More on preference and freedom

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Abstract. The paper seeks to formalize the notion of effective freedom or the freedom to realize meaningful choices. The definition of meaningful choice used in this paper is based on the preference orderings that a reasonable person may have. I argue that only alternatives that can be selected by a reasonable person from the set of all possible alternatives provide a meaningful choice. I discuss this approach and provide an axiomatization of the cardinality rule and two lexicographic versions of this rule in this context.

1 Introduction

Choosing for oneself and shaping ones own life is essential for a meaningful human life. With this idea in mind this paper explores some issues regarding the intrinsic value of freedom of choice. The paper considers an agent who faces alternative feasible sets, A, B, etc. Each one of these sets is a non-empty subset of some given universal set of alternatives X. Confronted with one of these sets, the agent has to choose exactly one among the possible alternatives. The problem considered is how to rank different sets according to the freedom that they offer. The paper focuses on the capacity of the feasible sets to provide meaningful choices. I shall discriminate between the alternatives that constitute meaningful choices and those that do not. I use as a reference the set of preferences that a reasonable person may have and the alternatives that she may choose in the universal set of alternatives.

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Sen (1991) pointed out the need for introducing preferences into the analysis of the freedom. According to his idea of effective freedom, an individual is free if she has access to alternatives that she regards as valuable in terms of some criteria. These criteria may be her own preferences or the preferences that a reasonable person in her place may have.

Jones and Sugden (1982) first suggested the use of the preferences of a reasonable person as a reference point for the evaluation of the freedom that a set of alternatives offers. According to them, "if any reasonable person would be indifferent between two particular alternatives, then offering choice contributes little to diversity." Pattanaik and Xu (1998) take the role of preferences a step forward. For them, the intrinsic value of freedom of choice should be judged "not [in terms of] the preferences that the agent actually has, nor [in terms of] ... his future preference ordering, but [in terms of] the preference orderings that a reasonable person in the agent's situation can possibly have." The model they propose has the virtue of capturing effective freedom without collapsing into an indirect utility ranking.

In comparing two opportunity sets, A and B, Pattanaik and Xu (1998) concentrate on $\bar{A}$ and $\bar{B}$, where $\bar{A}$ is the set of all alternatives in A which reasonable persons may choose from the feasible set A, and similarly for $\bar{B}$. This paper, however, follows a different approach. I first consider the set $\bar{X}$ of all alternatives in the universal set X that reasonable people will choose if the universal set was feasible. Then I concentrate on $\bar{X} \cap A$ and $\bar{X} \cap B$ when comparing A and B. The intuitive difference between the procedure of Pattanaik and Xu (1998) and the procedure discussed here can be illustrated with an example.

Consider the case where the universal set X contains four alternatives: life imprisonment, being beheaded, being hanged and being killed in the electric chair. Suppose that there are two sets of alternatives A and B to be ranked. The set A contains three alternatives: being beheaded, being hanged and being killed in the electric chair. The set B contains two alternatives: being hanged and being killed in the electric chair. It is plausible to assume that every reasonable person will prefer life imprisonment to any of the other alternatives. It is also reasonable to think that facing the possibility of death, we can find a reasonable person who may prefer any of the possible methods of execution figuring in X or may be indifferent between them.

The first rule characterized by Pattanaik and Xu (1998) considers $\bar{A}$ and $\bar{B}$ to be the relevant sets of alternatives when comparing A and B. Under this rule, one set offers more freedom than another if it contains a larger number of 'relevant' alternatives. Therefore, A is regarded as offering more freedom than B if and only if $\bar{A}$'s cardinality is greater than $\bar{B}$'s. In a second rule Pattanaik and Xu (1998) compare the number of alternatives in the intersection of A and the choices of a reasonable person in $A \cup B$, that is, the number of alternatives in $\overline{A \cup B} \cap A$, with that in $\overline{A \cup B} \cap B$. Again, A is considered as offering more freedom than B if the cardinality of the first is greater than that of the second. Both characterizations give importance in terms of freedom of choice to alternatives that are irrelevant to the agent. They are irrelevant in the sense that they would never be chosen by a reasonable person if the entire set X was feasible. Irrelevant alternatives are unable to fulfill a vital project because no reasonable person will ever choose them. A reasonable person will never choose the alternative of being beheaded at dawn as the alternative that helps her in "shaping his [her] life in accordance with some overall plan," and only a "being with the capacity so to shape his life can have or strive for a meaningful life" (Nozick 1974, p. 50).

