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A New Method for finding Amicable Pairs

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ABSTRACT. Let $\sigma(x)$ denote the sum of all divisors of the (positive) integer x. An *amicable pair* is a pair of integers (m, n) with $m < n$ such that $\sigma(m) = \sigma(n) = m + n$. A new method for finding amicable pairs is presented, based on the following observation of Erdős: For given s, let $x_1, x_2, \ldots$ be solutions of the equation $\sigma(x) = s$; then any pair (x_i, x_j) for which $x_i + x_j = s$ is amicable. An algorithm is described and applied to yield 116 new amicable pairs (m, n) with $2 \times 10^{11} < m < 5 \times 10^{11}$.

1. Introduction

Let m be a positive integer and let $\sigma(m)$ be the sum of all divisors of m. An *amicable pair* is a pair of positive integers (m, n), $m < n$, which satisfies $\sigma(m) = \sigma(n) = m + n$. Let $g = \gcd(m, n)$ and write $m = gM$, $n = gN$. The amicable pair (m, n) is called *regular* if $\gcd(g, M) = \gcd(g, N) = 1$, and M and N are squarefree. Other pairs are called *irregular* or *exotic*. The smallest regular amicable pair is $(220, 284) = (2^2 5 \cdot 11, 2^2 71)$ and the smallest irregular amicable pair is $(1184, 1210) = (2^5 37, 2 \cdot 5 \cdot 11^2)$. Below 10^{10} there are 1082 regular and 345 irregular amicable pairs (sometimes abbreviated to 'AP') [**11**].

Three essentially different methods to find amicable pairs are known. The first is a numerical method, the second and third are algebraic methods. In the first, a number m is chosen, $n := \sigma(m) - m$ is computed and, if $n > m$, $t := \sigma(n) - n$ is computed. If $t = m$, (m, n) is an amicable pair. By letting m run through a given interval, we can trace *all* amicable pairs (m, n) with m in that interval. In the second method, an assumption is made about the prime structure of m and n, for example, $m = 2^k pq$, $n = 2^k r$, where p, q and r are mutually different prime numbers. This yields two equations which k, p, q and r must satisfy. In the third method, amicable pairs are *constructed* from special numbers called

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breeders [**3**], which may be amicable numbers themselves [**10**]. The first method is exhaustive and CPU time-consuming (cf. [**11**]), whereas the second and the third are nonexhaustive and relatively inexpensive (cf. [**1**] and [**5**]). Thousands of amicable pairs are known [**1, 12**]. An exhaustive list of amicable pairs below 10^{10} was published in [**11**]. This bound was extended recently by Moews and Moews to 10^{11} [**8**] and to 2.01×10^{11} [**9**].

In this paper we present and apply a new method to find amicable pairs, the principle of which stems from Erdős: If $x_1, x_2, \ldots$ are solutions of the equation $\sigma(x) = s$, then any pair (x_i, x_j) for which $x_i + x_j = s$ is an amicable pair. Heuristically, values of s for which $\sigma(x) = s$ has *many* solutions have an increased chance to yield amicable pairs.

2. An algorithm to solve $\sigma(x) = s$ and derive amicable pairs

We are going to find solutions $x = x_1, x_2, \ldots$ of $\sigma(x) = s$ for suitable s. We restrict our search to amicable pairs with members which are divisible by 2 or by 3. We exclude the possibility that one member is divisible by 6 since in that case the other member must be *odd* [**6**]: no such amicable pair is known, and if it exists, it probably is very large. Only recently, amicable pairs have been found with members coprime to 6 [**2, 4**], but these are very large compared with the numbers in our search interval.

