beta(u, v) = \lambda\beta(u, v)^{\tau}$ for all $u, v \in V$. Then up to similarity one of the following holds for all $u, v \in V$.*

(i) $\beta(u, v) = 0$.

(ii) $\sigma = 1$*,* $\lambda = -1$ *and* $\beta(v, v) = 0$*.*
(iii) $\sigma^2 = 1 \neq \sigma$ *and* $\lambda = 1$*, i.e.* $\beta(v, u) = \beta(u, v)^\sigma$*.*
(iv) $\sigma = 1$ *and* $\lambda = 1$*, i.e.* $\beta(v, u) = \beta(u, v)$*.*

In characteristic 2 *Case (ii) also satisfies Case (iv). Otherwise all cases are mutually exclusive.*

Definition 3.1.9. A σ-sesquilinear form β is
(i) an **alternating** or **symplectic form** if β satisfies Case (ii),
(ii) a **σ-Hermitian form** or **unitary form** if β satisfies Case (iii), or
(iii) **symmetric bilinear** in Case (iv).

Note that unitary forms give rise to unitary groups, symplectic forms to symplectic groups and quadratic forms to orthogonal groups. If V is equipped with a σ-sesquilinear form β such that $\beta(u, v) = 0$ for all $u, v \in V$ then we say that V is equipped with the zero form which gives rise to linear groups.

The isometry groups of σ-sesquilinear, symplectic and quadratic forms over finite fields will be considered in a bit more detail in the following sections.

Let $\operatorname{diag}(a, \ldots, a, b, \ldots, b)$ denote a diagonal matrix with $n/2$ a's and $n/2$ b's along the diagonal. Similarly let $\operatorname{antidiag}(a, \ldots, a, b, \ldots, b)$ be an antidiagonal matrix with $n/2$ a's and $n/2$ b's along the antidiagonal.

3.1.1 Unitary Groups

Unitary groups preserve a non-degenerate σ-Hermitian form β, where σ is a field automorphism of order 2. Hence unitary groups only exist over $\mathbb{F}_{q^2}$ and $\sigma : x \mapsto x^q$. Furthermore, we will use $F = I_n$ as our standard unitary form matrix and denote the isometry group of F by $\mathrm{GU}_n(q)$ over $\mathbb{F}_{q^2}$. Note that all isometry groups of non-degenerate σ-Hermitian forms over a given finite field are conjugate.

We will frequently use the superscript $^\pm$, e.g. $\mathrm{GL}_n^\pm(q)$, to denote a linear or unitary group. Here the $+$ sign corresponds to the linear and the $-$ sign to a unitary group. Furthermore, if we use $\mathbb{F}_{q^u}$ then we mean $\mathbb{F}_{q^2}$ if the group on $\mathbb{F}_{q^u}$ is a unitary group and $\mathbb{F}_q$ otherwise.

3.1.2 Symplectic Groups

In this thesis we will use $F = \operatorname{antidiag}(1, \ldots, 1, -1, \ldots, -1)$ as our standard symplectic form matrix. Note that the isometry groups of any two non-degenerate symplectic forms on V over $\mathbb{F}_q$ are conjugate. We will denote the isometry group of F by $\mathrm{Sp}_n(q)$. Note that symplectic groups only exist in even dimension.

3.1.3 Orthogonal Groups in Odd Dimension

We will first consider isometry groups of quadratic forms over $\mathbb{F}_q$ in odd dimension.

Theorem 3.1.10 ([36, Thm 11.9, p.143])**.** *Let Q be a quadratic form on $\mathbb{F}_q^n$. If q is even and n is odd then the isometry group of Q is isomorphic to a symplectic group of dimension $n-1$.*

Hence we will only consider odd dimensional orthogonal groups in odd characteristic in which case it suffices to define the polar form of Q. We will use $F = I_n$ as our standard non-degenerate symmetric bilinear form matrix and denote the isometry group of F by $\mathrm{GO}_n^\circ(q)$. Note that there are two isometry classes of non-degenerate symmetric bilinear forms in odd dimension depending on whether the determinant of the form matrix is square or non-square in $\mathbb{F}_q^\times$. These two isometry classes lie in the same similarity class.

3.1.4 Orthogonal Groups in Even Dimension

In even dimension there are two isometry classes of non-degenerate quadratic forms which lie in two distinct similarity classes. Let β be the polar form of Q. Then β and Q uniquely determine each other over a field of odd characteristic.

Definition 3.1.11. Let q be odd. Let β be a non-degenerate symmetric bilinear form over $\mathbb{F}_q$ in even dimension. Then β has **plus-type** if it is isometric to our standard form matrix $\mathrm{antidiag}(1, \ldots, 1)$. Otherwise β has **minus-type**. Similarly, a non-degenerate quadratic form Q in even dimension and even characteristic is of plus-type if Q is isometric to our standard form matrix $\mathrm{antidiag}(1, \ldots, 1, 0, \ldots, 0)$ and of minus-type otherwise. We denote the isometry group of a standard form of plus-type by $\mathrm{GO}_n^+(q)$ and the isometry group of a standard form of minus-type (see Table 3.1.1 on p.25) by $\mathrm{GO}_n^-(q)$.

Note that if we want to talk about an orthogonal group of plus- and minus-type at the same time, we use the superscript $^\pm$, e.g. $\mathrm{GO}_n^\pm(q)$. Furthermore, if an orthogonal group does not preserve our standard form matrices as defined above but some other non-degenerate symmetric bilinear or quadratic form B then we will denote this group by $\mathrm{GO}_n^{\pm}(q, B)$. Furthermore, if we want to talk about an arbitrary orthogonal group in even or odd dimension we will use the the superscript $^{\epsilon}$, e.g. $\mathrm{GO}_n^{\epsilon}(q)$, where $\epsilon \in \{\circ, +, -\}$.

We will need to introduce the concept of the discriminant of a form matrix as this will enable us to determine whether an orthogonal group in even dimension and odd characteristic is of plus- or minus-type.

Definition 3.1.12. Let q be odd and let β be a non-degenerate symmetric bilinear form over $\mathbb{F}_q^n$ with form matrix B. Then β has **square discriminant** if $\det(B)$ is square in $\mathbb{F}_q^{\times}$ and **non-square discriminant** otherwise.

Lemma 3.1.13 ([8, Thm 1.5.42(ii), p.24])**.** *Let n be even and q be odd and suppose that $\mathrm{GO}_n^{\pm}(q, B)$ preserves a non-degenerate symmetric bilinear form with matrix B. Then $\mathrm{GO}_n^{\pm}(q, B)$ is of plus-type if and only if either* $\det(B)$ *is square in $\mathbb{F}_q^{\times}$ and $n(q-1)/4$ is even or* $\det(B)$ *is not a square in $\mathbb{F}_q^{\times}$ and $n(q-1)/4$ is odd. Otherwise it is of minus-type.*

3.1.5 Standard Forms and Definitions of Classical Groups

Let G be the isometry group of a zero, a σ-Hermitian, a symplectic or a quadratic form on $\mathbb{F}_q^n$ as described in the previous sections. The following table, Table 3.1.1, gives the form matrices preserved by G that we will use as our **standard form matrices**. It is taken from [8, Table 1.1, p.25]. In odd characteristic we will give the matrix of the polar form of a quadratic form instead of the form matrix of the quadratic form itself. Note that μ in the last line of the table is such that the polynomial $x^2 + x + \mu$ is irreducible over $\mathbb{F}_q$. Let $m = n/2$.

Table 3.1.1: Standard Classical Forms

Case	**Conditions**	**Form Type**	**Isom.Gp**	**Form**
L	—	zero	$\mathrm{GL}_n(q)$	$0_{n\times n}$
U	—	σ-Hermitian	$\mathrm{GU}_n(q)$	I_n
S	—	alternating	$\mathrm{Sp}_n(q)$	$\mathrm{antidiag}(1, \dots, 1, -1, \dots, -1)$
$\mathbf{O}^{\circ}$	qn odd	symmetric	$\mathrm{GO}_n^{\circ}(q)$	I_n
$\mathbf{O}^{+}$	q odd, n even	symmetric	$\mathrm{GO}_n^{+}(q)$	$\mathrm{antidiag}(1, \dots, 1)$
$\mathbf{O}^{-}$	q odd, n even	symmetric	$\mathrm{GO}_n^{-}(q)$	I_n if $n \equiv 2 \pmod 4$, $q \equiv 3 \pmod 4$ $\mathrm{diag}(\omega, 1, \dots, 1)$ otherwise
$\mathbf{O}^{+}$	q, n even	quadratic	$\mathrm{GO}_n^{+}(q)$	$\mathrm{antidiag}(1, \dots, 1, 0, \dots, 0)$
$\mathbf{O}^{-}$	q, n even	quadratic	$\mathrm{GO}_n^{-}(q)$	$\mathrm{antidiag}(1, \dots, 1, 0, \dots, 0)$ $+ \mathrm{E}_{m,m} + \mu\mathrm{E}_{m+1,m+1}$

Let $V = \mathbb{F}_{q^u}^n$ for some q, some n and some $u \in \{1, 2\}$. Suppose that V is equipped with either a zero, a unitary or an alternating form β or a quadratic form Q. Let $u = 2$ if V is equipped with a unitary form and let $u = 1$ otherwise. In general we will define a series of subgroups corresponding to each of the β (and Q). Let

$$\Omega \leqslant S \leqslant G \leqslant C \leqslant \Gamma \leqslant \mathrm{A}. \tag{3.1.1}$$

Then G is the group of isometries of β (or Q) as defined in the previous sections. The special group $S \trianglelefteq G$ consists of all determinant 1 matrices preserving β (or Q). The commutator subgroup of S is denoted by Ω. Note that $S = \Omega$ except in the orthogonal case. We will discuss the orthogonal case in the next section. Furthermore, let C be the similarity group of V. We can show that $C = \mathrm{N}_{\mathrm{GL}_n(q)}(\Omega)$. Let Γ be the group of all semi-isometries of β (Q). If β is identically zero then the split extension of Γ by the inverse transpose map $\gamma = -\mathrm{T}$ equals A. Otherwise $A = \Gamma$ since $n \geqslant 13$.

We will now consider the groups Ω. The automorphisms of G are then defined in Section 3.2.

3.1.6 The Quasisimple Classical Groups

In the linear, unitary and symplectic case Ω equals the respective special group. We will now consider the orthogonal case.

Let $\epsilon \in \{\circ, +, -\}$. Let $\mathrm{GO}_n^\epsilon(q)$ be the isometry group preserving our standard forms as in Table 3.1.1. Unlike in the linear, unitary or symplectic case, the special orthogonal group $\mathrm{SO}_n^\epsilon(q)$ is never isomorphic to Ω in dimension $\geqslant 13$. Instead Ω is a normal subgroup of $\mathrm{SO}_n^\epsilon(q)$ of index 2, which we will denote by $\Omega_n^\epsilon(q)$. In the following we will describe $\Omega_n^\epsilon(q)$ and give a way to determine whether an element of $\mathrm{SO}_n^\epsilon(q)$ lies in $\Omega_n^\epsilon(q)$.

For this we will need to define the concept of the spinor norm and the quasideterminant of an element.

Definition 3.1.14. Let V be a vector space equipped with a quadratic form Q and polar form β. Let $v \in V$ such that $Q(v) \neq 0$. Then the map $\mathrm{r}_v : V \mapsto V$ defined by $(x)\mathrm{r}_v = x - Q(v)^{-1}\beta(v, x)v$ is called a **reflection**.

Lemma 3.1.15 ([26, Prop 2.5.6, p.28])**.** *Let Q be a non-degenerate quadratic form. If $\mathrm{GO}_n^\epsilon(q, Q) \ncong \mathrm{GO}_4^+(2)$ then $\mathrm{GO}_n^\epsilon(q, Q)$ is generated by reflections.*

Definition 3.1.16. Let Q be a non-degenerate quadratic form and let β be the polar form of Q. Let $g = \prod_{i=1}^k \mathrm{r}_{v_i} \in \mathrm{GO}_n^\epsilon(q, Q)$, where $\mathrm{GO}_n^\epsilon(q, Q) \ncong \mathrm{GO}_4^+(2)$. If q is odd then the **spinor norm** of g, $\mathbf{sp}(\boldsymbol{g})$, is $+1$ if $\prod_{i=1}^k \beta(v_i, v_i)$

is a square in $\mathbb{F}_q^\times$ and -1 otherwise. If q is even then the **quasideterminant** of g is $+1$ if k is even and -1 otherwise.

Lemma 3.1.17 ([36, Thm 11.50, p.164])**.** *The spinor norm map and the quasideterminant map are homomorphisms.*

Definition 3.1.18. Let $\boldsymbol{\Omega_n^\epsilon(q)}$ be the subgroup of $\mathrm{GO}_n^\epsilon(q)$ consisting of all elements of $\mathrm{SO}_n^\epsilon(q)$ with spinor norm 1 in odd characteristic and of all elements of quasideterminant 1 in even characteristic.

The following lemma will help to calculate the spinor norm or quasideterminant of elements in $\mathrm{GO}_n^\epsilon(q)$.

Lemma 3.1.19 ([8, Prop 1.6.11, p.28])**.** *Let B be a non-degenerate bilinear form matrix in odd characteristic or a non-degenerate quadratic form matrix in even characteristic and let $g \in \mathrm{GO}_n^\epsilon(q, B)$. Let $A := I_n - g$ and let $k := \mathrm{rank}(A)$. If q is odd, let B be the the matrix preserved by $\mathrm{SO}_n^\epsilon(q, B)$. Then the matrix M over $\mathbb{F}_q$ with rows forming a basis of a complement of the nullspace of A has dimension $k \times n$. Furthermore,*

(i) if q is even and $g \notin \mathrm{GO}_4^+(2, B)$ then g has quasideterminant 1 if k is even and -1 otherwise;

(ii) if q is odd and $\det(MABM^{\mathrm{T}})$ is a square in $\mathbb{F}_q^\times$ then the spinor norm if g is 1. If $\det(MABM^{\mathrm{T}})$ is not square then the spinor norm is -1.

Lemma 3.1.20. *Let q be odd and let $\lambda \in \mathbb{F}_q^\times$. Let $g \in \mathrm{GO}_n^\pm(q, B)\backslash\mathrm{SO}_n^\pm(q, B)$ for some non-degenerate symmetric bilinear form matrix B. If λ is square in $\mathbb{F}_q^\times$, then $\mathrm{sp}(g, \lambda B) = \mathrm{sp}(g, B)$, where $\mathrm{sp}(g, \lambda B)$ is the spinor norm of g with respect to λB. If λ is non-square, then $\mathrm{sp}(g, \lambda B) = -\mathrm{sp}(g, B)$. The spinor norm of elements of $\mathrm{SO}_n^\pm(q, B)$ is well defined.*

Proof. Let β be the form associated with B. Then $\mathrm{GO}_n^\pm(q, B)$ also preserves $\lambda\beta$ for any $\lambda \in \mathbb{F}_q^\times$. Let $g \in \mathrm{GO}_n^\pm(q, B)$ and suppose that $g = \prod_{i=1}^k \mathrm{r}_{v_i}$, where r_{v_i} is a reflection for all i. Then the spinor norm of g equals 1 if and only if $\prod_{i=1}^k \beta(v_i, v_i)$ is a square in $\mathbb{F}_q^\times$ by Definition 3.1.19. Equivalently, the spinor norm of g is 1 if and only if an even number of these $\beta(v_i, v_i)$'s are non-square. Let λ be non-square. Then $\beta(v, v)$ is square if and only if $\lambda\beta(v, v)$ is non-square.

If $g \in \mathrm{GO}_n^\pm(q, B)\backslash\mathrm{SO}_n^\pm(q, B)$ then g is generated by an odd number of reflections. If $\mathrm{sp}(g) = 1$ then an even number of the $\beta(v_i, v_i)$'s is non-square and an odd number square. However, if we consider $\lambda\beta(v_i, v_i)$ then $\mathrm{sp}(g) = -1$ as we now have an odd number of $\lambda\beta(v_i, v_i)$'s that are non-square.

Since $\Omega_n^{\pm}(q,B)$ is simple and $\mathrm{SO}_n^{\pm}(q,B)$ only has one subgroup of index 2 it follows that the spinor norm map is well-defined on elements of $\mathrm{SO}_n^{\pm}(q,B)$. $\square$

We will now determine the spinor norm of $-I_n \in \Omega_n^{\pm}(q,B)$, for any non-degenerate symmetric bilinear form matrix B. The result follows directly from Lemma 3.1.19.

Lemma 3.1.21. *The element $-I \in \Omega_n^{+}(q,B)$ if and only if $4 \mid n$ or ($n \equiv 2 \pmod 4$ and $q \equiv 1 \pmod 4$) and $-I \in \Omega_n^{-}(q,B)$ if and only if $n \equiv 2 \pmod 4$ and $q \equiv 3 \pmod 4$.*

Lemma 3.1.22 ([26, Thm 2.1.3, p.16])**.** *Let $\Omega \in \{\mathrm{SL}_n^{\pm}(q), \mathrm{Sp}_n(q), \Omega_n^{\epsilon}(q)\}$. In dimension $\geqslant 13$ all such Ω are quasisimple.*

Finally, note that we will denote the projective version of Ω either by $\overline{\Omega}$ or, if we talk about a specific quasisimple group, we will use $\mathrm{L}_n(q)$, $\mathrm{U}_n(q)$, $\mathrm{S}_n(q)$ or $\mathrm{O}_n^{\epsilon}(q)$.

3.2 Outer Automorphisms of Classical Groups

In general there are three possible types of automorphisms that generate the outer automorphism groups of projective simple classical groups $\bar{\Omega}$, where $\bar{\Omega} \in \{\mathrm{L}_n^{\pm}(q), \mathrm{S}_n(q), \mathrm{O}_n^{\epsilon}(q)\}$. These are the diagonal, graph (or duality) and field automorphisms. (See [8, Section 1.7, p.32] for more details.) Note that we will define these outer automorphisms with respect to our standard classical forms as in Table 3.1.1 (p.25). Here, (a,b) denotes the greatest common divisor of a and b.

Definition 3.2.1. We call the matrices inducing the outer automorphisms that are given in this section our **standard representative matrices** for their respective outer automorphisms.

Note that with the exception of the orthogonal groups in even dimension the field automorphisms are generally generated by an outer automorphism ϕ which acts on the matrices $(a_{ij})_{n\times n}$ by sending each entry to its p^{th}-power, where p is the characteristic of the underlying field $\mathbb{F}_q$. In other words, ϕ sends every matrix $(a_{ij})_{n\times n}$ to $(a_{ij}^p)_{n\times n}$. Therefore, ϕ replaces each eigenvalue by its p^{th}-power.

Assume throughout that $n \geqslant 13$ to avoid exceptions in smaller dimensions and let $q = p^e$.

3.2.1 Case L:

Diagonal automorphisms: The diagonal automorphism δ is induced by conjugation by the diagonal matrix $\mathrm{diag}(\omega, 1, 1, \ldots, 1) \in \mathrm{GL}_n(q)$ where ω is a primitive element of $\mathbb{F}_q^\times$. In $\mathrm{Out}(\mathrm{L}_n(q))$ it has order $(q-1, n)$ and projectively extends $\mathrm{L}_n(q)$ to $\mathrm{PGL}_n(q)$.

Field automorphisms: Here ϕ has order e in $\mathrm{Out}(\mathrm{L}_n(q))$ and projectively ϕ extends $\mathrm{PGL}_n(q)$ to $\mathrm{P\Gamma L}_n(q)$.

Graph automorphisms: The graph or duality automorphism γ acts on elements $A \in \mathrm{GL}_n(q)$ by sending them to their inverse transposes, i.e $A^\gamma = (A^{-1})^{\mathrm{T}}$. Projectively it has order 2 and extends $\mathrm{P\Gamma L}_n(q) \cong \langle \mathrm{L}_n(q), \delta, \phi \rangle$ to $\mathrm{Aut}(\mathrm{L}_n(q))$.

3.2.2 Case U:

Diagonal automorphisms: The diagonal automorphism δ is induced by conjugation by the matrix $\mathrm{diag}(\omega_{q^2}^{q-1}, 1, 1, \ldots, 1) \in \mathrm{GU}_n(q)$, where ω_{q^2} is a primitive element of $\mathbb{F}_{q^2}^\times$. Its order in $\mathrm{Out}(\mathrm{U}_n(q))$ is $(q+1, n)$ and projectively it extends $\mathrm{U}_n(q)$ to $\mathrm{PGU}_n(q)$.

Field automorphisms: Here ϕ has order $2e$ in $\mathrm{Out}(\mathrm{U}_n(q))$ and projectively extends $\mathrm{PGU}_n(q)$ to $\mathrm{P\Gamma L}_n(q) = \mathrm{Aut}(\mathrm{U}_n(q))$.

Graph automorphisms: The graph (or duality) automorphism γ acts as inverse transpose on the elements of $\mathrm{GU}_n(q)$. Projectively it has order 2. Because of our choice of standard form, $\gamma = \phi^e$.

3.2.3 Case S:

Diagonal automorphisms: If q is even then the diagonal automorphism δ is trivial. If q is odd, then δ is induced by conjugation by the diagonal matrix $\mathrm{diag}(\omega, \ldots, \omega, 1, \ldots, 1) \in \mathrm{CSp}_n(q) \backslash \mathrm{Sp}_n(q)$, where ω is a primitive element of $\mathbb{F}_q^\times$. Projectively it extends $\mathrm{S}_n(q)$ to $\mathrm{PCSp}_n(q)$ and has order 2 in $\mathrm{Out}(\mathrm{S}_n(q))$.

Field automorphisms: Here the field automorphism ϕ has order e in $\mathrm{Out}(\mathrm{S}_n(q))$ and projectively extends $\mathrm{PCSp}_n(q)$ to $\mathrm{PC\Gamma Sp}_n(q) = \mathrm{Aut}(\mathrm{S}_n(q))$ when $n \geqslant 13$.

Graph automorphisms: There are no graph automorphisms of $\mathrm{S}_n(q)$ in dimension $\geqslant 13$.

3.2.4 Case O°:

Diagonal automorphisms: The diagonal automorphism δ is induced by an element of $\mathrm{SO}_n^{\circ}(q)\backslash\Omega_n^{\circ}(q)$. Projectively it extends $\mathrm{O}_n^{\circ}(q)$ to $\mathrm{PSO}_n^{\circ}(q)$ and has order 2 in $\mathrm{Out}(\mathrm{O}_n^{\circ}(q))$.

Field automorphisms: Here ϕ has order e in $\mathrm{Out}(\mathrm{O}_n^{\circ}(q))$ and projectively extends $\mathrm{PSO}_n^{\circ}(q) = \mathrm{PCGO}_n^{\circ}(q)$ to $\mathrm{PC\Gamma O}_n^{\circ}(q) = \mathrm{Aut}(\mathrm{O}_n^{\circ}(q))$.

Graph automorphisms: There are no graph automorphisms in this case.

3.2.5 Case $\mathrm{O}^{\pm}$:

Let F be our standard non-degenerate symmetric bilinear form or our standard quadratic form preserved by $\Omega_n^{\pm}(q)$ as in Table 3.1.1 (p.25).

Diagonal automorphisms: When q is even then the diagonal automorphisms are trivial.

Assume that q is odd. The diagonal automorphism δ' exists only when F has square discriminant. It is induced by an element of $\mathrm{SO}_n^{\pm}(q)\backslash\Omega_n^{\pm}(q)$ and projectively extends $\mathrm{O}_n^{\pm}(q)$ to $\mathrm{PSO}_n^{\pm}(q)$. Furthermore it has order 2 in $\mathrm{Out}(\mathrm{O}_n^{\pm}(q))$.

The diagonal automorphism δ extends $\mathrm{PGO}_n^{\pm}(q)$ to $\mathrm{PCGO}_n^{\pm}(q)$ projectively and exists for all orthogonal groups in even dimension and odd characteristic. Depending on whether $\Omega_n^{\pm}(q)$ preserves a form of plus- or minus-type, it has slightly different properties.

We will first consider $\Omega = \Omega_n^{+}(q)$. Then δ is induced by the element $\delta = \mathrm{diag}(\omega, \ldots, \omega, 1, \ldots, 1) \in \mathrm{CGO}_n^{+}(q)\backslash\mathrm{GO}_n^{+}(q)$, where ω is a primitive element of $\mathbb{F}_q^{\times}$. Note that $\det(\delta) = \omega^{n/2}$ and that $\delta F \delta^{\mathrm{T}} = \omega F$. We can show that if F has non-square discriminant then δ has order 2 in $\mathrm{Out}(\bar{\Omega})$. If $n \equiv 2 \pmod 4$ and F has square discriminant then δ has order 4.

Now let $\Omega = \Omega_n^{-}(q)$. Then δ depends on the discriminant of F. Let $a, b \in \mathbb{F}_q$ such that $a^2 + b^2 = \omega$ for some primitive element ω of $\mathbb{F}_q^{\times}$. Let $X = \left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$ and $Y = \left(\begin{smallmatrix} 0 & \omega \\ -1 & 0 \end{smallmatrix}\right)$. If F has square discriminant (i.e. if $F = I_n$) then δ is induced by $\delta = \mathrm{diag}(X, \ldots, X)$ whereas if F has non-square discriminant (i.e. if $F = \mathrm{diag}(\omega, 1, \ldots, 1)$) then $\delta = \mathrm{diag}(Y, X, \ldots, X)$. Similarly to the case $\Omega_n^{+}(q)$ we have $\delta F \delta^{\mathrm{T}} = \omega F$ and $\det(\delta) = \omega^{n/2}$. We can show that δ has order 4 in $\mathrm{Out}(\bar{\Omega})$ if F has square discriminant and order 2 otherwise.

Graph automorphisms: Here the graph automorphism γ has order 2 in $\mathrm{Out}(\Omega_n^{\pm}(q))$. If q is odd then γ is induced by an element of $\mathrm{GO}_n^{\pm}(q)\backslash\mathrm{SO}_n^{\pm}(q)$ and projectively extends $\mathrm{PSO}_n^{\pm}(q)$ to $\mathrm{PGO}_n^{\pm}(q)$. If q is even then γ is induced by an element of $\mathrm{SO}_n^{\pm}\backslash\Omega_n^{\pm}(q)$ and projectively extends $\mathrm{O}_n^{\pm}(q)$ to $\mathrm{PSO}_n^{\pm}(q) = \mathrm{PGO}_n^{\pm}(q)$. In both cases γ is induced by a reflection with spinor norm 1.

Field automorphisms: There are two different field automorphisms ϕ and φ depending on Ω and F.

If $\Omega = \Omega_n^+(q)$ or if $\Omega = \Omega_n^-(q)$ with q odd and F of square discriminant then ϕ is defined by sending every matrix $(a_{ij})_{n\times n}$ to $(a_{ij}^p)_{n\times n}$, where $(a_{ij})_{n\times n} \in \mathrm{GO}_n^{\pm}(q)$. It is undefined otherwise. Projectively, ϕ extends $\mathrm{PCGO}_n^{\pm}(q)$ to $\mathrm{PC\Gamma O}_n^{\pm}(q)$ and has order e in $\mathrm{Out}(\bar{\Omega})$.

If $\Omega = \Omega_n^-(q)$ with q odd and F of non-square discriminant then $F^\phi \neq F$ and therefore $\Omega^\phi \neq \Omega$. Let $f = \mathrm{diag}(\omega^{(p-1)/2}, 1, \ldots, 1)$. Then $fFf^{\mathrm{T}} = F^\phi$ which implies that $(\Omega)^{\phi f} = \Omega$. Hence we define the field automorphism φ to be ϕ followed by conjugation by f. Furthermore, $\varphi^e = \gamma$ in $\mathrm{Out}(\bar{\Omega})$. Projectively, φ extends $\mathrm{PCGO}_n^-(q)$ to $\mathrm{PC\Gamma O}_n^-(q)$. Note that if $q = p^e$ with q and e odd then by [7] there exists a form matrix that is fixed by ϕ. Since δ and γ have order 2, ϕ centralises both δ and γ. Furthermore, ϕ has order e and projectively extends $\mathrm{PCGO}_n^-(q)$ to $\mathrm{PC\Gamma O}_n^-(q)$.

Similarly, if $\Omega = \Omega_n^-(2^e)$ with e even then we can find a matrix f such that $\Omega^{\phi f} = \Omega$ and we define φ to be ϕ followed by conjugation by f (see [26, Section 2.8, p.36] for more details.) Again, $\varphi^e = \gamma$ in $\mathrm{Out}(\bar{\Omega})$ and projectively φ extends $\mathrm{PCGO}_n^-(2^e) = \mathrm{PGO}_n^-(2^e)$ to $\mathrm{PC\Gamma O}_n^-(2^e)$. Note that if e is odd then we can find a form matrix of Ω that is stabilised by ϕ by [7]. Then ϕ extends $\mathrm{PCGO}_n^-(2^e) = \mathrm{PGO}_n^-(2^e)$ to $\mathrm{PC\Gamma O}_n^-(2^e)$.

3.2.6 Presentations of Out($\bar{\Omega}$)

The following presentations of the outer automorphism groups of simple projective classical groups preserving our standard forms are taken from [8, p.36/37]. Let $q = p^e$ and assume that $n \geqslant 13$.

Case L:

$$\langle \delta, \gamma, \phi \mid \delta^{(q-1,n)} = \gamma^2 = \phi^e = [\gamma, \phi] = 1,\ \delta^\gamma = \delta^{-1},\ \delta^\phi = \delta^p \rangle.$$

Case U:

$$\langle \delta, \phi, \gamma \mid \delta^{(q+1,n)} = \phi^{2e} = \gamma^2 = 1,\ \delta^\gamma = \delta^{-1},\ \phi^e = \gamma,\ \delta^\phi = \delta^p \rangle.$$

Case S:

$$\langle \delta, \phi \mid \delta^{(q-1,2)} = \phi^e = [\delta, \phi] = 1 \rangle.$$

Case $\mathbf{O}^\circ$ (p odd):

$$\langle \delta, \phi \,|\, \delta^2 = \phi^e = [\delta, \phi] = 1 \rangle.$$

Case $\mathbf{O}^\pm$:

$\mathrm{O}_n^+(2^e)$:

$$\langle \gamma, \phi \,|\, \gamma^2 = \phi^e = [\gamma, \phi] = 1 \rangle.$$

$\mathrm{O}_n^-(2^e)$:

$$\langle \gamma, \varphi \,|\, \gamma^2 = 1, \varphi^e = \gamma \rangle.$$

$\mathrm{O}_n^+(p^e)$, $4 \mid n$, p odd:

$$\langle \delta', \gamma, \delta, \phi \,|\, \delta'^2 = \gamma^2 = \delta^2 = 1, (\delta\gamma)^2 = \delta', \phi^e = [\delta, \phi] = [\gamma, \phi] = 1 \rangle.$$

$\mathrm{O}_n^+(p^e)$, $n \equiv 2 \pmod 4$, $p^e \equiv 1 \pmod 4$:

$$\langle \delta', \gamma, \delta, \phi \,|\, \delta'^2 = \gamma^2 = 1, \delta^2 = \delta', \delta^\gamma = \delta^{-1}, \phi^e = [\gamma, \phi] = 1, \delta^\phi = \delta^p \rangle.$$

$\mathrm{O}_n^+(p^e)$, $n \equiv 2 \pmod 4$, $p^e \equiv 3 \pmod 4$:

$$\langle \gamma, \delta, \phi \,|\, \gamma^2 = \delta^2 = [\delta, \gamma] = \phi^e = [\gamma, \phi] = [\delta, \phi] = 1 \rangle.$$

$\mathrm{O}_n^-(p^e)$, $4 \mid n$ or ($p^e \equiv 1 \pmod 4$ and e even):

$$\langle \gamma, \delta, \varphi \,|\, \gamma^2 = \delta^2 = [\delta, \gamma] = [\delta, \varphi] = 1, \varphi^e = \gamma \rangle.$$

$\mathrm{O}_n^-(p^e)$, $n \equiv 2 \pmod 4$, $p^e \equiv 1 \pmod 4$ and e odd:

$$\langle \gamma, \delta, \phi \,|\, \gamma^2 = \delta^2 = \phi^e = [\delta, \gamma] = [\delta, \phi] = [\gamma, \phi] = 1 \rangle.$$

$\mathrm{O}_n^-(p^e)$, $n \equiv 2 \pmod 4$, $p^e \equiv 3 \pmod 4$:

$$\langle \delta', \gamma, \delta, \phi \,|\, \delta'^2 = \gamma^2 = 1, \delta^2 = \delta', \delta^\gamma = \delta^{-1}, \phi^e = [\gamma, \phi] = [\delta, \phi] = 1 \rangle.$$

Note that the above presentations imply that if q is odd and $\delta^2 = 1$ then $\langle \delta, \gamma \rangle \cong \mathrm{C}_2 \times \mathrm{C}_2$. Whereas if q is odd and $\delta^2 = \delta' \neq 1$ then $\langle \delta, \gamma \rangle \cong \mathrm{D}_8$.

3.2.7 Useful Properties of Automorphisms

In this section we will state some properties of automorphisms of classical groups that will be useful later on. We will first consider the conformal groups which play an important part in this thesis.

Let $\Omega \in \{\mathrm{SL}_n^{\pm}(q), \mathrm{Sp}_n(q), \Omega_n^{\epsilon}(q)\}$. Then the conformal group C equals $\mathrm{N}_{\mathrm{GL}_n(q^u)}(\Omega)$ by [8, Lemma 1.8.9, p.41]. The following table, Table 3.2.1, gives Ω, the conformal group C of Ω and the size of C for each type of classical group. This table was compiled using the information in [8, Section 1.7, p.32]. Let $\delta, \gamma \in \mathrm{Out}(\bar{\Omega})$ and denote their matrix representations by δ and γ respectively.

Table 3.2.1: Conformal groups

Case	$\boldsymbol{\Omega}$	$\boldsymbol{C}$	$\lvert\boldsymbol{C}\rvert$
L	$\mathrm{SL}_n(q)$	$\mathrm{GL}_n(q) = \langle \mathrm{SL}_n(q), \mathrm{Z}(\mathrm{GL}_n(q)), \delta\rangle$	$\lvert\mathrm{GL}_n(q)\rvert$
U	$\mathrm{SU}_n(q)$	$\langle \mathrm{SU}_n(q), \mathrm{Z}(\mathrm{GL}_n(q^2)), \delta\rangle$	$\frac{\lvert\mathrm{GU}_n(q)\rvert(q^2-1)}{(q+1)}$ $= \lvert\mathrm{GU}_n(q)\rvert(q-1)$
S	$\mathrm{Sp}_n(q)$	$\langle \mathrm{Sp}_n(q), \mathrm{Z}(\mathrm{GL}_n(q)), \delta\rangle$	$\lvert\mathrm{Sp}_n(q)\rvert(q-1)$
$\mathbf{O}^{\circ}$	$\Omega_n^{\circ}(q)$	$\langle \Omega_n^{\circ}(q), \mathrm{Z}(\mathrm{GL}_n(q)), \delta\rangle$	$\frac{\lvert\mathrm{GO}_n^{\circ}(q)\rvert(q-1)}{2}$
$\mathbf{O}^{\pm}$	$\Omega_n^{\pm}(q)$	$\langle \Omega_n^{\pm}(q), \mathrm{Z}(\mathrm{GL}_n(q)), \delta, \gamma\rangle$	$\lvert\mathrm{GO}_n^{\pm}(q)\rvert(q-1)$

Lemma 3.2.2. *Let $A \in \mathrm{GL}_n(\mathbb{C})$, $n \geqslant 3$, be of order m. Then duality sends the trace of A to its complex conjugate.*

Proof. By Lemma 2.2.7, $\mathrm{Trace}(A) = \sum \lambda_i \mathrm{z}_m^i$ for some roots of unity $\mathrm{z}_m^i \in \mathbb{C}$ with multiplicity λ_i. Then $\mathrm{Trace}(A^{\mathrm{T}}) = \sum \lambda_i \mathrm{z}_m^i$ and $\mathrm{Trace}(A^{-\mathrm{T}}) = \sum \lambda_i \mathrm{z}_m^{-i} = \sum \lambda_i \overline{\mathrm{z}_m^i} = \overline{\sum \lambda_i \mathrm{z}_m^i}$, where $\bar{a}$ denote the complex conjugate of $a \in \mathbb{C}$. ☐

Lemma 3.2.3. *Let q be odd, let $g \in \mathrm{GO}_n^+(q, F)$ and let $\lambda \in \mathbb{F}_q^{\times}$ be non-square. Then $\mathrm{sp}(g^{\delta}, F) = \mathrm{sp}(g, \lambda F)$, where $\mathrm{sp}(g, F)$ is the spinor norm of g with respect to F.*

Proof. If $|\delta'| = 2$ then we know that $\mathrm{GO}_n^+(q, F)$ has 3 subgroups of index 2, namely $\mathrm{SO}_n^+(q, F) = \Omega_n^+(q, F).\langle\delta'\rangle$, $\Omega_n^+(q, F).\langle\gamma\rangle$ and $\Omega_n^+(q, F).\langle\gamma\delta'\rangle$. If $g \in \mathrm{SO}_n^+(q, F)$ then $\mathrm{sp}(g, F) = \mathrm{sp}(g, \lambda F)$ by Lemma 3.1.20. Furthermore, since $\Omega_n^+(q, F) \trianglelefteq \mathrm{CGO}_n^+(q, F)$ and $\mathrm{SO}_n^+(q, F) \trianglelefteq \mathrm{CGO}_n^+(q, F)$ it follows that $\mathrm{sp}(g^{\delta}, F) = \mathrm{sp}(g, F)$.

Hence it remains to show that if $g \in \Omega_n^+(q,F).\langle\gamma\rangle\backslash\Omega_n^+(q,F)$ or $g \in \Omega_n^+(q,F).\langle\gamma\delta'\rangle\backslash\Omega_n^+(q,F)$ then $\mathrm{sp}(g^\delta,F) = \mathrm{sp}(g,\lambda F)(= -\mathrm{sp}(g,F))$. I.e. we want to show that $(\Omega_n^+(q,F).\langle\gamma\rangle)^\delta = \Omega.\langle\gamma\delta'\rangle$. If $4 \mid n$ then $\delta\gamma\delta\gamma = \delta'$ which implies that $\gamma^\delta = \delta^2\delta^{-1}\gamma\delta = \delta'\gamma$. If $n \equiv 2 \pmod 4$ and $q \equiv 1 \pmod 4$ then $\gamma\delta\gamma = \delta^{-1}$ from which it follows that $\delta^{-1}\gamma\delta = \gamma\delta\gamma\gamma\delta = \gamma\delta'$.

Finally, if $|\delta'| = 1$ then $\mathrm{GO}_n^+(q,F)$ has only one index 2 subgroup, namely $\mathrm{SO}_n^+(q,F)$ and $\mathrm{sp}(g,F) = \mathrm{sp}(g,\lambda F) = \mathrm{sp}(g^\delta,F)$ for all $g \in \mathrm{GO}_n^+(q,F)$. $\square$

The following lemma will be particularly useful in Chapter 4.

Lemma 3.2.4 ([8, Lemma 1.8.10, p.41])**.**

(i) Let $H, G \leqslant \mathrm{GU}_n(q)$ such that they are both absolutely irreducible and conjugate in $\mathrm{GL}_n(q^2)$. Then G and H are also conjugate in $\mathrm{GU}_n(q)$.

(ii) If G and H are two absolutely irreducible subgroups of $\mathrm{Sp}_n(q)$ or $\mathrm{GO}^\epsilon(q)$ that are conjugate in $\mathrm{GL}_n(q)$ then G and H are conjugate in $\mathrm{CSp}_n(q)$ or $\mathrm{CGO}_n^\epsilon(q)$ respectively.

3.3 Schur Indicator and Quotient Space

Related to classical groups is the Schur indicator of a representation.

Definition 3.3.1. Let $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ be an absolutely irreducible representation of some finite group G. Then the **Schur indicator** indicates which form $G\rho$ preserves. If $G\rho$ preserves a zero or a unitary form then the Schur indicator is $\circ$. If $G\rho$ preserves a symplectic form then the Schur indicator is $-$ and if $G\rho$ is a subgroup of an orthogonal group then the Schur indicator is $+$.

We will now define the quotient space of a vector space and show how a symmetric bilinear form can act on such a quotient space.

Definition 3.3.2. Let V be a vector space and let $U \leqslant V$ be a subspace of V. Then V/U is a vector space with elements of the form $v + U$ for all $v \in V$ and V/U is a **quotient space**.

Lemma 3.3.3. *Let V be a vector space on K with associated symmetric bilinear form β. Let $x \in V$ and let $V/\langle x\rangle$ be a quotient space with associated symmetric bilinear form β'. Assume that $\beta(x,v) = 0$ for all $v \in V$ and that if there exists $y \in V$ such that $\beta(y,v) = 0$ for all $v \in V$ then $y = x$. Let $v, w \in V$. Then $\beta'(u + \langle x\rangle, v + \langle x\rangle) = \beta(u,v)$.*

Proof. It is clear that

$$\begin{aligned}\beta(u+\lambda x, v+\lambda x) &= \beta(u,v)+\beta(u,\lambda x)+\beta(\lambda x,v)+\beta(\lambda x,\lambda x)\\ &= \beta(u,v) \text{ for all } \lambda\in K\backslash\{0\}.\end{aligned}$$

Hence $\beta'(u+\langle x\rangle, v+\langle x\rangle) = \beta(u,v)$. Furthermore, $u+\langle x\rangle = v+\langle x\rangle$ if and only if $v-u\in\langle x\rangle$. This holds if and only if $\beta(v-u,t)=0$ for all $t\in V$ which is the case if and only if $\beta(v,t)=\beta(u,t)$ for all $t\in V$. Hence our map is well defined. □

3.4 Maximality

Let S be a finite simple group and let $S\leqslant T\leqslant \mathrm{Aut}(S)$. In this section we will define different types of maximal subgroups of T.

Definition 3.4.1. ([40]) Let S be a simple finite group, let $S\leqslant T\leqslant \mathrm{Aut}(S)$ and let M be maximal in T.

(i) If $S\leqslant M$ then M is a **triviality**.
(ii) If $S\cap M$ is maximal in S then M is **ordinary maximal**.
(iii) If $S\cap M\neq S$ is non-maximal in S then M is a **novelty**.

As trivialities correspond to maximal subgroups of soluble groups T/S they will be omitted from all our tables.

Lemma 3.4.2. *Let S be a finite simple group and let $S\trianglelefteq T\leqslant \mathrm{Aut}(S)$. Suppose that $H<S$ is maximal and let $N=\mathrm{N}_T(H)$. Then N is maximal in T if $NS/S=T/S$. If $NS/S<T/S$ then N is not maximal.*

Proof. First suppose that $NS/S<T/S$. Then there exists a maximal subgroup M such that $NS\leqslant M<T$ and in particular $S\leqslant M$. Hence M is a triviality and since $S\nleqslant N$ it follows that $N<M<T$ is not maximal.

Now suppose that $NS/S=T/S$ and suppose that there exists a maximal subgroup M such that $N<M<T$. Then $MS=T$ and it follows that $S\nleqslant M$. Hence M is either ordinary maximal or a novelty. Suppose first that M is ordinary maximal. Then $S\cap M$ is maximal in S. It follows that $H=S\cap N\leqslant S\cap M<S$ which implies that $S\cap M=H$ since H is maximal in S as well. Hence $S\cap M=S\cap N$ and $M=N$, a contradiction. Hence no such M exists and N is maximal. We get a similar contradiction if we assume that M is a novelty. □

3.5 Aschbacher's Theorem

Aschbacher's Theorem which classifies the maximal subgroups of a classical group is crucial for this thesis. Note that trivialities are excluded from Aschbacher's Theorem.

Theorem 3.5.1 (Aschbacher's Theorem (approximate version))**.** *Let A be as in (3.1.1). Let H be a subgroup of A. Then H is either a geometric or an $\mathscr{S}$-subgroup of A.*

The maximal subgroups of A that are of geometric type were determined by Kleidman and Liebeck ([26]). Their results are stated in Chapter 11. We will not give any detailed description of the geometric subgroups as this is not required for this thesis.

Definition 3.5.2. ([8, Def 2.1.3, p.56]) Let $G \leqslant T$, where $\Omega \leqslant T \leqslant A$, A is as in (3.1.1) and $\Omega \in \{\mathrm{SL}_n^{\pm}(q), \mathrm{Sp}_n(q), \Omega_n^{\epsilon}(q)\}$. Then we say that G is an **$\mathscr{S}$-subgroup** in T if $G/(G \cap \mathrm{Z}(\mathrm{GL}_n(q^u)))$ is almost simple and the following conditions all hold:

(i) $\Omega \nleqslant G$;
(ii) G^{∞} is absolutely irreducible;
(iii) there does not exist any $g \in \mathrm{GL}_n(q^u)$ such that $(G^{\infty})^g$ is defined over a proper subfield of $\mathbb{F}_{q^u}$;
(iv) G^{∞} preserves a non-zero unitary form if and only if $\Omega = \mathrm{SU}_n(q)$;
(v) G^{∞} preserves a non-zero quadratic form if and only if $\Omega = \Omega_n^{\epsilon}(q)$; and
(vi) G^{∞} preserves a non-zero symplectic form and no non-zero quadratic form if and only if $\Omega = \mathrm{Sp}_n(q)$.

We will often divide $\mathscr{S}$-subgroups into $\mathscr{S}_1$-subgroups and $\mathscr{S}_2$-subgroups.

Definition 3.5.3. Let G be an $\mathscr{S}$-subgroup of some classical group $\mathrm{CL}_n(q^u)$ in characteristic p. Then G is an **$\mathscr{S}_2$-subgroup** of $\mathrm{CL}_n(q^u)$ if G^{∞} is isomorphic to a group of Lie type in characteristic p. We say that G has **defining characteristic** in this case. Otherwise G in an **$\mathscr{S}_1$-subgroup** and we say that G has **cross characteristic**.

In Chapter 4 we will introduce the theory needed to determine the $\mathscr{S}_1$-maximal subgroups. In Chapters 5, 6 and 7 we will then find the maximal $\mathscr{S}_1$-subgroups in dimension 13, 14 and 15 respectively. In Chapter 8 we will develop the theory behind $\mathscr{S}_2$-subgroups before determining the $\mathscr{S}_2$-maximal subgroups in Chapter 9. Finally, in Chapter 10, we will identify containments between $\mathscr{S}_1$- and $\mathscr{S}_2$-subgroups and between geometric and $\mathscr{S}$-subgroups. The tables with the maximal subgroups can be found in Chapter 11.

3.6 Simple versus Quasisimple

Our general aim is to determine the maximal subgroups of the almost simple classical groups. However, we will find the maximal subgroups of the quasisimple classical groups Ω and their extensions by outer automorphisms as it is easier to work with matrices than with permutations. The following lemmas show that we can easily deduce the maximal subgroups of $\Omega/\mathrm{Z}(\Omega).\bar{R}$ where $\bar{R} \leqslant \mathrm{Out}(\Omega/\mathrm{Z}(\Omega))$ once we know the maximal subgroups of $\Omega.R$, where $R \leqslant \mathrm{Out}(\Omega)$.

Lemma 3.6.1. *Let G be quasisimple and let $\bar{G} = G/\mathrm{Z}(G)$. If H is a maximal subgroup of G then $\mathrm{Z}(G) \leqslant H$. Furthermore H is maximal in G if and only if $\bar{H} = H/\mathrm{Z}(G)$ is maximal in $\bar{G}$.*

Proof. Suppose that $Z = \mathrm{Z}(G) \nleqslant H$. Then $G = \langle H, Z\rangle = HZ$ as Z is central. It follows that $G = G' = (HZ)' \leqslant H$ which gives the required contradiction. Therefore $Z \leqslant H$ when H is maximal. By the Correspondence Theorem H is maximal in G if and only if H/Z is maximal in G/Z. □

Lemma 3.6.2 ([8, Lemma 1.3.4, p.8])**.** *Let $G = Z.S$ be quasisimple where $Z = \mathrm{Z}(G)$ and S is non-abelian simple. Let $\alpha \in \mathrm{Aut}(G)$ be non-trivial. Then α is a non-trivial automorphism of G/Z, i.e. $\mathrm{Aut}(G)$ embeds in $\mathrm{Aut}(S)$.*

Furthermore, in all the dimensions we are interested in every $\beta \in \mathrm{Out}(\bar{\Omega})$ corresponds to some outer automorphism of Ω.

Lemma 3.6.3. *In dimension $13, 14$ and 15 all outer automorphisms $\beta \in \mathrm{Out}(\bar{\Omega})$ correspond to some $\beta' \in \mathrm{Out}(\Omega)$.*

Proof. By [26, Thm 5.1.4, p.173] the Schur multiplier of $\bar{\Omega}$ consists of scalars and therefore any $\beta \in \mathrm{Out}(\bar{\Omega})$ corresponds to some $\beta' \in \mathrm{Out}(\Omega)$. □

The following lemma will come in useful when we want to deduce information about the outer automorphisms of a classical group.

Lemma 3.6.4. *Let T be a finite group, let $R \leqslant T$ and let $\alpha \in \mathrm{Out}(T)$ be non-trivial with $\alpha = b\mathrm{Inn}(T)$. Then $\{R^g \mid g \in T\}^\alpha = \{R^g \mid g \in T\}$ if and only if there exists a non-trivial $a \in \mathrm{Aut}(T)$ such that $R^a = R$ and $a^{-1}b \in \mathrm{Inn}(T)$.*

Proof. First suppose that there exists $\alpha \in \mathrm{Out}(T)$ such that $\{R^g \mid g \in T\}^\alpha = \{R^g \mid g \in T\}$. Then for all $g \in T$ there exists $h \in T$ such that $(R^g)^b = R^h$ which implies that $((R^g)^b)^{h^{-1}} = R$. Hence $(R^{g_1})^b = R$ for some $g_1 \in T$ since $\mathrm{Inn}(T) \lhd \mathrm{Aut}(T)$. Since $g_1 \in T$ it follows that conjugation by $a = g_1 b$ induces an automorphism in $b\mathrm{Inn}(T)$ by quotienting by $\mathrm{Z}(T)$. Hence $g_1 b$ maps to $\alpha \in \mathrm{Out}(T)$ and $a^{-1}b \in \mathrm{Inn}(T)$.

Now suppose that there exists $a \in \mathrm{Aut}(T)$ such that $R^a = R$. Then $R^{ag} = R^g$ for all $g \in T$. Therefore $R^{g_2 a} = R^g$ for some $g_2 \in T$ and it follows that $\{R^g\}^{a\mathrm{Inn}(T)} = \{R^g\}$. Then $\alpha = a\mathrm{Inn}(T) \in \mathrm{Out}(T)$. $\square$

4 $\mathscr{S}$-Maximal Subgroups - The Cross Characteristic Case

The aim of this thesis is to find the maximal subgroups of the almost simple classical groups in dimension 13, 14 and 15. In this and Chapter 8 we will look at methods used to determine the maximal subgroups that lie in the last class in Aschbacher's Theorem, which is denoted by $\mathscr{S}$ (see Definition 3.5.2).

Let G be an $\mathscr{S}$-subgroup of some classical group. Then for each such G there exists a non-abelian simple group S such that $S \trianglelefteq G/\mathrm{Z}(G) \leqslant \mathrm{Aut}(S)$. (Note that $\mathrm{Z}(G)$ consists of scalar matrices of G since G is absolutely irreducible.) This group S can be of cross or defining characteristic (see Definition 3.5.3).

In this chapter we will look at the theory behind the cross characteristic subgroups, which we will also call $\mathscr{S}_1$-subgroups. In Chapter 8 we will then discuss $\mathscr{S}_2$-subgroups in more detail, although a lot of the theory developed in this chapter will also be relevant for $\mathscr{S}_2$-subgroups.

Definition 4.0.1. Let T be a classical group. We say that a subgroup $G < T$ is an **$\mathscr{S}_i$-maximal** subgroup of T if G is an $\mathscr{S}_i$-subgroup and if G is maximal among the $\mathscr{S}_i$-subgroups of T.

Let G be quasisimple. If G has a faithful absolutely irreducible representation of cross characteristic in dimension less than 250 then it appears in the tables by Hiß and Malle ([18]). The groups with such a representation in dimension 13, 14 and 15 together with their extensions by their outer automorphisms are our potential $\mathscr{S}_1$-maximal subgroups. Our aim is therefore to find the normalisers of these quasisimple groups within the classical groups in question.

Throughout this chapter let ρ be a faithful absolutely irreducible representation of G unless otherwise stated. Let $n \geqslant 13$ and let $G \cong G\rho$ be an $\mathscr{S}_1$-subgroup of a classical group $\mathrm{CL}_n(q^u)$. Here $u = 2$ in the unitary case and 1 otherwise. Let $\Omega \in \{\mathrm{SL}_n^{\pm}(q), \mathrm{Sp}_n(q), \Omega_n^{\epsilon}(q)\}$, where $\epsilon \in \{\circ, +, -\}$, and let $C = \mathrm{N}_{\mathrm{GL}_n(q^u)}(\Omega)$ be the conformal group of Ω. Then $G\rho \leqslant \Omega$ as the following lemma shows.

Lemma 4.0.2. *Let $n \geqslant 13$ and let ρ be an absolutely irreducible representation of G such that $G\rho \leqslant \mathrm{CL}_n(q^u)$. If G is quasisimple then $G\rho \leqslant \Omega$, where $\Omega \in \{\mathrm{SL}_n^{\pm}(q), \mathrm{Sp}_n(q), \Omega_n^{\epsilon}(q)\}$.*

Proof. If G is quasisimple then $G\rho \leqslant \Omega$ is also quasisimple. Hence $G\rho = (G\rho)^\infty \leqslant \mathrm{CL}_n(q^u)^\infty$. From the definition of classical groups and from Lemma 3.1.22 it follows that $\mathrm{CL}_n(q^u)^\infty = \Omega$ since $n \geqslant 13$. □

We will adopt the following convention throughout this chapter.

Convention 4.0.3. Our classical groups $\mathrm{CL}_n(q^u)$ preserve the respective standard form matrices as defined in Table 3.1.1 on p.25 unless otherwise stated.

Our first aim will be to find $\mathrm{N}_\Omega(G\rho)$ because this might be a possible $\mathscr{S}_1$-maximal subgroup of Ω (Section 4.2). Our main aim is, however, to find the maximal subgroups of the almost simple classical groups in dimensions 13, 14 and 15. Hence we also have to consider the actions of the outer automorphisms of $\bar{\Omega} \cong \Omega/\mathrm{Z}(\Omega)$ on $\mathrm{N}_\Omega(G\rho)/\mathrm{Z}(\Omega)$. We will conclude this chapter with a discussion about maximality of $\mathscr{S}_1$-subgroups (Section 4.9).

4.1 General Procedure

In this section the procedure is described which we will follow in order to find the maximal subgroups in cross characteristic.

To begin with we will list all potential $\mathscr{S}_1$-subgroups G that appear in [18]. By looking at the character values of the respective representations ρ of G in [12] and [24] we determine the character rings (see Definition 2.2.2) of ρ. Using this information, we can calculate the smallest fields over which these representations exist. In the case where the Schur indicator is ∘ we also need to identify the form preserved by $G\rho$. This could be a unitary or only the zero form. Using [12] and [24] again, we can then find $\mathrm{N}_C(G\rho)$, where $C = \mathrm{N}_{\mathrm{GL}_n(q^u)}(\Omega)$ is the conformal group of Ω. By considering the form, the determinants of the elements in $\mathrm{N}_C(G\rho)$, the field size and (in the orthogonal case only) the spinor norm, we can determine how much of $\mathrm{N}_C(G\rho)$ is contained in Ω. In other words, we find $\mathrm{N}_\Omega(G\rho)$.

Let $\bar{\beta}$ denote the outer automorphism of $\bar{\Omega}$ corresponding to $\beta \in \mathrm{Out}(\Omega)$. We also have to look at how elements of $\mathrm{Out}(\bar{\Omega})$ act on the $\bar{\Omega}$-conjugacy classes of $\mathrm{N}_\Omega(G\rho)/\mathrm{Z}(\Omega)$ corresponding to the conjugacy classes of $\mathrm{N}_\Omega(G\rho)$ in Ω. Once we have all this information we then have to decide which of these groups are indeed $\mathscr{S}_1$-maximal.

Let $\hat{\rho}$ be a characteristic 0 representation of G. Then by a p-modular reduction of $\hat{\rho}$ we mean that for all $\hat{g} \in G\hat{\rho}$, each entry of $\hat{g}$ is reduced modulo p. In most cases our representation ρ in characteristic p arises as a p-modular reduction of a characteristic 0 representation $\hat{\rho}$ with character ring R. Hence if we consider a specific representation $\hat{\rho}$ of characteristic 0 then we need to check that we can indeed reduce $\hat{\rho}$ modulo p.

Let $\hat{g} \in G\hat{\rho} \leqslant \mathrm{GL}_n(\mathbb{C})$ be arbitrary. Assuming that a p-modular reduction of $\hat{g}$ exists, it is clear that none of the denominators of the entries of $\hat{g}$ is divisible by p. The denominators may however be products of other prime numbers p_i and we cannot reduce $\hat{g}$ modulo p_i for any such p_i.

Definition 4.1.1. If p_i is a prime number that appears in the prime factorisation of the denominator of some entry of $\hat{g} \in \mathrm{G}\hat{\rho} \leqslant \mathrm{GL}_n(\mathbb{C})$, then p_i is an **exceptional prime**.

Let $p_1, \ldots, p_k$ be the exceptional primes of $G\hat{\rho}$. Then the matrix entries of all $\hat{g} \in G\hat{\rho}$ form a subring of $R[\frac{1}{p_1}, \ldots, \frac{1}{p_k}]$ and we cannot reduce $\hat{\rho}$ modulo p_i for any such prime number p_i. Hence to show that we can reduce a characteristic 0 representation $\hat{\rho}$ modulo p we have to show that for all $g \in G$ the entries of $g\hat{\rho}$ lie in $R[\frac{1}{p_1}, \ldots, \frac{1}{p_k}]$ where $p \neq p_i$ for all i. From the next lemma it follows that it is in fact sufficient to look at the lowest common multiples of the denominators of the entries of the matrices generating $G\hat{\rho}$.

Lemma 4.1.2. *Let $G\hat{\rho} = \langle \hat{g}_1, \ldots, \hat{g}_m \rangle$ and let a_i denote the lowest common multiple of the denominators of the entries of $\hat{g}_i$, where $i \in \{1, \ldots, m\}$. Let $\mathcal{A}$ denote the set containing the prime divisors of all a_i and let R be the character ring of $\hat{\rho}$. Then $\mathcal{A}$ contains all exceptional primes.*

Proof. Let $\hat{g}$ be an arbitrary element of $G\hat{\rho}$. We have to show that the denominator of any entry of $\hat{g}$ only contains elements of $\mathcal{A}$ and of R as its prime divisors. Let $\hat{g}_i \in \{\hat{g}_1, \ldots, \hat{g}_m\}$ for some i. Then $\hat{g}_i = \frac{1}{a_i}\hat{g}_i'$ where all entries of $\hat{g}_i'$ lie in R. Since $\hat{g}$ is the product of some $\hat{g}_j$ it can be written as $\hat{g} = \frac{1}{a}\hat{g}'$ where a is the product of the respective a_j and $\hat{g}'$ only contains entries in R. Therefore $\mathcal{A}$ contains all exceptional primes. $\square$

4.2 Determining the Preserved Form and Finding $\mathrm{N}_\Omega(G\rho)$

In this section we state a few results that will simplify the process of finding the normaliser of $G\rho$ in Ω. We will start by finding a way to determine the preserved form when the Schur indicator is $\circ$.

Lemma 4.2.1 ([8, Lemma 4.4.1 and Cor 4.4.2, p.167])**.** *Suppose G has an absolutely irreducible representation ρ over $\mathbb{F}_{q^2}$ of Schur indicator $\circ$. Then $G\rho$ preserves a non-degenerate unitary form if and only if the action of the* *field automorphism $\sigma : x \mapsto x^q$ on the Brauer characters is the same as complex conjugation.*

Suppose additionally that the character ring of $\hat{\rho}$ is generated over $\mathbb{Z}$ by the quadratic irrationalities $\hat{a}_1, \ldots, \hat{a}_r$ and let a_i denote the p-modular reduction of $\hat{a}_i$. Furthermore let ρ denote the p-modular reduction of $\hat{\rho}$. Then $G\rho$ preserves a unitary form if and only if:

(i) $\hat{a}_i \in \mathbb{R} \leftrightarrow a_i \in \mathbb{F}_q$*; and*

(ii) $\hat{a}_i \in \mathbb{C}\backslash\mathbb{R} \leftrightarrow a_i \in \mathbb{F}_{q^2}\backslash\mathbb{F}_q$ *for all* $1 \leqslant i \leqslant r$.

In the next part of this chapter we will now consider the normaliser of $G\rho$ in C and Ω.

Lemma 4.2.2 ([8, Lemma 4.4.3, p.168])**.** *Let ρ be an absolutely irreducible representation of a quasisimple group G. Let $G\rho \leqslant \Omega$ and let C be the conformal group of Ω. Then the outer automorphisms of G that stabilise ρ are induced by elements of* $\mathrm{N}_C(G\rho)$.

Remark 4.2.3. The normaliser of $G\rho$ in C is generated by the following elements:

(i) elements in $\mathrm{Z}(C)$, which are elements that centralise $G\rho$;

(ii) elements that lie in the inner automorphism group $\mathrm{Inn}(G\rho)$ of $G\rho$;

(iii) outer automorphisms of $G\rho$ that are induced by elements of $\mathrm{N}_C(G\rho)$. These are the outer automorphisms that stabilise ρ by Lemma 4.2.2, i.e. they split ρ (see Section 2.2.1).

To find $\mathrm{N}_\Omega(G\rho)$ once we know $\mathrm{N}_C(G\rho)$ is fairly straightforward in most cases. We just need to decide how much of $\mathrm{N}_C(G\rho)$ lies in Ω.

Let $\alpha \in \mathrm{Aut}(G)\backslash\mathrm{Inn}(G)$, assume that α has prime order and suppose that α is induced by an element of $\mathrm{N}_C(G\rho)$ which does not centralise $G\rho$. By Section 2.2.1 we are therefore in the case where ρ splits into $|\alpha|$ distinct representations when extended to $G.\langle\alpha\rangle$.

In the following lemma we will consider the case when $|\alpha| = 2$. Then ρ splits into two representations ρ_1 and ρ_2. Let χ_1 and χ_2 denote the character values of ρ_1 and ρ_2 respectively. From Section 2.2.1 it follows that $\chi_2(g) = -\chi_1(g)$ for all $g \in G.\langle\alpha\rangle\backslash G$.

Lemma 4.2.4. *Let ρ be an absolutely irreducible representation of G of dimension n such that an outer automorphism α of G of order 2 is induced by an element $g \in \mathrm{N}_C(G\rho)\backslash G\rho$ of order 2. Let $\rho_{1,2}$ denote the two representations into which ρ splits when extended to $G.\langle\alpha\rangle$. Then* $\det(g\rho_i) = \pm 1$. *Furthermore, if n is even then* $\det(g\rho_1) = \det(g\rho_2)$*, whereas if n is odd then* $\det(g\rho_1) = -\det(g\rho_2)$.

Proof. Let $k = \chi_1(g)$ and let $-k = \chi_2(g)$, where χ_1 and χ_2 are the Brauer characters of ρ_1 and ρ_2 respectively. Let a be the multiplicity of the eigenvalue 1 and let b be the multiplicity of the eigenvalue -1 of $g\rho_1$. Furthermore, let a' and b' be the multiplicities of the eigenvalues 1 and -1 of $g\rho_2$ respectively. Then

$$\begin{aligned} a + b &= n \\ a \cdot 1 + b \cdot (-1) &= k \end{aligned}$$

which implies that $a = \frac{n+k}{2}$ and $b = \frac{n-k}{2}$. Similar calculations show that $a' = \frac{n-k}{2}$ and $b' = \frac{n+k}{2}$. We know that $\det(g\rho_1) = 1^a \cdot (-1)^b = \pm 1$. Furthermore, $k = n - 2b$ and hence $b' = n - b$. If follows that if n is even then b is even if and only if b' is even. If n is odd then either b is even or b' is even. □

4.3 Finding $\mathrm{N}_{\Omega.\langle\beta\rangle}(G\rho)$

Since we are interested in finding the maximal subgroups of almost simple classical groups, we now consider $\bar{\Omega}$ and its outer automorphisms. To this purpose suppose that $G\rho$ is an $\mathscr{S}$-subgroup of Ω, where ρ is a faithful representation of G. Let $\beta \in \mathrm{Out}(\bar{\Omega})$. From Lemma 3.6.3 we know that in dimensions 13, 14 and 15 this outer automorphism β always corresponds to an outer automorphism of Ω. Therefore we will abuse notation and say that β acts on Ω as well. Furthermore, β corresponds to a coset $b\bar{\Omega}$ where $b \in \mathrm{Aut}(\bar{\Omega})$. In the following we will usually identify β with a coset representative in $\mathrm{Aut}(\bar{\Omega})$ and say that $G\rho.\langle\beta\rangle \leqslant \mathrm{N}_{\Omega.\langle\beta\rangle}(G\rho)$. Similarly, if $\alpha \in \mathrm{Aut}(G)$ or $\beta \in \mathrm{Aut}(\Omega)$ induces a non-trivial outer automorphism, then we will just write $\alpha \in \mathrm{Out}(G)$ or $\beta \in \mathrm{Out}(\Omega)$.

We want to know whether $\mathrm{N}_{\Omega.\langle\beta\rangle}(G\rho)$ is maximal in $\Omega.\langle\beta\rangle$. Let M be a C-conjugacy class of subgroups isomorphic to $G\rho$ that splits into the conjugacy classes M_1 to M_k in Ω.

Lemma 4.3.1. *Let $G\rho \leqslant \Omega$ and $\beta \in \mathrm{Out}(\Omega)$. If β stabilises an Ω-conjugacy class of $G\rho$ then (abusing notation) β normalises $G\rho$.*

Proof. Let $c_i \in C$ and let $M_i = \{(G\rho)^{c_i h} \mid h \in \Omega\}$ be an Ω-conjugacy class of $G\rho$ that is stabilised by β. Let $\beta = b\mathrm{Inn}(\Omega)$. Then $M_i^\beta = M_i$ and therefore there exists an $h \in \Omega$ such that $(G\rho)^{c_i hb} = (G\rho)^{c_i}$. Since ρ^{c_i} is equivalent to ρ we can without loss of generality assume that $(G\rho)^{hb} = G\rho$ for the representation ρ we have chosen. Since $hb\mathrm{Inn}(\Omega) = \beta$, the result follows. □

We will now look at the conjugacy classes of $G\rho$ in C and Ω. For this we will need the following definition.

Definition 4.3.2. Two representations $\rho_1, \rho_2 : G \to \mathrm{GL}_n(q)$ are **weakly equivalent** if there exist $\alpha \in \mathrm{Aut}(G)$, $\phi \in \mathrm{Aut}(\mathbb{F}_q)$ and $*$, where $*$ is either the duality or the trivial automorphism, such that for all $g \in G$, $g\rho_1$ is equivalent to $((g^\alpha\rho_2)^\phi)^*$.

Lemma 4.3.3 ([8, Lemma 4.4.3, p.168])**.** *Let G be a quasisimple group and let $\{\rho_1, \dots, \rho_r\}$ denote (up to equivalence) all weakly equivalent absolutely irreducible representations of G of dimension n. Let $G\rho_i \leqslant \Omega$ and let C be the conformal group of Ω.*

(i) The orbits of $\mathrm{Out}(G)$ on $\{\rho_1, \dots, \rho_r\}$ are in natural bijection with the conjugacy classes into which C partitions $\{G\rho_1, \dots, G\rho_r\}$.

(ii) Each C-class of subgroups splits into $|C : \mathrm{N}_C(G\rho_i)\Omega|$ classes in Ω.

In the following section we will look at the general behaviour of the outer automorphisms of $\bar{\Omega}$ and how they act on the Ω-conjugacy classes of $G\rho$ before looking at each type of classical group individually.

Suppose there exists $\alpha \in \mathrm{Out}(G)$ such that ${}^\alpha\rho$ and ρ^β are equivalent for some $\beta \in \mathrm{Out}(\Omega)$. Then the following lemma shows that β permutes the Ω-conjugacy classes of $G\rho$ a single C-conjugacy class splits into.

Lemma 4.3.4. *Let M be a single C-conjugacy class of $G\rho$ and assume that M splits into the classes M_1 to M_t in Ω. Let $\beta \in \mathrm{Out}(\bar{\Omega})$ and assume that $G\rho \leqslant M_i$ for some i. If there exists $\alpha \in \mathrm{Out}(G)$ such that ${}^\alpha\rho$ and ρ^β are equivalent then $M_i^\beta = M_k$ for some k.*

Proof. By assumption there exists $\alpha \in \mathrm{Out}(G)$ such that ${}^\alpha\rho$ and ρ^β are equivalent. Hence there exists $t \in \mathrm{GL}_n(q^u)$ such that $(G\rho^\beta)^t = G\rho$. Therefore $(G\rho)^\beta \in M_k$ for some k by Lemma 3.2.4. Since M_k is a conjugacy class of Ω it follows that $(G\rho)^{\beta\Omega} \in M_k$ and therefore $M_i^\beta = M_k$. □

It follows that β acts on these Ω-conjugacy classes of $G\rho$.

Definition 4.3.5. Let $G\rho \leqslant \Omega$ and suppose that Ω is defined over $\mathbb{F}_{q^u}$. Let $g \in \mathrm{N}_C(G\rho)$. We say that g **properly normalises** G if $g \neq \lambda h$ for any $h \in \Omega$ and any scalar $\lambda \in \mathbb{F}_{q^u}^\times$.

4.3.1 Diagonal Automorphisms

We will first consider $\delta \in \mathrm{Out}(\bar{\Omega})$ with $d = |\delta|$. Then δ corresponds to conjugation by a matrix D, where D is as defined in Section 3.2. From the definition of δ it follows that $D \in C\backslash\Omega$.

Lemma 4.3.6. *Let $G\rho \leqslant \Omega$ be absolutely irreducible. Suppose that there exists a single conjugacy class M of $G\rho$ in C and suppose that all automorphisms of $G\rho$ that are induced in C are induced in Ω.*

(i) If $\Omega \neq \Omega_n^{\pm}(q)$ then δ acts transitively on the $d = |\delta|$ conjugacy classes M splits into in Ω.

(ii) If $\Omega = \Omega_n^{\pm}(q)$ then $\langle\delta, \gamma\rangle$ acts transitively on the $2d = 2|\delta|$ conjugacy classes M splits into in Ω.

Proof. The number of conjugacy classes M splits into in Ω follows by Section 3.2 and Lemma 4.3.3. Let $\bar{C} = C/\mathrm{Z}(C)$. Then $\bar{C} = \bar{\Omega}.\langle\delta, \gamma\rangle$ if $\Omega = \Omega_n^{\pm}(q)$ and $\bar{C} = \bar{\Omega}.\langle\delta\rangle$ otherwise. Since there exists a single C-conjugacy class, δ or $\langle\delta, \gamma\rangle$ have to act transitively on the $\bar{\Omega}$-conjugacy classes of $G\rho$. □

From the above lemma and Lemma 4.3.4 it follows that if there exist $\alpha \in \mathrm{Out}(G)$ and $\beta \in \mathrm{Out}(\bar{\Omega})$ such that ${}^{\alpha}\rho$ and ρ^{β} are equivalent then at least one Ω-conjugacy class is stabilised by $\beta\delta^k$ (or $\beta\gamma^i\delta^k$ in Case $\mathbf{O}^{\pm}$) for some k and i.

Lemma 4.3.7. *Let $\beta, \mu \in \mathrm{Out}(\bar{\Omega})$ be conjugate in $\mathrm{Out}(\bar{\Omega})$. Then β stabilises an Ω-conjugacy class of $G\rho$ if and only if μ stabilises a conjugacy class.*

Proof. Since $\mathrm{Out}(\bar{\Omega})$ acts transitively on the $\bar{\Omega}$-conjugacy classes of $G\rho$ all stabilisers are conjugate. □

4.3.2 Field Automorphisms

Here we will consider some general properties of the field automorphisms as defined in Section 3.2. Note that we will consider graph automorphisms separately for Cases **L, U** and $\mathbf{O}^{\pm}$.

Lemma 4.3.8. *Let $G\rho_1, G\rho_2 \leqslant \Omega$, where ρ_1 and ρ_2 are two non-equivalent absolutely irreducible faithful representations. Suppose that the associated field of Ω is $\mathbb{F}_{q^u}$ and suppose that ${}^{\alpha}\rho_1 = \rho_2$ for some outer automorphism $\alpha \in \mathrm{Out}(G)$ of order $|\phi|$. If $(\mathrm{Trace}(g\rho_1))^p = \mathrm{Trace}(g\rho_2)$ for all $g \in G$ then ${}^{\alpha}\rho_1$ is equivalent to ρ_1^{ϕ}.*

Proof. For ${}^{\alpha}\rho_1$ to be equivalent to ρ_1^{ϕ} we need to show that

$$\mathrm{Trace}((g\rho_1)^{\phi}) = \mathrm{Trace}(g^{\alpha}\rho_1) = \mathrm{Trace}(g\rho_2)$$

since two representations are equivalent if and only if they have the same character values. Since $\mathrm{Trace}((g\rho_1)^{\phi}) = \mathrm{Trace}((g\rho_1)^p) = (\mathrm{Trace}(g\rho_1))^p$ the result follows. □

We will now consider the Cases **L, U, S**, $\mathbf{O}^{\circ}$ and $\mathbf{O}^{\pm}$ separately.

4.4 Graph Automorphisms for Case L:

In this section we will look at the outer automorphisms of a linear group. Note that for the groups considered in this thesis it turns out that we will not be required to look at the field automorphisms of linear groups. Hence we only need to consider the duality automorphism γ as the diagonal automorphism δ was discussed in Section 4.3.1. There is an easy way to prove that there exists an $\alpha \in \mathrm{Out}(G)$ such that ${}^{\alpha}\rho$ is equivalent to ρ^{γ} as the following lemma shows.

Lemma 4.4.1. *Let $\rho_{1,2}$ be two absolutely irreducible representations of G such that* $\mathrm{Trace}(g\rho_1)$ *equals the complex conjugate of* $\mathrm{Trace}(g\rho_2)$ *for all $g \in G$. If there exists $\alpha \in \mathrm{Out}(G)$ of order 2 such that ${}^{\alpha}\rho_1 = \rho_2$, then ${}^{\alpha}\rho_1$ is equivalent to ρ_1^{γ}.*

Proof. This is clear by Lemma 3.2.2 for characteristic 0 representations. By the definition of Brauer characters this also holds for characteristic p representations using a similar argument as in the proof of Lemma 3.2.2. □

We will first consider the case when $d := (q-1, n)$ is odd.

Lemma 4.4.2 ([8, Lemma 4.6.1, p.189])**.** *Let $\rho : G \to \Omega = \mathrm{SL}_n(q)$ be an absolutely irreducible representation and suppose that d is odd. If there exists $\alpha \in \mathrm{Out}(G)$ such that ${}^{\alpha}\rho$ is equivalent to ρ^{γ}, then an Ω-conjugacy class of $G\rho$ is stabilised by γ in $\mathrm{Out}(\bar{\Omega})$.*

Now assume that d is even. In this case it is a bit more complicated to determine whether γ or $\gamma\delta$ stabilises an $\mathrm{SL}_n(q)$-conjugacy class of $G\rho$.

Lemma 4.4.3 ([8, Lemma 4.6.2, p.189])**.** *Assume that d is even and let $\rho : G \to \mathrm{SL}_n(q)$ be a faithful absolutely irreducible representation. Also assume that there exist $\alpha \in \mathrm{Out}(G)$ and $x \in \mathrm{GL}_n(q)$ such that $x^{-1}(g^{\alpha}\rho)x = (g\rho)^{\gamma}$ for all $g \in G$. If $\det(x)$ is a square in $\mathbb{F}_q^{\times}$ then an $\mathrm{SL}_n(q)$-conjugacy class of $G\rho$ is stabilised by γ in $\mathrm{Out}(\mathrm{L}_n(q))$. Otherwise it is stabilised by $\gamma\delta$.*

Remark 4.4.4. Finally, we have to consider characteristic 0 representations $\hat{\rho}$ of G that preserve only the zero form. Let R be the character ring of $\hat{\rho}$ and let K be the corresponding character field. Let $\{p_1, \ldots, p_k\}$ be the set of exceptional primes of $\hat{\rho}$ (see Definition 4.1.1).

Suppose we want to determine the action of γ on the p-modular reductions of $G\hat{\rho}$. For all the groups considered in this thesis we can find an $\hat{x} \in \mathrm{GL}_n(K)$ and some $\alpha \in \mathrm{Out}(G)$ such that $\hat{x}^{-1}(g\hat{\rho})^{-\mathrm{T}}\hat{x} = (g^{\alpha})\hat{\rho}$ for all $g \in G$. Furthermore, we can find $\lambda \in \mathbb{C}$ such that $\lambda\hat{x}$ and $(\lambda\hat{x})^{-1}$ only have entries in $R[\frac{1}{p_1}, \ldots, \frac{1}{p_k}]$ and $\det(\lambda\hat{x}) = p_1^{e_1} \ldots p_k^{e_k}$ for some $e_i \in \mathbb{N}^0$. Hence we can reduce $\lambda\hat{x}$ modulo p for all $p \neq p_i$. From this it follows that we can apply Lemma 4.4.3 to x, the p-modular reduction of $\hat{x}$. That way we can deduce the action of γ on the p-modular reductions of our characteristic 0 representation.

4.5 Field Automorphisms for Case U:

Here we will consider the behaviour of the outer automorphisms γ and ϕ of the unitary groups. Let $q = p^e$. Let $d := (q+1, n) = |\delta|$ and let $\mathrm{U}_n(q) = \mathrm{U}_n(q, I_n)$. Recall from Section 3.2.6 that

$$\mathrm{Out}(\mathrm{U}_n(q)) = \langle \delta, \phi, \gamma \,|\, \delta^d = \phi^{2e} = \gamma^2 = 1,\ \delta^{\gamma} = \delta^{-1},\ \phi^e = \gamma,\ \delta^{\phi} = \delta^p \rangle.$$

Note that by [7] the isomorphism type of $\langle \mathrm{U}_n(q, B), \phi \rangle$ depends on the choice of form B preserved by the unitary group when n is even and q is odd. Recall that in Section 3.2 we defined σ and ϕ for groups preserving our standard unitary form matrix I_n. The problem is that even though $\sigma = \phi^e$ and ϕ are automorphisms of $\mathrm{SU}_n(q, I_n)$ and stabilise this group, they do not necessarily stabilise $\mathrm{SU}_n(q, B)$ for every non-degenerate unitary form B. One reason is that these automorphisms might not fix the form at all and even if they fix B, then $\langle \mathrm{SU}_n(q, I_n), \phi \rangle$ (or $\langle \mathrm{SU}_n(q, I_n), \sigma \rangle$) is not necessarily isomorphic to $\langle \mathrm{SU}_n(q, B), \phi \rangle$ (or $\langle \mathrm{SU}_n(q, B), \sigma \rangle$).

To find the action of the field automorphism for any non-degenerate unitary form B we find a way to map $G\rho \leqslant \mathrm{SU}_n(q, B)$ to some isomorphic group $H \leqslant \mathrm{SU}_n(q, I_n)$. Let $A \in \mathrm{GL}_n(q^2)$ such that $(G\rho)^A = H \leqslant \mathrm{SU}_n(q, I_n)$. Since ρ^A is equivalent to ρ it is sufficient for our purpose to determine the action of the outer automorphisms of $\mathrm{U}_n(q, I_n)$ on H.

If we do not need to use any specific generators for $G\rho$ however than we can assume without loss of generality that $G\rho$ preserves our standard unitary form matrix I_n.

Assume throughout that each $\mathrm{C} = \mathrm{CGU}_n(q)$ conjugacy class of $G\rho$ splits into d conjugacy classes in $\mathrm{SU}_n(q)$. By [8, Lemma 4.6.3, p.190] there are two $\mathrm{Out}(\mathrm{U}_n(q))$-conjugacy classes containing elements of the form $\phi\delta^i$ when d is even and one such conjugacy class when d is odd. We will first consider the case when d is odd.

Lemma 4.5.1. *Let $\rho : G \to \Omega = \mathrm{SU}_n(q)$ be an absolutely irreducible representation. Suppose that d is odd and that there exists $\alpha \in \mathrm{Out}(G)$ such that ${}^{\alpha}\rho$ is equivalent to ρ^{ϕ}. Then an Ω-class of $G\rho$ is stabilised by $\langle\phi\rangle$ in $\mathrm{Out}(\bar{\Omega})$.*

Proof. This follows from [8, Lemma 4.6.3, p.190] and Lemma 4.3.7. □

Now we will look at the case when d is even. The following lemma gives us a way of deciding whether a subgroup of a unitary group is stabilised by ϕ or by $\phi\delta$.

Lemma 4.5.2 ([8, Lemma 4.6.5, p.191])**.** *Let $\rho : G \to \mathrm{GL}_n(q^2)$ be an absolutely irreducible representation such that $G\rho \leqslant \mathrm{SU}_n(q, B) \cong \mathrm{SU}_n(q)$, where B is some non-degenerate unitary form. Assume that $d = (q + 1, n)$ is even. Also assume that there exist $\alpha \in \mathrm{Out}(G)$ and $x \in \mathrm{GL}_n(q^2)$ such that $x^{-1}(g\rho)^{\phi}x = g^{\alpha}\rho$ for all $g \in G$ and $xBx^{\sigma\mathrm{T}} = \lambda B^{\phi}$ with $\lambda \in \mathbb{F}_q^{\times}$. Let $A \in \mathrm{GL}_n(q^2)$ such that $(G\rho)^A \leqslant \mathrm{SU}_n(q)$ and let $l = \sqrt{\det(x)}$. Then a conjugacy class of $(G\rho)^A$ in $\mathrm{SU}_n(q)$ is stabilised in $\mathrm{U}_n(q)$ by ϕ if and only if either*

$$\phi = \gamma \text{ and } \frac{1}{\lambda^{n/2}} l^{1+\sigma} \det(B) = 1$$

or

$$(\frac{1}{\lambda^{n/2}}) l^{1+\sigma} (\det(B))^{\frac{1-p}{2}} = 1.$$

Since the representation ρ^A is equivalent to ρ, we will just say that an $\mathrm{SU}_n(q)$-conjugacy class of $G\rho$ is stabilised by ϕ in $\mathrm{Out}(\mathrm{U}_n(q))$.

Remark 4.5.3. Again we have to consider characteristic 0 representations $\hat{\rho}$ of G preserving a unitary form. Let R be the character ring of $\hat{\rho}$ and let $\{p_1, \ldots, p_k\}$ be the set of exceptional primes of $G\hat{\rho}$. Let $\hat{B}$ be the positive definite σ-Hermitian form preserved by $\hat{\rho}$. Then there exists a complex matrix $\hat{A}$ such that $\hat{A}\hat{A}^{\sigma\mathrm{T}} = \hat{B}$ by [27]. The p-modular reduction μB of $\hat{\mu}\hat{B}$ is a non-degenerate unitary form if the entries of both $\hat{\mu}\hat{B}$ and $(\hat{\mu}\hat{B})^{-1}$ lie in $R[\frac{1}{p_1}, \ldots, \frac{1}{p_k}]$ for some scalar $\hat{\mu} \in \mathbb{C}$. Even if such a $\hat{\mu}$ exists, however, we may not be able to find a suitable $\hat{A}$ with entries only in $R[\frac{1}{p_1}, \ldots, \frac{1}{p_k}]$. From this it follows that $\hat{A}$ cannot necessarily be reduced modulo p and hence we may not be able to use Lemma 4.5.2.

However we can find an $\hat{x} \in \mathrm{GL}_n(\mathbb{C})$ such that $\hat{x}^{-1}(g\hat{\rho})^{\phi}\hat{x} = (g^{\alpha})\hat{\rho}$ for all $g \in G$. If the entries of $\lambda\hat{x}$ and $(\lambda\hat{x})^{-1}$ lie in $R[\frac{1}{p_1}, \ldots, \frac{1}{p_k}]$ for some scalar

λ and if $\phi = \gamma$, then we do not have to find $\hat{A}$ explicitly, as the following lemma shows.

