Consensus by High Degree of DeGroot Model for Multi-Agent Systems

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Abstract: Nonlinear distributions by the high degree of DeGroot model has been studied in this for consensus problem of multi-agent systems (MAS). The idea behind the convergence of nonlinear distribution is that when the degree of nonlinear distribution is increasing the number of iterations is in turn decreasing. From these viewpoints, the efficient aspects of the proposed nonlinearity model by high degree are that the resulting process is of fast convergence and the consensus could not depend on the kind of transition matrix.

Keywords: DeGroot model, consensus by high degree, consensus problem, multi-agent systems, transition matrix.

1. INTRODUCTION

In recent years, there have been widely attractive researches in distributed system problems of the group autonomous agents. In many literatures, convergence to a common value has been established for the question of consensus or agreement. [1]. Operate, negotiate and reach agreements are the most difficulties as well as famous challenges for MAS [2]. Consensus is one of the most important problems in research on MAS, which involves the statuses of agent and control planning in reaching an agreement via exchange of information.

The consensus problem demonstrates how these numerous autonomous agents (multi-agent systems) congregate to a consensus through their local interdictions. Moreover, the way that the word of agreement is being expressed shows that the entire cases of the autonomous agents are equal [3]. The interest in distributed systems is inspired by organizing and managing multi-agents in large-scale networks with access to information to reach agreement on a similar point of interest on a decision (value) or consensus convergence.

In general, all of these concerns have agents who interact with each other for information exchange [4]. The agreement between the agents is obtained by collaboration between agents in most of the current research on consensus issues [5]. One of the most difficult problems in the area of multi-agent systems is to anatomize the complex interaction strategy in the case of phenomena that are considered easily enough [6]. One of the structure complexity of the nonlinear consensus for MAS that when the communication of the interconnections among agents is stochastic [7]. In [8], a MAS was developed that can learn and handle micro units in real-time strategy games and use real-time version of NEAT to adapt for new cases. The states could be defined as views, principles, figures, beliefs, positions, speeds, among others, depending on the context [4].

Consensus use occurs in many research areas. In biology, for example, the dynamics of consensus are studied in the behavioral sense of fish and bird schools flocking [9]. Models of consensus can be used to analyze, forecast and explain flocking behaviour. Consensus problems arise in robotics and control systems in the communication and collaboration of agents in the network of sensors and robots, which it has considered a big issue in the applications of network environment [10], [11]. Consensus is applied in economics to reach an agreement on a common trust in the price process. In management science, the issue of consensus was studied for the management community [12]. Through sociology, it is used in primary societies for a common language and in social networks for the dynamics opinion [13]. It was also a widely covered topic of interest in computer science [14].

The consensus topic has a long history in DeGroot's work [15] and Berger discussed the necessary and appropriate conditions of the DeGroot model in [16]. A distributed network computing was also presented in [17]. Tsitsiklis, Bertsekas in [10], [18], studied the problems of asynchronous setup in parallel computing. Jadbabaie also studied the problem of consensus collocation [11]. Another consideration was the theoretical framework for solving the problem of consensus that Olfati-Saber and Murray investigated [19][20]. A general report that has surveyed relevant consensus problems of MAS was given by Ren [1]. Moreover, Ozdaglar Nedic [21] and Olshevsky [22] widely studied the solution domain related with the consensus problem. Cheng et al [23] have achieved a reaching agreement for MAS by increasing the fault-tolerance in distributed systems and decreasing the iterations of message. It was accomplished by a proposed algorithm using digital signature and grouping concept. The nonlinear dynamic systems are studied in [24] for leader-based consensus on neural network of MAS.