Algorithm A (*For given positive integer s, find solutions x of the equation $\sigma(x) = s$, where $2 \mid x$ or $3 \mid x$ but $6 \nmid x$ and find amicable pairs among pairs of solutions found*). If $\sigma(p^e) \mid s$ for a prime p and positive integer exponent e, and if the equation $\sigma(y) = s/\sigma(p^e)$ has a solution y which is coprime to p, then $x = p^e y$ is a solution of the equation $\sigma(x) = s$ by the multiplicativity of the σ-function. This suggests our recursive algorithm: it finds a divisor $\sigma(p^e)$ of s and solves $\sigma(y) = s/\sigma(p^e)$, with $\gcd(p, y) = 1$. Two tables T_1 and T_2 of triples $(p, e, \sigma(p^e))$ will be used (where $\sigma(p^e) = p^e + p^{e-1} + \cdots + 1$). The ith triple from table T_1 will be denoted by T_{1i}, and similarly for table T_2. Solutions found are stored in $x_1, x_2, \ldots$. Choose upper bounds B_1 and B_2 for the $\sigma(p^e)$-values admitted in tables T_1 and T_2, respectively.

- **A1** [Precomputation of tables of $\sigma(p^e)$-values]. Fill table T_1 with triples $(p, e, \sigma(p^e))$ for $p = 2, 3$ and for those integers $e = 1, 2, \ldots$ for which $\sigma(p^e) < B_1$. Similarly, fill table T_2 for all the primes $p = 5, 7, \ldots$ and integers $e = 1, 2, \ldots$ for which $\sigma(p^e) < B_2$, such that the $\sigma(p^e)$-values are in *increasing* order. Set $i_{\max}$ and $j_{\max}$ to the number of triples in tables T_1 and T_2, respectively.
- **A2** [Initialize]. Set $d \leftarrow 1$, $s_d \leftarrow s$, $i \leftarrow 0$, $n \leftarrow 0$. The current value of d indicates that the dth prime divisor of x is being looked for (so $d - 1$ prime divisors have been found so far). The integer s_d is the current value of s for which $\sigma(x) = s$ is being solved; j_d $(d \geq 2)$ remembers the location in table T_2 where a prime power divisor of x has been found;

p_d and e_d are the prime and the corresponding exponent in that prime power.

A3 [Select next triple from table T_1]. Set $i \leftarrow i+1$. If $i > i_{\max}$, goto step A8, otherwise set $(p,e,\sigma) \leftarrow T_{1i}$. If $\sigma \nmid s_1$, repeat step A3, otherwise set

$$p_1 \leftarrow p, e_1 \leftarrow e, d \leftarrow 2, s_2 \leftarrow s_1/\sigma, sq \leftarrow \sqrt{s_2}, j \leftarrow 0.$$

A4 [Select next triple from table T_2]. Set $j \leftarrow j+1$. If $j > j_{\max}$, goto step A5, otherwise set $(p,e,\sigma) \leftarrow T_{2j}$. If $p = p_l$ for some $l \in \{2,3,\ldots,d-1\}$, repeat step A4. If $\sigma > sq$, goto step A6. If $\sigma \mid s_d$, set

$$j_d \leftarrow j, p_d \leftarrow p, e_d \leftarrow e, s_{d+1} \leftarrow s_d/\sigma, sq \leftarrow \sqrt{s_{d+1}}, d = d+1.$$

Repeat step A4.

A5 [Check if $s_d - 1$ is prime]. If $s_d - 1$ is prime, set

$$n \leftarrow n+1, x_n \leftarrow (s_d - 1)\prod_{k=1}^{d-1} p_k^{e_k}.$$

Goto step A7.

A6 [Check if s_d occurs as σ-value in table T_2]. If $\exists l$ and $T_{2l} = (p,e,\sigma)$ with $\sigma = s_d$, set

$$n \leftarrow n+1, x_n \leftarrow p^e \prod_{k=1}^{d-1} p_k^{e_k}.$$

Goto step A7.

A7 [Decrease depth d]. Set $d \leftarrow d-1$. If $d = 1$, goto step A3, otherwise set $j \leftarrow j_d, sq \leftarrow \sqrt{s_d}$; return to step A4.