Lemma 4.5.4 ([8, Prop 4.6.6, p.193])**.** *Let $\hat{\rho} : G \to \mathrm{SU}_n(\hat{B}, \mathbb{C})$ be a characteristic 0 representation preserving some unitary form $\hat{B}$ such that G has an absolutely irreducible representation ρ with $G\rho \leqslant \mathrm{SU}_n(q, B)$ that arises as a p-modular reduction of $\hat{\rho}$. Furthermore, let $S = R[\frac{1}{p_1}, \ldots, \frac{1}{p_s}]$, where R is the character ring of $\hat{\rho}$ and the p_is are the exceptional primes with $p_i \neq p$ for all i. Assume that:*

(i) $\phi = \gamma$;

(ii) there exist $\alpha \in \mathrm{Aut}(G)$ and $\hat{x} \in \mathrm{GL}_n(\mathbb{C})$ such that $\hat{x}^{-1}(g\hat{\rho})^{\phi}\hat{x} = (g^{\alpha})\hat{\rho}$ for all $g \in G$;

(iii) $\hat{B}, \hat{B}^{-1}, \hat{x}$ and $\hat{x}^{-1}$ have entries in S; and

(iv) $\hat{r}\hat{\nu}^2$ with $\hat{r} \in \mathbb{R}$ gives a factorisation of $\det(\hat{x})$ in S.

Now let r be the p-modular reduction of $\hat{r}$ and let $\epsilon = 1$ if $\sqrt{r} \in \mathbb{F}_q^{\times}$ and $\epsilon = -1$ otherwise. Let $A \in \mathrm{GL}_n(q^2)$ such that $(G\rho)^A \leqslant \mathrm{SU}_n(q)$. If $\epsilon\,\mathrm{sgn}(\hat{r}) = 1$ then an $\mathrm{SU}_n(q)$-conjugacy class of $(G\rho)^A$ is stabilised by ϕ in $\mathrm{Out}(\mathrm{U}_n(q))$. If $\epsilon\,\mathrm{sgn}(\hat{r}) = -1$ then a conjugacy class is stabilised by $\phi\delta$.

4.6 Field Automorphisms for Case S:

Let $G\rho \leqslant \Omega = \mathrm{Sp}_n(q)$. In this thesis we can always assume that $G\rho$ preserves our standard symplectic form as in Table 3.1.1 (p.25). Recall from Section 3.2.6 that a presentation of the outer automorphism group of $\mathrm{S}_n(p^e)$ is given by

$$\mathrm{Out}(\mathrm{S}_n(p^e)) = \langle \delta, \phi \mid \delta^{(q-1,2)} = \phi^e = [\delta, \phi] = 1 \rangle \cong \mathrm{C}_{(q-1,2)} \times \mathrm{C}_e.$$

As for Cases **L** and **U** suppose that there exists $\alpha \in \mathrm{Out}(G)$ such that ${}^{\alpha}\rho$ is equivalent to ρ^{ϕ}. (Recall that we can use Lemma 4.3.8 to determine whether ${}^{\alpha}\rho$ is equivalent to ρ^{ϕ}.) Furthermore, assume that a single $C = \mathrm{CSp}_n(q)$ conjugacy class of $G\rho$ splits into $(q-1, 2)$ conjugacy classes in $\mathrm{Sp}_n(q)$.

From this it follows that the two conjugacy classes are stabilised by $\phi\delta^i$ for some $i \in \{0, 1\}$. The following lemmas are sufficient for this thesis to determine the class stabiliser of an Ω-conjugacy class of $G\rho$.

Lemma 4.6.1 ([8, Lemma 4.6.7, p.195])**.** *Let $\mathrm{C\Gamma Sp}_n(q) = \mathrm{CSp}_n.\langle\phi\rangle$. All involutions in $\mathrm{PC\Gamma Sp}_n(q)$ lie in $\mathrm{S}_n(q).\langle\phi\rangle \cup \mathrm{PCSp}_n(q)$ when q is odd.*

Lemma 4.6.2. *Let $G\rho \leqslant \mathrm{Sp}_n(p^2) = \Omega$ with q odd. Assume that there exists $\alpha \in \mathrm{Out}(G)$ of order 2 such that ${}^{\alpha}\rho$ is equivalent to ρ^{ϕ}. Furthermore, suppose that $\delta \notin \mathrm{N}_C(G\rho)$ and that projectively $\langle G, \alpha\rangle\backslash G$ contains involutions. Then an Ω-conjugacy class of $G\rho$ is stabilised by $\langle\phi\rangle$ in $\mathrm{Out}(\bar{\Omega})$.*

Proof. Note that since $\delta \notin \mathrm{N}_C(G\rho)$, no conjugacy class of $G\rho$ is stabilised by δ. Therefore an Ω-conjugacy class of $G\rho$ has stabiliser $\langle \phi\delta^i \rangle$ in $\bar{\Omega}$ for some $i \in \{0, 1\}$. Since this stabiliser induces α it follows that $G\rho.\langle \phi\delta^i \rangle$ contains involutions. By Lemma 4.6.1 the stabiliser is therefore $\langle \phi \rangle$. $\square$

4.7 Field Automorphisms for Case O°:

Now we consider the field automorphisms of orthogonal groups in odd dimension. Let $G\rho \leqslant \Omega = \Omega_n^\circ(q)$ and suppose that ${}^\alpha\rho$ is equivalent to ρ^ϕ for some $\alpha \in \mathrm{Out}(G)$. Again we can assume that $G\rho$ preserves our standard form as in Table 3.1.1 (p.25). By Section 3.2.6,

$$\mathrm{Out}(\mathrm{O}_n^\circ(p^e)) = \langle \delta, \phi \,|\, \delta^2 = \phi^e = [\delta, \phi] = 1 \rangle \cong \mathrm{C}_2 \times \mathrm{C}_e$$

and δ is induced by a matrix in $\mathrm{SO}_n^\circ(q) \backslash \Omega_n^\circ(q)$.

The following lemmas help to determine whether $\langle \phi \rangle$ or $\langle \phi\delta \rangle$ stabilises an Ω-conjugacy class of $G\rho$.

Lemma 4.7.1 ([8, Lemma 4.9.40, p.239])**.** *Let* $\mathrm{C\Gamma O}_n^\circ(q)$ *denote the group* $\mathrm{CO}_n^\circ(q).\langle \phi \rangle$ *and let* $\mathrm{CSO}_n^\circ(q) = \mathrm{CO}_n^\circ(q) \cap \mathrm{SL}_n(q)$*. Then all involutions in* $\mathrm{PC\Gamma O}_n^\circ(q)$ *lie in* $\mathrm{PCSO}_n^\circ(q) \cup \mathrm{O}_n^\circ(q).\langle \phi \rangle$*.*

Lemma 4.7.2. *Let* $G\rho \leqslant \Omega_n^\circ(p^2) = \Omega$*. Assume that* $\delta \notin \mathrm{N}_C(G\rho)$ *and suppose that there exists* $\alpha \in \mathrm{Out}(G)$ *such that* ${}^\alpha\rho$ *is equivalent to* ρ^ϕ*, where* $|\phi| = |\alpha| = 2$*. Also suppose that projectively* $\langle G, \alpha \rangle \backslash G$ *contains involutions. Then an* Ω*-conjugacy class of* $G\rho$ *is stabilised by* $\langle \phi \rangle$ *in* $\mathrm{Out}(\bar{\Omega})$*.*

Proof. The proof is very similar to the proof of Lemma 4.6.2. $\square$

We will now consider the case when ϕ induces an outer automorphism of $G\rho$ of order 4. Then $\mathrm{Out}(\mathrm{O}_n^\circ(p^4)) \cong \mathrm{C}_2 \times \mathrm{C}_4$.

Let $\Omega = \Omega_n^\circ(p^4)$. It is clear by Lemma 4.3.4 and Lemma 4.3.6 that if there exists $\alpha \in \mathrm{Out}(G)$ of order 4 such that ${}^\alpha\rho$ is equivalent to ρ^ϕ then at least one and hence both of the Ω-conjugacy classes of $G\rho$ are either stabilised by $\langle \phi \rangle$ or by $\langle \phi\delta \rangle$ in $\mathrm{Out}(\bar{\Omega})$. Hence we can assume without loss of generality that the Ω-conjugacy class of $G\rho$ is stabilised by $\phi\delta^i$ for some i.

Lemma 4.7.3. *Let* $G\rho \leqslant \Omega = \Omega_n^\circ(p^4)$ *and suppose that there exist* $\alpha \in \mathrm{Out}(G)$ *of order* 4 *and* $x \in \mathrm{GL}_n(p^4)$ *such that* $x^{-1}(g\rho^\phi)x = g^\alpha\rho$ *for all* $g \in G$*. Suppose that there exists a single* C *conjugacy class of* $G\rho$ *that splits into* 2 *conjugacy classes in* Ω*. Then there exists* $\lambda \in \mathbb{F}_{p^4}^\times$ *such that*

$\lambda x \in \mathrm{SO}_n^\circ(p^4)$. Furthermore, if $\mathrm{sp}(\lambda x) = 1$ then the Ω-conjugacy class of $G\rho$ is stabilised by ϕ in $\mathrm{Out}(\bar{\Omega})$. Otherwise the conjugacy class is stabilised by $\phi\delta$.

Proof. By assumption the conjugacy class $G\rho^\Omega$ is stabilised by $\phi\delta^i$ for some $i \in \{0, 1\}$. Furthermore, $G\rho^{\phi x} = G\rho$ which implies that $\phi x \in \mathrm{Aut}(\Omega)$ by Lemma 3.2.4(ii). Since $G\rho^\Omega$ is only stabilised by $\phi\delta^i$ we can deduce from Lemma 3.6.4 that ϕx equals $\phi\delta^i$ in $\mathrm{Out}(\Omega)$. Hence $x = \lambda^{-1}\delta^i g$ for some scalar λ and some $g \in \mathrm{Inn}(\Omega)$. In particular this implies that $\lambda x \in \mathrm{SO}_n^\circ(p^4)$. Then $\lambda x \in \Omega$ if and only if $i = 0$, i.e. if and only if $\mathrm{sp}(\lambda x) = 1$. □

4.8 Outer Automorphisms for Case $\mathrm{O}^\pm$:

The theory of the outer automorphisms of the orthogonal groups in even dimensions is quite complex as there are more automorphisms to consider. Also, the outer automorphism group depends on the type of the orthogonal group and on the discriminant of the preserved form matrix.

4.8.1 Field Automorphisms for Case $\mathrm{O}^\pm$

Let B be a non-degenerate symmetric bilinear form matrix. We will first consider the field automorphisms of $\mathrm{O}_n^\pm(p^2, B)$ in odd characteristic.

Field Automorphisms in Odd Characteristic

The following lemma holds for all orthogonal groups in even dimension (in fact it can be easily adapted to hold for all quasisimple classical groups). Later on in this section we will only consider orthogonal groups of plus-type with dimension $n \equiv 2 \pmod 4$ though as this is what is needed for this thesis.

Recall from Section 3.2.6 that when $n \equiv 2 \pmod 4$ and p is odd the presentation of the outer automorphism group of $\mathrm{O}_n^+(p^2)$ is:

$$\mathrm{Out}(\mathrm{O}_n^+(p^2)) = \langle \gamma, \delta, \phi \mid \delta^4 = \gamma^2 = 1, \delta^\gamma = \delta^{-1}, \phi^2 = [\gamma, \phi] = 1, \delta^\phi = \delta^p \rangle,$$

since $p^2 \equiv 1 \pmod 4$.

Furthermore, we can show that $\mathrm{Out}(\mathrm{O}_n^+(p^2)) \cong \langle \gamma, \delta \rangle \times \langle \nu \rangle$, where $\langle \gamma, \delta \rangle \cong \mathrm{D}_8$, $\nu = \phi$ if $p \equiv 1 \pmod 4$ and $\nu = \gamma\phi$ if $p \equiv 3 \pmod 4$.

Lemma 4.8.1. *Let n be even and q be odd. Let $\rho : G \to \mathrm{GL}_n(q)$ be an absolutely irreducible representation and assume that $G\rho \leqslant \Omega_n^\pm(q, B)$ for some non-degenerate symmetric bilinear form matrix B. Also suppose that ${}^\alpha\rho$ is equivalent to ρ^ϕ for some $\alpha \in \mathrm{Out}(G)$. Then:*

(i) there exists $A \in \mathrm{GL}_n(q)$ such that $AFA^{\mathrm{T}} = B$, where F is our standard form matrix of the same sign as B as given in Table 3.1.1. Furthermore, $G\rho \cong (G\rho)^A \leqslant \Omega_n^{\pm}(q, F) = \Omega_n^{\pm}(q)$;

(ii) there exists $x \in \mathrm{GL}_n(q)$ such that $x^{-1}(g\rho)^{\phi}x = (g^{\alpha})\rho$ for all $g \in G$.

Proof. (i) In even dimension all non-degenerate orthogonal forms of the same sign are isometric by [8, Thm 1.5.3, p.20]. Hence there exists $A \in \mathrm{GL}_n(q)$ such that $AFA^{\mathrm{T}} = B$. Furthermore, if $g \in \Omega_n^{\pm}(q, B)$ then $gBg^{\mathrm{T}} = B$. From this it follows that $AA^{-1}gAA^{-1}BA^{-\mathrm{T}}A^{\mathrm{T}}g^{\mathrm{T}}A^{-\mathrm{T}}A^{\mathrm{T}} = B$. Hence $g^AF(g^A)^{\mathrm{T}} = F$. Since conjugation by the matrix A induces an isomorphism we deduce that $G\rho \cong G\rho^A \leqslant \Omega_n^{\pm}(q)$.

(ii) This follows since ${}^{\alpha}\rho$ is equivalent to ρ^{ϕ}. □

From now on we will only consider the case when $G\rho$ preserves an orthogonal form of plus-type. Hence we do not have to deal with the field automorphism φ. The following is based on [8, Lemma 4.6.7, p.195].

Lemma 4.8.2. *Suppose that n is even and p is odd. Then any involution in $\mathrm{PC\Gamma O}_n^+(p^2)$ lies in either a $\langle\delta\rangle$-conjugate of $\mathrm{O}_n^+(p^2).\langle\phi, \gamma\rangle$ or in $\mathrm{PCGO}_n^+(p^2)$.*

Proof. Note that elements in $\mathrm{GO}_n^+(p^2)$ preserve our standard form $F = \mathrm{antidiag}(1, \ldots, 1)$ but they also preserves any scalar multiple of F. Let $g \in \mathrm{GO}_n^+(p^2)\backslash\mathrm{SO}_n^+(p^2)$. By Lemma 3.1.20, $\mathrm{sp}(g, F) = \mathrm{sp}(g, \lambda F)$ if λ is square in $\mathbb{F}_{p^2}$ and by Lemma 3.2.3, $\mathrm{sp}(g^{\delta}, F) = \mathrm{sp}(g, \lambda F)$ if λ is non-square in $\mathbb{F}_{p^2}^{\times}$. Since we only want to determine the containment of g up to conjugation by δ, we can assume without loss of generality that elements in $\mathrm{GO}_n^+(p^2)$ preserve F.

Let $g \in \mathrm{PC\Gamma\Omega}_n^+(p^2)$ be of order 2. Then either g is the image of some $A \in \mathrm{CGO}_n^+(p^2)$ in which case $g \in \mathrm{PCGO}_n^+(p^2)$. Otherwise we can take g to be the image of some $A\sigma$, where $\sigma \in \mathrm{C\Gamma O}_n^+(p^2)\backslash\mathrm{CGO}_n^+(p^2)$ induces by conjugation a field automorphism of order 2. We want to show that we have $g \in \mathrm{O}_n^+(p^2).\langle\phi, \gamma\rangle$.

We find that for some $\lambda \in \mathbb{F}_{p^2}^{\times}$

$$AA^{\sigma} = (A\sigma)^2 = \lambda I, \tag{4.8.1}$$

since projectively $A\sigma$ has order 2. Since $(AA^{\sigma})^A = A^{\sigma}A = \lambda I$ and $(AA^{\sigma})^{\sigma} = A^{\sigma}A = \lambda^{\sigma}I$ it follows that $\lambda \in \mathbb{F}_p^{\times}$. Also, $AFA^{\mathrm{T}} = \mu F$ for some $\mu \in \mathbb{F}_{q^2}^{\times}$ since $A \in \mathrm{CGO}_n^+(p^2)$. Furthermore, note that $F = F^{\sigma}$. Hence, $(AFA^{\mathrm{T}})^{\sigma} = A^{\sigma}FA^{\sigma\mathrm{T}} = \mu^{\sigma}F$. From this it follows that

$$(AA^{\sigma})F(AA^{\sigma})^{\mathrm{T}} = \mu^{\sigma+1}F = \lambda^2 F$$

since $AA^\sigma = \lambda I$ by (4.8.1). Using the fact that $\lambda \in \mathbb{F}_p^\times$ it follows from Lemma 2.1.8 that μ is a square in $\mathbb{F}_{p^2}^\times$.

Next we want to show that (up to conjugacy) $g \in \mathrm{PGO}_n^+(p^2).\langle\phi\rangle$ in this case. This holds if we can show that $\nu A \in \mathrm{GO}_n^+(p^2)$ for some scalar ν. Let $\nu = \mu^{-\frac{1}{2}}$ which exists since μ is square. Then $(\nu A)F(\nu A)^{\mathrm{T}} = \nu^2\mu F = F$ as required.

Furthermore we will show that νA has spinor norm 1. Let $E = \nu A \in \mathrm{GO}_n^+(p^2)$. First of all note that $EE^\sigma = \nu^{1+p}AA^\sigma = (\mu^{1+p})^{-1/2}\lambda I = (\lambda^2)^{-1/2}\lambda I = \pm I$ depending on the square root of λ^2. Also, $\mathrm{sp}(E) = 1$ if and only if $\tau = \prod_{i=1}^k \beta(v_i, v_i)$, as defined in Definition 3.1.16, is square in $\mathbb{F}_{p^2}^\times$. Since $\pm I$ has spinor norm 1 in $\mathrm{SO}_n^+(p^2)$, it follows from the fact that the spinor norm map is a homomorphism that τ^{p+1} is a square in $\mathbb{F}_{p^2}$. By Lemma 2.1.8, τ is a square in $\mathbb{F}_{p^2}$ as well and hence $\mathrm{sp}(E) = 1$. Note that $A = \nu^{-1}E$ and hence conjugation by A corresponds to conjugation by E. It follows that $\mathrm{sp}(A) = 1$. Therefore, up to conjugation by δ, $g \in \mathrm{O}_n^+(p^2).\langle\phi,\gamma\rangle$. □

For the remainder of this subsection we will only consider the case when $n \equiv 2 \pmod 4$.

Lemma 4.8.3. *Let $\Omega = \Omega_n^+(p^2)$ with $n \equiv 2 \pmod 4$ and p odd. Assume that there exists $\alpha \in \mathrm{Out}(G)$ of order 2 such that ${}^\alpha\rho$ is equivalent to ρ^ϕ. Suppose that $\delta \notin \mathrm{N}_C(G\rho)$ and that projectively $\langle G,\alpha\rangle\backslash G$ contains involutions. Then an Ω-conjugacy class of $G\rho$ has either stabiliser $\langle\phi\rangle$ or $\langle\phi\gamma\rangle$ in* $\mathrm{Out}(\bar{\Omega})$.

Proof. The proof is similar to the proof of Lemma 4.6.2. □

Lemma 4.8.4. *Assume that $G\rho \leqslant \Omega = \Omega_n^+(p^2)$, where p is odd and $n \equiv 2 \pmod 4$. Assume that $\mathrm{N}_C(G\rho) = \mathrm{Z}(C)G\rho$ and that all images of the representations that are weakly equivalent to ρ lie in the same C-conjugacy class. Assume there exists $\alpha \in \mathrm{Out}(G)$ such that ${}^\alpha\rho$ is equivalent to ρ^ϕ and assume that projectively $G.\langle\alpha\rangle\backslash G$ contains involutions. Let $c \in C$ and assume that the Ω-conjugacy class of $(G\rho)^c$ is stabilised by $\phi\gamma^k$ in* $\mathrm{Out}(\bar{\Omega})$ *for some $k \in \{0,1\}$. Without loss of generality we can let $c \in \{1,\delta\}$.*

Proof. Let $\mathrm{Out}(\bar{\Omega}) = D \times \langle\nu\rangle$, where $D = \langle\delta,\gamma\rangle$, $\nu = \phi$ if $p \equiv 1 \pmod 4$ and $\nu = \phi\gamma$ if $p \equiv 3 \pmod 4$.

By Lemma 4.8.3 the Ω-conjugacy class of $(G\rho)^c$ is stabilised by $H = \langle\phi\gamma^k\rangle$. Furthermore, we know that $\mathrm{Out}(\bar{\Omega})$ acts transitively on the cosets of H.

If $H = \langle \nu \rangle$ then it is clear that all cosets of H are stabilised by $\langle \nu \rangle$. It follows that all Ω-conjugacy classes of G are stabilised by H and hence we can let $c = 1$.

If $H = \langle \gamma\nu \rangle$ then the cosets H, γH, $\delta^2 H$, $\delta^2\gamma H$ are stabilised by H, whereas the cosets δH, $\delta\gamma H$, $\delta^{-1}H$ and $\delta^{-1}\gamma H$ are stabilised by H^δ. It follows that either the Ω-conjugacy class of $G\rho$ or the Ω-conjugacy class of $(G\rho)^\delta$ is stabilised by $\gamma\nu$. Hence without loss of generality $c \in \{1, \delta\}$ in this case. □

Our aim is now to determine whether a conjugacy class is stabilised by ϕ or $\phi\gamma$. To do so we will first of all find an expression for x, where x conjugates $(g\rho)^\phi$ to $g^\alpha\rho$ for all $g \in G$.

Convention 4.8.5. Assume that $G\rho \leqslant \Omega_n^+(p^2, B)$, where B is some non-degenerate symmetric bilinear form matrix of plus-type, p is odd and $n \equiv 2 \pmod 4$ unless otherwise stated. Let C be the conformal group of $\Omega_n^+(p^2, B)$. Furthermore, let $F = \mathrm{antidiag}(1, \ldots, 1)$, our standard non-degenerate symmetric bilinear form matrix of plus-type, and let $A \in \mathrm{GL}_n(p^2)$ such that $(G\rho)^A \leqslant \Omega_n^+(p^2, F) = \Omega_n^+(p^2)$ and $AFA^{\mathrm{T}} = B$. Also assume that $\mathrm{N}_C(G\rho) = \mathrm{Z}(C)G\rho$ and that all images of the representations that are weakly equivalent to ρ lie in the same C-conjugacy class.

Note that our convention implies that there exist eight $\Omega_n^+(p^2)$-conjugacy classes of $(G\rho)^A$ and that the stabiliser of $G\rho$ has order 2.

Lemma 4.8.6. *Recall our Convention 4.8.5. Suppose that there exists $x \in \mathrm{GL}_n(p^2)$ such that $x^{-1}(g\rho)^\phi x = (g^\alpha)\rho$ for all $g \in G$. Also suppose that $(G\rho)^{Ac}$ is stabilised by $\phi\gamma^k h$ for some $c \in \{1, \delta\}$, some $k \in \{0, 1\}$ and some $h \in \Omega_n^+(p^2)$. Let $y = A^\phi c^\phi \gamma^k h c^{-1} A^{-1}$. Then $yBy^{\mathrm{T}} = B^\phi$ if $c = 1$ and $yBy^{\mathrm{T}} = \omega^{\phi-1}B^\phi$ for some primitive element $\omega \in \mathbb{F}_{p^2}^\times$ if $c = \delta$. Furthermore, $y = \mu x$ for some scalar $\mu \in \mathbb{F}_{p^2}^\times$.*

Proof. By assumption $(G\rho^{Ac})^{\phi\gamma^k h} = G\rho^{Ac}$. Hence $((G\rho)^\phi)^{A^\phi c^\phi \gamma^k h c^{-1} A^{-1}} = G\rho$ and so $(G\rho)^{\phi y} = G\rho$ by definition. Furthermore, $(G\rho)^{\phi x} = G\rho$ and x is only defined up to multiplication by some element $n \in \mathrm{N}_C(G\rho)$. However by our convention, we know that either $n \in \mathrm{G}\rho$ in which case it is an inner automorphism and we can ignore it or $n \in \mathrm{Z}(C)$, i.e. a scalar matrix since C is irreducible. Hence, without loss of generality, $y = \mu x$ for some $\mu \in \mathbb{F}_{p^2}^\times$.

Now let $c = \delta$. Then

$$\begin{aligned}
yBy^{\mathrm{T}} &= A^{\phi}c^{\phi}\gamma^{k}hc^{-1}A^{-1}BA^{-\mathrm{T}}c^{-\mathrm{T}}h^{\mathrm{T}}\gamma^{k\mathrm{T}}c^{\phi\mathrm{T}}A^{\phi\mathrm{T}} \\
&= A^{\phi}c^{\phi}\gamma^{k}hc^{-1}Fc^{-\mathrm{T}}h^{\mathrm{T}}\gamma^{k\mathrm{T}}c^{\phi\mathrm{T}}A^{\phi\mathrm{T}} \text{ since } AFA^{\mathrm{T}} = B \\
&= \omega^{-1}A^{\phi}c^{\phi}Fc^{\phi\mathrm{T}}A^{\phi\mathrm{T}} \text{ since } h, \gamma \in \mathrm{GO}_n^+(p^2) \text{ and } cFc^{\mathrm{T}} = \omega F \\
&= \omega^{\phi-1}A^{\phi}FA^{\phi\mathrm{T}} \text{ since } F = F^{\phi} \\
&= \omega^{\phi-1}B^{\phi}.
\end{aligned}$$

If $c = 1$ then it is straightforward to show that $yBy^{\mathrm{T}} = B^{\phi}$. □

In particular, the previous lemma implies that we can find an $x \in \mathrm{GL}_n(p^2)$ such that $x^{-1}(g\rho)^{\phi}x = g^{\alpha}\rho$ for all $g \in G$ and $xBx^{\mathrm{T}} = B^{\phi}$ since $\omega^{\phi-1}$ is square in $\mathbb{F}_{p^2}$.

Lemma 4.8.7. *Recall our Convention 4.8.5. Assume that there exists $x \in \mathrm{GL}_n(p^2)$ such that $x^{-1}(g\rho)^{\phi}x = g^{\alpha}\rho$ for all $g \in G$ and $xBx^{\mathrm{T}} = B^{\phi}$. Furthermore, suppose that $(G.\langle\alpha\rangle)\rho \nleqslant C$ and that projectively $G.\langle\alpha\rangle\backslash G$ contains involutions. Let $c \in \{1, \delta\}$. Then the $\Omega_n^+(p^2)$-conjugacy class of $(G\rho)^{Ac}$ is stabilised by $\langle\phi\rangle$ in $\mathrm{Out}(\mathrm{O}_n^+(p^2))$ if and only if $\det(A^{1-\phi})\det(x) = 1$. Otherwise the $\Omega_n^+(p^2)$-conjugacy class of $(G\rho)^{Ac}$ is stabilised by $\langle\phi\gamma\rangle$.*

Proof. By Lemma 4.8.6 there exists $y = A^{\phi}c^{\phi}\gamma^{k}hc^{-1}A^{-1}$ for some $h \in \Omega_n^+(p^2)$ and $k \in \{0, 1\}$ such that $y = \mu x$ for some $\mu \in \mathbb{F}_{p^2}$.

First, let $c = \delta$. If $xBx^{\mathrm{T}} = B^{\phi}$ it is straightforward to show that $x = \omega^{(1-p)/2}y = \omega^{(1-p)/2}A^{\phi}c^{\phi}\gamma^{k}hc^{-1}A^{-1}$. Furthermore, the conjugacy class of $(G\rho)^{Ac}$ is stabilised ϕ if and only if $k = 0$. Note that $\gamma^{k} = \omega^{(p-1)/2}c^{-\phi}A^{-\phi}xAch^{-1}$ and hence a conjugacy class is stabilised by ϕ if and only if $\det(\gamma^{k}) = \det(\omega^{(p-1)/2}c^{-\phi}A^{-\phi}xAch^{-1}) = 1$. Since $\det(c^{1-\phi}) = (\det(c))^{1-\phi} = \omega^{(1-p)n/2}$, it follows that $\det(\gamma^{k}) = \det(A^{1-\phi})\det(x)$. This holds similarly when $c = 1$. □

Next we consider the case when the characteristic of the representation is 0. Let $\Omega_n(\mathbb{C}, \hat{B})$ be the commutator subgroup of $\mathrm{GO}_n(\mathbb{C}, \hat{B})$ which is the subgroup of all matrices in $\mathrm{GL}_n(\mathbb{C})$ that preserve some given non-degenerate quadratic form matrix $\hat{B}$. Also let $\bar{\Omega}_n(\mathbb{C}) \cong \Omega_n(\mathbb{C})/\mathrm{Z}(\Omega_n(\mathbb{C}))$.

Lemma 4.8.8. *Let $\hat{F} = antidiag(1, \ldots, 1) \in \mathrm{GL}_n(\mathbb{C})$. Then for all non-degenerate symmetric bilinear form matrices $\hat{B}$ over $\mathbb{C}$ there exists some $\hat{A} \in \mathrm{GL}_n(\mathbb{C})$ such that $\hat{A}\hat{F}\hat{A}^{\mathrm{T}} = \hat{B}$. Furthermore, if $G\hat{\rho} \leqslant \Omega_n(\mathbb{C}, \hat{B})$, then $(G\hat{\rho})^{\hat{A}} \leqslant \Omega_n(\mathbb{C}, \hat{F})$ where $\hat{\rho}$ is an absolutely irreducible representation of characteristic 0.*

Proof. By [1, Thm 20.10, p.85] all non-degenerate quadratic forms over $\mathbb{C}$ are congruent to each other. Furthermore, if the characteristic of the field does not equal 2, then a quadratic form Q and a symmetric bilinear form β uniquely determine each other. Let $\hat{B}$ the form matrix of such a symmetric bilinear form β. Hence for all such $\hat{B}$ there exists some $\hat{A} \in \mathrm{GL}_n(\mathbb{C})$ such that $\hat{A}\hat{F}\hat{A}^{\mathrm{T}} = \hat{B}$.

Now let $\hat{g} \in G\hat{\rho}$. Then $\hat{g}\hat{B}\hat{g}^{\mathrm{T}} = \hat{B}$ from which it follows that $\hat{g}^{\hat{A}}\hat{F}(\hat{g}^{\hat{A}})^{\mathrm{T}} = \hat{F}$. Therefore, $G\hat{\rho} \cong (G\hat{\rho})^{\hat{A}} \leqslant \Omega_n(\mathbb{C}, \hat{F})$. □

Remark 4.8.9. If a quadratic form in characteristic p acts on a vector space V, then V can be written as the sum of hyperbolic lines and a 2-dimensional space W. If the quadratic form is of plus-type, then W is a hyperbolic line as well but if the quadratic form is of minus-type, then W does not contain any singular vectors ([36, p.138]). In characteristic 0, W always contains a singular vector. This follows from the fact that $\mathbb{C}$ is algebraically closed and so any quadratic equation in 2 variables has a non-zero root.

In comparison, if $V = \mathbb{F}_q^2$, q odd, then $x^2 + y^2$ and $\nu x^2 + y^2$ are two non-equivalent orthogonal forms on V, where $\nu \in \mathbb{F}_q^{\times}$ is non-square. If $q \equiv 1 \pmod 4$, then $-1 = \tau^2$ is square in $\mathbb{F}_q^{\times}$ by Lemma 2.1.7 and hence $(1, \tau)$ is a singular vector of the orthogonal form $x^2 + y^2$, whereas there are no singular vectors when the orthogonal form is given by $\nu x^2 + y^2$. This works similarly when $q \equiv 3 \pmod 4$ ([9]).

If q is even then the vector $(0, 1)$ is a singular vector of the quadratic form $Q = xy$ whereas there is no non-zero vector $(x, y) \in \mathbb{F}_q^2$ such that the quadratic form $Q((x, y)) = x^2 + xy + y^2 = 0$ ([36, p.139]).

Field Automorphisms in Characteristic 2

We will now consider the case when the field has characteristic 2. Recall that $F = \mathrm{antidiag}(1, \dots, 1, 0, \dots, 0)$ is our standard quadratic form preserved by $\Omega_n^+(2^i)$ by Table 3.1.1. Let $\Omega_n^+(2^i, F) = \Omega_n^+(2^i)$. For this thesis it is sufficient to consider the field automorphisms of $\Omega_n^+(2^2)$.

Lemma 4.8.10. *Assume that $G\rho \leqslant \Omega_n^+(2^2) = \Omega$ and let α be an outer automorphism of G of order 2. Suppose that $|\mathrm{Out}(G)| = 2$. If ${}^{\alpha}\rho$ is equivalent to ρ^{ϕ}, then the Ω-conjugacy class of $G\rho$ is either stabilised by $\langle\phi\rangle$ or by $\langle\phi\gamma\rangle$ in $\mathrm{Out}(\bar{\Omega})$.*

Proof. Since $\mathrm{Out}(\bar{\Omega}) = \langle \gamma, \phi \mid \gamma^2 = \phi^2 = [\gamma, \phi] = 1 \rangle \cong \mathrm{C}_2 \times \mathrm{C}_2$ it is clear that the Ω-conjugacy class of $G\rho$ is stabilised by either ϕ or by $\phi\gamma$. □

Lemma 4.8.11. *Suppose $G\rho \leqslant \Omega_n^+(2^2) = \Omega$ where $|\mathrm{Out}(G)| = 2$. Let α be an outer automorphism of order 2 of G. Assume that there exists $x \in \mathrm{GL}_n(2^2)$ such that $x^{-1}(g\rho)^{\phi}x = g^{\alpha}\rho$ for all $g \in G$. Then there exists $\lambda \in \mathbb{F}_4^{\times}$ such that $\lambda x \in \mathrm{SO}_n^+(2^2)$. If the quasideterminant of λx equals 1 then the $\Omega_n^+(2^2)$-conjugacy class of $G\rho$ has stabiliser $\langle\phi\rangle$ in $\mathrm{Out}(\bar{\Omega})$. Otherwise the conjugacy class has stabiliser $\langle\phi\gamma\rangle$.*

Proof. By Lemma 4.8.10 we know that the Ω-conjugacy class of $G\rho$ is stabilised by $\phi\gamma^i$ where $i \in \{0, 1\}$. Hence, $G\rho^{\phi\gamma^i h} = G\rho$ for some $h \in \Omega$.

Furthermore, we know that there exists $x \in \mathrm{GL}_n(2^2)$ such that $G\rho^{\phi x} = G\rho$. By assumption $|\mathrm{Out}(G)| = 2$ and it follows from Lemma 3.6.4 that $\phi\gamma h$ equals ϕx in $\mathrm{Out}(\Omega)$. Hence there exists $\lambda \in \mathbb{F}_4$ such that $\lambda x \in \mathrm{SO}_n^+(2^2)$. Finally, note that $\gamma^i h$ has quasideterminant 1 if and only if $i = 0$. Hence the Ω-conjugacy class of $G\rho$ is stabilised by ϕ if and only if λx has quasideterminant 1. $\square$

4.8.2 Diagonal and Graph Automorphisms for Case $\mathrm{O}^{\pm}$

Finally we will consider diagonal and graph automorphisms of an orthogonal group $\Omega = \Omega_n^{\pm}(p, B)$ preserving any non-degenerate symmetric bilinear form B. We will begin with the case when p is odd. Let C denote the respective conformal group of Ω.

We will only consider the case when $n \equiv 2 \pmod 4$ and $q = p$. By Section 3.2.6 the presentation of the outer automorphism group of $\mathrm{O}_n^{\pm}(p, B)$ is given by

$$\mathrm{Out}(\mathrm{O}_n^{\pm}(p, B)) = \langle \delta', \gamma, \delta \mid \delta'^2 = \gamma^2 = 1, \delta^2 = \delta', \delta^{\gamma} = \delta^{-1} \rangle$$

when the discriminant of B is square in $\mathbb{F}_p$ and by

$$\mathrm{Out}(\mathrm{O}_n^{\pm}(p, B)) = \langle \gamma, \delta \mid \gamma^2 = \delta^2 = [\delta, \gamma] = 1 \rangle$$

when the discriminant of B is non-square in $\mathbb{F}_p$.

Lemma 4.8.12. *Let $G\rho \leqslant \Omega_n^{\pm}(p, B)$, where B is the form matrix of some non-degenerate symmetric bilinear form. Let $n \equiv 2 \pmod 4$, let p be odd and let b be the discriminant of B. Let $g \in \mathrm{CGO}_n^{\pm}(p, B)$, assume that g properly normalises $G\rho$ and assume that g induces an outer automorphism α of order 2 of $G\rho$. Let $A \in \mathrm{GL}_n(q)$ such that $AFA^{\mathrm{T}} = B$, where F is our standard form matrix as given in Table 3.1.1 and let $\Omega = \Omega_n^{\pm}(p, F)$. Let $\lambda \in \mathbb{F}_p$ such that $gBg^{\mathrm{T}} = \lambda B$. Then an Ω-conjugacy class of $(G\rho)^A$ is stabilised by:*

(i) $\langle\gamma\rangle$ *in* $\mathrm{Out}(\bar{\Omega})$ *if* $g \in \mathrm{GO}_n^{\pm}(p,B)\backslash\mathrm{SO}_n^{\pm}(p,B)$*;*

(ii) $\langle\delta'\rangle$ *in* $\mathrm{Out}(\bar{\Omega})$ *if* $g \in \mathrm{SO}_n^{\pm}(p,B)\backslash\Omega_n^{\pm}(p,B)$ *and b is square in* $\mathbb{F}_p$*;*

(iii) $\langle\gamma\delta\rangle$ *in* $\mathrm{Out}(\bar{\Omega})$ *if* $g \in \mathrm{CGO}_n^{\pm}(p,B)\backslash\mathrm{GO}_n^{\pm}(p,B)$ *and b is square in* $\mathbb{F}_p$*;*

(iv) $\langle\delta\rangle$ *in* $\mathrm{Out}(\bar{\Omega})$ *if* $g \in \mathrm{CGO}_n^{\pm}(p,B)\backslash\mathrm{GO}_n^{\pm}(p,B)$*, b is non-square in* $\mathbb{F}_p$ *and* $\det(g) = \lambda^{n/2}$*;*

(v) $\langle\gamma\delta\rangle$ *in* $\mathrm{Out}(\bar{\Omega})$ *if* $g \in \mathrm{CGO}_n^{\pm}(p,B)\backslash\mathrm{GO}_n^{\pm}(p,B)$*, b is non-square in* $\mathbb{F}_p$ *and* $\det(g) = -\lambda^{n/2}$*.*

Proof. Since $g \in \mathrm{CGO}_n^{\pm}(p,B)$ it follows that $g^A \in \mathrm{CGO}_n^{\pm}(p)$ by Lemma 4.8.1. Furthermore, g^A induces some $\beta \in \mathrm{Out}(\mathrm{O}_n^{\pm}(p))$. It is clear that $\det(g^A) = \det(g)$. If $g \in \mathrm{GO}_n^{\pm}(p,B)\backslash\mathrm{SO}_n^{\pm}(p,B)$ then $\mathrm{sp}(g^A)$ does not necessarily equal $\mathrm{sp}(g)$. However, if $\mathrm{sp}(g^A) \neq \mathrm{sp}(g)$ then $\mathrm{sp}(g^A) = \mathrm{sp}(g^\delta)$ by Lemma 3.2.3. Since we are only interested in the conjugacy classes of the outer automorphisms stabilising $G\rho$ by Lemma 4.3.7 we can without loss of generality assume that g induces $\beta \in \mathrm{Out}(\bar{\Omega})$.

(i) If $g \in \mathrm{GO}_n^{\pm}(p,B)\backslash\mathrm{SO}_n^{\pm}(p,B)$ then g induces γ if g has spinor norm 1 by Section 3.2.5 and Lemma 3.6.4. Now consider the case when $\mathrm{sp}(g) = -1$. If b is square then g induces $\gamma\delta'$ but $\delta^{-1}\gamma\delta = \gamma\delta'$ in $\mathrm{Out}(\bar{\Omega})$ and hence γ and $\gamma\delta'$ are conjugate in $\mathrm{Out}(\bar{\Omega})$. From this it follows by Lemma 4.3.7 that an Ω-conjugacy class of $G\rho$ is stabilised by $\langle\gamma\rangle$. If b is non-square then $\mathrm{sp}(-g) = 1$ by Lemma 3.1.21 and $-g$ normalises $G\rho$ as well. Hence we again have that an Ω-conjugacy class of $G\rho$ is stabilised by $\langle\gamma\rangle$.

(ii) This is obvious by Lemma 3.6.4.

(iii) Since α has order 2 and $g \in \mathrm{CGO}_n^{\pm}(p,B)\backslash\mathrm{GO}_n^{\pm}(p,B)$ it follows that $\beta \in \{\gamma\delta, \gamma\delta^3\}$ when b is square in $\mathbb{F}_p$. Since $\delta^{-1}(\gamma\delta)\delta = \gamma\delta^3$ we can deduce from Lemma 4.3.7 that an Ω-conjugacy class of $G\rho$ is stabilised by $\langle\gamma\delta\rangle$.

(iv), (v) When b is non-square and $g \in \mathrm{CGO}_n^{\pm}(p,B)\backslash\mathrm{GO}_n^{\pm}(p,B)$ induces an outer automorphism α of order 2 then $\alpha \in \{\delta, \delta\gamma\}$. Furthermore, δ and $\delta\gamma$ are not conjugate since they commute in $\mathrm{Out}(\bar{\Omega})$. Let D be the matrix corresponding to δ as defined in Section 3.2.5. Then $DFD^T = \omega F$ for some primitive element $\omega \in \mathbb{F}_p^\times$ and $\det(D) = \omega^{n/2}$. Hence $(ADA^{-1})B(A^{-\mathrm{T}}D^{\mathrm{T}}A^{\mathrm{T}}) = \omega B$ and there exists $\mu \in \mathbb{F}_p$ such that $\mu^{-1}g(ADA^{-1})^{-1}$ stabilises B. If $\det(\mu^{-1}g(ADA^{-1})^{-1}) = 1$, then g is induced by δ and a conjugacy class is stabilised by δ. If $\det(\mu^{-1}g(ADA^{-1})^{-1}) = -1$ then a conjugacy class is stabilised by $\delta\gamma$. By definition $gBg^{\mathrm{T}} = \lambda B$, where λ is non-square. Hence $\lambda = \omega^{2k+1}$ for some k. It is straightforward to show that $\mu = \omega^k$. Hence $\det(g) = \pm\det(\mu ADA^{-1}) = \pm\lambda^{n/2}$. □

Since $(G\rho)^A$ is equivalent to $G\rho$ we will usually just say that an Ω-conjugacy class of $G\rho$ is stabilised by β in $\mathrm{Out}(\bar{\Omega})$.

Now we consider the case when $q = 2$. Then $\mathrm{Out}(\mathrm{O}_n^{\pm}(2)) = \langle \gamma \mid \gamma^2 = 1 \rangle$ and the following is obvious.

Lemma 4.8.13. *Let $G\rho \leqslant \Omega_n^{\pm}(2)$. Let $g \in \mathrm{SO}_n^{\pm}(2) \backslash \Omega_n^{\pm}(2)$ and assume that g properly normalises $G\rho$. Then a conjugacy class of $G\rho$ in $\Omega_n^{\pm}(2)$ is stabilised by $\langle \gamma \rangle$ in $\mathrm{Out}(\mathrm{O}_n^{\pm}(2))$.*

Remark 4.8.14. *Let $\hat{\rho}$ be a characteristic 0 representation of G such that $G\hat{\rho}$ preserves an orthogonal form $\hat{B}$ for all $g \in G$. Suppose that the entries of $g\hat{\rho}$ lie in $R[\frac{1}{p_1}, \ldots, \frac{1}{p_k}]$, where R is the character ring of $\hat{\rho}$ and $p_1, \ldots, p_k$ are the exceptional primes of this representation. If there exists $\mu \in \mathbb{C}$ such that the entries of $\hat{\mu}\hat{B}$ and $(\hat{\mu}\hat{B})^{-1}$ lie in $R[\frac{1}{p_1}, \ldots, \frac{1}{p_k}]$ then we can reduce $G\hat{\rho}$ modulo p for any odd $p \neq p_i$. Furthermore, for each such p-modular reduction B of $\hat{B}$ there exists $A \in \mathrm{GL}_n(\mathbb{F}_q)$ such that $ABA^{\mathrm{T}} = F$, where F is our standard form matrix as defined in Table 3.1.1. Hence we can always use Lemma 4.8.12 since we never have to find A explicitly.*

4.9 $\mathscr{S}_1$-Maximality

Finally, suppose that we have found all potential $\mathscr{S}_1$-maximal subgroups in $\Omega.R$, where $\Omega \in \{\mathrm{SL}_n^{\pm}(q), \mathrm{Sp}_n(q), \Omega_n^{\epsilon}(q)\}$ and $R \leqslant \mathrm{Out}(\bar{\Omega})$. The final step is to show whether any of these $\mathscr{S}_1$-subgroups are contained in any other $\mathscr{S}_1$-subgroup preserving the same form, because in this case they cannot be maximal. The following is based on [8, Section 4.8, p.211].

Lemma 4.9.1. *Let $H_1\rho_1$ and $H_2\rho_2$ be two $\mathscr{S}_1$-subgroups of Ω preserving the same form. Assume that $H_1 \leqslant H_2$ and assume that there does not exist an element $g \in \mathrm{GL}_n(q^u)$ such that $(H_1\rho_2)^g$ is defined over a proper subfield of $\mathbb{F}_{q^u}$. If ρ_2 reduces to an absolutely irreducible representation of H_1^{∞} such that ρ_2 and ρ_1 are equivalent on H_1^{∞} then $H_1\rho_1$ cannot be maximal as an $\mathscr{S}_1$-subgroup.*

Proof. By definition, H_1 and H_2 are almost simple extensions of quasisimple groups. Therefore, $H_1 = H_1^{\infty}.R_1$ and $H_2 = H_2^{\infty}.R_2$, where R_1 and R_2 are subgroups of the outer automorphism groups of H_1^{∞} and H_2^{∞} respectively such that $H_1\rho_1 = \mathrm{N}_{\Omega}(H_1^{\infty}\rho_1)$ and $H_2\rho = \mathrm{N}_{\Omega}(H_2^{\infty}\rho_2)$. By assumption $H_1 \leqslant H_2$ and therefore ρ_2 is also a representation of H_1.

We want to show that ρ_2 is an absolutely irreducible representation of H_1. By assumption $H_1^{\infty}\rho_2$ does not stabilise any non-zero subspace of $(\mathbb{F}_{q^r})^n$ for any r. From this it follows that $(H_1^{\infty}.R_1)\rho_2 \geqslant H_1^{\infty}\rho_2$ does not stabilise any subspace either. Since ρ_1 is equivalent to ρ_2 on H_1^∞ it follows that $H_1\rho_1$ is not $\mathscr{S}_1$-maximal. $\square$

For many groups it is straightforward to show that $H_1^\infty\rho$ is irreducible. There are two easy methods which work in most cases.

Lemma 4.9.2. *Let ρ be a faithful absolutely irreducible representation of H_2^∞ of dimension n and suppose that $H_1^\infty \leqslant H_2^\infty$.*

(i) If H_1^∞ has no non-trivial absolutely irreducible representation of dimension smaller than n then $H_1^\infty\rho$ is absolutely irreducible since it cannot be split into smaller parts.

(ii) If H_1^∞ has non-trivial absolutely irreducible representations $\rho_1, \ldots, \rho_k$ of dimensions n_i, $i \in \{1, \ldots, k\}$, smaller than n with $\sum n_i = n$ but there exists some $g \in H_1^\infty$ such that $\sum Trace(g\rho_i) \neq Trace(g\rho)$ for every such set of representations $\rho_1, \ldots, \rho_k$, then $H_1^\infty\rho$ is irreducible.

5 Maximal $\mathscr{S}_1$-Subgroups in Dimension 13

In this chapter we are going to determine the $\mathscr{S}_1$-maximal subgroups G in dimension 13. We will follow the procedure described in Chapter 4. Here $\Omega \in \{\mathrm{SL}^{\pm}_{13}(q), \Omega^{\circ}_{13}(q)\}$. Furthermore, we will denote the conformal group of Ω by C.

5.1 $\mathscr{S}_1$-Subgroups in Dimension 13

Let G be a quasisimple group with an absolutely irreducible representation ρ of dimension 13 in cross characteristic. All such groups are listed in Table 5.1.1 on p.62. We are interested in finding the extensions by automorphisms of $G\rho$ which might be $\mathscr{S}_1$-maximal in some classical group. The table also contains some useful information about these groups which we will need later.

In the first column of the table we can find the name of the group followed by its order and Schur indicator. Column '$\#\rho$' gives the number of weakly equivalent representations of ρ that do not lie in the same equivalence class (see Definition 4.3.2). The outer automorphisms that stabilise these representations are given in the 'Stab' column. The characteristics over which these representations occur can then be found in the column 'Charc'. Here 0 stands for all prime numbers that do not divide the order of G. We also require the character rings of the representations (see Definition 2.2.2) which is given in the column 'ChR'. In the final column we state the size of the outer automorphism group of G. The list of groups G and the characteristics of the representations where taken from [18], whereas the information in the other columns is mostly from [12, 24]. The ordinary and Brauer character tables of A_{14} and A_{15} and the Brauer character tables of $\mathrm{S}_6(3)$ are not contained in [12, 24] and so GAP was used to determine these character tables.

Comments on the character ring column in Table 5.1.1

(i) The algebraic irrationalities of the 13-dimensional absolutely irreducible representations of $\mathrm{U}_3(4)$ are b_5 and z_5, but $\mathrm{b}_5 = \mathrm{z}_5 + \mathrm{z}_5^4$ and hence the character ring is $\mathbb{Z}[\mathrm{z}_5]$.

(ii) The respective rows of the character table of $\mathrm{S}_6(3)$ contain algebraic conjugates of b_{27}, z_3 and i_3. However, all of them are elements of $\mathbb{Z}[\mathrm{z}_3]$ since $\mathrm{i}_3 = \mathrm{z}_3 - \mathrm{z}_3^2$ and $\mathrm{b}_{27} = \frac{1}{2}(-1+\sqrt{-27}) = \frac{1}{2}(-1+\mathrm{i}_3) + \mathrm{i}_3 = \mathrm{b}_3 + \mathrm{i}_3 = \mathrm{z}_3 + \mathrm{z}_3 - \mathrm{z}_3^2 = 2\mathrm{z}_3 - \mathrm{z}_3^2$.

(iii) The algebraic irrationalities of the 13-dimensional absolutely irreducible representations of $S_4(5)$ are, apart from algebraic conjugates of b_5, algebraic conjugates of r_5, but $r_5 = 1 + 2b_5$.

Further information regarding the irrationalities can be found in Table 2.2.1 (p.19).

Table 5.1.1: Potential $\mathscr{S}_1$-maximal subgroups in dimension 13

Gp	**Order**	**Ind**	$\#\rho$	**Stab**	**Charc**	**ChR**	**|Out|**
$L_2(27)$	$2^2 \cdot 3^3 \cdot 7 \cdot 13$	$\circ$	2	3	$0, 2, 7, 13\,(\neq 3)$	$\mathbb{Z}[b_{27}]$	6
$S_6(3)$	$2^9 \cdot 3^9 \cdot 5 \cdot 7 \cdot 13$	$\circ$	2	1	$0, 2, 5, 7, 13\,(\neq 3)$	$\mathbb{Z}[z_3]$	2
$U_3(4)$	$2^6 \cdot 3 \cdot 5^2 \cdot 13$	$\circ$	4	1	$0, 3, 13\,(\neq 2, 5)$	$\mathbb{Z}[z_5]$	4
A_7	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	$+$	1	2	$3, 5$	$\mathbb{Z}$	2
A_8	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	$+$	1	2	$3, 5$	$\mathbb{Z}$	2
A_{14}	$2^{10} \cdot 3^5 \cdot 5^2 \cdot 7^2$ $\cdot 11 \cdot 13$	$+$	1	2	$0, 3, 5, 11, 13\,(\neq 2, 7)$	$\mathbb{Z}$	2
A_{15}	$2^{10} \cdot 3^6 \cdot 5^3 \cdot 7^2$ $\cdot 11 \cdot 13$	$+$	1	2	$3, 5$	$\mathbb{Z}$	2
$L_2(13)$	$2^2 \cdot 3 \cdot 7 \cdot 13$	$+$	1	2	$0, 3\,(\neq 2, 7, 13)$	$\mathbb{Z}$	2
$L_2(25)$	$2^3 \cdot 3 \cdot 5^2 \cdot 13$	$+$	2	2_2	$0, 3, 13\,(\neq 2, 5)$	$\mathbb{Z}$	2^2
$L_3(3)$	$2^4 \cdot 3^3 \cdot 13$	$+$	1	2	$0, 13\,(\neq 2, 3)$	$\mathbb{Z}$	2
$S_4(5)$	$2^6 \cdot 3^2 \cdot 5^4 \cdot 13$	$+$	2	1	$0, 3, 13\,(\neq 2, 5)$	$\mathbb{Z}[b_5]$	2
J_2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	$+$	2	1	3	$\mathbb{Z}[b_5]$	2

Theorem 5.1.1. *Let G be an $\mathscr{S}_1$-subgroup of $\Omega \in \{SL^{\pm}_{13}(q), \Omega^{\circ}_{13}(q)\}$. Then G is contained in Table 5.1.1.*

Proof. See the tables in [18]. □

5.2 Schur Indicator $\circ$

By Table 5.1.1, the potential $\mathscr{S}_1$-maximal subgroups of quasisimple linear and unitary classical groups in dimension 13 are extensions of $L_2(27)$, $S_6(3)$ and $U_3(4)$. Recall from Section 3.2.6 that when $q = p^2$ the field automorphism ϕ and the graph automorphism γ in $\mathrm{Out}(U_n(p))$ are equal. We will usually use γ to denote this automorphism.

Proposition 5.2.1 ($\mathrm{L}_2(27)$).

(i) If $p \equiv 1 \pmod 3$, then there are $(p-1, 13)$ conjugacy classes of $\mathscr{S}_1$-subgroups of $\mathrm{SL}_{13}(p)$ *isomorphic to* $\mathrm{L}_2(27).3$*, with class stabiliser* $\langle\gamma\rangle$ *in* $\mathrm{Out}(\mathrm{L}_{13}(p))$*.*

(ii) If $p \equiv 2 \pmod 3$, then there are $(p+1, 13)$ conjugacy classes of $\mathscr{S}_1$-subgroups of $\mathrm{SU}_{13}(p)$ *isomorphic to* $\mathrm{L}_2(27).3$*, with class stabiliser* $\langle\gamma\rangle$ *in* $\mathrm{Out}(\mathrm{U}_{13}(p))$*.*

Proof. Let $G = \mathrm{L}_2(27)$. Then $\mathrm{Out}(G) = 6$ and the character ring of a 13-dimensional absolutely irreducible representation ρ of G is $\mathbb{Z}[\mathrm{b}_{27}]$ by Table 5.1.1. Therefore $G \leqslant \mathrm{SL}_{13}(p)$ when $p \equiv 1 \pmod 3$ and $G \leqslant \mathrm{SU}_{13}(p)$ when $p \equiv 2 \pmod 3$ by Table 2.2.1 and Lemma 4.2.1. There are two weakly equivalent 13-dimensional representations and their stabiliser is generated by an automorphism of G of order 3 by Table 5.1.1.

Hence, $\mathrm{L}_2(27)$ extends to a subgroup of shape $\mathrm{L}_2(27).3$ inside $\mathrm{GL}_{13}(p^e)$ for some e. Let ρ_1, ρ_2, ρ_3 denote the 3 representations ρ splits into on $G.3$. Since the ρ_i have character ring $\mathbb{Z}[\mathrm{z}_3]$ and Schur indicator $\circ$ it follows that $\mathrm{L}_2(27).3 \leqslant \mathrm{GL}_{13}(p)$ if $p \equiv 1 \pmod 3$ and $\mathrm{L}_2(27).3 \leqslant \mathrm{GU}_{13}(p)$ if $p \equiv 2 \pmod 3$.

Now we have to find the determinants of all the matrices that lie in $(\mathrm{L}_2(27).3)\rho_i \backslash \mathrm{L}_2(27)\rho_i$. Since .3 is a cyclic extension it is sufficient to calculate the determinant of an element $h \in G.3\backslash G$ of order 3. Then h lies in the conjugacy class $3C$.

By Lemma 2.2.7 the eigenvalues of h are third roots of unity. By [12, 24], $\mathrm{Trace}(h\rho_1) = 1$. Suppose the eigenvalue 1 exists with multiplicity a, the eigenvalue z_3 with multiplicity b and z_3^2 with multiplicity c. We know that there have to be 13 eigenvalues in total since it is a representation of dimension 13. Therefore,

$$\begin{aligned} a + b + c &= 13 \\ a \cdot 1 + b \cdot \mathrm{z}_3 + c \cdot \mathrm{z}_3^2 &= 1 \end{aligned}$$

for $a, b, c \in \mathbb{N}$. Hence,

$$b(1 - z_3) + c(1 - z_3^2) = 12,$$

which implies that $c = b$ since the imaginary parts of $b\mathrm{z}_3$ and $c\mathrm{z}_3^2$ have to cancel. Since $\mathrm{z}_3 \cdot \mathrm{z}_3^2 = 1$, the determinant of $h\rho_1$ is $1^a \cdot \mathrm{z}_3^b \cdot (\mathrm{z}_3^2)^b = 1$. Hence, $\mathrm{L}_2(27).3 \leqslant \mathrm{SL}_{13}(p)$ if $p \equiv 1 \pmod 3$ and $\mathrm{L}_2(27).3 \leqslant \mathrm{SU}_{13}(p)$ if $p \equiv 2 \pmod 3$.

Note that two weakly equivalent representations are fused by automorphisms of order 2 and 6 ([12, 24]). It follows that there exists a single conjugacy class of $\mathscr{S}_1$-subgroups isomorphic to $\mathrm{L}_2(27).3$ in $\mathrm{GL}_{13}(p)$ or $\mathrm{CGU}_{13}(p)$ by Lemma 4.2.2 and [12, 24]. Also, $\mathrm{L}_2(27).3$ is scalar-normalising and δ acts transitively on the conjugacy classes of $\mathrm{L}_2(27).3$ in $\mathrm{SL}^{\pm}_{13}(p)$ by Lemma 4.3.6. Furthermore, by Lemma 4.3.3(ii), there exist $(p \mp 1, 13)$ conjugacy classes of $G.3$ in $\mathrm{SL}_{13}(p)$.

Finally, we have to consider the action of $\gamma \in \mathrm{Out}(\mathrm{L}^{\pm}_{13}(p))$ on $G.3$. Note that the automorphism $\alpha \in \mathrm{Out}(\mathrm{L}_2(27))$ of order 2 sends the representations to their complex conjugates since b_{27}^{**} is the complex conjugate of b_{27} ([12, 24]). From this it follows that ${}^{\alpha}\rho$ is equivalent to ρ^{γ} by Lemma 4.4.1. Since d is odd, γ stabilises an $\mathrm{SL}_{13}(p)$-conjugacy class of $G.3$ by Lemma 4.4.2 and similarly γ stabilises an $\mathrm{SU}_{13}(p)$-conjugacy class of $G.3$ by Lemma 4.5.1. □

Proposition 5.2.2 ($\mathrm{S}_6(3)$)**.**

(i) If $p \equiv 1 \pmod 3$*, then* $\mathrm{SL}_{13}(p)$ *has* $(p-1, 13)$ *conjugacy classes of* $\mathscr{S}_1$*-subgroups isomorphic to* $\mathrm{S}_6(3)$*, with class stabiliser* $\langle\gamma\rangle$ *in* $\mathrm{Out}(\mathrm{L}_{13}(p))$*.*

(ii) If $p \equiv 2 \pmod 3$*, then* $\mathrm{SU}_{13}(p)$ *has* $(p+1, 13)$ *conjugacy classes of* $\mathscr{S}_1$*-subgroups isomorphic to* $\mathrm{S}_6(3)$*, with class stabiliser* $\langle\gamma\rangle$ *in* $\mathrm{Out}(\mathrm{U}_{13}(p))$*.*

Proof. Let $G = \mathrm{S}_6(3)$. Then $\mathrm{Out}(G) = 2$ and the character ring of a 13-dimensional absolutely irreducible representation ρ of G is $\mathbb{Z}[\mathrm{z}_3]$ by Table 5.1.1. Hence $\mathrm{S}_6(3)$ preserves a unitary form if $p \equiv 2 \pmod 3$ and no non-zero form if $p \equiv 1 \pmod 3$ by Lemma 4.2.1 and Table 2.2.1.

There are two weakly equivalent representations of G that are fused by the non-trivial outer automorphism α of G. Therefore there is one conjugacy class of G in C by Lemma 4.2.2, and G is scalar-normalising. Hence the number of conjugacy classes of subgroups isomorphic to $\mathrm{S}_6(3)$ in $\mathrm{SL}^{\pm}_{13}(p)$ is $d = (13, q \mp 1)$ respectively by Lemma 4.3.3(ii). Furthermore, by Lemma 4.3.6, the diagonal automorphism δ of $\mathrm{SL}^{\pm}_{13}(p)$ acts transitively on these conjugacy classes.

By looking at the character tables of $\mathrm{S}_6(3)$ in [12] and using GAP for the Brauer character tables we see that two non-equivalent 13-dimensional absolutely irreducible representations are algebraic conjugates of each other. Hence we have to prove that the algebraic conjugate of each irrationality given here is indeed its complex conjugate. We can see that the algebraic conjugate of i_3 is $-\mathrm{i}_3$ which is indeed its complex conjugate, the algebraic conjugate of z_3 is z_3^2 which is also its complex conjugate and the algebraic conjugate of $\mathrm{b}_{27} = \frac{1}{2}(-1 + i\sqrt{27})$ is $\frac{1}{2}(-1 - i\sqrt{27})$. Again this is the complex conjugate of b_{27}. Hence ${}^{\alpha}\rho$ is equivalent to ρ^{γ} by Lemma 4.4.1. Since d is odd, the result follows by Lemma 4.4.2 in Case **L** and by Lemma 4.5.1 in Case **U**. □

Proposition 5.2.3 ($\mathrm{U}_3(4)$)**.**

(i) *If* $p \equiv 1 \pmod 5$*, then there are* $(p-1, 13)$ *conjugacy classes of* $\mathscr{S}_1$*-subgroups of* $\mathrm{SL}_{13}(p)$ *isomorphic to* $\mathrm{U}_3(4)$*, with class stabiliser* $\langle\gamma\rangle$ *in* $\mathrm{Out}(\mathrm{L}_{13}(p))$*.*

(ii) *If* $p \equiv 2, 3 \pmod 5$*,* $p \neq 2$*, then there are* $(p^2+1, 13)$ *conjugacy classes of* $\mathscr{S}_1$*-subgroups of* $\mathrm{SU}_{13}(p^2)$ *isomorphic to* $\mathrm{U}_3(4)$*, with class stabiliser* $\langle\phi\rangle$ *in* $\mathrm{Out}(\mathrm{U}_{13}(p^2))$*.*

(iii) *If* $p \equiv 4 \pmod 5$*, then there are* $(p+1, 13)$ *conjugacy classes of* $\mathscr{S}_1$*-subgroups of* $\mathrm{SU}_{13}(p)$ *isomorphic to* $\mathrm{U}_3(4)$*, with class stabiliser* $\langle\gamma\rangle$ *in* $\mathrm{Out}(\mathrm{U}_{13}(p))$*.*

Proof. Let $G = \mathrm{U}_3(4)$. Then $\mathrm{Out}(G) = 4$ and the relevant 13-dimensional absolutely irreducible representations ρ of G have character ring the p-modular reduction of $\mathbb{Z}[\mathrm{z}_5]$ by Table 5.1.1. By Table 2.2.1, $\mathrm{z}_5 \in \mathbb{F}_p$ if and only if $p \equiv 1 \pmod 5$ and it is obvious that in this case $G \leqslant \mathrm{SL}_{13}(p)$.

Let a be the Brauer character value of $g \in G\rho$ and let $\overline{a}$ denote the complex conjugate of a. If $a \in \mathbb{Z}$, then $a^q = a = \overline{a}$. If $a = \mathrm{z}_5^i \in \mathbb{C}$, then $a^q = (\mathrm{z}_5^i)^q = \mathrm{z}_5^{4i} = \mathrm{z}_5^{-i} = \overline{a}$, where $q = p$ if $p \equiv 4 \pmod 5$ and $q = p^2$ when $p \equiv 2, 3 \pmod 5$. Hence $G \leqslant \mathrm{SU}_{13}(p)$ if $p \equiv 4 \pmod 5$ and $G \leqslant \mathrm{SU}_{13}(p^2)$ when $p \equiv 2, 3 \pmod 5$ by Lemma 4.2.1.

There are up to equivalence four weakly equivalent representations with trivial stabiliser by Table 5.1.1. Hence G is scalar-normalising and since the outer automorphism of order 4 of $\mathrm{U}_3(4)$ fuses the representations there is a single conjugacy class of subgroups G in C by Lemma 4.2.2. Therefore, $|C : \mathrm{N}_C(G\rho)\Omega| = (p-1, 13)$ in Case **L** and $(p+1, 13)$ or $(p^2+1, 13)$ in Case **U** by Lemma 4.3.3.

To find the stabiliser of one of these classes we will first consider the case when $p \equiv 1, 4 \pmod 5$. Then $\mathrm{Out}(\mathrm{L}^{\pm}_{13}(p)) = \langle\delta, \gamma\rangle \cong \mathrm{C}_2$ or $\mathrm{D}_{13\times 2}$. Since $\mathrm{Out}(G) = 4$ there exists $\alpha \in \mathrm{Out}(G)$ with $|\alpha| = 2$. By looking at [12] and [24], we see that α sends the representations to their complex conjugates and therefore ${}^{\alpha}\rho$ is equivalent to ρ^{γ}. Since $d = (p \mp 1, 13)$ is odd, the result follows from Lemma 4.4.2 in Case **L** and from Lemma 4.5.1 in Case **U**.

If $p \equiv 2, 3 \pmod 5$, we want to determine whether $\phi \in \mathrm{Out}(\mathrm{U}_{13}(p^2))$ corresponds to an outer automorphism α of $\mathrm{U}_3(4)$ of order 4. By Lemma 4.3.8 we need to show that for all $p \equiv 2, 3 \pmod 5$ there exist two representations

ρ_i and ρ_j such that $\mathrm{Trace}(g\rho_j) = (\mathrm{Trace}((g\rho_i))^p$ for all $g \in G$. By looking at [12, 24], we can see that this is indeed the case. In fact, ${}^{\alpha}\rho_1 = \rho_3$ when $p \equiv 3 \pmod 5$ and ${}^{\alpha}\rho_1 = \rho_4$ when $p \equiv 3 \pmod 5$. Here ρ_1 corresponds to the representation denoted by χ_3 in [12], ρ_3 to χ_5 and ρ_4 to χ_6. The result now follows from Lemma 4.5.1. □

Maximality

The final step is now to show whether any of the three $\mathscr{S}_1$-subgroups $\mathrm{U}_3(4)$, $\mathrm{S}_6(3)$ and $\mathrm{L}_2(27).3$ is contained in one of the others as an $\mathscr{S}_1$-subgroup. By Lagrange's theorem the only possible containment is $\mathrm{L}_2(27).3$ in $\mathrm{S}_6(3)$.

Proposition 5.2.4.
Let $d := (p-1, 13)$ in Case **L** *and let $d := (p+1, 13)$ in Case* **U**.

(i) No extension of $d \times \mathrm{L}_2(27).3$ is ever $\mathscr{S}_1$-maximal in any extension of $\mathrm{SL}^{\pm}_{13}(p)$.

(ii) If $p \neq 3$, then $d \times \mathrm{S}_6(3)$ is $\mathscr{S}_1$-maximal in $\mathrm{SL}^{\pm}_{13}(p)$.

(iii) If $p \neq 2, 5$, then $\mathrm{N}_{\mathrm{SL}^{\pm}_{13}(q)}(\mathrm{U}_3(4))$ is $\mathscr{S}_1$-maximal in $\mathrm{SL}^{\pm}_{13}(q)$.

Proof. (i) By [12], the group $\mathrm{L}_2(27).3$ is a subgroup of $\mathrm{S}_6(3)$. Also, the smallest non-trivial absolutely irreducible representations of $\mathrm{L}_2(27)$ have dimension 13 in characteristic $0, 2, 7$ and 13 by [12, 24]. Hence $\mathrm{L}_2(27).3$ is never $\mathscr{S}_1$-maximal in $\mathrm{SL}^{\pm}_{13}(p)$ by Lemma 4.9.2. Furthermore, $\mathrm{L}_2(27).6$ is a subgroup of $\mathrm{S}_6(3).2$ by [12]. Hence, no extension of $\mathrm{L}_2(27)$ is ever $\mathscr{S}_1$-maximal in any extension of $\mathrm{SL}^{\pm}_{13}(p)$.

(ii) and (iii) follow since the group orders are not divisors of each other. □

5.3 Schur Indicator +

Now we consider the quasisimple groups whose 13-dimensional absolutely irreducible representations preserve an orthogonal form. By Table 5.1.1 these groups are A_7, A_8 and A_{15} in characteristics 3 and 5, A_{14}, $\mathrm{L}_2(13)$, $\mathrm{L}_2(25)$, $\mathrm{L}_3(3)$ and $\mathrm{S}_4(5)$ in various characteristics and J_2 in characteristic 3 only.

The outer automorphisms of $\Omega^{\circ}_n(q, B)$ are independent of the preserved non-degenerate symmetric bilinear form B. Hence, even if we work computationally with a representation ρ of G such that $G\rho$ does not preserve our standard form matrix F as given in Table 3.1.1, we will give the stabiliser of $G\rho$ with respect to $\Omega^{\circ}_{13}(q, F) = \Omega^{\circ}_{13}(q)$.

Proposition 5.3.1 (A_7 in characteristics 3 and 5)**.**

(i) There is a unique conjugacy class of $\mathscr{S}_1$-subgroups of $\Omega^{\circ}_{13}(3)$ isomorphic to A_7, with class stabiliser $\langle\delta\rangle$ in $\mathrm{Out}(\mathrm{O}^{\circ}_{13}(3))$.

(ii) There are two conjugacy classes of $\mathscr{S}_1$-subgroups of $\Omega^{\circ}_{13}(5)$ isomorphic to $\mathrm{A}_7.2$, with trivial class stabiliser in $\mathrm{Out}(\mathrm{O}^{\circ}_{13}(5))$.

Proof. Let $G = \mathrm{A}_7$ and let $\Omega = \Omega^{\circ}_{13}(p)$. Then $\mathrm{Out}(G) = 2$ and the unique (up to equivalence) absolutely irreducible representation ρ of G has character ring $\mathbb{Z}$. Hence $\mathrm{A}_7 \leqslant \Omega$ when $p = 3, 5$. Also ρ splits into ρ_1 and ρ_2 under an outer automorphism of order 2 of G by Table 5.1.1. By [24], $G.2$ preserves an orthogonal form and has character ring $\mathbb{Z}$. Using Lemma 4.2.4 it is straightforward to show that $G.2 \leqslant \mathrm{SO}^{\circ}_{13}(p)$. We will use Magma and the specific absolutely irreducible 13-dimensional representations of G given in [6] (file a7d13comp) to determine the spinor norm of elements of $G.2\backslash G$. Note that two elements $x, y \in G$ are standard generators of G if y lies in class $3A$, y has order 5 and xy has order 7 by [6].

Since $G\rho$ preserves an orthogonal form we know that there exists a form matrix B that is induced by a symmetric bilinear form. Then $(g\rho)B(g\rho)^{\mathrm{T}} = B$ for all $g \in G$. Rearranging we get $B^{-1}(g\rho)B = (g\rho)^{-\mathrm{T}}$ for all $g\rho \in G\rho$ and by using the GHom command in Magma we can find a matrix homomorphism B that sends $g\rho$ to $(g\rho)^{-\mathrm{T}}$ by conjugation for all $g\rho \in G\rho$. Furthermore we have to find an element $g \in (G.2)\rho_1\backslash G\rho$. By using the automorphism of order 2 that sends x to x^{-1} and y to y ([6]) we can generate a group $H = \langle x^{-1}, y\rangle$ isomorphic to $G\rho$ and again using the GHom command we can then find an element $g \in (G.2)\rho_1\backslash G\rho$ such that $g^{-1}Hg = G\rho$. Using the SpinorNorm command in Magma we see that $\mathrm{sp}(g) = -1$ in characteristic 3 and $\mathrm{sp}(g) = 1$ in characteristic 5.

Hence, in characteristic 3, $\mathrm{A}_7.2 \nleqslant \Omega^{\circ}_{13}(3)$. Furthermore, there exists a unique C-conjugacy class of $\mathscr{S}_1$-subgroups G and $|C : \mathrm{N}_C(G)\Omega| = 1$ conjugacy class of G in $\Omega^{\circ}_{13}(3)$ by Lemma 4.3.3(ii). This conjugacy class is stabilised by δ since $\mathrm{S}_7 \leqslant \mathrm{SO}^{\circ}_{13}(3)$.

In characteristic 5, $\mathrm{A}_7.2 \leqslant \Omega^{\circ}_{13}(5)$. Since $G.2$ is scalar-normalising there are 2 conjugacy classes in $\Omega^{\circ}_{13}(5)$ by Lemma 4.3.3. Furthermore G has no further outer automorphisms and therefore the stabiliser of these classes is trivial. □

Proposition 5.3.2 (A_8 in characteristics 3 and 5)**.**

(i) There is a unique conjugacy class of $\mathscr{S}_1$-subgroups of $\Omega^\circ_{13}(3)$ isomorphic to A_8*, with class stabiliser* $\langle\delta\rangle$ *in* $\mathrm{Out}(\mathrm{O}^\circ_{13}(3))$.

(ii) There are two conjugacy classes of $\mathscr{S}_1$-subgroups of $\Omega^\circ_{13}(5)$ isomorphic to $\mathrm{A}_8.2$*, with trivial class stabiliser in* $\mathrm{Out}(\mathrm{O}^\circ_{13}(5))$.

Proof. The proof is very similar to the proof of Proposition 5.3.1. Again we use Magma to find the spinor norm of elements (file a8d13comp). Let ρ be a 13-dimensional absolutely irreducible representation of S_8. If $g \in \mathrm{S}_8\rho\backslash\mathrm{A}_8\rho$ then $\mathrm{sp}(g) = -1$ in characteristic 3 and $\mathrm{sp}(g) = 1$ in characteristic 5. □

Proposition 5.3.3 (A_{14})**.**

(i) If $p \equiv 1, 3, 9, 19, 25, 27 \pmod{28}$ *then there are two conjugacy classes of $\mathscr{S}_1$-subgroups of $\Omega^\circ_{13}(p)$ isomorphic to* $\mathrm{A}_{14}.2$*, with trivial class stabiliser in* $\mathrm{Out}(\mathrm{O}^\circ_{13}(p))$.

(ii) If $p \equiv 5, 11, 13, 15, 17, 23 \pmod{28}$ *then there exists a unique conjugacy class of $\mathscr{S}_1$-subgroups of $\Omega^\circ_{13}(p)$ isomorphic to* A_{14}*, with class stabiliser* $\langle\delta\rangle$ *in* $\mathrm{Out}(\mathrm{O}^\circ_{13}(p))$.

Proof. Let $G = \mathrm{A}_{14}$, let $\Omega = \Omega^\circ_{13}(p)$ and let ρ be the unique (up to equivalence) absolutely irreducible representation of dimension 13 of G. By Table 5.1.1 and [12, 24] we know that $G.2 \leqslant \mathrm{GO}^\circ_{13}(p)$. Furthermore, by Lemma 4.2.4, either $G.2\rho_1 \leqslant \mathrm{SO}^\circ_{13}(p)$ or $G.2\rho_2 \leqslant \mathrm{SO}^\circ_{13}(p)$, where ρ_1 and ρ_2 are the two representations into which ρ splits when extended to $G.2$. Hence $G.2 \leqslant \mathrm{SO}^\circ_{13}(p)$. To find the spinor norm of an element $g \in G\rho.2\backslash G\rho$ we will use Magma as for Proposition 5.3.1 (see file a14d13comp). Let B be the symmetric bilinear form preserved by $G\rho$.

Let $A := I_{13} - g$ and let M be the matrix whose rows form a basis of a complement of the nullspace of A using Magma again. We find that $\det(MABM^{\mathrm{T}}) = 2^{12} \cdot 7$ which is a square in $\mathbb{F}_p$ if and only if 7 is a square in $\mathbb{F}_p$. By Table 2.2.1 and Lemma 3.1.19 this implies that $\mathrm{A}_{14}.2 \leqslant \Omega$ if and only if $p \equiv 1, 3, 9, 19, 25, 27 \pmod{28}$. Otherwise $\mathrm{A}_{14}.2 \leqslant \mathrm{SO}^\circ_{13}(p)$ but $\mathrm{A}_{14}.2 \nleqslant \Omega$.

If $\mathrm{A}_{14}.2 \leqslant \Omega$ then $G.2$ is scalar-normalising and by Lemma 4.3.3 there exist 2 conjugacy classes of $\mathscr{S}_1$-subgroups isomorphic to $\mathrm{A}_{14}.2$ in Ω. Since A_{14} has no further non-trivial outer automorphisms the stabiliser of these classes is trivial.

If $p \cong 5, 11, 13, 15, 17, 23 \pmod{28}$ then $G.2 \nleqslant \Omega$ and there is a unique C-conjugacy class of $\mathscr{S}_1$-subgroups A_{14} in C and hence a unique class in Ω. This class is stabilised in $\mathrm{Out}(\bar{\Omega})$ by $\langle\delta\rangle$ since $\mathrm{S}_{14} \leqslant \mathrm{SO}^{\circ}_{13}(p)$. □

Proposition 5.3.4 (A_{15} in characteristics 3 and 5)**.**

(i) There are two conjugacy classes of $\mathscr{S}_1$-subgroups of $\Omega^{\circ}_{13}(3)$ isomorphic to S_{15}, with trivial class stabiliser in $\mathrm{Out}(\mathrm{O}^{\circ}_{13}(3))$.

(ii) There is a unique conjugacy class of $\mathscr{S}_1$-subgroups of $\Omega^{\circ}_{13}(5)$ isomorphic to A_{15}, with class stabiliser $\langle\delta\rangle$ in $\mathrm{Out}(\mathrm{O}^{\circ}_{13}(5))$.

Proof. The proof is very similar to the proof of Proposition 5.3.1. The information about this group was taken from GAP and [6]. See file a15d13comp for the Magma commands. □

Proposition 5.3.5 ($\mathrm{L}_2(13)$)**.**

(i) If $p \equiv 1, 3, 9, 19, 25, 27 \pmod{28}$ then there are two conjugacy classes of $\mathscr{S}_1$-subgroups of $\Omega^{\circ}_{13}(p)$ isomorphic to $\mathrm{L}_2(13).2$, with trivial class stabiliser in $\mathrm{Out}(\mathrm{O}^{\circ}_{13}(p))$.

(ii) If $p \equiv 5, 11, 13, 15, 17, 23 \pmod{28}$ then there is a unique conjugacy class of $\mathscr{S}_1$-subgroups of $\Omega^{\circ}_{13}(p)$ isomorphic to $\mathrm{L}_2(13)$, with class stabiliser $\langle\delta\rangle$ in $\mathrm{Out}(\mathrm{O}^{\circ}_{13}(p))$.

Proof. Let $G = \mathrm{L}_2(13)$ and let $\Omega = \Omega^{\circ}_{13}(p)$. Following the same procedure as for Proposition 5.3.3 it is straightforward to show that $G.2 \leqslant \mathrm{SO}^{\circ}_{13}(p)$. Furthermore, using Magma and Lemma 3.1.19, we can show that $G.2 \leqslant \Omega^{\circ}_{13}(p)$ if and only if 7 is a square in $\mathbb{F}_p$ (see file l213d13comp). The number of conjugacy classes follows from Lemma 4.3.3. □

Proposition 5.3.6 ($\mathrm{L}_2(25)$)**.**

(i) If $p \equiv 1, 4 \pmod 5$, then there exist two conjugacy classes of $\mathscr{S}_1$-subgroups of $\Omega^{\circ}_{13}(p)$ isomorphic to $\mathrm{L}_2(25).2_2$, with trivial class stabiliser in $\mathrm{Out}(\mathrm{O}^{\circ}_{13}(p))$.

(ii) If $p \equiv 2, 3 \pmod 5$, $p \neq 2$, then there exists a single conjugacy class of $\mathscr{S}_1$-subgroups of $\Omega^{\circ}_{13}(p)$ isomorphic to $\mathrm{L}_2(25)$, with class stabiliser $\langle\delta\rangle$ in $\mathrm{Out}(\mathrm{O}^{\circ}_{13}(p))$.

Proof. Let $G = \mathrm{L}_2(25)$. Then $\mathrm{Out}(G) = 2^2$ by [12]. Up to equivalence there are two absolutely irreducible 13-dimensional representations ρ_1 and ρ_2 of G that are weakly equivalent. Since the character ring of ρ_1 and ρ_2 is $\mathbb{Z}$ we can deduce that $\mathrm{L}_2(25) \leqslant \Omega^{\circ}_{13}(p) = \Omega$ $(p \neq 2, 5)$. Since the two representations are fused by the 2_1 outer automorphism of G by Table 5.1.1, there is a single conjugacy class in C. The 2_2 outer automorphism on the other hand splits the representations ρ_1 and ρ_2 and therefore $G.2_2 \leqslant \mathrm{N}_C(G\rho)$. By [12, 24] and Lemma 4.2.4, $\mathrm{L}_2(25).2_2 \leqslant \mathrm{SO}^{\circ}_{13}(p)$.

To show that $\mathrm{L}_2(25).2_2 \leqslant \Omega$ we have to check the spinor norm of the elements. Computations in Magma (file l225d13comp) show that the spinor norm is 1 if and only if 5 is a square in $\mathbb{F}_p$. If 5 is not square then the 2_2 outer automorphism induces $\delta \in \mathrm{Out}(\bar{\Omega})$. The number of conjugacy classes follows from Lemma 4.3.3. □

Proposition 5.3.7 ($\mathrm{L}_3(3)$)**.**

(i) If $p \equiv 1, 11 \pmod{12}$ *then* $\Omega^{\circ}_{13}(p)$ *has 2 conjugacy classes of* $\mathscr{S}_1$*-subgroups isomorphic to* $\mathrm{L}_3(3).2$*. Both classes have trivial class stabiliser in* $\mathrm{Out}(\mathrm{O}^{\circ}_{13}(p))$*.*

(ii) If $p \equiv 5, 7 \pmod{12}$ *then there exists a single conjugacy class of* $\mathscr{S}_1$*-subgroups of* $\Omega^{\circ}_{13}(p)$ *isomorphic to* $\mathrm{L}_3(3)$*, with class stabiliser* $\langle\delta\rangle$ *in* $\mathrm{Out}(\mathrm{O}^{\circ}_{13}(p))$*.*

Proof. Let $G = \mathrm{L}_3(3)$ and let $\Omega = \Omega^{\circ}_{13}(p)$. Then $\mathrm{Out}(G) = 2$ and there is (up to equivalence) one absolutely irreducible 13-dimensional representation ρ of G. This representation is split by the outer automorphism of order 2 of G by Table 5.1.1. Since the character ring is $\mathbb{Z}$ we have $\mathrm{L}_3(3) \leqslant \Omega$ if $p \neq 2, 3$. By [12, 24] and Lemma 4.2.4, $G.2 \leqslant \mathrm{SO}^{\circ}_{13}(p)$.

To find the spinor norm of the elements of $G.2 \backslash G$ we will use Lemma 3.1.19 and Magma (see file l33d13comp). We find that $G.2 \leqslant \Omega$ if and only if 3 is a square in $\mathbb{F}_p$. The number of conjugacy classes can be calculated using Lemma 4.3.3. □

Proposition 5.3.8 ($\mathrm{S}_4(5)$)**.**

(i) If $p \equiv 1, 4 \pmod 5$ *then there exist two conjugacy classes of* $\mathscr{S}_1$*-subgroups of* $\Omega^{\circ}_{13}(p)$ *isomorphic to* $\mathrm{S}_4(5)$*, with trivial class stabiliser in* $\mathrm{Out}(\mathrm{O}^{\circ}_{13}(p))$*.*

(ii) If $p \equiv 2, 3 \pmod 5$*,* $p \neq 2$*, then there are two conjugacy classes of* $\mathscr{S}_1$*-subgroups of* $\Omega^{\circ}_{13}(p^2)$ *isomorphic to* $\mathrm{S}_4(5)$*, with class stabiliser* $\langle\phi\rangle$ *in* $\mathrm{Out}(\mathrm{O}^{\circ}_{13}(p^2))$*.*

Proof. Let $G = \mathrm{S}_4(5)$. Then $\mathrm{Out}(G) = 2$ and there are up to equivalence two weakly equivalent 13-dimensional absolutely irreducible representations ρ_1 and ρ_2 of G with character ring $\mathbb{Z}[\mathrm{b}_5]$ by [12, 24] and Table 5.1.1.