The idea that one should consider only those options which are meaningful in the context of "shaping ones own life" is captured formally in this paper using $\overline{X} \cap A$ and $\overline{X} \cap B$ as the relevant alternatives when comparing A and B. In the example both A and B are declared indifferent in terms of the freedom they offer because no reasonable person will choose an alternative in these sets if her choice is not constrained.

Choosing for oneself and shaping ones own life is essential for a meaningful human life. The alternatives that can be chosen by a reasonable person in a society where no constraints exist are the ones that allow the agents to shape their lives and provide them with full control over themselves. If only the composition of the particular set is relevant for measuring the freedom that this set offers, then a person who has to choose between three different ways to die can have more freedom than another person who is able to choose between life and death. Also, a slave who has to choose among hundreds of escape plans can have more freedom than her master who decides among fewer options, even when the master's options are more real and desirable, in the eyes of any reasonable person, than the slave's dreams of freedom.

The focus on the set of alternatives a reasonable person may choose from the universal set can generate situations that may seem puzzling. This is because alternatives that are not feasible have a role in evaluating the freedom that a set offers. Let us consider a situation where all the reasonable persons in a society believe that a particular woman, a, is the most desirable mate. These preferences may seem inadequate when dealing with the problem of choosing a reasonable mate. However they are not far from the preferences that children seem to have over Christmas presents. In that case the sets $A = \{c, d, b\}$ and $B = \{d, b\}$ are indifferent in terms of the number of alternatives that intersect with $\overline{X}$. Even if a is unanimously considered the best possible choice, it may seem hard to claim that both A and B contain as much freedom as the empty set. However, if the social consensus is such that only a can fulfill the individuals' vital project, even if the selection of some other alternative can provide some utility, a set where a is not included will offer no valuable alternative in terms of freedom, i.e. no freedom at all.

This example is, somehow, pathological. A reasonable person may have any preference that is not contradictory or illogical. In the example it is not unthinkable that any woman can be the best mate for some reasonable person. The freedom attributed to a set depends on the preferences conceived as reasonable. Therefore the comparison between two sets will be paradoxical only if the preferences attributed to reasonable people are so.

The rest of the paper is organized as follows. The next section introduces some notation and definitions. Section 3 contains the main characterization result. The characterized rule provides a complete order of the sets of alternatives. In Sect. 4 two lexicographic versions of the previous rule are considered. The relation between the original approach and its lexicographic versions is also discussed. The paper finishes with some final remarks in Sect. 5.

2 Notation and definitions

Let X be the finite universal set of alternatives. At any given time, the agent faces a non-empty subset of X. Let Z be the set of all non-empty subsets of X. The elements of Z are the feasible sets that the agent may face. Among these feasible alternatives she chooses exactly one. Let $\succsim$ be a binary relation defined over Z. For all $A, B \in Z$, $[A \succsim B]$ means that A offers at least as much freedom as B. For all $A, B \in Z$, $[A \succ B \text{ iff } A \succsim B \text{ and } \neg(B \succsim A)]$ and $[A \sim B \text{ iff } A \succsim B \text{ and } B \succsim A]$.

The reference set of preference orderings over X is denoted as $\wp = (R_1, \ldots, R_n)$. A preference ordering over X is a reflexive, complete and transitive weak preference relation, 'at least as good as', over X. $\wp$ will be interpreted as the set of all possible preference orderings over X that a reasonable person may have. I denote by $\max(A)$ the set of all alternatives x in A such that x is a best alternative in X for some ordering in $\wp$. Let $\mathscr{P}(\wp) = \max(X)$. I will call $\mathscr{P}(\wp)$ the set of relevant alternatives in X.

The binary relation $\succsim$ over Z may satisfy several properties.