A8 [Sort solutions and check pair sums]. Sort the solutions $x_1, x_2, \ldots, x_n$ in increasing order and find amicable pairs, i.e., pairs (x_i, x_j) with $x_i + x_j = s$. Notice that this algorithm finds amicable pairs (m,n), if any, with $m < s/2$ and $n > s/2$, where m and n are in the neighborhood of $s/2$.

Application of this algorithm with $B_1 = 70$ and $B_2 = 100$ to $s = 504 = 2^3 3^2 7$ yields the five solutions $x_1 = 2\cdot 11\cdot 13$, $x_2 = 2\cdot 167$, $x_3 = 2^2 5\cdot 11$, $x_4 = 2^2 71$, $x_5 = 2^5 7$, in this order, and the amicable pair $(x_3, x_4) = (220, 284)$, since $x_3 + x_4 = s$. The equation $\sigma(x) = 504$ has five more solutions, viz., $x = 192, 246, 415, 451$, and 503, but the first two of them are divisible by 6, and the other three have a smallest prime divisor > 3, and such solutions are excluded from our algorithm.

3. Applying Algorithm A to suitable numbers s

Inspection of the pair sums of known amicable pairs reveals that in many cases these sums have only small prime divisors. In particular, among the 1427 amicable pairs below 10^{10} there are 37 *pairs* of amicable pairs with the same pair sum (and no such *triples*), and in these 37 pair sums the largest occurring prime is 37. Suggested by the frequencies of occurrence of these primes in the 37 pair sums, and by the exponent ranges of these primes (cf. Table 5 in [**11**]), we have generated two sets, S_1 and S_2, of s-values as input for Algorithm A, as follows. Let

$$S = \{s = 2^{i_1}3^{i_2}5^{i_3}7^{i_4}11^{i_5}13^{i_6}17^{i_7}19^{i_8}23^{i_9}31^{i_{10}} \mid$$

$6 \le i_1 \le 14,\ 3 \le i_2 \le 6,\ 0 \le i_3 \le 3,\ 0 \le i_4 \le 2,\ 0 \le i_k \le 1$ for $k = 5, \ldots, 10\}$; then $S_1 := \{s \in S \mid 4 \times 10^{10} < s < 2 \times 10^{11}\}$ and $S_2 := \{s \in S \mid 4 \times 10^{11} < s < 10^{12}\}$. Algorithm A was applied to all elements of S_1 and S_2, with $B_1 = 260$ and $B_2 = 10^6$.

From the 3582 elements in S_1, Algorithm A found 200 numbers s which yielded one or more amicable pairs, viz., 2 each yielded three APs, 8 each yielded two APs, and 190 each yielded one AP, hence a total of 212 APs (m, n) with $m < 10^{11}$. Computing time was about 13 CPU hours on a 33 MHZ SGI workstation. All the 212 APs found occur in the list by Moews and Moews [**8**]. This list contains *four* triples of amicable pairs having the same pair sum, viz., $2^{14}3^6 5^2 7 \cdot 31$, $2^{12}3^4 5^4 7^2 11$, $2^{13}3^4 5 \cdot 11 \cdot 13^2 19$, and $2^{10}3^5 5^2 7 \cdot 11 \cdot 13 \cdot 19$. The first and fourth of these pair sums belong to our set S_1, and our algorithm found the two corresponding triples of APs.

In the set S_2, which contains 1850 elements, Algorithm A found, after about 30 CPU hours of computing time, 118 numbers s which yielded one or more APs, viz., 1 yielded three APs, 7 each yielded two APs, and 110 each yielded one AP, hence a total of 127 APs. They are in the interval $[1.88 \times 10^{11}, 5 \times 10^{11}]$. Of these, 102 (80%) are regular, and 25 (20%) are irregular. Moews and Moews [**9**] found that there are 970 amicable pairs in the interval $[10^{11}, 2 \times 10^{11}]$, 778 (80%) of them being regular and 192 (20%) being irregular, so it appears that our algorithm finds a selection of regular and irregular amicable pairs in a ratio which is representative for *all* the amicable pairs in the corresponding interval. The other known nonexhaustive methods usually only find regular amicable pairs and very few irregular pairs.