Hence if $p \equiv 1, 4 \pmod 5$, then $G \leqslant \Omega_{13}^{\circ}(p) = \Omega$ by Table 5.1.1 and Table 2.2.1. Also there is no outer automorphism of G that splits these representations. But the two representations are fused by the outer automorphism of G of order 2 ([12, 24]) which implies that there is a single conjugacy class in C and hence there exist two conjugacy classes of G in Ω by Lemma 4.3.3. Since there are no field automorphisms and the diagonal automorphism acts transitively on the two Ω-conjugacy classes by Lemma 4.3.6, the stabiliser of these classes is trivial.

If $p \equiv 2, 3 \pmod 5$, $p \neq 2$, then $\mathrm{S}_4(5)$ can only be realised in an orthogonal group with field size at least p^2 by Table 5.1.1 and Table 2.2.1. Again, there are two conjugacy classes of $\mathscr{S}_1$-subgroups isomorphic to $\mathrm{S}_4(5)$ in $\Omega = \Omega_{13}^{\circ}(p^2)$. Let ϕ be the field automorphism of order 2 of Ω. Let α be an outer automorphism of order 2 of G such that α fuses ρ_1 and ρ_2. We can show that ${}^{\alpha}\rho_1$ is equivalent to ρ_1^{ϕ} using Lemma 4.3.8. Since $G.2\backslash G$ contains involutions, an Ω-conjugacy class of $\mathrm{S}_4(5)$ is stabilised by $\langle\phi\rangle \in \mathrm{Out}(\bar{\Omega})$ by Lemma 4.7.2. □

Proposition 5.3.9 (J_2).
There are exactly 2 *conjugacy classes of* $\mathscr{S}_1$*-subgroups of* $\Omega_{13}(3^2)$ *isomorphic to* J_2*, with class stabiliser* $\langle\phi\rangle$ *in* $\mathrm{Out}(\mathrm{O}_{13}^{\circ}(9))$.

Proof. Let $G = \mathrm{J}_2$. Then $\mathrm{Out}(G) = 2$ by [12]. Since the character ring of the 13-dimensional absolutely irreducible representations ρ of G is $\mathbb{Z}[\mathrm{b}_5]$ by Table 5.1.1 it follows that $\mathrm{J}_2 \leqslant \Omega_{13}^{\circ}(9) = \Omega$ by Table 2.2.1. Up to equivalence there are two weakly equivalent representations of G which are fused by an outer automorphism α of order 2 of J_2 by [24]. Therefore, $\mathrm{N}_C(G)\Omega = \mathrm{Z}(C)\Omega$ which implies that there are two conjugacy classes of G in Ω by Lemma 4.3.3.

Finally we want to know how the field automorphism ϕ of order 2 of Ω acts on these conjugacy classes. Using Lemma 4.3.8 we can show that ρ^{ϕ} is equivalent to ${}^{\alpha}\rho$, where $\alpha \in \mathrm{Out}(G)$ of order 2. Since $\mathrm{J}_2.2\backslash\mathrm{J}_2$ contains involutions, the class stabiliser of G in Ω is $\langle\phi\rangle$ by Lemma 4.7.2. □

Maximality

Finally we have to consider possible containments between all of the $\mathscr{S}_1$-subgroups of orthogonal groups in dimension 13. The following proposition summarises the $\mathscr{S}_1$-containments of the $\mathscr{S}_1$-subgroups of orthogonal groups in dimension 13.

Proposition 5.3.10.

(i) *No extension of* A_7 *is* $\mathscr{S}_1$*-maximal in any extension of* $\Omega^\circ_{13}(p)$.

(ii) *The group* A_8 *is not* $\mathscr{S}_1$*-maximal in* $\Omega^\circ_{13}(3)$ *but* $\mathrm{A}_8.2$ *is* $\mathscr{S}_1$*-maximal in* $\Omega^\circ_{13}(3).\langle\delta\rangle$. *Furthermore,* $\mathrm{A}_8.2$ *is not* $\mathscr{S}_1$*-maximal in* $\Omega^\circ_{13}(5)$.

(iii) *Let* $G = \mathrm{A}_{14}$ *with* $p \neq 2, 7$. *If* $p = 3, 5$ *then no extension of* G *is* $\mathscr{S}_1$*-maximal in any extension of* $\Omega^\circ_{13}(p)$. *Otherwise* $\mathrm{N}_{\Omega^\circ_{13}(p)}(G)$ *is always* $\mathscr{S}_1$*-maximal in* $\Omega^\circ_{13}(p)$.

(iv) *If* $p = 3, 5$ *then* $\mathrm{N}_{\Omega^\circ_{13}(p)}(\mathrm{A}_{15})$ *is* $\mathscr{S}_1$*-maximal in* $\Omega^\circ_{13}(p)$.

(v) *No extension of* $\mathrm{L}_2(13)$ *is* $\mathscr{S}_1$*-maximal in any extension of* $\Omega^\circ_{13}(p)$.

(vi) *If* $p \equiv 1, 4 \pmod 5$ *then no extension of* $\mathrm{L}_2(25).2_2$ *is* $\mathscr{S}_1$*-maximal in any extension of* $\Omega^\circ_{13}(p)$. *If* $p \equiv 2, 3 \pmod 5$, $p \neq 2$, *then* $\mathrm{L}_2(25)$ *is* $\mathscr{S}_1$*-maximal in* $\Omega^\circ_{13}(p)$.

(vii) *If* $p \neq 2, 3$, *then* $\mathrm{N}_{\Omega^\circ_{13}(p)}(\mathrm{L}_3(3))$ *is* $\mathscr{S}_1$*-maximal in* $\Omega^\circ_{13}(p)$.

(viii) *If there exists an* $\mathscr{S}_1$*-subgroup* $G = \mathrm{S}_4(5)$ *of* $\Omega^\circ_{13}(q)$ *then* G *is* $\mathscr{S}_1$*-maximal.*

(ix) *The group* J_2 *is* $\mathscr{S}_1$*-maximal in* $\Omega^\circ_{13}(9)$.

Proof. (i) Using Magma (file s1dim13cont) it is straightforward to show that the $\mathscr{S}_1$-subgroup A_7 is a subgroup of the $\mathscr{S}_1$-subgroup A_8 in dimension 13. Furthermore, $\mathrm{S}_7 \leqslant \mathrm{S}_8$ and hence no extension of A_7 is ever $\mathscr{S}_1$-maximal.

(ii) Using Magma (file s1dim13cont) we can show that the 13-dimensional absolutely irreducible representation of A_{15} has an absolutely irreducible subgroup isomorphic to A_8 in both characteristic 3 and 5. Hence $\mathrm{S}_8 \leqslant \mathrm{A}_{15} \leqslant \Omega^\circ_{13}(5)$ is never maximal. However $\mathrm{S}_{15} \leqslant \Omega^\circ_{13}(3)$ whereas $\mathrm{S}_8 \nleqslant \Omega^\circ_{13}(3)$. To show that $\mathrm{A}_8.2$ is $\mathscr{S}_1$-maximal in $\Omega^\circ_{13}(3).\langle\delta\rangle$ we have to show that $\mathrm{A}_8.2$ is not contained in any other $\mathscr{S}_1$-subgroup. By Lagrange's theorem the only other possible containment not yet considered is whether A_8 is a subgroup of J_2. We can show (file s1dim13cont) that A_8 is not a subgroup of J_2.

(iii) The only $\mathscr{S}_1$-subgroup that can contain A_{14} is A_{15} by Lagrange's theorem. The smallest non-trivial absolutely irreducible representation in characteristic 3 and 5 of A_{14} has dimension 13 by GAP and the character values of the respective absolutely irreducible 13-dimensional representations match. Furthermore, $S_{14} \leqslant S_{15}$. Hence no extension of A_{14} is $\mathscr{S}_1$-maximal in characteristic 3 and 5 by Lemma 4.9.2.

(iv) The group $A_{15}(.2)$ is always $\mathscr{S}_1$-maximal since it is not contained in any of the other $\mathscr{S}_1$-subgroups in $\Omega_{13}^{\circ}(p)$ by Lagrange's theorem.

(v) By [12], $L_2(13)$ is a subgroup of A_{14}. Let ρ be an absolutely irreducible representation of dimension 13 of $L_2(13)$. All character values of ρ are integers and there is no combination of absolutely irreducible representations of $L_2(13)$ of dimension less than 13 that gives only integer character values. Furthermore, the character values of the absolutely irreducible representation of dimension 13 of A_{14} correspond to ρ and $L_2(13).2 \leqslant S_{14}$ by [6].

(vi) By Lagrange's theorem we have to check whether $L_2(25).2_2$ is an $\mathscr{S}_1$-subgroup of $S_4(5)$, A_{14} or A_{15}.
By [12], $L_2(25).2_2$ is a subgroup of $S_4(5)$ and has trivial stabiliser in $\Omega_{13}^{\circ}(p)$. Since the character values in dimension 13 match and since the smallest non-trivial absolutely irreducible representation of $L_2(25)$ is of dimension 13, we know by Lemma 4.9.1 and Lemma 4.9.2 that this is sufficient to show that $L_2(25).2_2$ is an $\mathscr{S}_1$-subgroup of $S_4(5)$ if $p \equiv 1, 4 \pmod 5$. If $p \equiv 2, 3 \pmod 5$ then $L_2(25).2_2 \leqslant \Omega_{13}^{\circ}(p)$ but $S_4(5) \nleqslant \Omega_{13}^{\circ}(p)$.
Furthermore, $L_2(25).2_2$ cannot be a subgroup of either A_{14} or A_{15} since its smallest permutation representation acts on 26 points by [17].

(vii) Since the smallest permutation representation of $L_3(3).2$ is of dimension 26 by [6], $L_3(3).2$ is not a subgroup of $A_{14}(.2)$ or $A_{15}(.2)$. Therefore, if $p \equiv 1, 11 \pmod{12}$ then $L_3(3).2$ is $\mathscr{S}_1$-maximal in $\Omega_{13}^{\circ}(p)$. By [12], $L_3(3)$ is a subgroup of A_{13} however and hence a subgroup of A_{14}. But by [12, 24], $L_3(3)$ has elements of order 2 with character value -3 in dimension 13 whereas no element of order 2 of A_{14} (or A_{15}) has character value -3 by GAP. Therefore $L_3(3)$ is not an $\mathscr{S}_1$-subgroup of A_{14} (or A_{15}). There are no other possible containments by Lagrange's theorem and the result follows.

(viii) Since $S_4(5)$ cannot be a subgroup of any of the other $\mathscr{S}_1$-subgroups by Lagrange's theorem, it has to be $\mathscr{S}_1$-maximal.

(ix) By Lagrange's theorem the only possible $\mathscr{S}_1$-subgroups containing J_2 would be A_{14} or A_{15} but A_{14} and A_{15} are subgroups of $\Omega_{13}^{\circ}(3)$ whereas J_2 is a subgroup of $\Omega_{13}^{\circ}(9)$ and not of $\Omega_{13}^{\circ}(3)$. □

6 Maximal $\mathscr{S}_1$-Subgroups in Dimension 14

To calculate the maximal $\mathscr{S}_1$-subgroups in dimension 14, we will follow the same pattern as in Chapter 5. We will first find all potential maximal subgroups G and then we will determine in which classical group G and its extension by outer automorphisms sits.

6.1 $\mathscr{S}_1$-Subgroups in Dimension 14

We start by writing down all potential $\mathscr{S}_1$-maximal subgroups, as given in [18]. Table 6.1.1 contains all such groups G. Please see Section 5.1 for a description of how to read the table.

Note that the irrationalities appearing as entries of the 14-dimensional representations in the character tables of $2.\mathrm{S}_6(3)$ are $\mathrm{b}_{27}, \mathrm{z}_3$ and i_3 by [12, 24]. Since $\mathrm{i}_3 = \mathrm{z}_3 - \mathrm{z}_3^2$ and $\mathrm{b}_{27} = \frac{1}{2}(-1 + \sqrt{-27}) = 2\mathrm{z}_3 - \mathrm{z}_3^2$ it follows that the character ring of $2.\mathrm{S}_6(3)$ is $\mathbb{Z}[\mathrm{z}_3]$.

Also note that $2.\mathrm{L}_2(13)$ has two non-equivalent absolutely irreducible representations ρ_1 and ρ_2 of dimension 14 with character ring $\mathbb{Z}[\mathrm{r}_3]$ by [12, 24]. In Lemma 6.1.2 we shall show that ρ_1 and ρ_2 are weakly equivalent if and only if $p \equiv 5, 7 \pmod{12}$.

Let $H \in \{\mathrm{A}_7, \mathrm{L}_2(13), 2.\mathrm{L}_2(13)\}$. By [12, 24] there exist two absolutely irreducible representations of H in dimension 14 that are not weakly equivalent. We denote the images of H under these two representations by H_1 and H_2.

Theorem 6.1.1. *Let G be an $\mathscr{S}_1$-subgroup of $\Omega \in \{\mathrm{SL}^{\pm}_{14}(q), \mathrm{Sp}_{14}(q), \Omega^{\pm}_{14}(q)\}$. Then G is contained in Table 6.1.1.*

Proof. See the tables in [18]. □

Table 6.1.1: Possible $\mathscr{S}_1$-maximal subgroups in dimension 14

Gp	Order	Ind	$\#\rho$	Stab	Charc	ChR	$\lvert$Out$\rvert$
$2.\mathrm{L}_2(27)$	$2^3 \cdot 3^3 \cdot 7 \cdot 13$	$\circ$	2	3	$0, 7, 13\,(\neq 2, 3)$	$\mathbb{Z}[\mathrm{b}_{27}]$	6
$2.\mathrm{S}_6(3)$	$2^{10} \cdot 3^9 \cdot 5 \cdot 7 \cdot 13$	$\circ$	2	1	$0, 5, 7, 13\,(\neq 2, 3)$	$\mathbb{Z}[\mathrm{z}_3]$	2
$\mathrm{Sz}(8)$	$2^6 \cdot 5 \cdot 7 \cdot 13$	$\circ$	2	3	$0, 5, 7, 13\,(\neq 2)$	$\mathbb{Z}[\mathrm{i}_1]$	3
$2.\mathrm{A}_7$	$2^4 \cdot 3^2 \cdot 5 \cdot 7$	$-$	2	1	$0, 5, 7\,(\neq 2, 3)$	$\mathbb{Z}[\mathrm{r}_2]$	2
$2.\mathrm{L}_2(13)_1$	$2^3 \cdot 3 \cdot 7 \cdot 13$	$-$	1	2	$0, 3, 7\,(\neq 2, 13)$	$\mathbb{Z}$	2
$2.\mathrm{L}_2(13)_2$	$2^3 \cdot 3 \cdot 7 \cdot 13$	$-$	1, 1 or 2[a]	2	$0, 7\,(\neq 2, 3, 13)$	$\mathbb{Z}[\mathrm{r}_3]$	2
$\mathrm{L}_2(29)$	$2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 29$	$-$	2	1	2	$\mathbb{Z}[\mathrm{b}_{29}]$	2
$2.\mathrm{L}_2(29)$	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 29$	$-$	2	1	$0, 3, 5, 7\,(\neq 2, 29)$	$\mathbb{Z}[\mathrm{b}_{29}]$	2
$2.\mathrm{J}_2$	$2^8 \cdot 3^3 \cdot 5^2 \cdot 7$	$-$	1	2	$0, 3, 5, 7\,(\neq 2)$	$\mathbb{Z}$	2
A_{7_1}	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	$+$	1	2	$0, 7\,(\neq 2, 3, 5)$	$\mathbb{Z}$	2
A_{7_2}	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	$+$	1	2	$0, 7\,(\neq 2, 3, 5)$	$\mathbb{Z}$	2
A_7	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	$+$	1	2	2	$\mathbb{Z}$	2
A_8	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	$+$	1	2	$0, 7\,(\neq 2, 3, 5)$	$\mathbb{Z}$	2
A_{15}	$2^{10} \cdot 3^6 \cdot 5^3 \cdot 7^2$ $\cdot 11 \cdot 13$	$+$	1	2	$0, 7, 11, 13\,(\neq 2, 3, 5)$	$\mathbb{Z}$	2
A_{15}	$2^{10} \cdot 3^6 \cdot 5^3 \cdot 7^2$ $\cdot 11 \cdot 13$	$+$	1	2	2	$\mathbb{Z}$	2
A_{16}	$2^{14} \cdot 3^6 \cdot 5^3 \cdot 7^2$ $\cdot 11 \cdot 13$	$+$	1	2	2	$\mathbb{Z}$	2
$\mathrm{L}_2(13)_1$	$2^2 \cdot 3 \cdot 7 \cdot 13$	$+$	1	2	$0, 7\,(\neq 2, 3, 13)$	$\mathbb{Z}$	2
$\mathrm{L}_2(13)_2$	$2^2 \cdot 3 \cdot 7 \cdot 13$	$+$	1	2	$0, 7\,(\neq 2, 3, 13)$	$\mathbb{Z}$	2
$\mathrm{L}_2(13)$	$2^2 \cdot 3 \cdot 7 \cdot 13$	$+$	1	2	2	$\mathbb{Z}$	2
$\mathrm{S}_6(2)$	$2^9 \cdot 3^4 \cdot 5 \cdot 7$	$+$	1	1	3	$\mathbb{Z}$	1
$\mathrm{U}_3(3)$	$2^5 \cdot 3^3 \cdot 7$	$+$	1	2	$0, 7\,(\neq 2, 3)$	$\mathbb{Z}$	2
$\mathrm{U}_3(3)$	$2^5 \cdot 3^3 \cdot 7$	$+$	1	2	2	$\mathbb{Z}$	2
$\mathrm{G}_2(3)$	$2^6 \cdot 3^6 \cdot 7 \cdot 13$	$+$	1	2	$0, 7, 13\,(\neq 2, 3)$	$\mathbb{Z}$	2
$\mathrm{G}_2(3)$	$2^6 \cdot 3^6 \cdot 7 \cdot 13$	$+$	1	2	2	$\mathbb{Z}$	2
J_1	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$	$+$	1	1	11	$\mathbb{Z}[\mathrm{b}_5, \mathrm{c}_{19}]$	1
J_2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	$+$	1	2	5	$\mathbb{Z}$	2
J_2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	$+$	2	1	$0, 2, 7\,(\neq 3, 5)$	$\mathbb{Z}[\mathrm{b}_5]$	2

[a] If $p \equiv 1, 11 \pmod{12}$ then there are two representations that are not weakly equivalent whereas if $p \equiv 5, 7 \pmod{12}$ then the two representations are weakly equivalent under a field automorphism of $\mathrm{Sp}_{14}(p^2)$.

Lemma 6.1.2. *Let* $G = 2.\mathrm{L}_2(13)$ *and let* ρ_1 *and* ρ_2 *be two non-equivalent absolutely irreducible* 14*-dimensional representations of* G *with character ring* $\mathbb{Z}[\mathrm{r}_3]$*. Then* ρ_1 *and* ρ_2 *are weakly equivalent if and only if* $p \equiv 5, 7 \pmod{12}$.

Proof. By Table 6.1.1, $G \leqslant \mathrm{Sp}_{14}(q)$. Hence ρ_1 and ρ_2 are weakly equivalent if and only if there exists $\alpha \in \mathrm{Out}(G)$ or a non-trivial $\phi \in \mathrm{Out}(\mathrm{S}_{14}(q))$ such that $g\rho_1$ and $(g^\alpha \rho_2)^\phi$ are equivalent for all $g \in G$ by Definition 4.3.2. Since the only non-trivial automorphism of $\mathrm{Out}(G)$ stabilises the representations we know that α is trivial.

Furthermore, ϕ is non-trivial if and only if the smallest e such that $G\rho_i \leqslant \mathrm{Sp}_{14}(p^e)$ is strictly greater than 1. By Table 6.1.1 and Table 2.2.1 this is the case if and only if $p \equiv 5, 7 \pmod{12}$. In this case $G\rho_i \leqslant \mathrm{Sp}_{14}(p^2)$.

To prove that $(g\rho_1)$ and $(g\rho_2)^\phi$ are equivalent for all $g \in G$ if $p \equiv 5, 7 \pmod{12}$ we have to prove that $\chi_1(g^\phi) = \chi_2(g)$ for all $g \in G$ where χ_1 and χ_2 are the associated characters of ρ_1 and ρ_2 respectively.

Note that all conjugacy classes with the exception of $12A$ and $12B$ are fixed under ϕ since $\chi_1(g) = \chi_2(g) \in \mathbb{Z}$ by see [12, 24]. Hence it remains to show that $\chi_1(g^\phi) = (\chi_1(g))^p = \chi_2(g)$ for all g that lie in one of the conjugacy classes $12A$ or $12B$. If $p \equiv 5 \pmod{12}$ it is straightforward to check that $(\chi_1(g))^p = \mathrm{r}_3^p = -(\mathrm{z}_{12}^{3\cdot 5} + 2\mathrm{z}_{12}^{7\cdot 5}) = -\mathrm{r}_3 = \chi_2(g)$ for all $g \in 12A$. This holds similarly for $g \in 12B$. If $p \equiv 7 \pmod{12}$ then a similar argument shows that $\chi_1(g) = \chi_2(g^\phi)$ for all $g \in G$. $\square$

Information regarding the algebraic irrationalities can be found in Table 2.2.1 (p.19). Let $\Omega \in \{\mathrm{SL}_{14}^{\pm}(q), \mathrm{Sp}_{14}(q), \Omega_{14}^{\pm}(q)\}$ and let C denote the conformal group of Ω.

6.2 Schur Indicator ○

The potential maximal $\mathscr{S}_1$-subgroups in dimension 14 with Schur indicator ○ are $2.\mathrm{L}_2(27)$, $2.\mathrm{S}_6(3)$ and $\mathrm{Sz}(8)$.

Proposition 6.2.1 ($2.\mathrm{L}_2(27)$)**.**

(i) *If* $p \equiv 1 \pmod 3$*, then* $\mathrm{SL}_{14}(p)$ *has* $(p-1, 14)$ *conjugacy classes of* $\mathscr{S}_1$*-subgroups isomorphic to* $2.\mathrm{L}_2(27).3$*. If* $p \equiv 1, 7 \pmod{24}$ *then the class stabiliser is* $\langle\gamma\rangle$ *in* $\mathrm{Out}(\mathrm{L}_{14}(p))$ *and if* $p \equiv 13, 19 \pmod{24}$ *then the class stabiliser is* $\langle\gamma\delta\rangle$ *in* $\mathrm{Out}(\mathrm{L}_{14}(p))$*.*

(ii) *If* $p \equiv 2 \pmod 3$*,* $p \neq 2$*, then* $\mathrm{SU}_{14}(p)$ *has* $(p+1, 14)$ *conjugacy classes of* $\mathscr{S}_1$*-subgroups isomorphic to* $2.\mathrm{L}_2(27).3$*. If* $p \equiv 17, 23 \pmod{24}$ *then the class stabiliser is* $\langle\gamma\rangle$ *in* $\mathrm{Out}(\mathrm{U}_{14}(p))$ *and if* $p \equiv 5, 11 \pmod{24}$ *then the class stabiliser is* $\langle\gamma\delta\rangle$*.*

Proof. Let $G = 2.\mathrm{L}_2(27)$. Then $\mathrm{Out}(G) = 6$ and up to equivalence there exist two weakly equivalent 14-dimensional representations ρ of G with character ring $\mathbb{Z}[\mathrm{b}_{27}]$ by Table 6.1.1. From this it follows by Table 2.2.1 and Lemma 4.2.1 that $G \leqslant \mathrm{SL}_{14}(p)$ if $p \equiv 1 \pmod 3$ and $G \leqslant \mathrm{SU}_{14}(p)$ if $p \equiv 2 \pmod 3$, $p \neq 2$.

The stabiliser of ρ is the outer automorphism of G of order 3. Using [12, 24] it is straightforward to show that $G.3 \leqslant \mathrm{SL}_{14}(p)$ if $p \equiv 1 \pmod 3$ and $G.3 \leqslant \mathrm{SU}_{14}(p)$ if $p \equiv 2 \pmod 3$.

Since the two non-equivalent representations are fused by the outer automorphism α of order 2 of G there exists one conjugacy class of G in C. Hence, by Lemma 4.3.3, there exist $(p \mp 1, 14)$ conjugacy classes of subgroups isomorphic to $2.\mathrm{L}_2(27).3$ in $\mathrm{SL}^{\pm}_{14}(p)$ respectively.

Furthermore, α acts by complex conjugation on the two non-equivalent representations of dimension 14 of G. It follows that ${}^{\alpha}\rho$ is equivalent to ρ^{γ} by Lemma 4.4.1.

First consider Case **L**. We want to find a matrix $x \in \mathrm{GL}_{14}(p)$ such that $x^{-1}(g^{\alpha}\rho)x = (g\rho)^{\gamma}$ for all $g \in G$ in order to use Lemma 4.4.3. Since we do not want to find such an x for each possible prime p individually, we will work with a characteristic 0 representation $\hat{\rho}$ and show that it can be reduced modulo p for all $p \equiv 1 \pmod 3$ (see file 2l227d14comp).

Note that the character field corresponding to the ring $\mathbb{Z}[\mathrm{b}_{27}]$ is $\mathbb{Q}[\mathrm{z}_3]$ since $\mathrm{b}_{27}^{-1} = -\frac{1}{14} - \frac{3}{14}\mathrm{i}_3$, $\mathrm{b}_{27} = 2\mathrm{z}_3 - \mathrm{z}_3^2$ and $\mathrm{i}_3 = \mathrm{z}_3 - \mathrm{z}_3^2$.

Let a and b be standard generators of $2.\mathrm{L}_2(27)\hat{\rho}$. It is straightforward to check that the denominators of the entries of a and b are only divisible by 2. Hence we can reduce $G\hat{\rho} = \langle a, b\rangle$ modulo p for all $p \neq 2, 3$ by Lemma 4.1.2.

Furthermore, α acts on a and b by sending a to a^{-1} and b to b^{-1} by [19]. Using the GHom command in Magma we can find a matrix $\hat{x} \in \mathrm{GL}_{14}(\mathbb{C})$ such that $\hat{x}^{-1}(g^{\alpha}\hat{\rho})\hat{x} = (g\hat{\rho})^{-\mathrm{T}}$ for all $g \in G$. Note that $\hat{x}$ has entries in $\mathbb{Z}[\mathrm{i}_3]$.

The lowest common multiples of the denominators of the entries of $\hat{x}$ and $\hat{x}^{-1}$ are 1 and 6 respectively and $\det(\hat{x}) = -2^{12} \cdot 3^6 \cdot (1 + \mathrm{i}_3)$. Hence we can reduce $\hat{x}$ modulo p for all $p \neq 2, 3$. It follows from Remark 4.4.4 that we can use Lemma 4.4.3 to determine whether an $\mathrm{SL}_{14}(p)$-conjugacy class of subgroups isomorphic to $2.\mathrm{L}_2(27).3$ is stabilised by γ or $\gamma\delta$ in $\mathrm{Out}(\mathrm{L}_{14}(p))$.

We find that $\det(\hat{x})$ is a square in $\mathbb{F}_p$ if and only if $-(1 + \mathrm{i}_3)$ is a square in $\mathbb{F}_p$.Note that $-(1 + \mathrm{i}_3) = \frac{1}{4}(\mathrm{r}_2 - \mathrm{i}_6)^2$ and we can show that $\frac{1}{2}(\mathrm{r}_2 - \mathrm{i}_6) \in \mathbb{F}_p$ if and only if $p \equiv 1, 7 \pmod{24}$ using Table 2.2.1. Hence it follows from Lemma 4.4.3 that $2.\mathrm{L}_2(27).3$ is stabilised by $\langle\gamma\rangle$ if $p \equiv 1, 7 \pmod{24}$ and by $\langle\gamma\delta\rangle$ if $p \equiv 13, 19 \pmod{24}$.

In Case **U** we have to find a matrix $\hat{B}$ of the unitary form preserved by $G\hat{\rho}$ and we hope to show that $\hat{B}$ can be reduced modulo p for all $p \equiv 2 \pmod 3$, $p \neq 2$. As in Case **L** there exists $\hat{x} \in \mathrm{GL}_{14}(\mathbb{C})$ such that $\hat{x}^{-1}(g^{\alpha}\hat{\rho})\hat{x} = (g\hat{\rho})^{-\mathrm{T}}$ for all $g \in G$ and $\hat{x}$ is reducible modulo p for all p we are interested in. Hence we can apply Lemma 4.5.4 (see file 2l227d14comp).

Note that we can use the same generators a and b as in Case **L**. Using Magma we can find a form matrix $\hat{B}$ and a matrix $\hat{x}$ both with entries in $\mathbb{Z}[\mathrm{i}_3]$.

Let $S = \mathbb{Z}[\mathrm{b}_{27}, \frac{1}{2}, \frac{1}{3}]$. Then $\hat{B}$, $\hat{B}^{-1}$, $\hat{x}$ and $\hat{x}^{-1}$ have entries in S since $\frac{1}{2}(1+\mathrm{i}_3) = \frac{1}{3}(2+\mathrm{b}_{27})$. Furthermore, $\det(\hat{x}) = -2^{13} \cdot 3^6 \cdot \frac{1}{2}(1+\mathrm{i}_3) = -2^{13} \cdot 3^5 \cdot (2+\mathrm{b}_{27}) = 2^{13} \cdot 3^4 \cdot (1-\mathrm{b}_{27})^2$ since $-3(2+\mathrm{b}_{27}) = (1-\mathrm{b}_{27})^2$. This gives a factorisation in S.

Let $\hat{r} = 2^{13} \cdot 3^4$ with p-modular reduction r. Then $\hat{r} > 0$ and $\sqrt{r} \in \mathbb{F}_p$ if and only if $\sqrt{2} \in \mathbb{F}_p$. So by Lemma 4.5.4 an $\mathrm{SU}_{14}(p)$-conjugacy class of $G.3$ is stabilised by γ if $\mathrm{r}_2 \in \mathbb{F}_p$ and by $\gamma\delta$ otherwise. By Table 2.2.1, $\mathrm{r}_2 \in \mathbb{F}_p$ if and only if $p \equiv 1, 7 \pmod 8$. Since we are in Case **U** and $p \equiv 2 \pmod 3$, $p \neq 2$, we deduce that a conjugacy class of $G.3$ is stabilised by γ if $p \equiv 17, 23 \pmod{24}$ and by $\gamma\delta$ if $p \equiv 5, 11 \pmod{24}$. □

Proposition 6.2.2 ($2.\mathrm{S}_6(3)$)**.**

(i) If $p \equiv 1 \pmod 3$, then $\mathrm{SL}_{14}(p)$ has $(p-1, 14)$ conjugacy classes of $\mathscr{S}_1$-subgroups isomorphic to $2.\mathrm{S}_6(3)$. If $p \equiv 1, 7 \pmod{24}$ then the class stabiliser is $\langle\gamma\rangle$ in $\mathrm{Out}(\mathrm{L}_{14}(p))$ and if $p \equiv 13, 19 \pmod{24}$ then the class stabiliser is $\langle\gamma\delta\rangle$.

(ii) If $p \equiv 2 \pmod 3$, $p \neq 2$, then $\mathrm{SU}_{14}(p)$ has $(p+1, 14)$ conjugacy classes of $\mathscr{S}_1$-subgroups isomorphic to $2.\mathrm{S}_6(3)$. If $p \equiv 17, 23 \pmod{24}$ then the class stabiliser is $\langle\gamma\rangle$ in $\mathrm{Out}(\mathrm{U}_{14}(p))$ and if $p \equiv 5, 11 \pmod{24}$ then the class stabiliser is $\langle\gamma\delta\rangle$.

Proof. Let $G = 2.\mathrm{S}_6(3)$. By Table 6.1.1, $\mathrm{Out}(G) = 2$ and the character ring of the 14-dimensional absolutely irreducible representations of G is $\mathbb{Z}[\mathrm{z}_3]$. Hence it follows from Lemma 4.2.1 and Table 2.2.1 that G preserves a unitary form if and only if $p \equiv 2 \pmod 3$. Again by Table 6.1.1, G has (up to equivalence) two weakly equivalent absolutely irreducible representations ρ in dimension 14. There is one conjugacy class of G in C since the non-trivial outer automorphism α of G fuses these two representations. Hence the number of conjugacy classes of subgroups isomorphic to $2.\mathrm{S}_6(3)$ in $\mathrm{SL}^{\pm}_{14}(p)$ is $(14, q \mp 1)$ respectively by Lemma 4.3.3.

Now we want to determine how $\gamma \in \mathrm{Out}(\mathrm{L}^{\pm}_{14}(p))$ acts on the $\mathrm{SL}^{\pm}_{14}(p)$-conjugacy classes of G. Using Lemma 4.4.1 it is straightforward to show that ${}^{\alpha}\rho$ is equivalent to ρ^{γ}.

Let $\hat{\rho}$ be the 14-dimensional absolutely irreducible representation of G in characteristic 0 given in [19]. Then we can find a matrix $\hat{x} \in \mathrm{GL}_{14}(\mathbb{C})$

such that $\hat{x}^{-1}(g^{\alpha}\hat{\rho})\hat{x} = (g\hat{\rho})^{\gamma}$ for all $g \in G$. Here $\det(\hat{x}) = 2^{13} \cdot \mathrm{z}_3$ and $\hat{x}$ is reducible modulo p for all $p \neq 2, 3$ (file 2s63d14comp).

Hence, in Case **L** we have to determine when $2\mathrm{z}_3 = -2 + \frac{2}{3}(2 + \mathrm{b}_{27}) = -1 + \mathrm{i}_3 = \frac{1}{4}(\mathrm{r}_2 + \mathrm{i}_6)^2$ (using $\frac{1}{2}(1 + \mathrm{i}_3) = \frac{1}{3}(2 + \mathrm{b}_{27})$) is square in $\mathbb{F}_p$. Using Table 2.2.1 we can show that $2\mathrm{z}_3$ is square in $\mathbb{F}_p$ if and only if $p \equiv 1, 7 \pmod{24}$. It follows from Lemma 4.4.3 that an $\mathrm{SL}_{14}(p)$-conjugacy class of G is stabilised by $\langle\gamma\rangle$ when $p \equiv 1, 7 \pmod{24}$ and by $\langle\gamma\delta\rangle$ when $p \equiv 13, 19 \pmod{24}$.

For Case **U** note that the outer automorphism ϕ of $\mathrm{U}_{14}(p)$ acts on z_3 by sending z_3 to $-\mathrm{z}_3 - 1$. Furthermore, we can find a form matrix $\hat{B}$ preserved by $G\hat{\rho}$ that can be reduced modulo p for all $p \neq 2, 3$. All entries of $\hat{x}$, $\hat{x}^{-1}$, $\hat{B}$ and $\hat{B}^{-1}$ lie in $S = \mathbb{Z}[\mathrm{z}_3, \frac{1}{2}, \frac{1}{3}]$. Furthermore, $\det(\hat{x}) = 2^{13} \cdot \mathrm{z}_3 = 2^{13} \cdot (1 + \mathrm{z}_3)^2$ gives a factorisation in S.

Let $\hat{r} = 2^{13}$ and let r denote the p-modular reduction of $\hat{r}$. Then $\sqrt{r} \in \mathbb{F}_p$ if and only if $\sqrt{2} \in \mathbb{F}_p$. By Table 2.2.1, $\mathrm{r}_2 \in \mathbb{F}_p$ if and only if $p \equiv 1, 7 \pmod{8}$. By Lemma 4.5.4 it follows that an $\mathrm{SU}_{14}(p)$-conjugacy class of G is stabilised by γ if $p \equiv 17, 23 \pmod{24}$ and by $\gamma\delta$ if $p \equiv 5, 11 \pmod{24}$. □

Proposition 6.2.3 (Sz(8))**.**

(i) If $p \equiv 1 \pmod 4$*, then there are* $2 \cdot (p - 1, 14)$ *conjugacy classes of* $\mathscr{S}_1$*-subgroups of* $\mathrm{SL}_{14}(p)$ *isomorphic to* $\mathrm{Sz}(8).3$ *in* $\mathrm{SL}_{14}(p)$*, with trivial class stabiliser in* $\mathrm{Out}(\mathrm{L}_{14}(p))$*.*

(ii) If $p \equiv 3 \pmod 4$*, then there are* $2 \cdot (p + 1, 14)$ *conjugacy classes of* $\mathscr{S}_1$*-subgroups of* $\mathrm{SU}_{14}(p)$ *isomorphic to* $\mathrm{Sz}(8).3$*, with trivial class stabiliser in* $\mathrm{Out}(\mathrm{U}_{14}(p))$*.*

Proof. Let $G = \mathrm{Sz}(8)$ with $\mathrm{Out}(G) = 3$. By Table 6.1.1 there are up to equivalence 2 weakly equivalent 14-dimensional absolutely irreducible representations of G with character ring $\mathbb{Z}[\mathrm{i}]$. It follows from Table 2.2.1 and Lemma 4.2.1 that $G \leqslant \mathrm{SL}_{14}(p)$ when $p \equiv 1 \pmod 4$ and $G \leqslant \mathrm{SU}_{14}(p)$ when $p \equiv 3 \pmod 4$.

Furthermore, $G\rho.3 \leqslant \mathrm{N}_C(G\rho)$ by Table 6.1.1. Using [12, 24] it follows that $G.3 \leqslant \mathrm{SL}_{14}(p)$ if $p \equiv 1 \pmod 4$ and $G.3 \leqslant \mathrm{SU}_{14}(p)$ if $p \equiv 3 \pmod 4$.

The two non-equivalent 14-dimensional absolutely irreducible representations of G are not fused by any outer automorphism of G. Hence there are two conjugacy classes of G in C by Lemma 4.2.2. By Lemma 4.3.3 each of these classes splits into $(p \mp 1, 14)$ conjugacy classes in $\mathrm{SL}^{\pm}_{14}(p)$ respectively. Note that γ fuses two non-equivalent representations and hence there is one $\mathrm{Aut}(\mathrm{L}^{\pm}_{14}(p))$-conjugacy class.

Furthermore, the conjugacy classes of G in $\mathrm{SL}_{14}^{\pm}(p)$ have trivial stabiliser since $\mathrm{Out}(G) = 3$ and since $\delta \in \mathrm{Out}(\mathrm{L}_{14}^{\pm}(p))$ acts transitively on the conjugacy classes by Lemma 4.3.6. □

Maximality

Now we determine the possible $\mathscr{S}_1$-containments between the groups $2.\mathrm{L}_2(27).3$, $2.\mathrm{S}_6(3)$ and $\mathrm{Sz}(8).3$.

Proposition 6.2.4.
Let $d := (p-1, 14)$ in Case **L** *and let $d := (p+1, 14)$ in Case* **U***.*

(i) No extension of $d \circ 2.\mathrm{L}_2(27)$ is $\mathscr{S}_1$-maximal in any extension of $\mathrm{SL}_{14}^{\pm}(p)$.

(ii) If $p \neq 2, 3$ then $d \circ 2.\mathrm{S}_6(3)$ is $\mathscr{S}_1$-maximal in $\mathrm{SL}_{14}^{\pm}(p)$.

(iii) If $p \neq 2$ then $d \times \mathrm{Sz}(8).3$ is $\mathscr{S}_1$-maximal in $\mathrm{SL}_{14}^{\pm}(p)$.

Proof. (i) By [6], $\mathrm{L}_2(27).3$ is a subgroup of $\mathrm{S}_6(3)$ and it follows that $2.\mathrm{L}_2(27).3$ is a subgroup of $2.\mathrm{S}_6(3)$. Furthermore, the smallest non-trivial absolutely irreducible representations of $2.\mathrm{L}_2(27)$ have dimension 14. Hence, $2.\mathrm{L}_2(27).3$ is an $\mathscr{S}_1$-subgroup of $2.\mathrm{S}_6(3)$ by Lemma 4.9.2. Furthermore, $\mathrm{L}_2(27).6$ is a subgroup of $\mathrm{S}_6(3)$ by [12] from which it follows that no extension of $3.\mathrm{L}_2(27)$ is $\mathscr{S}_1$-maximal in dimension 14.

(ii) By Lagrange's theorem neither $2.\mathrm{L}_2(27).3$ nor $\mathrm{Sz}(8).3$ could contain $2.\mathrm{S}_6(3)$.

(iii) By Lagrange's theorem, $\mathrm{Sz}(8).3$ could be a subgroup of $2.\mathrm{S}_6(3)$. Looking at the character tables of $2.\mathrm{S}_6(3)$ and $\mathrm{Sz}(8)$ in [12, 24], however, it can easily be seen that $\mathrm{Sz}(8)$ cannot be an $\mathscr{S}_1$-subgroup of $2.\mathrm{S}_6(3)$ in dimension 14 as the respective character values do not match. □

6.3 Schur Indicator −

By Table 6.1.1 the potential 14-dimensional $\mathscr{S}_1$-maximal subgroups with Schur indicator − are $2.\mathrm{A}_7$, $2.\mathrm{L}_2(13)_{1,2}$, $\mathrm{L}_2(29)$, $2.\mathrm{L}_2(29)$ and $2.\mathrm{J}_2$.

Proposition 6.3.1 ($2.\mathrm{A}_7$)**.**

(i) If $p \equiv 1, 7 \pmod 8$, then there are 2 conjugacy classes of $\mathscr{S}_1$-subgroups of $\mathrm{Sp}_{14}(p)$ isomorphic to $2.\mathrm{A}_7$, which have trivial class stabiliser in $\mathrm{Out}(\mathrm{S}_{14}(p))$.

(ii) If $p \equiv 3, 5 \pmod 8$*,* $p \neq 3$*, then there are* 2 *conjugacy classes of* $\mathscr{S}_1$*-subgroups of* $\mathrm{Sp}_{14}(p^2)$ *isomorphic to* $2.\mathrm{A}_7$*, with class stabiliser* $\langle \phi \rangle$ *in* $\mathrm{Out}(\mathrm{S}_{14}(p^2))$*.*

Proof. Let $G = 2.\mathrm{A}_7$. Then $\mathrm{Out}(G) = 2$ and there are (up to equivalence) two weakly equivalent absolutely irreducible 14-dimensional representations ρ of G with character ring $\mathbb{Z}[\mathrm{r}_2]$ by Table 6.1.1. It follows from Table 2.2.1 that $2.\mathrm{A}_7 \leqslant \mathrm{Sp}_{14}(p)$ when $p \equiv 1, 7 \pmod 8$ and $2.\mathrm{A}_7 \leqslant \mathrm{Sp}_{14}(p^2)$ when $p \equiv 3, 5 \pmod 8$, $p \neq 3$. Since the two representations are fused by the outer automorphism α of order 2 of G, there exist two conjugacy classes of G in $\mathrm{Sp}_{14}(q)$ by Lemma 4.3.3.

Since the field automorphism of $\mathrm{Sp}_{14}(p)$ is trivial, the stabiliser of these conjugacy classes is trivial when $p \equiv 1, 7 \pmod 8$.

If $p \equiv 3, 5 \pmod 8$ then, using Lemma 4.3.8, ${}^{\alpha}\rho$ is equivalent to ρ^{ϕ} since the p-modular reduction of $\mathrm{r}_2^t = \mathrm{z}_8^{7t} + \mathrm{z}_8^t = -\mathrm{r}_2$ when $t \in \{3, 5\}$. Projectively $G.2 \backslash G$ contains involutions and hence a conjugacy class of $2.\mathrm{A}_7$ in $\mathrm{Sp}_{14}(p^2)$ is stabilised by $\langle \phi \rangle$ by Lemma 4.6.2. □

Proposition 6.3.2 ($2.\mathrm{L}_2(13)_1$)**.**
Let $2.\mathrm{L}_2(13)_1$ *be the image of an absolutely irreducible* 14*-dimensional representation* ρ *of* $2.\mathrm{L}_2(13)$ *with character ring* $\mathbb{Z}$*.*

(i) If $p \equiv 1, 7 \pmod 8$*, then there exist two conjugacy classes of* $\mathscr{S}_1$*-subgroups of* $\mathrm{Sp}_{14}(p)$ *weakly equivalent to* $2.\mathrm{L}_2(13)_1.2$*, with trivial class stabiliser in* $\mathrm{Out}(\mathrm{S}_{14}(p))$*.*

(ii) If $p \equiv 3, 5 \pmod 8$*,* $p \neq 13$*, then there is a single conjugacy class of* $\mathscr{S}_1$*-subgroups* G *of* $\mathrm{Sp}_{14}(p)$ *weakly equivalent to* $2.\mathrm{L}_2(13)_1$*, with class stabiliser* $\langle \delta \rangle$ *in* $\mathrm{Out}(\mathrm{S}_{14}(p))$*.*

Proof. Let $G = 2.\mathrm{L}_2(13)_1$. Then $\mathrm{Out}(G) = 2$ by [12] and $G \leqslant \mathrm{Sp}_{14}(p)$, $p \neq 2, 13$, since the character ring of ρ is $\mathbb{Z}$. By [12, 24] the 14-dimensional absolutely irreducible representation ρ' of $G.2$ preserves a symplectic form and has character ring $\mathbb{Z}[\mathrm{r}_2]$. Therefore, $G.2 \leqslant \mathrm{Sp}_{14}(p)$ if $p \equiv 1, 7 \pmod 8$ and $G.2 \leqslant \mathrm{Sp}_{14}(p^2)$ if $p \equiv 3, 5 \pmod 8$ by Table 2.2.1.

If $G.2 \leqslant \mathrm{Sp}_{14}(p)$ then there are no other non-trivial outer automorphisms of G that could stabilise the representation. Hence there exists one conjugacy class of G in the conformal group by Table 6.1.1 which splits into two classes in $\mathrm{Sp}_{14}(p)$ by Lemma 4.3.3.

Now let $p \equiv 3, 5 \pmod 8$. Then ρ' is defined over $\mathbb{F}_{p^2}$, i.e. $G.2 \leqslant \mathrm{Sp}_{14}(p^2)$. Let $g \in \mathrm{Sp}_{14}(p^2) \backslash \mathrm{Sp}_{14}(p)$ such that g induces the outer automorphism α of order 2 of G. Let $C = \mathrm{CSp}_{14}(p)$. We also know that

$G.2 \leqslant \mathrm{N}_C(G)$ by Lemma 4.2.2. Hence α is also induced by conjugation by an element $g' \in C$. We are going to show that g' induces $\delta \in \mathrm{Out}(\mathrm{S}_{14}(p))$.

Since α is induced by both g and g', $g'g^{-1}$ stabilises G. Hence $g'g^{-1} = \lambda I$ for some scalar λ by Lemma 2.2.9. Note that $\lambda \in \mathbb{F}_{p^2} \backslash \mathbb{F}_p$ and in particular that $\lambda I \neq \pm I$.

Let B be the matrix of the symplectic form preserved by $\mathrm{Sp}_{14}(p^2)$. Then $g^{\mathrm{T}}Bg = B$, but $(g')^{\mathrm{T}}Bg' = (\lambda g)^{\mathrm{T}}B(\lambda g) = \lambda^2 B \neq B$. Hence, $\mathrm{N}_C(G)$ is generated by G, scalars and δ.

Furthermore, there is a single conjugacy class of G in C by Table 6.1.1. By Lemma 4.3.3(ii) this class splits into $|C : \mathrm{N}_C(G)\mathrm{Sp}_{14}(p)|$ classes in $\mathrm{Sp}_{14}(p)$. It follows that $\mathrm{Sp}_{14}(p)$ has a single conjugacy class of $\mathscr{S}_1$-subgroups isomorphic to G. $\square$

Proposition 6.3.3 ($2.\mathrm{L}_2(13)_2$).
Let $2.\mathrm{L}_2(13)_2$ be the image of an absolutely irreducible 14-dimensional representation of $2.\mathrm{L}_2(13)$ with character ring $\mathbb{Z}[\mathrm{r}_3]$.

(i) If $p \equiv 1, 23 \pmod{24}$, then there exist 4 conjugacy classes of $\mathscr{S}_1$-subgroups of $\mathrm{Sp}_{14}(p)$ weakly equivalent to $2.\mathrm{L}_2(13)_2.2$, with trivial class stabiliser in $\mathrm{Out}(\mathrm{S}_{14}(p))$. Furthermore, there exist two $\mathrm{Aut}(\mathrm{S}_{14}(p))$-conjugacy classes of $2.\mathrm{L}_2(13)_2.2$.

(ii) If $p \equiv 5, 7, 17, 19 \pmod{24}$, then there are 4 conjugacy classes of $\mathscr{S}_1$-subgroups of $\mathrm{Sp}_{14}(p^2)$ isomorphic to $2.\mathrm{L}_2(13)_2.2$, with trivial class stabiliser in $\mathrm{Out}(\mathrm{S}_{14}(p^2))$.

(iii) If $p \equiv 11, 13 \pmod{24}$, $p \neq 13$, then there are 2 conjugacy classes of $\mathscr{S}_1$-subgroups G of $\mathrm{Sp}_{14}(p)$ isomorphic to $2.\mathrm{L}_2(13)_2$, with class stabiliser $\langle\delta\rangle$ in $\mathrm{Out}(\mathrm{S}_{14}(p))$. Furthermore, there are two $\mathrm{Aut}(\mathrm{S}_{14}(p))$-conjugacy classes of $2.\mathrm{L}_2(13)_2$.

Proof. Let $G = 2.\mathrm{L}_2(13)_2$ with $\mathrm{Out}(G) = 2$. Recall that by Lemma 6.1.2 two non-equivalent 14-dimensional absolutely irreducible representations of $2.\mathrm{L}_2(13)$ with character ring $\mathbb{Z}[\mathrm{r}_3]$ are weakly equivalent if and only if $p \equiv 5, 7 \pmod{12}$. Since these two representations share the same main properties, we only look at the first of the two representations, ρ_1, because we do not get any new results by looking at the second representation ρ_2. We only have to consider ρ_2 when we calculate the number of conjugacy classes.

Since ρ_1 has character ring $\mathbb{Z}[\mathrm{r}_3]$ by Table 6.1.1, $G \leqslant \mathrm{Sp}_{14}(p)$ if and only if $p \equiv 1, 11 \pmod{12}$, $p \neq 13$, by Table 2.2.1. Otherwise, when $p \equiv 5, 7 \pmod{12}$, then $G \leqslant \mathrm{Sp}_{14}(p^2)$.

By [12, 24] $G.2 \leqslant \mathrm{Sp}_{14}(p^e)$ for some e and the character ring of this extension is $\mathbb{Z}[\mathrm{r}_2, \mathrm{y}_{24}]$. Therefore, the matrices of $G.2\backslash G$ have entries only in $\mathbb{F}_p$ if $p \equiv 1, 11 \pmod{12}$ and $\mathrm{r}_2, \mathrm{y}_{24} \in \mathbb{F}_p$, i.e. if $p \equiv 1, 23 \pmod{24}$. Otherwise they have entries from $\mathbb{F}_{p^2}$.

If $p \equiv 5, 7 \pmod{12}$ then $G \leqslant \mathrm{Sp}_{14}(p^2)$ and $G.2 \leqslant \mathrm{Sp}_{14}(p^2)$. It is clear that $G.2$ is scalar normalising and since there are two weakly equivalent representations that do not lie in the same orbit of $\mathrm{Out}(G)$, there are $2 \cdot 2 = 4$ conjugacy classes of groups isomorphic to $2.\mathrm{L}_2(13)_2.2$ in $\mathrm{Sp}_{14}(p^2)$ using Lemma 4.2.4, Lemma 4.2.2 and Lemma 4.3.3. There are no other outer automorphisms of G and it follows that the stabiliser of these conjugacy classes has to be trivial.

If $G \leqslant \mathrm{Sp}_{14}(p)$ and $G.2 \leqslant \mathrm{Sp}_{14}(p)$ then $p \equiv 1, 23 \pmod{24}$. Since ρ_1 and ρ_2 are not weakly equivalent, the group $\mathrm{Sp}_{14}(p)$ has $2 \cdot 2$ conjugacy classes of $\mathscr{S}_1$-subgroups isomorphic to $2.\mathrm{L}_2(13)_2.2$ by Lemma 4.2.4 and Lemma 4.3.3. These conjugacy classes lie in two distinct $\mathrm{Aut}(\mathrm{Sp}_{14}(p))$-classes.

If $p \equiv 11, 13 \pmod{24}$, $p \neq 13$, then $G \leqslant \mathrm{Sp}_{14}(p)$ but $G.2 \nleqslant \mathrm{Sp}_{14}(p)$. Using a similar argument as in Proposition 6.3.2, we can show that a conjugacy class of G in $\mathrm{Sp}_{14}(p)$ is stabilised by δ. Hence $\mathrm{Sp}_{14}(p)$ contains $2 \cdot 1$ conjugacy classes of groups isomorphic to G. □

Proposition 6.3.4 ($\mathrm{L}_2(29)$ in characteristic 2)**.**
There is a single conjugacy class of $\mathscr{S}_1$-subgroups of $\mathrm{Sp}_{14}(4)$ *isomorphic to* $\mathrm{L}_2(29)$*, with class stabiliser* $\langle\phi\rangle$ *in* $\mathrm{Out}(\mathrm{S}_{14}(4))$*.*

Proof. Let $G = \mathrm{L}_2(29)$. Then $\mathrm{Out}(G) = 2$ and there are (up to equivalence) two 14-dimensional absolutely irreducible representations of G. They are weakly equivalent and have character ring $\mathbb{Z}[\mathrm{b}_{29}]$ by [24]. Hence $\mathrm{L}_2(29) \leqslant \mathrm{Sp}_{14}(2^2)$ by Table 2.2.1. By [24] these two representations are fused by the outer automorphism α of order 2 of G. Hence there is one conjugacy class of G in $\mathrm{Sp}_{14}(2^2)$ by Lemma 4.3.3. Furthermore, ${}^{\alpha}\rho$ is equivalent to ρ^{ϕ} by Lemma 4.3.8 and [24] which implies that this single conjugacy class has to be stabilised by ϕ. □

Proposition 6.3.5 ($2.\mathrm{L}_2(29)$)**.**

(i) *If* $p \equiv 1, 4, 5, 6, 7, 9, 13, 16, 20, 22, 23, 24, 25, 28 \pmod{29}$ *then there are* 2 *conjugacy classes of* $\mathscr{S}_1$*-subgroups of* $\mathrm{Sp}_{14}(p)$ *isomorphic to* $2.\mathrm{L}_2(29)$*, with trivial class stabiliser in* $\mathrm{Out}(\mathrm{S}_{14}(p))$*.*

(ii) *If* $p \equiv 2, 3, 8, 10, 11, 12, 14, 15, 17, 18, 19, 21, 26, 27 \pmod{29}$*,* $p \neq 2$*, then* $\mathrm{Sp}_{14}(p^2)$ *has* 2 *conjugacy classes of* $\mathscr{S}_1$*-subgroups isomorphic to* $2.\mathrm{L}_2(29)$*, with class stabiliser* $\langle\phi\rangle$ *in* $\mathrm{Out}(\mathrm{S}_{14}(p^2))$*.*

Proof. Let $G = 2.\mathrm{L}_2(29)$. Then $\mathrm{Out}(G) = 2$ and G has (up to equivalence) two weakly equivalent absolutely irreducible representations ρ of dimension 14 with character ring $\mathbb{Z}[\mathrm{b}_{29}]$ by Table 6.1.1. It follows that $G \leqslant \mathrm{Sp}_{14}(p)$ if $p \equiv 1, 4, 5, 6, 7, 9, 13, 16, 20, 22, 23, 24, 25, 28 \pmod{29}$ and $G \leqslant \mathrm{Sp}_{14}(p^2)$ otherwise by Table 2.2.1.

These two representations are fused by the outer automorphism α of order 2 of G. Hence there are two conjugacy classes of G in the respective symplectic group by Lemma 4.3.3. If $G \leqslant \mathrm{Sp}_{14}(p)$ then the field automorphism of $\mathrm{Sp}_{14}(p)$ is trivial and therefore the stabiliser of the conjugacy classes of G is trivial.

If $p \equiv 2, 3, 8, 10, 11, 12, 14, 15, 17, 18, 19, 21, 26, 27 \pmod{29}$, $p \neq 2$, then ${}^{\alpha}\rho$ is equivalent to ρ^{ϕ} using Lemma 4.3.8 and [12, 24]. Furthermore, projectively $2.\mathrm{L}_2(29).2 \backslash 2.\mathrm{L}_2(29)$ contains involutions which implies by Lemma 4.6.2 that an $\mathrm{Sp}_{14}(p^2)$-conjugacy classes of G is stabilised by $\langle \phi \rangle$. □

Proposition 6.3.6 ($2.\mathrm{J}_2$)**.**

(i) If $p \equiv 1, 7 \pmod 8$ then there are 2 conjugacy classes of $\mathscr{S}_1$-subgroups of $\mathrm{Sp}_{14}(p)$ isomorphic to $2.\mathrm{J}_2.2$, which have trivial class stabiliser in $\mathrm{Out}(\mathrm{S}_{14}(p))$.

(ii) If $p \equiv 3, 5 \pmod 8$ then there exists a single conjugacy class of $\mathscr{S}_1$-subgroups of $\mathrm{Sp}_{14}(p)$ isomorphic to $2.\mathrm{J}_2$, which has class stabiliser $\langle \delta \rangle$ in $\mathrm{Out}(\mathrm{S}_{14}(p))$.

Proof. Let $G = 2.\mathrm{J}_2$. Then $\mathrm{Out}(G) = 2$ and since the character ring of the unique (up to equivalence) absolutely irreducible representation of dimension 14 of G is $\mathbb{Z}$ by [12, 24], it follows that $G \leqslant \mathrm{Sp}_{14}(p)$, $p \neq 2$. Furthermore, $G.2$ preserves a symplectic form and has character ring $\mathbb{Z}[\mathrm{r}_2]$. It follows that $G.2 \leqslant \mathrm{Sp}_{14}(p)$ when $p \equiv 1, 7 \pmod 8$. In this case there are 2 conjugacy classes of $G.2$ in $\mathrm{Sp}_{14}(p)$ by Lemma 4.3.3 which have trivial stabiliser.

If $p \equiv 3, 5 \pmod 8$ then we can show using a similar argument as in Proposition 6.3.2 that an $\mathrm{Sp}_{14}(p)$-conjugacy class of G is stabilised by $\langle \delta \rangle$.

Furthermore, there is a single conjugacy class of G in $\mathrm{Sp}_{14}(p)$ by Lemma 4.3.3. □

Maximality

Finally, we want to determine whether any of these groups with Schur Indicator $-$ could be contained in one another as $\mathscr{S}_1$-subgroups.

Proposition 6.3.7.
Let G be an $\mathscr{S}_1$-subgroup of $\mathrm{Sp}_{14}(q)$. Then $\mathrm{N}_{\mathrm{Sp}_{14}(q)}(G)$ is $\mathscr{S}_1$-maximal in $\mathrm{Sp}_{14}(q)$.

Proof. The only $\mathscr{S}_1$-subgroup of $\mathrm{Sp}_{14}(q)$ in even characteristic is $\mathrm{L}_2(29)$ which implies that $\mathrm{L}_2(29)$ is $\mathscr{S}_1$-maximal. In odd characteristic the only possible containment by Lagrange's theorem is $2.\mathrm{A}_7$ in $2.\mathrm{J}_2$. Using Magma (file s1dim14cont) we can show that A_7 is not a subgroup of J_2 and hence $2.\mathrm{A}_7$ cannot be a subgroup of $2.\mathrm{J}_2$. $\square$

6.4 Schur Indicator +

The groups G to consider here are A_7, A_8, A_{15}, A_{16}, $\mathrm{L}_2(13)$, $\mathrm{S}_6(2)$, $\mathrm{U}_3(3)$, $\mathrm{G}_2(3)$, J_1 and J_2.

Let q be odd and let $G\rho \leqslant \Omega^{\pm}_{14}(q, B)$, where B is the matrix of a non-degenerate symmetric bilinear form. The following table, Table 6.4.1, will help to determine when $\det(B)$ is square in $\mathbb{F}_p$ and hence whether B is of type + or $-$ by Lemma 3.1.13. It will also be useful for spinor norm calculations. It shows whether an element a is a square in $\mathbb{F}_p$. The symbol '$\square$' indicates that a is square, i.e. $\sqrt{a} \in \mathbb{F}_p$, for given primes p and '$\boxtimes$' indicates that a is not a square.

In this section we will mostly use Magma to determine the behaviour of the outer automorphisms of $\mathrm{O}^{\pm}_{14}(q)$. Hence we will usually work with a specific absolutely irreducible representation ρ of G such that $G\rho \leqslant \Omega^{\pm}_{14}(q, B)$ for some non-degenerate symmetric bilinear form B. With the exception of J_2 we only have to look at $\delta, \gamma \in \mathrm{Out}(\mathrm{O}^{\pm}_{14}(p, B))$. Since δ and γ are independent of the preserved form we will state our final results with respect to our standard form matrices as in Table 3.1.1 (p.25). In case J_2 we will determine the behaviour of the field automorphisms with respect to our standard form matrices.

Table 6.4.1: Square root containments

a	$\sqrt{a}$	$\square/\boxtimes$
3	$r_3 = -(z_{12}^3 + 2z_{12}^7)$	$\square$ $p \equiv 1, 11 \pmod{12}$
		$\boxtimes$ $p \equiv 5, 7 \pmod{12}$
5	$r_5 = 1 + 2z_5 + 2z_5^4$	$\square$ $p \equiv 1, 4 \pmod{5}$
		$\boxtimes$ $p \equiv 2, 3 \pmod{5}$
13	$r_{13} = 1 + 2z_{13} + 2z_{13}^3 + 2z_{13}^4$	$\square$ $p \equiv 1, 3, 4, 9, 10, 12 \pmod{13}$
	$+2z_{13}^9 + 2z_{13}^{10} + 2z_{13}^{12}$	$\boxtimes$ $p \equiv 2, 5, 6, 7, 8, 11 \pmod{13}$
15	r_{15}	$\square$ $p \equiv 1, 7, 11, 17, 43, 49, 53, 59 \pmod{60}$
		$\boxtimes$ $p \equiv 13, 19, 23, 29, 31, 37, 41, 47 \pmod{60}$
39	r_{39}	$\square$ $p \equiv 1, 5, 7, 19, 23, 25, 31, 35, 41, 49, 61,$
		$67, 89, 95, 107, 115, 121, 125, 131,$
		$133, 137, 149, 151, 155 \pmod{156}$
		$\boxtimes$ $p \equiv 11, 17, 29, 37, 43, 47, 53, 55, 59, 71,$
		$73, 77, 79, 83, 85, 97, 101, 103, 109,$
		$113, 119, 127, 139, 145 \pmod{156}$
$\frac{1}{2} - \frac{r_3}{4}$	$(\frac{r_2}{4} - \frac{r_6}{4}) = \frac{1}{4}(2z_{24}^3 + z_{24}^9 + 2z_{24}^{11}$	$\square$ $p \equiv 1, 23 \pmod{24}$
	$+2z_{24}^{17} + z_{24}^{21})$	$\boxtimes$ $p \equiv 5, 7, 11, 13, 17, 19 \pmod{24}$
$\frac{1}{2} + \frac{r_3}{4}$	$(\frac{r_2}{4} + \frac{r_6}{4})$	$\square$ $p \equiv 1, 23 \pmod{24}$
	$= \frac{1}{4}(-z_{24}^9 - 2z_{24}^{11} - 2z_{24}^{17} + z_{24}^{21})$	$\boxtimes$ $p \equiv 5, 7, 11, 13, 17, 19 \pmod{24}$

Proposition 6.4.1 (A_{7_1}).
Let A_{7_1} *be the image of a* 14*-dimensional absolutely irreducible representation* ρ *of* A_7 *whose associated character value for all elements of order* 6 *is* 2*.*

(i) If $p \equiv 1, 49 \pmod{60}$ *then there are eight conjugacy classes of* $\mathscr{S}_1$*-subgroups of* $\Omega_{14}^+(p)$ *weakly equivalent to* $\mathrm{A}_{7_1}.2$*, with trivial class stabiliser in* $\mathrm{Out}(\mathrm{O}_{14}^+(p))$*.*

(ii) If $p \equiv 7, 19, 31, 43 \pmod{60}$ *then there are four conjugacy classes of* $\mathscr{S}_1$*-subgroups of* $\Omega_{14}^+(p)$ *weakly equivalent to* $\mathrm{A}_{7_1}.2$*, with trivial class stabiliser in* $\mathrm{Out}(\mathrm{O}_{14}^+(p))$*.*

(iii) If $p \equiv 13, 37 \pmod{60}$ *then there exist four conjugacy classes of* $\mathscr{S}_1$*-subgroups of* $\Omega_{14}^+(p)$ *weakly equivalent to* A_{7_1}*, with class stabiliser* $\langle \delta' \rangle$ *in* $\mathrm{Out}(\mathrm{O}_{14}^+(p))$*.*

(iv) If $p \equiv 11, 59 \pmod{60}$ *then there are eight conjugacy classes of* $\mathscr{S}_1$*-subgroups of* $\Omega^-_{14}(p)$ *weakly equivalent to* $\mathrm{A}_{7_1}.2$*, with trivial class stabiliser in* $\mathrm{Out}(\mathrm{O}^-_{14}(p))$*.*

(v) If $p \equiv 17, 29, 41, 53 \pmod{60}$ *then there are four conjugacy classes of* $\mathscr{S}_1$*-subgroups of* $\Omega^-_{14}(p)$ *weakly equivalent to* $\mathrm{A}_{7_1}.2$*, with trivial class stabiliser in* $\mathrm{Out}(\mathrm{O}^-_{14}(p))$*.*

(vi) If $p \equiv 23, 47 \pmod{60}$ *then there are four conjugacy classes of* $\mathscr{S}_1$*-subgroups of* $\Omega^-_{14}(p)$ *weakly equivalent to* A_{7_1}*, with class stabiliser* $\langle \delta' \rangle$ *in* $\mathrm{Out}(\mathrm{O}^-_{14}(p))$*.*

Proof. Let $G = \mathrm{A}_{7_1}$. Then $\mathrm{Out}(G) = 2$ and by Table 6.1.1, $G \leqslant \Omega^{\pm}_{14}(p)$ since the character ring of ρ is $\mathbb{Z}$. We find the form matrix B preserved by $G\rho$ using Magma (file a71d14comp). Since B has determinant 3 times a square it follows that B has square discriminant if and only if $p \equiv 1, 11 \pmod{12}$ by Table 6.4.1. Hence $G \leqslant \Omega^+_{14}(p, B)$ if $p \equiv 1, 7 \pmod{12}$ and $G \leqslant \Omega^-_{14}(p, B)$ if $p \equiv 5, 11 \pmod{12}$ by Lemma 3.1.13.

Now we consider $G\rho.2 = \langle G\rho, g \rangle$. Computer calculations show that g has determinant 1 and preserves the form B. It follows that $G.2 \leqslant \mathrm{SO}^{\pm}_{14}(p, B)$. Note that $G.2 \cong \langle G, -g \rangle \leqslant \mathrm{SO}^{\pm}_{14}(p, B)$ as well. Hence if the spinor norm of g or of $-g$ is 1 then $G.2 \leqslant \Omega^{\pm}_{14}(p, B)$.

Computer calculations show that g has spinor norm 1 if and only if 5 is square in $\mathbb{F}_p$, and $-g$ has spinor norm 1 if and only if either both 3 and 5 are square or if they are both non-square in $\mathbb{F}_p$. Hence $G.2 \leqslant \Omega^{\pm}_{14}(p, B)$ if $p \equiv 1, 11, 19, 29, 31, 41, 49, 59 \pmod{60}$ or if $p \equiv 1, 7, 11, 17, 43, 49, 53, 59 \pmod{60}$ by Table 6.4.1. The number of $\Omega^{\pm}_{14}(p, B)$-conjugacy classes follows from Lemma 4.3.3.

If $G.2 \nleqslant \Omega^{\pm}_{14}(p, B)$ then 5 is non-square in $\mathbb{F}_p$ and 3 is square. Hence the discriminant of B is square and we can use Lemma 4.8.12(ii) to show that in this case a conjugacy class of G is stabilised by $\langle \delta' \rangle$. Therefore each C-conjugacy class of G splits into 4 conjugacy classes in $\Omega^{\pm}_{14}(p, B)$ by Lemma 4.3.3. □

Proposition 6.4.2 (A_{7_2})**.**
Let A_{7_2} *be the image of a* 14*-dimensional absolutely irreducible representation of* A_7 *that affords the character value* -1 *for all elements of order* 6*.*

(i) If $p \equiv 1, 17, 49, 53 \pmod{60}$ *then there are four conjugacy classes of* $\mathscr{S}_1$*-subgroups of* $\Omega^+_{14}(p)$ *weakly equivalent to* A_{7_2}*, with class stabiliser* $\langle \gamma \rangle$ *in* $\mathrm{Out}(\mathrm{O}^+_{14}(p))$*.*

(ii) If $p \equiv 19, 23, 31, 47 \pmod{60}$ *then there are two conjugacy classes of* $\mathscr{S}_1$*-subgroups of* $\Omega^+_{14}(p)$ *weakly equivalent to* A_{7_2}*, with class stabiliser* $\langle\gamma\rangle$ *in* $\mathrm{Out}(\mathrm{O}^+_{14}(p))$.

(iii) If $p \equiv 7, 11, 41, 59 \pmod{60}$ *then there exist four conjugacy classes of* $\mathscr{S}_1$*-subgroups of* $\Omega^-_{14}(p)$ *weakly equivalent to* A_{7_2}*, with class stabiliser* $\langle\gamma\rangle$ *in* $\mathrm{Out}(\mathrm{O}^-_{14}(p))$.

(iv) If $p \equiv 13, 29, 37, 43 \pmod{60}$ *then there exist two conjugacy classes of* $\mathscr{S}_1$*-subgroups of* $\Omega^-_{14}(p)$ *weakly equivalent to* A_{7_2}*, with class stabiliser* $\langle\gamma\rangle$ *in* $\mathrm{Out}(\mathrm{O}^-_{14}(p))$.

Proof. Let $G = \mathrm{A}_{7_2}$. Then $\mathrm{Out}(G) = 2$ and $G \leqslant \Omega^\pm_{14}(p)$ by Table 6.1.1. Computer calculations (file a72d14comp) show that the determinant of the form matrix B preserved by G is 15 times a square. From this it follows by Lemma 3.1.13 and Table 6.4.1 that $G \leqslant \Omega^+_{14}(p, B)$ if $p \equiv 1,\ 17,\ 19,\ 23,\ 31, 47, 49, 53 \pmod{60}$ and $G \leqslant \Omega^-_{14}(p, B)$ if $p \equiv 7, 11, 13, 29, 37, 41, 43, 59 \pmod{60}$. Furthermore, the matrix inducing the outer automorphism of G sits inside $\mathrm{GO}^\pm_{14}(p, B) \backslash \mathrm{SO}^\pm_{14}(p, B)$. Therefore an $\Omega^\pm_{14}(p, B)$-conjugacy class of G is stabilised by $\langle\gamma\rangle$ by Lemma 4.8.12(i). The number of conjugacy classes follows from Lemma 4.3.3. □

Proposition 6.4.3 (A_7 in characteristic 2)**.**
There are two conjugacy classes of $\mathscr{S}_1$*-subgroups of* $\Omega^-_{14}(2)$ *isomorphic to* $\mathrm{A}_7.2$*, with trivial stabiliser in* $\mathrm{Out}(\mathrm{O}^-_{14}(2))$.

Proof. Computer calculations (file a7ch2d14comp) show that $\mathrm{A}_7.2 \leqslant \Omega^-_{14}(2)$ in characteristic 2. Furthermore, since $\mathrm{Out}(\mathrm{A}_7) = 2$, the stabiliser of any of the 2 conjugacy classes of $\mathrm{A}_7.2$ in $\Omega^-_{14}(2)$ has to be trivial. □

Proposition 6.4.4 (A_8)**.**

(i) If $p \equiv 1, 17, 49, 53 \pmod{60}$ *then there are four conjugacy classes of* $\mathscr{S}_1$*-subgroups of* $\Omega^+_{14}(p)$ *weakly equivalent to* A_8*, with class stabiliser* $\langle\gamma\rangle$ *in* $\mathrm{Out}(\mathrm{O}^+_{14}(p))$.

(ii) If $p \equiv 19, 23, 31, 47 \pmod{60}$ *then there are two conjugacy classes of* $\mathscr{S}_1$*-subgroups of* $\Omega^+_{14}(p)$ *weakly equivalent to* A_8*, with class stabiliser* $\langle\gamma\rangle$ *in* $\mathrm{Out}(\mathrm{O}^+_{14}(p))$.

(iii) If $p \equiv 7, 11, 41, 59 \pmod{60}$ *then there exist four conjugacy classes of* $\mathscr{S}_1$*-subgroups of* $\Omega^-_{14}(p)$ *weakly equivalent to* A_8*, with class stabiliser* $\langle\gamma\rangle$ *in* $\mathrm{Out}(\mathrm{O}^-_{14}(p))$.

(iv) If $p \equiv 13, 29, 37, 43 \pmod{60}$ *then there exist two conjugacy classes of* $\mathscr{S}_1$*-subgroups of* $\Omega^-_{14}(p)$ *weakly equivalent to* A_8*, with class stabiliser* $\langle\gamma\rangle$ *in* $\mathrm{Out}(\mathrm{O}^-_{14}(p))$*.*

Proof. Let $G = \mathrm{A}_8$. Then $\mathrm{Out}(G) = 2$ and $\mathrm{A}_8 \leqslant \Omega^\pm_{14}(p)$ by Table 6.1.1. Let ρ be an absolutely irreducible 14-dimensional representation of G. Computer calculations (a8d14comp) show that the matrix B of the form preserved by $G\rho$ has determinant a square times 15. By Lemma 3.1.13 and Table 6.4.1, $\mathrm{A}_8 \leqslant \Omega^+_{14}(p, B)$ if $p \equiv 1, 17, 19, 23, 31, 47, 49, 53 \pmod{60}$ and $\mathrm{A}_8 \leqslant \Omega^-_{14}(p, B)$ if $p \equiv 7, 11, 13, 29, 37, 41, 43, 59 \pmod{60}$. Also, using Magma again, we can show that the matrix g inducing the outer automorphism of G sits inside $\mathrm{GO}^\pm_{14}(p, B) \backslash \mathrm{SO}^\pm_{14}(p, B)$. Hence it follows from Lemma 4.8.12(i) that a conjugacy class of G in $\Omega^\pm_{14}(p, B)$ is stabilised by $\langle\gamma\rangle$. The number of conjugacy classes follows from Lemma 4.3.3. □

Proposition 6.4.5 (A_{15})**.**

(i) If $p \equiv 1, 17, 49, 53 \pmod{60}$ *then there are four conjugacy classes of* $\mathscr{S}_1$*-subgroups of* $\Omega^+_{14}(p)$ *weakly equivalent to* A_{15}*, with class stabiliser* $\langle\gamma\rangle$ *in* $\mathrm{Out}(\mathrm{O}^+_{14}(p))$*.*

(ii) If $p \equiv 19, 23, 31, 47 \pmod{60}$ *then there are two conjugacy classes of* $\mathscr{S}_1$*-subgroups of* $\Omega^+_{14}(p)$ *weakly equivalent to* A_{15}*, with class stabiliser* $\langle\gamma\rangle$ *in* $\mathrm{Out}(\mathrm{O}^+_{14}(p))$*.*

(iii) If $p \equiv 7, 11, 41, 59 \pmod{60}$ *then there exist four conjugacy classes of* $\mathscr{S}_1$*-subgroups of* $\Omega^-_{14}(p)$ *weakly equivalent to* A_{15}*, with class stabiliser* $\langle\gamma\rangle$ *in* $\mathrm{Out}(\mathrm{O}^-_{14}(p))$*.*

(iv) If $p \equiv 13, 29, 37, 43 \pmod{60}$ *then there exist two conjugacy classes of* $\mathscr{S}_1$*-subgroups of* $\Omega^-_{14}(p)$ *weakly equivalent to* A_{15}*, with class stabiliser* $\langle\gamma\rangle$ *in* $\mathrm{Out}(\mathrm{O}^-_{14}(p))$*.*

(v) There is a unique conjugacy class of $\mathscr{S}_1$*-subgroups of* $\Omega^+_{14}(2)$ *isomorphic to* A_{15} *with class stabiliser* $\langle\gamma\rangle$ *in* $\mathrm{Out}(\mathrm{O}^+_{14}(2))$*.*

Proof. Let $G = \mathrm{A}_{15}$. Then $\mathrm{Out}(G) = 2$ and $G \leqslant \Omega^\pm_{14}(p)$ since the character ring of a 14-dimensional absolutely irreducible representation of G is $\mathbb{Z}$ by Table 6.1.1.

In odd characteristic computer calculations (file a15d14comp) show that the discriminant of the preserved form matrix B is square if and only if 15 is square in $\mathbb{F}_p$. Hence it follows from Lemma 3.1.13 and Table 6.4.1 that

$A_{15} \leqslant \Omega_{14}^{+}(p,B)$ if $p \equiv 1, 17, 19, 23, 31, 47, 49, 53 \pmod{60}$. Otherwise A_{15} is a subgroup of $\Omega_{14}^{-}(p,B)$ ($p \neq 2,3,5$).

Using Magma again we can show that the matrix g inducing the outer automorphism of order 2 of G sits inside $\mathrm{GO}_{14}^{\pm}(p,B)\backslash\mathrm{SO}_{14}^{\pm}(p,B)$. Therefore a conjugacy class of G is stabilised by $\langle\gamma\rangle$ by Lemma 4.8.12(i). The number of conjugacy classes of G in $\Omega_{14}^{\pm}(p,B)$ follows from Lemma 4.3.3.

If $p = 2$ then computer calculations (file a15d14comp) show that $A_{15} \leqslant \Omega_{14}^{+}(2)$. Furthermore, the matrix inducing the non-trivial outer automorphism of A_{15} sits inside $\mathrm{SO}_{14}^{+}(2)\backslash\Omega_{14}^{+}(2)$. Therefore there is exactly one conjugacy class of G in $\Omega_{14}^{+}(2)$ by Lemma 4.3.3 which is stabilised by $\langle\gamma\rangle$. □

Proposition 6.4.6 (A_{16} in characteristic 2)**.**
There is a unique conjugacy class of $\mathscr{S}_1$-subgroups of $\Omega_{14}^{+}(2)$ isomorphic to A_{16}, with class stabiliser $\langle\gamma\rangle$ in $\mathrm{Out}(\mathrm{O}_{14}^{+}(2))$.

Proof. Computer calculations in Magma (file a16d14comp) show that $A_{16} \leqslant \Omega_{14}^{+}(2)$. Furthermore, the matrix inducing the non-trivial outer automorphism of A_{16} is an element of $\mathrm{SO}_{14}^{+}(2)\backslash\Omega_{14}^{+}(2)$. It follows that there exists one $\Omega_{14}^{+}(2)$-conjugacy class of A_{16} by Lemma 4.3.3 which is stabilised by $\langle\gamma\rangle$. □

Proposition 6.4.7 ($L_2(13)_1$)**.**
Let $L_2(13)_1$ be the image of a 14-dimensional absolutely irreducible representation of $L_2(13)$ whose associated character value of all elements of order 2 is 2.

(i) If $p \equiv 1, 5, 25, 41, 49, 61, 89, 121, 125, 133, 137, 149 \pmod{156}$ there exist four conjugacy classes of $\mathscr{S}_1$-subgroups of $\Omega_{14}^{+}(p)$ weakly equivalent to $L_2(13)_1$, with class stabiliser $\langle\gamma\rangle$ in $\mathrm{Out}(\mathrm{O}_{14}^{+}(p))$.

(ii) If $p \equiv 11, 43, 47, 55, 59, 71, 79, 83, 103, 119, 127, 139 \pmod{156}$ there exist two conjugacy classes of $\mathscr{S}_1$-subgroups of $\Omega_{14}^{+}(p)$ weakly equivalent to $L_2(13)_1$, with class stabiliser $\langle\gamma\rangle$ in $\mathrm{Out}(\mathrm{O}_{14}^{+}(p))$.

(iii) If $p \equiv 7, 19, 23, 31, 35, 67, 95, 107, 115, 131, 151, 155 \pmod{156}$ there exist four conjugacy classes of $\mathscr{S}_1$-subgroups of $\Omega_{14}^{-}(p)$ weakly equivalent to $L_2(13)_1$, with class stabiliser $\langle\gamma\rangle$ in $\mathrm{Out}(\mathrm{O}_{14}^{-}(p))$.

(iv) If $p \equiv 17, 29, 37, 53, 73, 77, 85, 97, 101, 109, 113, 145 \pmod{156}$ there exist two conjugacy classes of $\mathscr{S}_1$-subgroups of $\Omega_{14}^{-}(p)$ weakly equivalent to $L_2(13)_1$, with class stabiliser $\langle\gamma\rangle$ in $\mathrm{Out}(\mathrm{O}_{14}^{-}(p))$.

Proof. Let $G = \mathrm{L}_2(13)_1$. Then $\mathrm{Out}(G) = 2$ and by Table 6.1.1, $G \leqslant \Omega^{\pm}_{14}(p)$ for all $p \neq 2, 3, 13$. Furthermore, computer calculations (file l2131d14comp) show that the discriminant of the preserved form matrix B is square if and only if $3 \cdot 13$ is square in $\mathbb{F}_p$. The type of orthogonal form now follows from Lemma 3.1.13 and Table 6.4.1.

Computer calculations show as well that the matrix g which induces the outer automorphism of order 2 of G sits inside $\mathrm{GO}^{\pm}_n(p, B)\backslash\mathrm{SO}^{\pm}_n(p, B)$. Therefore a conjugacy class of G is stabilised by $\langle\gamma\rangle$ by Lemma 4.8.12(i). The number of conjugacy classes follows from Lemma 4.3.3. □

Proposition 6.4.8 ($\mathrm{L}_2(13)_2$)**.**
Let $\mathrm{L}_2(13)_2$ *be the image of a* 14*-dimensional absolutely irreducible representation of* $\mathrm{L}_2(13)$ *whose associated character value of all elements of order* 2 *is* -2.

(i) *If* $p \equiv 1, 25, 49, 61, 121, 133 \pmod{156}$*, then there are four conjugacy classes of* $\mathscr{S}_1$*-subgroups of* $\Omega^{+}_{14}(p)$ *weakly equivalent to* $\mathrm{L}_2(13)_2$*, with class stabiliser* $\langle\gamma\rangle$ *in* $\mathrm{Out}(\mathrm{O}^{+}_{14}(p))$.

(ii) *If* $p \equiv 11, 47, 59, 71, 83, 119 \pmod{156}$*, then there are two conjugacy classes of* $\mathscr{S}_1$*-subgroups of* $\Omega^{+}_{14}(p)$ *weakly equivalent to* $\mathrm{L}_2(13)_2$*, with class stabiliser* $\langle\gamma\rangle$ *in* $\mathrm{Out}(\mathrm{O}^{+}_{14}(p))$.

(iii) *If* $p \equiv 5, 41, 89, 125, 137, 149 \pmod{156}$*, then there are four conjugacy classes of* $\mathscr{S}_1$*-subgroups of* $\Omega^{+}_{14}(p)$ *weakly equivalent to* $\mathrm{L}_2(13)_2$*, with class stabiliser* $\langle\gamma\delta\rangle$ *in* $\mathrm{Out}(\mathrm{O}^{+}_{14}(p))$.

(iv) *If* $p \equiv 43, 55, 79, 103, 127, 139 \pmod{156}$*, then there are two conjugacy classes of* $\mathscr{S}_1$*-subgroups of* $\Omega^{+}_{14}(p)$ *weakly equivalent to* $\mathrm{L}_2(13)_2$*, with class stabiliser* $\langle\gamma\delta\rangle$ *in* $\mathrm{Out}(\mathrm{O}^{+}_{14}(p))$.

(v) *If* $p \equiv 23, 35, 95, 107, 131, 155 \pmod{156}$*, then there are four conjugacy classes of* $\mathscr{S}_1$*-subgroups of* $\Omega^{-}_{14}(p)$ *weakly equivalent to* $\mathrm{L}_2(13)_2$*, with class stabiliser* $\langle\gamma\rangle$ *in* $\mathrm{Out}(\mathrm{O}^{-}_{14}(p))$.

(vi) *If* $p \equiv 37, 73, 85, 97, 109, 145 \pmod{156}$*, then there are two conjugacy classes of* $\mathscr{S}_1$*-subgroups of* $\Omega^{-}_{14}(p)$ *weakly equivalent to* $\mathrm{L}_2(13)_2$*, with class stabiliser* $\langle\gamma\rangle$ *in* $\mathrm{Out}(\mathrm{O}^{-}_{14}(p))$.