Definition 1. *A binary relation $\succsim$ over Z satisfies:*

1. *indifference of no-choice situations (INS) iff, for all $x, y \in X$, $\{x\} \sim \{y\}$;*
2. *simple non-dominance (SND) iff, for all $x, y \in X$, if $\#\max(\{x\}) = \#\max(\{y\})$, then $\{x\} \sim \{y\}$;*
3. *inclusion monotonicity (IMON) iff, for all $A, B \in Z$ if $A \supseteq B$ and $\max(A \backslash B) \neq \varnothing$, then $A \succ B$. If $A \supseteq B$ and $\max(A \backslash B) = \varnothing$, then $A \sim B$;*
4. *composition (COM) iff, for all $A, B, C, D \in Z$, such that $A \cap C = B \cap D = \varnothing$ and $A, B, C, D \subseteq \mathscr{P}(\wp)$,*

$$[A \succsim B \text{ and } C \succsim D] \rightarrow [A \cup C \succsim B \cup D], \quad \text{and,}$$

$$[A \succsim B \text{ and } C \succ D] \rightarrow [A \cup C \succ B \cup D].$$

The previous axioms are versions of traditional ones adapted to the context of the paper, which provides them with a new meaning. The INS is a classical axiom. However, we can think of situations where INS fails to capture the idea of effective freedom. For example, consider two situations, such that in each case you must read a book. In the first case the book considered is a telephone directory. In the second, the book is one that a reasonable person may have chosen from the set of all the available books ever written. INS will rank both sets as indifferent, but, intuitively, the two sets seem to offer very different degrees of freedom. The second set contains a readable book. However, no reasonable person will consider the telephone directory readable.

Therefore, this first set is empty from a reader's point of view and thus less preferred than the second set. INS, however, cannot capture this situation where we compare a set containing a book and an empty set. In the model, the spirit of INS is captured by SND when the alternatives compared are both relevant. The cardinality rule characterized in the next section does not satisfy INS but satisfies SND.

IMON adapts a preference independent axiom, one that Sen (1991) called weak dominance. IMON requires that set inclusion of relevant alternatives implies preference. It restricts the role of non-relevant alternatives and excludes them from consideration when a set is compared with one of its subsets.

The COM axiom was originally defined by Sen (1991). It requires only that $A \cap C = B \cap D = \emptyset$. As Pattanaik and Xu (1998) remark, there may be differences in the contributions that sets C and D give to A and B. To avoid this problem they impose the restriction that all alternatives in $A \cup C$ and in $B \cup D$ are 'relevant' alternatives in their sense. In line with the same approach, I simplify the axiom for the context defined in this paper. It is enough that the sets considered are subsets of $\mathscr{P}(\wp)$ and it is not necessary to postulate restrictions on their unions.

3 The result

In this section I characterize the binary relation defined by the cardinality rule in terms of relevant alternatives. This is, for all $A, B \in Z$,

$$A \succsim^* B \leftrightarrow \#\max(A) \geqq \#\max(B).$$

That is, a set A will be declared preferred to B if and only if the number of relevant alternatives that A contains is bigger than the number of relevant alternatives contained in B.

Proposition 1. $\succsim$ *satisfies SND, COM and IMON if and only if* $\succsim \, = \, \succsim^*$.

Proof. The necessity part of the proposition is straightforward; I prove only the sufficiency part. This proof has two stages. Let $\succsim$ satisfy SND, COM and IMON. First, I show that:

$$\text{for all } A, B \in Z, \text{ if } \#\max(A) = \#\max(B), \text{ then } A \sim B. \tag{1}$$

Suppose $A, B \in Z$ and $\#\max(A) = \#\max(B) = g$. Let $\max(A) = \{a_1, \ldots, a_g\}$ and $\max(B) = \{b_1, \ldots, b_g\}$. By SND,

$$\{a_1\} \sim \{b_1\} \tag{2}$$

and

$$\{a_2\} \sim \{b_2\}. \tag{3}$$

$\{a_1\} \cap \{b_1\} = \{a_2\} \cap \{b_2\} = \emptyset$ and, further $\max(\{a_1\}) = \{a_1\}, \max(\{a_2\}) = \{a_2\}$ and $\max(\{b_1\}) = \{b_1\}, \max(\{b_2\}) = \{b_2\}$, since $a_1, a_2 \in \max(A)$ and b_1,