The begging studies, however, have built on the conjecture that linear protocols are the dynamics related to agent consensus. This conjecture cannot always be satisfied because physical engineering systems are of a particular kind of consensus problem[25], [26]. It is not sufficient to agree that their actions can be modified through an unbounded value for these physical systems [27]. In turn, this suggests the creation of consensus protocols to ensure that the initial general state is bound [25], [26]. In addition, the produce protocol is running and can be utilized to develop the performance of the consensus for dynamic algorithm [28], [29]. Hence, the motivation of this work is to design and analyze a non-linear MAS consensus protocol. Therefore, there are background in turn motivates us to design a nonlinear consensus protocol for consensus problem for MAS. There is still significant difficulty in designing for a nonlinear system, however, which also motivates us to try a more effective method for assessing the stability of nonlinear systems. The challenge of constructing a nonlinear system therefore requires research effort, which is a motivation for us to explore and examine the stability of nonlinear systems. The current concern is to explore possible nonlinear models with faster convergence to achieve optimal consensus, but with relatively low complexity and more flexible system conditions. Indeed, a lot of research, like [30], [31], [40], [41], [32]–[39] have presented nonlinear stochastic control for convergence to the average.

2. METHODOLOGY

In the DeGroot linear distribution [12], it has been considered the group of x_i agents $(x_i = (x_1, x_2, \dots, x_m))$. The initial state for each agent is $x_i^0 = (x_1^0, x_2^0, \dots, x_m^0)$. It has one transition matrix P_{ij} $(P_{ij} \geq 0)$ to update statuses of all agents where i contacts j for updating (see Figure 2.1). Therefore, the limit behavior of the trajectories has been studied of each initial x_i^0 states using DeGroot linear distribution by $x_i^1 = \sum_{j=1}^{n} p_{ij}\, x_i^0$. DeGroot's linear distribution for consensus problem general operator in MAS is as follows:

$$x_i^{(t+1)} = \sum_{i=1}^{m} p_{ij}\, x_i^t, \qquad (2.1)$$

where p_{ij} is the transition matrix, x_i^t represents the states of agents (column vectors) and t the number iterations to reach consensus, which means:

$$x_i^{(t+1)} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{pmatrix} \begin{pmatrix} x_1^{(t)} \\ x_2^{(t)} \\ \vdots \\ x_m^{(t)} \end{pmatrix} \qquad (2.2)$$

The stochastic distribution cases of DeGroot allow consensus being attained if only if all states of agents $x_i^{(t+1)}$ converge to the same limit as $t \to \infty$. Then, the evaluation of DeGroot linear model will be as follows:

$$V(x_i^{(t+1)}) \begin{cases} x_1^{(t+1)} = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m \\ x_2^{(t+1)} = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m \\ \vdots \;=\; \vdots \;+\; \vdots \;+ \ddots +\; \vdots \\ x_3^{(t+1)} = a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mm}x_m \end{cases} \qquad (2.3)$$

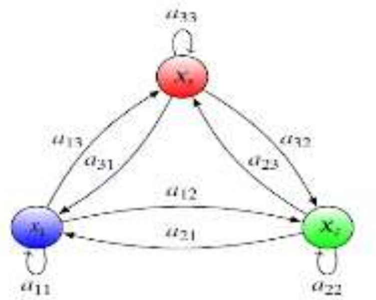

Figure 2.1: The structure of DeGroot linear distribution for multi-agent systems.

Definition 2.9: Let V be DSQO and $x^0 \in S^{m-1}$. $\mathrm{x}^0 \in \mathrm{S}^{\mathrm{m-1}}$.The sequence $\{x^0, V(x^0), V^2(x^0), \dots, V^n(x^0)\}$ $\{\mathrm{x}^0, \mathrm{V}(\mathrm{x}^0), \mathrm{V}^2(\mathrm{x}^0), \dots\}$ is called the trajectory of DSQO starting at $x^0\mathrm{x}^0$. where $V^2(x^0) = V(V(x^0))$. Usually, it can put $V(x^0) = x^0$, [34], [42]. $\mathrm{V}^0(\mathrm{x}^0) = \mathrm{x}^0$.The $\omega(x^0)\omega(\mathrm{x}^0)$ is donated the set of limit points of the trajectory and it is said to be the ω- limit set of the trajectory.