(vii) *If* $p \equiv 7, 19, 31, 67, 115, 151 \pmod{156}$ *there are four conjugacy classes of* $\mathscr{S}_1$*-subgroups of* $\Omega^{-}_{14}(p)$ *isomorphic to* $\mathrm{L}_2(13)_2$*, with class stabiliser* $\langle\gamma\delta\rangle$ *in* $\mathrm{Out}(\mathrm{O}^{-}_{14}(p))$.

(viii) *If* $p \equiv 17, 29, 53, 77, 101, 113 \pmod{156}$ *there are two conjugacy classes of* $\mathscr{S}_1$*-subgroups of* $\Omega^-_{14}(p)$ *isomorphic to* $\mathrm{L}_2(13)_2$*, with class stabiliser* $\langle\gamma\delta\rangle$ *in* $\mathrm{Out}(\mathrm{O}^-_{14}(p))$*.*

Proof. Let $G = \mathrm{L}_2(13)_2$. Then $\mathrm{Out}(G) = 2$ and $G \leqslant \Omega^{\pm}_{14}(p)$ by Table 6.1.1. Furthermore the discriminant of the preserved form matrix B is square if and only if 39 is square in $\mathbb{F}_p$ (file l2132d14comp). The type of orthogonal form preserved by G follows from Lemma 3.1.13 and Table 6.4.1.

Computations in Magma show that the matrix g inducing the outer automorphism of order 2 of G is an element of $\mathrm{GO}^{\pm}_{14}(p, B)\backslash\mathrm{SO}^{\pm}_{14}(p, B)$ if 3 is a square in $\mathbb{F}_p$. Otherwise $g \in \mathrm{CGO}^{\pm}_{14}(p, B)\backslash\mathrm{GO}^{\pm}_{14}(p, B)$ with $\det(g) = -3^7$.

If 3 is square in $\mathbb{F}_p$, i.e. if $p \equiv 1, 11 \pmod{12}$ by Table 6.4.1, then a conjugacy class of G in $\Omega^{\pm}_{14}(p, B)$ is stabilised by $\langle\gamma\rangle$ by Lemma 4.8.12(i). The number of conjugacy classes follows from Lemma 4.3.3.

If 3 is not a square in $\mathbb{F}_p$ then we can show that $gBg^{\mathrm{T}} = 3B$. Hence this conjugacy class is stabilised by $\langle\gamma\delta\rangle$ by Lemma 4.8.12 and the number of conjugacy classes follows from Lemma 4.3.3. □

Proposition 6.4.9 ($\mathrm{L}_2(13)$ in characteristic 2)**.**
There is a unique conjugacy class of $\mathscr{S}_1$*-subgroups of* $\Omega^+_{14}(2)$ *isomorphic to* $\mathrm{L}_2(13)$*, with class stabiliser* $\langle\gamma\rangle$ *in* $\mathrm{Out}(\mathrm{O}^+_{14}(2))$*.*

Proof. In characteristic 2 computer calculations (file l213ch2d14comp) show that $\mathrm{L}_2(13) \leqslant \Omega^+_{14}(2)$. Furthermore, the matrix inducing the non-trivial outer automorphism of $\mathrm{L}_2(13)$ sits inside $\mathrm{SO}^+_{14}(2)\backslash\Omega^+_{14}(2)$. From this it follows by Lemma 4.8.13 and Lemma 4.3.3 that the unique conjugacy class of G in $\Omega^+_{14}(2)$ is stabilised by $\langle\gamma\rangle$. □

Proposition 6.4.10 ($\mathrm{S}_6(2)$ in characteristic 3)**.**
There are four conjugacy classes of $\mathscr{S}_1$*-subgroups of* $\Omega^+_{14}(3)$ *isomorphic to* $\mathrm{S}_6(2)$*, with trivial class stabiliser in* $\mathrm{Out}(\mathrm{O}^+_{14}(3))$*.*

Proof. Let $G = \mathrm{S}_6(2)$. Computer calculations (file s62d14comp) show that $G \leqslant \Omega^+_{14}(3)$. Since $|\mathrm{Out}(G)| = 1$, any proper normaliser of G has to be trivial and there are four conjugacy classes of G in $\Omega^+_{14}(3)$ by Lemma 4.3.3. □

Proposition 6.4.11 ($\mathrm{U}_3(3)$)**.**

(i) *If* $p \equiv 1 \pmod{12}$ *then there are* 8 *conjugacy classes of* $\mathscr{S}_1$*-subgroups of* $\Omega^+_{14}(p)$ *isomorphic to* $\mathrm{U}_3(3).2$*, which have trivial class stabiliser in* $\mathrm{Out}(\mathrm{O}^+_{14}(p))$*.*

(ii) If $p \equiv 7 \pmod{12}$ *then there are* 4 *conjugacy classes of* $\mathscr{S}_1$*-subgroups of* $\Omega_{14}^{+}(p)$ *isomorphic to* $\mathrm{U}_3(3).2$*, which have trivial class stabiliser in* $\mathrm{Out}(\mathrm{O}_{14}^{+}(p))$.

(iii) If $p \equiv 5 \pmod{12}$ *then there are* 4 *conjugacy classes of* $\mathscr{S}_1$*-subgroups of* $\Omega_{14}^{-}(p)$ *isomorphic to* $\mathrm{U}_3(3).2$*, which have trivial class stabiliser in* $\mathrm{Out}(\mathrm{O}_{14}^{-}(p))$.

(iv) If $p \equiv 11 \pmod{12}$ *then there are* 8 *conjugacy classes of* $\mathscr{S}_1$*-subgroups of* $\Omega_{14}^{-}(p)$ *isomorphic to* $\mathrm{U}_3(3).2$*, which have trivial class stabiliser in* $\mathrm{Out}(\mathrm{O}_{14}^{-}(p))$.

(v) There are 2 *conjugacy classes of* $\mathscr{S}_1$*-subgroups of* $\Omega_{14}^{-}(2)$ *isomorphic to* $\mathrm{U}_3(3).2$*, with trivial class stabiliser in* $\mathrm{Out}(\mathrm{O}_{14}^{-}(2))$.

Proof. Let $G = \mathrm{U}_3(3)$. Then $\mathrm{Out}(G) = 2$ and $G \leqslant \Omega_{14}^{\pm}(p)$ by Table 6.1.1.

In odd characteristic we find using Magma (file u33d14comp) that the discriminant of the preserved form matrix B is a square if and only if 3 is square in $\mathbb{F}_p$ ($p \neq 2, 3$). From this it follows by Lemma 3.1.13 and Table 6.4.1 that $G \leqslant \Omega_{14}^{+}(p, B)$ if $p \equiv 1, 7 \pmod{12}$. Otherwise $G \leqslant \Omega_{14}^{-}(p, B)$.

Furthermore, the matrix g inducing the outer automorphism of order 2 of G sits inside $\mathrm{SO}_{14}^{\pm}(p, B)$ and $-g$ has always spinor norm 1. Therefore $G.2 \leqslant \Omega_{14}^{\pm}(p, B)$ for all p ($p \neq 2, 3$). By Lemma 4.3.3 each C-conjugacy class of subgroups isomorphic to G splits into 4 or 8 conjugacy classes in $\Omega_{14}^{\pm}(p, B)$. Also, since G does not afford any further outer automorphisms the stabiliser of any of these conjugacy classes is trivial.

If $p = 2$ then $\mathrm{U}_3(3).2 \leqslant \Omega_{14}^{-}(2)$ using Magma (file u33d14comp). By Lemma 4.3.3, there are 2 conjugacy classes of subgroups isomorphic to G in $\Omega_{14}^{-}(2)$. Since G does not afford any other non-trivial outer automorphism, the stabiliser of these classes has to be trivial. □

Proposition 6.4.12 ($\mathrm{G}_2(3)$)**.**

(i) If $p \equiv 1 \pmod{24}$ *then there are* 8 *conjugacy classes of* $\mathscr{S}_1$*-subgroups of* $\Omega_{14}^{+}(p)$ *isomorphic to* $\mathrm{G}_2(3).2$*, which have trivial class stabiliser in* $\mathrm{Out}(\mathrm{O}_{14}^{+}(p))$.

(ii) If $p \equiv 13 \pmod{24}$ *then there are* 4 *conjugacy classes of* $\mathscr{S}_1$*-subgroups of* $\Omega_{14}^{+}(p)$ *isomorphic to* $\mathrm{G}_2(3)$*. The class stabiliser in* $\mathrm{Out}(\mathrm{O}_{14}^{+}(p))$ *is* $\langle \delta' \rangle$.

(iii) If $p \equiv 7, 19 \pmod{24}$ *then there exist* 2 *conjugacy classes of* $\mathscr{S}_1$*-subgroups of* $\Omega_{14}^{+}(p)$ *isomorphic to* $\mathrm{G}_2(3)$*, with class stabiliser* $\langle\delta\rangle$ *in* $\mathrm{Out}(\mathrm{O}_{14}^{+}(p))$*.*

(iv) If $p \equiv 23 \pmod{24}$ *then there are* 8 *conjugacy classes of* $\mathscr{S}_1$*-subgroups of* $\Omega_{14}^{-}(p)$ *isomorphic to* $\mathrm{G}_2(3).2$*, which have trivial class stabiliser in* $\mathrm{Out}(\mathrm{O}_{14}^{-}(p))$*.*

(v) If $p \equiv 11 \pmod{24}$ *then there are* 4 *conjugacy classes of* $\mathscr{S}_1$*-subgroups of* $\Omega_{14}^{-}(p)$ *isomorphic to* $\mathrm{G}_2(3)$*. The class stabiliser in* $\mathrm{Out}(\mathrm{O}_{14}^{-}(p))$ *is* $\langle\delta'\rangle$*.*

(vi) If $p \equiv 5, 17 \pmod{24}$ *then there* 2 *conjugacy classes of* $\mathscr{S}_1$*-subgroups of* $\Omega_{14}^{-}(p)$ *isomorphic to* $\mathrm{G}_2(3)$*, with class stabiliser* $\langle\delta\rangle$ *in* $\mathrm{Out}(\mathrm{O}_{14}^{-}(p))$*.*

(vii) There are 2 *conjugacy classes of* $\mathscr{S}_1$*-subgroups of* $\Omega_{14}^{-}(2)$ *isomorphic to* $\mathrm{G}_2(3).2$*, which have trivial class stabiliser in* $\mathrm{Out}(\mathrm{O}_{14}^{-}(2))$*.*

Proof. Let $G = \mathrm{G}_2(3)$. Then $G \leqslant \Omega_{14}^{\pm}(p)$ and $\mathrm{Out}(G) = 2$ by Table 6.1.1.

In odd characteristic calculations in Magma (file g23d14comp) show that the discriminant of the preserved form matrix B is square if and only if 3 is square in $\mathbb{F}_p$ ($p \neq 2, 3$). Therefore, by Table 6.4.1 and Lemma 3.1.13, $G \leqslant \Omega_{14}^{+}(p, B)$ if and only if $p \equiv 1, 7 \pmod{12}$. If $p \equiv 5, 11 \pmod{12}$ then G preserves an orthogonal form of minus-type.

Computer calculations show that the matrix g inducing the outer automorphism of order 2 of G sits inside $\mathrm{CGO}_{14}^{\pm}(p, B)$ and preserves the form up to scalar multiplication by 3. It follows that if 3 is a square in $\mathbb{F}_p$ then $\mathrm{r}_3^{-1}g$ has determinant 1, preserves B and induces the outer automorphism of order 2 of G. Furthermore, $\pm\mathrm{r}_3^{-1}g$ has spinor norm 1 if and only if $\frac{1}{2} \mp \frac{\mathrm{r}_3}{4}$ is square in $\mathbb{F}_p$ which holds if and only if $p \equiv 1, 23 \pmod{24}$ by Table 6.4.1.

Hence, if $p \equiv 1, 23 \pmod{24}$ then $G.2 \leqslant \Omega_{14}^{\pm}(p, B)$ and there are 8 conjugacy classes of $G.2$ in $\Omega_{14}^{\pm}(p, B)$ by Lemma 4.3.3. If $p \equiv 11, 13 \pmod{24}$ then $\mathrm{r}_3^{-1}g \in \mathrm{SO}_{14}^{\pm}(p, B) \backslash \Omega_{14}^{\pm}(p, B)$ and it follows that an $\Omega_{14}^{\pm}(p, B)$-conjugacy class of G is stabilised by $\langle\delta'\rangle$ by Lemma 4.8.12(ii). There are 4 such conjugacy classes by Lemma 4.3.3.

If $p \equiv 5, 7, 17, 19 \pmod{24}$ then $g \in \mathrm{CGO}_{14}^{\pm}(p, B) \backslash \mathrm{GO}_{14}^{\pm}(p, B)$ and we can show that $\det(g) = 3^7$. There are 2 $\Omega_{14}^{\pm}(p)$-conjugacy classes of G by Lemma 4.3.3 and the class stabiliser is $\langle\delta\rangle$ by Lemma 4.8.12(iv).

If $p = 2$ then $\mathrm{G}_2(3).2 \leqslant \Omega_{14}^{-}(2)$ (file g23d14comp) and there are two conjugacy classes of G in $\Omega_{14}^{-}(2)$ by Lemma 4.3.3. Furthermore, since G

does not afford any further non-trivial outer automorphism, the stabiliser in $\mathrm{Out}(\mathrm{O}_{14}^{-}(2))$ of these $\Omega_{14}^{-}(2)$-conjugacy classes of G has to be trivial. $\square$

Proposition 6.4.13 (J_1)**.**
There are eight conjugacy classes of $\mathscr{S}_1$-subgroups of $\Omega_{14}^{-}(11)$ isomorphic to J_1, with trivial class stabiliser in $\mathrm{Out}(\mathrm{O}_{14}^{-}(11))$.

Proof. Let $G = \mathrm{J}_1$. Then computer calculations (file j1d14comp) show that $G \leqslant \Omega_{14}^{-}(11)$. Since the outer automorphism group of G is trivial, every conjugacy class has trivial stabiliser. There are 8 conjugacy classes of G in $\Omega_{14}^{-}(11)$ by Lemma 4.3.3. $\square$

Proposition 6.4.14 (J_2 in characteristic $\neq 2, 3, 5$)**.**

(i) If $p \equiv 1, 49 \pmod{60}$ then there are eight conjugacy classes of $\mathscr{S}_1$-subgroups of $\Omega_{14}^{+}(p)$ isomorphic to J_2, with trivial class stabiliser in $\mathrm{Out}(\mathrm{O}_{14}^{+}(p))$.

(ii) If $p \equiv 19, 31 \pmod{60}$ then there are four conjugacy classes of $\mathscr{S}_1$-subgroups of $\Omega_{14}^{+}(p)$ isomorphic to J_2, with trivial class stabiliser in $\mathrm{Out}(\mathrm{O}_{14}^{+}(p))$.

(iii) If $p \equiv 7, 13, 37, 43 \pmod{60}$ then there are eight conjugacy classes of $\mathscr{S}_1$-subgroups of $\Omega_{14}^{+}(p^2)$ isomorphic to J_2, with class stabiliser $\langle \phi \rangle$ in $\mathrm{Out}(\mathrm{O}_{14}^{+}(p^2))$.

(iv) If $p \equiv 17, 23, 47, 53 \pmod{60}$ then there are eight conjugacy classes of $\mathscr{S}_1$-subgroups of $\Omega_{14}^{+}(p^2)$ isomorphic to J_2, with class stabiliser $\langle \phi\gamma \rangle$ in $\mathrm{Out}(\mathrm{O}_{14}^{+}(p^2))$.

(v) If $p \equiv 11, 59 \pmod{60}$ then there are eight conjugacy classes of $\mathscr{S}_1$-subgroups of $\Omega_{14}^{-}(p)$ isomorphic to J_2, with trivial class stabiliser in $\mathrm{Out}(\mathrm{O}_{14}^{-}(p))$.

(vi) If $p \equiv 29, 41 \pmod{60}$ then there are four conjugacy classes of $\mathscr{S}_1$-subgroups of $\Omega_{14}^{-}(p)$ isomorphic to J_2, with trivial class stabiliser in $\mathrm{Out}(\mathrm{O}_{14}^{-}(p))$.

Proof. Let $G = \mathrm{J}_2$ which implies that $\mathrm{Out}(G) = 2$. By Table 6.1.1 the 14-dimensional absolutely irreducible representations ρ of G have character ring $\mathbb{Z}[\mathrm{b}_5]$ and Schur indicator + from which it follows by Table 2.2.1 that $G \leqslant \Omega_{14}^{\pm}(p)$ if $p \equiv 1, 4 \pmod 5$ and $G \leqslant \Omega_{14}^{\pm}(p^2)$ if $p \equiv 2, 3 \pmod 5$.

Furthermore, the discriminant of the form matrix B preserved by $G\rho$ is square if and only if 3 is square in $\mathbb{F}_p$ (file j2d14comp). Hence if $G \leqslant$

$\Omega^{\pm}_{14}(p,B)$ then G preserves an orthogonal form of plus-type if $p \equiv 1, 19, 31, 49 \pmod{60}$ by Table 6.4.1 and Lemma 3.1.13. If $G \nleqslant \Omega^{\pm}_{14}(p,B)$ then $p^2 \equiv 1 \pmod 4$ and the discriminant of the form matrix is always square. Hence, G preserves an orthogonal form of plus-type in this case.

By [12, 24] the outer automorphism of G can only be induced by an element that does not sit inside $C = \mathrm{CGO}^{\pm}_{14}(p^t,B)$, $t \in \{1,2\}$ since it fuses the representations. Therefore there are either 4 or 8 conjugacy classes of G in $\Omega^{\pm}_{14}(p^t,B)$ by Lemma 4.3.3.

If $G \leqslant \Omega^{\pm}_{14}(p,B)$ then $\Omega^{\pm}_{14}(p)$ does not have any non-trivial field automorphism. It follows that the stabiliser of these conjugacy classes of G in $\Omega^{\pm}_{14}(p,B)$ is trivial.

If $p \equiv 7, 13, 17, 23, 37, 43, 47, 53 \pmod{60}$, i.e. if $G \leqslant \Omega^{+}_{14}(p^2,B)$, then there are eight $\Omega^{+}_{14}(p^2,B)$-conjugacy classes of G by Lemma 4.3.3. Furthermore, using Lemma 4.3.8, we can show that ${}^{\alpha}\rho$ is equivalent to ρ^{ϕ}, where α is the outer automorphism of order 2 of G. To find the stabiliser of these conjugacy classes, we work in characteristic 0 first.

Let $\hat{\rho}$ be an absolutely irreducible 14-dimensional representation of G. Then computer calculations in Magma (file j2d14comp) show that $G\hat{\rho}$ preserves a bilinear form $\hat{B}$ with $\det(\hat{B}) = 3^{13}$. Hence $G\hat{\rho} \leqslant \Omega_{14}(K,\hat{B})$ for some characteristic 0 field K. Here $K \leqslant \mathbb{Q}(w)$, where $w^4 + w^3 + w^2 + w + 1 = 0$, from which it follows that w is a fifth root of unity. Furthermore, all elements of K are of the form $a + b(w^2 + w^3)$, $a, b \in \mathbb{Q}$. Let $\hat{\phi} : w \mapsto w^2$. Then $(a + b(w^2 + w^3))\hat{\phi} = (a - b) - b(w^2 + w^3)$ and it is straightforward to check that $\hat{\phi}$ is a field automorphism of K.

Computationally we can also find $\hat{x}_1 \in \mathrm{GL}_{14}(K)$ such that $\hat{x}_1^{-1}(g\hat{\rho})^{\hat{\phi}}\hat{x}_1 = g^{\alpha}\hat{\rho}$ for all $g \in G$. Furthermore, if we let $\hat{x} = \hat{x}_1/(w^2 + w^3)$, then $\hat{x}\hat{B}\hat{x}^{\mathrm{T}} = \hat{B}^{\hat{\phi}}$ with $\det(\hat{x}) = 1$. Let $\hat{F} = \mathrm{antidiag}(1, \ldots, 1)$. We also know that there exists $\hat{A} \in \mathrm{GL}_{14}(\mathbb{C})$ such that $\hat{A}\hat{F}\hat{A}^{\mathrm{T}} = \hat{B}$ and $(G\hat{\rho})^{\hat{A}} \leqslant \Omega^{+}_{14}(\mathbb{C},\hat{F})$ by Lemma 4.8.8. From this it follows that $\det(\hat{A})^2 = -\det(\hat{B}) = -3^{13}$.

Since we want to use Lemma 4.8.7, we need to consider the p-modular reductions of $\hat{A}$, $\hat{B}$, $\hat{F}$ and $\hat{x}$. Note that this is straightforward for $\hat{B}$, $\hat{F}$ and $\hat{x}$ as these matrices can easily be calculated. We do not know $\hat{A}$ explicitly though and hence we do not know whether we can actually reduce $\hat{A}$ modulo p for all the prime numbers p we are interested in.

Let B, F, x and ρ denote the p-modular reductions of $\hat{B}$, $\hat{F}$, $\hat{x}$ and $\hat{\rho}$ respectively where B is the form matrix of a bilinear form of plus-type. Therefore we know that there exists some $A' \in \mathrm{GL}_{14}(p^2)$ such that $A'FA'^{\mathrm{T}} = B$ by Lemma 4.8.1. Furthermore, $\det(A')^2$ is equal to the p-modular reduction of -3^{13}.

Note that in characteristic 0, we defined $\hat{\phi}$ on K by sending w to w^2. However, we could have defined $\hat{\phi}$ by sending w to w^3 as well as this induces the same field automorphism on K since

$$\begin{aligned}[a + b(w^2 + w^3)]\hat{\phi} &= a + b((w^2)^2 + (w^3)^2) \\ &= a + b(w^4 + w) \\ &= a + b((w^3)^3 + (w^2)^3).\end{aligned}$$

In characteristic p, a field automorphism ϕ sends w to w^p, where $p \equiv 2, 3 \pmod 5$ in this case. Therefore, without loss of generality, we can define the p-modular reduction ϕ of $\hat{\phi}$ by sending w to w^2 if $p \equiv 2 \pmod 5$ and by sending w to w^3 if $p \equiv 3 \pmod 5$. Hence considering the p-modular reduction of $\hat{B}^{\hat{\phi}}$ we find that $xBx^{\mathrm{T}} = B^{\phi}$. Also note that $\det(x) = 1$ and $x^{-1}(g\rho)^{\phi}x = g^{\alpha}\rho$ for all $g \in G$.

Note as well that projectively $(G.2)\rho\backslash G\rho$ contains involutions. Hence, if $\det(A')^{1-\phi} = 1$ then an $\Omega^+_{14}(p^2)$-conjugacy class of $(G\rho)^{A'}$ is stabilised by ϕ in $\mathrm{Out}(\Omega^+_{14}(p^2))$ by Lemma 4.8.7. Taking the square root, we find that $\det(A')$ is a p-modular reduction of $\epsilon \cdot (\sqrt{3}) \cdot 3^6\mathrm{i}$, where $\epsilon \in \{\pm 1\}$. Therefore, we need to calculate the p-modular reduction of $((\sqrt{3}) \cdot 3^6)^{1-p} \cdot \mathrm{i}^{1-p}$ since $\epsilon^{1-p} = 1$.

First of all note that $\mathrm{i}^{1-p} = 1$ if and only if $1-p \equiv 0 \pmod 4$. This is the case if and only if $p \equiv 1 \pmod 4$. Furthermore, $((\sqrt{3}) \cdot 3^6)^{1-p} = \pm 1$ if and only if $((\sqrt{3}) \cdot 3^6)^{p-1} = \pm 1$ and $((\sqrt{3}) \cdot 3^6)^{p-1} = 1$ if and only if $(\sqrt{3}) \cdot 3^6 \in \mathbb{F}_p$. This is the case if and only if $\sqrt{3} \in \mathbb{F}_p$. By Table 2.2.1, $\sqrt{3} \in \mathbb{F}_p$ if and only if $p \equiv 1, 11 \pmod{12}$. Hence, $\det(A'^{1-p}) = 1$ if and only if $p \equiv 1, 7 \pmod{12}$.

Therefore a conjugacy class of $G\rho$ is stabilised by ϕ if $p \equiv 7, 13, 37, 43 \pmod{60}$ and by $\phi\gamma$ if $p \equiv 17, 23, 47, 53 \pmod{60}$. □

Proposition 6.4.15 (J_2 in characteristic 2)**.**
There are two conjugacy classes of $\mathscr{S}_1$-subgroups of $\Omega^+_{14}(4)$ isomorphic to J_2, with class stabiliser $\langle\phi\gamma\rangle$ in $\mathrm{Out}(\mathrm{O}^+_{14}(4))$.

Proof. Let $G = \mathrm{J}_2$. Using Magma (see file j2ch2d14comp) we can find a 14-dimensional absolutely irreducible representation of J_2 that preserves our standard quadratic form matrix $\mathrm{antidiag}(1, \ldots, 1, 0, \ldots, 0)$ (see Table 3.1.1). Hence $\mathrm{J}_2 \leqslant \Omega^+_{14}(4)$ and there are two $\Omega^+_{14}(4)$-conjugacy classes of G by Lemma 4.3.3.

Let α be the automorphism of order 2 of J_2. Then the matrix $x \in \mathrm{GL}_{14}(4)$ that conjugates $g^{\alpha}\rho$ to $(g\rho)^{\phi}$ for all $g \in G$ has quasideterminant -1. Hence, by Lemma 4.8.11, an $\Omega^+_{14}(2^2)$-conjugacy class of J_2 has stabiliser $\phi\gamma$. □

Proposition 6.4.16 (J$_2$ in characteristic 5)**.**
There are four conjugacy classes of $\mathscr{S}_1$-subgroups of $\Omega^-_{14}(5)$ isomorphic to J$_2$.2*, with trivial class stabiliser in* Out(O$^-_{14}(5)$)*.*

Proof. Let G = J$_2$. Then Out(G) = 2 and J$_2 \leqslant \Omega^\pm_{14}(5)$ since the 14-dimensional absolutely irreducible representation of G in characteristic 5 has character ring $\mathbb{Z}$ by Table 6.1.1. Furthermore computer calculations (file j2ch5d14comp) show that in fact J$_2$.2 $\leqslant \Omega^-_{14}(5)$. By Lemma 4.3.3 there are 4 conjugacy classes of J$_2$.2 in $\Omega^-_{14}(5)$. Since J$_2$.2 does not afford any non-trivial outer automorphism the stabiliser of these classes has to be trivial. □

Maximality

Finally, we want to show which of these groups are $\mathscr{S}_1$-maximal in $\Omega^\pm_{14}(q)$.

Proposition 6.4.17.
In Case $\mathbf{O}^+$ *let* $d := 1$ *if* q *even or if* $q \equiv 3 \pmod 4$ *and* $d := 2$ *if* $q \equiv 1 \pmod 4$*. In Case* $\mathbf{O}^-$ *let* $d := 1$ *if* q *even or if* $q \equiv 1 \pmod 4$ *and* $d := 2$ *if* $q \equiv 3 \pmod 4$*.*

(i) *If* $p \neq 2,3,5$ *then* $\mathrm{N}_{\Omega^\pm_{14}(p)}(\mathrm{A}_{7_1})$ *is* $\mathscr{S}_1$*-maximal in* $\Omega^\pm_{14}(p)$*.*

(ii) *No extension of* $d \times \mathrm{A}_{7_2}$ *is* $\mathscr{S}_1$*-maximal in any extension of* $\Omega^\pm_{14}(p)$*.*

(iii) *The group* A$_7$.2 *is* $\mathscr{S}_1$*-maximal in* $\Omega^-_{14}(2)$*.*

(iv) *No extension of* $d \times \mathrm{A}_8$ *is* $\mathscr{S}_1$*-maximal in any extension of* $\Omega^\pm_{14}(p)$*.*

(v) *If* $p \neq 2,3,5$ *then* $\mathrm{N}_{\Omega^\pm_{14}(p)}(\mathrm{A}_{15})$ *is* $\mathscr{S}_1$*-maximal in* $\Omega^\pm_{14}(p)$*. No extension of* A$_{15}$ *is* $\mathscr{S}_1$*-maximal in any extension of* $\Omega^+_{14}(2)$*.*

(vi) *The group* A$_{16}$ *is* $\mathscr{S}_1$*-maximal in* $\Omega^+_{14}(2)$*.*

(vii) *If* $p \neq 2,3,13$ *then* $\mathrm{N}_{\Omega^\pm_{14}(p)}(\mathrm{L}_2(13)_1)$ *is* $\mathscr{S}_1$*-maximal in* $\Omega^\pm_{14}(p)$*.*

(viii) *If* $p \neq 2,3,13$ *then* $\mathrm{N}_{\Omega^\pm_{14}(p)}(\mathrm{L}_2(13)_2)$ *is* $\mathscr{S}_1$*-maximal in* $\Omega^\pm_{14}(p)$*.*

(ix) *The group* L$_2$(13) *is* $\mathscr{S}_1$*-maximal in* $\Omega^+_{14}(2)$*.*

(x) *The group* S$_6$(2) *is* $\mathscr{S}_1$*-maximal in* $\Omega^+_{14}(3)$*.*

(xi) *No extension of* $d \times \mathrm{U}_3(3).2$ *is ever* $\mathscr{S}_1$*-maximal in any extension of* $\Omega^\pm_{14}(p)$*.*

(xii) If $p \neq 3$ then $\mathrm{N}_{\Omega^{\pm}_{14}(p)}(\mathrm{G}_2(3))$ *is* $\mathscr{S}_1$*-maximal in* $\Omega^{\pm}_{14}(p)$.

(xiii) The group $2 \times \mathrm{J}_1$ *is* $\mathscr{S}_1$*-maximal in* $\Omega^{-}_{14}(11)$.

(xiv) If there exists an $\mathscr{S}_1$*-subgroup* $G \cong d \times \mathrm{J}_2$ *in* $\Omega^{\pm}_{14}(q)$ *then* G *is* $\mathscr{S}_1$*-maximal.*

Proof. (i) We find that A_7 could be a subgroup of A_8, A_{15} and J_2 by Lagrange's theorem. It is clear that $\mathrm{A}_7 \leqslant \mathrm{A}_8$ and $\mathrm{A}_7 \leqslant \mathrm{A}_{15}$ but $\mathrm{A}_7 \nleqslant \mathrm{J}_2$ using Magma (file s1dim14cont). Furthermore, note that A_{7_1} is not an $\mathscr{S}_1$-subgroup of A_8 as A_{7_1} has character value 2 for all elements of order 6 whereas the 14-dimensional absolutely irreducible representation of A_8 has character values -1 or 0 for all elements of order 6 ([12, 24]). Now we consider a possible containment of A_{7_1} in A_{15}. We can show that A_{15} has 4 conjugacy classes of subgroups isomorphic to A_7, two of which are conjugate in S_{15} (file s1dim14cont). Hence these 4 conjugacy classes correspond to 3 non-equivalent 14-dimensional representations of A_7. By looking at the traces of elements of order 6 of groups contained in these conjugacy classes, we find that $\mathrm{A}_{7_1} \nleqslant \mathrm{A}_{15}$ in dimension 14.

(ii) It is clear that A_7 is a subgroup of A_8 in an abstract way. Furthermore, the character values of A_{7_2} correspond to the character values of the 14-dimensional absolutely irreducible representation ρ of A_8 and there are no reducible representations of A_7 that afford the same character values as ρ by [14, 24]. Furthermore, A_7 and A_8 preserve the same orthogonal form in dimension 14, $\mathrm{S}_7 \leqslant \mathrm{S}_8$ and both A_7 and A_8 are stabilised by γ in $\mathrm{Out}(\mathrm{O}^{\pm}_{14}(p))$.

(iii) By looking at the respective character tables ([14, 24]) it straightforward to see that in even characteristic A_7 can only be an $\mathscr{S}_1$-subgroup of A_{15} or A_{16} in dimension 14. However A_{15} and A_{16} preserve an orthogonal form of plus-type whereas A_7 preserves an orthogonal form of minus-type.

(iv) It is clear that $\mathrm{A}_8 \leqslant \mathrm{A}_{15}$ in an abstract way and so we need to check whether A_8 is also on $\mathscr{S}_1$-subgroup of A_{15} in dimension 14. Note that A_{15} has 3 conjugacy classes of subgroups isomorphic to A_8, two of which are conjugate in S_{15} (file s1dim14cont). Since $\mathrm{S}_{15} \leqslant \mathrm{GL}_{14}(p)$ by Proposition 6.4.5, we can deduce that these two conjugacy classes correspond to equivalent representations of A_8. Looking at the traces of elements of groups contained in these conjugacy classes we see that A_8 is an $\mathscr{S}_1$-subgroup of A_{15}. Furthermore, $\mathrm{S}_8 \leqslant \mathrm{S}_{15}$ and A_8 and A_{15} have the same class stabiliser in $\mathrm{Out}(\mathrm{O}^{\pm}_{14}(p))$.

(v) The only group that could contain A_{15} is A_{16} but A_{16} only has a 14-dimensional absolutely irreducible representation in characteristic 2. Hence $d \times A_{15}$ is $\mathscr{S}_1$-maximal in odd characteristic. If the characteristic is 2 then we can show using Magma (file s1dim14cont) that A_{15} is an $\mathscr{S}_1$-subgroup of A_{16}. Furthermore, $S_{15} \leqslant S_{16}$ and both A_{15} and A_{16} have class stabiliser $\langle \gamma \rangle$.

(vi) By Lagrange's theorem A_{16} has to be $\mathscr{S}_1$-maximal.

(vii) By [12], $L_2(13)$ is a subgroup of both A_{15} and $G_2(3)$ and there are no other possible containments by Lagrange's theorem. Looking at the character tables of these three groups ([12, 24, 14]), it is clear that $L_2(13)_1$ can only be an $\mathscr{S}_1$-subgroup of A_{15}. Using Magma we see that A_{15} has exactly one conjugacy class A of subgroups isomorphic to $L_2(13)$ (file s1dim14cont). By looking at the traces of elements of A we see that this conjugacy class does not correspond to an absolutely irreducible 14-dimensional representation of $L_2(13)$. Hence $d \times L_2(13)_1$ is not an $\mathscr{S}_1$-subgroup of $d \times A_{15}$ and hence has to be $\mathscr{S}_1$-maximal.

(viii) Looking at the character tables [12, 24, 14], we can see that $L_2(13)_2$ could only be an $\mathscr{S}_1$-subgroup of $G_2(3)$. We can easily check that $G_2(3)$ has exactly one conjugacy class of subgroups isomorphic to $L_2(13)$ (file s1dim14cont). Looking at elements of order 3 it follows that $G_2(3)$ has no conjugacy class of subgroups corresponding to $L_2(13)_2$.

(ix) Computer calculations show that neither A_{15}, A_{16} nor $G_2(3)$ have any absolutely irreducible subgroups isomorphic to $L_2(13)$ in characteristic 2 (see file s1dim14cont). Hence $L_2(13)$ is $\mathscr{S}_1$-maximal in $\Omega^+_{14}(2)$.

(x) In characteristic 3 there are no 14-dimensional $\mathscr{S}_1$-subgroups of $\Omega^+_{14}(3)$ that could contain $S_6(3)$ by Lagrange's theorem.

(xi) By [12], $U_3(3).2$ is a subgroup of $G_2(3)$. Furthermore the character values of the two 14-dimensional absolutely irreducible representations of $U_3(3)$ and $G_2(3)$ coincide and there are no reducible representations of $U_3(3)$ with character values corresponding to the 14-dimensional absolutely irreducible representation of $G_2(3)$. Finally, $\mathrm{Out}(U_3(3)) = 2$ and hence no extension of $d \times U_3(3)$ is $\mathscr{S}_1$-maximal.

(xii) The only possible containment is $G_2(3) \leqslant A_{15}, A_{16}$ but the smallest permutation representation of $G_2(3)$ acts on 351 points by [17].

(xiii) By Lagrange's theorem $2 \times J_1$ can not be contained in any of the other $\mathscr{S}_1$-subgroups and hence has to be maximal.

(xiv) Note that A_{15} and A_{16} are the only $\mathscr{S}_1$-subgroups that could contain J_2 by Lagrange's theorem. It is straightforward to check using Magma (file s1dim14cont) that J_2 is not a subgroup of A_{16} and hence not a subgroup of A_{15}. □

7 Maximal $\mathscr{S}_1$-Subgroups in Dimension 15

To determine the $\mathscr{S}_1$-maximal subgroups of classical groups in dimension 15 we will again use the theory developed in Chapter 4. This chapter has the same structure as the previous two chapters, Chapter 5 and Chapter 6.

7.1 $\mathscr{S}_1$-Subgroups in Dimension 15

The following table, Table 7.1.1, contains information about the potential 15-dimensional $\mathscr{S}_1$-maximal subgroups. It was put together using the paper by Hiß and Malle ([18]), the ATLAS ([12]), the Brauer character tables ([24]), and GAP ([14]). For a description on how to read this table please see Section 5.1.

Note that $3.\mathrm{A}_7$ and $\mathrm{U}_4(2)$ have two 15-dimensional absolutely irreducible representations that are not weakly equivalent and we denote the images of these representations by the subscripts $_1$ and $_2$.

The following lemma gives the number of weakly equivalent representations of $\mathrm{L}_2(16)$.

Lemma 7.1.1. *Let $G = \mathrm{L}_2(16)$. If $p \equiv 1, 2, 4, 8, 9, 13, 15, 16 \pmod{17}$ then G has (up to equivalence) two sets of four weakly equivalent 15-dimensional representations, whereas if $p \equiv 3, 5, 6, 7, 10, 11, 12, 14 \pmod{17}$ then there are (up to equivalence) eight weakly equivalent representations of G.*

Proof. Let ρ_i be one of the 15-dimensional representations with characters χ_2, χ_3, χ_4 or χ_5 and let ρ_j denote one of the 15-dimensional representations with characters χ_6, χ_7, χ_8 or χ_9 as given in [12, 24].

To show that ρ_i and ρ_j are weakly equivalent we need to show that there exists $\alpha \in \mathrm{Out}(G)$ or $\phi \in \mathrm{Out}(\mathrm{O}^\circ_{15}(q))$ such that $g\rho_i$ and $(g^\alpha \rho_j)^\phi$ are equivalent for all $g \in G$ by Definition 4.3.2 and Table 7.1.1. Using [12, 24], it is straightforward to show that the outer automorphism $\alpha \in \mathrm{Out}(G)$ of order 4 fuses the four representations ρ_i and it also fuses the four representations ρ_j.

Furthermore, ϕ is non-trivial if and only if the smallest e such that $G\rho_i \leqslant \Omega^\circ_{15}(p^e)$ is strictly greater than 1. By Table 7.1.1 and Table 2.2.1 we find that $e = 2$ when $p \equiv 4, 13 \pmod{17}$, $e = 4$ when $p \equiv 2, 8, 9, 15 \pmod{17}$ and $e = 8$ when $p \equiv 3, 5, 6, 7, 10, 11, 12, 14 \pmod{17}$.

We can show that if $e \leqslant 4$, then ϕ permutes the representations of the form ρ_i among each other and similarly for ρ_j. When $e = 8$ however, then for all i we can find a j such that $g\rho_i = (g\rho_j)^\phi$ for all $g \in G$. $\square$

Table 7.1.1: Potential $\mathscr{S}_1$-maximal subgroups in dimension 15

Gp	**Order**	**Ind**	**#ρ**	**Stab**	**Charc**	**ChR**	**Out**
$3.\mathrm{A}_6$	$2^3 \cdot 3^3 \cdot 5$	$\circ$	2	2_3	$0, 5\ (\neq 2, 3)$	$\mathbb{Z}[\mathrm{z}_3]$	2^2
$3.\mathrm{A}_{7_1}$	$2^3 \cdot 3^3 \cdot 5 \cdot 7$	$\circ$	2	1	$0, 5, 7\ (\neq 2, 3)$	$\mathbb{Z}[\mathrm{z}_3]$	2
$3.\mathrm{A}_{7_2}$	$2^3 \cdot 3^3 \cdot 5 \cdot 7$	$\circ$	2	1	$0, 5\ (\neq 2, 3, 7)$	$\mathbb{Z}[\mathrm{z}_3]$	2
$3.\mathrm{A}_7$	$2^3 \cdot 3^3 \cdot 5 \cdot 7$	$\circ$	2	1	2	$\mathbb{Z}[\mathrm{z}_3]$	2
$\mathrm{L}_2(31)$	$2^5 \cdot 3 \cdot 5 \cdot 31$	$\circ$	2	1	$0, 2, 3, 5\ (\neq 31)$	$\mathbb{Z}[\mathrm{b}_{31}]$	2
$3.\mathrm{L}_3(4)$	$2^6 \cdot 3^3 \cdot 5 \cdot 7$	$\circ$	6	2_1	$0, 5, 7\ (\neq 2, 3)$	$\mathbb{Z}[\mathrm{z}_3]$	$2 \times \mathrm{S}_3$
$3_1.\mathrm{U}_4(3)$	$2^7 \cdot 3^7 \cdot 5 \cdot 7$	$\circ$	2	2_2	$0, 2, 5, 7\ (\neq 3)$	$\mathbb{Z}[\mathrm{z}_3]$	2^2 [b]
M_{12}	$2^6 \cdot 3^3 \cdot 5 \cdot 11$	$\circ$	2	1	3	$\mathbb{Z}[\mathrm{b}_{11}]$	2
$3.\mathrm{M}_{22}$	$2^7 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11$	$\circ$	2	1	2	$\mathbb{Z}[\mathrm{b}_{11}, \mathrm{z}_3]$	2
A_7	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	+	1	2	$0, 3, 5\ (\neq 2, 7)$	$\mathbb{Z}$	2
A_{16}	$2^{14} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$	+	1	2	$0, 3, 5, 7, 11, 13\ (\neq 2)$	$\mathbb{Z}$	2
A_{17}	$2^{14} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17$	+	1	2	17	$\mathbb{Z}$	2
$\mathrm{L}_2(16)$	$2^4 \cdot 3 \cdot 5 \cdot 17$	+	4, 4 or 8 [a]	1	$0, 3, 5\ (\neq 2, 17)$	$\mathbb{Z}[\mathrm{y}_{17}]$	4
$\mathrm{L}_2(16)$	$2^4 \cdot 3 \cdot 5 \cdot 17$	+	1	4	17	$\mathbb{Z}$	4
$\mathrm{L}_2(29)$	$2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 29$	+	2	1	$0, 3, 5, 7\ (\neq 2, 29)$	$\mathbb{Z}[\mathrm{b}_{29}]$	2
$\mathrm{L}_3(4)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	+	3	2^2	3	$\mathbb{Z}$	$2 \times \mathrm{S}_3$
$\mathrm{S}_6(2)$	$2^9 \cdot 3^4 \cdot 5 \cdot 7$	+	1	1	$0, 5, 7\ (\neq 2, 3)$	$\mathbb{Z}$	1
$\mathrm{U}_4(2)_1$	$2^6 \cdot 3^4 \cdot 5$	+	1	2	$0, 5\ (\neq 2, 3)$	$\mathbb{Z}$	2
$\mathrm{U}_4(2)_2$	$2^6 \cdot 3^4 \cdot 5$	+	1	2	$0, 5\ (\neq 2, 3)$	$\mathbb{Z}$	2

[a] If $p \equiv 1, 2, 4, 8, 9, 13, 15, 16 \pmod{17}$ then there are (up to equivalence) two sets of four weakly equivalent representations, whereas if $p \equiv 3, 5, 6, 7, 10, 11, 12, 14 \pmod{17}$ then (up to equivalence) all eight 15-dimensional representations are weakly equivalent.
[b] Note that $\mathrm{Out}(\mathrm{U}_4(3)) = \mathrm{D}_8$ but $\mathrm{Out}(3_1.\mathrm{U}_4(3)) = 2^2$ by [12, p.xx and p.52].

Information regarding the algebraic irrationalities that appear in dimension 15 can be found in Table 2.2.1 on p.19.

Theorem 7.1.2. *Let G be an $\mathscr{S}_1$-subgroup of $\Omega \in \{\mathrm{SL}^{\pm}_{15}(q), \Omega^{\circ}_{15}(q)\}$. Then G is contained in Table 7.1.1.*

Proof. See the tables in [18]. $\square$

7.2 Schur Indicator $\circ$

In this section we will look at the groups $3.\mathrm{A}_6$, $3.\mathrm{A}_{7_{1,2}}$, $3.\mathrm{A}_7$, $\mathrm{L}_2(31)$, $3.\mathrm{L}_3(4)$, $3_1.\mathrm{U}_4(3)$, M_{12} and $3.\mathrm{M}_{22}$ as they have Schur indicator $\circ$ by Table 7.1.1.

Proposition 7.2.1 ($3.\mathrm{A}_6$)**.**

(i) If $p \equiv 1 \pmod 3$ then $\mathrm{SL}_{15}(p)$ has $(p-1, 15)$ conjugacy classes of $\mathscr{S}_1$-subgroups isomorphic to $3.\mathrm{A}_6.2_3$ which have class stabiliser $\langle\gamma\rangle$ in $\mathrm{Out}(\mathrm{L}_{15}(p))$.

(ii) If $p \equiv 2 \pmod 3$*,* $p \neq 2$*, then there exist* $(p+1, 15)$ *conjugacy classes of* $\mathscr{S}_1$*-subgroups of* $\mathrm{SU}_{15}(p)$ *isomorphic to* $3.\mathrm{A}_6.2_3$*, with class stabiliser* $\langle\gamma\rangle$ *in* $\mathrm{Out}(\mathrm{U}_{15}(p))$*.*

Proof. Let $G = 3.\mathrm{A}_6$. Then $\mathrm{Out}(G) = 2^2$ and the character ring of a 15-dimensional absolutely irreducible representation ρ of G is $\mathbb{Z}[\mathrm{z}_3]$ by Table 7.1.1. Hence, it follows from Lemma 4.2.1 and Table 2.2.1 that $G \leqslant \mathrm{SL}_{15}(p)$ if $p \equiv 1 \pmod 3$ and $G \leqslant \mathrm{SU}_{15}(p)$ if $p \equiv 2 \pmod 3$ $(p \neq 2)$.

Furthermore, ρ splits into ρ_1 and ρ_2 and $(G.2_3)\rho_i$ has Schur indicator $\circ$ and character ring $\mathbb{Z}[\mathrm{z}_3]$. Hence $(G.2_3)\rho_i$ preserves only the zero form if $p \equiv 1 \pmod 3$ and a unitary form otherwise. Note that $(G.2_3)\rho_i \cong \langle G, g\rangle$ where g is some element of order 4 in $G.2_3\backslash G$ with $g^2 \in G$. Here $\mathrm{Trace}(g\rho_2) = \mathrm{Trace}(g^2\rho) = -1$. A straightforward calculation shows that $G.2_3 \leqslant \mathrm{SL}^{\pm}_{15}(p)$.

Since $\mathrm{Out}(G) = 2^2$ and both the 2_1 and the 2_2 outer automorphisms of G fuse the two weakly-equivalent representations of G we find that there is one conjugacy class of $G.2_3$ in the respective conformal group C by Lemma 4.3.3. Then C splits into $(p \mp 1, 15)$ classes in $\mathrm{SL}^{\pm}_{15}(p)$ respectively by Lemma 4.3.3.

If $p \equiv 1 \pmod 3$ then it is straightforward to show using Lemma 4.4.1 and Lemma 4.4.2 that γ stabilises one of these conjugacy classes. If $p \equiv 2 \pmod 3$ then this follows from Lemma 4.4.1 and Lemma 4.5.1. ☐

Proposition 7.2.2 ($3.\mathrm{A}_{7_1}$)**.**
Let $3.\mathrm{A}_{7_1}$ *be the image of a* 15*-dimensional absolutely irreducible representation* ρ *of* $3.\mathrm{A}_7$ *which has character value* -1 *for all involutions.*

(i) If $p \equiv 1 \pmod 3$*, then there are* $(p-1, 15)$ *conjugacy classes of* $\mathscr{S}_1$*-subgroups of* $\mathrm{SL}_{15}(p)$ *weakly equivalent to* $3.\mathrm{A}_{7_1}$*, with class stabiliser* $\langle\gamma\rangle$ *in* $\mathrm{Out}(\mathrm{L}_{15}(p))$*.*

(ii) If $p \equiv 2 \pmod 3$*,* $p \neq 2$*, then* $\mathrm{SU}_{15}(p)$ *has* $(p+1, 15)$ *conjugacy classes of* $\mathscr{S}_1$*-subgroups weakly equivalent to* $3.\mathrm{A}_{7_1}$*, with class stabiliser* $\langle\gamma\rangle$ *in* $\mathrm{Out}(\mathrm{U}_{15}(p))$*.*

Proof. Let $G = 3.\mathrm{A}_{7_1}$. Then $\mathrm{Out}(G) = 2$ and the character ring of ρ is $\mathbb{Z}[\mathrm{z}_3]$ by Table 7.1.1. Hence it follows from Lemma 4.2.1 and Table 2.2.1 that $G \leqslant \mathrm{SL}_{15}(p)$ if $p \equiv 1 \pmod 3$ and $G \leqslant \mathrm{SU}_{15}(p)$ if $p \equiv 2 \pmod 3$ $(p \neq 2)$. By Lemma 4.3.3 there are $(15, p \mp 1)$ conjugacy classes of G in $\mathrm{SL}^{\pm}_{15}(p)$ since two weakly equivalent representations of G are fused by the outer automorphism α of order 2 of G. By Lemma 4.4.1, ${}^{\alpha}\rho$ is equivalent to ρ^{γ}. Hence, by Lemma 4.4.2 and Lemma 4.5.1, one of these conjugacy classes is stabilised by γ. ☐

Proposition 7.2.3 ($3.\mathrm{A}_{7_2}$)**.**
Let $3.\mathrm{A}_{7_2}$ *correspond to the image of a* 15*-dimensional absolutely irreducible representation of* $3.\mathrm{A}_7$ *which has character value* 3 *for all involutions.*

(i) If $p \equiv 1 \pmod 3$*, then there are* $(p-1, 15)$ *conjugacy classes of* $\mathscr{S}_1$*-subgroups of* $\mathrm{SL}_{15}(p)$ *weakly equivalent to* $3.\mathrm{A}_{7_2}$*, with class stabiliser* $\langle\gamma\rangle$ *in* $\mathrm{Out}(\mathrm{L}_{15}(p))$*.*

(ii) If $p \equiv 2 \pmod 3$*,* $p \neq 2$*, then there are* $(p+1, 15)$ *conjugacy classes of* $\mathscr{S}_1$*-subgroups of* $\mathrm{SU}_{15}(p)$ *weakly equivalent to* $3.\mathrm{A}_{7_2}$*, with class stabiliser* $\langle\gamma\rangle$ *in* $\mathrm{Out}(\mathrm{U}_{15}(p))$*.*

Proof. Similar to Proposition 7.2.3. □

Proposition 7.2.4 ($3.\mathrm{A}_7$ in characteristic 2)**.**
There are three conjugacy classes of $\mathscr{S}_1$*-subgroups of* $\mathrm{SU}_{15}(2)$ *isomorphic to* $3.\mathrm{A}_7$*, with class stabiliser* $\langle\gamma\rangle$ *in* $\mathrm{Out}(\mathrm{U}_{15}(2))$*.*

Proof. Let $G = 3.\mathrm{A}_7$ with $\mathrm{Out}(G) = 2$. Since the character ring of a 15-dimensional absolutely irreducible representation ρ of G is $\mathbb{Z}[\mathrm{z}_3]$ by Table 7.1.1, G is a subgroup of $\mathrm{SU}_{15}(2)$ by Lemma 4.2.1 and Table 2.2.1. Furthermore, the two weakly equivalent representations of G are fused by an outer automorphism α of order 2 of G. Hence there are $(3, 15) = 3$ conjugacy classes of G in $\mathrm{SU}_{15}(2)$ by Lemma 4.3.3. Using Lemma 4.4.1 it is straightforward to show that ${}^{\alpha}\rho$ is equivalent to ρ^{γ}. Hence, by Lemma 4.5.1, the class stabiliser is $\langle\gamma\rangle$. □

Proposition 7.2.5 ($\mathrm{L}_2(31)$)**.**

(i) If $p \equiv 1, 2, 4, 5, 7, 8, 9, 10, 14, 16, 18, 19, 20, 25, 28 \pmod{31}$*, then there are* $(p-1, 15)$ *conjugacy classes of* $\mathscr{S}_1$*-subgroups of* $\mathrm{SL}_{15}(p)$ *isomorphic to* $\mathrm{L}_2(31)$*, with class stabiliser* $\langle\gamma\rangle$ *in* $\mathrm{Out}(\mathrm{L}_{15}(p))$*.*

(ii) If $p \equiv 3, 6, 11, 12, 13, 15, 17, 21, 22, 23, 24, 26, 27, 29, 30 \pmod{31}$*, then there are* $(p+1, 15)$ *conjugacy classes of* $\mathscr{S}_1$*-subgroups of* $\mathrm{SU}_{15}(p)$ *isomorphic to* $\mathrm{L}_2(31)$*, with class stabiliser* $\langle\gamma\rangle$ *in* $\mathrm{Out}(\mathrm{U}_{15}(q))$*.*

Proof. Let $G = \mathrm{L}_2(31)$. Then $\mathrm{Out}(G) = 2$ and the character ring of the 15-dimensional absolutely irreducible representations of G is $\mathbb{Z}[\mathrm{b}_{31}]$ by Table 7.1.1. From Table 2.2.1 and Lemma 4.2.1 it follows therefore that $G \leqslant \mathrm{SL}_{15}(p)$ when $\mathrm{b}_{31} \in \mathbb{F}_p$ and $G \leqslant \mathrm{SU}_{15}(p)$ when $\mathrm{b}_{31} \in \mathbb{F}_{p^2} \backslash \mathbb{F}_p$. Since the two weakly equivalent representations of G are fused by the non-trivial outer automorphism of G, there are $(p \mp 1, 15)$ conjugacy classes of G in $\mathrm{SL}^{\pm}_{15}(p)$ by Lemma 4.3.3.

Now note that $\mathrm{b}_{31}^{**} = \sum_{r=1}^{15} \mathrm{z}_{31}^{-r^2}$ which is the complex conjugate of b_{31}. Hence the character values of two weakly equivalent representations are complex conjugates of each other by [12, 24] and therefore at least one of the conjugacy classes of G in $\mathrm{SL}^{\pm}_{15}(p)$ is stabilised by γ by Lemma 4.4.1, Lemma 4.4.2 and Lemma 4.5.1. □

Proposition 7.2.6 ($3.\mathrm{L}_3(4)$)**.**

(i) *If* $p \equiv 1 \pmod 3$*, then* $\mathrm{SL}_{15}(p)$ *has* $(p-1, 15)$ *conjugacy classes of* $\mathscr{S}_1$*-subgroups isomorphic to* $3.\mathrm{L}_3(4).2_1 \cong \mathrm{SL}_3(4).2_1$*, with class stabiliser* $\langle\gamma\rangle$ *in* $\mathrm{Out}(\mathrm{L}_{15}(q))$*.*

(ii) *If* $p \equiv 2 \pmod 3$*,* $p \neq 2$*, then* $\mathrm{SU}_{15}(p)$ *has* $(p+1, 15)$ *conjugacy classes of* $\mathscr{S}_1$*-subgroups isomorphic to* $3.\mathrm{L}_3(4).2_1 \cong \mathrm{SL}_3(4).2_1$*, with class stabiliser* $\langle\gamma\rangle$ *in* $\mathrm{Out}(\mathrm{U}_{15}(q))$*.*

Proof. Let $G = 3.\mathrm{L}_3(4)$. Then $\mathrm{Out}(G) = 2 \times \mathrm{S}_3$ and there are (up to equivalence) 6 weakly equivalent 15-dimensional absolutely irreducible representations of G with character ring $\mathbb{Z}[\mathrm{z}_3]$ by [12, 24]. Hence, by Lemma 4.2.1 and Table 2.2.1, $G \leqslant \mathrm{SL}_{15}(p)$ if and only if $p \equiv 1 \pmod 3$. Furthermore, by [12, 24] and Lemma 4.2.4 we find that $G.2_1 \leqslant \mathrm{SL}_{15}(p)$ when $p \equiv 1 \pmod 3$ and $G.2_1 \leqslant \mathrm{SU}_{15}(p)$ when $p \equiv 2 \pmod 3$ $(p \neq 2)$.

Let z be the central element of $2 \times \mathrm{S}_3$. Then z corresponds to the 2_1 automorphism in [12, 24]. Let b be an element of order 2 of S_3. Without loss of generality we can let b and zb correspond to the 2_2 and the 2_3 automorphism of G respectively. Then b', b'' and bz', bz'' are conjugates of b and z respectively under automorphisms of S_3. Hence over $G.2_1$, b and bz (and any images under automorphisms of them) are induced by $\gamma \in \mathrm{Out}(\mathrm{L}^{\pm}_{15}(q))$ by Lemma 4.4.1. The number of conjugacy classes follows from Lemma 4.3.3 and the stabiliser of the conjugacy classes follows now from Lemma 4.4.2 and 4.5.1. □

Proposition 7.2.7 ($3_1.\mathrm{U}_4(3)$ in characteristic not 2)**.**

(i) *If* $p \equiv 1 \pmod 3$ *then* $\mathrm{SL}_{15}(p)$ *has* $(p-1, 15)$ *conjugacy classes of* $\mathscr{S}_1$*-subgroups isomorphic to* $3_1.\mathrm{U}_4(3).2_2$*, with class stabiliser* $\langle\gamma\rangle$ *in* $\mathrm{Out}(\mathrm{L}_{15}(p))$*.*

(ii) *If* $p \equiv 2 \pmod 3$ *then* $\mathrm{SU}_{15}(p)$ *has* $(p+1, 15)$ *conjugacy classes of* $\mathscr{S}_1$*-subgroups isomorphic to* $3_1.\mathrm{U}_4(3).2_2$*, with class stabiliser* $\langle\gamma\rangle$ *in* $\mathrm{Out}(\mathrm{U}_{15}(p))$*.*

Proof. Let $G = 3_1.\mathrm{U}_4(3)$. Then $\mathrm{Out}(G) = 2^2$ and the character ring of the absolutely irreducible 15-dimensional representations of G is $\mathbb{Z}[\mathrm{z}_3]$ by Table 7.1.1. Hence $G \leqslant \mathrm{SL}_{15}(p)$ when $p \equiv 1 \pmod 3$ and $G \leqslant \mathrm{SU}_{15}(p)$ when $p \equiv 2 \pmod 3$ by Lemma 4.2.1 and Table 2.2.1. Since the two non-equivalent representations are fused by the 2_1 automorphism of G there is one conjugacy class of G in the respective conformal group which splits into $(p \mp 15)$ classes in $\mathrm{SL}^{\pm}_{15}(p)$ respectively by Lemma 4.3.3. Furthermore it follows from [12, 24], Lemma 4.2.4 and Lemma 4.2.1 that $G.2_2 \leqslant \mathrm{SL}_{15}(p)$ when $p \equiv 1 \pmod 3$ and $G.2_2 \leqslant \mathrm{SU}_{15}(p)$ when $p \equiv 2 \pmod 3$. Finally, by Lemma 4.4.1, Lemma 4.4.2 and Lemma 4.5.1, at least one of these conjugacy classes is stabilised by γ. □

Proposition 7.2.8 (M_{12} in characteristic 3)**.**
There is one conjugacy class of $\mathscr{S}_1$-subgroups of $\mathrm{SL}_{15}(3)$ *isomorphic to* M_{12}*, with class stabiliser $\langle\gamma\rangle$ in* $\mathrm{Out}(\mathrm{L}_{15}(3))$.

Proof. Let $G = \mathrm{M}_{12}$, so $\mathrm{Out}(G) = 2$. Then the character ring of the 15-dimensional absolutely irreducible representations of G is $\mathbb{Z}[\mathrm{b}_{11}]$ by Table 7.1.1. From Lemma 4.2.1 and Table 2.2.1 it follows therefore that $G \leqslant \mathrm{SL}_{15}(3)$. Since the non-trivial outer automorphism of G fuses the two 15-dimensional absolutely irreducible weakly equivalent representations of G it follows that there is $(3-1, 15) = 1$ conjugacy class of G in $\mathrm{SL}_{15}(3)$ by Lemma 4.3.3. Furthermore, the class stabiliser is $\langle\gamma\rangle$ by Lemma 4.4.1 and Lemma 4.4.2. □

Proposition 7.2.9 ($3.\mathrm{M}_{22}$)**.**
There are three conjugacy classes of $\mathscr{S}_1$-subgroups of $\mathrm{SU}_{15}(2)$ *isomorphic to* $3.\mathrm{M}_{22}$*, with class stabiliser $\langle\gamma\rangle$ in* $\mathrm{Out}(\mathrm{U}_{15}(2))$.

Proof. Let $G = 3.\mathrm{M}_{22}$. Then $\mathrm{Out}(G) = 2$ and the character ring of the absolutely irreducible 15-dimensional representations of G is $\mathbb{Z}[\mathrm{b}_{11}, \mathrm{z}_3]$ by Table 7.1.1. Therefore, $G \leqslant \mathrm{SU}_{15}(2)$ by Table 2.2.1 and Lemma 4.2.1. Furthermore, the non-trivial outer automorphism of G fuses the two weakly equivalent representations and hence there are $(2+1, 15) = 3$ conjugacy classes by Lemma 4.3.3. Finally, one can show that the character values of the two non-equivalent 15-dimensional absolutely irreducible representations of G are complex conjugates of each other and hence, by Lemma 4.4.1 and Lemma 4.5.1, the class stabiliser is $\langle\gamma\rangle$. □

Maximality

Finally we will show which of these groups are $\mathscr{S}_1$-maximal in $\mathrm{SL}^{\pm}_{15}(q)$.

Proposition 7.2.10.
Let $d := (p-1, 15)$ in Case **L** *and let $d := (p+1, 15)$ in Case* **U**.

(i) *No extension of $d \circ 3.\mathrm{A}_6$ is $\mathscr{S}_1$-maximal in any extension of $\mathrm{SL}^{\pm}_{15}(p)$.*

(ii) *If $p \neq 2, 3$ then $d \circ 3.\mathrm{A}_{7_1}$ is not $\mathscr{S}_1$-maximal in $\mathrm{SL}^{\pm}_{15}(p)$ but $d \circ 3.\mathrm{A}_{7_1}.\langle\gamma\rangle$ is $\mathscr{S}_1$-maximal in $\mathrm{SL}^{\pm}_{15}(p).\langle\gamma\rangle$.*

(iii) *If $p \neq 2, 3, 7$ then $d \circ 3.\mathrm{A}_{7_2}$ is $\mathscr{S}_1$-maximal in $\mathrm{SL}^{\pm}_{15}(p)$.*

(iv) *The group $3.\mathrm{A}_7$ is not $\mathscr{S}_1$-maximal in $\mathrm{SU}_{15}(2)$ but $3.\mathrm{A}_7.\langle\gamma\rangle$ is $\mathscr{S}_1$-maximal in $\mathrm{SU}_{15}(2).\langle\gamma\rangle$.*

(v) *If $p \neq 31$ then $d \times \mathrm{L}_2(31)$ is $\mathscr{S}_1$-maximal in $\mathrm{SL}^{\pm}_{15}(p)$.*

(vi) *If $p \neq 2, 3$ then $d \circ 3.\mathrm{L}_3(4).2_1$ is $\mathscr{S}_1$-maximal in $\mathrm{SL}^{\pm}_{15}(p)$.*

(vii) *If $p \neq 3$ then $d \circ 3_1.\mathrm{U}_4(3).2_2$ is $\mathscr{S}_1$-maximal in $\mathrm{SL}^{\pm}_{15}(p)$.*

(viii) *The group M_{12} is $\mathscr{S}_1$-maximal in $\mathrm{SL}_{15}(3)$.*

(ix) *The group $3.\mathrm{M}_{22}$ is $\mathscr{S}_1$-maximal in $\mathrm{SU}_{15}(2)$.*

Proof. (i) By Lagrange's theorem $3.\mathrm{A}_6$ can be a subgroup of $3.\mathrm{A}_7$, $3.\mathrm{L}_3(4)$ or of $3_1.\mathrm{U}_4(3)$. By [12], $3.\mathrm{A}_6.2_3$ is a subgroup of $3.\mathrm{L}_3(4).2_1$. Let ρ be an absolutely irreducible 15-dimensional representation of $3.\mathrm{L}_3(4)$. Looking at the character values of the subgroups of $3.\mathrm{L}_3(4)\rho$ isomorphic to $3.\mathrm{A}_6$, we see that $3.\mathrm{A}_6$ is indeed an absolutely irreducible subgroup of $3.\mathrm{L}_3(4)\rho$ (see file s1dim15cont). Furthermore, $\mathrm{A}_6.2^2 \leqslant \mathrm{L}_3(4).2^2$ by [12] and hence no extension of $3.\mathrm{A}_6$ is $\mathscr{S}_1$-maximal.

(ii) By Lagrange's theorem $3.\mathrm{A}_7$ can be a subgroup of $3.\mathrm{L}_3(4)$ and of $3_1.\mathrm{U}_4(3)$ in odd characteristic. By Magma (file s1dim15cont) $3.\mathrm{A}_7 \leqslant 3_1.\mathrm{U}_4(3)$. Let ρ be an absolutely irreducible 15-dimensional representation of $3_1.\mathrm{U}_4(3)$. We can show using [12, 24], that $3.\mathrm{A}_7\rho = 3.\mathrm{A}_{7_1}$. By [12] $\mathrm{S}_7 \leqslant \mathrm{U}_4(3).2_2$ but $3.\mathrm{A}_{7_1}.2 \nleqslant \mathrm{SL}^{\pm}_{15}(p)$ whereas $3_1.\mathrm{U}_4(3).2_2 \leqslant \mathrm{SL}^{\pm}_{15}(p)$. Hence $d \circ 3.\mathrm{A}_{7_1}.\langle\gamma\rangle$ is $\mathscr{S}_1$-maximal in $\mathrm{SL}^{\pm}_{15}(p).\langle\gamma\rangle$.

(iii) Using [12, 24], $3.\mathrm{A}_{7_2}$ is not an $\mathscr{S}_1$-subgroup of $3.\mathrm{L}_3(4)$ and by (ii), $3.\mathrm{A}_{7_2}$ is not an $\mathscr{S}_1$-subgroup of $3_1.\mathrm{U}_4(3)$ either. There are no other possible containments by Lagrange's theorem.

(iv) In characteristic 2 we can use Magma (see file s1dim15cont) to show that $3.\mathrm{A}_7$ is an absolutely irreducible subgroup of the $\mathscr{S}_1$-subgroup $3.\mathrm{M}_{22}$ in dimension 15. However $\mathrm{S}_7 \not\leqslant \mathrm{M}_{22}.2$ by [12] and hence $d \circ 3.\mathrm{A}_7$ extends to a novelty under $\langle \gamma \rangle$. If $3.\mathrm{A}_7 \leqslant 3_1.U_4(3)$ then $3.\mathrm{A}_7$ would also extend to a novelty by (ii).

(v) By Lagrange's theorem, $\mathrm{L}_2(31)$ cannot be contained in any of the other 15-dimensional $\mathscr{S}_1$-subgroups of $\mathrm{SL}^{\pm}_{15}(p)$.

(vi) By [12], $\mathrm{L}_3(4)$ is a subgroup of $\mathrm{U}_4(3)$ but $\mathrm{L}_3(4).2_1$ is not a subgroup of $\mathrm{U}_4(3).2_2$. Hence $3.\mathrm{L}_3(4).2_1$ cannot be a subgroup of $3_1.\mathrm{U}_4(3).2_2$. Furthermore, note that $3.\mathrm{L}_3(4)$ cannot be an $\mathscr{S}_1$-subgroup of $3.\mathrm{M}_{22}$ as the $\mathscr{S}_1$-subgroup $3.\mathrm{M}_{22}$ only exists in characteristic 2. There are no other containments possible by Lagrange's theorem.

(vii) By Lagrange's theorem, $\mathrm{N}_{\mathrm{SL}^{\pm}_{15}(p)}(3_1.\mathrm{U}_4(3))$ has to be $\mathscr{S}_1$-maximal.

(viii) By Lagrange's theorem, M_{12} could be a subgroup of $3.\mathrm{M}_{22}$. However in dimension 15, M_{12} is an $\mathscr{S}_1$-subgroup in characteristic 3, whereas the $\mathscr{S}_1$-subgroup $3.\mathrm{M}_{22}$ only exists in characteristic 2.

(ix) The $\mathscr{S}_1$-subgroup $3.\mathrm{M}_{22}$ cannot be a subgroup of any of the other 15-dimensional $\mathscr{S}_1$-subgroups of $\mathrm{SL}^{\pm}_{15}(q)$ by Lagrange's theorem. □

7.3 Schur Indicator +

Here we will determine the behaviour of the groups A_7, A_{16}, A_{17}, $\mathrm{L}_2(16)$, $\mathrm{L}_2(29)$, $\mathrm{L}_3(4)$, $\mathrm{S}_6(2)$ and $\mathrm{U}_4(2)_{1,2}$.

Proposition 7.3.1 (A_7).
If $p \neq 2, 7$ then $\Omega^{\circ}_{15}(p)$ has 2 conjugacy classes of $\mathscr{S}_1$-subgroups isomorphic to $\mathrm{A}_7.2$ with trivial stabiliser in $\mathrm{Out}(\mathrm{O}^{\circ}_{15}(p))$.

Proof. Let $G = \mathrm{A}_7$ with $\mathrm{Out}(G) = 2$. By Table 7.1.1, $G \leqslant \Omega^{\circ}_{15}(p)$ when $p \neq 2, 7$. Furthermore, computer calculations (file a7d15comp) show that in fact $\mathrm{A}_7.2 \leqslant \Omega^{\circ}_{15}(p)$ for all $p \neq 2, 7$. There is up to equivalence one absolutely irreducible 15-dimensional representation of G and hence there are 2 conjugacy classes of G in $\Omega^{\circ}_{15}(p)$ by Lemma 4.3.3. Since $G.2$ does not afford any non-trivial outer automorphism, the stabiliser of the conjugacy classes has to be trivial. □

Proposition 7.3.2 (A_{16})**.**

(i) *If* $p \equiv 1, 7 \pmod 8$*, then there are* 2 *conjugacy classes of* $\mathscr{S}_1$*-subgroups of* $\Omega^{\circ}_{15}(p)$ *isomorphic to* $\mathrm{A}_{16}.2$*, with trivial stabiliser in* $\mathrm{Out}(\mathrm{O}^{\circ}_{15}(p))$*.*

(ii) *If* $p \equiv 3, 5 \pmod 8$ *then there exists a single conjugacy class of* $\mathscr{S}_1$*-subgroups of* $\Omega^{\circ}_{15}(p)$ *isomorphic to* A_{16}*, with class stabiliser* $\langle\delta\rangle$ *in* $\mathrm{Out}(\mathrm{O}^{\circ}_{15}(p))$*.*

Proof. Let $G = \mathrm{A}_{16}$. Then $\mathrm{Out}(G) = 2$ and by Table 7.1.1 it is clear that $G \leqslant \Omega^{\circ}_{15}(p)$. Computer calculations (file a16d15comp) and Table 2.2.1 show that $G.2 \leqslant \Omega^{\circ}_{15}(p)$ if and only if $p \equiv 1, 7 \pmod 8$. Otherwise the element inducing the non-trivial outer automorphism of G sits inside $\mathrm{SO}^{\circ}_{15}(p)\backslash\Omega^{\circ}_{15}(p)$ and hence is induced by δ. The number of conjugacy classes follows from Table 7.1.1 and Lemma 4.3.3. □

Proposition 7.3.3 (A_{17} in characteristic 17)**.**
There exist two conjugacy classes of $\mathscr{S}_1$*-subgroups of* $\Omega^{\circ}_{15}(17)$ *isomorphic to* $\mathrm{A}_{17}.2$*, with trivial stabiliser in* $\mathrm{Out}(\mathrm{O}^{\circ}_{15}(17))$*.*

Proof. Let $G = \mathrm{A}_{17}$. Then $\mathrm{Out}(G) = 2$ and computer calculations (file a17d15comp) show that $\mathrm{S}_{17} \leqslant \Omega^{\circ}_{15}(17)$. Hence there are 2 conjugacy classes of G in $\Omega^{\circ}_{15}(17)$ by Table 7.1.1 and Lemma 4.3.3. □

Proposition 7.3.4 ($\mathrm{L}_2(16)$ in characteristic not 17)**.**

(i) *If* $p \equiv 1, 16 \pmod{17}$*, then there exist* 4 *conjugacy classes of* $\mathscr{S}_1$*-subgroups of* $\Omega^{\circ}_{15}(p)$ *isomorphic to* $\mathrm{L}_2(16)$*, which have trivial stabiliser in* $\mathrm{Out}(\mathrm{O}^{\circ}_{15}(p))$*. Furthermore, there exist two* $\mathrm{Aut}(\mathrm{O}^{\circ}_{15}(p))$*-classes of* $\mathrm{L}_2(16)$*.*

(ii) *If* $p \equiv 4, 13 \pmod{17}$*, then there exist* 4 *conjugacy classes of* $\mathscr{S}_1$*-subgroups of* $\Omega^{\circ}_{15}(p^2)$ *isomorphic to* $\mathrm{L}_2(16)$*, which have class stabiliser* $\langle\phi\rangle$ *in* $\mathrm{Out}(\mathrm{O}^{\circ}_{15}(p^2))$*. Furthermore, there are two* $\mathrm{Aut}(\mathrm{O}^{\circ}_{15}(p^2))$*-classes of* $\mathrm{L}_2(16)$*.*

(iii) *If* $p \equiv 2, 8, 9, 15 \pmod{17}$*,* $p \neq 2$*, then* $\Omega^{\circ}_{15}(p^4)$ *has* 4 *conjugacy classes of* $\mathscr{S}_1$*-subgroups isomorphic to* $\mathrm{L}_2(16)$ *which have class stabiliser* $\langle\phi\rangle$ *in* $\mathrm{Out}(\mathrm{O}^{\circ}_{15}(p^4))$*. Furthermore, there exist two* $\mathrm{Aut}(\mathrm{O}^{\circ}_{15}(p^4))$*-classes of* $\mathrm{L}_2(16)$*.*

(iv) *If* $p \equiv 3, 5, 6, 7, 10, 11, 12, 14 \pmod{17}$*, then* $\Omega^{\circ}_{15}(p^8)$ *has* 4 *conjugacy classes of* $\mathscr{S}_1$*-subgroups isomorphic to* $\mathrm{L}_2(16)$*, with class stabiliser* $\langle\phi^2\rangle$ *in* $\mathrm{Out}(\mathrm{O}^{\circ}_{15}(p^8))$*.*

Proof. Let $G = \mathrm{L}_2(16)$, so $\mathrm{Out}(G) = 4$. In dimension 15 there are (up to equivalence) either two sets of four weakly equivalent absolutely irreducible representations or 8 weakly equivalent absolutely irreducible representations ρ of G with character ring $\mathbb{Z}[\mathrm{y}_{17}]$ by Table 7.1.1 and Lemma 7.1.1. Hence $G \leqslant \Omega^{\circ}_{15}(p^e)$, where e follows from Table 2.2.1. Furthermore, there are either 1 or 2 conjugacy classes of G in $\mathrm{CO}^{\circ}_{15}(p^e)$ by Lemma 4.3.3 depending on e. If $e = 8$, then there exist 2 classes and otherwise there exists 1 conjugacy class by Lemma 4.3.3. By Lemma 4.3.3 each of these conjugacy classes splits into 2 classes in $\Omega^{\circ}_{15}(p^e)$. Note that there are two $\mathrm{Aut}(\mathrm{O}^{\circ}_{15}(p^e))$-conjugacy classes of G when $e < 8$.

If $p \equiv 1, 16 \pmod{17}$ then $\Omega^{\circ}_{15}(p)$ has no non-trivial outer automorphism that could induce any non-trivial outer automorphism of G and hence the stabiliser of the conjugacy classes is trivial.

If $p \equiv 4, 13 \pmod{17}$ then it follows by Lemma 4.7.2 that the $\Omega^{\circ}_{15}(p^2)$-conjugacy classes of G are stabilised by ϕ in $\mathrm{Out}(\mathrm{O}^{\circ}_{15}(p^2))$ since $G.2\backslash G$ contains involutions.

If $p \equiv 2, 8, 9, 15 \pmod{17}$ then ${}^{\alpha}\rho$ is equivalent to ρ^{ϕ} using Lemma 4.3.8, where α is the outer automorphism of order 4 of G. Furthermore, we can find a matrix $x \in \mathrm{GL}_{15}(p^4)$ with $x^{-1}(g\rho)^{\phi}x = g^{\alpha}\rho$ for all $g \in G$ and $xFx^{\mathrm{T}} = F$, where F is our standard form matrix preserved by $G\rho$ (see file l216d15comp). Hence, by Lemma 4.7.3, if $\mathrm{sp}(x) = 1$ then all conjugacy classes of $G\rho$ are stabilised by ϕ. Using Magma it is straightforward to show that the spinor norm of x is indeed 1 in all relevant characteristics.

If $p \equiv 3, 5, 6, 7, 10, 11, 12, 14 \pmod{17}$ then ${}^{\alpha}\rho$ is equivalent to ρ^{ϕ^2} using Lemma 4.3.8 again. Furthermore, when $p \neq 3, 5$, then there exists a matrix $x \in \mathrm{GL}_{15}(p^8)$ of spinor norm 1 such that $x^{-1}(g\rho)^{\phi^2}x = g^{\alpha}\rho$ for all $g \in G$ and $xFx^{\mathrm{T}} = F$. Hence, by Lemma 4.7.3, a conjugacy class is stabilised by $\langle\phi^2\rangle$ in $\mathrm{Out}(\mathrm{O}^{\circ}_{15}(p^2))$. If $p = 3, 5$, then our computer calculations show that there exists a matrix $x \in \mathrm{SO}^{\circ}_{15}(p^8)$ such that $x^{-1}(g\rho)^{\phi^{-2}}x = g^{\alpha}\rho$ for all $g \in G$. Using a similar approach as in Lemma 4.7.3, we can show that $G\rho$ is stabilised by ϕ^2 since x has spinor norm 1. $\square$

Proposition 7.3.5 ($\mathrm{L}_2(16)$ in characteristic 17)**.**
There exist two conjugacy classes of $\mathscr{S}_1$-subgroups of $\Omega^{\circ}_{15}(17)$ *isomorphic to* $\mathrm{L}_2(16).4$*, with trivial stabiliser in* $\mathrm{Out}(\mathrm{O}^{\circ}_{15}(17))$*.*

Proof. Let $G = \mathrm{L}_2(16)$. Then $\mathrm{Out}(G) = 4$ and from Table 7.1.1 it follows that $G \leqslant \Omega^{\circ}_{15}(17)$. Computer calculations (l216ch17d15comp) show that $\mathrm{L}_2(16).4 \leqslant \Omega^{\circ}_{15}(17)$. The number of conjugacy classes now follows from Table 7.1.1 and Lemma 4.3.3. $\square$

Proposition 7.3.6 ($\mathrm{L}_2(29)$)**.**

(i) If $p \equiv 1, 4, 5, 6, 7, 9, 13, 16, 20, 22, 23, 24, 25, 28 \pmod{29}$ *then* $\Omega^\circ_{15}(p)$ *has 2 conjugacy classes of* $\mathscr{S}_1$*-subgroups isomorphic to* $\mathrm{L}_2(29)$*, with trivial stabiliser in* $\mathrm{Out}(\mathrm{O}^\circ_{15}(p))$.

(ii) If $p \equiv 2, 3, 8, 10, 11, 12, 14, 15, 17, 18, 19, 21, 26, 27 \pmod{29}$*,* $p \neq 2$*, then* $\Omega^\circ_{15}(p^2)$ *has 2 conjugacy classes of* $\mathscr{S}_1$*-subgroups isomorphic to* $\mathrm{L}_2(29)$*, with class stabiliser* $\langle \phi \rangle$ *in* $\mathrm{Out}(\mathrm{O}^\circ_{15}(p^2))$.

Proof. Let $G = \mathrm{L}_2(29)$. Then $\mathrm{Out}(G) = 2$ and the character ring of an absolutely irreducible 15-dimensional representation ρ of G is $\mathbb{Z}[\mathrm{b}_{29}]$ by Table 7.1.1. Hence it follows from Table 2.2.1 that $G \leqslant \Omega^\circ_{15}(p)$ if and only if $\mathrm{b}_{29} \in \mathbb{F}_p$. Otherwise $G \leqslant \Omega^\circ_{15}(p^2)$. Furthermore, the outer automorphism α of order 2 of G fuses two weakly equivalent 15-dimensional absolutely irreducible representations of G. Hence the number of conjugacy classes follows from Lemma 4.3.3.

If $\mathrm{b}_{29} \in \mathbb{F}_p$, then the stabiliser of these conjugacy classes is trivial. Now suppose that $\mathrm{b}_{29} \in \mathbb{F}_{p^2} \backslash \mathbb{F}_p$. To show that ${}^{\alpha}\rho$ is equivalent to ρ^{ϕ} it is sufficient to show that $\mathrm{b}_{29}^p = \mathrm{b}_{29}^*$ by Lemma 4.3.8 and [12, 24]. This is indeed the case. Since $G.2 \backslash G$ contains involutions it follows from Lemma 4.7.2 that the class stabiliser is ϕ. □

Proposition 7.3.7 ($\mathrm{L}_3(4)$ in characteristic 3)**.**
There is a single conjugacy class of $\mathscr{S}_1$*-subgroups of* $\Omega^\circ_{15}(3)$ *isomorphic to* $\mathrm{L}_3(4).2_2$*, with class stabiliser* $\langle \delta \rangle$ *in* $\mathrm{Out}(\mathrm{O}^\circ_{15}(3))$.

Proof. Let $G = \mathrm{L}_3(4)$. Then $\mathrm{Out}(G) = 2 \times \mathrm{S}_3$. Since the 15-dimensional absolutely irreducible representations of G have Schur indicator $+$ and character ring $\mathbb{Z}$, it follows that $G \leqslant \Omega^\circ_{15}(3)$. Furthermore, $G.2^2$ has an absolutely irreducible 15-dimensional representation with character ring $\mathbb{Z}$ and Schur indicator $+$. Computer calculations (file l34d15comp) show that only $G.2_2 \leqslant \Omega^\circ_{15}(3)$. The other two outer automorphisms 2_1 and 2_2 are induced by elements in $\mathrm{SO}^\circ_{15}(3) \backslash \Omega^\circ_{15}(3)$. Note that the 2_1 outer automorphism is induced by the central element z of $2 \times \mathrm{S}_3$. Let b be an element of S_3 of order 2. Without loss of generality we can assume that b induces the 2_2 outer automorphism and bz the 2_3 outer automorphism of G. Hence over $G.2_2$, the 2_1 and the 2_3 automorphism are induced by $\delta \in \mathrm{Out}(\mathrm{O}^\circ_{15}(3))$. Finally, there is one conjugacy class in the conformal group as the three weakly equivalent representation are fused by outer automorphisms of G. Hence there is one conjugacy class of G in $\Omega^\circ_{15}(3)$ by Lemma 4.3.3. □

Proposition 7.3.8 ($\mathrm{S}_6(2)$)**.**
If $p \neq 2, 3$ then $\Omega^\circ_{15}(p)$ has 2 conjugacy classes of $\mathscr{S}_1$-subgroups isomorphic to $\mathrm{S}_6(2)$, with trivial stabiliser in $\mathrm{Out}(\mathrm{O}^\circ_{15}(p))$.

Proof. Since $\mathrm{Out}(\mathrm{S}_6(2)) = 1$ the result follows immediately from Table 7.1.1 and Lemma 4.3.3. □

Proposition 7.3.9 ($\mathrm{U}_4(2)_1$)**.**
Let $\mathrm{U}_4(2)_1$ be the image of a 15 dimensional absolutely irreducible representation of $\mathrm{U}_4(2)$ with character value -1 for all involutions.

If $p \neq 2, 3$ then there are two conjugacy classes of $\mathscr{S}_1$-subgroups of $\Omega^\circ_{15}(p)$ weakly equivalent to $\mathrm{U}_4(2)_1.2$, with trivial stabiliser in $\mathrm{Out}(\mathrm{O}^\circ_{15}(p))$.

Proof. Since $\mathrm{Out}(\mathrm{U}_4(2)) = 2$ this follows from Table 7.1.1, computer calculations (file u421d15comp) and Lemma 4.3.3. □

Proposition 7.3.10 ($\mathrm{U}_4(2)_2$)**.**
Let $\mathrm{U}_4(2)_2$ be to the image of a 15-dimensional absolutely irreducible representation of $\mathrm{U}_4(2)$ whose involutions have character values 7 or 3.