The first eight of the 127 APS found from S_2 have their smaller member $< 2.01 \times 10^{11}$ and, therefore, occur in the list by Moews and Moews [**9**]. Two others occur in [**12**], and one in [**7**]. This leaves a total of 116 APs (m, n) with $2.01 \times 10^{11} < m < 5 \times 10^{11}$, which seem to be new. A list of these APs is available from the author upon request. The new *triple* of APs having the same pair sum was found for $s = 2^{14}3^6 5^2 7 \cdot 11 \cdot 19$, and the three corresponding amicable pairs are: $(3^3 \cdot 5 \cdot 11 \cdot 29 \cdot 113 \cdot 167 \cdot 263, 3^3 \cdot 5 \cdot 11 \cdot 131 \cdot 839 \cdot 1367)$, $(3 \cdot 5 \cdot 7 \cdot 19 \cdot 29 \cdot 43 \cdot 86183, 3 \cdot 5 \cdot 7 \cdot 19 \cdot 89 \cdot 113 \cdot 11087)$, and $(2^3 \cdot 31 \cdot 43 \cdot 113 \cdot 419 \cdot 431, 2^3 \cdot 37 \cdot 47 \cdot 167 \cdot 197 \cdot 479)$.

Let N_s be the number of solutions x of $\sigma(x) = s$ found with our algorithm. If $N_s \approx \sqrt{s}$, and if solutions are "randomly" distributed in $[1, s]$, we have a reasonable chance to find a pair (x_1, x_2) among the N_s solutions for which $x_1 + x_2 = s$. For the numbers s in S_1 and S_2, we found values of $N_s/\sqrt{s}$ in the range $(0.01, 0.43)$. The maximum value we found was 0.4224, for $s = 2^{12}3^6 5^2 7^2 11 \cdot 13$ with $N_s = 305536$, but no amicable pair resulted from this s. The second-largest quotient was 0.388, for $s = 2^{13}3^5 5^2 7^2 11 \cdot 19$ with $N_s = 277222$. This s yielded two new amicable pairs, viz. $(3^3 5 \cdot 11^2 23 \cdot 223 \cdot 2969, 3^3 5 \cdot 11 \cdot 197 \cdot 839 \cdot 1063)$ and

$(2 \cdot 5 \cdot 13 \cdot 37 \cdot 109 \cdot 419 \cdot 1151, 2 \cdot 5 \cdot 29 \cdot 31 \cdot 43 \cdot 113 \cdot 5879)$. In order to get some feeling for the behavior of the quotient $N_s/\sqrt{s}$, we have computed it for $s = i!$, $i = 8, \ldots, 15$, and for a few other values of s. The results as given in Table 1 show an increasing tendency for the quotient, which gives some support to our expectation that many (but much larger) smooth numbers s (i.e., only with small prime factors) exist for which the quotient exceeds 1, and these numbers can be expected to yield many amicable numbers. For $s = 3 \cdot 14!$, $s = 4 \cdot 14!$, and $s = 15!$ Algorithm A yielded two APs, one AP, and one AP, respectively; for the other values of s in Table 1 no AP was found.

Table 1. N_s and $N_s/\sqrt{s}$ for various values of s

s	11!	12!	13!	14!	$2 \cdot 14!$	$3 \cdot 14!$	$4 \cdot 14!$	$8 \cdot 14!$	$12 \cdot 14!$	15!
N_s	1665	6999	25656	110137	163869	200965	236219	331105	449253	442439
$N_s/\sqrt{s}$	0.264	0.320	0.325	0.373	0.392	0.393	0.400	0.397	0.439	0.387

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