3. PROPOSED WORK

In this section, a high degree for the agents' status of the DeGroot model is proposed for consensus problem in MAS.

Refer to equation (2.1), suppose that i agents have degree n, $(x^n)_i^t$, where $n \geq 1$.
DeGroot's linear distribution for consensus problem general operator in MAS is as follows:

$$x_i^{(t+1)} = \sum_{i=1}^{m} p_{ij}\, (x^n)_i^t, \qquad (3.1)$$

where p_{ij} is the transition matrix, $(x^n)_i^t$ are the states of agents (column vectors), n is the degree of the state which could be $n \geq 1$ and t is the number of iterations to reach a consensus. which means:

$$x_i^{(t+1)} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{pmatrix} \begin{pmatrix} (x^n)_1^t \\ (x^n)_2^t \\ \vdots \\ (x^n)_m^t \end{pmatrix} \quad (3.2)$$

The stochastic distribution cases of DeGroot contain the condition that the consensus is attained if all states of agents $x_i^{(t+1)}$ converge to the same limit.

4. RESULT AND SIMULATION

In this section, the linear distribution of the DeGroot model with higher is studied.

Considering the initial values for all cases for example are:

$$x_1 = 0.1 \quad , \quad x_2 = 0.7, \quad x_3 = 0.2$$

It can note that the results are generalized for any initial values from zero to one $(0 \leq x_1^0 \leq 1)$.

Then, considering the transition matrices for each protocol in all cases as follows:

4.1 The nonlinear distribution A by higher degree of DeGroot's linear when $n = 12$.

DeGroot model: $P_{ij} = \{a_{ij} \geq 0, \ , \sum_{j=1}^{m} P_{ij,k} = 1, \forall i, j = 1, \dots, m\}$.

1. Transition matrix of normal non symmetric:

$$p_{ij} = \begin{pmatrix} 0.7 & 0.4 & 0.4 \\ 0.4 & 0.2 & 0.6 \\ 0.9 & 0.9 & 0.7 \end{pmatrix}$$

2. Transition matrix of normal symmetric:

$$p_{ij} = \begin{pmatrix} 0.9 & 0.4 & 0.6 \\ 0.4 & 0.8 & 0.5 \\ 0.6 & 0.5 & 0.2 \end{pmatrix}$$

3. Transition matrix of stochastic non symmetric:

$$p_{ij} = \begin{pmatrix} 0.25 & 0.5 & 0.25 \\ 0.1 & 0.15 & 0.75 \\ 0.3 & 0.1 & 0.6 \end{pmatrix}$$

4. Transition matrix of stochastic symmetric:

$$p_{ij} = \begin{pmatrix} 0.4 & 0.5 & 0.1 \\ 0.5 & 0.4 & 0.1 \\ 0.1 & 0.1 & 0.8 \end{pmatrix}$$

5. Transition matrix of doubly stochastic non symmetric:

$$p_{ij} = \begin{pmatrix} 0.4 & 0.05 & 0.55 \\ 0.1 & 0.85 & 0.05 \\ 0.5 & 0.1 & 0.4 \end{pmatrix}$$

6. Transition matrix of doubly stochastic symmetric:

$t \to \infty$. Then, the evaluation of the linear operator of DeGroot will be as follows:

$$V\left(x_i^{(t+1)}\right) \begin{cases} x_1^{(t+1)} = a_{11}x_1^n + a_{12}x_2^n + \cdots + a_{1m}x_m^n \\ x_2^{(t+1)} = a_{21}x_1^n + a_{22}x_2^n + \cdots + a_{2m}x_m^n \\ \vdots \quad = \ \vdots \ + \ \vdots \ + \ddots + \ \vdots \\ x_3^{(t+1)} = a_{m1}x_1^n + a_{m2}x_2^n + \dots + a_{mm}x_m^n \end{cases} \quad (3.3)$$

$$p_{ij} = \begin{pmatrix} 0.9 & 0.05 & 0.05 \\ 0.05 & 0.3 & 0.65 \\ 0.05 & 0.65 & 0.3 \end{pmatrix}$$