If $p \neq 2, 3$ then there are two conjugacy classes of $\mathscr{S}_1$-subgroups of $\Omega^\circ_{15}(p)$ weakly equivalent to $\mathrm{U}_4(2)_2.2$, with trivial stabiliser in $\mathrm{Out}(\mathrm{O}^\circ_{15}(p))$.

Proof. Let $G = \mathrm{U}_4(2)_2$. Then $\mathrm{Out}(G) = 2$ and the result follows from computer calculations (file u422d15comp), Table 7.1.1 and Lemma 4.3.3. □

Maximality

Here we will determine which of these groups are indeed $\mathscr{S}_1$-maximal in any extension by outer automorphisms of $\Omega^\circ_{15}(q)$.

Proposition 7.3.11.

(i) If $p \neq 2, 7$ then $\mathrm{A}_7.2$ is $\mathscr{S}_1$-maximal in $\Omega^\circ_{15}(p)$.

(ii) If $p \neq 2, 17$ then $\mathrm{N}_{\Omega^\circ_{15}(p)}(\mathrm{A}_{16})$ is $\mathscr{S}_1$-maximal in $\Omega^\circ_{15}(p)$. No extension of A_{16} is $\mathscr{S}_1$-maximal in any extension of $\Omega^\circ_{15}(17)$.

(iii) The group $\mathrm{A}_{17}.2$ is $\mathscr{S}_1$-maximal in $\Omega^\circ_{15}(17)$.

(iv) If there exists an $\mathscr{S}_1$-subgroup $G = \mathrm{L}_2(16)$ of $\Omega^\circ_{15}(q)$, $q \neq 17$, then G is $\mathscr{S}_1$-maximal.

(v) No extension of $\mathrm{L}_2(16)$ is $\mathscr{S}_1$-maximal in any extension of $\Omega^\circ_{15}(17)$.

(vi) If there exists an $\mathscr{S}_1$-subgroup $G = \mathrm{L}_2(29)$ of $\Omega^\circ_{15}(q)$ then G is $\mathscr{S}_1$-maximal.

(vii) The group $\mathrm{L}_3(4).2_2$ is $\mathscr{S}_1$-maximal in $\Omega^\circ_{15}(3)$.

(viii) If $p \neq 2, 3$ then $\mathrm{S}_6(2)$ is $\mathscr{S}_1$-maximal in $\Omega^\circ_{15}(p)$.

(ix) If $p \neq 2, 3$ then $\mathrm{U}_4(2)_1.2$ is $\mathscr{S}_1$-maximal in $\Omega^\circ_{15}(p)$.

(x) No extension of $\mathrm{U}_4(2)_2.2$ is $\mathscr{S}_1$-maximal in any extension of $\Omega^\circ_{15}(p)$.

Proof. (i) By Lagrange's theorem, A_7 could be a subgroup of A_{16}, A_{17}, $\mathrm{L}_3(4)$ or $\mathrm{S}_6(2)$. Using Magma (file s1dim15cont) we can show that A_7 is not a subgroup of $\mathrm{L}_3(4)$. Since S_8 is a subgroup of $\mathrm{S}_6(2)$ ([12]), $\mathrm{A}_7.2$ is a subgroup of $\mathrm{S}_6(2)$. Using Magma again we can show that the 15-dimensional absolutely irreducible representations of $\mathrm{S}_6(2)$ induce reducible representations of A_7. We can also show that the 15-dimensional absolutely irreducible representations of A_{16} and A_{17} induce reducible representations of A_7.

(ii) It is clear that $\mathrm{A}_{16} \leqslant \mathrm{A}_{17}$ and that this is the only possible containment. However, A_{17} is only an $\mathscr{S}_1$-subgroup of $\Omega^\circ_{15}(q)$ in characteristic 17 and hence A_{16} is $\mathscr{S}_1$-maximal in the other characteristics. Comparing the Brauer character tables ([14]) of A_{17} and A_{16} and using Lemma 4.9.2 it follows that A_{16} is indeed an $\mathscr{S}_1$-subgroup of A_{17} in characteristic 17. Furthermore, $\mathrm{S}_{16} \leqslant \mathrm{S}_{17}$ and both have trivial stabiliser in $\mathrm{Out}(\mathrm{O}^\circ_{15}(17))$.

(iii) There are no containments possible by Lagrange's theorem.

(iv) The only other $\mathscr{S}_1$-subgroup that could contain $\mathrm{L}_2(16)$ is A_{17}. But A_{17} is only an $\mathscr{S}_1$-subgroup of $\Omega^\circ_{15}(q)$ in characteristic 17.

(v) Using Magma (file s1dim15cont), we find that A_{17} has (up to conjugacy) two subgroups isomorphic to $\mathrm{L}_2(16).4$ in dimension 15 and both of these subgroups are absolutely irreducible. Furthermore, $\mathrm{L}_2(16).4$ has trivial stabiliser.

(vi) By Lagrange's theorem, $\mathrm{L}_2(29)$ cannot be an $\mathscr{S}_1$-subgroup of any of the other $\mathscr{S}_1$-subgroups of $\Omega^\circ_{15}(q)$.

(vii) The possible groups that could contain $\mathrm{L}_3(4)$ by Lagrange's theorem are A_{16}, A_{17} and $\mathrm{S}_6(2)$. However, the smallest permutation representation of $\mathrm{L}_3(4)$ has degree 21 ([17]) and $\mathrm{S}_6(2)$ is not an $\mathscr{S}_1$-subgroup of $\Omega^\circ_{15}(3)$.

(viii) Since the smallest permutation representation of $\mathrm{S}_6(2)$ has degree 28 by [17], $\mathrm{S}_6(2)$ cannot be a subgroup of A_{16} or A_{17} which are the only possible containments by Lagrange's theorem.

(ix) The group $U_4(2)_1.2$ can be a subgroup of A_{16}, A_{17} and $S_6(2)$ by Lagrange's theorem. However, the smallest permutation representation of $U_4(2)$ acts on 27 points by [17]. Hence $U_4(2)$ is not a subgroup of either A_{16} or A_{17}. Even though $U_4(2).2$ is a subgroup of $S_6(2)$ the character values of the respective 15-dimensional absolutely irreducible representations do not match by [12, 24]. Hence $U_4(2)_1.2$ is $\mathscr{S}_1$-maximal.

(x) By [12], $U_4(2).2$ is a subgroup of $S_6(2)$. Furthermore the respective character values match and there is no combination of irreducible representations of $U_4(2)$ of dimension smaller than 15 that could have been induced by the absolutely irreducible 15-dimensional representations of $S_6(2)$. Hence $U_4(2)_2.2$ is a subgroup of $S_6(2)$ in dimension 15 by Lemma 4.9.2. □

8 $\mathscr{S}$-Maximal Subgroups - The Defining Characteristic Case

In this chapter we will develop the theory necessary to calculate the maximal $\mathscr{S}_2$-subgroups. Recall from Definition 3.5.3 that $\mathscr{S}_2$-subgroups are groups of Lie type in characteristic p. Since their representations arise from those of their associated algebraic groups, we will give a very short introduction to the theory behind algebraic groups. For a more thorough introduction see e.g. [15]. The book by Malle and Testerman [30] also gives an exposition of algebraic groups and gives more details regarding their representations. Another standard textbook covering this topic is by Carter [11].

We will briefly discuss representations of $\mathscr{S}_2$-subgroups and their associated weights in Section 8.1. Most of these representations arise as symmetric or exterior powers (Section 8.2) or as adjoint modules (Section 8.4). In Section 8.3 we then consider the representations of $\mathrm{SL}_2(q)$ and in Section 8.5 we will consider the behaviour of the outer automorphisms of $\Omega \in \{\mathrm{SL}_n^{\pm}(q), \mathrm{Sp}_n(q), \Omega_n^{\epsilon}(q)\}$ acting on the $\mathscr{S}_2$-subgroups.

8.1 Algebraic Groups and Highest Weight Theory

In this section we will briefly discuss the theory behind the highest weight of an algebraic group. This is a vector associated with each representation of an algebraic group. Note that the theory behind this is fairly complex and therefore we will only concentrate on the groups that are needed for this thesis. The following is based on [30].

We will first define an algebraic group. To do so we will need to set up some notation first.

Definition 8.1.1. Let K be an algebraically closed field. Let

$$X(J) = \{(x_1, \dots, x_n) \in K^n \mid f(x_1, \dots, x_n) = 0 \text{ for all } f \in J\},$$

where $J \lhd K[T_1, \dots, T_n]$ is an ideal. Then $X(J)$ is called an **algebraic set**. These algebraic sets in K^n form the closed sets of a topology on K^n, called the **Zariski topology** (see [15, Section 1.1, p.1]). An **affine variety** is an algebraic set together with the induced Zariski topology.

Definition 8.1.2. Let X, Y be two affine varieties. Then a map $\mu : X \to Y$ which can be defined by polynomial functions in the coordinates is said to be a **morphism**.

Now we have the necessary setup to give a definition of an algebraic group.

Definition 8.1.3. Let X be an affine variety and let X be equipped with a group structure such that multiplication and inversion are morphisms of varieties. Then X is a **linear algebraic group**. Note that any linear algebraic group can be embedded as a subgroup of $\mathrm{GL}_n(K)$ for some n and some algebraically closed field K by [30, Thm 1.7, p.5].

Definition 8.1.4. A **morphism of algebraic groups** is a group homomorphism that is also a morphism of varieties.

From now on let K be an algebraically closed field and let A be a linear algebraic group. Note that we will often just say that A is an algebraic group.

Definition 8.1.5. A subgroup T of an algebraic group A is called a **torus** if it is isomorphic to D_m for some m, where D_m is the group generated by the diagonal matrices of $\mathrm{GL}_m(K)$. A torus is a **maximal torus** if it is maximal among the tori of A with respect to inclusion.

Definition 8.1.6. A **character** of an algebraic group A is a morphism from A into $K^\times$. The set of characters is denoted by $\boldsymbol{X(A)}$. A **cocharacter** of A is a morphism $\mu : K^\times \to A$. The set of cocharacters is denoted by $\boldsymbol{Y(A)}$.

Characters and cocharacters will play an important part in finding the highest weight of a group, partly due to the following lemma.

Lemma 8.1.7 ([30, Prop 3.6, p.23])**.** *Let T be a torus of an algebraic group A with character set $X(T)$ and cocharacter set $Y(T)$. Let $\langle , \rangle$ be a map $X(T) \times Y(T) \to \mathbb{Z}$ such that $\chi(\mu(t)) = t^{\langle \chi,\mu \rangle}$ for all $\chi \in X(T)$, $\mu \in Y(T)$ and $t \in K^\times$. Then any homomorphism $X(T) \to \mathbb{Z}$ is of the form $\chi \mapsto \langle \chi, \gamma \rangle$ for some $\gamma \in Y(T)$ and any homomorphism $Y(T) \to \mathbb{Z}$ is of the form $\gamma \mapsto \langle \chi, \gamma \rangle$ for some $\chi \in X(T)$.*

We will now consider a special subset of the set of characters.

Definition 8.1.8. Let T be a maximal torus of an algebraic group A and let $\chi \in X(T)$. Let $\mathfrak{g}$ be the Lie algebra of A (see [30, Section 7, p.44]) and let $\mathfrak{g}_\chi = \{v \in \mathfrak{g} \mid t^{-1}vt = \chi(t)v \text{ for all } t \in T\}$. Then the set $\Phi(A) := \{\chi \in X(T) \mid \chi \neq 0, \mathfrak{g}_\chi \neq 0\}$ is the **set of roots** of A.

A subset $\Delta \subseteq \Phi$ is a **base** of Φ if for any $\beta \in \Phi$ we can find an integral linear combination such that $\beta = \sum_{\alpha\in\Delta} c_\alpha \alpha$ with either all $c_\alpha \leqslant 0$ or all $c_\alpha \geqslant 0$, $c_\alpha \in \mathbb{Z}$.

By [30, Lemma 8.15 and Lemma 8.19, p.60] for each root α_i there exists a unique **coroot** $\check{\alpha}_i$ such that $\langle \alpha_i, \check{\alpha}_i \rangle = 2$, using the map $X(T) \times Y(T) \to \mathbb{Z}$ as in Lemma 8.1.7.

We say that $\lambda \in X(T)$ is **dominant** if $\langle \lambda, \check{\alpha} \rangle \geqslant 0$ for all $\alpha \in \Delta$.

Example 8.1.9. Let $H = \mathrm{SL}_3(K)$, where K is an algebraically closed field. Then a maximal torus T of H is given by $T := \langle \mathrm{diag}(\alpha, \beta, \alpha^{-1}\beta^{-1}) \mid \alpha, \beta \in \mathrm{K}^{\times} \rangle$. Furthermore,

$$\Phi(H) = \{\chi_{ij} \mid 1 \leqslant i, j \leqslant 3, i \neq j\},$$

where $\chi_{ij}(t) = t_i t_j^{-1}$ for all $t = \mathrm{diag}(t_1, t_2, t_3) \in T$. Then $\Delta(H) = \{\chi_{12}, \chi_{23}\}$ with $\chi_{12}(t) = \alpha\beta^{-1}$ and $\chi_{23}(t) = \alpha\beta^2$ by [30, Example 8.2(2), p.51 and Example 9.8, p.67].

From now on we will only consider the groups that are actually needed in this thesis. Let $K = \overline{\mathbb{F}_p}$ and let $P \in \{\mathrm{SL}_n^{\pm}(K), \mathrm{Sp}_n(K), \Omega_n(K), \mathrm{G}_2(K)\}$. Then P is a linear algebraic group by [30].

We now have the necessary setup to consider representations of P.

Definition 8.1.10. Let $\rho : P \to \mathrm{GL}(V)$ be a representation such that V is a finite dimensional vector space over K and ρ is a morphism of algebraic groups. Then ρ is a **rational representation**. Let $T \leqslant P$ be a maximal torus, let $\chi \in X(T)$ and let

$$V_\chi = \{v \in V \mid v(t\rho) = \chi(t)v \text{ for all } t \in T\}.$$

If $V_\chi \neq 0$ then χ is a **weight** of V.

We also need to define a very important subgroup of an algebraic group called a Borel subgroup. For a more general definition see [30, Def 6.3, p.37].

Definition 8.1.11. Let $U_n := \left\{ \begin{pmatrix} * & & 0 \\ & \ddots & \\ * & & * \end{pmatrix} \in \mathrm{GL}_n(K) \right\}$ be the group of all lower triangular matrices of $\mathrm{GL}_n(K)$. Then $B = U_n \cap P$ is a **Borel subgroup** of P.

Definition 8.1.12. Let $B \leqslant P$ be as in Definition 8.1.11 and let $\rho : P \to \mathrm{GL}(V)$ be a rational representation. Then there exists $v^+ \in V \backslash \{0\}$ such that $\langle v^+ \rangle$ is invariant under $B\rho$ (see [30, Thm 4.1, p.26]). Then v^+ is a **maximal vector** of V with respect to B.

Note that $\langle v^+ \rangle$ is stabilised by every maximal torus $T\rho$ of $B\rho$ and hence $\langle v^+ \rangle \in V_\lambda$ for some $\lambda \in X(T)$.

Lemma 8.1.13 ([30, Thm 15.9, p.125])**.** *Let $\rho : P \to \mathrm{GL}(V)$ be a rational representation and let v^+ be a maximal vector of V (with respect to B). Let λ be the weight associated with v^+. Then λ is dominant.*

Now we can finally define the highest weight of P with respect to a rational representation ρ.

Definition 8.1.14. Let $\rho : P \to \mathrm{GL}(V)$ be a rational representation, where $V = \langle (P\rho)v^+ \rangle$ for some maximal vector $v^+ \in V$ (with respect to a fixed Borel subgroup B). If $\langle v^+ \rangle \leqslant V_\lambda$ then λ is the **highest weight** of ρ. We denote an irreducible KP-module with highest weight λ by $\boldsymbol{L(\lambda)}$.

Definition 8.1.15. Let $\Delta = \{\alpha_1, \dots, \alpha_l\}$ be a base of the root system Φ of P with respect to a maximal torus T. Let $\check{\Delta} = \{\check{\alpha}_1, \dots, \check{\alpha}_l\}$ be the unique set of coroots of Δ. In particular that implies that $\langle \alpha_i, \check{\alpha}_i \rangle = 2$ for all i. Furthermore, let $R = \{\lambda_1, \dots, \lambda_l\}$ be a set of characters such that $\langle \lambda_i, \check{\alpha}_j \rangle = \delta_{ij}$. Then the λ_i are **fundamental dominant weights**. Note that this implies that a weight is dominant if it is a non-negative linear combination of the fundamental dominant weights.

Remark 8.1.16. The highest weight λ of a representation can be written as $\lambda = \sum a_i \lambda_i$, where $a_i \in \mathbb{N}^0$. In the tables of [29] the highest weight of a rational representation ρ is denoted by $(a_l, a_{l-1}, \dots, a_1)$, where $(0, \dots, 0, 1)$ is the highest weight of the natural representation of P.

Example 8.1.17. Let $H = \mathrm{SL}_3(K)$. Recall from Example 8.1.9 that every element t of a maximal torus T of H is of the form $t = \mathrm{diag}(\alpha, \beta, \alpha^{-1}\beta^{-1})$, where $\alpha, \beta \in K^\times$ and $\Delta(H) = \{\chi_{12}, \chi_{23}\}$, where $\chi_{12}(t) = \alpha\beta^{-1}$ and $\chi_{23}(t) = \alpha\beta^2$ for all $t \in T$. Then $\check{\Delta} = \{\check{\chi}_{12}, \check{\chi}_{23}\}$, where $\check{\chi}_{12}(a) = \mathrm{diag}(a, a^{-1}, 1)$ and $\check{\chi}_{23}(a) = \mathrm{diag}(1, a, a^{-1})$ for all $a \in K^\times$. To find the fundamental dominant weight λ_1 recall from Definition 8.1.15 that $\langle \lambda_1, \check{\chi}_{12} \rangle = 1$ and $\langle \lambda_1, \check{\chi}_{23} \rangle = 0$. Hence $a^{\langle \lambda_1, \check{\chi}_{12} \rangle} = \lambda_1(\check{\chi}_{12}(a)) = \lambda_1(\mathrm{diag}(a, a^{-1}, 1)) = a^1$ and $a^{\langle \lambda_1, \check{\chi}_{23} \rangle} = \lambda_1(\check{\chi}_{23}(a)) = \lambda_1(\mathrm{diag}(1, a, a^{-1})) = a^0$ from which it follows that $\lambda_1(t) = \alpha$ for all $t \in T$. We can show similarly that $\lambda_2(t) = \alpha\beta$ for all $t \in T$.

The following lemmas will be useful when it comes to determining irreducibility of representations and later for maximality calculations.

Lemma 8.1.18 ([30, Thm 15.17, p.128])**.**

(i) Two irreducible rational representations ρ_1 and ρ_2 of P of highest weights μ_1 and μ_2 are equivalent if and only if $\mu_1 = \mu_2$.

(ii) If λ is a dominant weight of a rational representation ρ of P then there exists an irreducible rational representation of P with highest weight λ.

Lemma 8.1.19 ([30, Prop 14.12, p.127])**.** *Suppose that M_1 and M_2 are two irreducible modules with highest weights μ_1 and μ_2 respectively. Then the tensor product $M_1 \otimes M_2$ contains an irreducible subquotient (quotient of submodule) with highest weight $\mu_1 + \mu_2$.*

Definition 8.1.20. Let $\lambda = \sum a_i \lambda_i$ be a dominant weight of P. Then λ is ***m*-restricted** if $0 \leqslant a_i \leqslant m - 1$ for some $m \in \mathbb{N} \backslash \{0\}$.

Theorem 8.1.21 (Steinberg's Tensor Product Thm, [35])**.** *Let ϕ be a field automorphism of $K = \overline{\mathbb{F}_p}$, raising elements to their p^{th}-power. Furthermore, let $\lambda_0, \lambda_1, \ldots, \lambda_n$ be p-restricted weights associated with a KP-module. Then as KP-modules,*

$$L(\lambda_0 + p\lambda_1 + \ldots + p^n \lambda_n) \cong L(\lambda_0) \otimes^{\phi} L(\lambda_1) \otimes \ldots \otimes^{\phi^r} L(\lambda_r).$$

Note that so far we have only considered representations of P defined over $\overline{\mathbb{F}_p}$. However we are interested in finding the representations of the finite group $P(q) \in \{\mathrm{SL}_n^{\pm}(q), \mathrm{Sp}_n(q), \Omega_n^{\epsilon}(q), \mathrm{G}_2(q)\}$. Here the $P(q)$ consists of the fixed points of P under the map sending matrix entries to their q^{th}-powers. We have to find a way to restrict our representations of P to representations of $P(q)$.

Theorem 8.1.22 ([35])**.** *Any module in characteristic p for $P(q)$ is isomorphic to the restriction of $L(\lambda)$ to the module of $P(q)$, where λ is a q-restricted weight of P.*

Lemma 8.1.23. *Let $n \in \{13, 14, 15\}$. Let $P(q)$ have an absolutely irreducible representation $\rho : P(q) \to \Omega \in \{\mathrm{SL}_n^{\pm}(q^t), \mathrm{Sp}_n(q^t), \Omega_n^{\epsilon}(q^t)\}$ that is absolutely tensor indecomposable. Let C be the conformal group associated with Ω. Then $P(q)\rho$ is conjugate in C to a representation listed in [29].*

Proof. This follows from the proof of [8, Cor 5.1.10, p.272]. $\square$

We will also need the following.

Definition 8.1.24. A group $G \leqslant \mathrm{GL}_n(q)$ is **self-dual** if there exists $h \in \mathrm{GL}_n(q)$ such that $h^{-1} g^{-\mathrm{T}} h = g$ for all $g \in G$.

Lemma 8.1.25 ([31, Thm 8.11, p.106])**.** *Let $\rho : \mathrm{SL}_n(K) \to \mathrm{GL}(V)$ be an irreducible representation of $\mathrm{SL}_n(K)$ with highest weight $\lambda = \sum_{i=1}^{l} a_i \lambda_i$. Here $l = n - 1$ is the Lie rank of $\mathrm{SL}_n(K)$ and the λ_i are uniquely determined by the roots $\alpha_i = \chi_{i,i+1}$ which form a basis of the root system of $\mathrm{SL}_n(K)$. Then $\mathrm{SL}_n(K)\rho$ is self-dual if and only if $a_i = a_{l+1-i}$ for all i.*

Lemma 8.1.26 ([3, Thm 8.11, p.106]). *If a group $G \leqslant \mathrm{GL}_n(q)$ is self-dual and q is odd, then G preserves a non-degenerate symmetric bilinear or antisymmetric bilinear form.*

Lemma 8.1.27 ([8, Prop 5.1.12, p.273]). *Over $K = \overline{\mathbb{F}_p}$ all representations of $\mathrm{Sp}_n(K)$ and $\Omega_n^{\epsilon}(K)$ are self-dual.*

8.2 Exterior and Symmetric Powers

In many instances the representations of algebraic groups arise as symmetric or exterior powers of their natural modules. In this section we will define exterior and symmetric powers and state some useful results. For a more in-depth introduction see [8, Section 5.2.1, p.276].

Let G be a group, K a field and let V_r be a KG-module. Suppose that V_r has K-basis $(e_1, \ldots, e_r)$. We will define the symmetric and exterior powers as submodules of the tensor power module $V_r^{\otimes k} = V_r \otimes \ldots \otimes V_r$ with k factors.

Definition 8.2.1. Let $\epsilon(\pi)$ be the sign of the permutation $\pi \in \mathrm{S}_k$ and let $(u_1 \otimes \ldots \otimes u_k)\pi = u_{1\pi} \otimes \ldots \otimes u_{k\pi}$ for all $u_i \in V_r$. The k^{th}**-exterior power** of V_r is

$$\Lambda^k(V_r) := \langle \sum_{\pi \in \mathrm{S}_k} \epsilon(\pi)(u_1 \otimes u_2 \otimes \ldots \otimes u_k)\pi \mid u_i \in V_r \rangle_K.$$

We denote the image of $u_1 \otimes u_2 \otimes \ldots \otimes u_k$ in $\Lambda^k(V_r)$ by $u_1 \wedge u_2 \wedge \ldots \wedge u_k$. Then the standard basis of $\Lambda^k(V_r)$ is $(e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_k} \mid 1 \leqslant i_1 < \ldots < i_k \leqslant r)$ ordered lexicographically.

Now let $\mathrm{char}(K) > k$ or let $\mathrm{char}(K) = 0$. The k^{th}**-symmetric power** of V_r is

$$\mathrm{S}^k(V_r) := \langle \sum_{\pi \in \mathrm{S}_k} (u_1 \otimes u_2 \otimes \ldots \otimes u_k)\pi \mid u_i \in V_r \rangle_K.$$

We denote the image of $u_1 \otimes u_2 \otimes \ldots \otimes u_k$ in $\mathrm{S}^k(V_r)$ by $u_1 u_2 \ldots u_k$. The standard basis of $\mathrm{S}^k(V_r)$ is $(e_{i_1} e_{i_2} \ldots e_{i_k} \mid 1 \leqslant i_1 \leqslant \ldots \leqslant i_k \leqslant r)$ ordered lexicographically.

Lemma 8.2.2. *The dimension of the symmetric power module $\mathrm{S}^k(V_r)$ is $\binom{r+k-1}{k}$.*

Proof. The symmetric power $\mathrm{S}^k(V_r)$ has basis $(e_{i_1} e_{i_2} \ldots e_{i_k} \mid 1 \leqslant i_1 \leqslant i_2 \leqslant \ldots \leqslant i_k \leqslant r)$ ordered lexicographically. It follows that to find the dimension of $\mathrm{S}^k(V_r)$ we have to count the number of multisets of size k with entries from $[1, \ldots, r]$. Here a multiset is an extension of the concept of a set in the sense that a multiset may contain multiple occurences of the same element. By [13, p.39] this number is given by $\binom{r+k-1}{k}$. $\square$

Lemma 8.2.3. *The dimension of the exterior power module $\Lambda^k(V_r)$ is $\binom{r}{k}$.*

Proof. The exterior power $\Lambda^k(V_r)$ has basis $(e_{i_1} \wedge \ldots \wedge e_{i_k} \mid 1 \leqslant i_1 < \ldots < i_k \leqslant r)$. Hence any such basis element can be constructed by choosing k basis elements of V_r and then rearranging them such that the subscripts appear in ascending order. It follows that $\dim(\Lambda^k(V_r)) = \binom{r}{k}$. $\square$

Lemma 8.2.4 ([8, Prop 5.2.4, p.277])**.** *Suppose that G preserves the form β which is either σ-Hermitian, alternating or symmetric. Then on $\Lambda^2(V_k)$ or $\mathrm{S}^2(V_k)$ G preserves the forms*

$$\beta^{2-}(e_i \wedge e_j, e_k \wedge e_l) = \beta(e_i, e_k)\beta(e_j, e_l) - \beta(e_i, e_l)\beta(e_j, e_k)$$

and

$$\beta^{2+}(e_i e_j, e_k e_l) = \beta(e_i, e_k)\beta(e_j, e_l) + \beta(e_i, e_l)\beta(e_j, e_k)$$

respectively. Furthermore, if β is symmetric or alternating then both β^{2-} and β^{2+} are symmetric, whereas if β is σ-Hermitian then so are β^{2-} and β^{2+}.

Lemma 8.2.5 ([8, Prop 5.2.5, p.278])**.** *Let G be a group in odd characteristic and let χ be a complex or Brauer character of G. Then for all $g \in G$ $(\chi(g)^2 + \chi(g^2))/2$ and $(\chi(g)^2 - \chi(g^2))/2$ are the character values on g of the symmetric and exterior squares of the $\mathbb{C}G$-modules corresponding to χ.*

We will conclude this section by calculating the fundamental dominant weights of $\mathrm{SL}_2(K)$ on $\mathrm{S}^n(V_2)$ and $\mathrm{G}_2(K)$ on $\Lambda^2(V_7)$, where $K = \overline{\mathbb{F}_p}$.

Lemma 8.2.6.
Let $K = \overline{\mathbb{F}_p}$.

- *(i) Let Φ be a root system of $\mathrm{SL}_2(K)$ with respect to a maximal torus $T = \langle \mathrm{diag}(\alpha, \alpha^{-1}) \mid \alpha \in K^\times \rangle$. Then $\Delta = \{\chi_{12}\}$, $\check{\chi}_{12}(a) = \mathrm{diag}(a, a^{-1})$ for all $a \in K^\times$ and $\lambda_1(t) = \alpha$ for all $t \in T$.*
- *(ii) Let K have characteristic $\geqslant m + 1$. The highest weight of the rational representation ρ of $\mathrm{SL}_2(K)$ on $\mathrm{S}^m(V_2)$ is (m).*

Proof. (i) It is clear that a maximal torus of $\mathrm{SL}_2(K)$ is given by $T = \langle \mathrm{diag}(\alpha, \alpha^{-1}) \mid \alpha \in K^\times \rangle$ and that $B = \langle \left(\begin{smallmatrix} a & 0 \\ b & a^{-1} \end{smallmatrix}\right) \mid a, b \in K \rangle$ is a Borel subgroup of $\mathrm{SL}_2(K)$. Furthermore, $\Phi = \{\chi_{12}, \chi_{21}\}$ and $\Delta = \{\chi_{12}\}$, where $\chi_{12}(t) = \alpha^2$

for all $t \in T$. Then $\check{\chi}_{12}(a) = \mathrm{diag}(a, a^{-1})$ for all $a \in K^\times$. Furthermore, $a^{\langle \lambda_1, \check{\chi}_{12} \rangle} = \lambda_1(\check{\chi}_{12}(a)) = \lambda_1(\mathrm{diag}(a, a^{-1})) = a^1$ which implies that $\lambda_1(t) = \alpha$ for all $t \in T$.

(ii) By [8, Lemma 5.3.1, p.280], the representation ρ of $\mathrm{SL}_2(K)$ acting on $\mathrm{S}^m(V_2)$ is absolutely irreducible and has dimension $m+1$. Let T be a maximal torus and B be a Borel subgroup of $\mathrm{SL}_2(K)$. It is straightforward to show that

$$T\rho = \langle \mathrm{diag}(\alpha^m, \alpha^{m-2}, \ldots, \alpha^{-m}) \mid \alpha \in K^\times \rangle$$

and that we can choose B such that $B\rho$ consists of lower triangular matrices. Then $B\rho$ stabilises the subspace $\langle v^+ \rangle = \langle (1, 0, \ldots, 0) \rangle$. We know that $v^+ \in V_\chi = \langle v \in V \mid v(t\rho) = \chi(t)v \text{ for all } t \in T \rangle$ for some non-zero weight χ. Here $v^+(t\rho) = \alpha^m v^+$ for all $t \in T$. Hence the weight connected to v^+ is χ where $\chi(t) = \alpha^m$ for all $t \in T$. This weight is dominant by Lemma 8.1.13 and since ρ is irreducible, (m) is the highest weight of ρ by Lemma 8.1.18. □

Lemma 8.2.7. *(i) A maximal torus T of the group G_2 is of the form $T = \langle \mathrm{diag}(a^2b, ab, a, 1, a^{-1}, a^{-1}b^{-1}, a^{-2}b^{-1}) \mid a, b \in K^\times \rangle$, where K is an algebraically closed field.*

(ii) Let Φ be the root system of G_2 with respect to T and let Δ be a base of Φ. Then $\Delta = \{\chi_{12}, \chi_{23}\}$, where $\chi_{12}(t) = a$ and $\chi_{23}(t) = b$ for all $t \in T$. Furthermore, we find that $\check{\Delta} = \{\check{\chi}_{12}, \check{\chi}_{23}\}$ with $\check{\chi}_{12}(a) = \mathrm{diag}(a, a^{-1}, a^2, 1, a^{-2}, a, a^{-1})$ and $\check{\chi}_{23}(a) = \mathrm{diag}(1, a, a^{-1}, 1, a, a^{-1}, 1)$ for all $a \in K^\times$. We also have that $R = \{\lambda_1, \lambda_2\}$ with $\lambda_1(t) = a^2b$ and $\lambda_2(t) = a^3b^2$ for all $t \in T$.

Proof. (i) Note that a maximal torus T of G_2 has two generators. We will first work over $\mathbb{F}_q$. If $\mathrm{G}_2(q)$ is defined over $\mathbb{F}_q$ then T is isomorphic to $\mathrm{C}_{q-1} \times \mathrm{C}_{q-1}$. We also know that T has to preserve a non-degenerate symmetric bilinear form B since $\mathrm{G}_2(q) \leqslant \mathrm{SO}_7^\circ(q, B)$ by [8, Prop 5.7.2, p.305]. Let $B = \mathrm{antidiag}(1, -1, 1, \frac{p-1}{2}, 1, -1, 1)$ since this is the form preserved by the generators of $\mathrm{G}_2(q)$ given in [22].

The first generator of T is $t_1 = \mathrm{diag}(\omega, \omega^{-1}, \omega^2, 1, \omega^{-2}, \omega, \omega^{-1})$ of order $q-1$ where ω is a primitive element of $\mathbb{F}_q^\times$ as given in [22]. Let g_2 be the second generator of $\mathrm{G}_2(q)$ given in [22]. Then $l := t_1^{(q-1)/2} \cdot g_2^{-1} \cdot t_1^{(q-1)/2} \cdot g_2 = \mathrm{diag}(1, -1, -1, 1, -1, -1, 1)$. Hence we can take the second generator of T to be $t_2 = l \cdot t_1 = \mathrm{diag}(\omega, \omega^{(q-3)/2}, \omega^{(q+3)/2}, 1, \omega^{(-q-3)/2}, \omega^{(3-q)/2}, \omega^{-1})$.

Now let $t = \mathrm{diag}(\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta)$ be an arbitrary element of T. Since $tBt^{\mathrm{T}} = B$ it follows that $\alpha = \eta^{-1}, \beta = \zeta^{-1}, \gamma = \epsilon^{-1}$ and $\delta^2 = 1$. Furthermore $\det(t) = 1$ and hence $\delta = 1$. Adding to this, by looking at t_1 and t_2 we can deduce that $\alpha = \beta\gamma$. Hence $t = \mathrm{diag}(\beta\gamma, \beta, \gamma, 1, \gamma^{-1}, \beta^{-1}, \beta^{-1}\gamma^{-1})$ is an arbitrary element of T. Now let $\beta = ab$ and let $\gamma = a$. Note that this convention is purely to get the same results as in [30]. It follows that $t = \mathrm{diag}(a^2b, ab, a, 1, a^{-1}, a^{-1}b^{-1}, a^{-2}b^{-1})$ is an arbitrary element of $T \leqslant \mathrm{G}_2(q)$.

We can restrict the natural module of $\mathrm{G}_2(K)$ to $\mathrm{G}_2(q)$ by considering the fixed points of $\mathrm{G}_2(K)$ under a map sending the matrix entries to their q^{th} power. Hence we know that any torus element of $T \leqslant \mathrm{G}_2(K)$ has to satisfy the same rules as over $\mathbb{F}_q$. It follows that over K an arbitrary element of T is of the form $t = \mathrm{diag}(a^2b, ab, a, 1, a^{-1}, a^{-1}b^{-1}, a^{-2}b^{-1})$, where $a, b \in K^\times$.

(ii) Let $t = \mathrm{diag}(t_1, t_2, t_3, t_4, t_5, t_6, t_7)$, $t_i \in K^\times$ and let $\chi_{ij}(t) = t_i t_j^{-1}$. Since $\mathrm{G}_2 = \mathrm{G}_2(K)$ is a subgroup of $\mathrm{SL}_7(K)$, its root system is a subsystem of the root system of $\mathrm{SL}_7(K)$ which is $\{\chi_{ij} \mid 1 \leqslant i, j \leqslant 7, i \neq j\}$ by [30, Example 8.2(2), p.51]. To determine the root system of G_2 we need to find the elements of the Lie Algebra $\mathfrak{so}(7, K, J)$ of $\mathrm{SO}_7(K, J)$, where J is a non-degenerate symmetric bilinear form matrix. By [16, Section 1.2.2, p.14], $A = (a_{ij}) \in \mathfrak{so}(7, K, J)$ if and only if $A^{\mathrm{T}}J = -JA$. If we let $J = \mathrm{antidiag}(1, -1, 1, \mu, 1, -1, 1)$ for some $\mu \in K^\times$ then $a_{71} = a_{62} = a_{53} = a_{44} = a_{35} = a_{26} = a_{17} = 0$. Let $S = \{(1,7), (7,1), (2,6), (6,2), (3,5), (5,3), (4,4)\}$. It is clear that χ_{ij} is not a root of G_2 when $(i, j) \in S$. Furthermore, by [30, Example 9.5(4), p.65] G_2 has exactly 12 roots and we can show that

$$\Phi = \{\chi_{ij} \mid 1 \leqslant i, j \leqslant 7, i \neq j, (i, j) \notin S\}$$

is a root system of G_2, where each of these 12 roots appears multiple times.

The base of this root system is given by $\Delta = \{\chi_{12}, \chi_{23}\}$, where $\chi_{12}(t) = a$ and $\chi_{23}(t) = b$ for all $t \in T = \langle \mathrm{diag}(a^2b, ab, a, 1, a^{-1}, a^{-1}b^{-1}, a^{-2}b^{-1}) \mid a, b \in K^\times \rangle$.

Now we also have to find $\check{\chi}_{12}$ such that $\langle \chi_{12}, \check{\chi}_{12} \rangle = 2$ and, by [30, Example 15.21(1)], $\langle 3\chi_{12} + 2\chi_{23}, \check{\chi}_{12} \rangle = 0$. The first condition $\langle \chi_{12}, \check{\chi}_{12} \rangle = 2$ implies that $\chi_{12}(\check{\chi}_{12}(\alpha)) = \alpha^2$ for all $\alpha \in K^\times$. Furthermore, $\check{\chi}_{12}(\alpha) = \mathrm{diag}(\alpha^i, \alpha^j, \alpha^{i-j}, 1, \alpha^{j-i}, \alpha^{-j}, \alpha^{-i})$ for some i and j. Hence $\chi_{12}(\check{\chi}_{12}(\alpha)) = \alpha^{i-j} = \alpha^2$. Therefore, $\check{\chi}_{12}(\alpha) = \mathrm{diag}(\alpha^4\beta, \alpha^2\beta, \alpha^2, 1, \alpha^{-2}, \alpha^{-2}\beta^{-1}, \alpha^{-4}\beta^{-1})$ for some β yet to be determined. For the second condition to be satisfied we additionally need $(\chi_{12}(\check{\chi}_{12}(\alpha)))^3(\chi_{23}(\check{\chi}_{12}(\alpha)))^2 = \alpha^0$. We find that $(\alpha^4\beta\alpha^{-2}\beta^{-1})^3(\alpha^2\beta\alpha^{-2}) = \alpha^0$ if and only if $\beta = \alpha^{-3}$. Hence $\check{\chi}_{12}(a) = \mathrm{diag}(a, a^{-1}, a^2, 1, a^{-2}, a, a^{-1})$ for all $a \in K^\times$.

Similarly, we want to find $\check{\chi}_{23}$ such that $\langle \chi_{23}, \check{\chi}_{23} \rangle = 2$ and $\langle 3\chi_{12} + 2\chi_{23}, \check{\chi}_{23} \rangle = 1$ by [30, Example 15.21(1), p.129]. It is straightforward to show that $\check{\chi}_{23}(a) = \mathrm{diag}(1, a, a^{-1}, 1, a, a^{-1}, 1)$ for all $a \in K^\times$.

Finally, we need to find the fundamental dominant weights λ_1 and λ_2 of G_2 such that $\langle \lambda_1, \check{\chi}_{12} \rangle = \langle \lambda_2, \check{\chi}_{23} \rangle = 1$ and $\langle \lambda_2, \check{\chi}_{12} \rangle = \langle \lambda_1, \check{\chi}_{23} \rangle = 0$. A straightforward calculation shows that $\lambda_1(t) = a^2b$ and $\lambda_2(t) = a^3b^2$. (This also agrees with the results in [30, Example 15.21(1), p.129].) $\square$

8.3 $\mathbf{SL_2}(q) = \mathbf{Sp_2}(q)$

In this section we will consider the 13-, 14- and 15-dimensional $\mathscr{S}_2$-subgroups isomorphic to $\mathrm{SL}_2(q)$ and $\mathrm{L}_2(q)$. For a more detailed version of the following see [8, Section 5.3, p.80]. Note that since $\mathrm{SL}_2(2)$ and $\mathrm{SL}_2(3)$ are soluble, we will assume throughout that $q \geqslant 4$. For $m \geqslant 1$, let V_{m+1} be the $(m+1)$-dimensional module $\mathrm{S}^m(V_2)$.

Since $\mathrm{SL}_2(q) = \mathrm{Sp}_2(q)$, it follows that $\mathrm{SL}_2(q)$ preserves a bilinear form β. Furthermore, a representation of $\mathrm{SL}_2(q)$ on $\mathrm{S}^m(V_2)$ preserves the form

$$\begin{aligned}\beta^m(&e_{i_1}e_{i_2}\dots e_{i_m}, e_{j_1}e_{j_2}\dots e_{j_m})\\ &= \sum_{\pi\in \mathrm{S}_m} \beta(e_{i_1}, e_{j_1\pi})\beta(e_{i_2}, e_{j_2\pi})\dots\beta(e_{i_m}, e_{j_m\pi}).\end{aligned}$$

By Lemma 8.1.27, V_{m+1} is self-dual.

Let V_1 be the trivial module of $\mathrm{SL}_2(q)$. Let $m \in \mathbb{N}$ with $0 \leqslant m \leqslant q-1$ and let $a_0, a_1, \dots, a_s \in \{0, \dots, p-1\}$ such that $m = a_0 + a_1p + \dots + a_sp^s$ with $a_s \neq 0$. Then let

$$M(m) := V_{a_0+1} \otimes V_{a_1+1}^{\phi} \otimes \dots \otimes V_{a_s+1}^{\phi^s}$$

with dimension $(a_0+1)(a_1+1)\dots(a_s+1) \leqslant m+1$.

Theorem 8.3.1 ([5, p.588])**.** *If $0 \leqslant m \leqslant q-1$ then each $M(m)$ is an absolutely irreducible module for $\mathrm{SL}_2(q)$ and furthermore $M(i)$ is not isomorphic to $M(j)$ for any $0 \leqslant i < j \leqslant q-1$. Conversely, for each absolutely irreducible module for $\mathrm{SL}_2(q)$ there exists some $0 \leqslant m \leqslant q-1$ such that the module is isomorphic to $M(m)$.*

First we will consider the p-restricted modules with $p \geqslant n = \dim(M(m))$. We then get the following.

Lemma 8.3.2 ([8, Prop 5.3.6, p.283])**.**
Let δ_H and ϕ_H denote the generating diagonal and field automorphism of a group H respectively.

(i) If $n > 2$ is even and $p \geqslant n$, then there exists a single conjugacy class of $\mathscr{S}_2$-subgroups of $\mathrm{Sp}_n(q)$ isomorphic to $\mathrm{SL}_2(q)$. This conjugacy class is stabilised by $\delta_{\mathrm{Sp}_n(q)}$ and $\phi_{\mathrm{Sp}_n(q)}$ which induce $\delta_{\mathrm{SL}_2(q)}$ and $\phi_{\mathrm{SL}_2(q)}$ respectively.

(ii) *If* $n \equiv \pm 3 \pmod 8$ *and* $p \geqslant n$ *then there exists a single conjugacy class of* $\mathscr{S}_2$*-subgroups of* $\Omega_n^\circ(q)$ *isomorphic to* $\mathrm{L}_2(q)$*. The automorphisms* $\delta_{\Omega_n^\circ(q)}$ *and* $\phi_{\Omega_n^\circ(q)}$ *induce* $\delta_{\mathrm{SL}_2(q)}$ *and* $\phi_{\mathrm{SL}_2(q)}$ *respectively and stabilise the conjugacy class.*

(iii) *If* $n \equiv \pm 1 \pmod 8$ *and* $p \geqslant n$ *then there exist two conjugacy classes of* $\mathscr{S}_2$*-subgroups of* $\Omega_n^\circ(q)$ *isomorphic to* $\mathrm{L}_2(q).2 = \mathrm{PGL}_2(q)$*. Furthermore,* $\phi_{\Omega_n^\circ(q)}$ *stabilises the conjugacy classes and induces* $\phi_{\mathrm{L}_2(q)}$ *whereas* $\delta_{\Omega_n^\circ(q)}$ *interchanges the conjugacy classes.*

For the p-unrestricted modules we will need the following corollary.

Corollary 8.3.3 ([8, Cor 5.3.3, p.281])**.** *Let* $q = p^e$ *and let* $m = a_0 + a_1 p + \ldots + a_{e-1}p^{e-1}$*, where* $0 \leqslant a_i \leqslant p-1$ *for all* i*. Then the minimal field over which* $M(m) = V_{a_0+1} \otimes \ldots \otimes V_{a_{e-1}+1}^{\phi^{e-1}}$ *can be realised is* $\mathbb{F}_{p^f}$ *if and only if*

(i) $f \mid e$

(ii) f *is minimal such that* $a_i = a_j$ *whenever* $i \equiv j \pmod f$*.*

Finally, we can show that there are in fact no maximal absolutely irreducible p-unrestricted representations of $\mathrm{SL}_2(q)$ or $\mathrm{L}_2(q)$ in dimensions 13, 14 or 15. Note that there might be $\mathscr{S}_2$-maximal subgroups isomorphic to $\mathrm{SL}_2(q)$ or $\mathrm{L}_2(q)$ but since they cannot be maximal we will not consider them any further.

Lemma 8.3.4. *Let* $q = p^e$*. There are no* p*-unrestricted representations of* $\mathrm{SL}_2(q)$ *or* $\mathrm{L}_2(q)$ *that induce maximal subgroups in dimensions* 13, 14 *and* 15*.*

Proof. The proof follows the same outline as the proof of [8, Thm 5.3.9, p.287]. Let $M(m) = V_{a_0+1} \otimes \ldots \otimes V_{a_{e-1}+1}^{\phi^{e-1}}$ with $0 \leqslant a_i \leqslant p-1$ and $m = a_0 + a_1 p + \ldots + a_{e-1}p^{e-1}$. By Lemma 8.3.1 we need to consider all such $M(m)$ with dimension $(a_0+1)(a_1+1)\ldots(a_s+1) \in \{13, 14, 15\}$.

Note that each p-unrestricted module $M(m)$ with only one non-trivial tensor factor is an algebraic conjugate of a p-restricted module and hence gives a subgroup that is conjugate to one of the groups considered in Lemma 8.3.2 using a similar argument as in Lemma 8.1.23. Hence we assume that M has at least 2 non-trivial tensor factors which is only possible in dimensions 14 and 15. In dimension 14 we have $a_0 = 1$, $a_1 = 6$ and $M(m) = V_2 \otimes V_7^\phi$. In dimension 15 we get $a_0 = 2$, $a_1 = 4$ and $M(m) = V_3 \otimes V_5^\phi$. (Note that a_0 and a_1 could be interchanged in both cases.) By [8, Prop 5.1.14, p.274], either $M(m)$ can be written over a proper subfield $\mathbb{F}_{p^{e/t}}$ for some $t > 1$ or M preserves a form other then the induced symplectic or symmetric form.

To show that $M(m)$ cannot be written over a proper subfield we can use Corollary 8.3.3. In dimension 14 we have $m = 13 = 1+6p$ (or $m = 13 = 6+p$) and $e = 2$ which implies that f has to equal 1. Therefore, for Corollary 8.3.3 to hold, we would require a_0 to equal a_1 which is not the case. Similarly in dimension 15. Hence we can assume that $M(m)$ cannot be written over a proper subfield.

Now suppose that $M(m)$ preserves either a unitary form or, when $p = 2$, a quadratic form. By [8, Lemma 1.8.8, p.40] a representation of $\mathrm{SL}_2(q)$ can only preserve a unitary and a bilinear form at the same time if the representation can be written over a proper subfield of $\mathbb{F}_q$ - a contradiction. When $p = 2$, then $a_i \in \{0, 1\}$ and hence $M(m)$ has dimension a power of 2. Again this does not occur in dimensions 13, 14 or 15. □

8.4 Adjoint Module

In this section we will briefly introduce the concept of adjoint modules. For a more detailed discussion see [8, Section 5.4.1, p.293].

Definition 8.4.1. Let ρ be a representation of some group and let V be the module of ρ. Then V^* is the **dual module** acted on by $\rho^{-\mathrm{T}}$.

Let $G = \mathrm{GL}_t^{\pm}(q)$ and let $g \in G$. Let V be the natural module of G over $\mathbb{F}_{q^u}$ and let V^* be the dual module of V. Define a representation $\rho : G \to V \otimes V^*$ by $g\rho = g \otimes g^{-\mathrm{T}}$. Let M be the $\mathbb{F}_q\mathrm{GL}_t(q)$-module $M_{t\times t}(\mathbb{F}_q)$ or let M be the $\mathbb{F}_q\mathrm{GU}_t(q)$-module $M = \{A \in M_{t\times t}(\mathbb{F}_{q^2}) \mid A^{\mathrm{T}} = \mathrm{A}^{\sigma}\}$ corresponding to $G\rho$. Here σ sends the entries a_{ij} of A to a_{ij}^q.

Definition 8.4.2. Let M be as above. Let U be the submodule of M consisting of all matrices of trace 0 and let U' be the submodule of M consisting of all scalar matrices. Then the **adjoint module** W is

$$W = U/(U \cap U').$$

Lemma 8.4.3 ([8, Lemma 5.4.10, p.294])**.** *Let $G = \mathrm{GL}_t^{\pm}(q)$ and let W be as in Definition 8.4.2. If $p \mid t$ then W has dimension $t^2 - 2$. Otherwise W has dimension $t^2 - 1$. Furthermore, W is absolutely irreducible as an $\mathbb{F}_q\mathrm{SL}_t^{\pm}(q)$-module.*

By [8, p.294] we can define a quadratic form Q on M by

$$Q(A) = \sum_{1 \leqslant i < j \leqslant n} (a_{ij}a_{ji} - a_{ii}a_{jj}).$$

Throughout the reminder of this section let B be the form matrix of the polar form β of Q. Let E be the subset of M containing the matrices with all diagonal entries equal to 0 and let D be the set of diagonal matrices of M.

Lemma 8.4.4 ([8, Lemma 5.4.11, p.295])**.**
Let $G = \mathrm{GL}_t^{\pm}(q)$ and let M, U, U', W, E, D and β be as defined above. Then:

(i) $W \cong E \perp (D \cap U)/U'$;

(ii) *E is a non-degenerate space of plus-type if $G = \mathrm{GL}_t(q)$ and a non-degenerate space of type $(-1)^{\binom{t}{2}}$ if $G = \mathrm{GU}_t(q)$;*

(iii) *If $p \mid t$ then β is degenerate.*

Hence we have to determine the type of orthogonal form of the non-degenerate space $(D \cap U)/U'$. We will only have to consider the case when $p = 2$ and $t = 4$.

Lemma 8.4.5. *Let $G = \mathrm{GL}_t^{\pm}(q)$ and let M, U, U' and D be as above. If $p = 2$ and $t = 4$, then $(D \cap U)/U'$ is a non-degenerate space of minus-type in any odd extension of $\mathbb{F}_2$ and an orthogonal space of plus-type in any even extension of $\mathbb{F}_2$.*

Proof. By the proof of [8, Lemma 5.4.11(iv), p.295], $D \cap U = \langle d_1, d_2, d_3 \rangle$, where $d_j = E_{j,j} - E_{4,4}$ for all $1 \leqslant j \leqslant 3$ and $E_{i,j} = (a_{lk})$ is a matrix with $a_{lk} = 1$ if $l = i$, $k = j$ and $a_{lk} = 0$ otherwise. Furthermore, the form matrix with respect to this basis $\{d_1, d_2, d_3\}$ is

$$B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

and $Q(d_j) = 1$ for all j.

Let $U' = \langle d \rangle$, where $d = \mathrm{diag}(1,1,1,1) = d_1 + d_2 + d_3$. It is clear that $(D \cap U)/U' = \langle d_1 + U', d_2 + U' \rangle$. We now have to find the quadratic form Q' with polar form β' on $(D \cap U)/U'$. We have

$$\begin{aligned} Q(d_j + \alpha d) &= Q(d_j) + Q(\alpha d) + \beta(d_j, \alpha d) \\ &= 1 + \alpha^2 Q(d) + \alpha(\beta(d_j, d_1) + \beta(d_j, d_2) + \beta(d_j, d_3)) \\ &= 1 + 6\alpha^2 + 2\alpha \\ &= 1 \text{ for all } \alpha \in \mathbb{F}_{2^i}. \end{aligned}$$

If $k \neq j$ then $\beta'(d_j + U', d_k + U') = \beta(d_j, d_k) = 1$ by Lemma 3.3.3. Hence, the matrix of Q' is $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ which is of minus-type if and only if $x^2 + x + 1$ is irreducible in $\mathbb{F}_{2^i}$ by [8, Prop 1.5.42(iii), p.24].

It is clear that $x^2 + x + 1$ is irreducible in $\mathbb{F}_2$ and from [28, Cor 3.47, p.100] it follows that x^2+x+1 is irreducible in any odd extension of $\mathbb{F}_2$ and reducible otherwise. $\square$

Corollary 8.4.6. *Let $G = \mathrm{SL}_4^{\pm}(2^i)$ and let U, U', W, E and D be as above. Then the adjoint module W of G preserves an orthogonal form of plus-type if and only if i is even and an orthogonal form of minus-type otherwise.*

Proof. By Lemma 8.4.4, $W \cong E \perp (D \cap U)/U'$ and E is a space of type $k_1 = +$. Let k_2 be the type of the space $(D \cap U)/U'$. By [8, Prop 1.5.42(iv), p.24], W has type k_1k_2. The result follows from Lemma 8.4.5. $\square$

Lemma 8.4.7 ([8, Lemma 5.4.13, p.297])**.** *Let ρ be the adjoint representation of $\mathrm{SL}_t^{\pm}(q)$ and let d generate the diagonal automorphisms of $\mathrm{SL}_t^{\pm}(q)$. Then $d\rho \in \mathrm{SO}_n^{\epsilon}(q, B)$. Furthermore, $d\rho \in \Omega_n^{\epsilon}(q, B)$ if and only if t is odd or q is even.*

Lemma 8.4.8 ([8, Lemma 5.4.14, p.297])**.** *Suppose that $G = \mathrm{SL}_t^{\pm}(q)$ has an adjoint representation ρ of dimension n. Then $\gamma \in \mathrm{Out}(\mathrm{L}_t^{\pm}(q))$ is induced by an element $g \in \mathrm{GO}_n^{\epsilon}(q, B)$. If $\binom{t}{2}$ is even or if qn is odd then $g \in \Omega_n^{\epsilon}(q, B)$ or $-g \in \Omega_n^{\epsilon}(q, B)$. If $\binom{t}{2}$ and q are odd and n is even then $g \in \mathrm{GO}_n^{\epsilon}(q, B)\backslash\mathrm{SO}_n^{\epsilon}(q, B)$. If $\binom{t}{2}$ is odd and q is even then we can show that $g \in \mathrm{SO}_n^{\epsilon}(q, B)\backslash\Omega_n^{\epsilon}(q, B)$.*

8.5 Outer Automorphisms

Let $\Omega \in \{\mathrm{SL}_n^{\pm}(q), \mathrm{Sp}_n(q), \Omega_n^{\epsilon}(q)\}$, where $\epsilon \in \{\circ, +, -\}$, and let $\rho : G \to \Omega$ be an absolutely irreducible representation of a quasisimple group G such that $G\rho$ is an $\mathscr{S}_2$-subgroup of Ω. In this section we will consider how the outer automorphisms of Ω act on $G\rho$. To avoid confusion we will denote an outer automorphism $\beta \in \mathrm{Out}(H)$ for some group H by β_H.

Lemma 8.5.1 ([8, Prop 5.1.9, p.272])**.** *Let G be quasisimple and let $\Omega \in \{\mathrm{SL}_n^{\pm}(q), \mathrm{Sp}_n(q), \Omega_n^{\epsilon}(q)\}$, where $\epsilon \in \{\circ, +, -\}$. Let $\rho : G \to \Omega$ be a representation such that $G\rho$ is an $\mathscr{S}_2$-subgroup of Ω.*

(i) Let $\delta \in \mathrm{Out}(G)$. Then $(G.\langle\delta\rangle)\rho \leqslant C$, the conformal group of Ω.

(ii) Let $\phi_G \in \mathrm{Out}(G)$ and let $\phi_\Omega \in \mathrm{Out}(\Omega)$. Then ${}^{\phi_G}\rho$ is equivalent to ρ^{ϕ_Ω}.

(iii) Let $G = \mathrm{SL}_t(q)$ and let $\Omega = \mathrm{SL}_n(q)$. Then ${}^{\gamma_G}\rho$ is equivalent to ρ^{γ_Ω}.

We will need to consider the Cases **L** , **U** and $\mathbf{O}^+$ separately.

Cases L and U

We will consider the unitary and linear cases first. The following lemma is an adaptation of [8, Lemma 5.9.1, p.310].

Lemma 8.5.2. *Let $G = \mathrm{SL}_t^{\pm}(q)$ and let $\rho : G \to \mathrm{SL}_n^{\pm}(q)$ be a representation of G with module $U \in \{\mathrm{S}^2(V_t), \mathrm{S}^4(V_t), \Lambda^2(V_t)\}$. Let $n = \dim(U)$ and let A denote an $n \times t^m$ matrix, $m \in \{2, 4\}$, whose rows form the basis vectors for U as a subspace of $V_t^{\otimes m}$.*

(i) If $AA^{\mathrm{T}}D = I_n$, then $D \in M_{n\times n}(p)$.

(ii) For $g \in \mathrm{SL}_t^{\pm}(q)$ the action matrix $M(g)$ of g on U is $Ag^{\otimes m}A^{\mathrm{T}}D$.

(iii) If $G = \mathrm{SL}_t^{\pm}(q)$ and n is odd, then $\phi_{\mathrm{L}_n^{\pm}(q)}$ is contained in the stabiliser of $G\rho$ in $\mathrm{Out}(\mathrm{L}_n^{\pm}(q))$. Furthermore, $\phi^{\pm}_{\mathrm{L}_n(q)}$ induces ϕ_G.

(iv) If $G = \mathrm{SL}_t(q)$ and n is odd, then $\gamma_{\mathrm{L}_n(q)}$ is contained in the stabiliser of $G\rho$ in $\mathrm{Out}(\mathrm{L}_n(q))$. Furthermore, $\gamma_{\mathrm{L}_n(q)}$ induces γ_G.

Proof. (i)-(iii) It is straightforward to see that the proof of [8, Lemma 5.9.1, p.310] extends to all our cases.

(iv) By the proof of [8, Lemma 5.9.1(iv), p.310] we know that $M(g^{\gamma_G}) = D^{-1}M(g)^{\gamma_{\mathrm{L}_n(q)}}D$. The result follows now from Lemma 4.4.2. □

Case $\mathbf{O}^+$

In Case $\mathbf{O}^+$ we have the added difficulty that there are up to 2 isomorphism classes of groups $\langle \Omega_n^+(p^f, B), \phi\rangle$ when f is even by [7]. However if f is odd then there is only one isomorphism class.

Lemma 8.5.3. *Let G be a quasisimple group. Let ρ be an absolutely irreducible representation of G such that $G\rho$ is an $\mathscr{S}_2$-subgroup of $\Omega_n^+(p^f, B)$, where f is even and B is a non-degenerate symmetric bilinear form matrix (or quadratic form matrix if p is even) of plus-type. Assume that $B \in M_{n\times n}(p)$. Let F be our standard symmetric or quadratic form matrix of plus-type. Then:*

(i) There exists $x \in \mathrm{GL}_n(p^f)$ such that $(G\rho)^x \leqslant \Omega_n^+(p^f, F)$;

(ii) If p is odd then $\langle (G\rho)^x, \phi\rangle$ and $\langle G\rho, \phi\rangle$ lie in the same isomorphism class if and only if $\det(B)\cdot\det(F)$ is a square in $\mathbb{F}_p^\times$. If $\det(B)\cdot\det(F)$ is not square and $(G\rho)^x$ is stabilised by ϕ then $G\rho$ is stabilised by $\phi\gamma$;

(iii) If p is even then $\langle (G\rho)^x, \phi\rangle$ and $\langle G\rho, \phi\rangle$ lie in the same isomorphism class if and only if $(x^{-1})^{\phi}x$ has quasideterminant 1. If the quasideterminant is -1 and $(G\rho)^x$ is stabilised by ϕ then $G\rho$ is stabilised by $\phi\gamma$.

Proof. (i) This follows by Lemma 4.8.1 for odd p and it is clear that Lemma 4.8.1 also holds for even p.

(ii) and (iii) follow from [7, Prop 12]. $\square$

Note that in our final results we will always give the stabiliser of $(G\rho)^x$.

8.6 Maximality

The following lemma is useful when we want to determine containments between the various $\mathscr{S}_2$-subgroups.

Lemma 8.6.1. *Let $\mathrm{L}_t^{\pm}(q)\rho_n$ be a reducible subgroup of $\mathrm{L}_n^{\pm}(q)$ where $2 \leqslant t < n$. Furthermore let ρ be a representation of $\mathrm{L}_n^{\pm}(q)$ such that $\mathrm{L}_n^{\pm}(q)\rho$ acts on $\mathrm{S}^2(V_n)$ or $\Lambda^2(V_n)$. Then $(\mathrm{L}_t^{\pm}(q)\rho_n)\rho$ is reducible.*

Proof. Let $\langle e_1, \dots, e_n\rangle$ be an n-dimensional vector space acted on by $\mathrm{L}_n^{\pm}(q)$. Without loss of generality we can assume that $\mathrm{L}_t^{\pm}(q)\rho_n$ fixes the subspace $\langle e_1, \dots, e_r\rangle$, where $t \leqslant r \leqslant n-1$ since $\mathrm{L}_t^{\pm}(q)$ is irreducible but $\mathrm{L}_t^{\pm}(q)\rho_n$ is not. Let $\{e_i \otimes e_j - e_j \otimes e_i \,|\, 1 \leqslant i < j \leqslant n\}$ be a basis of $\Lambda^2(V_n)$ and let $\{e_i \otimes e_i \,|\, 1 \leqslant i \leqslant n\} \cup \{e_i \otimes e_j + e_j \otimes e_i \,|\, 1 \leqslant i < j \leqslant n\}$ be a basis for $\mathrm{S}^2(V_n)$. Then it follows that $(\mathrm{L}_t^{\pm}(q)\rho_n)\rho$ fixes the subspace $\langle e_i \otimes e_j - e_j \otimes e_i \,|\, 1 \leqslant i < j \leqslant r\rangle$ in the exterior square case. In the symmetric square case $(\mathrm{L}_t^{\pm}(q)\rho_n)\rho$ fixes the subspace $\langle (e_i \otimes e_i \,|\, 1 \leqslant i \leqslant r), (e_i \otimes e_j + e_j \otimes e_i \,|\, 1 \leqslant i < j \leqslant r)\rangle$. $\square$

9 Maximal $\mathscr{S}_2$-Subgroups in Dimension 13, 14 and 15

In this chapter we will determine the maximal $\mathscr{S}_2$-subgroups of the classical groups in dimension 13, 14 and 15. We will start by finding the potential $\mathscr{S}_2$-candidates. Rather than looking at the dimensions individually as we did for the $\mathscr{S}_1$-candidates, we will group the representations according to their behaviour as in Chapter 8.

We will start by determining the representations of $\mathrm{SL}_2(q)$. This is followed by finding the $\mathscr{S}_2$-subgroups that act on exterior or symmetric power modules. We will then discuss adjoint modules before determining the maximal $\mathscr{S}_2$-subgroups.

9.1 $\mathscr{S}_2$-candidates

The following table, Table 9.1.1, gives the potential $\mathscr{S}_2$-maximal subgroups G taken from [29] with the exception of $\mathrm{L}_2(q)$ whose information was taken from Section 8.3.

The first column gives the dimension in which such a group G has an absolutely irreducible representation in defining characteristic. This is followed by its Lie name as it appear in the tables of [29]. The corresponding classical group name is then given which is also the name which we will use throughout this chapter. The 'Weight' column shows the highest weight of G in the respective dimensions (see Definition 8.1.14). The final column then gives the characteristics in which these groups appear. The results in the final column were partially taken from [29] and partially determined by inspection of the highest weights.

Note that $\mathrm{Sp}_6(q)$ has two absolutely irreducible representations with distinct weights in dimension 14. We will denote the images of these representations by $\mathrm{Sp}_6(q)_1$ and $\mathrm{Sp}_6(q)_2$. Similarly, we will denote the images of the two 15-dimensional absolutely irreducible representations of $\mathrm{SL}_3^{\pm}(q)$ with distinct weights by $\mathrm{SL}_3^{\pm}(q)_1$ and $\mathrm{SL}_3^{\pm}(q)_2$.

Let $E_{i,j} = (a_{lk})$ be a square matrix with $a_{lk} = 1$ if $l = i$, $k = j$ and $a_{lk} = 0$ otherwise.

Theorem 9.1.1. *Let G be an $\mathscr{S}_2$-subgroup of $\Omega \in \{\mathrm{SL}_n^{\pm}(q), \mathrm{Sp}_{14}(q), \Omega_n^{\epsilon}(q)\}$, $n \in \{13, 14, 15\}$ that is potentially maximal. Then G is contained in Table 9.1.1.*

Proof. This follows from the tables in [29] and Section 8.3. □

Table 9.1.1: $\mathscr{S}_2$-Subgroups in Dimension 13, 14 and 15

Dim	**Lie Name**		**Weight**	**Char**
13	A_1	$\mathrm{L}_2(q)$	(12)	$\geqslant 13$
	B_2	$\mathrm{O}_5^{\circ}(q)$	$(0,2)$	5
	C_3	$\mathrm{S}_6(q)$	$(0,1,0)$	3
14	A_1	$\mathrm{L}_2(q)$	(13)	$\geqslant 17$
	A_3	$\mathrm{L}_4(q)$	$(1,0,1)$	2
	${}^2\mathrm{A}_3$	$\mathrm{U}_4(q)$	$(1,0,1)$	2
	B_2	$\mathrm{O}_5^{\circ}(q)$	$(0,2)$	$\neq 2,5$
	C_{3_1}	$\mathrm{S}_6(q)$	$(0,1,0)$	$\neq 3$
	C_{3_2}	$\mathrm{S}_6(q)$	$(1,0,0)$	$\neq 2$
	G_2	$\mathrm{G}_2(q)$	$(1,0)$	$\neq 3$
15	A_1	$\mathrm{L}_2(q)$	(14)	$\geqslant 17$
	A_{2_1}	$\mathrm{L}_3(q)_1$	$(0,4)$	$\neq 2,3$
	${}^2\mathrm{A}_{2_1}$	$\mathrm{U}_3(q)_1$	$(0,4)$	$\neq 2,3$
	A_{2_2}	$\mathrm{L}_3(q)_2$	$(1,2)$	$\neq 2$
	${}^2\mathrm{A}_{2_2}$	$\mathrm{U}_3(q)_2$	$(1,2)$	$\neq 2$
	A_3	$\mathrm{L}_4(q)$	$(1,0,1)$	$\neq 2$
	${}^2\mathrm{A}_3$	$\mathrm{U}_4(q)$	$(1,0,1)$	$\neq 2$
	A_4	$\mathrm{L}_5(q)$	$(0,0,0,2)$	$\neq 2$
	${}^2\mathrm{A}_4$	$\mathrm{U}_5(q)$	$(0,0,0,2)$	$\neq 2$
	A_5	$\mathrm{L}_6(q)$	$(0,0,0,1,0)$	all
	${}^2\mathrm{A}_5$	$\mathrm{U}_6(q)$	$(0,0,0,1,0)$	all

9.2 $\mathbf{SL_2}(\boldsymbol{q}) = \mathbf{Sp_2}(\boldsymbol{q})$

The results in this section follow directly from Lemma 8.3.2 and Lemma 8.3.4.

Proposition 9.2.1 ($\mathrm{SL}_2(q)$)**.**

(i) If $p \geqslant 13$ then there is one conjugacy class of $\mathscr{S}_2$-subgroups of $\Omega_{13}^{\circ}(q)$ isomorphic to $\mathrm{L}_2(q)$, with class stabiliser $\langle \delta, \phi \rangle$ in $\mathrm{Out}(\mathrm{O}_{13}^{\circ}(q))$.

(ii) If $p \geqslant 14$ then there is one conjugacy class of $\mathscr{S}_2$-subgroups of $\mathrm{Sp}_{14}(q)$ isomorphic to $\mathrm{SL}_2(q)$, with class stabiliser $\langle \delta, \phi \rangle$ in $\mathrm{Out}(\mathrm{S}_{14}(q))$.

(iii) If $p \geqslant 15$ then there are two conjugacy classes of $\mathscr{S}_2$-subgroups of $\Omega_{15}^{\circ}(q)$ isomorphic to $\mathrm{L}_2(q).2$, with class stabiliser $\langle \phi \rangle$ in $\mathrm{Out}(\mathrm{O}_{15}^{\circ}(q))$.

Proof. This follows from Lemma 8.3.2. $\square$

9.3 Exterior and Symmetric Powers

Let G be one of the groups appearing in Table 9.1.1 and let ρ_i be an arbitrary representation of G of dimension i. Let $G\rho_n \leqslant \Omega \in \{\mathrm{SL}_n^{\pm}(q), \mathrm{Sp}_n(q), \Omega_n^{\epsilon}(q)\}$, where $n \in \{13, 14, 15\}$, and suppose that the module associated with ρ_n is a subquotient of a symmetric or exterior power module. (Note that we will often just say that ρ_n acts on a subquotient of a symmetric or exterior power.) Let $G\rho_i$ be generated by g_i and h_i, let b_i denote the preserved form matrix and let d_i be an automorphism of G that lies in the conformal group of Ω. Here δ_Ω stands for any generating diagonal automorphism of Ω, whereas d_i may or may not denote a diagonal automorphism δ_G of G.

9.3.1 Dimension 13

Proposition 9.3.1 ($\mathrm{Sp}_6(3^i)$)**.**
There is a single conjugacy class of $\mathscr{S}_2$-subgroups of $\Omega_{13}^{\circ}(3^i)$ isomorphic to $\mathrm{S}_6(3^i)$*, with class stabiliser $\langle \delta, \phi \rangle$ in* $\mathrm{Out}(\mathrm{O}_{13}^{\circ}(3^i))$.

Proof. Let $G = \mathrm{Sp}_6(3^i)$. By [26, Table 5.4.A, p.199] there exists a 15-dimensional representation ρ_{15} of G that acts on $\Lambda^2(V_6)$, a 14-dimensional representation of G acting on a submodule of $\Lambda^2(V_6)$ and a 13-dimensional representation acting on a subquotient of $\Lambda^2(V_6)$.

We will first find $b_{15} = \beta^{-2}$ (see Lemma 8.2.4), $g_{15} = g_6\rho_{15}$, h_{15} and d_{15} before calculating the matrices for the 14-dimensional representation (see file s2sp61comp for the explicit matrices). By Lemma 8.2.4 the matrix of b_{15} is given by $b_{15} = c + c^{\mathrm{T}}$, where $c = \frac{1}{2}(E_{5,5} + E_{8,8} + E_{10,10}) - (E_{1,15} + E_{2,14} + E_{6,13}) + E_{3,12} + E_{4,9} + E_{7,11}$. For a basis of $\Lambda^2(V_6)$ we choose the set $\{e_i \wedge e_j \,|\, 1 \leqslant i < j \leqslant 6\}$ ordered lexicographically. Then we can show that b_{15}, g_{15}, h_{15} and d_{15} preserve the 14-dimensional subspace

$$\begin{aligned} W =& \langle e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, e_1 \wedge e_5, e_1 \wedge e_6 - e_3 \wedge e_4, e_2 \wedge e_3, e_2 \wedge e_4, \\ & e_2 \wedge e_5 - e_3 \wedge e_4, e_2 \wedge e_6, e_3 \wedge e_5, e_3 \wedge e_6, e_4 \wedge e_5, e_4 \wedge e_6, e_5 \wedge e_6\rangle. \end{aligned}$$

Therefore we can calculate the respective 14-dimensional matrices g_{14}, h_{14} and d_{14} induced by δ_G using this basis of W. We can also find $b_{14} = a + a^{\mathrm{T}}$, where $a = E_{5,5} + E_{8,8} + E_{3,11} + E_{4,9} + E_{5,8} + E_{7,10} - (E_{1,14} + E_{2,13} + E_{6,12})$, which is an orthogonal form matrix with $\det(b_{14}) = 3$. It follows that over $\mathbb{F}_{3^i}$ the form is not non-degenerate.

It is straightforward to show that when $v = (00001001000000)$ then $vb_{14}w^{\mathrm{T}} = 0$ for all $w \in \mathbb{F}_{3^i}^{14}$. Hence b_{14} induces a non-degenerate form on the quotient module $\mathbb{F}_{3^i}^{14}/\langle v \rangle$ by Lemma 3.3.3. From this it follows that $\mathrm{Sp}_3(3^i)$

preserves a 13-dimensional orthogonal form by Lemma 3.3.3. Furthermore, we can find a d_{13} induced by δ_G with determinant 1 and spinor norm -1.

Finally, $-I_6\rho_{13} = I_{13}$ which implies that the kernel of the representation is $\langle \pm I_6 \rangle$ and therefore the image of the representation ρ_{13} is isomorphic to $\mathrm{S}_6(3^i)$. Since there is only one conjugacy class of $G\rho_{13}$ in $\Omega^{\circ}_{13}(3^i, b_{13})$ by Lemma 8.1.23 and Lemma 4.3.3, it follows that this class is stabilised by $\phi_{\Omega^{\circ}_{13}(3^i)}$. □

Proposition 9.3.2 ($\Omega^{\circ}_5(5^i)$)**.**
There is a unique conjugacy class of $\mathscr{S}_2$-subgroups of $\Omega^{\circ}_{13}(5^i)$ isomorphic to $\Omega^{\circ}_5(5^i)$, with class stabiliser $\langle \delta, \phi \rangle$ in $\mathrm{Out}(\mathrm{O}^{\circ}_{13}(5^i))$.

Proof. Let $G = \Omega^{\circ}_5(5^i)$. It is straightforward to check that G has a 14-dimensional representation ρ_{14} acting on a submodule of $\mathrm{S}^2(V_5)$ (see file s2o5comp). We find that $G\rho_{14}$ preserves the bilinear form $b_{14} = c + c^{\mathrm{T}}$, where $c = \frac{3}{2}(E_{5,5} + E_{8,8}) + 2(E_{1,14} + E_{5,8} + E_{6,12}) + E_{2,13} + E_{4,9} + \frac{1}{2}(E_{3,11} + E_{7,10})$ with determinant 5. Hence b_{14} is degenerate in characteristic 5. We can show that b_{14} induces a non-degenerate form on the 13-dimensional quotient space $\mathbb{F}_{5^i}^{14}/\langle(00001001000000)\rangle$ from which it follows that $G\rho_{13}$ preserves an orthogonal form by Lemma 3.3.3. Furthermore, we can show that d_{13} induced by δ_G has determinant 1 and spinor norm -1 by Lemma 3.1.19. Finally, there is a single conjugacy class of G in $\Omega^{\circ}_{13}(5^i, b_{13})$ by Lemma 8.1.23 and Lemma 4.3.3 and hence this conjugacy class has to be stabilised by $\phi_{\Omega^{\circ}_{13}(5^i)}$.

Finally, we have to show that ρ_{13} is indeed absolutely irreducible. By [29], there exists a 13-dimensional absolutely irreducible representation τ of $\Omega^{\circ}_5(5^i)$ with weight $(0, 2)$ which implies that $G\tau$ acts absolutely irreducible on a subquotient of $V_5 \otimes V_5$. Since ρ_{13} acts on a subquotient of $V_5 \otimes V_5$ as well, we know that ρ_{13} is equivalent to τ and hence absolutely irreducible. □

9.3.2 Dimension 14

Proposition 9.3.3 ($\mathrm{Sp}_6(q)_1$ in characteristic $p \neq 2, 3$)**.**
Let ρ_{14} be a representation of $G = \mathrm{Sp}_6(q)$ with highest weight $(0, 1, 0)$ and let $G\rho_{14} = \mathrm{S}_6(q)_1$.

- *(i) If $p \equiv 1 \pmod{12}$ then $\Omega^{+}_{14}(p^i)$ has 4 conjugacy classes of $\mathscr{S}_2$-subgroups ot type $\mathrm{S}_6(p^i)_1$, with class stabiliser $\langle \delta', \phi \rangle$ in $\mathrm{Out}(\mathrm{O}^{+}_{14}(p^i))$.*
- *(ii) If $p \equiv 7 \pmod{12}$ and i is odd, then $\Omega^{+}_{14}(p^i)$ has 4 conjugacy classes of $\mathscr{S}_2$-subgroups of type $\mathrm{PCSp}_6(p^i)_1$, which have class stabiliser $\langle \phi \rangle$ in*

$\mathrm{Out}(\mathrm{O}^+_{14}(p^i))$. *If i is even, then $\Omega^+_{14}(p^i)$ has 4 conjugacy classes of $\mathscr{S}_2$-subgroups of type* $\mathrm{S}_6(p^i)_1$, *with class stabiliser* $\langle \delta', \phi \rangle$ *in* $\mathrm{Out}(\mathrm{O}^+_{14}(p^i))$.

(iii) If $p \equiv 5 \pmod{12}$ and i is odd, then $\Omega^-_{14}(p^i)$ has 4 conjugacy classes of $\mathscr{S}_2$-subgroups of type $\mathrm{PCSp}_6(p^i)_1$, *which have class stabiliser* $\langle \phi \rangle$ *in* $\mathrm{Out}(\mathrm{O}^-_{14}(p^i))$. *If i is even, then $\Omega^+_{14}(p^i)$ has 4 conjugacy classes of $\mathscr{S}_2$-subgroups of type* $\mathrm{S}_6(p^i)_1$, *with class stabiliser* $\langle \delta', \phi\gamma \rangle$ *in* $\mathrm{Out}(\mathrm{O}^+_{14}(p^i))$.

(iv) If $p \equiv 11 \pmod{12}$ and i is odd, then $\Omega^-_{14}(p^i)$ has 4 conjugacy classes of $\mathscr{S}_2$-subgroups of type $\mathrm{S}_6(p^i)_1$, *which have class stabiliser* $\langle \delta', \phi \rangle$ *in* $\mathrm{Out}(\Omega^-_{14}(p^i))$. *If i is even, then $\Omega^+_{14}(p^i)$ has 4 conjugacy classes of $\mathscr{S}_2$-subgroups of type* $\mathrm{S}_6(p^i)_1$, *with class stabiliser* $\langle \delta', \phi\gamma \rangle$ *in* $\mathrm{Out}(\mathrm{O}^+_{14}(p^i))$.

Proof. Let $\Omega = \Omega^\pm_{14}(q)$. By Table 9.1.1, $\mathrm{Sp}_6(q)_1$ is only defined for characteristic $\geqslant 5$ since we treat the characteristic 2 case separately. By [26, Table 5.4.A, p.199] there exists a 14-dimensional irreducible representation ρ_{14} of G acting on a submodule of $\Lambda^2(V_6)$. By Proposition 9.3.1 the form matrix preserved by $G\rho_{14}$ is $b_{14} = c + c^{\mathrm{T}}$, where $c = E_{5,5} + E_{8,8} + E_{3,11} + E_{4,9} + E_{5,8} + E_{7,10} - (E_{1,14} + E_{2,13} + E_{6,12})$. This is an orthogonal form matrix with $\det(b_{14}) = 3$. Hence, by Lemma 3.1.13, $\mathrm{Sp}_6(q)_1$ preserves a form of plus-type if and only if $q \equiv 1, 7 \pmod{12}$.

We can also show that $-I_6\rho_{15} = I_{15}$ corresponds to $I_{14} \in \Omega^\pm_{14}(q)$ in dimension 14. This implies that the kernel of ρ_{14} consists of the elements $\pm I_6$ and hence $G\rho_{14}$ is isomorphic to $\mathrm{S}_6(q)_1$.

Calculations show that $\det(d_{14}) = \omega^{14}$ for some primitive element $\omega \in \mathbb{F}_q^\times$. Hence $\frac{1}{\omega}d_{14}$ has determinant 1 and preserves b_{14}. Using Lemma 3.1.19 we find that $\frac{1}{\omega}d_{14}$ has spinor norm 1 if and only if $(\omega^{-1} - 1)^6\omega$ is square in $\mathbb{F}_q^\times$ (see file s2s61comp). Since this is never the case in odd characteristic, the spinor norm of $\frac{1}{\omega}d_{14}$ is always -1. Hence $\langle \mathrm{S}_6(q), \frac{1}{\omega}d_{14} \rangle \leqslant \mathrm{SO}^\pm_{14}(q, b_{14})$.

If $\det(b_{14}) = 3$ is square in $\mathbb{F}_q$, then $\frac{1}{\omega}d_{14}$ is induced by δ'_Ω by the definition of δ'. If 3 is not a square in $\mathbb{F}_q$ then $-\frac{1}{\omega}d_{14}$ has spinor norm 1 by Lemma 3.1.21. Therefore $-\frac{1}{\omega}d_{14} \in \Omega^\pm_{14}(q, b_{14})$ in this case. Note however that $-\frac{1}{\omega}d_{14} \notin \Omega^\pm_{14}(p^{2i}, b_{14})$ for any $i \geqslant 1$. The number of conjugacy classes follows from Lemma 8.1.23 and Lemma 4.3.3.

Let $K = \Omega^\pm_{14}(p^i, b_{14})$. It is straightforward to show that $b_{14}^{\phi_K} = b_{14}$, $h_{14}^{\phi_K} = h_{14}$ and $g_{14}^{\phi_K} = g_{14}^p$. Hence $G\rho_{14} \leqslant K$ is stabilised by ϕ_K. By Lemma 4.8.1, there exists $x \in \mathrm{GL}_{14}(p^i)$ such that $(G\rho_{14})^x \leqslant \Omega^\pm_{14}(p^i)$ preserves our respective standard form matrices. It follows from Lemma 8.5.3(ii) that $(G\rho_{14})^x$ is stabilised by $(\phi\gamma)_\Omega$ if and only if i is even and -3 is not a square in $\mathbb{F}_p^\times$ and by ϕ_Ω otherwise. By Table 2.2.1, -3 is not a square if and only if $p \equiv 5, 11 \pmod{12}$. $\square$

Proposition 9.3.4 ($\mathrm{Sp}_6(2^i)_1$ in characteristic 2)**.**
Let ρ_{14} be a representation of $G = \mathrm{Sp}_6(2^i)$ with highest weight $(0,1,0)$ and let $G\rho_{14} = \mathrm{Sp}_6(2^i)_1$.

If i is odd then there exist two conjugacy classes of $\mathscr{S}_2$-subgroups of $\Omega^-_{14}(p^i)$ of type $\mathrm{Sp}_6(2^i)_1$, with class stabiliser $\langle\phi\rangle$ in $\mathrm{Out}(\mathrm{O}^-_{14}(2^i))$. If i is even then there exist two conjugacy classes of $\mathscr{S}_1$-subgroups of $\Omega^+_{14}(2^i)$ of type $\mathrm{Sp}_6(2^i)_1$, with class stabiliser $\langle\phi\gamma\rangle$ in $\mathrm{Out}(\mathrm{O}^+_{14}(2^i))$.

Proof. Let $\Omega = \Omega^{\pm}_{14}(2^i)$. Using a similar method as for $\mathrm{Sp}_6(q)_1$, $p \neq 2,3$, we can show that $\mathrm{Sp}_6(2^i)_1$ acts on a submodule of $\Lambda^2(V_6)$ and preserves the symmetric bilinear form matrix $b_{14} := c + c^{\mathrm{T}}$, where $c = E_{1,14} + E_{2,13} + E_{3,11} + E_{4,9} + E_{5,8} + E_{6,12} + E_{7,10}$ (see file s2sp61ch2comp). This form implies that our 14-dimensional vector space V can be written as an orthogonal sum ([36, p.56]) as follows:

$$\begin{aligned} V :=& \langle e_1 \wedge e_2, e_5 \wedge e_6\rangle \perp \langle e_1 \wedge e_3, e_4 \wedge e_6\rangle \perp \langle e_1 \wedge e_4, e_3 \wedge e_6\rangle \\ &\perp \langle e_1 \wedge e_5, e_2 \wedge e_6\rangle \perp \langle e_2 \wedge e_3, e_4 \wedge e_5\rangle \perp \langle e_2 \wedge e_4, e_3 \wedge e_5\rangle \\ &\perp \langle e_1 \wedge e_6 + e_3 \wedge e_4, e_2 \wedge e_5 + e_3 \wedge e_4\rangle, \end{aligned}$$

where $\{e_i \wedge e_j \mid 1 \leqslant i < j \leqslant 6\}$ is a basis of $\Lambda^2(V_6)$ ordered lexicographically.