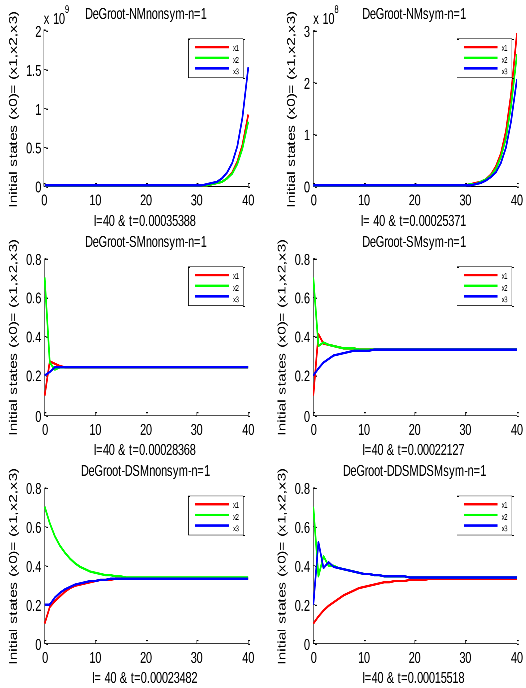

Fig 4.1: The convergence of DeGroot linear distribution when n=1 with NMnonsym, NMsym, SMnonsym, SMsym, DSMnonsym and DSMsym.

As it can see in the Figure 4.1, the limit behavior of trajectories of DeGroot linear distribution is diverging in the case of NM nonsym and sym. However, it converges in the cases of SM and DSM when the matrix is nonsym and sym. Meanwhile, the limit behavior converges to the same limit in the case of SM nonsym and it converges to the center in the cases of SM sym, DSM nonsym and DSM sym. This means that, when the matrix is SM nonsym the limit converges to the same value that depends on sum of each column while the limit converges to the center when the matrix is DSM (the stochastic symmetric matrix is also doubly stochastic matrix). Further, it can be obtained that if the matrix is non-stochastic then the limit never converges.

4.2 The nonlinear distribution A by higher degree of DeGroot's linear when $n \geq 2$.

4.2.1 DeGroot's linear when $n = 2$.

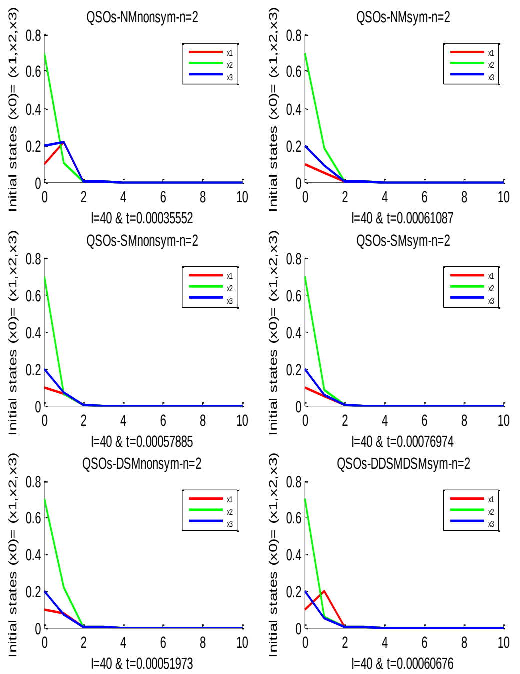

Fig 4.2: The convergence of DeGroot linear distribution when n=2 with NMnonsym, NMsym, SMnonsym, SMsym, DSMnonsym and DSMsym.