To find the sign of the orthogonal form, we have to determine the number of hyperbolic lines. Note that there are at least 6 hyperbolic lines. Let Q be the quadratic form associated with b_{14} and let $\langle a, b\rangle$ be any summand of the orthogonal sum above. If $Q(a + \alpha b) = 0$ for some $\alpha \in \mathbb{F}_{2^i}$ then, by [36, Thm 7.3, p.56], $\langle a, b\rangle$ forms a hyperbolic line. However, using Magma we can show that $\mathrm{Sp}_6(2)$ preserves an orthogonal form of minus-type. Hence, there exists a summand $\langle a, b\rangle$ which does not contain any singular vectors in $\mathbb{F}_2$. Without loss of generality we can then assume that $Q(a) = Q(b) = 1$ which holds in all field extensions of $\mathbb{F}_2$. Since $b_{14}(a, b) = 1$, we can use [8, Prop 1.5.42(iii), p.24] to show that the quadratic form on this 2-dimensional subspace is of minus-type if and only if $x^2 + x + 1$ is irreducible in the respective field.

By [28, Cor 3.47, p.100], we find that an irreducible polynomial of degree n over a field $\mathbb{F}_q$ remains irreducible over any extension $\mathbb{F}_{q^k}$ of $\mathbb{F}_q$ if and only if k and n are coprime. From this it follows that $x^2 + x + 1$ is irreducible in $\mathbb{F}_{2^i}$ for all odd i. Hence $\mathrm{Sp}_6(2^i)$ preserves an orthogonal form of plus-type in dimension 14 if and only if i is even.

Let $K = \Omega^{\pm}_{14}(2^i, Q)$. It is clear that ϕ_K stabilises $\mathrm{Sp}_6(2^i)_1$ since $b_{14}^{\phi_K} = b_{14}$, $h_{14}^{\phi_K} = h_{14}$ and $g_{14}^{\phi_K} = g_{14}^2$. However, b_{14} is not our standard form. We can find $x \in \mathrm{GL}_{14}(4)$ such that $(\mathrm{Sp}_6(4)\rho_{14})^x$ preserves our standard form (see file s2sp61ch2comp). We can show that $(x^{-1})^{\phi}x$ has quasideterminant -1 and hence it follows from Lemma 8.5.3 that $\mathrm{Sp}_6(4)_1$ is stabilised by $(\gamma\phi)_\Omega$. Since $\mathrm{Sp}_6(2^i)_1$ contains $\mathrm{Sp}_6(4)_1$ as a subgroup when i is even it follows that $\mathrm{Sp}_6(2^i)_1$ is stabilised by $(\phi\gamma)_\Omega$ for all even i. If i is odd then $\mathrm{Sp}_6(2^i)_1$ is stabilised by ϕ_Ω since there is only one isomorphism class of groups $\langle \Omega^-_{14}(2^i, B), \phi\rangle$ by [7], where B is a non-degenerate quadratic form. $\square$

Proposition 9.3.5 ($\mathrm{Sp}_6(q)_2$)**.**
Let ρ_{14} be a representation of $G = \mathrm{Sp}_6(q)$ with highest weight $(1,0,0)$ and let $\mathrm{Sp}_6(q)_2 = G\rho_{14}$.

For p odd there exists one conjugacy class of $\mathscr{S}_2$-subgroups of $\mathrm{Sp}_{14}(q)$ of type $\mathrm{Sp}_6(q)_2$ which has class stabiliser $\langle\delta,\phi\rangle$ in $\mathrm{Out}(\mathrm{S}_{14}(q))$.

Proof. There exists a 20-dimensional representation of G that acts on $\Lambda^3(V_6)$ and, by [26, Table 5.4.A, p.199], there also exists a 14-dimensional irreducible representation that acts on a submodule of $\Lambda^3(V_6)$. Using a similar method as for $\mathrm{Sp}_6(q)_1$ we can show (see file s2sp62comp) that $\mathrm{Sp}_6(q)_2$ preserves the symplectic form $b_{14} = c - c^T$, where $c = E_{2,13} + E_{5,11} - (E_{1,14} + E_{7,10}) + 2(E_{3,12} + E_{4,9} + E_{6,8})$. Furthermore, the diagonal automorphism δ_G of G induces $d_{14} = \mathrm{diag}(\omega^3, \omega^2, \omega^2, \omega^2, \omega^2, \omega^2, \omega, \omega, \omega, \omega^2, \omega, \omega, \omega, 1)$ with determinant ω^{21}, where ω is a primitive element of $\mathbb{F}_q^\times$. From this it follows that projectively d_{14} has determinant ω^7 and preserves b_{14} up to multiplication by ω. Furthermore, the number of $\Omega = \mathrm{Sp}_{14}(q)$ conjugacy classes can be determined using Lemma 8.1.23 and Lemma 4.3.3. We can show that there is only one conjugacy class which has to be stabilised by ϕ_Ω. Finally, since $-I_6\rho_{14} = -I_{14}$, the kernel of ρ_{14} is trivial and hence $\mathrm{Sp}_6(q)_2 \leqslant \mathrm{Sp}_{14}(q)$. $\square$

Proposition 9.3.6 ($\Omega_5^\circ(q)$)**.**

(i) *If $p \equiv 1, 9 \pmod{20}$ then there are 4 conjugacy classes of $\mathscr{S}_2$-subgroups of $\Omega^+_{14}(p^i)$ isomorphic to $\Omega^\circ_5(p^i)$, which have class stabiliser $\langle\delta',\phi\rangle$ in $\mathrm{Out}(\mathrm{O}^+_{14}(p^i))$.*

(ii) *Let $p \equiv 3, 7 \pmod{20}$. If i is odd then there are 4 conjugacy classes of $\mathscr{S}_2$-subgroups of $\Omega^+_{14}(p^i)$ isomorphic to $\mathrm{SO}^\circ_5(p^i)$, with class stabiliser $\langle\phi\rangle$ in $\mathrm{Out}(\mathrm{O}^+_{14}(p^i))$. If i is even then there are 4 conjugacy classes of $\mathscr{S}_2$-subgroups of $\Omega^+_{14}(p^i)$ isomorphic to $\Omega^\circ_5(p^i)$, with class stabiliser $\langle\delta',\phi\rangle$ in $\mathrm{Out}(\mathrm{O}^+_{14}(p^i))$.*

(iii) *Let $p \equiv 13, 17 \pmod{20}$. If i is odd then there are 4 conjugacy classes of $\mathscr{S}_2$-subgroups of $\Omega^-_{14}(p^i)$ isomorphic to $\mathrm{SO}^\circ_5(p^i)$, with class stabiliser*

$\langle \phi \rangle$ in $\mathrm{Out}(\mathrm{O}_{14}^{-}(p^i))$. If i is even then there are 4 conjugacy classes of $\mathscr{S}_2$-subgroups of $\Omega_{14}^{+}(p^i)$ isomorphic to $\Omega_5^{\circ}(p^i)$, with class stabiliser $\langle \delta', \phi\gamma \rangle$ in $\mathrm{Out}(\mathrm{O}_{14}^{+}(p^i))$.

(iv) Let $p \equiv 11, 19 \pmod{20}$. If i is odd then there are 4 conjugacy classes of $\mathscr{S}_2$-subgroups of $\Omega_{14}^{-}(p^i)$ isomorphic to $\Omega_5^{\circ}(p^i)$, with class stabiliser $\langle \delta', \phi \rangle$ in $\mathrm{Out}(\mathrm{O}_{14}^{-}(p^i))$. If i is even then there are 4 conjugacy classes of $\mathscr{S}_2$-subgroups of $\Omega_{14}^{+}(p^i)$ isomorphic to $\Omega_5^{\circ}(p^i)$, with class stabiliser $\langle \delta', \phi\gamma \rangle$ in $\mathrm{Out}(\mathrm{O}_{14}^{+}(p^i))$.

Proof. Let $G = \Omega_5^{\circ}(q)$, where $p \neq 2, 5$ and let $\Omega = \Omega_{14}^{\pm}(q)$. It is straightforward to show that G has a 14-dimensional representation ρ_{14} acting on a submodule of $\mathrm{S}^2(V_5)$ (see file s2o5comp). We need to show that ρ_{14} is irreducible though. Since G has weight $(0, 2)$ in dimension 14 we know, by Lemma 8.1.19, that G acts irreducibly on a subquotient of $V_5 \otimes V_5$. Since $\mathrm{S}^2(V_5)$ is a submodule of $V_5 \otimes V_5$, we know that the 14-dimensional representation we have found must be absolutely irreducible.

We can show that $G\rho_{14}$ preserves the bilinear form b_{14} with determinant 5 as in Proposition 9.3.2. Hence the type of the preserved orthogonal form follows from Lemma 3.1.13 and Table 2.2.1.

Furthermore, we know that $\langle G, \delta_G \rangle \rho_{14} \leqslant \Omega_{14}^{\pm}(q, b_{14})$ if and only if $d_{14} = \mathrm{diag}(\omega, 1, 1, 1, \omega^{-1})\rho_{14}$ is an element of $\Omega_{14}^{\pm}(q)$, where ω is a primitive element of $\mathbb{F}_q^{\times}$. A straightforward calculation shows that

$$d_{14} = \mathrm{diag}(\omega^2, \omega, \omega, \omega, 1, 1, 1, 1, \omega^{-1}, 1, \omega^{-1}, 1, \omega^{-1}, \omega^{-2})$$

with $\det(d_{14}) = 1$ and that d_{14} preserves b_{14}. Furthermore, using Lemma 3.1.19, d_{14} has spinor norm 1 if and only if $\omega^5(1-\omega^{-2})^2(1-\omega^{-1})^6$ is square in $\mathbb{F}_q^{\times}$ which can never be the case.

By the definition of δ'_{Ω}, d_{14} is induced by $\delta'_{\Omega} \in \mathrm{Out}(\mathrm{O}_{14}^{\pm}(q))$ if 5 is a square in $\mathbb{F}_q^{\times}$. When 5 is not square, we can show using Lemma 3.1.21 that $-d_{14} \in \Omega_{14}^{\pm}(q, b_{14})$, where $q = p^i$ with i odd. Hence $\langle G\rho_{14}, -d_{14} \rangle \leqslant \Omega_{14}^{\pm}(p^i, b_{14})$ in this case.

It is straightforward to show that $\phi_{\Omega_{14}^{\pm}(q,b_{14})}$ stabilises $G\rho_{14}$. Let $H \cong G\rho_{14}$ preserve our standard form matrices. We can use Lemma 8.5.3 to show when H is stabilised by ϕ_{Ω} or by $(\phi\gamma)_{\Omega}$. □

Proposition 9.3.7 ($\mathrm{G}_2(q)$)**.**

(i) *No extension of* $\mathrm{G}_2(2^i)$ *is* $\mathscr{S}_2$*-maximal in any extension of* $\Omega^{\pm}_{14}(2^i)$*.*

(ii) *If* $p \equiv 1 \pmod{12}$ *then there are* 8 *conjugacy classes of* $\mathscr{S}_2$*-subgroups of* $\Omega^+_{14}(p^i)$ *isomorphic to* $\mathrm{G}_2(p^i)$*, with class stabiliser* $\langle\phi\rangle$ *in* $\mathrm{Out}(\mathrm{O}^+_{14}(p^i))$*.*

(iii) *If* $p \equiv 7 \pmod{12}$ *and* i *is odd, then there exist* 4 *conjugacy classes of* $\mathscr{S}_2$*-subgroups of* $\Omega^+_{14}(p^i)$ *isomorphic to* $\mathrm{G}_2(p^i)$*, with class stabiliser* $\langle\phi\rangle$ *in* $\mathrm{Out}(\mathrm{O}^+_{14}(p^i))$*. If* i *is even then there exist* 8 *such conjugacy classes, with class stabiliser* $\langle\phi\rangle$ *in* $\mathrm{Out}(\mathrm{O}^+_{14}(p^i))$*.*

(iv) *If* $p \equiv 5 \pmod{12}$ *and* i *is odd, then there exist* 4 *conjugacy classes of* $\mathscr{S}_2$*-subgroups of* $\Omega^-_{14}(p^i)$ *isomorphic to* $\mathrm{G}_2(p^i)$*, with class stabiliser* $\langle\phi\rangle$ *in* $\mathrm{Out}(\mathrm{O}^-_{14}(p^i))$*. If* i *is even then there exist* 8 *conjugacy classes of* $\mathscr{S}_2$*-subgroups of* $\Omega^+_{14}(p^i)$ *isomorphic to* $\mathrm{G}_2(p^i)$*, with class stabiliser* $\langle\phi\gamma\rangle$ *in* $\mathrm{Out}(\mathrm{O}^+_{14}(p^i))$*.*

(v) *If* $p \equiv 11 \pmod{12}$ *and* i *is odd, then there exist* 8 *conjugacy classes of* $\mathscr{S}_2$*-subgroups of* $\Omega^-_{14}(p^i)$ *isomorphic to* $\mathrm{G}_2(p^i)$*, with class stabiliser* $\langle\phi\rangle$ *in* $\mathrm{Out}(\mathrm{O}^-_{14}(p^i))$*. If* i *is even then there exist* 8 *conjugacy classes of* $\mathscr{S}_2$*-subgroups of* $\Omega^+_{14}(p^i)$ *isomorphic to* $\mathrm{G}_2(p^i)$*, with class stabiliser* $\langle\phi\gamma\rangle$ *in* $\mathrm{Out}(\mathrm{O}^+_{14}(p^i))$*.*

Proof. We will first consider the even characteristic case. By [8, Prop 5.7.1, p.305], $\mathrm{G}_2(2^i)$ is an $\mathscr{S}_2$-subgroup of $\mathrm{Sp}_6(2^i)$. Furthermore, there exists a 14-dimensional absolutely irreducible representation ρ of $\mathrm{Sp}_6(2^i)$ acting on a submodule of $\Lambda^2(V_6)$ by Proposition 9.3.4. Using Magma, we can show that the 14-dimensional representation of $\mathrm{G}_2(2)$ acting on a submodule of $\Lambda^2(V_6)$ is absolutely irreducible. Hence the image of the 14-dimensional representation of $\mathrm{G}_2(2^i)$ is absolutely irreducible for all i and therefore is an $\mathscr{S}_2$-subgroup of $\mathrm{Sp}_6(2^i)\rho$. Furthermore, $\mathrm{G}_2(2^i).\langle\phi_{\mathrm{Sp}_6(2^i)}\rangle \leqslant \mathrm{Sp}_6(2^i).\langle\phi_{\mathrm{Sp}_6(2^i)}\rangle$ and $\mathrm{Sp}_6(2^i).\langle\phi_{\mathrm{Sp}_6(2^i)}\rangle \leqslant \Omega^{\pm}_{14}(2^i).\langle\beta\rangle$, where $\beta \in \{\phi_{\Omega^{\pm}_{14}(2^i)}, (\phi\gamma)_{\Omega^{\pm}_{14}(2^i)}\}$ by Proposition 9.3.4. It follows that no extension of $\mathrm{G}_2(2^i)\rho$ is $\mathscr{S}_2$-maximal.

Now let $G = \mathrm{G}_2(q)$, where $p \neq 2, 3$ and let $\Omega = \Omega^{\pm}_{14}(q)$. Then there exists a 7-dimensional representation of G which is absolutely irreducible by [22] and there exists a 21-dimensional representation of G acting on $\Lambda^2(V_7)$. Furthermore, we can find a 14-dimensional representation ρ_{14} of G acting on a submodule of $\Lambda^2(V_7)$ (see s2g2comp). To show that ρ_{14} is absolutely irreducible we will determine its highest weight. Note that the results in Lemma 8.2.7 also apply to the finite group G by Theorem 8.1.22.

Let B be a Borel subgroup of the natural representation of $\mathrm{G}_2(K)$ over an algebraically closed field K and let

$$T = \langle \mathrm{diag}(a^2b, ab, a, 1, a^{-1}, a^{-1}b^{-1}, a^{-2}b^{-1}) \mid a, b \in K^\times \rangle$$

be a maximal torus of $\mathrm{G}_2(K)$ as given in Lemma 8.2.7. Then we can choose B such that $B\rho_{14}$ consists of lower triangular matrices and stabilises the subspace $\langle v^+ \rangle = \langle (1, 0, \ldots, 0) \rangle$. Furthermore,

$$T\rho_{14} = \langle \mathrm{diag}(a^3b^2, a^3b, a^2b, ab, a, 1, b, 1, a^{-1}, b^{-1}, a^{-1}b^{-1}, \\ a^{-2}b^{-1}, a^{-3}b^{-1}, a^{-3}b^{-2}) \rangle.$$

Hence the weight associated with v^+ is a^3b^2 which is dominant by Lemma 8.1.13. Furthermore there exists an absolutely irreducible representation of $\mathrm{G}_2(K)$ with a^3b^2 as its highest weight by Lemma 8.1.18. Note that $a^3b^2 = \lambda_2$, where λ_2 is one of the fundamental dominant weights of $\mathrm{G}_2(K)$ by Lemma 8.2.7. By [29] the irreducible representation of $\mathrm{G}_2(K)$ with highest weight $(1, 0)$ has dimension 14 and hence $\mathrm{G}_2(q)\rho_{14}$ is absolutely irreducible using Lemma 8.1.22.

We find that $G\rho_{14}$ preserves the orthogonal form $b_{14} = c + c^{\mathrm{T}}$, where $c = E_{1,14} + E_{7,10} - (E_{2,13} + E_{6,6} + E_{6,8} + E_{8,8}) + 3(E_{5,9} - E_{4,11}) + \frac{3}{4}E_{3,12}$. It follows that $\det(b_{14}) = 2^{-4}3^7$ and hence G preserves an orthogonal form of plus-type if either $p \equiv 1, 7 \pmod{12}$ or ($p \equiv 5, 11 \pmod{12}$ and $q = p^{2i}$) and a form of minus-type otherwise by Lemma 3.1.13. The number of conjugacy classes follows from Lemma 8.1.23 and Lemma 4.3.3. It is straightforward to show that $\phi_{\Omega^{\pm}_{14}(q, b_{14})}$ stabilises $G\rho_{14}$. Let $H \cong G\rho_{14}$ preserve our standard form matrices. We can use Lemma 8.5.3 to show whether H is stabilised by ϕ_Ω or $(\phi\gamma)_\Omega$. ☐

9.3.3 Dimension 15

Proposition 9.3.8 ($\mathrm{SL}^{\pm}_3(q)_1$).
Let ρ_{15} be an absolutely irreducible representation of $G = \mathrm{SL}^{\pm}_3(q)$ with highest weight $(0, 4)$ and let $G\rho_{15} = \mathrm{SL}^{\pm}_3(q)_1$.

(i) If $p \geqslant 5$, then there are exactly $t = (5, q-1)$ conjugacy classes of $\mathscr{S}_2$-subgroups of $\mathrm{SL}_{15}(q)$ of type $\mathrm{SL}_3(q)_1$, with class stabiliser $\langle \delta^t, \gamma, \phi \rangle$ in $\mathrm{Out}(\mathrm{L}_{15}(q))$.

(ii) If $p \geqslant 5$, then there are exactly $t = (5, q+1)$ conjugacy classes of $\mathscr{S}_2$-subgroups of $\mathrm{SU}_{15}(q)$ of type $\mathrm{SU}_3(q)_1$, with class stabiliser $\langle \delta^t, \phi \rangle$ in $\mathrm{Out}(\mathrm{U}_{15}(q))$.

Proof. Let $\Omega = \mathrm{SL}^{\pm}_{15}(q)$. We can find a 15-dimensional representation ρ of G such that $G\rho$ acts on the symmetric power module $\mathrm{S}^4(V_3)$ (see file s2sl31comp in the linear and file s2su31comp in the unitary case). We need to show that ρ is absolutely irreducible and hence equivalent to ρ_{15}. Note that $G\rho$ acting on $\mathrm{S}^4(V_3)$ has weight $(0,4)$ since it is a submodule of $V_3 \otimes V_3 \otimes V_3 \otimes V_3$. Furthermore, ρ_{15} has weight $(0,4)$. Hence the representation ρ we have found is indeed equivalent to ρ_{15}. It follows that $p \geqslant 5$ and that the module is not self-dual. In particular that implies that $G\rho_{15}$ preserves either only the zero or an Hermitian form.

If $G = \mathrm{SL}_3(q)$ and $q = p^e$ with e odd, then $G\rho_{15}$ preserves only the zero form. So suppose now that e is even and let $\tau_\Omega = \phi_\Omega^{e/2}$. We want to show that $\tau_\Omega \neq \gamma_\Omega$ and hence that $\mathrm{SL}_3(p^e)$ never preserves a unitary form in dimension 15 when e is even. By [8, Prop 5.1.9, p.272], $\phi_\Omega^{e/2}$ sends $(0,4)$ to $(0, 4p^{e/2})$, whereas γ_Ω sends $(0,4)$ to $(4,0)$. Hence $\mathrm{SL}_3(q)\rho_{15}$ preserves only the zero form.

If $G = \mathrm{SU}_3(q)$ then we can show using Magma that $G\rho_{15}$ preserves a unitary form (see file s2su31comp). Note that the G we have chosen for our computer calculations preserves the unitary form $\mathrm{antidiag}(1,1,1)$ as given in Magma. Since all isometry groups of non-degenerate unitary forms are conjugate it does not matter which unitary form G preserves.

Now consider the kernel of ρ_{15} and let $\lambda I_3 \in \mathrm{Z}(G)$. Then $\lambda I_3 \rho_{15} = \lambda^4 I_{15}$. Since $\det(\lambda I_3) = 1$ it follows that $\lambda \in \{1, \mathrm{z}_3, \mathrm{z}_3^{-1}\}$ and so the only possibility for λ^4 to equal 1 is when $\lambda = 1$. Hence $\ker(\rho_{15}) = 1$ and ρ_{15} is a faithful representation of G. Let $d_3 = \mathrm{diag}(\omega, 1, 1)$ in Case **L** and let $d_3 = \mathrm{diag}(\omega^{-1}, 1, \omega^q)$ in Case **U** induce the diagonal automorphism of G, where ω is a primitive element of $\mathbb{F}_q^\times$ or $\mathbb{F}_{q^2}^\times$ respectively. Then d_{15} has determinant ω^{20} or $\omega^{20(q-1)}$ and projectively d_{15} has determinant ω^5 or $\omega^{5(q-1)}$. We want to show that projectively d_{15}^i never has determinant 1 for any $1 \leqslant i < (q \pm 1, 3) = |\delta_G|$, i.e. we will show that $w^{-1} d_{15}^i$ never has determinant 1.

We will consider Case **L** first. Suppose that $\mu(\omega^{-1} d_{15}^i)$ has determinant 1 for some i and some scalar $\mu \in \mathbb{F}_q^\times$. Then $\mu^{15}\omega^{5i} = 1$ and hence $(\mu^5)^3 = \omega^{-3i}\omega^{-2i}$. From this it follows that ω^{2i} is a cube which is a contradiction unless i is a multiple of 3 or $3 \nmid q-1$. However $i < (q-1, 3) \leqslant 3$ by definition and if $3 \nmid q-1$ then $|\delta_{\mathrm{SL}_3(q)}| = 1$.

Similarly in Case **U** this implies that $\omega^{2i(q-1)}$ is a cube which only holds if 3 divides $q-1$, i.e. $3 \nmid q+1$, since $i < (q+1, 3) \leqslant 3$. If 3 divides $q-1$ however then $(q+1,3) = 1$ and the diagonal automorphism of $\mathrm{SU}_3(q)$ is trivial. Therefore, no non-trivial diagonal automorphism of $\mathrm{SL}^{\pm}_3(q)_1$ is induced by an element of $\mathrm{SL}^{\pm}_{15}(q)$.

Finally, using Lemma 4.3.3 and Lemma 8.1.23, there are $(5, q \mp 1)$ conjugacy classes of G in $\mathrm{SL}^{\pm}_{15}(q)$ respectively. Hence, δ^t_Ω stabilises $G\rho_{15}$. By Lemma 8.5.2, $\mathrm{SL}^{\pm}_3(q)_1$ is stabilised by ϕ_Ω and γ_Ω. $\square$

Proposition 9.3.9 ($\mathrm{SL}^{\pm}_3(q)_2$).
Let ρ_{15} be an absolutely irreducible representation of $G = \mathrm{SL}^{\pm}_3(q)$ with highest weight $(1, 2)$ and let $G\rho_{15} = \mathrm{SL}^{\pm}_3(q)_2$. Let $t = (5, q-1)$ in the linear case and let $t = (5, q+1)$ in the unitary case.

(i) If $p \geqslant 3$, then there are exactly t conjugacy classes of $\mathscr{S}_2$-subgroups of $\mathrm{SL}_{15}(q)$ of type $\mathrm{SL}_3(q)_2$, with class stabiliser $\langle \delta^t, \gamma, \phi \rangle$ in $\mathrm{Out}(\mathrm{L}_{15}(q))$.

(ii) If $p \geqslant 3$, then there are exactly t conjugacy classes of $\mathscr{S}_2$-subgroups of $\mathrm{SU}_{15}(q)$ of type $\mathrm{SU}_3(q)_2$, with class stabiliser $\langle \delta^t, \phi \rangle$ in $\mathrm{Out}(\mathrm{U}_{15}(q))$.

Proof. Let $\Omega = \mathrm{SL}^{\pm}_{15}(q)$. It follows from Lemma 8.1.19 that $\mathrm{SL}^{\pm}_3(q)_2$ acts on a subquotient of the 18-dimensional module $V_3^* \otimes \mathrm{S}^2(V_3)$, where V_3^* denotes the dual module of V_3. Let ρ_{18} be a representation of $\mathrm{SL}^{\pm}_3(q)$ such that ρ_{18} acts on $V_3^* \otimes \mathrm{S}^2(V_3)$.

Since the module of $\mathrm{SL}^{\pm}_3(q)_2$ is not self-dual it carries only the zero or a unitary form. If $\mathrm{SL}_3(q)_2$ preserves a unitary form then $(\mathrm{SL}_3(q)_2)^{\gamma_\Omega} = ((\mathrm{SL}_3(q)_2)^{\phi_\Omega^{e/2}}$, where $q = p^e$. But γ_Ω sends the weight $(1, 2)$ to $(2, 1)$ whereas $\phi_\Omega^{e/2}$ sends $(1, 2)$ to $(p^{e/2}, 2p^{e/2})$ by [8, Prop 5.1.9, p.272]. It follows that $\mathrm{SL}(q)_2$ always preserves no other than the zero form, whereas computer calculations (see file s2su32comp) show that $\mathrm{SU}_3(q)_2$ preserves a unitary form.

Now let $\lambda I_3 \in G$. It is straightforward to show that $\lambda I_3 \rho_{15} = \lambda I_{15}$ and hence $\ker(\rho_{15}) = I_3$ which implies that ρ_{15} is a faithful representation of G.

Note that d_{15} induced by δ_G has determinant ω^5 or $\omega^{5(1-q)}$ in the linear or unitary case respectively. Using the same argument as in Proposition 9.3.8, we can show that d^i_{15} never has determinant 1 for any $1 \leqslant i < (q \mp 1, 3) = |\delta_3|$. Furthermore, the number of conjugacy classes follows from Lemma 8.1.23, Lemma 4.3.3 and the fact that δ^t_Ω stabilises the representation.

Let $\Omega = \mathrm{L}^{\pm}_{15}(q)$. As our final step we have to consider the action of $\gamma_{\mathrm{L}_{15}(q)}$ and ϕ_Ω in $\mathrm{Out}(\Omega)$ on $\mathrm{SL}^{\pm}_3(q)_2$. By Lemma 4.4.2 and Lemma 8.5.1 it is clear that $\gamma_{\mathrm{L}_{15}(q)}$ stabilises $\mathrm{SL}_3(q)_2$. Furthermore, by Lemma 8.5.1, $\rho_{15}^{\phi_{\mathrm{U}_{15}(q)}}$ is equivalent to ${}^{\phi_{\mathrm{U}_3(q)}}\rho_{15}$ and hence it follows from Lemma 4.5.1, that $\phi_{\mathrm{U}_{15}(q)}$ stabilises $\mathrm{SU}_3(q)_2$. Hence it remains to consider the action of $\phi_{\mathrm{L}_{15}(q)}$.

We can show that $\mathrm{SL}_3(q)\rho_{18}$ preserves the 15-dimensional subspace

$$W = \langle f_1 - 2f_{15}, f_2 - f_{17}, f_3 + \tfrac{q-1}{2} f_{18}, f_4, f_5, f_6, f_7, f_8 - f_{15}, f_9, f_{10} - 2f_{17}, \\ f_{11} + \tfrac{q-1}{2} f_{18}, f_{12}, f_{13}, f_{14}, f_{16} \rangle,$$

where the f_i are the 18-dimensional standard basis vectors of $\mathbb{F}_q^{18}$. Now let A denote the 15×18 matrix whose rows are the basis vectors of W as a subspace of $V_3^* \otimes \mathrm{S}^2(V_3)$ (see file s2sl32comp). Then A has entries in $\mathbb{F}_p$ and hence $AA^{\mathrm{T}}D = I_{15}$ for some D implies that $D \in M_{15\times 15}(p)$. Furthermore, using a similar approach as in Lemma 8.5.2 we can show that $g \in \mathrm{SL}_3(q)$ acts on the 15-dimensional subspace as $A(g^* \otimes g_2)A^{\mathrm{T}}D$, where g_2 is the action matrix of g on $\mathrm{S}^2(V_3)$. Hence

$$(A(g^* \otimes g_2)A^{\mathrm{T}}D)^{\phi_{\mathrm{L}_{15}(q)}} = A((g^{\phi_{\mathrm{L}_3(q)}})^* \otimes g_2^{\phi_R})A^{\mathrm{T}}D,$$

where $R = \mathrm{SL}_6(q)$. By [8, Lemma 5.9.1, p.310], the automorphism ϕ_R induces $\phi_{\mathrm{SL}_3(q)}$.

Hence $\mathrm{SL}_3^{\pm}(q)_2$ is stabilised by γ_Ω and ϕ_Ω. □

Proposition 9.3.10 ($\mathrm{SL}_5^{\pm}(q)$)**.**

(i) For odd q, there are $t = (3, q-1)$ conjugacy classes of $\mathscr{S}_2$-subgroups G of $\mathrm{SL}_{15}(q)$ isomorphic to $\mathrm{SL}_5(q)$, with class stabiliser $\langle \delta^t, \gamma, \phi \rangle$ in $\mathrm{Out}(\mathrm{L}_{15}(q))$.

(ii) For odd q, there are $t = (3, q+1)$ conjugacy classes of $\mathscr{S}_2$-subgroups G of $\mathrm{SU}_{15}(q)$ isomorphic to $\mathrm{SU}_5(q)$, with class stabiliser $\langle \delta^t, \phi \rangle$ in $\mathrm{Out}(\mathrm{U}_{15}(q))$.

Proof. Let $\Omega = \mathrm{SL}_{15}^{\pm}(q)$. By [26, Table 5.4.A, p.199] there exists a 15-dimensional representation ρ_{15} of $\mathrm{SL}_5^{\pm}(q)$ such that $\mathrm{SL}_5^{\pm}(q)\rho_{15}$ acts irreducibly on $\mathrm{S}^2(V_5)$. Furthermore, by Table 9.1.1 this representation has weight $(0,0,0,2)$ and is not self-dual by Lemma 8.1.25. Hence the image of ρ_{15} preserves either only the zero or a unitary form. In the case $G = \mathrm{SU}_5(q)$, $G\rho_{15}$ preserves a unitary form by Lemma 8.2.4. If $G = \mathrm{SL}_5(p^e)$ then it is clear that if e is odd, $G\rho_{15}$ cannot preserve a unitary form. Hence assume that e is even. Then it follows from [8, Prop 5.1.9, p.272] that $\phi_\Omega^{e/2} \neq \gamma_\Omega$ and hence $\mathrm{SL}_5(q)\rho_{15}$ never preserves a unitary form.

Note that if $\lambda I_5 \in \mathrm{SL}_5^{\pm}(q)$ then $\lambda^5 = 1$ whereas $(\lambda I_5)\rho_{15} = \lambda^2 I_{15} \in \ker(\rho_{15})$ if and only if $\lambda = \pm 1$. Hence ρ_{15} is a faithful representation of $\mathrm{SL}_5^{\pm}(q)$.

Now consider the case $G = \mathrm{SL}_5(q)$ and let d_5 generate the diagonal automorphisms of G. Then $d_{15} = \mathrm{diag}(\omega^2, \omega, \omega, \omega, \omega, 1, \ldots, 1)$ with determinant ω^6, where ω is primitive in $\mathbb{F}_q^\times$. Suppose that projectively $d_{15}^i \in \mathrm{SL}_{15}(q)$ for some i. This holds if we can find $\mu I_{15} \in \mathrm{SL}_{15}$ such that $\det(\mu d_{15}^i) = 1$ for some $\mu \in \mathbb{F}_q^\times$. Then $\mu^{15} = \omega^{-6i}$ and $(\mu^3)^5 = \omega^{-5i}\omega^{-i}$ which implies that we require ω^i to be a fifth power. This holds either if $5 \mid i$ or if $5 \nmid q-1$. Since $|\delta_G| = (5, q-1)$ it follows that $i \leqslant 4$. Furthermore, if $5 \nmid q-1$ then δ_G is trivial. Hence $d_{15}^i \notin \mathrm{SL}_{15}(q)$ for any $1 \leqslant i \leqslant 4$. Since $\det(d_{15}^i) = \omega^6$, the diagonal automorphism of G is induced by $\delta^6_{\mathrm{L}_{15}(q)} \in \mathrm{Out}(\mathrm{L}_{15}(q))$.

If $G = \mathrm{SU}_5(q)$, then $d_{15} = \mathrm{diag}(\omega^{2(q-1)}, \omega^{q-1}, \omega^{q-1}, \omega^{q-1}, \omega^{q-1}, 1, \ldots, 1)$. If projectively $d_{15}^i \in \mathrm{SU}_{15}(q)$ for some $1 \leqslant i \leqslant 4$, then there exists $\mu \in \mathbb{F}_{q^2}^\times$ such that $\det(\mu d_{15}^i) = \mu^{15}\omega^{6i(q-1)} = 1$. Similarly to the linear case this holds if and only if $\omega^{i(q-1)}$ is a fifth power. Again $i \leqslant 4$ and $\omega = \lambda^5$ for some $\lambda \in \mathbb{F}_{q^2}^\times$ if and only if δ_G is trivial. So assume that $5 \mid (q-1)$ but then $5 \nmid (q+1)$ and hence δ_G is trivial again. It follows that δ_G is induced by $\delta^6_{\mathrm{U}_{15}(q)} \in \mathrm{Out}(\mathrm{U}_{15}(q))$.

Finally, there is one conjugacy class of $\mathrm{SL}_5^\pm(q)$ in the respective conformal group C of $\mathrm{SL}_{15}^\pm(q)$ by Lemma 8.1.23 and each such class splits into $t = |C : \mathrm{N}_C(\mathrm{SL}_5^\pm(q)\rho_{15})|$ classes in $\mathrm{SL}_{15}^\pm(q)$ by Lemma 4.3.3. In the linear case $t = (3, q-1)$ since $|\langle \delta^6_{\mathrm{L}_{15}(q)} \rangle| = \frac{(15,q-1)}{(6,(15,q-1))} = \frac{(15,q-1)}{(3,q-1)}$. Similarly we can show that in the unitary case $t = (3, q+1)$.

The action of γ_Ω and ϕ_Ω follows from Lemma 8.5.2. □

Proposition 9.3.11 ($\mathrm{SL}_6^\pm(q)$).
Let $t_l = (5, q-1)$ and let $t_u = (5, q+1)$.

(i) If $p \geqslant 3$ then there are t_l conjugacy classes of $\mathscr{S}_2$-subgroups G of $\mathrm{SL}_{15}(q)$ isomorphic to $\frac{(q-1,6)}{2}.\mathrm{L}_6(q).2$, with class stabiliser $\langle \delta^{t_l}, \gamma, \phi \rangle$ in $\mathrm{Out}(\mathrm{L}_{15}(q))$.

(ii) If $p \geqslant 3$ then there are t_u conjugacy classes of $\mathscr{S}_2$-subgroups G of $\mathrm{SU}_{15}(3^i)$ isomorphic to $\frac{(q+1,6)}{2}.\mathrm{U}_6(q).2$, with class stabiliser $\langle \delta^{t_u}, \phi \rangle$ in $\mathrm{Out}(\mathrm{U}_{15}(q))$.

(iii) If $p = 2$ then there are t_l conjugacy classes of $\mathscr{S}_2$-subgroups of $\mathrm{SL}_{15}(2^i)$ isomorphic to $\mathrm{SL}_6(2^i)$, with class stabiliser $\langle \delta^{t_l}, \gamma, \phi \rangle$ in $\mathrm{Out}(\mathrm{L}_{15}(2^i))$. There are also t_u conjugacy classes of $\mathscr{S}_2$-subgroups of $\mathrm{SU}_{15}(2^i)$ isomorphic to $\mathrm{SU}_6(2^i)$, with class stabiliser $\langle \delta^{t_u}, \phi \rangle$ in $\mathrm{Out}(\mathrm{U}_{15}(2^i))$.

Proof. Let $\Omega = \mathrm{SL}_{15}^\pm(q)$. By [26, Table 5.4.A, p.199] there exists a 15-dimensional absolutely irreducible representation ρ_{15} of $\mathrm{SL}_6^\pm(q)$ such that

$\mathrm{SL}_6^{\pm}(q)\rho_{15}$ acts on the exterior power $\Lambda^2(V_6)$ and by Table 9.1.1 has weight $(0,0,0,1,0)$. Hence the weight is not self-dual by Lemma 8.1.25 and the image of ρ_{15} preserves either only the zero or a unitary form. It follows by [8, Prop 5.1.9, p.272] and Lemma 8.2.4 that $\mathrm{SU}_6(q)\rho_{15}$ preserves a unitary form whereas $\mathrm{SL}_6(q)\rho_{15}$ preserves no non-zero form.

To find $\ker(\rho_{15})$, let $\lambda I_6 \in \mathrm{SL}_6^{\pm}(q)$, i.e. $\lambda \in \{\pm 1, \mathrm{z}_6^{\pm 1}, \mathrm{z}_3^{\pm 1}\}$. Then $\lambda I_6 \rho_{15} = \lambda^2 I_{15}$ from which it follows that $\lambda I_6 \in \ker(\rho_{15})$ if and only if $\lambda = \pm 1$. Therefore, $\frac{(q\mp 1,6)}{2}.\mathrm{L}_6^{\pm}(q) \leqslant \mathrm{SL}_{15}^{\pm}(q)$ if q is odd and $\mathrm{SL}_6^{\pm}(q) \leqslant \mathrm{SL}_{15}^{\pm}(q)$ if q is even.

We will first assume that q is odd. We want to determine whether $d_6^i\rho_{15} \leqslant \mathrm{SL}_{15}^{\pm}(q)$ for any i. First let $G = \mathrm{SL}_6(q)$. Then δ_G^i is induced by $d_6^i = \mathrm{diag}(\omega^i, 1, 1, 1, 1, 1)$, where ω is a primitive element of $\mathbb{F}_q^{\times}$. Furthermore, $d_{15}^i = \mathrm{diag}(\omega^i, \omega^i, \omega^i, \omega^i, \omega^i, 1, \dots, 1)$ with determinant ω^{5i}. Now suppose that there exists $\mu I_{15} \in \mathrm{GL}_{15}(q)$ such that $\det(\mu d_{15}^i) = 1$. Then $\mu^{15}\omega^{5i} = 1$ and so $(\mu^5)^3 = \omega^{-3i}\omega^{-2i}$. This implies that ω^{2i} needs to be a cube which holds if and only if $3 \mid i$ or $3 \nmid q-1$. We will first consider the case when $3 \nmid q-1$. Then $|\delta_G| = (6, q-1) = 2$ and $d_{15}^i \in \mathrm{SL}_{15}(q)$ for all i. Hence $\mathrm{L}_6(q).\langle\delta_G\rangle \leqslant \mathrm{SL}_{15}(q)$ in this case. If $3 \mid q-1$ then $1 \leqslant i < |\delta_G| = (6, q-1) = 6$. It follows that the cubes of the diagonal automorphisms of $\mathrm{L}_6(q)$ are induced by elements of $\mathrm{SL}_{15}(q)$ and $3.\mathrm{L}_6(q).\langle\delta_G^3\rangle \leqslant \mathrm{SL}_{15}(q)$. Furthermore, since $\det(d_{15}) = \omega^5$ the class stabiliser of $G\rho_{15}$ is induced by $\delta^5_{\mathrm{L}_{15}(q)} \in \mathrm{GL}_{15}(q)$. The number of conjugacy classes follows from Lemma 8.1.23 and Lemma 4.3.3.

Now consider the case $G = \mathrm{SU}_6(q)$. Then δ_G is induced by $d_6 = \mathrm{diag}(\omega^{q-1}, 1, \dots, 1)$, where ω is a primitive element of $\mathbb{F}_{q^2}^{\times}$, and it follows that $d_{15} = \mathrm{diag}(\omega^{q-1}, \omega^{q-1}, \omega^{q-1}, \omega^{q-1}, \omega^{q-1}, 1, \dots, 1)$ with determinant $\omega^{5(q-1)}$. Similarly to the linear case we find that if $3 \nmid q+1$ then $\mathrm{U}_6(q).\langle\delta_6\rangle \leqslant \mathrm{SU}_{15}(q)$ and if $3 \mid q+1$ then $3.\mathrm{U}_6(q).\langle\delta_6^3\rangle \leqslant \mathrm{SU}_{15}(q)$. Furthermore, since $\det(d_{15}) = \omega^{5(q-1)}$ the class stabiliser is generated by $\delta^5_{\mathrm{U}_{15}(q)} \in \mathrm{CGU}_{15}(q)$. The number of conjugacy classes follows from Lemma 4.3.3 and Lemma 8.1.23.

Now let $p = 2$. To consider the diagonal automorphisms, let $G = \mathrm{SL}_6(2^j)$. Assume that there exists $\mu I_{15} \in \mathrm{GL}_{15}(2^j)$ such that $\det(\mu d_{15}^i) = \mu^{15}\omega^{5i} = 1$. Again this implies that we require ω^{2i} to be a cube. First note that $i < 3$ since $|\delta_G| = (6, 2^j - 1) = (3, 2^j - 1)$. Suppose that there exists $\lambda \in \mathbb{F}_{2^j}$ such that $\lambda^3 = \omega^2$. Since all elements are square in characteristic 2, this implies in particular that there exists some $\nu \in \mathbb{F}_{2^j}$ such that $\nu^2 = \lambda$. Hence we want to find ν such that $\nu^3 = \omega$. This is possible if and only if

$(3, 2^j - 1) = 1$ but then $|\delta_G| = 1$. Therefore no diagonal automorphism of G is induced by an element of $\mathrm{SL}_{15}(2^j)$. Let $t = (2^j - 1, 5)$. Then there are t conjugacy classes of G in $\mathrm{GL}_{15}(2^j)$. The normaliser of G in $\mathrm{GL}_{15}(2^j)$ is generated by G itself, $\delta^t_{\mathrm{L}_{15}(q)}$ and scalars.

If $G = \mathrm{SU}_6(2^j)$ then a diagonal automorphism of G is induced by an element of $\mathrm{SU}_{15}(2^j)$ if and only if we can find $\mu \in \mathbb{F}_{2^j}$ such that $\mu^{15} = \omega^{5i(2^j-1)}$ for some i. Again $i < 3$. If $3 \nmid 2^j - 1$ then we can show using a similar argument as in the linear case that it is possible to find $\nu \in \mathbb{F}_{2^{2j}}$ such that $\nu^3 = \omega$ if and only if $(3, 2^j + 1) = 1$. Hence no automorphism of G is induced by an element of $\mathrm{SU}_{15}(2^j)$ in this case. If $3 \mid 2^j - 1$ then $3 \nmid 2^j + 1$ and hence the diagonal automorphism of G has order 1 in this case giving the same result as before. Again δ_G is induced by $\delta^5_{\mathrm{U}_{15}(q)}$.

Finally, by Lemma 8.5.2 we know that ϕ_Ω and γ_Ω stabilise $\mathrm{SL}^{\pm}_6(q)\rho_{15}$. □

9.4 Adjoint Modules

In this section we will consider the representations that act on adjoint modules. The theory behind this was given in Section 8.4.

9.4.1 Dimension 14

Proposition 9.4.1 ($\mathrm{L}^{\pm}_4(2^i)$).

(i) If i is odd there are two conjugacy classes of $\mathscr{S}_2$-subgroups of $\Omega^-_{14}(2^i)$ isomorphic to $\mathrm{SL}_4(2^i).2$, with class stabiliser $\langle\phi\rangle$ in $\mathrm{Out}(\mathrm{O}^-_{14}(2^i))$. If i is even then there are two conjugacy classes of $\mathscr{S}_2$-subgroups of $\Omega^+_{14}(2^i)$ isomorphic to $\mathrm{SL}_4(2^i).2$, with class stabiliser $\langle\phi\rangle$ in $\mathrm{Out}(\mathrm{O}^+_{14}(2^i))$.

(ii) If i is odd there are two conjugacy classes of $\mathscr{S}_2$-subgroups of $\Omega^-_{14}(2^i)$ isomorphic to $\mathrm{SU}_4(2^i).2$, with class stabiliser $\langle\phi\rangle$ in $\mathrm{Out}(\mathrm{O}^-_{14}(2^i))$. If i is even there are two conjugacy classes of $\mathscr{S}_2$-subgroups of $\Omega^+_{14}(2^i)$ isomorphic to $\mathrm{SU}_4(2^i).2$, with class stabiliser $\langle\phi\rangle$ in $\mathrm{Out}(\mathrm{O}^+_{14}(2^i))$.

Proof. Let $G = \mathrm{SL}^{\pm}_4(2^i)$, let $\Omega = \Omega^{\pm}_{14}(2^i, B)$ for some non-degenerate symmetric bilinear form B and let ρ_{14} be the adjoint representation of G. The type of orthogonal form preserved by $G\rho_{14}$ follows from Corollary 8.4.6. By Lemma 8.4.8, the automorphism $\gamma \in \mathrm{Out}(\mathrm{L}^{\pm}_4(2^i))$ is induced by an element of Ω. By Lemma 8.5.1, ϕ_Ω induces either ϕ_G or $(\phi\gamma)_G$. Since γ stabilises $G\rho_{14}$ it follows that $G\rho_{14}$ is always stabilised by ϕ_Ω. □

9.4.2 Dimension 15

Proposition 9.4.2 ($\mathrm{L}_4^{\pm}(q)$ in odd characteristic)**.**

(i) If $p \geqslant 3$ then $\Omega_{15}^{\circ}(q)$ has a single conjugacy class of $\mathscr{S}_2$-subgroups isomorphic to $\mathrm{L}_4(q).2$, with class stabiliser $\langle \delta, \phi \rangle$ in $\mathrm{Out}(\mathrm{O}_{15}^{\circ}(q))$.

(ii) If $p \geqslant 3$ then $\Omega_{15}^{\circ}(q)$ has a single conjugacy classes of $\mathscr{S}_2$-subgroups isomorphic to $\mathrm{U}_4(q).2$, with class stabiliser $\langle \delta, \phi \rangle$ in $\mathrm{Out}(\mathrm{O}_{15}^{\circ}(q))$.

Proof. Let $\Omega = \Omega_{15}^{\circ}(q)$. By [8, Lemma 5.4.10, p.294] there exists a 15-dimensional adjoint representation ρ_{15} of $\mathrm{SL}_4^{\pm}(q)$ in odd characteristic. Then $\mathrm{SL}_4^{\pm}(q)\rho_{15}$ preserves an orthogonal form. By Lemma 8.4.7 the diagonal automorphism of $\mathrm{SL}_4^{\pm}(q)\rho_{15} \cong \mathrm{L}_4^{\pm}(q)$ sits inside $\mathrm{SO}_{15}^{\circ}(q)\backslash\Omega$ and hence is induced by δ_{Ω}.

Furthermore, it follows from Lemma 8.4.8 that the duality automorphism of $\mathrm{SL}_4^{\pm}(q)\rho$ is induced by an element of Ω. By Lemma 8.5.1, no element of $\mathrm{CGO}_{15}^{\circ}(q)$ induces the field automorphism of $\mathrm{SL}_4^{\pm}(q)\rho_{15}$. By Lemma 4.3.3 and Lemma 8.1.23 there is one conjugacy class of $\mathrm{SL}_4^{\pm}(q)\rho_{15}$ in Ω and this class is therefore stabilised by ϕ_{Ω}. $\square$

9.5 $\mathscr{S}_2$-Maximality

In this section we will determine which of the $\mathscr{S}_2$-subgroups in Table 9.1.1 are $\mathscr{S}_2$-maximal in dimension 13, 14 and 15.

Proposition 9.5.1 ($\mathscr{S}_2$-maximal subgroups in dimension 13)**.**

(i) The group $\mathrm{S}_6(3^i)$ is $\mathscr{S}_2$-maximal in $\Omega_{13}^{\circ}(3^i)$.

(ii) The group $\Omega_5^{\circ}(5^i)$ is $\mathscr{S}_2$-maximal in $\Omega_{13}^{\circ}(5^i)$.

(iii) In characteristic $\geqslant 13$ the group $\mathrm{L}_2(q)$ is $\mathscr{S}_2$-maximal in $\Omega_{13}^{\circ}(q)$.

Proof. All the 13-dimensional $\mathscr{S}_2$-subgroups occur in different characteristics. $\square$

Proposition 9.5.2 ($\mathscr{S}_2$-maximal subgroups in dimension 14)**.**

(i) In characteristic $\geqslant 17$ the group $\mathrm{SL}_2(q)$ is $\mathscr{S}_2$-maximal in $\mathrm{Sp}_{14}(q)$.

(ii) In odd characteristic $\mathrm{Sp}_6(q)_2$ is $\mathscr{S}_2$-maximal in $\mathrm{Sp}_{14}(q)$.

(iii) The groups $\mathrm{SL}_4^{\pm}(2^i).2$ are $\mathscr{S}_2$-maximal in $\Omega_{14}^{\pm}(2^i)$.

(iv) If $p \neq 3$ then $\mathrm{N}_{\Omega^{\pm}_{14}(q)}(\mathrm{S}_6(q)_1)$ is $\mathscr{S}_2$-maximal in $\Omega^{\pm}_{14}(q)$.

(v) If $p \neq 2, 5$ then $\mathrm{N}_{\Omega^{\pm}_{14}(q)}(\Omega^{\circ}_5(q))$ is $\mathscr{S}_2$-maximal in $\Omega^{\pm}_{14}(q)$.

(vi) If $p \neq 2, 3$ then $\mathrm{N}_{\Omega^{\pm}_{14}(q)}(\mathrm{G}_2(q))$ is $\mathscr{S}_2$-maximal in $\Omega^{\pm}_{14}(q)$. No extension of $\mathrm{G}_2(2^i)$ is $\mathscr{S}_2$-maximal in any extension of $\Omega^{\pm}_{14}(2^i)$.

Proof. (i) The only (up to equivalence) $\mathscr{S}_2$-subgroup that can contain $\mathrm{SL}_2(q)$ is $\mathrm{Sp}_6(q)_2$. By [8, Section 8.2, p.377] $\mathrm{Sp}_6(q)$ has only one absolutely irreducible subgroup H isomorphic to $\mathrm{SL}_2(q)$. Let ρ_{14} be the absolutely irreducible representation of $\mathrm{Sp}_6(q)$. We are going to calculate the highest weight of $H\rho_{14}$ (see Section 8.1). Note that we can work over $\mathbb{F}_q$ by Lemma 8.1.22.

It is straightforward to calculate the maximal torus T of H and its Borel group B using Lemma 8.2.6. Then

$$T = \langle \mathrm{diag}(\alpha^5, \alpha^3, \alpha, \alpha^{-1}, \alpha^{-3}, \alpha^{-5}) \mid \alpha \in \mathbb{F}_q^{\times} \rangle$$

and we can let B consist of lower triangular matrices. It can be shown that

$$T\rho_{14} = \langle \mathrm{diag}(\alpha^9, \alpha^7, \alpha^5, \alpha^3, \alpha^3, \alpha, \alpha, \alpha^{-1}, \alpha^{-3}, \alpha^{-1}, \alpha^{-3}, \alpha^{-5}, \alpha^{-7}, \alpha^{-9}) \rangle$$

and that $B\rho_{14}$ consists of lower triangular matrices. Hence $B\rho_{14}$ stabilises the subspace $\langle (1, 0, \ldots, 0) \rangle$ with weight $\chi(t) = \alpha^9$ for all $t \in T$. By Table 9.1.1 the 14-dimensional absolutely irreducible representation ρ of $\mathrm{SL}_2(q)$ has highest weight (13). Hence $H\rho_{14}$ is not equivalent to $\mathrm{SL}_2(q)\rho$. It follows that $\mathrm{SL}_2(q)\rho$ is $\mathscr{S}_2$-maximal.

(ii) By Lagrange's theorem $\mathrm{Sp}_6(q)_2$ has to be $\mathscr{S}_2$-maximal.

(iii) By [8, Section 8.2, p.377], $\mathrm{SL}^{\pm}_4(2^i)$ is not a subgroup of $\mathrm{G}_2(2^i)$ or $\mathrm{Sp}_6(2^i)$.

(iv) By Lagrange's theorem $\mathrm{Sp}_6(q)$ is not a subgroup of $\mathrm{G}_2(q)$.

(v) By [8, Section 8.2, p.377], $\Omega^{\circ}_5(q)$ is not a subgroup of $\mathrm{G}_2(q)$ or $\mathrm{Sp}_6(q)$.

(vi) Since the natural representation of $\mathrm{G}_2(q)$ has dimension 7 in odd characteristic, it can not be a subgroup of any of the other $\mathscr{S}_2$-subgroups in dimension 14. In even characteristic the result follows from Proposition 9.3.7. □

Proposition 9.5.3 ($\mathscr{S}_2$-maximal subgroups in dimension 15)**.**

(i) No extension of $\mathrm{N}_{\mathrm{SL}^{\pm}_{15}(q)}(\mathrm{SL}^{\pm}_3(q)_2)$ is $\mathscr{S}_2$-maximal in any extension of $\mathrm{SL}^{\pm}_{15}(q)$.

(ii) If $p \neq 2,3$ then $\mathrm{N}_{\mathrm{SL}^{\pm}_{15}(q)}(\mathrm{SL}^{\pm}_3(q)_1)$ *is* $\mathscr{S}_2$*-maximal in* $\mathrm{SL}^{\pm}_{15}(q)$.

(iii) If $p \neq 2$ then $\mathrm{N}_{\mathrm{SL}^{\pm}_{15}(q)}(\mathrm{SL}^{\pm}_5(q))$ *is* $\mathscr{S}_2$*-maximal in* $\mathrm{SL}^{\pm}_{15}(q)$.

(iv) If $p \geqslant 3$ then $\mathrm{N}_{\mathrm{SL}^{\pm}_{15}(q)}((3, q \mp 1).\mathrm{L}^{\pm}_6(q))$ *is* $\mathscr{S}_2$*-maximal in* $\mathrm{SL}^{\pm}_{15}(q)$.

Furthermore, $\mathrm{N}_{\mathrm{SL}^{\pm}_{15}(2^i)}(\mathrm{SL}^{\pm}_6(2^i))$ *is* $\mathscr{S}_2$*-maximal in* $\mathrm{SL}^{\pm}_{15}(2^i)$.

(v) If $p \geqslant 17$ then $\mathrm{N}_{\Omega^{\circ}_{15}(q)}(\mathrm{L}_2(q))$ *is* $\mathscr{S}_2$*-maximal in* $\Omega^{\circ}_{15}(q)$.

(vi) If $p \geqslant 3$ then $\mathrm{N}_{\Omega^{\circ}_{15}(q)}(\mathrm{L}^{\pm}_4(q))$ *is* $\mathscr{S}_2$*-maximal in* $\Omega^{\circ}_{15}(q)$.

Proof. (i) By [8, Table 8.25, p.389 and Table 8.27, p.391], $\mathrm{SL}^{\pm}_6(q)$ has an irreducible subgroup H isomorphic to $\mathrm{SL}^{\pm}_3(q)$. Let ρ_{15} be a 15-dimensional absolutely irreducible representation of $\mathrm{SL}^{\pm}_6(q)$. We are going to to show that $H\rho_{15}$ is of type $\mathrm{SL}^{\pm}_3(q)_2$. To do so we will find the highest weight of $H\rho_{15}$. By Lemma 8.1.18 if the highest weight of $H\rho_{15}$ is $(1,2)$ then $\mathrm{SL}^{\pm}_3(q)_2$ is an $\mathscr{S}_2$-subgroup of $\mathrm{SL}^{\pm}_6(q)$ equivalent to $H\rho_{15}$ and hence can never be maximal.

We will first consider $\mathrm{SL}_3(q)$. Note that we can use our result of $\mathrm{SL}_3(\overline{\mathbb{F}_p})$ in Section 8.1 since by Theorem 8.1.22 we get equivalent results for $\mathrm{SL}_3(q)$. By Example 8.1.9 the maximal torus of $\mathrm{SL}_3(q)$ is given by $T = \langle \mathrm{diag}(\alpha, \beta, \alpha^{-1}\beta^{-1}) \rangle$ for $\alpha, \beta \in \mathbb{F}_q^{\times}$. Furthermore, by Example 8.1.17, $\lambda_1(t) = \alpha$ and $\lambda_2(t) = \alpha\beta$ and the Borel subgroup can be chosen to consist of lower triangular matrices. Then there exists a 6-dimensional representation ρ_6 of $\mathrm{SL}_3(q)$ acting on $\mathrm{S}^2(V_3)$ which is irreducible by [8, Prop 5.4.5, p.291] and therefore, without loss of generality, $H = \mathrm{SL}_3(q)\rho_6$. Then $T\rho_6 = \langle \mathrm{diag}(\alpha^2, \alpha\beta, \beta^{-1}, \beta^2, \alpha^{-1}, \alpha^{-2}\beta^{-2}) \rangle$ is the maximal torus of H. Furthermore, $B\rho_6$ consists of lower triangular matrices. Since $\mathrm{SL}_6(q)\rho_{15}$ acts on $\Lambda^2(V_6)$ we will now determine $(T\rho_6)\rho_{15}$ and $(B\rho_6)\rho_{15}$. A straightforward calculation shows that

$$\begin{aligned}(T\rho_6)\rho_{15} = \langle \mathrm{diag}(&\alpha^3\beta, \alpha^2\beta^{-1}, \alpha^2\beta^2, \alpha, \beta^{-2}, \alpha, \alpha\beta^3, \beta, \alpha^{-1}\beta^{-1}, \beta,\\ &\beta^{-1}\alpha^{-1}, \alpha^{-2}\beta^{-3}, \alpha^{-1}\beta^2, \alpha^{-2}, \alpha^{-3}\beta^{-2}) \rangle\end{aligned}$$

and $(B\rho_6)\rho_{15}$ is again generated by lower triangular matrices. Hence $(B\rho_6)\rho_{15}$ stabilises the 1-dimensional subspace $\langle v^+ \rangle = \langle (1, 0, \ldots, 0) \rangle$. By Definition 8.1.10, the weight related to v^+ is $\lambda = \chi((t\rho_6)\rho_{15}) = \alpha^3\beta$ for all $t \in T$ which is dominant by Lemma 8.1.13. Hence, by Lemma 8.1.18 there exists up to isomorphism a unique irreducible representation of H with highest weight $\lambda = 2\lambda_1 + \lambda_2$. Since there exists a 15-dimensional representation ρ of $\mathrm{SL}_3(q)$ with highest weight $(1,2)$, ρ_{15} has to be equivalent to ρ and hence $\mathrm{SL}_3(q)_2$ is not $\mathscr{S}_2$-maximal.

Let $\Omega = \mathrm{SL}_{15}(q)$. Recall from Proposition 9.3.11 that $(q-1,3).\mathrm{L}_6(q).2$ is stabilised by $\langle \delta_\Omega^t, \gamma_\Omega, \phi_\Omega\rangle$ in odd characteristic, where $t=(q-1,5)$. Let $K = \mathrm{SL}_6(q)$. Then $G = \mathrm{SL}_3(q)$ has stabiliser $\langle \delta_K^2, \phi_K, \gamma_K\rangle$ in $\mathrm{Out}(\mathrm{L}_6(q))$ if $q \equiv \pm 1 \pmod 8$ and stabiliser $\langle \delta_K^2, \phi_K, (\gamma\delta)_K\rangle$ if $q \equiv \pm 3 \pmod 8$ by [8, Section 8.2, p.377]. We are going to show that $(q-1,3).\mathrm{L}_6(q).\mathrm{Out}(\mathrm{L}_6(q))$ is contained in $\mathrm{SL}_{15}(q).\langle \delta_\Omega^t, \gamma_\Omega, \phi_\Omega\rangle$. In particular this implies that $\mathrm{SL}_3(q)_2$ never extends to a novelty. Let $r = (3, q-1)$. Then $|\delta_\Omega| = (5, q-1)r$ and $|\delta_K| = 2r$ from which it follows that $|\mathrm{Out}(\mathrm{L}_6)| = |\langle \delta_K, \phi_K, \gamma_K\rangle| = 2r \cdot e \cdot 2$. Furthermore, $|\langle \delta_\Omega^t, \gamma_\Omega, \phi_\Omega\rangle| = r \cdot 2 \cdot e$. We also know that $(q-1,3).\mathrm{L}_6(2).2 \leqslant \mathrm{SL}_{15}(q)$ which proves our claim.

We can similarly show that $\mathrm{SU}_3(q)_2$ is not $\mathscr{S}_2$-maximal. The maximal torus of $\mathrm{SU}_3(q)$ is $T = \langle \mathrm{diag}(\alpha, \beta, \alpha^{-1}\beta^{-1}) \mid \alpha\alpha^\sigma = \beta\beta^\sigma = 1, \alpha, \beta \in \mathbb{F}_{q^2}^\times\rangle$. As in the linear case no extension of $\mathrm{SU}_3(q)_2$ is ever $\mathscr{S}_2$-maximal in any extension of $\mathrm{SU}_{15}(q)$.

(ii) By [8, Section 8.2, p.377], $\mathrm{SL}_6^\pm(q)$ has only one (up to equivalence) irreducible subgroup H isomorphic to $\mathrm{SL}_3^\pm(q)$. Let ρ_{15} be an absolutely irreducible 15-dimensional representation of $\mathrm{SL}_6^\pm(q)$. By (i) $H\rho_{15}$ is of type $\mathrm{SL}_3^\pm(q)_2$. Furthermore, again by [8, Section 8.2, p.377], $\mathrm{SL}_5^\pm(q)$ has no irreducible subgroups isomorphic to $\mathrm{SL}_3^\pm(q)$. By Lemma 8.6.1 any reducible subgroup of either $\mathrm{SL}_6(q)$ or $\mathrm{SL}_5(q)$ remains reducible in their respective 15-dimensional representations.

(iii) By [8, Section 8.2, p.377] there are no irreducible subgroups of $\mathrm{SL}_6^\pm(q)$ isomorphic to $\mathrm{SL}_5^\pm(q)$. Hence by Lemma 8.6.1 the 15-dimensional absolutely irreducible representation of $\mathrm{SL}_6^\pm(q)$ acting on $\Lambda^2(V_6)$ has no irreducible subgroups isomorphic to $\mathrm{SL}_5^\pm(q)$.

(iv) The group $\mathrm{SL}_6^\pm(q)$ is the largest $\mathscr{S}_2$-subgroup of $\mathrm{SL}_{15}^\pm(q)$.

(v) The only other $\mathscr{S}_2$-subgroup of $\Omega_{15}^\circ(q)$ is $\mathrm{SL}_4^\pm(q).2$. By [8, Section 8.2, p.377] the only subgroups of $\mathrm{SL}_4^\pm(q)$ that could contain an $\mathscr{S}_2$-subgroup isomorphic to $\mathrm{L}_2(q)$ are the classical groups $\Omega_4^\pm(q)$, $\mathrm{Sp}_4(q)$ and $\mathrm{SU}_4(q)$. However by looking at the respective tables of $\Omega_4^-(q)$, $\mathrm{Sp}_4(q)$ and $\mathrm{SU}_4(q)$ in [8, Section 8.2, p.377] we see that there is no containment. Furthermore, $\Omega_4^+(q) = \mathrm{SL}_2(q) \circ \mathrm{SL}_2(q)$ does not contain $\mathrm{L}_2(q)$ either.

(vi) The group $\mathrm{SL}_4^\pm(q)$ is the largest $\mathscr{S}_2$-subgroup of $\Omega_{15}^\circ(q)$. □

10 Containments

In this chapter we will find all maximal subgroups of the quasisimple classical groups, and their extensions by outer automorphisms, in dimension 13, 14 and 15. We will first determine the $\mathscr{S}$-subgroups that are potentially maximal before identifying containments between the maximal geometric subgroups and these $\mathscr{S}$-subgroups.

Definition 10.0.1. An $\mathscr{S}_2$-subgroup G is **potentially maximal** in some extension of $\Omega \in \{\mathrm{SL}_n^{\pm}(q), \mathrm{Sp}_n(q), \Omega_n^{\epsilon}(q)\}$ if G is $\mathscr{S}_2$-maximal but G is not the image of a p-unrestricted representation of some cover of $\mathrm{L}_2(q)$ which might be $\mathscr{S}_2$-maximal but is not maximal overall.

10.1 $\mathscr{S}$-Maximals

We will look at each dimens ion individually. The 13, 14 and 15-dimensional $\mathscr{S}_1$-subgroups are considered in Chapter 5, Chapter 6 and Chapter 7 respectively. The potentially maximal $\mathscr{S}_2$-subgroups are given in Chapter 9.

Definition 10.1.1. A group G is an $\mathscr{S}_1$**-novelty** if G is an $\mathscr{S}_1$-subgroup of $\Omega \in \{\mathrm{SL}_n^{\pm}(q), \mathrm{Sp}_n(q), \Omega_n^{\epsilon}(q)\}$ and if G is a novelty among the $\mathscr{S}_1$-subgroups.

A subgroup is $\mathscr{S}$**-maximal** if it is either $\mathscr{S}_1$- or potentially $\mathscr{S}_2$-maximal and maximal among the union of the $\mathscr{S}_1$- and $\mathscr{S}_2$-subgroups.

We will also need the following lemma.

Lemma 10.1.2. *Let $\Delta \in \{\Lambda^2(V_6), \Lambda^2(V_7), \Lambda^3(V_6), \mathrm{S}^2(V_5), \mathrm{S}^4(V_3), \mathrm{S}^{12}(V_2), \mathrm{S}^{13}(V_2), \mathrm{S}^{14}(V_2), V_3^* \otimes \mathrm{S}^2(V_3), V_4^* \otimes V_4\}$ be a module, where t is the dimension of the underlying vector space V_t. Let H be the quasisimple preimage of one of the defining characteristic representations ρ given in Table 9.1.1 with natural representation τ of dimension t. Then $H\rho \leqslant \Omega \in \{\mathrm{SL}_n^{\pm}(q), \mathrm{Sp}_n(q), \Omega_n^{\epsilon}(q)\}$, $n \in \{13, 14, 15\}$, and $H\rho$ acts on a subquotient of Δ. Let G be a quasisimple subgroup of $H\rho$. If $G \leqslant H\rho$ and G is an $\mathscr{S}_1$-subgroup of Ω then $G_1\tau = (G\rho^{-1})\tau$ is absolutely irreducible as a subgroup of $H\tau$.*

Proof. Let ρ' be a representation of H such that $H\rho'$ acts on Δ and assume that $G_1\tau$ is reducible. We are going to show that G is reducible in this case.

We will first consider the case $\Delta = \Lambda^2(V_6)$. Let $\{e_1, \ldots, e_6\}$ be a basis of V_6. Then $\{e_i \otimes e_j - e_j \otimes e_i \,|\, 1 \leqslant i < j \leqslant 6\}$ is a basis of $\Lambda^2(V_6)$. Without loss of generality suppose that $(G_1\tau)$ fixes the subspace $\{e_1, \ldots, e_r\}$, $r \leqslant 5$. Then $(G_1\tau)\rho'$ fixes the subspace $\langle e_i \otimes e_j - e_j \otimes e_i \,|\, 1 \leqslant i < j \leqslant r\rangle$, which has dimension at most $\binom{5}{2} = 10$. Hence $(G_1\tau)\rho$ fixes a subspace of at most dimension 10 which implies that G is reducible.

Now let $\Delta^- = \Lambda^k(V_{t-1})$ if $\Delta = \Lambda^k(V_t)$. Similarly, let $\Delta^- = \mathrm{S}^k(V_{t-1})$, $V_2^* \otimes \mathrm{S}_2(V_2)$ or $V_3^* \otimes V_3$. It is straightforward to show that $(G_1\tau)\rho$ stabilises a subspace of at most $\dim(\Delta^-)$ if $G_1\tau$ is reducible. The following table shows that unless $\Delta = \Lambda^2(V_7)$, G is reducible if $G_1\tau$ is reducible.

Δ	$\dim(\Delta)$	$\dim(\Delta^-)$	$\dim(\rho)$
$\Lambda^2(V_6)$	$\binom{6}{2} = 15$	$\binom{5}{2} = 10$	13, 14, 15
$\Lambda^2(V_7)$	$\binom{7}{2} = 21$	$\binom{6}{2} = 15$	14
$\Lambda^3(V_6)$	$\binom{6}{3} = 20$	$\binom{5}{3} = 10$	14
$\mathrm{S}^2(V_5)$	$\binom{6}{2} = 15$	$\binom{5}{2} = 10$	13, 14,15
$\mathrm{S}^4(V_3)$	$\binom{6}{4} = 15$	$\binom{5}{4} = 5$	15
$\mathrm{S}^{12}(V_2)$	13	1	13
$\mathrm{S}^{13}(V_2)$	14	1	14
$\mathrm{S}^{14}(V_2)$	15	1	15
$V_3^* \otimes \mathrm{S}^2(V_3)$	$3 \times \binom{4}{2} = 18$	$2 \times \binom{3}{2} = 6$	15
$V_4^* \otimes V_4$	$4 \times 4 = 16$	$3 \times 3 = 9$	14, 15

Now let $H = \mathrm{G}_2(q)$, $p \geqslant 5$, and let $\Delta = \Lambda^2(V_7)$. Then ρ has dimension 14 and $H\rho \leqslant \Omega_{14}^{\pm}(q)$. Furthermore, $\dim(\Lambda^2(V_7)) = 21$, $\dim(\Lambda^2(V_6)) = 15$ and $\dim(\Lambda^2(V_5)) = 10$. Hence we have to show that G cannot be not absolutely irreducible if $G_1\tau$ stabilises a 6-dimensional subspace.

Let π be an absolutely irreducible 6-dimensional representation of some cover G_1^* of a group isomorphic to G. By Proposition 6.4.17 and [8, Thm 4.3.3, p.162], we can deduce that the only possibilities for $G \leqslant \Omega_{14}^{\pm}(q)$ and G_1^* with absolutely irreducible 6-dimensional representation are $G \in \{\mathrm{A}_7, \mathrm{L}_2(13)_{1,2}\}$ and $G_1^* \in \{\mathrm{A}_7, 3.\mathrm{A}_7, 6.\mathrm{A}_7, 2.\mathrm{L}_2(13)\}$.

By looking at the maximal subgroups of $\mathrm{G}_2(q)$ ([8, Table 8.41, p.397]) we see that G needs to be isomorphic to a subgroup of $\mathrm{GL}_2(q)$, $\mathrm{PGL}_2(q)$, $\mathrm{SL}_3^{\pm}(q)$, $2^3.\mathrm{L}_3(2)$ or J_1 if $G \leqslant H\rho$. By [6], G cannot be isomorphic to a subgroup of J_1 or $2^3.\mathrm{L}_3(2)$. Furthermore, by looking at the tables in [8, Section 8.2, p.377], we see that in fact the only possible containment is $(3.\mathrm{A}_7)\tau \leqslant \mathrm{G}_2(5)$. Computer calculations (file s1ins2cont) show that $(3.\mathrm{A}_7\tau)\rho$ is reducible.

Finally, we will assume that $G_1\tau$ is irreducible but not absolutely irreducible. Then there exists some $s > 1$ such that $G_1\tau$ is reducible in $\mathrm{GL}_t(q^s)$ and hence $(G_1\tau)\rho$ is reducible over $\mathbb{F}_{q^s}$. In particular this implies that G is not absolutely irreducible, which proves our claim. $\square$

10.1.1 $\mathscr{S}$-Maximals in Dimension 13

The following table, Table 10.1.1 gives all 13-dimensional $\mathscr{S}_1$- and $\mathscr{S}_2$-maximal subgroups G that are potentially maximal. The groups are taken from Theorem 5.1.1, Proposition 5.2.4, Proposition 5.3.10 and Proposition 9.5.1.

The table shows whether G is $\mathscr{S}_1$- or $\mathscr{S}_2$-maximal, it gives the Schur indicator of G, G^∞, the order of G^∞ and the characteristics in which G occurs. If the column 'PermRep' is non-empty then some extension of G^∞ is $\mathscr{S}$-maximal. This column gives the degree of the smallest permutation representation of H, where $H = G^\infty/\mathrm{Z}(G^\infty)$. This is mostly useful for determining whether an $\mathscr{S}$-maximal subgroup is contained in a geometric subgroup. If there is an $*$ next to a group name, then G is an $\mathscr{S}_1$-novelty.

The degrees of the smallest permutation representations were taken from [6] and [17]. The remaining information comes from Table 5.1.1 and Table 9.1.1 with the exception of the group orders of the $\mathscr{S}_2$-subgroups which comes from [17].

Table 10.1.1: Potential $\mathscr{S}$-maximal subgroups in dimension 13

$\mathscr{S}_i$	**Ind**	**Gp**	**Order**	**Charc**	**PermRep**
$\mathscr{S}_1$	$\circ$	$\mathrm{S}_6(3)$	$2^9 \cdot 3^9 \cdot 5 \cdot 7 \cdot 13$	$\neq 3$	364
$\mathscr{S}_1$	$\circ$	$\mathrm{U}_3(4)$	$2^6 \cdot 3 \cdot 5^2 \cdot 13$	$\neq 2, 5$	65
$\mathscr{S}_1$	$+$	A_8^*	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	3	8
$\mathscr{S}_1$	$+$	A_{14}	$2^{10} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13$	$\neq 2, 3, 5, 7$	14
$\mathscr{S}_1$	$+$	A_{15}	$2^{10} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$	$3, 5$	15
$\mathscr{S}_1$	$+$	$\mathrm{L}_2(25)$	$2^3 \cdot 3 \cdot 5^2 \cdot 13$	$\equiv 2, 3 \pmod 5, \neq 2$	26
$\mathscr{S}_1$	$+$	$\mathrm{L}_3(3)$	$2^4 \cdot 3^3 \cdot 13$	$\neq 2, 3$	13
$\mathscr{S}_1$	$+$	$\mathrm{S}_4(5)$	$2^6 \cdot 3^2 \cdot 5^4 \cdot 13$	$\neq 2, 5$	156
$\mathscr{S}_1$	$+$	J_2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	3	—
$\mathscr{S}_2$	$+$	$\mathrm{L}_2(q)$	$\frac{1}{2}q(q^2-1)$	$\geqslant 13$	$\geqslant 14$
$\mathscr{S}_2$	$+$	$\mathrm{S}_6(3^i)$	$\frac{1}{2}3^{9i}\prod_{j=1}^{3}(3^{2ij}-1)$	3	$\geqslant 364$
$\mathscr{S}_2$	$+$	$\Omega_5^\circ(5^i)$	$\frac{1}{2}5^{4i}\prod_{j=1}^{2}(5^{2ij}-1)$	5	$\geqslant 156$

Theorem 10.1.3. *Let G be an $\mathscr{S}_1$- or potentially $\mathscr{S}_2$-maximal subgroup of any extension of $\Omega \in \{\mathrm{SL}_{13}^{\pm}(q), \Omega_{13}^\circ(q)\}$. Then G^∞ is contained in Table 10.1.1.*

Proof. This can be seen using Theorem 5.1.1, Proposition 5.2.4, Proposition 5.3.10, Theorem 9.1.1 and Proposition 9.5.1. ☐

Proposition 10.1.4 ($\mathscr{S}$-maximal subgroups in dimension 13)**.**
No extension of J_2 *is* $\mathscr{S}$*-maximal in any extension of* $\Omega_{13}^{\circ}(9)$ *and no extension of the* $\mathscr{S}_2$*-subgroup* $\mathrm{L}_2(13)$ *is* $\mathscr{S}$*-maximal in any extension of* $\Omega_{13}^{\circ}(13)$*. There are no other containments between the* $\mathscr{S}_1$*-maximal subgroups and the* $\mathscr{S}_2$*-maximal subgroups considered in Theorem 10.1.3.*

Proof. First we will consider possible containments of $\mathscr{S}_2$-subgroups in $\mathscr{S}_1$-subgroups. By looking at Table 10.1.1, using Lagrange's theorem and by considering the characteristics in which the respective representations occur, it is clear that $\mathrm{L}_2(13) \leqslant \mathrm{A}_{14} \leqslant \Omega_{13}^{\circ}(13)$ is the only possible containment. Using Magma (file s2ins1cont), we can show that the image of an absolutely irreducible 13-dimensional representation of A_{14} in characteristic 13 has an absolutely irreducible subgroup isomorphic to $\mathrm{L}_2(13)$. Both A_{14} and $\mathrm{L}_2(13)$ are stabilised by $\delta \in \mathrm{Out}(\mathrm{O}_{13}^{\circ}(13))$ and $\mathrm{L}_2(13).2 \leqslant \mathrm{S}_{14}$ by [6]. Hence, no extension of $\mathrm{L}_2(13)$ is maximal in any extension of $\Omega_{13}^{\circ}(13)$.

Now we will consider potential containments of $\mathscr{S}_1$-subgroups in $\mathscr{S}_2$-subgroups. The 13-dimensional $\mathscr{S}_2$-subgroups in Table 10.1.1 preserve an orthogonal form and a cover of each one can be realised in dimension $\leqslant 6$. Let G be an $\mathscr{S}_1$-subgroup in Table 10.1.1. If G is contained in one of the $\mathscr{S}_2$-subgroups in Table 10.1.1 then G needs to have Schur indicator + in dimension 13 and there needs to exist an absolutely irreducible 2-dimensional representation of G, or an absolutely irreducible 5-dimensional representation of G preserving an orthogonal form or an absolutely irreducible 6-dimensional representation of G preserving a symplectic form by Lemma 10.1.2. By looking at [8, Thm 4.3.3, p.162], we see that the only possible containment is $\mathrm{J}_2 \leqslant \mathrm{S}_6(9) \leqslant \Omega_{13}^{\circ}(9)$.

By Table [8, Table 8.29, p.392], there exists an $\mathscr{S}_1$-subgroup H of $\mathrm{Sp}_6(9)$ isomorphic to $2.\mathrm{J}_2$ with class stabiliser $\langle \phi_{\mathrm{Sp}_6(9)} \rangle$. Let ρ_{13} be a 13-dimensional absolutely irreducible representation of $\mathrm{Sp}_6(9)$ acting on a subquotient of $\Lambda^2(V_6)$ as in Proposition 9.3.1. Computer calculations (file s1ins2cont) show that $H\rho_{13} \cong \mathrm{J}_2$ is absolutely irreducible. By Proposition 5.3.9 an absolutely irreducible subgroup of $\Omega = \Omega_{13}^{\circ}(9)$ isomorphic to J_2 has stabiliser $\langle \phi_\Omega \rangle$. By Proposition 9.3.1, $(\mathrm{Sp}_6(9).\langle \delta_{\mathrm{Sp}_6(9)}, \phi_{\mathrm{Sp}_6(9)} \rangle)\rho_{13} \leqslant \Omega.\langle \delta_\Omega, \phi_\Omega \rangle$ since δ_Ω induces $\delta_{\mathrm{Sp}_6(9)}$ and ϕ_Ω induces either $\phi_{\mathrm{Sp}_6(9)}$ or $(\phi\delta)_{\mathrm{Sp}_6(9)}$. It follows that no extension of J_2 is ever $\mathscr{S}$-maximal in any extension of Ω. $\square$

10.1.2 $\mathscr{S}$-Maximals in Dimension 14

In dimension 14 we will consider the $\mathscr{S}$-maximal subgroups depending on the type of form they preserve. Note that there are no $\mathscr{S}_2$-maximal subgroups of $\mathrm{SL}_{14}^{\pm}(q)$.

The following table, Table 10.1.2 contains the $\mathscr{S}_1$- and potentially $\mathscr{S}_2$-maximal subgroups. Apart from whether G is $\mathscr{S}_1$- or $\mathscr{S}_2$-maximal, the Schur indicator of G, G^∞, the order of G^∞ and the characteristics in which G exists, the table also give the smallest permutation representation of H, where $G^\infty = \mathrm{Z}(H).H$. If the degree is not given then no extension of G^∞ is $\mathscr{S}$-maximal in any characteristic. The information in the table comes from [6], [17], Table 6.1.1 and Table 9.1.1.

Table 10.1.2: Potential $\mathscr{S}$-maximal subgroups in dimension 14

$\mathscr{S}_i$	**Ind**	**Gp**	**Order**	**Charc**	**PermRep**
$\mathscr{S}_1$	$\circ$	$2.\mathrm{S}_6(3)$	$2^{10} \cdot 3^9 \cdot 5 \cdot 7 \cdot 13$	$\neq 2, 3$	364
$\mathscr{S}_1$	$\circ$	$\mathrm{Sz}(8)$	$2^6 \cdot 5 \cdot 7 \cdot 13$	$\neq 2$	65
$\mathscr{S}_1$	$-$	$2.\mathrm{A}_7$	$2^4 \cdot 3^2 \cdot 5 \cdot 7$	$\neq 2, 3$	7
$\mathscr{S}_1$	$-$	$2.\mathrm{L}_2(13)_1$	$2^3 \cdot 3 \cdot 7 \cdot 13$	$\neq 2, 13$	14
$\mathscr{S}_1$	$-$	$2.\mathrm{L}_2(13)_2$	$2^3 \cdot 3 \cdot 7 \cdot 13$	$\neq 2, 3, 13$	14
$\mathscr{S}_1$	$-$	$\mathrm{L}_2(29)$	$2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 29$	2	30
$\mathscr{S}_1$	$-$	$2.\mathrm{L}_2(29)$	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 29$	$\neq 2, 29$	30
$\mathscr{S}_1$	$-$	$2.\mathrm{J}_2$	$2^8 \cdot 3^3 \cdot 5^2 \cdot 7$	$\neq 2$	100
$\mathscr{S}_2$	$-$	$\mathrm{SL}_2(q)$	$q(q^2-1)$	$\geqslant 17$	$\geqslant 18$
$\mathscr{S}_2$	$-$	$\mathrm{Sp}_6(q)$	$q^9 \prod_{j=1}^{3}(q^{2j}-1)$	$\neq 2$	$\geqslant 364$
$\mathscr{S}_1$	$+$	A_{7_1}	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	$\neq 2, 3, 5$	7
$\mathscr{S}_1$	$+$	A_7	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	2	7
$\mathscr{S}_1$	$+$	A_{15}	$2^{10} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$	$\neq 2, 3, 5$	15
$\mathscr{S}_1$	$+$	A_{16}	$2^{14} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$	2	16
$\mathscr{S}_1$	$+$	$\mathrm{L}_2(13)_1$	$2^2 \cdot 3 \cdot 7 \cdot 13$	$\neq 2, 3, 13$	14
$\mathscr{S}_1$	$+$	$\mathrm{L}_2(13)_2$	$2^2 \cdot 3 \cdot 7 \cdot 13$	$\neq 2, 3, 13$	14
$\mathscr{S}_1$	$+$	$\mathrm{L}_2(13)$	$2^2 \cdot 3 \cdot 7 \cdot 13$	2	14
$\mathscr{S}_1$	$+$	$\mathrm{S}_6(2)$	$2^9 \cdot 3^4 \cdot 5 \cdot 7$	3	28
$\mathscr{S}_1$	$+$	$\mathrm{G}_2(3)$	$2^6 \cdot 3^6 \cdot 7 \cdot 13$	$\neq 3$	351
$\mathscr{S}_1$	$+$	J_1	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$	11	—
$\mathscr{S}_1$	$+$	J_2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	$\neq 3$	—
$\mathscr{S}_2$	$+$	$\mathrm{L}_4(2^i)$	$2^{6i} \prod_{j=2}^{4}(2^{ij}-1)$	2	$\geqslant 8$
$\mathscr{S}_2$	$+$	$\mathrm{U}_4(2^i)$	$2^{6i} \prod_{j=2}^{4}(2^{ij}-(-1)^j)$	2	$\geqslant 27$
$\mathscr{S}_2$	$+$	$\Omega_5^\circ(q)$	$\frac{1}{2}q^4 \prod_{j=1}^{2}(q^{2j}-1)$	$\neq 2, 5$	$\geqslant 27$
$\mathscr{S}_2$	$+$	$\mathrm{S}_6(q)$	$\frac{1}{(2,q-1)}q^9 \prod_{j=1}^{3}(q^{2j}-1)$	$\neq 3$	$\geqslant 28$
$\mathscr{S}_2$	$+$	$\mathrm{G}_2(q)$	$q^6(q^2-1)(q^6-1)$	$\neq 2, 3$	$\geqslant 3906$

Theorem 10.1.5. *Let G be $\mathscr{S}_1$-maximal or potentially $\mathscr{S}_2$-maximal in any extension of $\Omega \in \{\mathrm{SL}^{\pm}_{14}(q), \mathrm{Sp}_{14}(q), \Omega^{\pm}_{14}(q)\}$. Then G^{∞} is contained in Table 10.1.2.*

Proof. This follows from Theorem 6.1.1, Proposition 6.2.4, Proposition 6.3.7, Proposition 6.4.17, Theorem 9.1.1 and Proposition 9.5.1. □

Proposition 10.1.6 ($\mathscr{S}$-maximal subgroups in dimension 14 - Case **L**/**U**)**.** *All $\mathscr{S}_1$-maximal subgroups in Proposition 6.2.4 are $\mathscr{S}$-maximal.*

Proof. All $\mathscr{S}_1$-maximal subgroups are $\mathscr{S}$-maximal since there are no $\mathscr{S}_2$-subgroups of $\mathrm{SL}^{\pm}_{14}(q)$ by Proposition 9.5.2. □

Proposition 10.1.7 ($\mathscr{S}$-maximal subgroups in dimension 14 - Case **S**)**.** *All $\mathscr{S}_1$-subgroups and all $\mathscr{S}_2$-maximal subgroups in Theorem 10.1.5 preserving a symplectic form are $\mathscr{S}$-maximal with the following exceptions:*

(i) If $p = \pm 3, \pm 27, \pm 29, \pm 35, \pm 43, \pm 51 \pmod{104}$ then $2.\mathrm{L}_2(13)_1$ is not $\mathscr{S}$-maximal in $\mathrm{Sp}_{14}(p)$ but extends to a novelty under $\langle \delta \rangle$.

(ii) If $p \equiv 11, 19, 21, 29 \pmod{40}$ then $2.\mathrm{J}_2$ is not $\mathscr{S}$-maximal in $\mathrm{Sp}_{14}(p)$ but extends to a novelty under $\langle \delta \rangle$. Furthermore, no extension of $2.\mathrm{J}_2$ is $\mathscr{S}$-maximal in any extension of $\mathrm{Sp}_{14}(5)$.

Proof. We will first determine whether any of the $\mathscr{S}_2$-maximal subgroups $\mathrm{SL}_2(q)$, $p \geqslant 17$, or $\mathrm{Sp}_6(q)_2$ are not $\mathscr{S}$-maximal. First note that the only $\mathscr{S}_1$-subgroups with order divisible by a prime $\geqslant 17$ are $\mathrm{L}_2(29)$ and $2.\mathrm{L}_2(29)$ which have order divisible by 29. However neither of these groups is isomorphic to an $\mathscr{S}_1$-subgroup in characteristic 29. Hence $\mathrm{SL}_2(q)$ is $\mathscr{S}$-maximal. Furthermore, $\mathrm{Sp}_6(q)_2$ has order divisible by q^9 and none of the $\mathscr{S}_1$-subgroups in Table 10.1.2 with Schur indicator $-$ has order divisible by q^9. It follows that $\mathrm{Sp}_6(q)_2$ is $\mathscr{S}$-maximal as well.

Now we will show which of the $\mathscr{S}_1$-subgroups are $\mathscr{S}$-maximal. By Lemma 10.1.2, we only have to consider $\mathscr{S}_1$-subgroups with an absolutely irreducible 2 or 6 dimensional representation. These groups in the relevant characteristics are

(i) $2.\mathrm{L}_2(13) \leqslant \mathrm{Sp}_6(p) \leqslant \mathrm{Sp}_{14}(p)$ and

(ii) $2.\mathrm{J}_2 \leqslant \mathrm{Sp}_6(p) \leqslant \mathrm{Sp}_{14}(p)$

by [8, Thm 4.3.3, p.162].

From now on let ρ_{14} be an absolutely irreducible 14-dimensional representation of $\mathrm{Sp}_6(q)$ acting on a submodule of $\Lambda^3(V_6)$ as in Proposition 9.3.5.

We will first consider $\mathbf{2.L_2(13)}$. There exists an irreducible subgroup H of $\mathrm{Sp}_6(q)$ isomorphic to $2.\mathrm{L}_2(13)$ by Table [8, Table 8.29, p.392]. Furthermore, $H \leqslant \mathrm{Sp}_6(p)$ if and only if $p \equiv \pm1, \pm3, \pm4 \pmod{13}$ and has trivial class stabiliser. Otherwise $H \leqslant \mathrm{Sp}_6(p^2)$.

Computer calculations (file s1ins2cont) show that $H\rho_{14}$ is equivalent to $2.\mathrm{L}_2(13)_1$ using [12]. By Proposition 6.3.2, $2.\mathrm{L}_2(13)_1.2 \leqslant \mathrm{Sp}_{14}(p)$ with trivial class stabiliser if $p \equiv 1, 7 \pmod 8$. It follows that $2.\mathrm{L}_2(13)_1.2$ is $\mathscr{S}$-maximal in this case. If $p \equiv 3, 5 \pmod 8$ then $2.\mathrm{L}_2(13)_1 \leqslant \mathrm{Sp}_{14}(p)$ and has class stabiliser $\langle \delta_{\mathrm{Sp}_{14}(p)} \rangle$. Hence, if $p \equiv \pm1, \pm3, \pm4 \pmod{13}$ and $p \equiv 3, 5 \pmod 8$ then $2.\mathrm{L}_2(13)_1$ is not $\mathscr{S}$-maximal in $\mathrm{Sp}_{14}(p)$ but extends to a novelty under $\langle \delta_{\mathrm{Sp}_{14}(p)} \rangle$.

Finally we will consider $\mathbf{2.J_2}$. By [8, Table 8.29, p.392], $\mathrm{Sp}_6(q)$ has an absolutely irreducible subgroup H isomorphic to $2.\mathrm{J}_2$. Here $H \leqslant \mathrm{Sp}_6(p)$ with trivial stabiliser if $p \equiv 1, 4 \pmod 5$, $H \leqslant \mathrm{Sp}_6(5)$ with class stabiliser $\langle \delta_{\mathrm{Sp}_6(q)} \rangle$ and $H \leqslant \mathrm{Sp}_6(p^2)$ otherwise. By Proposition 6.3.6, there exists an $\mathscr{S}_1$-subgroup G of $\mathrm{Sp}_{14}(p)$ isomorphic to $2.\mathrm{J}_2$. Here $G.2 \leqslant \mathrm{Sp}_{14}(p)$ if $p \equiv 1, 7 \pmod 8$ and $G \leqslant \mathrm{Sp}_{14}(p)$ with class stabiliser $\langle \delta_{\mathrm{Sp}_{14}(p)} \rangle$ if $p \equiv 3, 5 \pmod 8$. It follows that $\mathrm{N}_{\mathrm{Sp}_{14}(p)}(G)$ is $\mathscr{S}$-maximal if $p \equiv 1, 7 \pmod 8$ or if ($p \equiv 3, 5 \pmod 8$ and $p \equiv 2, 3 \pmod 5$).

Hence assume from now on that ($p \equiv 3, 5 \pmod 8$ and $p \equiv 1, 4 \pmod 5$) or $p = 5$. We can show using Magma (file s1ins2cont) and [12] that $2.\mathrm{J}_2\rho_{14}$ is equivalent to G. If ($p \equiv 3, 5 \pmod 8$ and $p \equiv 1, 4 \pmod 5$) then $2.\mathrm{J}_2$ has class stabiliser $\langle \delta_{\mathrm{Sp}_{14}(q)} \rangle$ in $\mathrm{Out}(\mathrm{S}_{14}(q))$ but trivial stabiliser in $\mathrm{Out}(\mathrm{S}_6(q))$. Hence G extends to a novelty in this case. If $p = 5$ then we know that $2.\mathrm{J}_2.2 \leqslant \mathrm{Sp}_6(q).\langle \delta_{\mathrm{Sp}_6(q)} \rangle$. Furthermore, it follows from Proposition 9.3.5 that $\mathrm{Sp}_6(q).\langle \delta_{\mathrm{Sp}_6(q)}, \phi_{\mathrm{Sp}_6(q)} \rangle \leqslant \mathrm{Sp}_{14}(q).\langle \delta_{\mathrm{Sp}_{14}(q)}, \phi_{\mathrm{Sp}_{14}(q)} \rangle$. Therefore no extension of G is ever $\mathscr{S}$-maximal in characteristic 5. □

Proposition 10.1.8 ($\mathscr{S}$-maximal subgroups in dimension 14 - Case $\mathbf{O}^{\pm}$). *All $\mathscr{S}_1$-maximal subgroups and all $\mathscr{S}_2$-maximal subgroups considered in Theorem 10.1.9 that preserve an orthogonal form are $\mathscr{S}$-maximal with the exception of the following groups:*

(i) No extension of $\Omega_5^\circ(3).2$ is $\mathscr{S}$-maximal in any extension of $\Omega_{14}^+(3)$.

(ii) No extension of A_{7_1} is $\mathscr{S}$-maximal in any extension of $\Omega_{14}^+(7)$.

(iii) No extension of J_1 is $\mathscr{S}$-maximal in any extension of $\Omega_{14}^-(11)$.

(iv) If $p \equiv 1, 3, 4, 9, 10, 12 \pmod{13}$ then $\mathrm{L}_2(13)_1$ is not $\mathscr{S}$-maximal in $\Omega_{14}^{\pm}(p)$ but extends to a novelty under $\langle \gamma \rangle$.

(v) *If* $p \equiv 1, 3, 4, 9, 10, 12 \pmod{13}$ *then* $\mathrm{L}_2(13)_2$ *is not* $\mathscr{S}$*-maximal in* $\Omega^{\pm}_{14}(p)$ *but* $\mathrm{L}_2(13)_2.\langle\gamma\rangle$ *and* $\mathrm{L}_2(13)_2.\langle\gamma\delta\rangle$ *are* $\mathscr{S}$*-maximal in* $\Omega^{\pm}_{14}(p).\langle\gamma\rangle$ *and* $\Omega^{\pm}_{14}(p).\langle\gamma\delta\rangle$ *respectively.*

(vi) *If* $p \neq 3, 5$ *then no extension of* J_2 *is* $\mathscr{S}$*-maximal in any extension of* $\Omega^{\pm}_{14}(q)$*.*

Proof. We will first consider possible containments of $\mathscr{S}_2$-subgroups in $\mathscr{S}_1$-subgroups. The $\mathscr{S}_2$-subgroups to consider are $\mathrm{SL}^{\pm}_4(2^i).2$, $\mathrm{S}_6(q)_1$, $\Omega^{\circ}_5(q)$ and $\mathrm{G}_2(q)$.

By Lagrange's theorem $\mathbf{L^{\pm}_4(2).2}$ could be a subgroup of A_{15} or A_{16}. However we know that $\mathrm{L}^{\pm}_4(2).2 \leqslant \Omega^{-}_{14}(2)$, whereas $\mathrm{A}_{15}, \mathrm{A}_{16} \leqslant \Omega^{+}_{14}(2)$. There are no other containments possible by Lagrange's theorem which implies that $\mathrm{L}^{\pm}_4(2).2$ is $\mathscr{S}$-maximal.

Similarly, $\mathrm{S}_6(2)$ could be a subgroup of A_{15} or A_{16}. Again we find that $\mathrm{S}_6(2) \leqslant \Omega^{-}_{14}(2)$, whereas $\mathrm{A}_{15}, \mathrm{A}_{16} \leqslant \Omega^{+}_{14}(2)$. Since $q^9 \mid |\mathrm{S}_6(q)|$ there does not exist any other $\mathscr{S}_1$-subgroup that could contain $\mathbf{S_6(\boldsymbol{q})_1}$.

The 14-dimensional $\mathscr{S}_2$-subgroup $\boldsymbol{\Omega^{\circ}_5(p^i)}$ does not exist in characteristic 2 or 5. In characteristic 3 computer calculations (file s2ins1cont) show that $\Omega^{\circ}_5(3).2 \leqslant \mathrm{S}_6(2) \leqslant \Omega^{+}_{14}(3)$ and that $\Omega^{\circ}_5(3)$ is absolutely irreducible. Furthermore, $\Omega^{\circ}_5(3).2$ has trivial stabiliser and hence no extension of $\Omega^{\circ}_5(3)$ is $\mathscr{S}$-maximal. None of the other $\mathscr{S}_1$-subgroups has order divisible by p^4 for $p \geqslant 3$ with the exception of $\mathrm{G}_2(3)$ which has order divisible by 3^4. However, $\mathrm{G}_2(3)$ has no cross characteristic representation in characteristic 3. Hence, any extension of $\Omega^{\circ}_5(q)$, $q \neq 3$, is $\mathscr{S}$-maximal.

Finally, since none of the relevant $\mathscr{S}_1$-maximal subgroups has order divisible by q^6, when $p \geqslant 5$, it follows that $\mathbf{G_2(\boldsymbol{q})}$ is always $\mathscr{S}$-maximal in $\Omega^{\pm}_{14}(q)$.

Next we will consider the potential containments of $\mathscr{S}_1$-subgroups in $\mathscr{S}_2$-subgroups. Let G be an $\mathscr{S}_1$-subgroup with Schur indicator + in Table 10.1.2 and let K be an $\mathscr{S}_2$-subgroup with Schur indicator + in Table 10.1.2 such that K has a natural representation ρ in dimension t. If $G \leqslant K$ then there has to exist an absolutely irreducible t-dimensional representation τ of some cover of G such that $G\tau \leqslant K\rho$ by Lemma 10.1.2.

The only possible containments by [8, Thm 4.3.3, p.162] are

(i) $\mathrm{A}_7 \leqslant \mathrm{SL}_4(2).2 \leqslant \Omega^{-}_{14}(2)$,
(ii) $\mathrm{A}_{7_{1,2}}.2 \leqslant \Omega^{\circ}_5(7).2 \leqslant \Omega^{+}_{14}(7)$,
(iii) $\mathrm{J}_1 \leqslant \mathrm{G}_2(11) \leqslant \Omega^{-}_{14}(11)$,
(iv) $\mathrm{L}_2(13)_{1,2} \leqslant \mathrm{S}_6(p)_1 \leqslant \Omega^{\pm}_{14}(p)$,

(v) $\mathrm{J}_2 \leqslant \mathrm{S}_6(q)_1 \leqslant \Omega^{\pm}_{14}(q)$ and

(vi) $\mathrm{L}_2(13)_{1,2} \leqslant \mathrm{G}_2(p) \leqslant \Omega^{\pm}_{14}(p)$.

Note that $\mathrm{L}_2(13)$ is not an irreducible subgroup of $\mathrm{S}_6(2)_1$ in dimension 14 since $\mathrm{L}_2(13) \leqslant \Omega^{+}_{14}(2)$ whereas $\mathrm{S}_6(2)_1 \leqslant \Omega^{-}_{14}(2)$.

Computer calculations (file s1ins2cont) show that $\mathbf{A_7}$ is not an irreducible subgroup of $\mathrm{SL}_4(2) \leqslant \Omega^{-}_{14}(2)$. However, we can show using Magma that $\mathbf{A_{7_1}.2}$ is an absolutely irreducible subgroup of $\Omega^{\circ}_5(7).2 \leqslant \Omega^{+}_{14}(7)$ (file s1ins2cont). Since $\mathrm{A}_{7_1}.2$ has trivial stabiliser it follows that no extension of A_{7_1} is ever $\mathscr{S}$-maximal in characteristic 7.

Further calculations in Magma (file s1ins2cont) show that $\mathbf{J_1}$ is an absolutely irreducible subgroup of $\mathrm{G}_2(11) \leqslant \Omega^{-}_{14}(11)$ with trivial stabiliser. Hence no extension of J_1 is $\mathscr{S}$-maximal in any extension of $\Omega^{-}_{14}(11)$.

From now on let ρ_{14} be a 14-dimensional absolutely irreducible representation of $\mathrm{Sp}_6(q)$ acting on a submodule of $\Lambda^2(V_6)$ (see Proposition 9.3.3 and Proposition 9.3.4).

By [8, Table 8.29, p.392] there exists an $\mathscr{S}_1$-subgroup H of $\mathrm{Sp}_6(q)$ isomorphic to $2.\mathbf{L_2(13)}$. Here $q = p$ if $p \equiv 1,3,4,9,10,12 \pmod{13}$ in which case H has trivial stabiliser and $q = p^2$ otherwise. Computations in Magma (file s1ins2cont) show that $H\rho_{14}$ is equivalent to $\mathrm{L}_2(13)_1$. By Proposition 6.4.7, $\mathrm{L}_2(13)_1 \leqslant \Omega = \Omega^{\pm}_{14}(p)$ and has stabiliser $\langle \gamma_\Omega \rangle$. It follows that if $p \equiv 2,5,6,7,8,11 \pmod{13}$ then $\mathrm{L}_2(13)_1$ is $\mathscr{S}$-maximal. If $p \equiv 1,3,4,9,10,12 \pmod{13}$ then $\mathrm{Sp}_6(q)\rho_{14}$ and $\mathrm{L}_2(13)_1$ preserve the same orthogonal form. It follows that $\mathrm{L}_2(13)_1$ is not $\mathscr{S}$-maximal in this case but extends to a novelty under $\langle \gamma_\Omega \rangle$.

We will now consider $\mathbf{J_2}$. By [8, Table 8.29 and Table 8.30, p.392] there exists an $\mathscr{S}_1$-subgroup H of $\mathrm{Sp}_6(q)$ isomorphic to $2.\mathrm{J}_2$ (J_2 in characteristic 2). Here $q = p$ and H has trivial stabiliser if $p \equiv 1,4 \pmod 5$, $q = p^2$ and H has stabiliser $\langle \phi_{\mathrm{Sp}_6(q)} \rangle$ if $p \equiv 2,3 \pmod 5$ and $q = p$ and H has stabiliser $\langle \delta_{\mathrm{Sp}_6(q)} \rangle$ if $p = 5$. Since $\mathrm{J}_2.2 \leqslant \Omega^{-}_{14}(5)$ by Proposition 6.4.16, it follows that $\mathrm{J}_2.2$ is $\mathscr{S}$-maximal in characteristic 5. Hence assume now that $p \neq 3,5$.

Let G be an $\mathscr{S}_1$-subgroup of $\Omega^{\pm}_{14}(q)$ isomorphic to J_2. Computer calculations (file s1ins2cont) show that $H\rho_{14}$ is equivalent to G by looking at the character values ([12]). Furthermore, G and $\mathrm{Sp}_6(q)\rho_{14}$ preserve the same orthogonal form. By Propositions 6.4.14 and 6.4.15, $G \leqslant \Omega^{\pm}_{14}(p)$ if and only if $p \equiv 1,4 \pmod 5$ and $G \leqslant \Omega^{+}_{14}(p^2)$ otherwise. Hence G is not $\mathscr{S}$-maximal in $\Omega^{\pm}_{14}(q)$.

If $p \equiv 1,4 \pmod 5$ then G has trivial stabiliser and if $p \equiv 2,3 \pmod 5$ then G and $\mathrm{Sp}_6(p^2)\rho_{14}$ are both stabilised by $\langle \phi_{\Omega^{+}_{14}(p^2)} \rangle$ if $p \equiv 7,13,37,43 \pmod{60}$ and by $\langle (\phi\gamma)_{\Omega^{+}_{14}(p^2)} \rangle$ if $p \equiv 17,23,47,53 \pmod{60}$ or if $p = 2$.

Furthermore, $\mathrm{Sp}_6(p^2)\rho_{14}$ is stabilised by $\langle \delta'_{\Omega^+_{14}(p^2)}\rangle$. It is clear that $\delta'_{\Omega^+_{14}(p^2)}$ induces $\delta_{\mathrm{Sp}_6(p^2)}$ and that $\phi_{\Omega^+_{14}(p^2)}$ and $(\phi\gamma)_{\Omega^+_{14}(p^2)}$ induce either $\phi_{\mathrm{Sp}_6(p^2)}$ or $(\phi\delta)_{\mathrm{Sp}_6(p^2)}$. Hence, $\mathrm{S}_6(p^2).\mathrm{Out}(\mathrm{S}_6(p^2)) \leqslant \Omega^+_{14}(p^2).\langle \delta'_{\Omega^+_{14}(p^2)}, \mu\rangle$, where $\mu \in \{\phi_{\Omega^+_{14}(p^2)}, (\phi\gamma)_{\Omega^+_{14}(p^2)}\}$. It follows that no extension of G is $\mathscr{S}$-maximal if $p \neq 3, 5$.

There also exists an absolutely irreducible subgroup H of $\mathrm{G}_2(p)$ isomorphic to $\mathbf{L_2(13)}$ if $p \equiv 1, 3, 4, 9, 10, 12 \pmod{13}$, $p \neq 3$, by [8, Table 8.41, p.397]. Computer calculations (file s1ins2cont) show that H acts on a 14-dimensional submodule of $\Lambda^2(V_7)$. Let ρ'_{14} denote this 14-dimensional representation of H. Looking at the character values of $H\rho'_{14}$ we find that $H\rho'_{14}$ is equivalent to $\mathrm{L}_2(13)_2$. Furthermore, H has trivial stabiliser in dimension 7 whereas $\mathrm{L}_2(13)_2$ has stabiliser $\langle \gamma_{\Omega^{\pm}_{14}(p)}\rangle$ or $\langle (\gamma\delta)_{\Omega^{\pm}_{14}(p)}\rangle$. Hence $\mathrm{L}_2(13)_2$ extends to a novelty. □

10.1.3 $\mathscr{S}$-Maximals in Dimension 15

Let G be one of the $\mathscr{S}_1$-maximal subgroups in Proposition 7.2.10 or Proposition 7.3.11 or let G be an $\mathscr{S}_2$-maximal subgroup in Proposition 9.5.3. Then G^∞ together with its Schur indicator, group order and the characteristics in which G exists is given in Table 10.1.3. The column 'PermRep' gives the degree of the smallest permutation representation of $H = G^\infty/\mathrm{Z}(G^\infty)$ if some extension of G^∞ is $\mathscr{S}$-maximal. Otherwise the entry is left empty. The $*$ next to a group name signifies that G is an $\mathscr{S}_1$-novelty.

The content of the table was taken from Table 7.1.1, Table 9.1.1, [6] and [17].

Theorem 10.1.9. *Let G be an $\mathscr{S}_1$- or $\mathscr{S}_2$-maximal subgroup of any extension of $\Omega \in \{\mathrm{SL}^{\pm}_{15}(q), \Omega^{\circ}_{15}(q)\}$ and suppose that G is potentially maximal. Then G^∞ is contained in Table 10.1.3.*

Proof. This follows from Theorem 7.1.2, Proposition 7.2.10, Proposition 7.3.11, Theorem 9.1.1 and Proposition 9.5.1. □

Table 10.1.3: Potential $\mathscr{S}$-maximal subgroups in dimension 15

$\mathscr{S}_i$	**Ind**	**Gp**	**Order**	**Charc**	**PermRep**
$\mathscr{S}_1$	$\circ$	$3.\mathrm{A}^*_{7_1}$	$2^3 \cdot 3^3 \cdot 5 \cdot 7$	$\neq 2, 3$	—
$\mathscr{S}_1$	$\circ$	$3.\mathrm{A}_{7_2}$	$2^3 \cdot 3^3 \cdot 5 \cdot 7$	$\neq 2, 3, 7$	7
$\mathscr{S}_1$	$\circ$	$3.\mathrm{A}^*_7$	$2^3 \cdot 3^3 \cdot 5 \cdot 7$	2	—
$\mathscr{S}_1$	$\circ$	$\mathrm{L}_2(31)$	$2^5 \cdot 3 \cdot 5 \cdot 31$	$\neq 31$	32
$\mathscr{S}_1$	$\circ$	$3.\mathrm{L}_3(4)$	$2^6 \cdot 3^3 \cdot 5 \cdot 7$	$\neq 2, 3$	—
$\mathscr{S}_1$	$\circ$	$3_1.\mathrm{U}_4(3)$	$2^7 \cdot 3^7 \cdot 5 \cdot 7$	$\neq 3$	—
$\mathscr{S}_1$	$\circ$	M_{12}	$2^6 \cdot 3^3 \cdot 5 \cdot 11$	3	—
$\mathscr{S}_1$	$\circ$	$3.\mathrm{M}_{22}$	$2^7 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11$	2	—
$\mathscr{S}_2$	$\circ$	$\mathrm{SL}_3(q)$	$q^3 \prod_{j=2}^{3}(q^j - 1)$	$\geqslant 5$	$\geqslant 31$
$\mathscr{S}_2$	$\circ$	$\mathrm{SU}_3(q)$	$q^6 \prod_{j=2}^{3}(q^j - (-1)^j)$	$\geqslant 5$	$\geqslant 50$
$\mathscr{S}_2$	$\circ$	$\mathrm{SL}_5(q)$	$q^{10} \prod_{j=2}^{5}(q^j - 1)$	$\neq 2$	$\geqslant 121$
$\mathscr{S}_2$	$\circ$	$\mathrm{SU}_5(q)$	$q^{10} \prod_{j=2}^{5}(q^j - (-1)^j)$	$\neq 2$	$\geqslant 2440$
$\mathscr{S}_2$	$\circ$	$(3, q-1).\mathrm{L}_6(q)$	$\frac{1}{2}q^{15} \prod_{j=2}^{6}(q^j - 1)$	all	$\geqslant 63$
$\mathscr{S}_2$	$\circ$	$(3, q-1).\mathrm{U}_6(q)$	$\frac{1}{2}q^{15} \prod_{j=2}^{6}(q^j - (-1)^j)$	all	$\geqslant 672$
$\mathscr{S}_1$	$+$	A_7	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	$\neq 2, 7$	—
$\mathscr{S}_1$	$+$	A_{16}	$2^{14} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$	$\neq 2, 17$	16
$\mathscr{S}_1$	$+$	A_{17}	$2^{14} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17$	17	17
$\mathscr{S}_1$	$+$	$\mathrm{L}_2(16)$	$2^4 \cdot 3 \cdot 5 \cdot 17$	$\neq 2, 17$	17
$\mathscr{S}_1$	$+$	$\mathrm{L}_2(29)$	$2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 29$	$\neq 2, 29$	30
$\mathscr{S}_1$	$+$	$\mathrm{L}_3(4)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	3	21
$\mathscr{S}_1$	$+$	$\mathrm{S}_6(2)$	$2^9 \cdot 3^4 \cdot 5 \cdot 7$	$\neq 2, 3$	28
$\mathscr{S}_1$	$+$	$\mathrm{U}_4(2)_1$	$2^6 \cdot 3^4 \cdot 5$	$\neq 2, 3$	—
$\mathscr{S}_2$	$+$	$\mathrm{L}_2(q)$	$\frac{1}{2}q(q^2 - 1)$	$\geqslant 17$	$\geqslant 18$
$\mathscr{S}_2$	$+$	$\mathrm{L}_4(q)$	$\frac{1}{(4,q-1)}q^6 \prod_{j=2}^{4}(q^j - 1)$	$\neq 2$	$\geqslant 40$
$\mathscr{S}_2$	$+$	$\mathrm{U}_4(q)$	$\frac{1}{(4,q+1)}q^6 \prod_{j=2}^{4}(q^j - (-1)^j)$	$\neq 2$	$\geqslant 112$

Before considering containments we will first prove the following lemma as it will be very useful later on.

Lemma 10.1.10. *Let ρ be a 15-dimensional absolutely irreducible representation of $\mathrm{SL}_6^{\pm}(p)$ acting on the exterior square $\Lambda^2(V_6)$ as in Proposition 9.3.11. Then $(\mathrm{SL}_6^{\pm}(p).\mathrm{Out}(\mathrm{L}_6^{\pm}(p)))\rho \leqslant \mathrm{SL}_{15}^{\pm}(p).\langle \delta^t_{\mathrm{SL}_{15}^{\pm}(p)}, \gamma_{\mathrm{SL}_{15}^{\pm}(p)}\rangle$, where $t = (p, 5 \mp 1)$.*

Proof. By Proposition 9.3.11, $\mathrm{SL}_6^{\pm}(2)\rho \cong \mathrm{SL}_6^{\pm}(2)$ and $\mathrm{SL}_6^{\pm}(p)\rho \cong (p \mp 1, 3).\mathrm{L}_6^{\pm}(p)$ in odd characteristic.

Suppose first that p is odd and let $r = (3, p \mp 1)$. Then $(r.\mathrm{L}_6^{\pm}(p)).2)\rho \leqslant \mathrm{SL}_{15}^{\pm}(p)$ and $(r.\mathrm{L}_6^{\pm}(p).2)\rho$ is stabilised by $\langle \delta^t_{\mathrm{SL}_{15}^{\pm}(p)}, \gamma_{\mathrm{SL}_{15}^{\pm}(p)} \rangle$. Furthermore, $|\mathrm{Out}(\mathrm{L}_6^{\pm}(p))| = 2r \cdot 2$ and $|\langle \delta_{\mathrm{SL}_{15}^{\pm}(p)}, \gamma_{\mathrm{SL}_{15}^{\pm}(p)} \rangle| = r \cdot 2$. It follows that $(r.\mathrm{L}_6^{\pm}(p).\mathrm{Out}(\mathrm{L}_6^{\pm}(p)))\rho$ is a subgroup of $\mathrm{SL}_{15}^{\pm}(q).\langle \delta^t_{\mathrm{SL}_{15}^{\pm}(p)}, \gamma_{\mathrm{SL}_{15}^{\pm}(p)} \rangle$.

In even characteristic we can similarly show that $\mathrm{SL}_6^{\pm}(2).\mathrm{Out}(\mathrm{L}_6^{\pm}(2)) \cong \mathrm{SL}_6^{\pm}(2).\langle \delta_{\mathrm{SL}_6^{\pm}(2)}, \gamma_{\mathrm{SL}_6^{\pm}(2)} \rangle$ is a subgroup of $\mathrm{SL}_{15}^{\pm}(2).\langle \delta_{\mathrm{SL}_{15}^{\pm}(2)}, \gamma_{\mathrm{SL}_{15}^{\pm}(2)} \rangle$. □

Proposition 10.1.11 ($\mathscr{S}$-maximal subgroups in dimension 15 - **L**/**U**).
Let $t = (p-1, 5)$ in Case **L** *and let $t = (p+1, 5)$ in Case* **U**. *All $\mathscr{S}_2$-maximal subgroups considered in Theorem 10.1.9 that preserve only the zero or a unitary form are $\mathscr{S}$-maximal in dimension* 15.

- *(i) No extensions of* $3.\mathrm{M}_{22}$ *or* $3.\mathrm{A}_7$ *are $\mathscr{S}$-maximal in any extension of* $\mathrm{SU}_{15}(2)$.
- *(ii) No extension of* M_{12} *is $\mathscr{S}$-maximal in any extension of* $\mathrm{SL}_{15}(3)$.
- *(iii) No extension of* $3.\mathrm{A}_{7_2}$ *is $\mathscr{S}$-maximal in any extension of* $\mathrm{SU}_{15}(5)$.
- *(iv) No extension of* $t \times 3.\mathrm{A}_{7_1}$ *is $\mathscr{S}$-maximal in any extension of* $\mathrm{SL}_{15}^{\pm}(p)$. *If $p \equiv 1, 7, 17, 23 \pmod{24}$, $p \neq 7$, then $t \times 3.\mathrm{A}_{7_2}$ is not $\mathscr{S}$-maximal in $\mathrm{SL}_{15}^{\pm}(p)$ but extends to a novelty under $\langle\gamma\rangle$.*
- *(v) No extension of* $t \times 3.\mathrm{L}_3(4).2_1$ *is $\mathscr{S}$-maximal in any extension of* $\mathrm{SL}_{15}^{\pm}(p)$.
- *(vi) No extension of* $t \times 3_1.\mathrm{U}_4(3).2_2$ *is $\mathscr{S}$-maximal in any extension of* $\mathrm{SL}_{15}^{\pm}(p)$.

All other $\mathscr{S}_1$-maximal subgroups with Schur indicator $\circ$ considered in Theorem 10.1.9 are $\mathscr{S}$-maximal.

Proof. To begin with we will prove that all $\mathscr{S}_2$-maximal subgroups given in Theorem 10.1.9 that preserve either only the zero or a unitary form are $\mathscr{S}$-maximal. The only $\mathscr{S}_2$-subgroup in even characteristic is $\mathrm{SL}_6^{\pm}(q)$ and $q^{15} \mid |\mathrm{SL}_6^{\pm}(q)|$. However none of the $\mathscr{S}_1$-subgroups with Schur indicator $\circ$ in Theorem 10.1.9 has order divisible by q^{15}. In characteristic $\geqslant 3$, we need to show that none of the $\mathscr{S}_1$-subgroups has order divisible by q^{10} since $q^{10} \mid |\mathrm{SL}_5^{\pm}(q)|$ and in characteristic $\geqslant 5$ we have to show that none of the $\mathscr{S}_1$-subgroups has order divisible by q^3 since $q^3 \mid |\mathrm{SL}_3^{\pm}(q)|$. By looking at Table 10.1.3 it is straightforward to see that the maximal $\mathscr{S}_2$-subgroups in Theorem 10.1.9 are all $\mathscr{S}$-maximal in Cases **L** and **U**.

We will now consider containments of $\mathscr{S}_1$- in $\mathscr{S}_2$-subgroups. By Lemma 10.1.2, Theorem 10.1.9, Table 10.1.3 and [8, Thm 4.3.3, p.162] we have the following potential containments:

(i) $3.\mathrm{A}_7 \leqslant \mathrm{SU}_3(5) \leqslant \mathrm{SU}_{15}(3)$,
(ii) $3.\mathrm{M}_{22} \leqslant \mathrm{SU}_6(2) \leqslant \mathrm{SU}_{15}(2)$,
(iii) $\mathrm{M}_{12} \leqslant \mathrm{L}_6(3) \leqslant \mathrm{SL}_{15}(3)$,
(iv) $3.\mathrm{A}_7 \leqslant (p \mp 1, 3).\mathrm{L}_6^{\pm}(p) \leqslant \mathrm{SL}_{15}^{\pm}(p)$,
(v) $3.\mathrm{L}_3(4) \leqslant (p \mp 1, 3).\mathrm{L}_6^{\pm}(p) \leqslant \mathrm{SL}_{15}^{\pm}(p)$ and
(vi) $3_1.\mathrm{U}_4(3) \leqslant (p \mp 1, 3).\mathrm{L}_6^{\pm}(p) \leqslant \mathrm{SL}_{15}^{\pm}(p)$.

By [8, Table 8.6, p.379], $\mathrm{SU}_3(5)$ has (up to equivalence) one $\mathscr{S}_1$-subgroup H isomorphic to $\mathbf{3.A_7}$. Computer calculations (file s1ins2cont) show that there exists an absolutely irreducible 15-dimensional representation ρ of H on $\mathrm{S}^4(V_3)$. Furthermore, by [24], $H\rho$ is equivalent to $3.\mathrm{A}_{7_2}$. By [8, Table 8.6, p.379], H has stabiliser $\langle \gamma_{\mathrm{SU}_3(5)} \rangle$ and $3.\mathrm{A}_{7_2}$ has stabiliser $\langle \gamma_{\mathrm{SU}_{15}(5)} \rangle$ by Proposition 7.2.3. By Proposition 9.3.9, $\mathrm{SU}_3(5)_2$ has stabiliser $\langle \delta_{\mathrm{SU}_{15}(5)}, \gamma_{\mathrm{SU}_{15}(5)} \rangle$. Since $\gamma_{\mathrm{SU}_{15}(5)}$ induces $\gamma_{\mathrm{SU}_3(5)}$ by Lemma 8.5.2 it follows that no extension of $3.\mathrm{A}_{7_2}$ is ever $\mathscr{S}$-maximal in characteristic 5.

Computer calculations (file s1ins2cont) show that there exists $\mathbf{3.M_{22}} \leqslant \mathrm{SU}_6(2) \leqslant \mathrm{SU}_{15}(2)$ which is absolutely irreducible in dimension 15 and is stabilised by $\langle \gamma_{\mathrm{SU}_6(2)} \rangle$ in $\mathrm{Out}(\mathrm{U}_6(2))$ and by $\langle \gamma_{\mathrm{SU}_{15}(2)} \rangle$ in $\mathrm{Out}(\mathrm{U}_{15}(2))$. By Lemma 10.1.10, $(\mathrm{SU}_6(2).\mathrm{Out}(\mathrm{U}_6(2)))\rho_{15} \leqslant \mathrm{SU}_{15}(2).\langle \delta_{\mathrm{SU}_{15}(2)}, \gamma_{\mathrm{SU}_{15}(2)} \rangle$ which implies that $3.\mathrm{M}_{22}$ is never $\mathscr{S}$-maximal.

Throughout the remainder of this proof let ρ_{15} be a 15-dimensional absolutely irreducible representation of $\mathrm{SL}_6^{\pm}(q)$ acting on $\Lambda^2(V_6)$ as in Proposition 9.3.11.

By [8, Table 8.25, p.389] there exists an $\mathscr{S}_1$-subgroup H of $\mathrm{SL}_6(3)$ isomorphic to $2.\mathbf{M_{12}}$ which has stabiliser $\langle (\gamma\delta)_{\mathrm{SL}_6(3)} \rangle$. Computer calculations (file s1ins2cont) show that $H\rho_{15}$ is absolutely irreducible and that $H\rho_{15} \cong \mathrm{M}_{12}$. Furthermore, the image G of a 15-dimensional absolutely irreducible representation of M_{12} has stabiliser $\langle \gamma_{\mathrm{SL}_{15}(3)} \rangle$ by Proposition 7.2.8. By Lemma 10.1.10, $G.\langle \gamma_{\mathrm{SL}_{15}(3)} \rangle \leqslant (\mathrm{SL}_6(3).\mathrm{Out}(\mathrm{L}_6(3)))\rho_{15} \leqslant \mathrm{SL}_{15}(3).\langle \gamma_{\mathrm{SL}_{15}(3)}, \delta_{\mathrm{SL}_{15}(3)} \rangle$. It follows that no extension of G is $\mathscr{S}$-maximal.

Now we will consider $\mathbf{3.A_7} \leqslant \mathrm{SL}_6^{\pm}(p)$. Let H be an $\mathscr{S}_1$-subgroup of $\mathrm{SL}_6^{\pm}(p)$ isomorphic to $3.\mathrm{A}_7$. Using Lemma 8.2.5 and [12, 24], we can show that $H\rho_{15}$ is absolutely irreducible in odd characteristic and computer calculations (file s1ins2cont) show that $H\rho_{15}$ is also absolutely irreducible in even characteristic. In odd characteristic we can furthermore show that $H\rho_{15}$ is equivalent to $3.\mathrm{A}_{7_1}$. By [8, Prop 4.7.10, p.206], H is either stabilised by $\langle (\gamma\delta)_{\mathrm{L}_6^{\pm}(p)} \rangle$ or by $\langle \gamma_{\mathrm{L}_6^{\pm}(p)} \rangle$. Furthermore, $3.\mathrm{A}_{7_1}$ has stabiliser $\langle \gamma_{\mathrm{L}_{15}^{\pm}(p)} \rangle$ and in even characteristic an $\mathscr{S}_1$-subgroup G of $\mathrm{SU}_{15}(2)$ isomorphic to $3.\mathrm{A}_7$ has stabiliser $\langle \gamma_{\mathrm{U}_{15}(2)} \rangle$. By Lemma 10.1.10, $r.\mathrm{L}_6^{\pm}(p).\mathrm{Out}(\mathrm{L}_6^{\pm}(p)) \leqslant \mathrm{SL}_{15}^{\pm}(p).\langle \delta_{\mathrm{SL}_{15}^{\pm}(p)}, \gamma_{\mathrm{SL}_{15}^{\pm}(p)} \rangle$, where $r = (3, \mp 1)$. This implies that no extension of $3.\mathrm{A}_{7_1}$ or of G is $\mathscr{S}$-maximal.

Continuing with $3.\mathrm{A}_7$, let H be an $\mathscr{S}_1$-subgroup of $\mathrm{SL}_6^{\pm}(p)$ isomorphic to $6.\mathrm{A}_7$. By [8, Table 8.25, p.389 and Table 8.27, p.391], we find that $H \leqslant \mathrm{SL}_6^{\pm}(p)$ if and only if $p \equiv 1, 7, 17, 23 \pmod{24}$. Using Magma (see file s1ins2cont) we can show that $H\rho_{15}$ is equivalent to $3.\mathrm{A}_{7_2}$. However, $3.\mathrm{A}_{7_2} \leqslant \mathrm{SL}_{15}^{\pm}(p)$ for all $p \neq 2, 3, 7$. Hence, $t \times 3.\mathrm{A}_{7_2}$ is $\mathscr{S}$-maximal if $p \equiv 5, 11, 13, 19$ (mod 24). Furthermore, H has trivial stabiliser whereas $3.\mathrm{A}_{7_2}$ is stabilised by $\langle \gamma_{\mathrm{SL}_{15}^{\pm}(p)} \rangle$. It follows that $3.\mathrm{A}_{7_2}$ extends to a novelty if $p \equiv 1, 7, 17, 23$ (mod 24), $p \neq 7$.

The next group to consider is $\mathbf{3.L_3(4)}$. By [8, Table 8.25, p.389 and Table 8.27, p.391] there exists an $\mathscr{S}_1$-subgroup H of $\mathrm{SL}_6^{\pm}(p)$ isomorphic to $6.\mathrm{L}_3(4)$. Here $H \leqslant \mathrm{SL}_6(p)$ if $p \equiv 1 \pmod 3$ and $H \leqslant \mathrm{SU}_6(p)$ if $p \equiv 2 \pmod 3$, $p \neq 2$. We also find that $H.2_1^- \leqslant \mathrm{SL}_6^{\pm}(p)$ if $p \equiv 1, 5, 19, 23 \pmod{24}$ with stabiliser $\langle \gamma_{\mathrm{SL}_6^{\pm}(p)} \rangle$. Otherwise $H \leqslant \mathrm{SL}_6^{\pm}(p)$ with stabiliser $\langle \delta^3_{\mathrm{SL}_6^{\pm}(p)}, \gamma_{\mathrm{SL}_6^{\pm}(p)} \rangle$, where $\delta^3_{\mathrm{SL}_6^{\pm}(p)}$ induces the 2_1 outer automorphism of H. By Proposition 7.2.6 there exists an absolutely irreducible subgroup G of $\mathrm{SL}_{15}^{\pm}(p)$ isomorphic to $3.\mathrm{L}_3(4)$. Here $G.2_1 \leqslant \mathrm{SL}_{15}(p)$ if $p \equiv 1 \pmod 3$ and $G.2_1 \leqslant \mathrm{SU}_{15}(p)$ if $p \equiv 2 \pmod 3$. In both cases $G.2_1$ is stabilised by $\langle \gamma_{\mathrm{SL}_{15}^{\pm}(p)} \rangle$. Computer calculations (file s1ins2cont) show that $H\rho_{15}$ is equivalent to G.

Let $H.2_1 = \langle H, A \rangle$. Then $H.2_1^- = \langle H, \mathrm{i}A \rangle$ by [12, p.xxiii]. Furthermore, $(\mathrm{i}A)\rho_{15} = \mathrm{i}^2(A\rho_{15}) = -A\rho_{15}$. Since $\ker(\rho_{15}) = \pm 1$ it follows that $(H.2_1^-)\rho_{15} \cong 3.\mathrm{L}_3(4).2_1$. Hence, if $H.2_1^- \leqslant \mathrm{SL}_6(p)$ then G is not $\mathscr{S}$-maximal. Now suppose that $H \leqslant \mathrm{SL}_6^{\pm}(p)$ but $H.2_1^- \nleqslant \mathrm{SL}_6(p)$. By Proposition 9.3.11, $(\mathrm{SL}_6^{\pm}(p).\langle \delta^3_{\mathrm{L}_6^{\pm}(p)} \rangle)\rho_{15} \leqslant \mathrm{SL}_{15}^{\pm}(p)$ in the cases we are interested in. Since $\delta^3_{\mathrm{L}_6^{\pm}(p)}$ induces the 2_1 outer automorphism of H, it follows that we always have $G.2_1 \leqslant (\mathrm{SL}_6^{\pm}(p).\langle \delta^3_{\mathrm{L}_6^{\pm}(p)} \rangle)\rho_{15} \leqslant \mathrm{SL}_{15}^{\pm}(p)$. Hence $G.2_1$ is not $\mathscr{S}$-maximal. By Lemma 10.1.10, no extension of $t \times 3.\mathrm{L}_3(4).2_1$ is ever $\mathscr{S}$-maximal.

Our final group to consider is $\mathbf{3_1.U_4(3)}$. We will first assume that p is odd. Then the proof works similarly to the proof of $3.\mathrm{L}_3(4)$. By [8, Table 8.25, p.389 and Table 8.27, p.391] there exists an $\mathscr{S}_1$-subgroup H of $\mathrm{SL}_6^{\pm}(p)$ isomorphic to $6_1.\mathrm{U}_4(3)$. Here either $H.2_2^- \leqslant \mathrm{SL}_6^{\pm}(p)$ with stabiliser $\langle \gamma_{\mathrm{SL}_6^{\pm}(p)} \rangle$ or $H \leqslant \mathrm{SL}_6^{\pm}(p)$ with stabiliser $\langle \delta^3_{\mathrm{SL}_6^{\pm}(p)}, \gamma_{\mathrm{SL}_6^{\pm}(p)} \rangle$, where $\delta^3_{\mathrm{SL}_6^{\pm}(p)}$ induces the 2_2 outer automorphism of H. By Proposition 7.2.7 there exists an $\mathscr{S}_1$-subgroup G of $\mathrm{SL}_{15}^{\pm}(p)$ isomorphic to $3_1.\mathrm{U}_4(3)$ and $G.2_2 \leqslant \mathrm{SL}_{15}^{\pm}(p)$ with stabiliser $\langle \gamma_{\mathrm{SL}_{15}^{\pm}(p)} \rangle$. Computer calculations (file s1ins2cont) show that $H\rho_{15}$ is equivalent to G. As for $3.\mathrm{L}_3(4)$ we can similarly show that $G.2_2 \leqslant (3.\mathrm{L}_6^{\pm}(p).\langle \delta^3_{\mathrm{L}_6^{\pm}(p)} \rangle)\rho_{15} \leqslant \mathrm{SL}_{15}^{\pm}(p)$. By Lemma 10.1.10, no extension of $t \times$

$3_1.\mathrm{U}_4(3).2_2$ is ever $\mathscr{S}$-maximal in $\mathrm{SL}^{\pm}_{15}(p)$ in odd characteristic.

In even characteristic there is an $\mathscr{S}_1$-subgroup H in $\mathrm{SU}_6(2)$ isomorphic to $3_1.\mathrm{U}_4(3)$ by [8, Table 8.27, p.391]. In fact we have $H.2_2 \leqslant \mathrm{SU}_6(2)$ which has stabiliser $\langle \gamma_{\mathrm{SU}_6(2)} \rangle$. Computations in Magma (file s1ins2cont) show that $H\rho_{15}$ is absolutely irreducible and hence $H\rho_{15}.2_2$ has stabiliser $\langle \gamma_{\mathrm{SU}_{15}(2)} \rangle$. It follows from Lemma 10.1.10 that no extension of $3_1.\mathrm{U}_4(3).2_2$ is $\mathscr{S}$-maximal in any extension of $\mathrm{SU}_{15}(2)$. □

Proposition 10.1.12 ($\mathscr{S}$-maximal subgroups in dimension 15 - Case $\mathbf{O}^{\circ}$). *All $\mathscr{S}_2$-maximal subgroups in Theorem 10.1.9 that preserve an orthogonal form are $\mathscr{S}$-maximal.*

(i) No extension of $\mathrm{A}_7.2$ is $\mathscr{S}$-maximal in any extension of $\Omega^{\circ}_{15}(p)$.

(ii) No extension of $\mathrm{U}_4(2)_1.2$ is $\mathscr{S}$-maximal in any extension of $\Omega^{\circ}_{15}(p)$.

All other $\mathscr{S}_1$-maximal subgroups with Schur indicator + in Theorem 10.1.9 are $\mathscr{S}$-maximal.

Proof. We will first consider potential containments of $\mathscr{S}_2$- in $\mathscr{S}_1$-subgroups. Here we have to consider $\mathrm{L}_2(q)$, $p \geqslant 17$, and $\mathrm{L}^{\pm}_4(q)$. By Lagrange's theorem $\mathbf{L_2(\boldsymbol{q})}$ could be a subgroup of $\mathrm{A}_{17} \leqslant \Omega^{\circ}_{15}(17)$, but $\mathrm{L}_2(q)$ acts on $q+1 > 17$ points and hence there is no containment. There are no other relevant $\mathscr{S}_1$-subgroups containing $\mathrm{L}_2(q)$.

By Lagrange's theorem $\mathrm{L}^{\pm}_4(3).2$ could be a subgroup of A_{16}. But looking at the degree of the smallest permutation representations of $\mathrm{L}^{\pm}_4(3)$ in Table 10.1.3, we see that there is no containment. There are no further containments since $q^6 \mid |\mathbf{L^{\pm}_4(\boldsymbol{q}).2}|$, q odd, and none of the other $\mathscr{S}_1$-subgroups has order divisible by $q^6 = p^{6i}$ and a cross characteristic 15-dimensional absolutely irreducible representation in characteristic p.

Now we will consider potential containments of $\mathscr{S}_1$-maximal in $\mathscr{S}_2$-maximal subgroups. By Lemma 10.1.2, Theorem 10.1.9, Table 10.1.3 and [8, Thm 4.3.3, p.162], we have the following potential containments:

(i) $\mathrm{L}_3(4) \leqslant \mathrm{U}_4(3) \leqslant \Omega^{\circ}_{15}(3)$,

(ii) $\mathrm{A}_7 \leqslant \mathrm{L}^{\pm}_4(p) \leqslant \Omega^{\circ}_{15}(p)$ and

(iii) $\mathrm{U}_4(2) \leqslant \mathrm{L}^{\pm}_4(p) \leqslant \Omega^{\circ}_{15}(p)$.

From now on let ρ_{15} be an absolutely irreducible adjoint representation of $\mathrm{SL}^{\pm}_4(q)$ in odd characteristic as in Proposition 9.4.2.

We will first consider $\mathbf{L_3(4)}$. By [8, Table 8.11, p.382] there exists an $\mathscr{S}_1$-subgroup H of $\mathrm{SU}_4(3)$ isomorphic to $4_2.\mathrm{L}_3(4)$ which has stabiliser $\langle \delta^2_{\mathrm{SU}_4(3)}, (\gamma\delta)_{\mathrm{SU}_4(3)} \rangle$. Here $\delta^2_{\mathrm{SU}_4(3)}$ induces the 2_2 outer automorphism of H. By Proposition 7.3.7 there exists an $\mathscr{S}_1$-subgroup G of $\Omega^{\circ}_{15}(3)$ isomorphic to

$L_3(4)$, where $G.2_2 \leqslant \Omega^{\circ}_{15}(3)$ and has stabiliser $\langle \delta_{\Omega^{\circ}_{15}(3)} \rangle$. Computer calculations (file s1ins2cont) show that $4_2.L_3(4)\rho_{15}$ is equivalent to G. By Proposition 9.4.2, $(SU_4(3).\langle \gamma_{U_4(3)} \rangle)\rho_{15} \leqslant \Omega^{\circ}_{15}(3)$ but $(SU_4(3).\langle \delta^2_{U_4(3)} \rangle)\rho_{15} \nleqslant \Omega^{\circ}_{15}(3)$. It follows that $G.2_2 \nleqslant U_4(3).2 \leqslant \Omega^{\circ}_{15}(3)$. Hence $L_3(4).2_2$ is $\mathscr{S}$-maximal in $\Omega^{\circ}_{15}(3)$.

The next group to consider is $\mathbf{A_7}$. By [8, Table 8.9, p.381 and Table 8.11, p.382], there exists an $\mathscr{S}_1$-subgroup H of $SL^{\pm}_4(p)$ isomorphic to $2.A_7$ with stabiliser $\langle \gamma_{SL^{\pm}_4(p)} \rangle$. Furthermore, there exists an $\mathscr{S}_1$-subgroup G of $\Omega^{\circ}_{15}(p)$ isomorphic to A_7 and $G.2 \leqslant \Omega^{\circ}_{15}(p)$ by Proposition 7.3.1. Computer calculations (file s1ins2cont) show that $H\rho_{15}$ is equivalent to G. Since $(SL^{\pm}_4(p).\langle \gamma_{SL^{\pm}_4(p)} \rangle)\rho_{15} \leqslant \Omega^{\circ}_{15}(p)$ by Proposition 9.4.2 it follows that $G.2 \leqslant (SL^{\pm}_4(p).\langle \gamma_{SL^{\pm}_4(p)} \rangle)\rho_{15} \leqslant \Omega^{\circ}_{15}(p)$. Hence no extension of $A_7.2$ is $\mathscr{S}$-maximal in any extension of $\Omega^{\circ}_{15}(p)$.

Finally we will consider $\mathbf{U_4(2)}$. By [8, Table 8.9, p.381 and Table 8.11, p.382], there exists an $\mathscr{S}_1$-subgroup H of $SL^{\pm}_4(p)$ isomorphic to $2.U_4(2)$ with stabiliser $\langle \gamma_{SL^{\pm}_4(p)} \rangle$. Computations in Magma (file s1ins2cont) and [12] show that $H\rho_{15}$ is equivalent to $U_4(2)_1$. By Proposition 7.3.9, $U_4(2)_1.2 \leqslant \Omega^{\circ}_{15}(p)$. From Proposition 9.4.2 it follows that $U_4(2)_1.2 \leqslant (SL^{\pm}_4(p).\langle \gamma_{SL^{\pm}_4(p)} \rangle)\rho_{15} \leqslant \Omega^{\circ}_{15}(p)$. Hence no extension of $U_4(2)_1.2$ is $\mathscr{S}$-maximal. □

10.2 $\mathcal{G}$-Subgroups

In this section we will give a short discussion of the tables in [26] which state the maximal subgroups of geometric type of the almost simple classical groups in dimension $\geqslant 12$. In particular, we will consider the novelty subgroups given in these tables. For the tables containing the maximal subgroups of geometric type in dimension 13, 14 and 15 see Chapter 11.

Let $\Omega \in \{SL^{\pm}_n(q), Sp_n(q), \Omega^{\epsilon}_n(q)\}$ unless otherwise stated and let $\bar{T} = T/Z(T)$ for any group T. Recall that for orthogonal groups in odd dimension we only consider the case when q is odd.

Definition 10.2.1. Let H be a subgroup of a classical group T, such that $\bar{\Omega} \leqslant \bar{T} \leqslant \mathrm{Aut}(\bar{\Omega})$. If H lies in one of the Aschbacher classes $\mathcal{C}_1$ to $\mathcal{C}_8$, then H is a **$\mathcal{G}$-subgroup** of T.

Roughly speaking (see e.g. [8, Table 2.1, p.55]) the geometric classes can be described as follows:

- $\mathcal{C}_1$: stabilisers of totally singular or non-singular subspaces,
- $\mathcal{C}_2$: stabilisers of decompositions $V = \bigoplus_{i=1}^{t} V_i$, $\dim(V_i) = n/t$,
- $\mathcal{C}_3$: stabilisers of extension fields of $\mathbb{F}_{q^u}$ of prime index dividing n,

- $\mathcal{C}_4$: stabilisers of tensor product decompositions $V = V_1 \otimes V_2$,
- $\mathcal{C}_5$: stabilisers of subfields of $\mathbb{F}_{q^u}$ of prime index,
- $\mathcal{C}_6$: normalisers of symplectic type or extraspecial groups in absolutely irreducible representations,
- $\mathcal{C}_7$: stabilisers of decompositions $V = \otimes_{i=1}^{t} V_i$, $\dim(V_i) = a$, $n = a^t$,
- $\mathcal{C}_8$: groups of similarities of non-degenerate classical forms.

Note that some of the outer automorphisms of classical groups are denoted slightly differently in [26] than in this thesis as defined in Section 3.2. For a more detailed discussion of the following see [8, p.57-58]. The main things to note are as follows.

In Case **L**, the graph automorphism $\gamma \in \mathrm{Out}(\mathrm{L}_n(q))$ is denoted by $\ddot{\iota}$ in [26].

Nothing changes in Cases **U** and **S**.

In Case $\mathbf{O}^{\circ}$, the outer automorphism $\ddot{r}_{\square}\ddot{r}_{\boxtimes}$ defined in [26] stands for the diagonal automorphism $\delta \in \mathrm{Out}(\mathrm{O}_n^{\circ}(q))$.

In Case $\mathbf{O}^{+}$, we find that $\ddot{r}_{\square}$ corresponds to $\gamma \in \mathrm{Out}(\mathrm{O}_n^{+}(q))$ and in dimension 14, when $q \equiv 1 \pmod 4$ then $\ddot{r}_{\square}\ddot{r}_{\boxtimes} = \delta' \in \mathrm{Out}(\mathrm{O}_n^{+}(q))$. Hence we can deduce that $\ddot{r}_{\boxtimes}\delta$ corresponds to $\gamma\delta^3$. Furthermore, in dimension 14,

$$\ker_{\ddot{\Gamma}}(\ddot{\gamma}) = \begin{cases} \langle \phi \rangle & \text{if } q \text{ is even} \\ \langle \delta, \phi \rangle & \text{if } q \equiv 3 \pmod 4 \\ \langle \delta, \delta', \phi \rangle & \text{if } q \equiv 1 \pmod 4 \end{cases}$$

by [26, Prop 2.7.4, p.38]. Now let $\Gamma(V, \mathbb{F}_q, Q)$ denote the group of semi-similarities of a vector space V with underlying field $\mathbb{F}_q$ and non-degenerate quadratic form Q. To determine $\ker_{\ddot{\Gamma}}(\ddot{\tau})$ when q is odd, note that by [26, Lemma 2.1.2, p.12], τ is a map from $\Gamma(V, \mathbb{F}_q, Q)$ to $\mathbb{F}_q^{\times}$ given by $\tau(g) = \lambda$ for all such $g \in \Gamma(V, \mathbb{F}_q, Q)$. Here λ is uniquely determined for any g by $Q(vg) = \lambda Q(v)^{\sigma}$, $\sigma \in \mathrm{Aut}(\mathbb{F}_q)$, for all $v \in V$. Hence in dimension 14 it is clear that projectively $\ker_{\Gamma}(\tau)$ equals

$$\ker_{\ddot{\Gamma}}(\ddot{\tau}) = \begin{cases} \langle \delta', \gamma, \phi \rangle & \text{if } q \equiv 1 \pmod 4 \\ \langle \gamma, \phi \rangle & \text{if } q \equiv 3 \pmod 4. \end{cases}$$

Finally, for Case $\mathbf{O}^{-}$ note that $\ddot{r}_{\square} = \gamma \in \mathrm{Out}(\mathrm{O}_n^{-}(q))$ and that in dimension 14, $\ddot{r}_{\square}\ddot{r}_{\boxtimes} = \delta'$ when $q \equiv 3 \pmod 4$ and $\ddot{\phi} = \varphi$ when $q \equiv 1 \pmod 4$.

Furthermore we find that, in dimension 14,

$$\ker_{\ddot{\Gamma}}(\ddot{\tau}) = \begin{cases} \langle \gamma, \varphi \rangle & \text{if } q \equiv 1 \pmod 4 \\ \langle \delta', \gamma, \phi \rangle & \text{if } q \equiv 3 \pmod 4). \end{cases}$$

Let H be a representative of a conjugacy class of a $\mathcal{G}$-subgroup. The following table lists all the geometric subgroups in dimension 13, 14 and 15 where $H_{\bar{\Omega}} = \bar{H} \cap \bar{\Omega}$ is not maximal in $\bar{\Omega}$ but H extends to a novelty in some extension by outer automorphisms. It gives the groups H and K with their respective stabilisers such that $H_{\bar{\Omega}} < K_{\bar{\Omega}} < \bar{\Omega}$. Note that there are always two distinct $K_{\bar{\Omega}}$ containing $H_{\bar{\Omega}}$ that are not conjugate in $\bar{\Omega}$.

Table 10.2.1: Geometric Novelty Groups

$a = k(n-k)$, $b = k(2n-3k)$, $d = (q-1, 2)$

Case	**Type of H**	**Stab of H**	**Type of K**	**Stab of K**
L	$(\mathrm{SL}_k(q) \times \mathrm{SL}_{n-k}(q)) : (q-1)$	$\langle \delta, \gamma, \phi \rangle$	$\mathrm{E}_q^a : (\mathrm{SL}_k(q) \times \mathrm{SL}_{n-k}(q)) : (q-1)$	$\langle \delta, \phi \rangle$
	$\mathrm{E}_q^b : (\mathrm{SL}_k(q)^2 \times \mathrm{SL}_{n-2k}(q)).(q-1)^2$	$\langle \delta, \gamma, \phi \rangle$	$\mathrm{E}_q^a : (\mathrm{SL}_k(q) \times \mathrm{SL}_{n-k}(q)) : (q-1)$	$\langle \delta, \phi \rangle$
O$^+$	$\mathrm{E}_q^{27} : (\frac{1}{d}\mathrm{GL}_6(q) \times \Omega_2^+(q)).d$	$\langle \delta, \gamma, \phi \rangle$	$\mathrm{E}_q^{21} : \frac{1}{d}\mathrm{GL}_7(q)$	$\langle \delta, \phi \rangle$
	$(\Omega_2^+(3) \times \Omega_{12}^+(3)).2^2$	$\langle \delta, \gamma, \phi \rangle$	$\Omega_{13}^\circ(3).2$	$\langle \delta', \gamma, \phi \rangle$
	$\Omega_2^+(5)^7.2^{12}.\mathrm{S}_7$	$\langle \delta, \gamma, \phi \rangle$	$2^{13}.\mathrm{A}_{14}$	$\langle \delta', \gamma, \phi \rangle$
	$\mathrm{SL}_7(q).\frac{(q-1)}{d}$	$\langle \delta, \gamma, \phi \rangle$	$\mathrm{E}_q^{21} : \frac{1}{d}\mathrm{GL}_7(q)$	$\langle \delta, \phi \rangle$
O$^-$	$(\Omega_2^+(3) \times \Omega_{12}^-(3)).2^2$	$\langle \delta, \gamma, \phi \rangle$	$\Omega_{13}^\circ(3).2$	$\langle \delta', \gamma, \phi \rangle$
	$\Omega_2^-(3)^7.2^{12}.\mathrm{S}_7$	$\langle \delta, \gamma, \phi \rangle$	$2^{13}.\mathrm{A}_{14}$	$\langle \delta', \gamma, \phi \rangle$

Let $H_{\bar{\Omega}} \leqslant K_{\bar{\Omega}} \leqslant \bar{\Omega} \leqslant T \leqslant \mathrm{Aut}(\bar{\Omega})$. The following lemma shows when $\mathrm{N}_T(H_{\bar{\Omega}})\bar{\Omega} = T$ but $\mathrm{N}_T(K_{\bar{\Omega}})\bar{\Omega} \neq T$, i.e. when $H_{\bar{\Omega}}$ extends to a novelty.

Lemma 10.2.2. *Let H and K be as given in Table 10.2.1. Suppose that $K_{\bar{\Omega}}$ has stabiliser $R < \mathrm{Out}(\bar{\Omega})$ and $H_{\bar{\Omega}}$ has stabiliser $\langle R, \kappa \rangle \leqslant \mathrm{Out}(\bar{\Omega})$ where $\kappa \notin R$ and $|\kappa| = 2$. Then $H_{\bar{\Omega}}.R'$ is never maximal in $\bar{\Omega}.R'$ for any subgroup $R' \leqslant R$. However $H_{\bar{\Omega}}.\langle \kappa, R' \rangle$ is maximal in $\bar{\Omega}.\langle \kappa, R' \rangle$ for any $R' \leqslant R$.*

Proof. Note that there exists one $\bar{\Omega}$-class of $H_{\bar{\Omega}}$ but two $\bar{\Omega}$-conjugacy classes of $K_{\bar{\Omega}}$. Denote the $\bar{\Omega}$-conjugacy class of $H_{\bar{\Omega}}$ by $[H_{\bar{\Omega}}] = \{H_{\bar{\Omega}}^x \mid x \in \bar{\Omega}\}$ and let $[K_1]$ and $[K_2]$ denote the two distinct conjugacy classes of $K_{\bar{\Omega}}$. Without loss of generality let K_1 and K_2 be the two representatives of $[K_1]$ and $[K_2]$ respectively that contain $H_{\bar{\Omega}}$. Then they are the unique members of their respective conjugacy classes that contain $H_{\bar{\Omega}}$. Let $T = \bar{\Omega}.R'$. Then R' fixes one and hence all the $\bar{\Omega}$-conjugacy classes. Since $H_{\bar{\Omega}} = \mathrm{N}_{\bar{\Omega}}(H_{\bar{\Omega}})$ by [26, Lemma 3.2.1, p.63] it follows that $H_{\bar{\Omega}}.R' = \mathrm{N}_T(H_{\bar{\Omega}})$. Therefore $\mathrm{N}_T(H_{\bar{\Omega}})$ fixes the class $[K_i]$ for all $i \in \{1, 2\}$. Furthermore since $H_{\bar{\Omega}}$ is normalised by $\mathrm{N}_T(H_{\bar{\Omega}})$, K_i has to be normalised by $\mathrm{N}_T(H_{\bar{\Omega}})$ since K_i is the unique member of its respective conjugacy class such that $H_{\bar{\Omega}} \leqslant K_i$. Hence $\mathrm{N}_T(H_{\bar{\Omega}}) \leqslant \mathrm{N}_T(K_i)$ and $H_{\bar{\Omega}}.R'$ is never maximal in $\bar{\Omega}.R'$.

If $T = \bar{\Omega}.\langle \kappa, R' \rangle$ then $\mathrm{N}_T(H_{\bar{\Omega}}) = H_{\bar{\Omega}}.\langle \kappa, R' \rangle$. Since $[K_1]^\kappa = [K_2]$ it follows that $\mathrm{N}_T(H_{\bar{\Omega}}) \nleqslant \mathrm{N}_T(K_i)$. By [26, p.68] all $\bar{\Omega}.\langle \kappa, R' \rangle$ novelties that occur in dimensions 13, 14 and 15 are maximal in $\bar{\Omega}.\langle \kappa, R' \rangle$. □

10.3 $\mathcal{G}$ and $\mathscr{S}$ Containments

We will first develop some theory that will be useful when it comes to determining the maximal subgroups of the quasisimple classical groups and their extensions by outer automorphisms. Let $\Omega \in \{\mathrm{SL}_n^{\pm}(q), \mathrm{Sp}_n(q), \Omega_n^{\epsilon}(q)\}$ with $n \in \{13, 14, 15\}$.

Lemma 10.3.1. *Let $G = \mathrm{Z}(G).S$ be quasisimple, where S is non-abelian simple, and let $G \leqslant H$. Then $K.S$ embeds in N_i/N_{i+1} for some i, where $H = N_0 \rhd N_1 \rhd \ldots \rhd N_t = 1$ is a composition series of H and $K \leqslant \mathrm{Z}(G)$.*

Proof. Consider the chain $G = N_0 \cap G \geqslant N_1 \cap G \geqslant \ldots \geqslant N_t \cap G = 1$. Let $a \in N_i \cap G$, $b \in N_{i-1} \cap G$. Then $b^{-1}ab \in N_i \cap G$ as $N_i \lhd N_{i-1}$ and $a, b \in G$. Hence we have the following chain $G = N_0 \cap G \trianglerighteq N_1 \cap G \trianglerighteq \ldots \trianglerighteq N_t \cap G = 1$. Since we have a chain of normal subgroups and all normal subgroups of G are contained in $\mathrm{Z}(G)$ there exists some i such that $N_i \cap G = G$ and $N_{i+1} \cap G \leqslant \mathrm{Z}(G)$. Hence $(N_i \cap G)/(N_{i+1} \cap G) \cong K.S$, where $K \leqslant \mathrm{Z}(G)$ and $K.S$ is quasisimple. Then $G \leqslant N_i$ and $N_{i+1} \cap (K.S) = 1$. It follows that $GN_{i+1}/N_{i+1} \leqslant N_i/N_{i+1}$. Furthermore,

$$\begin{aligned} GN_{i+1}/N_{i+1} &= (K.S)N_{i+1}/N_{i+1} \\ &\cong K.S/((K.S) \cap N_{i+1}) \text{ (2nd Isomorphism Thm)} \\ &= K.S. \end{aligned}$$

Hence $K.S$ embeds in N_i/N_{i+1}. □

Lemma 10.3.2. *Let $H = (\mathrm{GL}_1(q) \wr \mathrm{S}_n) \cap \Omega$ be a $\mathcal{C}_2$-subgroup of Ω in dimension n. Let $S \leqslant \mathrm{A}_n$ be non-abelian simple and assume that S is an $\mathscr{S}$-subgroup of Ω. Then S is not a subgroup of H.*

Proof. Assume that $S \leqslant H$. Since S is non-abelian simple, S is a subgroup of the non-abelian composition factor A_n of H by Lemma 10.3.1. However, A_n is reducible in dimension n and hence cannot contain an irreducible subgroup which leads to a contradiction. □

Lemma 10.3.3 ([8, Prop 5.3.10, p.287])**.** *Let $K = \mathrm{S}^n(V_2) \otimes \mathrm{S}^m(V_2)$ for $n \geqslant m \geqslant 1$, where $\mathrm{S}^k(V_2)$ is a symmetric power of a 2-dimensional vector space as in Section 8.3. Then K is reducible.*

We will also need to introduce the concept of an induced representation. These will be useful when we want to show whether a group G is a subgroup of group $H \in \mathcal{C}_2$.

Definition 10.3.4. Let X be a finite group and let $Y \leqslant X$. Let τ be a representation of Y with character χ. Then for all $g \in X$, the **induced character** χ^X is given by

$$\chi^X(g) = \frac{1}{|Y|} \sum_{x \in X} \chi^\circ(xgx^{-1}),$$

where

$$\chi^\circ(y) = \begin{cases} \chi(y) & \text{if } y \in Y \\ 0 & \text{otherwise.} \end{cases}$$

The representation corresponding to χ^X is the **induced representation** τ^X of X.

Theorem 10.3.5 ([23, Thm 5.8, p.65])**.** *Let $\rho : X \to \mathrm{GL}(V)$ be an irreducible characteristic 0 representation with character μ. Let $V = W_1 + \ldots + W_k$ be an imprimitive decomposition of V. Let $Y \leqslant X$ and suppose that $Y\rho$ is the stabiliser of W_i for some i. Let χ be the character of Y acting on W_i. Then $\mu = \chi^X$.*

Theorem 10.3.6 ([23, Thm 5.9, p.65])**.** *Let $Y \leqslant X$, where X is a finite group and let $\tau : Y \to \mathrm{GL}(W)$. Then $X\tau^X$ is an imprimitive matrix group acting on $V = W_1 + \ldots + W_k$, where $W_i \cong W$ and $k = |X : Y|$. Furthermore, $Y\tau$ stabilises one of these subspaces W_i.*

Lemma 10.3.7 ([23, p.64])**.** *Let $Y \leqslant X$ and let $\tau = \tau_1 + \tau_2$ be a reducible representation of Y. Then the induced representation $\tau^X = (\tau_1 + \tau_2)^X = \tau_1^X + \tau_2^X$ is reducible.*

We will now determine the maximal subgroups of T, where $\bar{\Omega} \leqslant \bar{T} \leqslant \mathrm{Aut}(\bar{\Omega})$ in dimension 13, 14 and 15. Let $X \leqslant T$ and let $X_\Omega = X \cap \Omega$. We will consider each dimension separately. By [26] all maximal geometric subgroups of T in dimension 13, 14 and 15 are maximal. Hence we only have to consider potential containments of $\mathscr{S}$-subgroups in $\mathcal{G}$-subgroups.

10.3.1 Maximal Subgroups in Dimension 13

We will start by determining the maximal subgroups of classical groups in dimension 13. We can show that there are no containments of $\mathscr{S}$- in $\mathcal{G}$-maximal subgroups.

Proposition 10.3.8. *All $\mathscr{S}$-maximal subgroups in Proposition 10.1.4 are maximal in their respective quasisimple classical groups.*

Proof. For an $\mathscr{S}$-subgroup G_Ω to sit inside a $\mathcal{G}$-subgroup H_Ω, the group H_Ω must lie in Class $\mathcal{C}_2$, $\mathcal{C}_4$, $\mathcal{C}_6$ or $\mathcal{C}_7$. This implies that we only have to consider groups H_Ω in Classes $\mathcal{C}_2$ and $\mathcal{C}_6$ in Cases **L** and **U** and groups in Class $\mathcal{C}_2$ in Case $\mathbf{O}^\circ$ by the tables in [26]. It is clear that there are no containments when H_Ω lies in Class $\mathcal{C}_6$ as no cover of either of $\mathrm{S}_6(3)$ or $\mathrm{U}_3(4)$ has a 2-dimensional absolutely irreducible representation.

We will now consider the possible containments when H_Ω lies in Class $\mathcal{C}_2$. Then $H_\Omega = (q \mp 1)^{12}.\mathrm{S}_{13}$ in Cases **L** and **U** respectively or $H_\Omega = 2^{12}.\mathrm{A}_{13}(.2)$ in Case $\mathbf{O}^\circ$. Let S be the simple part of $G_\Omega = \mathrm{Z}(G_\Omega).S$. Then $K.S$ has to be a subgroup of A_{13} for some $K \leqslant \mathrm{Z}(G_\Omega)$ by Lemma 10.3.1. By Lagrange's theorem and considering the smallest permutation representations of S (Table 10.1.1), the only possible containments are $G_\Omega = \mathrm{A}_8$ in $H_\Omega = 2^{12}.\mathrm{A}_{13}$ in characteristic 3 or $G_\Omega = \mathrm{L}_3(3)$ in $H_\Omega = 2^{12}.\mathrm{A}_{13}(.2)$, both in Case $\mathbf{O}^\circ$. However none of these containments are possible by Lemma 10.3.2. □

10.3.2 Maximal Subgroups in Dimension 14

In the following lemmas we will consider the potential containments of $\mathscr{S}$-subgroups in $\mathcal{G}$-subgroups. We will then prove in Proposition 10.3.12 that the $\mathscr{S}$-maximal subgroups in Proposition 10.1.6, Proposition 10.1.7 and Proposition 10.1.8 are in fact maximal.

Lemma 10.3.9. *Let $G_\Omega = \mathrm{SL}_2(q)$, $p \geqslant 17$, be an $\mathscr{S}$-maximal subgroup of $\mathrm{Sp}_{14}(q)$. Let $H_\Omega = \mathrm{GL}_7(q).2 \leqslant \mathrm{Sp}_{14}(q)$ lie in Class $\mathcal{C}_2$. Then G_Ω is not a subgroup of H_Ω.*

Proof. If $G_\Omega \leqslant H_\Omega$ then $(G_\Omega)^\infty \leqslant (H_\Omega)^\infty$. But $(G_\Omega)^\infty = \mathrm{SL}_2(q)$ is irreducible whereas $(H_\Omega)^\infty = \mathrm{SL}_7(q) \times \mathrm{SL}_7(q)$ is reducible, a contradiction. □

Lemma 10.3.10. *Let G be an $\mathscr{S}$-subgroup of a 14-dimensional quasisimple classical group Ω and let H be a $\mathcal{C}_2$-subgroup of Ω. If $G_\Omega/\mathrm{Z}(G_\Omega) \cong \mathrm{A}_7$, then $G_\Omega \nleqslant H_\Omega$.*

Proof. Suppose that G_Ω is imprimitive and preserves a decomposition into seven 2-dimensional subspaces. Since $G_\Omega = (2.)\mathrm{A}_7$ by Table 6.1.1, the only subgroups of G_Ω of index 7 are $K = (2.)\mathrm{A}_6$ by [12]. By [12, 24], K only has an absolutely irreducible 2-dimensional representation in characteristic 3 but G_Ω does not exist in characteristic 3 by Table 6.1.1. Hence in all our cases, K only has reducible 2-dimensional representations. Therefore, if G_Ω is an induced representation, then G_Ω is reducible by Lemma 10.3.7, a contradiction. $\square$

Lemma 10.3.11. *Let G_Ω be an $\mathscr{S}$-maximal subgroup of $\Omega^{\pm}_{14}(q)$ and let $H_\Omega \leqslant \Omega^{\pm}_{14}(q)$ lie in Class $\mathcal{C}_2$. If*

(i) $G_\Omega = \mathrm{L}_2(13)$ *or* $G_\Omega = \mathrm{G}_2(q)$, $H_\Omega = \mathrm{SO}^\circ_7(p)^2$,
(ii) $G_\Omega = \mathrm{L}_2(13)$ *or* $G_\Omega = \mathrm{G}_2(q)$, $H_\Omega = \Omega^\circ_7(q)^2.2^2.\mathrm{S}_2$,
(iii) $G_\Omega = \mathrm{L}_2(13)$ *or* $G_\Omega = \mathrm{G}_2(q)$, $H_\Omega = \mathrm{SL}_7(q).\frac{(q-1)}{(q-1,2)}$ *in Case* $\mathbf{O}^+$,
(iv) $G_\Omega = \mathrm{S}_6(2)$, $H_\Omega = \mathrm{SO}^\circ_7(3)^2$, *or*
(v) $G_\Omega = \mathrm{S}_6(2)$, $H_\Omega = \mathrm{SL}_7(3)$

then $G_\Omega \nleqslant H_\Omega$.

Proof. Since G_Ω is non-abelian simple it has to be a subgroup of either $\Omega^\circ_7(q)^2$ or $\mathrm{SL}_7(q)$. By [21], $\Omega^\circ_7(q)^2$ and $\mathrm{SL}_7(q)$ are reducible and hence cannot contain G_Ω. $\square$

Proposition 10.3.12. *All* 14*-dimensional $\mathscr{S}$-maximal subgroups in Proposition 10.1.6, Proposition 10.1.7 and Proposition 10.1.8 are maximal in their respective classical groups.*

Proof. Note that we only have to consider possible containments of $\mathscr{S}$-subgroups G_Ω in G_Ω-subgroups H_Ω when H_Ω lies in Class $\mathcal{C}_2$ or $\mathcal{C}_4$ by [26].

Suppose first that H_Ω lies in Class $\mathcal{C}_4$. Then H_Ω does not preserve an orthogonal form by [26]. In Cases **L**, **U** and **S** none of the $\mathscr{S}_1$-subgroups has a 2-dimensional absolutely irreducible representation in cross characteristic by [8, Thm 4.3.3, p.162]. The only $\mathscr{S}_2$-subgroup with a 2-dimensional absolutely irreducible representation is $\mathrm{SL}_2(q)$. Hence $G_\Omega = \mathrm{SL}_2(q)$ is a potential subgroup of $H_\Omega = \mathrm{Sp}_2(q) \circ \mathrm{GO}^\circ_7(q)$. By Section 8.3, $\mathrm{SL}_2(q)$ acts on the modules $\mathrm{S}^{n-1}(V_2)$ and therefore, G_Ω acts on $\mathrm{S}^{13}(V_2)$. If $G_\Omega \leqslant H_\Omega$ then G_Ω acts on $\mathrm{S}^1(V_2) \otimes \mathrm{S}^6(V_2)$ as well. This implies that $\mathrm{S}^1(V_2) \otimes \mathrm{S}^6(V_2) \cong \mathrm{S}^{13}(V_2)$. However, $\mathrm{S}^{13}(V_2)$ is irreducible whereas $\mathrm{S}^1(V_2) \otimes \mathrm{S}^6(V_2)$ is reducible by Lemma 10.3.3, a contradiction.

Now suppose that H_Ω lies in Class $\mathcal{C}_2$. In Cases **L** and **U** there are no $\mathscr{S}$-subgroups acting on 14 or less points by Table 10.1.2. There are also no groups with a 7-dimensional absolutely irreducible representation in the appropriate characteristics by [8, Thm 4.3.3, p.162]. Hence all $\mathscr{S}$-maximal subgroups preserving a unitary or zero form are maximal.

In Case **S** there are two potential containments of $\mathscr{S}$-subgroups in subgroups of type $\mathcal{C}_2$ by Lagrange's theorem and [29], namely $\mathrm{SL}_2(q)$ in $\mathrm{GL}_7(q).2$ and $2.\mathrm{A}_7$ in $\mathrm{Sp}_2(q)^7 : \mathrm{S}_7$. However it follows from Lemma 10.3.9 and Lemma 10.3.10 that these containments are not possible.

Finally, in Case $\mathbf{O}^{\pm}$ it follows from Lemma 10.3.2 that the images of absolutely irreducible 14-dimensional representations of $\mathrm{L}_2(13)$ and A_{7_1} are not subgroups of $H_\Omega = 2^{13}.\mathrm{A}_{14}(.2) \leqslant \Omega^{\pm}_{14}(p)$. All other potential containments are considered in Lemma 10.3.10 and Lemma 10.3.11. $\square$

10.3.3 Maximal Subgroups in Dimension 15

We will first consider a possible containment of an $\mathscr{S}$-subgroup in a $\mathcal{G}$-subgroup and then show that this is in fact the only containment.

Proposition 10.3.13. *In Cases* **L** *and* **U** *the $\mathscr{S}$-subgroup $G_\Omega = 3.\mathrm{A}_{7_2}$ is a subgroup of the $\mathcal{C}_2$-subgroup $H_\Omega = (p \mp 1)^{14}.\mathrm{S}_{15}$ in dimension 15. Furthermore, no extension of G_Ω is maximal.*

Proof. By [12] the only subgroup (up to conjugacy in $3.\mathrm{S}_7$) of $3.\mathrm{A}_7$ of index 15 is $K = 3 \times \mathrm{L}_2(7)$. We want to show that G_Ω is the image of the induced representation τ^{G_Ω}, where τ is a 1-dimensional representation of K. Using Magma (file sgcont) we can determine the character values of τ.

Let R_1, R_2 and R_3 be the conjugacy classes of K denoted by Class 3, 4 and 5 in the character tables of K (see file sgcont). Then elements of R_1, R_2 and R_3 contain elements of order 3 with character values 1, z_3 and z_3^{-1} with respect to τ. We can show that the only conjugacy classes of K that are conjugate in $3.\mathrm{A}_7$ are R_1, R_2 and R_3. Let $g \in P$, where P is the single conjugacy class of $3.\mathrm{A}_{7_2}$ containing the R_i,, and let χ be the character associated with τ. Let

$$\chi^\circ(g) = \begin{cases} \chi(g) & \text{if } g \in K \\ 0 & \text{otherwise} \end{cases}$$

as in Definition 10.3.4. Then $\sum_{x \in G_\Omega} \chi^\circ(xgx^{-1}) = \alpha(1 + \mathrm{z}_3 + \mathrm{z}_3^{-1})$ for some $\alpha \in \mathbb{N}\backslash\{0\}$. Note that P has 840 elements and $|R_i| = 56$. Hence $\alpha = |G_\Omega|\frac{3\cdot 56}{840} = \frac{|G_\Omega|}{5}$ and $\chi^{G_\Omega}(g) = \frac{1}{|K|}\sum_{x \in G_\Omega} \chi^\circ(xgx^{-1}) = \frac{|G_\Omega|}{5|K|} \cdot 0 = 0$ which is the indeed the character value of elements of P. Using similar calculations we can determine χ^{G_Ω}. We can show that G_Ω is indeed the image of τ^{G_Ω}. Hence G_Ω is an imprimitive matrix group acting on 15 1-dimensional subspaces.