4.2.2 DeGroot's linear when $n = 10$.

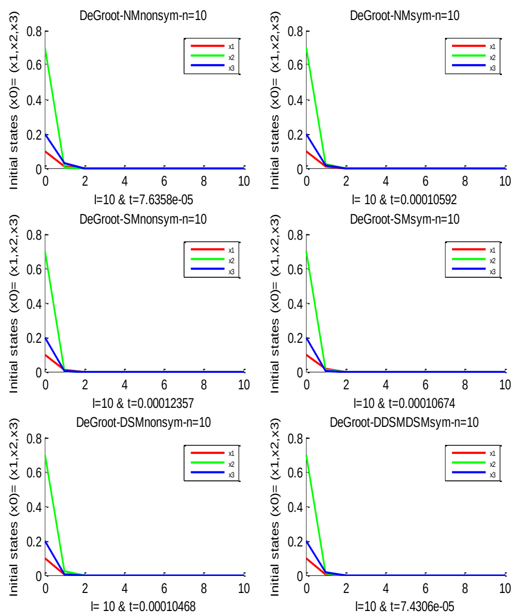

Fig 4.3: The convergence of DeGroot linear distribution when n=10 with NMnonsym, NMsym, SMnonsym, SMsym, DSMnonsym and DSMsym.

4.2.3 DeGroot's linear when $n = 100$.

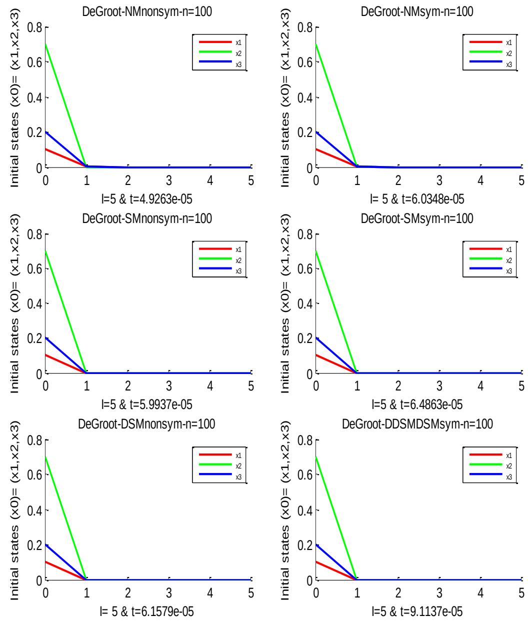

Figure 4.4: The convergence of DeGroot linear distribution when n=100 with NMnonsym, NMsym, SMnonsym, SMsym, DSMnonsym and DSMsym.

From the simulation analysis, the results are portrayed in the Figures [4.2 - 4.4]. The implication of the resulting analysis is that, it indicates the convergence to zero of the limit behavior of nonlinear distribution of DeGroot.

Hence, MAS reaches to a consensus in any case of the distribution of transition matrix under a DeGroot model with higher degree.

The efficiency of the proposed nonlinearity model by higher degree attains fast convergence to consensus compared to the DeGroot linear model [15], and may even take only one execution step. Furthermore, the most significant and efficient aspect of the proposed nonlinearity model by higher degree is that the consensus does not depend on the transition matrix.

4.2.3 The higher degrees of DeGroot's linear

The transition matrix for DeGroot model:

$$P_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

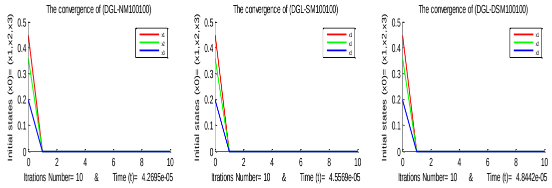

Figure 4.5: The convergence of the higher degrees of DG with zero's transition matrix.

5. CONCLUSION AND FUTURE WORK

In this paper, the nonlinear distributions by higher degree have been studied for the DeGroot with respect to the consensus problem in MAS. The presented investigation demonstrates that the proposed nonlinear distribution by higher degree of the DeGroot are attributed to more efficient convergence for the consensus problem in MAS. The nonlinear distribution by higher converges to zero under any distribution case of the transition matrices. The problem left open in this work is that the convergence to zero has sense in consensus problem in MAS in real application or not.

ACKNOWLEDGEMENT

We would like to thank International Islamic University Malaysia.

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