Looking at [12], we can see that $3.\mathrm{S}_7$ has no index 15 subgroup. In fact, A_7 has two conjugacy classes of subgroups isomorphic to $\mathrm{L}_2(7)$ which are fused under S_7. Hence $3.\mathrm{S}_{7_2} \nleqslant H_\Omega$. Let $t \in 3.\mathrm{S}_7 \backslash 3.\mathrm{A}_7$. Then

$$3.\mathrm{S}_{7_2} \cong \left\langle \left(\begin{smallmatrix} G_\Omega & 0 \\ 0 & {G_\Omega}^t \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & I_{15} \\ I_{15} & 0 \end{smallmatrix}\right) \right\rangle,$$

since the outer automorphism of order 2 of G_Ω is induced by γ. Furthermore, γ interchanges G_Ω and ${G_\Omega}^t$ which implies that ${G_\Omega}^t = {G_\Omega}^{-\mathrm{T}}$. Since

$$H_\Omega.\langle\gamma\rangle \cong \left\langle \left(\begin{smallmatrix} H_\Omega & 0 \\ 0 & {H_\Omega}^{-\mathrm{T}} \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & I_{15} \\ I_{15} & 0 \end{smallmatrix}\right) \right\rangle$$

it follows that $G_\Omega.\langle\gamma\rangle \leqslant H_\Omega.\langle\gamma\rangle$. □

Proposition 10.3.14. *In dimension* 15 *no extension of* $3.\mathrm{A}_{7_2}$ *is maximal in any extension of* $\mathrm{SL}^{\pm}_{15}(p)$*. All other* $\mathscr{S}$*-maximal subgroups are maximal in dimension* 15*.*

Proof. Note that we only have to consider possible containments of $\mathscr{S}$-subgroups G_Ω in H_Ω when H_Ω lies in Class $\mathcal{C}_2$ or $\mathcal{C}_4$ by [26].

First suppose that H_Ω lies in Class $\mathcal{C}_4$. In Cases **L** and **U** this implies that $H_\Omega = \mathrm{SL}^{\pm}_3 \circ \mathrm{SL}^{\pm}_5(q)$. However there is no $\mathscr{S}$-subgroup G_Ω that has both an absolutely irreducible representation in dimension 3 and 5 by [8, Thm 4.3.3, p.162] and [29]. In Case $\mathbf{O}^\circ$, $G_\Omega = \mathrm{L}_2(q).2$ is a potential subgroup of $H_\Omega = (\Omega^\circ_3(q) \times \Omega^\circ_5(q)).2$. By Section 8.3, G_Ω acts on $\mathrm{S}^{14}(V_2)$. If $G_\Omega \leqslant H_\Omega$ then G_Ω also has to act on the module $\mathrm{S}^2(V_2) \otimes \mathrm{S}^4(V_2)$. By Lemma 10.3.3, $\mathrm{S}^2(V_2) \otimes \mathrm{S}^4(V_2)$ is reducible whereas $\mathrm{S}^{14}(V_2)$ is not. Hence there are no containments of $\mathscr{S}$-subgroups in H_Ω when H_Ω lies in Class $\mathcal{C}_4$.

If H_Ω lies in Class $\mathcal{C}_2$ then H_Ω has shape $M \wr \mathrm{S}_t$ for some t by [8, Table 2.5, p.63], where $M \in \{\mathrm{GL}^{\pm}_m(q), \mathrm{GO}^\circ_m(q)\}$. Hence $K.S \leqslant G_\Omega$, where $K \leqslant \mathrm{Z}(G_\Omega)$ and $S \cong G_\Omega/\mathrm{Z}(G_\Omega)$, has to be a subgroup of A_t. By looking at the smallest permutation representations of the $\mathscr{S}$-subgroups (Table 10.1.3) and using Lagrange's theorem, we can see that the only possible containments is $G_\Omega = 3.\mathrm{A}_7$ in $H_\Omega = (p \mp 1)^{14}.\mathrm{S}_{15}$ in Cases **L** and **U**, which is considered in Lemma 10.3.13. □

11 Final Tables

Let $\Omega \in \{\mathrm{SL}_n(q), \mathrm{SU}_n(q), \mathrm{Sp}_n(q), \Omega_n^{\epsilon}(q)\}$, where $n \in \{13, 14, 15\}$, and let $\Omega \leqslant T \leqslant A$ as in (3.1.1) (p.26). Let $\bar{K} = K/\mathrm{Z}(K)$ be the projective version of any group K. Then $\bar{A} = \mathrm{Aut}(\bar{\Omega})$. Furthermore, let $q = p^i$ for some prime p and some $i \geqslant 1$ and let C denote the respective conformal group of Ω. Recall that the outer automorphisms of Ω where defined in Section 3.2. The tables in this chapter contain the maximal subgroups of T but it is straightforward to deduce the maximal subgroups of the almost simple classical groups $\bar{T}$.

The tables are ordered by dimension and within each dimension we will first give the tables in Case **L**, followed by Case **U**, Case **S** and Case **O**. In dimension 14 we will first state the results for Case $\mathbf{O}^+$ and then for Case $\mathbf{O}^-$. For each family of classical group there are two tables, the first one will give the maximal subgroups of T that are of geometric type and the second one the maximal subgroups that lie in class $\mathscr{S}$. These two types of tables have very similar but not identical structure.

In general each row in a table denotes an $\mathrm{Aut}(\Omega)$-class of a maximal subgroup K of Ω. Note that unless K is a novelty, $\bar{K}.\bar{R}$ is a maximal subgroup of $\bar{\Omega}.\bar{R}$ for all $\bar{R} \leqslant \mathrm{Out}(\bar{\Omega})$ if R stabilises K.

The $\mathcal{G}$-subgroup tables

Let H be a representative of a conjugacy class of maximal geometric subgroups. The first column of the tables states the Aschbacher class in which H lies followed by the structure of H. In the 'Notes' column we give restrictions on q and p and state whether H is a novelty which we will denote by Nu, where $u \in \mathbb{N}$. The column labelled 'c' then gives the number of conjugacy classes a single C-conjugacy class of H splits into in Ω. The stabiliser of H in $\mathrm{Out}(\bar{\Omega})$ is given in the final column. If the information regarding the stabiliser of a group is too long to be conveniently included in a table, we will denote this by Su, for some $u \in \mathbb{N}$. Information regarding novelty subgroups and stabilisers can be found in an auxiliary table directly following the respective table.

The $\mathscr{S}$-subgroup tables

The first column of the tables gives the maximal subgroups G of type $\mathscr{S}$. We will use the notation in the ATLAS [12] to denote these groups. This is followed by restrictions on q and p. If any of these groups G extends to a novelty subgroup then the table includes a column 'Notes'. Otherwise this column is omitted. Again we have a column giving the number of conjugacy classes a single C-conjugacy class of G splits into in Ω and a column giving the stabiliser of G in $\mathrm{Out}(\bar{\Omega})$. Further information about novelty subgroups and stabilisers can again be found in an auxiliary table directly following the respective table.

Table 11.0.1: Maximal subgroups of $\mathrm{SL}_{13}(q)$ of geometric type

$d := |\mathrm{Z}(\mathrm{SL}_{13}(q))| = (q-1,13)$, $|\delta| = d$, $|\phi| = e$, $|\gamma| = 2$, $q = p^e$.

$\mathcal{C}_i$	**Subgroup**	**Notes**	**c**	**Stab**
$\mathcal{C}_1$	$\mathrm{E}_q^{12} : \mathrm{GL}_{12}(q)$		2	$\langle\delta,\phi\rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{22} : (\mathrm{SL}_2(q) \times \mathrm{SL}_{11}(q)) : (q-1)$		2	$\langle\delta,\phi\rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{30} : (\mathrm{SL}_3(q) \times \mathrm{SL}_{10}(q)) : (q-1)$		2	$\langle\delta,\phi\rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{36} : (\mathrm{SL}_4(q) \times \mathrm{SL}_9(q)) : (q-1)$		2	$\langle\delta,\phi\rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{40} : (\mathrm{SL}_5(q) \times \mathrm{SL}_8(q)) : (q-1)$		2	$\langle\delta,\phi\rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{42} : (\mathrm{SL}_6(q) \times \mathrm{SL}_7(q)) : (q-1)$		2	$\langle\delta,\phi\rangle$
$\mathcal{C}_1$	$\mathrm{GL}_{12}(q)$	N1	1	$\langle\delta,\gamma,\phi\rangle$
$\mathcal{C}_1$	$(\mathrm{SL}_2(q) \times \mathrm{SL}_{11}(q)) : (q-1)$	N1	1	$\langle\delta,\gamma,\phi\rangle$
$\mathcal{C}_1$	$(\mathrm{SL}_3(q) \times \mathrm{SL}_{10}(q)) : (q-1)$	N1	1	$\langle\delta,\gamma,\phi\rangle$
$\mathcal{C}_1$	$(\mathrm{SL}_4(q) \times \mathrm{SL}_9(q)) : (q-1)$	N1	1	$\langle\delta,\gamma,\phi\rangle$
$\mathcal{C}_1$	$(\mathrm{SL}_5(q) \times \mathrm{SL}_8(q)) : (q-1)$	N1	1	$\langle\delta,\gamma,\phi\rangle$
$\mathcal{C}_1$	$(\mathrm{SL}_6(q) \times \mathrm{SL}_7(q)) : (q-1)$	N1	1	$\langle\delta,\gamma,\phi\rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{23} : (\mathrm{GL}_{11}(q) \times (q-1))$	N1	1	$\langle\delta,\gamma,\phi\rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{40} : (\mathrm{SL}_2(q)^2 \times \mathrm{SL}_9(q)) : (q-1)^2$	N1	1	$\langle\delta,\gamma,\phi\rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{51} : (\mathrm{SL}_3(q)^2 \times \mathrm{SL}_7(q)) : (q-1)^2$	N1	1	$\langle\delta,\gamma,\phi\rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{56} : (\mathrm{SL}_4(q)^2 \times \mathrm{SL}_5(q)) : (q-1)^2$	N1	1	$\langle\delta,\gamma,\phi\rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{55} : (\mathrm{SL}_5(q)^2 \times \mathrm{SL}_3(q)) : (q-1)^2$	N1	1	$\langle\delta,\gamma,\phi\rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{48} : \mathrm{GL}_6(q)^2$	N1	1	$\langle\delta,\gamma,\phi\rangle$
$\mathcal{C}_2$	$(q-1)^{12}.\mathrm{S}_{13}$	$q \geqslant 5$	1	$\langle\delta,\gamma,\phi\rangle$
$\mathcal{C}_3$	$\frac{q^{13}-1}{q-1}.13$		1	$\langle\delta,\gamma,\phi\rangle$
$\mathcal{C}_5$	$\mathrm{SL}_{13}(q_0).[(\frac{q-1}{q_0-1},13)]$	$q = q_0^r$, r prime	$(\frac{q-1}{q_0-1},13)$	$\langle\delta^c,\gamma,\phi\rangle$
$\mathcal{C}_6$	$((q-1,13) \circ 13^3).\mathrm{Sp}_2(13)$	$q = p \equiv 1 \pmod{13}$ or $(q = p^3 \,\&\, p \equiv 3,9 \pmod{13})$	$(q-1,13)$	$\langle\delta^c,\gamma,\phi\rangle$
$\mathcal{C}_8$	$\mathrm{SO}^\circ_{13}(q).(q-1,13)$	q odd	$(q-1,13)$	$\langle\delta^c,\gamma,\phi\rangle$
$\mathcal{C}_8$	$\mathrm{SU}_{13}(q^{1/2}).(q^{1/2}-1,13)$	q square	$(q^{1/2}-1,13)$	$\langle\delta^c,\gamma,\phi\rangle$

N1	Maximal under subgroups not contained in $\langle\delta,\phi\rangle$

Table 11.0.2: Maximal subgroups of $\mathrm{SL}_{13}(q)$ in Class $\mathscr{S}$

In all examples $q = p$. So $d := |\mathrm{Z}(\mathrm{SL}_{13}(p))| = (p-1,13)$, $|\delta| = d$, $|\phi| = 1$, $|\gamma| = 2$.

Subgroup	**Conditions on q**	**c**	**Stab**
$d \times \mathrm{S}_6(3)$	$q = p \equiv 1 \pmod 3$	d	$\langle\gamma\rangle$
$d \times \mathrm{U}_3(4)$	$q = p \equiv 1 \pmod 5$	d	$\langle\gamma\rangle$

Table 11.0.3: Maximal subgroups of $\mathrm{SU}_{13}(q)$ of geometric type

$d := |\mathrm{Z}(\mathrm{SU}_{13}(q))| = (q+1, 13)$, $|\delta| = d$, $|\phi| = 2e$, $q = p^e$.

$\mathcal{C}_i$	**Subgroup**	**Notes**	**c**	**Stab**
$\mathcal{C}_1$	$\mathrm{E}_q^{23} : \mathrm{SU}_{11}(q).(q^2-1)$		1	$\langle\delta, \phi\rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{40} : (\mathrm{SL}_2(q^2) \times \mathrm{SU}_9(q)).(q^2-1)$		1	$\langle\delta, \phi\rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{51} : (\mathrm{SL}_3(q^2) \times \mathrm{SU}_7(q)).(q^2-1)$		1	$\langle\delta, \phi\rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{56} : (\mathrm{SL}_4(q^2) \times \mathrm{SU}_5(q)).(q^2-1)$		1	$\langle\delta, \phi\rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{55} : (\mathrm{SL}_5(q^2) \times \mathrm{SU}_3(q)).(q^2-1)$		1	$\langle\delta, \phi\rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{48} : \mathrm{GL}_6(q^2)$		1	$\langle\delta, \phi\rangle$
$\mathcal{C}_1$	$\mathrm{GU}_{12}(q)$		1	$\langle\delta, \phi\rangle$
$\mathcal{C}_1$	$(\mathrm{SU}_2(q) \times \mathrm{SU}_{11}(q)).(q+1)$		1	$\langle\delta, \phi\rangle$
$\mathcal{C}_1$	$(\mathrm{SU}_3(q) \times \mathrm{SU}_{10}(q)).(q+1)$		1	$\langle\delta, \phi\rangle$
$\mathcal{C}_1$	$(\mathrm{SU}_4(q) \times \mathrm{SU}_9(q)).(q+1)$		1	$\langle\delta, \phi\rangle$
$\mathcal{C}_1$	$(\mathrm{SU}_5(q) \times \mathrm{SU}_8(q)).(q+1)$		1	$\langle\delta, \phi\rangle$
$\mathcal{C}_1$	$(\mathrm{SU}_6(q) \times \mathrm{SU}_7(q)).(q+1)$		1	$\langle\delta, \phi\rangle$
$\mathcal{C}_2$	$(q+1)^{12}.\mathrm{S}_{13}$		1	$\langle\delta, \phi\rangle$
$\mathcal{C}_3$	$\frac{q^{13}+1}{q+1}.13$		1	$\langle\delta, \phi\rangle$
$\mathcal{C}_5$	$\mathrm{SU}_{13}(q_0).[(\frac{q+1}{q_0+1}, 13)]$	$q = q_0^r$, r odd prime	$(\frac{q+1}{q_0+1}, 13)$	$\langle\delta^c, \phi\rangle$
$\mathcal{C}_5$	$\mathrm{SO}_{13}^{\circ}(q).[(q+1, 13)]$	q odd	$(q+1, 13)$	$\langle\delta^c, \phi\rangle$
$\mathcal{C}_6$	$((q+1, 13) \circ 13^3).\mathrm{Sp}_2(13)$	$(q = p^2 \,\&\, p \equiv 5, 8 \pmod{13})$ or $(q = p^6 \,\&\, p \equiv 2, 6, 7, 11 \pmod{13})$	$(q+1, 13)$	$\langle\delta^c, \phi\rangle$

Table 11.0.4: Maximal subgroups of $\mathrm{SU}_{13}(q)$ in Class $\mathscr{S}$

$d := |\mathrm{Z}(\mathrm{SU}_{13}(q))| = (q+1, 13)$, $|\delta| = d$, $|\phi| = 2e$, $q = p^e$, $\phi^e = \gamma$.

Subgroup	**Conditions on q**	**c**	**Stab**
$d \times \mathrm{S}_6(3)$	$q = p \equiv 2 \pmod 3$	d	$\langle\gamma\rangle$
$d \times \mathrm{U}_3(4)$	$q = p \equiv 4 \pmod 5$	d	$\langle\gamma\rangle$
	$q = p^2$, $p \equiv 2, 3 \pmod 5$, $p \neq 2$	d	$\langle\phi\rangle$

Table 11.0.5: Maximal subgroups of $\Omega^{\circ}_{13}(q)$ of geometric type

$d := |\mathrm{Z}(\Omega^{\circ}_{13}(q))| = 1$, $|\delta| = 2$, $|\phi| = e$, $q = p^e$ odd.

$\mathcal{C}_{\mathbf{i}}$	**Subgroup**	**Notes**	**c**	**Stab**
$\mathcal{C}_1$	$\mathrm{E}_q^{11} : (\frac{q-1}{2} \times \Omega^{\circ}_{11}(q)).2$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{19} : (\frac{1}{2}\mathrm{GL}_2(q) \times \Omega^{\circ}_{9}(q)).2$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{24} : (\frac{1}{2}\mathrm{GL}_3(q) \times \Omega^{\circ}_{7}(q)).2$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{26} : (\frac{1}{2}\mathrm{GL}_4(q) \times \Omega^{\circ}_{5}(q)).2$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{25} : (\frac{1}{2}\mathrm{GL}_5(q) \times \Omega^{\circ}_{3}(q)).2$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{21} : \frac{1}{2}\mathrm{GL}_6(q)$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_1$	$\Omega^{+}_{12}(q).2$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_1$	$\Omega^{-}_{12}(q).2$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_1$	$(\Omega^{+}_{2}(q) \times \Omega^{\circ}_{11}(q)).2^2$	$q \neq 3$	1	$\langle \delta, \phi \rangle$
$\mathcal{C}_1$	$(\Omega^{-}_{2}(q) \times \Omega^{\circ}_{11}(q)).2^2$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_1$	$(\Omega^{\circ}_{3}(q) \times \Omega^{+}_{10}(q)).2^2$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_1$	$(\Omega^{\circ}_{3}(q) \times \Omega^{-}_{10}(q)).2^2$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_1$	$(\Omega^{+}_{4}(q) \times \Omega^{\circ}_{9}(q)).2^2$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_1$	$(\Omega^{-}_{4}(q) \times \Omega^{\circ}_{9}(q)).2^2$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_1$	$(\Omega^{\circ}_{5}(q) \times \Omega^{+}_{8}(q)).2^2$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_1$	$(\Omega^{\circ}_{5}(q) \times \Omega^{-}_{8}(q)).2^2$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_1$	$(\Omega^{+}_{6}(q) \times \Omega^{\circ}_{7}(q)).2^2$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_1$	$(\Omega^{-}_{6}(q) \times \Omega^{\circ}_{7}(q)).2^2$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_2$	$2^{12}.\mathrm{A}_{13}$	$q = p \equiv \pm 3 \pmod 8$	1	$\langle \delta, \phi \rangle$
$\mathcal{C}_2$	$2^{12}.\mathrm{S}_{13}$	$q = p \equiv \pm 1 \pmod 8$	2	$\langle \phi \rangle$
$\mathcal{C}_5$	$\Omega^{\circ}_{13}(q_0)$	$q = q_0^r$, r odd prime	1	$\langle \delta, \phi \rangle$
$\mathcal{C}_5$	$\mathrm{SO}^{\circ}_{13}(q_0)$	$q = q_0^2$	2	$\langle \phi \rangle$

Table 11.0.6: Maximal subgroups of $\Omega^{\circ}_{13}(q)$ in Class $\mathscr{S}$

$d := |\mathrm{Z}(\Omega^{\circ}_{13}(q))| = 1$, $|\delta| = 2$, $|\phi| = e$, $q = p^e$ odd.

Subgroup	**Conditions on q**	**Notes**	**c**	**Stab**
A_8	$q = 3$	N1	1	$\langle\delta\rangle$
S_{14}	$q = p \equiv \pm1, \pm3, \pm9 \pmod{28}$, $p \neq 3$		2	
A_{14}	$q = p \equiv \pm5, \pm11, \pm13 \pmod{28}$, $p \neq 5$		1	$\langle\delta\rangle$
S_{15}	$q = 3$		2	
A_{15}	$q = 5$		1	$\langle\delta\rangle$
$\mathrm{L}_2(25)$	$q = p \equiv 2, 3 \pmod 5$, $p \neq 2$		1	$\langle\delta\rangle$
$\mathrm{L}_3(3).2$	$q = p \equiv 1, 11 \pmod{12}$		2	
$\mathrm{L}_3(3)$	$q = p \equiv 5, 7 \pmod{12}$		1	$\langle\delta\rangle$
$\mathrm{S}_4(5)$	$q = p \equiv 1, 4 \pmod 5$		2	
$\mathrm{S}_4(5)$	$q = p^2$, $p \equiv 2, 3 \pmod 5$, $p \neq 2$		2	$\langle\phi\rangle$
$\mathrm{S}_6(q)$	$p = 3$		1	$\langle\delta, \phi\rangle$
$\Omega^{\circ}_5(q)$	$p = 5$		1	$\langle\delta, \phi\rangle$
$\mathrm{L}_2(q)$	$p \geqslant 13$, $q \neq 13$		1	$\langle\delta, \phi\rangle$

N1	Maximal under $\langle\delta\rangle$

Table 11.0.7: Maximal subgroups of $\mathrm{SL}_{14}(q)$ of geometric type

$d := |\mathrm{Z}(\mathrm{SL}_{14}(q))| = (q-1,14)$, $|\delta| = d$, $|\phi| = e$, $|\gamma| = 2$, $q = p^e$.

$\mathcal{C}_\mathbf{i}$	**Subgroup**	**Notes**	**c**	**Stab**
$\mathcal{C}_1$	$\mathrm{E}_q^{13} : \mathrm{GL}_{13}(q)$		2	$\langle\delta,\phi\rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{24} : (\mathrm{SL}_2(q) \times \mathrm{SL}_{12}(q)) : (q-1)$		2	$\langle\delta,\phi\rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{33} : (\mathrm{SL}_3(q) \times \mathrm{SL}_{11}(q)) : (q-1)$		2	$\langle\delta,\phi\rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{40} : (\mathrm{SL}_4(q) \times \mathrm{SL}_{10}(q)) : (q-1)$		2	$\langle\delta,\phi\rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{45} : (\mathrm{SL}_5(q) \times \mathrm{SL}_9(q)) : (q-1)$		2	$\langle\delta,\phi\rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{48} : (\mathrm{SL}_6(q) \times \mathrm{SL}_8(q)) : (q-1)$		2	$\langle\delta,\phi\rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{49} : (\mathrm{SL}_7(q) \times \mathrm{SL}_7(q)) : (q-1)$		1	$\langle\delta,\gamma,\phi\rangle$
$\mathcal{C}_1$	$\mathrm{GL}_{13}(q)$	N1	1	$\langle\delta,\gamma,\phi\rangle$
$\mathcal{C}_1$	$(\mathrm{SL}_2(q) \times \mathrm{SL}_{12}(q)) : (q-1)$	N1	1	$\langle\delta,\gamma,\phi\rangle$
$\mathcal{C}_1$	$(\mathrm{SL}_3(q) \times \mathrm{SL}_{11}(q)) : (q-1)$	N1	1	$\langle\delta,\gamma,\phi\rangle$
$\mathcal{C}_1$	$(\mathrm{SL}_4(q) \times \mathrm{SL}_{10}(q)) : (q-1)$	N1	1	$\langle\delta,\gamma,\phi\rangle$
$\mathcal{C}_1$	$(\mathrm{SL}_5(q) \times \mathrm{SL}_9(q)) : (q-1)$	N1	1	$\langle\delta,\gamma,\phi\rangle$
$\mathcal{C}_1$	$(\mathrm{SL}_6(q) \times \mathrm{SL}_8(q)) : (q-1)$	N1	1	$\langle\delta,\gamma,\phi\rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{25} : (\mathrm{GL}_{12}(q) \times (q-1))$	N1	1	$\langle\delta,\gamma,\phi\rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{44} : (\mathrm{SL}_2(q)^2 \times \mathrm{SL}_{10}(q)) : (q-1)^2$	N1	1	$\langle\delta,\gamma,\phi\rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{57} : (\mathrm{SL}_3(q)^2 \times \mathrm{SL}_8(q)) : (q-1)^2$	N1	1	$\langle\delta,\gamma,\phi\rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{64} : (\mathrm{SL}_4(q)^2 \times \mathrm{SL}_6(q)) : (q-1)^2$	N1	1	$\langle\delta,\gamma,\phi\rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{65} : (\mathrm{SL}_5(q)^2 \times \mathrm{SL}_4(q)) : (q-1)^2$	N1	1	$\langle\delta,\gamma,\phi\rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{60} : (\mathrm{SL}_6(q)^2 \times \mathrm{SL}_2(q)) : (q-1)^2$	N1	1	$\langle\delta,\gamma,\phi\rangle$
$\mathcal{C}_2$	$(q-1)^{13}.\mathrm{S}_{14}$	$q \geqslant 5$	1	$\langle\delta,\gamma,\phi\rangle$
$\mathcal{C}_2$	$\mathrm{SL}_2(q)^7.(q-1)^6.\mathrm{S}_7$	$q \neq 2$	1	$\langle\delta,\gamma,\phi\rangle$
$\mathcal{C}_2$	$\mathrm{SL}_7(q)^2.(q-1).\mathrm{S}_2$		1	$\langle\delta,\gamma,\phi\rangle$
$\mathcal{C}_3$	$(\frac{(q-1,2)(q^7-1)}{q-1} \circ \mathrm{SL}_2(q^7)).7$		1	$\langle\delta,\gamma,\phi\rangle$
$\mathcal{C}_3$	$((q-1,7)(q+1) \circ \mathrm{SL}_7(q^2)).(q+1,7).2$		1	$\langle\delta,\gamma,\phi\rangle$
$\mathcal{C}_4$	$\mathrm{SL}_2(q) \circ \mathrm{SL}_7(q)$	$q \neq 2$	1	$\langle\delta,\gamma,\phi\rangle$
$\mathcal{C}_5$	$\mathrm{SL}_{14}(q_0).[(\frac{q-1}{q_0-1},14)]$	$q = q_0^r$, r prime	$(\frac{q-1}{q_0-1},14)$	$\langle\delta^c,\gamma,\phi\rangle$
$\mathcal{C}_8$	$(q-1,14)^{\cdot}\mathrm{S}_{14}(q)$	q odd	$(q-1,7)$	$\langle\delta^c,\gamma,\phi\rangle$
$\mathcal{C}_8$	$(q-1,14)^{\cdot}\mathrm{PCSp}_{14}(q)$	q even	$(q-1,7)$	$\langle\delta^c,\gamma,\phi\rangle$
$\mathcal{C}_8$	$\mathrm{SO}_{14}^+(q).(q-1,14)$	q odd	$(q-1,7)$	$\langle\delta^c,\gamma,\phi\rangle$
$\mathcal{C}_8$	$\mathrm{SO}_{14}^-(q).(q-1,14)$	q odd	$(q-1,7)$	S1
$\mathcal{C}_8$	$\mathrm{SU}_{14}(q^{1/2}).(q^{1/2}-1,14)$	q square	$(q^{1/2}-1,14)$	$\langle\delta^c,\gamma,\phi\rangle$

N1	Maximal under subgroups not contained in $\langle\delta,\phi\rangle$
S1	$\langle\delta^c,\gamma,\phi\rangle$ if $q \equiv 3 \pmod 4$
	$\langle\delta^c,\phi\delta^{(p-1)/2},\gamma\delta^{-1}\rangle$ if $q \equiv 1 \pmod 4$

Table 11.0.8: Maximal subgroups of $\mathrm{SL}_{14}(q)$ in Class $\mathscr{S}$

In all examples $q = p$. So $d := |\mathrm{Z}(\mathrm{SL}_{14}(p))| = (p-1, 14)$, $|\delta| = d$, $|\phi| = 1$, $|\gamma| = 2$.

Subgroup	**Conditions on q**	**c**	**Stab**
$d \circ 2.\mathrm{S}_6(3)$	$q = p \equiv 1, 7 \pmod{24}$	d	$\langle \gamma \rangle$
	$q = p \equiv 13, 19 \pmod{24}$	d	$\langle \gamma\delta \rangle$
$d \times \mathrm{Sz}(8).3$	$q = p \equiv 1 \pmod 4$	$2d$	

Table 11.0.9: Maximal subgroups of $\mathrm{SU}_{14}(q)$ of geometric type

$d := |\mathrm{Z}(\mathrm{SU}_{14}(q))| = (q+1, 14)$, $|\delta| = d$, $|\phi| = 2e$, $q = p^e$.

$\mathcal{C}_{\mathrm{i}}$	**Subgroup**	**Notes**	**c**	**Stab**
$\mathcal{C}_1$	$\mathrm{E}_q^{25} : \mathrm{SU}_{12}(q).(q^2-1)$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{44} : (\mathrm{SL}_2(q^2) \times \mathrm{SU}_{10}(q)).(q^2-1)$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{57} : (\mathrm{SL}_3(q^2) \times \mathrm{SU}_8(q)).(q^2-1)$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{64} : (\mathrm{SL}_4(q^2) \times \mathrm{SU}_6(q)).(q^2-1)$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{65} : (\mathrm{SL}_5(q^2) \times \mathrm{SU}_4(q)).(q^2-1)$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{60} : (\mathrm{SL}_6(q^2) \times \mathrm{SU}_2(q)).(q^2-1)$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{49} : \mathrm{SL}_7(q^2).(q-1)$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_1$	$\mathrm{GU}_{13}(q)$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_1$	$(\mathrm{SU}_2(q) \times \mathrm{SU}_{12}(q)).(q+1)$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_1$	$(\mathrm{SU}_3(q) \times \mathrm{SU}_{11}(q)).(q+1)$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_1$	$(\mathrm{SU}_4(q) \times \mathrm{SU}_{10}(q)).(q+1)$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_1$	$(\mathrm{SU}_5(q) \times \mathrm{SU}_9(q)).(q+1)$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_1$	$(\mathrm{SU}_6(q) \times \mathrm{SU}_8(q)).(q+1)$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_2$	$(q+1)^{13}.\mathrm{S}_{14}$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_2$	$\mathrm{SU}_2(q)^7.(q+1)^6.\mathrm{S}_7$	$q \neq 2$	1	$\langle \delta, \phi \rangle$
$\mathcal{C}_2$	$\mathrm{SU}_7(q)^2.(q+1).\mathrm{S}_2$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_2$	$\mathrm{SL}_7(q^2).(q-1).2$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_3$	$(\frac{(q+1,2)(q^7+1)}{q+1} \circ \mathrm{SU}_2(q^7)).7$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_4$	$\mathrm{SU}_2(q) \circ \mathrm{SU}_7(q)$	$q \neq 2$	1	$\langle \delta, \phi \rangle$
$\mathcal{C}_5$	$\mathrm{SU}_{14}(q_0).[(\frac{q+1}{q_0+1}, 14)]$	$q = q_0^r$, r odd prime	$(\frac{q+1}{q_0+1}, 14)$	$\langle \delta^c, \phi \rangle$
$\mathcal{C}_5$	$\mathrm{SO}_{14}^{+}(q).[(q+1, 14)]$	q odd	$(q+1, 7)$	S1
$\mathcal{C}_5$	$\mathrm{SO}_{14}^{-}(q).[(q+1, 14)]$	q odd	$(q+1, 7)$	S2
$\mathcal{C}_5$	$\mathrm{Sp}_{14}(q).[(q+1, 7)]$		$(q+1, 7)$	$\langle \delta^c, \phi \rangle$

S1	$\langle \delta^c, \phi \rangle$ if $q \equiv 1 \pmod 4$
	$\langle \delta^c, \phi\delta^{(p-1)/2} \rangle$ if $q \equiv 3 \pmod 4$
S2	$\langle \delta^c, \phi\delta^{(p-1)/2} \rangle$ if $q \equiv 1 \pmod 4$
	$\langle \delta^c, \phi \rangle$ if $q \equiv 3 \pmod 4$

Table 11.0.10: Maximal subgroups of $\mathrm{SU}_{14}(q)$ in Class $\mathscr{S}$

In all examples $q = p$. So $d := |\mathrm{Z}(\mathrm{SU}_{14}(p))| = (p+1, 14)$, $|\delta| = d$, $|\phi| = 2$, $\phi = \gamma$.

Subgroup	**Conditions on q**	**c**	**Stab**
$d \circ 2.\mathrm{S}_6(3)$	$q = p \equiv 17, 23 \pmod{24}$	d	$\langle \gamma \rangle$
	$q = p \equiv 5, 11 \pmod{24}$	d	$\langle \gamma\delta \rangle$
$d \times \mathrm{Sz}(8).3$	$q = p \equiv 3 \pmod 4$	$2d$	

Table 11.0.11: Maximal subgroups of $\mathrm{Sp}_{14}(q)$ of geometric type

$d := |\mathrm{Z}(\mathrm{Sp}_{14}(q))| = (q-1, 2)$, $|\delta| = d$, $|\phi| = e$, $q = p^e$.

$\mathcal{C}_\mathbf{i}$	**Subgroup**	**Notes**	**c**	**Stab**
$\mathcal{C}_1$	$\mathrm{E}_q^{13} : ((q-1) \times \mathrm{Sp}_{12}(q))$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{23} : (\mathrm{GL}_2(q) \times \mathrm{Sp}_{10}(q))$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{30} : (\mathrm{GL}_3(q) \times \mathrm{Sp}_8(q))$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{34} : (\mathrm{GL}_4(q) \times \mathrm{Sp}_6(q))$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{35} : (\mathrm{GL}_5(q) \times \mathrm{Sp}_4(q))$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{33} : (\mathrm{GL}_6(q) \times \mathrm{Sp}_2(q))$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{28} : \mathrm{GL}_7(q)$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_1$	$\mathrm{Sp}_2(q) \times \mathrm{Sp}_{12}(q)$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_1$	$\mathrm{Sp}_4(q) \times \mathrm{Sp}_{10}(q)$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_1$	$\mathrm{Sp}_6(q) \times \mathrm{Sp}_8(q)$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_2$	$\mathrm{Sp}_2(q)^7 : \mathrm{S}_7$	$q \neq 2$	1	$\langle \delta, \phi \rangle$
$\mathcal{C}_2$	$\mathrm{GL}_7(q).2$	q odd	1	$\langle \delta, \phi \rangle$
$\mathcal{C}_3$	$\mathrm{Sp}_2(q^7).7$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_3$	$\mathrm{GU}_7(q).2$	q odd	1	$\langle \delta, \phi \rangle$
$\mathcal{C}_4$	$\mathrm{Sp}_2(q) \circ \mathrm{GO}_7^\circ(q)$	q odd	1	$\langle \delta, \phi \rangle$
$\mathcal{C}_5$	$\mathrm{Sp}_{14}(q_0).2$	$q = q_0^2$ odd	2	$\langle \phi \rangle$
$\mathcal{C}_5$	$\mathrm{Sp}_{14}(q_0)$	$q = q_0^r$, r odd prime or q even	1	$\langle \delta, \phi \rangle$
$\mathcal{C}_8$	$\mathrm{GO}_{14}^+(q)$	q even	1	$\langle \phi \rangle$
$\mathcal{C}_8$	$\mathrm{GO}_{14}^-(q)$	q even	1	$\langle \phi \rangle$

Table 11.0.12: Maximal subgroups of $\mathrm{Sp}_{14}(q)$ in Class $\mathscr{S}$

$d := |\mathrm{Z}(\mathrm{Sp}_{14}(q))| = (q-1, 2)$, $|\delta| = d$, $|\phi| = e$, $q = p^e$.

Subgroup	**Conditions on q**	**Notes**	**c**	**Stab**
$2.\mathrm{A}_7$	$q = p \equiv 1, 7 \pmod 8$		2	
	$q = p^2$, $p \equiv 3, 5 \pmod 8$, $p \neq 3$		2	$\langle\phi\rangle$
$2.\mathrm{L}_2(13).2$	$q = p \equiv 1, 7 \pmod 8$		2	
$2.\mathrm{L}_2(13)$	$q = p \equiv \pm5, \pm11, \pm19, \pm21, \pm37, \pm45 \pmod{104}$		1	$\langle\delta\rangle$
$2.\mathrm{L}_2(13)$	$q = p \equiv \pm3, \pm27, \pm29, \pm35, \pm43, \pm51 \pmod{104}$	N1	1	$\langle\delta\rangle$
$2.\mathrm{L}_2(13).2$	$q = p \equiv 1, 23 \pmod{24}$		2	
	$q = p \equiv 1, 23 \pmod{24}$		2	
	$q = p^2$, $p \equiv 5, 7 \pmod{12}$		4	
$2.\mathrm{L}_2(13)$	$q = p \equiv 11, 13 \pmod{24}$, $p \neq 13$		1	$\langle\delta\rangle$
	$q = p \equiv 11, 13 \pmod{24}$, $p \neq 13$		1	$\langle\delta\rangle$
$\mathrm{L}_2(29)$	$q = 4$		1	$\langle\phi\rangle$
$2.\mathrm{L}_2(29)$	$q = p \equiv \pm1, \pm4, \pm5, \pm6, \pm7, \pm9, \pm13 \pmod{29}$		2	
	$q = p^2$, $p \equiv \pm2, \pm3, \pm8, \pm10, \pm11, \pm12, \pm14 \pmod{29}$, $p \neq 2$		2	$\langle\phi\rangle$
$2.\mathrm{J}_2.2$	$q = p \equiv 1, 7 \pmod 8$		2	
$2.\mathrm{J}_2$	$q = p \equiv 3, 13, 27, 37 \pmod{40}$		1	$\langle\delta\rangle$
$2.\mathrm{J}_2$	$q = p \equiv 11, 19, 21, 29 \pmod{40}$	N1	1	$\langle\delta\rangle$
$\mathrm{SL}_2(q)$	$p \geqslant 17$		1	$\langle\delta, \phi\rangle$
$\mathrm{Sp}_6(q)$	$p \geqslant 3$		1	$\langle\delta, \phi\rangle$

N1	Maximal under $\langle\delta\rangle$

Table 11.0.13: Maximal subgroups of $\Omega_{14}^{+}(q)$ of geometric type

$q = 2^e : |\mathrm{Z}(\Omega_{14}^{+}(q))| = 1,\ |\delta| = 1,\ |\gamma| = 2,\ |\delta'| = 1,\ |\phi| = e.$
$q = p^e \equiv 1 \pmod 4 : |\mathrm{Z}(\Omega_{14}^{+}(q))| = 2,\ |\delta| = 4,\ |\gamma| = 2,\ \delta^2 = \delta',\ |\phi| = e.$
$q = p^e \equiv 3 \pmod 4 : |\mathrm{Z}(\Omega_{14}^{+}(q))| = 1,\ |\delta| = 2,\ |\gamma| = 2,\ |\delta'| = 1,\ |\phi| = e.$
$t := (q-1, 2)$

$\mathcal{C}_{\mathrm{i}}$	**Subgroup**	**Notes**	**c**	**Stab**
$\mathcal{C}_1$	$\mathrm{E}_q^{12} : (\frac{q-1}{t} \times \Omega_{12}^{+}(q)).t$		1	$\langle\delta,\gamma,\phi\rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{21} : (\frac{1}{t}\mathrm{GL}_2(q) \times \Omega_{10}^{+}(q)).t$		1	$\langle\delta,\gamma,\phi\rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{27} : (\frac{1}{t}\mathrm{GL}_3(q) \times \Omega_{8}^{+}(q)).t$		1	$\langle\delta,\gamma,\phi\rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{30} : (\frac{1}{t}\mathrm{GL}_4(q) \times \Omega_{6}^{+}(q)).t$		1	$\langle\delta,\gamma,\phi\rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{30} : (\frac{1}{t}\mathrm{GL}_5(q) \times \Omega_{4}^{+}(q)).t$		1	$\langle\delta,\gamma,\phi\rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{27} : (\frac{1}{t}\mathrm{GL}_6(q) \times \Omega_{2}^{+}(q)).t$	N1	1	$\langle\delta,\gamma,\phi\rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{21} : \frac{1}{t}\mathrm{GL}_7(q)$		2	$\langle\delta,\phi\rangle$
$\mathcal{C}_1$	$\Omega_{13}^{\circ}(q).2$	q odd	2	$\langle\delta',\gamma,\phi\rangle$
$\mathcal{C}_1$	$(\Omega_2^{+}(3) \times \Omega_{12}^{+}(3)).2^2$	$q = 3$, N2	1	$\langle\delta,\gamma,\phi\rangle$
$\mathcal{C}_1$	$(\Omega_2^{+}(q) \times \Omega_{12}^{+}(q)).2^t$	$q \geqslant 4$	1	$\langle\delta,\gamma,\phi\rangle$
$\mathcal{C}_1$	$(\Omega_2^{-}(q) \times \Omega_{12}^{-}(q)).2^t$		1	$\langle\delta,\gamma,\phi\rangle$
$\mathcal{C}_1$	$(\Omega_3^{\circ}(q) \times \Omega_{11}^{\circ}(q)).2^2$	q odd	2	$\langle\delta',\gamma,\phi\rangle$
$\mathcal{C}_1$	$(\Omega_4^{+}(q) \times \Omega_{10}^{+}(q)).2^t$		1	$\langle\delta,\gamma,\phi\rangle$
$\mathcal{C}_1$	$(\Omega_4^{-}(q) \times \Omega_{10}^{-}(q)).2^t$		1	$\langle\delta,\gamma,\phi\rangle$
$\mathcal{C}_1$	$(\Omega_5^{\circ}(q) \times \Omega_{9}^{\circ}(q)).2^2$	q odd	2	$\langle\delta',\gamma,\phi\rangle$
$\mathcal{C}_1$	$(\Omega_6^{+}(q) \times \Omega_{8}^{+}(q)).2^t$		1	$\langle\delta,\gamma,\phi\rangle$
$\mathcal{C}_1$	$(\Omega_6^{-}(q) \times \Omega_{8}^{-}(q)).2^t$		1	$\langle\delta,\gamma,\phi\rangle$
$\mathcal{C}_1$	$\mathrm{Sp}_{12}(q)$	q even	1	$\langle\gamma,\phi\rangle$
$\mathcal{C}_2$	$2^{13}.\mathrm{S}_{14}$	$q = p \equiv 1 \pmod 8$	4	$\langle\gamma,\phi\rangle$
$\mathcal{C}_2$	$2^{13}.\mathrm{A}_{14}$	$q = p \equiv 5 \pmod 8$	2	$\langle\delta',\gamma,\phi\rangle$
$\mathcal{C}_2$	$\Omega_2^{+}(5)^7.2^{12}.\mathrm{S}_7$	$q = 5$, N2	1	$\langle\delta,\gamma,\phi\rangle$
$\mathcal{C}_2$	$\Omega_2^{+}(q)^7.2^{12}.\mathrm{S}_7$	$q \geqslant 7$	1	$\langle\delta,\gamma,\phi\rangle$
$\mathcal{C}_2$	$\Omega_7^{\circ}(q)^2.2^2.\mathrm{S}_2$	$q \equiv 1 \pmod 4$	2	$\langle\delta',\gamma,\phi\rangle$
$\mathcal{C}_2$	$\mathrm{SL}_7(q).\frac{(q-1)}{t}$	N1	1	$\langle\delta,\gamma,\phi\rangle$
$\mathcal{C}_2$	$\mathrm{SO}_7^{\circ}(q)^2$	$q \equiv 3 \pmod 4$	1	$\langle\delta,\gamma,\phi\rangle$
$\mathcal{C}_3$	$2 \times \Omega_7^{\circ}(q^2).2$	$q \equiv 1 \pmod 4$	2	S1
$\mathcal{C}_3$	$\Omega_7^{\circ}(q^2).2$	$q \equiv 3 \pmod 4$	1	$\langle\delta,\gamma,\phi\rangle$
$\mathcal{C}_5$	$\Omega_{14}^{+}(q_0)$	$q = q_0^r$, r odd prime or q even	1	$\langle\delta,\gamma,\phi\rangle$
$\mathcal{C}_5$	$\mathrm{SO}_{14}^{+}(q_0)$	$q = q_0^2$, $q_0 \equiv 1 \pmod 4$	2	$\langle\delta',\gamma,\phi\rangle$
$\mathcal{C}_5$	$\mathrm{SO}_{14}^{+}(q_0).2$	$q = q_0^2$, $q_0 \equiv 3 \pmod 4$	4	$\langle\gamma,\phi\rangle$
$\mathcal{C}_5$	$\Omega_{14}^{-}(q_0)$	$q = q_0^2$ even	1	$\langle\gamma,\phi\rangle$
$\mathcal{C}_5$	$\mathrm{SO}_{14}^{-}(q_0).2$	$q = q_0^2$, $q_0 \equiv 1 \pmod 4$	4	$\langle\gamma,\phi\rangle$
$\mathcal{C}_5$	$\mathrm{SO}_{14}^{-}(q_0)$	$q = q_0^2$, $q_0 \equiv 3 \pmod 4$	2	$\langle\delta',\gamma,\phi\rangle$

N1	Maximal under subgroups not contained in $\langle\delta, \phi\rangle$
N2	Maximal under subgroups not contained in $\langle\delta', \gamma, \phi\rangle$
S1	$\langle\gamma\delta, \gamma\delta^3, \phi\rangle$ if $p \equiv 1 \pmod 4$
	$\langle\gamma\delta, \gamma\delta^3, \gamma\phi\rangle$ if $p \equiv 3 \pmod 4$, $q \equiv 1 \pmod 4$

Table 11.0.14: Maximal subgroups of $\Omega^+_{14}(q)$ in Class $\mathscr{S}$

$q = 2^e : |Z(\Omega^+_{14}(q))| = 1 = d$, $|\delta| = 1$, $|\gamma| = 2$, $|\delta'| = 1$, $|\phi| = e$.
$q = p^e \equiv 1 \pmod 4 : |Z(\Omega^+_{14}(q))| = 2 = d$, $|\delta| = 4$, $|\gamma| = 2$, $\delta^2 = \delta'$, $|\phi| = e$.
$q = p^e \equiv 3 \pmod 4 : |Z(\Omega^+_{14}(q))| = 1 = d$, $|\delta| = 2$, $|\gamma| = 2$, $|\delta'| = 1$, $|\phi| = e$.

Subgroup	**Conditions on q**	**Notes**	**c**	**Stab**
$(d \times A_7).2$	$q = p \equiv 1, 7, 19, 31, 43, 49 \pmod{60}$, $p \neq 7$		$4d$	
$2 \times A_7$	$q = p \equiv 13, 37 \pmod{60}$		4	$\langle\delta'\rangle$
$d \times A_{15}$	$q = p \equiv 1, 17, 19, 23, 31, 47, 49, 53 \pmod{60}$		$2d$	$\langle\gamma\rangle$
A_{16}	$q = 2$		1	$\langle\gamma\rangle$
$d \times L_2(13)$	$q = p \equiv 5, 11, 41, 47, 59, 71, 83, 89, 119,$ $125, 137, 149 \pmod{156}$		$2d$	$\langle\gamma\rangle$
$d \times L_2(13)$	$q = p \equiv 1, 25, 43, 49, 55, 61, 79, 103, 121,$ $127, 133, 139 \pmod{156}$	N1	$2d$	$\langle\gamma\rangle$
$L_2(13)$	$q = p \equiv 11, 47, 59, 71, 83, 119 \pmod{156}$		2	$\langle\gamma\rangle$
$2 \times L_2(13)$	$q = p \equiv 1, 25, 49, 61, 121, 133 \pmod{156}$	N1	4	$\langle\gamma\rangle$
$2 \times L_2(13)$	$q = p \equiv 5, 41, 89, 125, 137, 149 \pmod{156}$		4	$\langle\gamma\delta\rangle$
$L_2(13)$	$q = p \equiv 43, 55, 79, 103, 127, 139 \pmod{156}$	N2	2	$\langle\gamma\delta\rangle$
$L_2(13)$	$q = 2$		1	$\langle\gamma\rangle$
$S_6(2)$	$q = 3$		4	
$(2 \times G_2(3)).2$	$q = p \equiv 1 \pmod{24}$		8	
$2 \times G_2(3)$	$q = p \equiv 13 \pmod{24}$		4	$\langle\delta'\rangle$
$G_2(3)$	$q = p \equiv 7, 19 \pmod{24}$		2	$\langle\delta\rangle$
$SL_4(q).2$	$q = 2^i$, i even		2	$\langle\phi\rangle$
$SU_4(q).2$	$q = 2^i$, i even		2	$\langle\phi\rangle$
$2 \times S_6(q)$	$q \equiv 1 \pmod{12}$		4	S1
$S_6(q).2$	$q \equiv 7 \pmod{12}$		4	$\langle\phi\rangle$
$S_6(q)$	$q = 2^i$, i even		2	$\langle\phi\gamma\rangle$
$2 \times \Omega^\circ_5(q)$	$q \equiv 1, 9 \pmod{20}$		4	S2
$\Omega^\circ_5(q).2$	$q \equiv 3, 7 \pmod{20}$, $q \neq 3$		4	$\langle\phi\rangle$
$d \times G_2(q)$	$q \equiv 1, 7 \pmod{12}$		$4d$	S3

S1	$\langle \delta', \phi \rangle$ if $p \equiv 1, 7 \pmod{12}$
	$\langle \delta', \phi\gamma \rangle$ if $p \equiv 5, 11 \pmod{12}$
S2	$\langle \delta', \phi \rangle$ if $p \equiv 1, 3, 7, 9 \pmod{20}$
	$\langle \delta', \phi\gamma \rangle$ if $p \equiv 11, 13, 17, 19 \pmod{20}$
S3	$\langle \phi \rangle$ if $p \equiv 1, 7 \pmod{12}$
	$\langle \phi\gamma \rangle$ if $p \equiv 5, 11 \pmod{12}$
N1	Maximal under $\langle \gamma \rangle$
N2	Maximal under $\langle \gamma\delta \rangle$

Table 11.0.15: Maximal subgroups of $\Omega_{14}^{-}(q)$ of geometric type

$q = 2^e : |\mathrm{Z}(\Omega_{14}^{-}(q))| = 1,\ |\delta| = 1,\ |\gamma| = 2,\ |\delta'| = 1,\ |\phi| = e.$
$q = p^e \equiv 1 \pmod 4 : |\mathrm{Z}(\Omega_{14}^{-}(q))| = 1,\ |\delta| = 2,\ |\gamma| = 2,\ |\delta'| = 1,\ |\varphi| = 2e,\ \varphi^e = \gamma.$
$q = p^e \equiv 3 \pmod 4 : |\mathrm{Z}(\Omega_{14}^{-}(q))| = 2,\ |\delta| = 4,\ |\gamma| = 2,\ \delta^2 = \delta',\ |\phi| = e.$
$t := (q-1, 2)$

$\mathcal{C}_\mathrm{i}$	**Subgroup**	**Notes**	**c**	**Stab**
$\mathcal{C}_1$	$\mathrm{E}_q^{12} : (\frac{q-1}{t} \times \Omega_{12}^{-}(q)).t$		1	S1
$\mathcal{C}_1$	$\mathrm{E}_q^{21} : (\frac{1}{t}\mathrm{GL}_2(q) \times \Omega_{10}^{-}(q)).t$		1	S1
$\mathcal{C}_1$	$\mathrm{E}_q^{27} : (\frac{1}{t}\mathrm{GL}_3(q) \times \Omega_{8}^{-}(q)).t$		1	S1
$\mathcal{C}_1$	$\mathrm{E}_q^{30} : (\frac{1}{t}\mathrm{GL}_4(q) \times \Omega_{6}^{-}(q)).t$		1	S1
$\mathcal{C}_1$	$\mathrm{E}_q^{30} : (\frac{1}{t}\mathrm{GL}_5(q) \times \Omega_{4}^{-}(q)).t$		1	S1
$\mathcal{C}_1$	$\mathrm{E}_q^{27} : (\frac{1}{t}\mathrm{GL}_6(q) \times \Omega_{2}^{-}(q)).t$		1	S1
$\mathcal{C}_1$	$\Omega_{13}^{\circ}(q).2$	q odd	2	S2
$\mathcal{C}_1$	$(\Omega_2^{+}(3) \times \Omega_{12}^{-}(3)).2^t$	$q = 3$, N1	1	S1
$\mathcal{C}_1$	$(\Omega_2^{+}(q) \times \Omega_{12}^{-}(q)).2^t$	$q \geqslant 3$	1	S1
$\mathcal{C}_1$	$(\Omega_2^{-}(q) \times \Omega_{12}^{+}(q)).2^t$		1	S1
$\mathcal{C}_1$	$(\Omega_3^{\circ}(q) \times \Omega_{11}^{\circ}(q)).2^2$	q odd	2	S2
$\mathcal{C}_1$	$(\Omega_4^{+}(q) \times \Omega_{10}^{-}(q)).2^t$		1	S1
$\mathcal{C}_1$	$(\Omega_4^{-}(q) \times \Omega_{10}^{+}(q)).2^t$		1	S1
$\mathcal{C}_1$	$(\Omega_5^{\circ}(q) \times \Omega_{9}^{\circ}(q)).2^2$	q odd	2	S2
$\mathcal{C}_1$	$(\Omega_6^{+}(q) \times \Omega_{8}^{-}(q)).2^t$		1	S1
$\mathcal{C}_1$	$(\Omega_6^{-}(q) \times \Omega_{8}^{+}(q)).2^t$		1	S1
$\mathcal{C}_1$	$\mathrm{Sp}_{12}(q)$	q even	1	$\langle\gamma, \varphi\rangle$
$\mathcal{C}_2$	$2^{13}.\mathrm{S}_{14}$	$q = p \equiv 7 \pmod 8$	4	$\langle\gamma\rangle$
$\mathcal{C}_2$	$2^{13}.\mathrm{A}_{14}$	$q = p \equiv 3 \pmod 8$	2	$\langle\delta', \gamma, \phi\rangle$
$\mathcal{C}_2$	$\Omega_2^{-}(q)^7.2^{12}.\mathrm{S}_7$	$q = 3$, N1	1	$\langle\delta, \delta', \gamma, \phi\rangle$
$\mathcal{C}_2$	$\Omega_2^{-}(q)^7.2^{6t}.\mathrm{S}_7$	$q \neq 3$	1	S1
$\mathcal{C}_2$	$\Omega_7^{\circ}(q)^2.2^2.\mathrm{S}_2$	$q \equiv 3 \pmod 4$	2	$\langle\delta', \gamma, \phi\rangle$
$\mathcal{C}_2$	$\mathrm{SO}_7^{\circ}(q)^2$	$q \equiv 1 \pmod 4$	1	$\langle\delta, \gamma, \varphi\rangle$
$\mathcal{C}_3$	$(\frac{q+1}{(q+1,2)} \circ \mathrm{SU}_7(q)).(q+1, 7)$		1	S1
$\mathcal{C}_3$	$\Omega_7^{\circ}(q^2).2$	$q \equiv 1 \pmod 4$	1	$\langle\delta, \gamma, \varphi\rangle$
$\mathcal{C}_3$	$2 \times \Omega_7^{\circ}(q^2).2$	$q \equiv 3 \pmod 4$	2	$\langle\delta, \delta'\phi\rangle$
$\mathcal{C}_5$	$\Omega_{14}^{-}(q_0)$	$q = q_0^r$, r odd prime	1	S1

N1	Maximal under subgroups not contained in $\langle\delta', \gamma, \phi\rangle$
S1	$\langle\gamma, \varphi\rangle$ if q even
	$\langle\delta, \gamma, \varphi\rangle$ if $q \equiv 1 \pmod 4$
	$\langle\delta, \delta', \gamma, \phi\rangle$ if $q \equiv 3 \pmod 4$
S2	$\langle\gamma, \varphi\rangle$ if $q \equiv 1 \pmod 4$
	$\langle\delta', \gamma, \phi\rangle$ if $q \equiv 3 \pmod 4$

Table 11.0.16: Maximal subgroups of $\Omega^-_{14}(q)$ in Class $\mathscr{S}$

$q = 2^e : |\mathrm{Z}(\Omega^-_{14}(q))| = 1 = d,\ |\delta| = 1,\ |\gamma| = 2,\ |\delta'| = 1,\ |\phi| = e.$
$q = p^e \equiv 1 \pmod 4) : |\mathrm{Z}(\Omega^-_{14}(q))| = 1 = d,\ |\delta| = 2,\ |\gamma| = 2,\ |\delta'| = 1,\ |\phi| = e.$
$q = p^e \equiv 3 \pmod 4) : |\mathrm{Z}(\Omega^-_{14}(q))| = 2 = d,\ |\delta| = 4,\ |\gamma| = 2,\ \delta^2 = \delta',\ |\phi| = e.$

Subgroup	**Conditions on q**	**Notes**	**c**	**Stab**
$(d \times \mathrm{A}_7).2$	$q = p \equiv 11, 17, 29, 41, 53, 59 \pmod{60}$		$4d$	
$2 \times \mathrm{A}_7$	$q = p \equiv 23, 47 \pmod{60}$		4	$\langle \delta' \rangle$
$\mathrm{A}_7.2$	$q = 2$		2	
$d \times \mathrm{A}_{15}$	$q = p \equiv 7, 11, 13, 29, 37, 41, 43, 59 \pmod{60}$		$2d$	$\langle \gamma \rangle$
$d \times \mathrm{L}_2(13)$	$q = p \equiv 7, 19, 31, 37, 67, 73, 85, 97, 109, 115, 145, 151 \pmod{156}$		$2d$	$\langle \gamma \rangle$
$d \times \mathrm{L}_2(13)$	$q = p \equiv 17, 23, 29, 35, 53, 77, 95, 101, 107, 113, 131, 155 \pmod{156}$	N1	$2d$	$\langle \gamma \rangle$
$\mathrm{L}_2(13)$	$q = p \equiv 37, 73, 85, 97, 109, 145 \pmod{156}$		2	$\langle \gamma \rangle$
$2 \times \mathrm{L}_2(13)$	$q = p \equiv 23, 35, 95, 107, 131, 155 \pmod{156}$	N1	4	$\langle \gamma \rangle$
$2 \times \mathrm{L}_2(13)$	$q = p \equiv 7, 19, 31, 67, 115, 151 \pmod{156}$		4	$\langle \gamma\delta \rangle$
$\mathrm{L}_2(13)$	$q = p \equiv 17, 29, 53, 77, 101, 113 \pmod{156}$	N2	2	$\langle \gamma\delta \rangle$
$(2 \times \mathrm{G}_2(3)).2$	$q = p \equiv 23 \pmod{24}$		8	
$2 \times \mathrm{G}_2(3)$	$q = p \equiv 11 \pmod{24}$		4	$\langle \delta' \rangle$
$\mathrm{G}_2(3)$	$q = p \equiv 5, 17 \pmod{24}$		2	$\langle \delta \rangle$
$\mathrm{G}_2(3).2$	$q = 2$		2	
$\mathrm{J}_2.2$	$q = 5$		4	
$\mathrm{SL}_4(q).2$	$q = 2^i$, i odd		2	$\langle \phi \rangle$
$\mathrm{SU}_4(q).2$	$q = 2^i$, i odd		2	$\langle \phi \rangle$
$2 \times \mathrm{S}_6(q)$	$q \equiv 11 \pmod{12}$		4	$\langle \delta', \phi \rangle$
$\mathrm{S}_6(q).2$	$q \equiv 5 \pmod{12}$		4	$\langle \phi \rangle$
$\mathrm{S}_6(q)$	$q = 2^i$, i odd		2	$\langle \phi \rangle$
$\Omega^\circ_5(q).2$	$q \equiv 13, 17 \pmod{20}$		4	$\langle \phi \rangle$
$2 \times \Omega^\circ_5(q)$	$q \equiv 11, 19 \pmod{20}$		4	$\langle \delta', \phi \rangle$
$d \times \mathrm{G}_2(q)$	$q \equiv 5, 11 \pmod{12}$		$4d$	$\langle \phi \rangle$

N1	Maximal under $\langle \gamma \rangle$
N2	Maximal under $\langle \gamma\delta \rangle$

Table 11.0.17: Maximal subgroups of $\mathrm{SL}_{15}(q)$ of geometric type

$d := |\mathrm{Z}(\mathrm{SL}_{15}(q))| = (q-1, 15)$, $|\delta| = d$, $|\phi| = e$, $|\gamma| = 2$, $q = p^e$.

$\mathcal{C}_\mathrm{i}$	**Subgroup**	**Notes**	**c**	**Stab**
$\mathcal{C}_1$	$\mathrm{E}_q^{14} : \mathrm{GL}_{14}(q)$		2	$\langle \delta, \phi \rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{26} : (\mathrm{SL}_2(q) \times \mathrm{SL}_{13}(q)) : (q-1)$		2	$\langle \delta, \phi \rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{36} : (\mathrm{SL}_3(q) \times \mathrm{SL}_{12}(q)) : (q-1)$		2	$\langle \delta, \phi \rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{44} : (\mathrm{SL}_4(q) \times \mathrm{SL}_{11}(q)) : (q-1)$		2	$\langle \delta, \phi \rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{50} : (\mathrm{SL}_5(q) \times \mathrm{SL}_{10}(q)) : (q-1)$		2	$\langle \delta, \phi \rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{54} : (\mathrm{SL}_6(q) \times \mathrm{SL}_9(q)) : (q-1)$		2	$\langle \delta, \phi \rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{56} : (\mathrm{SL}_7(q) \times \mathrm{SL}_8(q)) : (q-1)$		2	$\langle \delta, \phi \rangle$
$\mathcal{C}_1$	$\mathrm{GL}_{14}(q)$	N1	1	$\langle \delta, \gamma, \phi \rangle$
$\mathcal{C}_1$	$(\mathrm{SL}_2(q) \times \mathrm{SL}_{13}(q)) : (q-1)$	N1	1	$\langle \delta, \gamma, \phi \rangle$
$\mathcal{C}_1$	$(\mathrm{SL}_3(q) \times \mathrm{SL}_{12}(q)) : (q-1)$	N1	1	$\langle \delta, \gamma, \phi \rangle$
$\mathcal{C}_1$	$(\mathrm{SL}_4(q) \times \mathrm{SL}_{11}(q)) : (q-1)$	N1	1	$\langle \delta, \gamma, \phi \rangle$
$\mathcal{C}_1$	$(\mathrm{SL}_5(q) \times \mathrm{SL}_{10}(q)) : (q-1)$	N1	1	$\langle \delta, \gamma, \phi \rangle$
$\mathcal{C}_1$	$(\mathrm{SL}_6(q) \times \mathrm{SL}_9(q)) : (q-1)$	N1	1	$\langle \delta, \gamma, \phi \rangle$
$\mathcal{C}_1$	$(\mathrm{SL}_7(q) \times \mathrm{SL}_8(q)) : (q-1)$	N1	1	$\langle \delta, \gamma, \phi \rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{27} : (\mathrm{GL}_{13}(q) \times (q-1))$	N1	1	$\langle \delta, \gamma, \phi \rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{48} : (\mathrm{SL}_2(q)^2 \times \mathrm{SL}_{11}(q)) : (q-1)^2$	N1	1	$\langle \delta, \gamma, \phi \rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{63} : (\mathrm{SL}_3(q)^2 \times \mathrm{SL}_9(q)) : (q-1)^2$	N1	1	$\langle \delta, \gamma, \phi \rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{72} : (\mathrm{SL}_4(q)^2 \times \mathrm{SL}_7(q)) : (q-1)^2$	N1	1	$\langle \delta, \gamma, \phi \rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{75} : (\mathrm{SL}_5(q)^2 \times \mathrm{SL}_5(q)) : (q-1)^2$	N1	1	$\langle \delta, \gamma, \phi \rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{72} : (\mathrm{SL}_6(q)^2 \times \mathrm{SL}_3(q)) : (q-1)^2$	N1	1	$\langle \delta, \gamma, \phi \rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{63} : \mathrm{GL}_7(q)^2$	N1	1	$\langle \delta, \gamma, \phi \rangle$
$\mathcal{C}_2$	$(q-1)^{14}.\mathrm{S}_{15}$	$q \geqslant 5$	1	$\langle \delta, \gamma, \phi \rangle$
$\mathcal{C}_2$	$\mathrm{SL}_3(q)^5.(q-1)^4.\mathrm{S}_5$		1	$\langle \delta, \gamma, \phi \rangle$
$\mathcal{C}_2$	$\mathrm{SL}_5(q)^3.(q-1)^2.\mathrm{S}_3$		1	$\langle \delta, \gamma, \phi \rangle$
$\mathcal{C}_3$	$(\frac{(q-1,3)(q^5-1)}{q-1} \circ \mathrm{SL}_3(q^5)).\frac{(q^5-1,3)}{(q-1,3)}.5$		1	$\langle \delta, \gamma, \phi \rangle$
$\mathcal{C}_3$	$(\frac{(q-1,5)(q^3-1)}{q-1} \circ \mathrm{SL}_5(q^3)).\frac{(q^3-1,5)}{(q-1,5)}.3$		1	$\langle \delta, \gamma, \phi \rangle$
$\mathcal{C}_4$	$\mathrm{SL}_3(q) \circ \mathrm{SL}_5(q)$		1	$\langle \delta, \gamma, \phi \rangle$
$\mathcal{C}_5$	$\mathrm{SL}_{15}(q_0).[(\frac{q-1}{q_0-1}, 15)]$	$q = q_0^r$, r prime	$(\frac{q-1}{q_0-1}, 15)$	$\langle \delta^c, \gamma, \phi \rangle$
$\mathcal{C}_8$	$\mathrm{SO}_{15}^\circ(q).(q-1,15)$	q odd	$(q-1,15)$	$\langle \delta^c, \gamma, \phi \rangle$
$\mathcal{C}_8$	$\mathrm{SU}_{15}(q^{1/2}).(q^{1/2}-1,15)$	q square	$(q^{1/2}-1,15)$	$\langle \delta^c, \gamma, \phi \rangle$

N1	Maximal under subgroups not contained in $\langle \delta, \phi \rangle$

Table 11.0.18: Maximal subgroups of $\mathrm{SL}_{15}(q)$ in Class $\mathscr{S}$

$d := |\mathrm{Z}(\mathrm{SL}_{15}(q))| = (q-1, 15)$, $|\delta| = d$, $|\phi| = e$, $|\gamma| = 2$, $q = p^e$,
$t := (5, q-1)$, $r := (3, q-1)$.

Subgroup	**Conditions on q**	**c**	**Stab**
$d \times \mathrm{L}_2(31)$	$q = p \equiv 1, 2, 4, 5, 7, 8, 9, 10, 14, 16, 18, 19, 20, 25, 28 \pmod{31}$	d	$\langle \gamma \rangle$
$t \times \mathrm{SL}_3(q)$	$p \neq 2, 3$	t	$\langle \delta^t, \gamma, \phi \rangle$
$r \times \mathrm{SL}_5(q)$	$p \neq 2$	r	$\langle \delta^r, \gamma, \phi \rangle$
$(t \times r.\mathrm{L}_6(q)).2$	$p \neq 2$	t	$\langle \delta^t, \gamma, \phi \rangle$
$t \times \mathrm{SL}_6(q)$	$p = 2$	t	$\langle \delta^t, \gamma, \phi \rangle$

Table 11.0.19: Maximal subgroups of $\mathrm{SU}_{15}(q)$ of geometric type

$d := |\mathrm{Z}(\mathrm{SU}_{15}(q))| = (q+1, 15)$, $|\delta| = d$, $|\phi| = 2e$, $q = p^e$.

$\mathcal{C}_i$	**Subgroup**	**Notes**	**c**	**Stab**
$\mathcal{C}_1$	$\mathrm{E}_q^{27} : \mathrm{SU}_{13}(q).(q^2-1)$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{48} : (\mathrm{SL}_2(q^2) \times \mathrm{SU}_{11}(q)).(q^2-1)$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{63} : (\mathrm{SL}_3(q^2) \times \mathrm{SU}_9(q)).(q^2-1)$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{72} : (\mathrm{SL}_4(q^2) \times \mathrm{SU}_7(q)).(q^2-1)$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{75} : (\mathrm{SL}_5(q^2) \times \mathrm{SU}_5(q)).(q^2-1)$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{72} : (\mathrm{SL}_6(q^2) \times \mathrm{SU}_3(q)).(q^2-1)$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{63} : \mathrm{GL}_7(q^2)$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_1$	$\mathrm{GU}_{14}(q)$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_1$	$(\mathrm{SU}_2(q) \times \mathrm{SU}_{13}(q)).(q+1)$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_1$	$(\mathrm{SU}_3(q) \times \mathrm{SU}_{12}(q)).(q+1)$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_1$	$(\mathrm{SU}_4(q) \times \mathrm{SU}_{11}(q)).(q+1)$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_1$	$(\mathrm{SU}_5(q) \times \mathrm{SU}_{10}(q)).(q+1)$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_1$	$(\mathrm{SU}_6(q) \times \mathrm{SU}_9(q)).(q+1)$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_1$	$(\mathrm{SU}_7(q) \times \mathrm{SU}_8(q)).(q+1)$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_2$	$(q+1)^{14}.\mathrm{S}_{15}$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_2$	$\mathrm{SU}_3(q)^5.(q+1)^4.\mathrm{S}_5$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_2$	$\mathrm{SU}_5(q)^3.(q+1)^2.\mathrm{S}_3$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_3$	$(\frac{(q+1,3)(q^5+1)}{q+1} \circ \mathrm{SU}_3(q^5)).\frac{(q^5+1,3)}{(q+1,3)}.5$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_3$	$(\frac{(q+1,5)(q^3+1)}{q+1} \circ \mathrm{SU}_5(q^3)).\frac{(q^3+1,5)}{(q+1,5)}.3$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_4$	$\mathrm{SU}_3(q) \circ \mathrm{SU}_5(q)$		1	$\langle \delta, \phi \rangle$
$\mathcal{C}_5$	$\mathrm{SU}_{15}(q_0).[(\frac{q+1}{q_0+1}, 15)]$	$q = q_0^r$, r odd prime	$(\frac{q+1}{q_0+1}, 15)$	$\langle \delta^c, \phi \rangle$
$\mathcal{C}_5$	$\mathrm{SO}^{\circ}_{15}(q).[(q+1, 15)]$	q odd	$(q+1, 15)$	$\langle \delta^c, \phi \rangle$

Table 11.0.20: Maximal subgroups of $\mathrm{SU}_{15}(q)$ in Class $\mathscr{S}$

$d := |\mathrm{Z}(\mathrm{SU}_{15}(q))| = (q+1, 15)$, $|\delta| = d$, $|\phi| = 2e$, $\phi^e = \gamma$, $q = p^e$,
$t := (5, q+1)$, $r := (3, q+1)$.

Subgroup	**Conditions on *q***	**c**	**Stab**
$d \times \mathrm{L}_2(31)$	$q = p \equiv 3, 6, 11, 12, 13, 15, 17, 21, 22,$ $23, 24, 26, 27, 29, 30 \pmod{31}$	d	$\langle \gamma \rangle$
$t \times \mathrm{SU}_3(q)$	$p \neq 2, 3$	t	$\langle \delta^t, \phi \rangle$
$r \times \mathrm{SU}_5(q)$	$p \neq 2$	r	$\langle \delta^r, \phi \rangle$
$(t \times r.\mathrm{U}_6(q)).2$	$p \neq 2$	t	$\langle \delta^t, \phi \rangle$
$t \times \mathrm{SU}_6(q)$	$p = 2$	t	$\langle \delta^t, \phi \rangle$

Table 11.0.21: Maximal subgroups of $\Omega^{\circ}_{15}(q)$ of geometric type

$|\mathrm{Z}(\Omega^{\circ}_{15}(q))| = 1$, $|\delta| = 2$, $|\phi| = e$, $q = p^e$ odd.

$\mathcal{C}_\mathbf{i}$	**Subgroup**	**Notes**	**c**	**Stab**
$\mathcal{C}_1$	$\mathrm{E}_q^{13} : (\frac{(q-1)}{2} \times \Omega^{\circ}_{13}(q)).2$		1	$\langle\delta, \phi\rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{23} : (\frac{1}{2}\mathrm{GL}_2(q) \times \Omega^{\circ}_{11}(q)).2$		1	$\langle\delta, \phi\rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{30} : (\frac{1}{2}\mathrm{GL}_3(q) \times \Omega^{\circ}_{9}(q)).2$		1	$\langle\delta, \phi\rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{34} : (\frac{1}{2}\mathrm{GL}_4(q) \times \Omega^{\circ}_{7}(q)).2$		1	$\langle\delta, \phi\rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{35} : (\frac{1}{2}\mathrm{GL}_5(q) \times \Omega^{\circ}_{5}(q)).2$		1	$\langle\delta, \phi\rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{33} : (\frac{1}{2}\mathrm{GL}_6(q) \times \Omega^{\circ}_{3}(q)).2$		1	$\langle\delta, \phi\rangle$
$\mathcal{C}_1$	$\mathrm{E}_q^{28} : \frac{1}{2}\mathrm{GL}_7(q)$		1	$\langle\delta, \phi\rangle$
$\mathcal{C}_1$	$\Omega^{+}_{14}(q).2$		1	$\langle\delta, \phi\rangle$
$\mathcal{C}_1$	$\Omega^{-}_{14}(q).2$		1	$\langle\delta, \phi\rangle$
$\mathcal{C}_1$	$(\Omega^{+}_{2}(q) \times \Omega^{\circ}_{13}(q)).2^2$	$q \neq 3$	1	$\langle\delta, \phi\rangle$
$\mathcal{C}_1$	$(\Omega^{-}_{2}(q) \times \Omega^{\circ}_{13}(q)).2^2$		1	$\langle\delta, \phi\rangle$
$\mathcal{C}_1$	$(\Omega^{\circ}_{3}(q) \times \Omega^{+}_{12}(q)).2^2$		1	$\langle\delta, \phi\rangle$
$\mathcal{C}_1$	$(\Omega^{\circ}_{3}(q) \times \Omega^{-}_{12}(q)).2^2$		1	$\langle\delta, \phi\rangle$
$\mathcal{C}_1$	$(\Omega^{+}_{4}(q) \times \Omega^{\circ}_{11}(q)).2^2$		1	$\langle\delta, \phi\rangle$
$\mathcal{C}_1$	$(\Omega^{-}_{4}(q) \times \Omega^{\circ}_{11}(q)).2^2$		1	$\langle\delta, \phi\rangle$
$\mathcal{C}_1$	$(\Omega^{\circ}_{5}(q) \times \Omega^{+}_{10}(q)).2^2$		1	$\langle\delta, \phi\rangle$
$\mathcal{C}_1$	$(\Omega^{\circ}_{5}(q) \times \Omega^{-}_{10}(q)).2^2$		1	$\langle\delta, \phi\rangle$
$\mathcal{C}_1$	$(\Omega^{+}_{6}(q) \times \Omega^{\circ}_{9}(q)).2^2$		1	$\langle\delta, \phi\rangle$
$\mathcal{C}_1$	$(\Omega^{-}_{6}(q) \times \Omega^{\circ}_{9}(q)).2^2$		1	$\langle\delta, \phi\rangle$
$\mathcal{C}_1$	$(\Omega^{\circ}_{7}(q) \times \Omega^{+}_{8}(q)).2^2$		1	$\langle\delta, \phi\rangle$
$\mathcal{C}_1$	$(\Omega^{\circ}_{7}(q) \times \Omega^{-}_{8}(q)).2^2$		1	$\langle\delta, \phi\rangle$
$\mathcal{C}_2$	$2^{14}.\mathrm{A}_{15}$	$q = p \equiv \pm 3 \pmod 8$	1	$\langle\delta, \phi\rangle$
$\mathcal{C}_2$	$2^{14}.\mathrm{S}_{15}$	$q = p \equiv \pm 1 \pmod 8$	2	$\langle\phi\rangle$
$\mathcal{C}_2$	$\Omega^{\circ}_{3}(q)^5.2^8.\mathrm{S}_5$	$q \neq 3$	1	$\langle\delta, \phi\rangle$
$\mathcal{C}_2$	$\Omega^{\circ}_{5}(q)^3.2^4.\mathrm{S}_3$		1	$\langle\delta, \phi\rangle$
$\mathcal{C}_3$	$\Omega^{\circ}_{3}(q^5).5$		1	$\langle\delta, \phi\rangle$
$\mathcal{C}_3$	$\Omega^{\circ}_{5}(q^3).3$		1	$\langle\delta, \phi\rangle$
$\mathcal{C}_4$	$(\Omega^{\circ}_{3}(q) \times \Omega^{\circ}_{5}(q)).2$	$q \neq 3$	1	$\langle\delta, \phi\rangle$
$\mathcal{C}_5$	$\Omega^{\circ}_{15}(q_0)$	$q = q_0^r$, r odd prime	1	$\langle\delta, \phi\rangle$
$\mathcal{C}_5$	$\mathrm{SO}^{\circ}_{15}(q_0)$	$q = q_0^2$	2	$\langle\phi\rangle$

Table 11.0.22: Maximal subgroups of $\Omega_{15}^{\circ}(q)$ in Class $\mathscr{S}$

$|\mathrm{Z}(\Omega_{15}^{\circ}(q))| = 1$, $|\delta| = 2$, $|\phi| = e$, $q = p^e$ odd.

Subgroup	**Conditions on q**	**c**	**Stab**
$\mathrm{A}_{16}.2$	$q = p \equiv 1, 7 \pmod 8$, $p \neq 17$	2	
A_{16}	$q = p \equiv 3, 5 \pmod 8$	1	$\langle\delta\rangle$
$\mathrm{A}_{17}.2$	$q = 17$	2	
$\mathrm{L}_2(16)$	$q = p \equiv \pm 1 \pmod{17}$	2	
	$q = p \equiv \pm 1 \pmod{17}$	2	
	$q = p^2$, $p \equiv \pm 4 \pmod{17}$	2	$\langle\phi\rangle$
	$q = p^2$, $p \equiv \pm 4 \pmod{17}$	2	$\langle\phi\rangle$
	$q = p^4$, $p \equiv \pm 2, \pm 8 \pmod{17}$, $p \neq 2$	2	$\langle\phi\rangle$
	$q = p^4$, $p \equiv \pm 2, \pm 8 \pmod{17}$, $p \neq 2$	2	$\langle\phi\rangle$
	$q = p^8$, $p \equiv \pm 3, \pm 5, \pm 6, \pm 7 \pmod{17}$	4	$\langle\phi^2\rangle$
$\mathrm{L}_2(29)$	$q = p \equiv \pm 1, \pm 4, \pm 5, \pm 6, \pm 7, \pm 9, \pm 13 \pmod{29}$	2	
	$q = p^2$, $p \equiv \pm 2, \pm 3, \pm 8, \pm 10, \pm 11, \pm 12, \pm 14 \pmod{29}$, $p \neq 2$	2	$\langle\phi\rangle$
$\mathrm{L}_3(4).2_2$	$q = 3$	1	$\langle\delta\rangle$
$\mathrm{S}_6(2)$	$q = p \neq 2, 3$	2	
$\mathrm{L}_2(q).2$	$p \geqslant 17$	2	$\langle\phi\rangle$
$\mathrm{L}_4(q).2$	$p \neq 2$	1	$\langle\delta, \phi\rangle$
$\mathrm{U}_4(q).2$	$p \neq 2$	1	$\langle\delta, \phi\rangle$

References

[1] M. Aschbacher. Finite Group Theory. Cambridge studies in advanced mathematics 10, *Cambridge University Press, Cambridge*, 2000.

[2] M. Aschbacher. On the maximal subgroups of the finite classical groups. *Invent. Math.* **76** (1984), 469-514.

[3] N. Blackburn and B. Huppert. Finite Groups II. Grundlehren der mathematischen Wissenschaften 242. A Series of Comprehensive Studies in Mathematics, *Springer Verlag, Berlin*, 1982.

[4] W. Bosma, J. Cannon and C. Playoust. The Magma algebra system. I. The user language *J. Symbolic Comput.* **24** (1997), 235-265.

[5] R. Brauer and C. Nesbitt. On the modular characters of groups. *Ann. of Math.(2)* **42** (1941), 556-590.

[6] J.N. Bray, S.J. Nickerson, R.A. Parker, R.A. Wilson and others. ATLAS of Finite Group Representations.
<http://brauer.maths.qmul.ac.uk/Atlas/ >

[7] J.N. Bray, D.F. Holt and C.M. Roney-Dougal. Certain classical groups are not well-defined. *J. Group Theory* **12** (2009), 171-180.

[8] J.N. Bray, D.F. Holt and C.M. Roney-Dougal. The Maximal Subgroups of the Low-Dimensional Finite Classical Groups. London Mathematical Society Lecture Note Series 407. *Cambridge University Press, Cambridge*, 2013.

[9] P. Cameron. Notes on Classical Groups.
<http://www.maths.qmul.ac.uk/~pjc/class_gps/cg.pdf>

[10] J. J. Cannon and D. F. Holt. Computing maximal subgroups of finite groups. *J. Symbolic Comput.* **37** (2004), 589-609.

[11] R.W. Carter Simple groups of Lie type. *Wiley-Interscience, London-New York*, 1972.

[12] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker and R.A. Wilson. $\mathbb{ATLAS}$ of Finite Groups. *Oxford University Press, Clarendon Press, Oxford*, 1985.

[13] C. Chuan-Chong and K. Khee-Meng. Principles and techniques in combinatorics. *World Scientific, Singapore*, 1992.

[14] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.6.4; 2013.
<http://www.gap-system.org>

[15] M. Geck. An Introduction to Algebraic Geometry and Algebraic Groups. Oxford Graduate Texts in Mathematics 10, *Oxford University Press, Oxford*, 2003.

[16] R. Goodman and N.R. Wallach. Symmetry, representations and invariants. Graduate Texts in Mathematics 255, *Springer-Verlag, Dordrecht-New York*, 2009.

[17] R. Grizzard. Minimal Degrees of Finite Simple Groups of Lie Type.
<http://www.math.wisc.edu/∼grizzard/papers/table.pdf>

[18] G. Hiß and G. Malle. Low-dimensional representations of quasi-simple groups. *LMS J. Comput. Math.* **4** (2001), 22-63. Corrigienda: LMS J. Comput. Math. **5** (2002), 95-126.

[19] Private correspondence with Prof D.F. Holt, University of Warwick

[20] D.F. Holt and C.M. Roney-Dougal. Constructing maximal subgroups of classical groups. *LMS J. Comput. Math.* **8** (2005), 46-79.

[21] D.F. Holt and C.M. Roney-Dougal. Constructing maximal subgroups of orthogonal groups. *LMS J. Comput. Math.* **13** (2010), 164-191.

[22] R.B. Howlett, L.J. Rylands and D.E. Taylor. Matrix generators for exceptional groups of Lie type. *J. Symbolic Comput.* **31** (2000), 429-445.

[23] I.M. Isaacs. Character Theory of Finite Groups. *Academic Press, New York-London*, 1976.

[24] C. Jansen, K. Lux, R. Parker and R. Wilson. An ATLAS of Brauer Characters. *The Clarendon Press, Oxford University Press, New York*, 1995.

[25] P.B. Kleidman. The maximal subgroups of the low-dimensional classical groups. PhD Thesis, University of Cambridge, 1987.

[26] M. Kleidmann and M. Liebeck. The subgroup structure of the finite classical groups. London Math. Soc. Lecture Note Ser., 129. *Cambridge University Press, Cambridge*, 1990.

[27] D.W. Lewis. The isometry classification of hermitian forms over division algebras. *Linear Algebra and its Applications* **43** (1982), 245-272.

[28] R. Lidl and H. Niederreiter. Introduction to Finite Fields and Their Applications. *Cambridge University Press, Cambridge*, 2000.

[29] F. Lübeck. Small degree representations of finite Chevalley groups in defining characteristic. *LMS J. Comput. Math.* **4** (2001), 135-169.

[30] G. Malle and D. Testermann. Linear algebraic groups and finite groups of Lie type. Cambridge studies in advanced mathematics, 133. *Cambridge University Press, Cambridge*, 2011.

[31] A.L. ONishchik and E.B. Vinberg (Eds.). Lie Groups and Lie Algebras III: Structure of Lie Groups and Lie Algebras. Encylopaedia of Mathematical Sciences, 41. *Springer Verlag, Berlin-Heidelberg*, 1994.

[32] L.J. Rylands and D.E. Taylor. Matrix Generators for the Orthogonal Groups. *J. Symbolic Comput.* **25** (1998), 351-360.

[33] I. Schur. Untersuchungen über die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen. *J. Reine Angew. Math.* **132** (1907), 85-137.

[34] W. Stein. Elementary number theory: primes, congruences, and secrets. A computational approach. *Springer Verlag, New York*, 2009.

[35] R. Steinberg. Representations of algebraic groups. *Nagoya Maths. J.* **22** (1963), 33-56.

[36] D. E. Taylor. The geometry of the classical groups. *Heldermann Verlag, Berlin*, 1992.

[37] D.E. Taylor. Pairs of Generators for Matrix Groups. I *The Cayley Bulletin* **3** (1987), 76-85.

[38] A.V. Vasil'ev. Minimal permutation representations of finite simple exceptional groups of types G_2 and F_4. *Algebra i Logika* **35 (6)** (1996).

[39] S.H. Weintraub. Galois Theory *Springer Verlag, New York*, 2006.

[40] R.A. Wilson. Maximal subgroups of automorphism groups of simple groups. *J. London Math. Soc. (2)* **32** (1985), 460-466.

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