$b_2 \in \max(B)$. Hence by (2), (3) and COM, I have

$$\{a_1, a_2\} \sim \{b_1, b_2\}. \tag{4}$$

By SND, again,

$$\{a_3\} \sim \{b_3\}. \tag{5}$$

By (4), (5) and COM,

$$\{a_1, a_2, a_3\} \sim \{b_1, b_2, b_3\}. \tag{6}$$

Proceeding in this way, I finally have $\{a_1, \ldots, a_g\} \sim \{b_1, \ldots, b_g\}$, that is $\max(A) \sim \max(B)$. If $A = \max(A)$, then $A \sim \max(B)$. Suppose $\{A \backslash \max(A)\} \neq \varnothing$. Let $\{A \backslash \max(A)\} = \{\bar{a}_1, \ldots, \bar{a}_m\} \neq \varnothing$. It is clear that $T_1 = \max(A) \cup \{\bar{a}_1, \ldots, \bar{a}_m\}$ is such that $T_1 \subseteq A$ and $\max(A \backslash T_1) = \varnothing$. Then, by IMON, $T_1 \sim \max(B)$. Hence I have

$$A \sim \max(B). \tag{7}$$

Similarly, by IMON, from (7), we have $A \sim B$, which proves (1).

Next, I show:

$$\text{for all } A, B \in Z, \text{ if } \#\max(A) > \#\max(B), \text{ then } A \succ B. \tag{8}$$

Suppose $A, B \in Z$ and $\#\max(A) > \#\max(B)$. Let $\#\max(B) = g$ and $\#\max(A) = g + t$ (where $t > 0$). Further, let $\max(B) = \{b_1, \ldots, b_g\}$ and $\max(A) = \{a_1, \ldots, a_g, \ldots, a_{g+t}\}$. Note that $\max\{a_1, \ldots, a_g\} = \{a_1, \ldots, a_g\}$. Hence, by (1),

$$\{a_1, \ldots, a_g\} \sim B \tag{9}$$

since $\max(A) = \{a_1, \ldots, a_g, \ldots, a_{g+t}\}$ it is clear that $T_{g+1} = \max\{a_1, \ldots, a_g\} \cup \{a_{g+1}\}$ is such that $\{a_1, \ldots, a_g\} \subseteq T_{g+1}$ and $\max(T_{g+1} \backslash \{a_1, \ldots, a_g\}) \neq \varnothing$. Then by IMON and (9), it follows that

$$T_{g+1} \succ \{a_1, \ldots, a_g\}$$

and by (9)

$$T_{g+1} \succ \max(B). \tag{10}$$

Taking (10), adding $a_{g+2}, \ldots, a_{g+t}$ on the left hand side, and using IMON repeatedly, I have

$$\{a_1, \ldots, a_{g+t}\} \succ B. \tag{11}$$

Taking (11) and using an argument similar to the one used to establish (7), by IMON, I have $A \succ B$, which proves (8). (1) and (8) complete the proof of the sufficiency part of the proposition. ■

The binary relation $\succsim^*$ is transitive. Given $\succsim^*$, indifference between two sets may arise either because each of them has an empty intersection with $\mathscr{P}(\wp)$ or because the two intersections with $\mathscr{P}(\wp)$ have the same number of elements. In both cases, particularly the first one, a lexicographic version of the original rule may enrich the ranking allow us to discriminate between such sets.

4 A lexicographic approach

The set of preferences that a reasonable person may have should be rich enough and should take into account any valuable alternative in terms of freedom. Nevertheless, there may be situations where the aspirations of a reasonable person in X cannot be satisfied in the sets to be compared. Let us consider the example of a country where free press is not available and there is a consensus that free newspapers are the uniquely relevant choice. Even in that case it can be claimed that different sets of 'not free' newspapers may provide alternatives that deserve some value in terms of freedom.

Admitting this possibility, one way of adapting the previous approach may be to sequentially remove the first element in all the reasonable persons' preferences and compare the available sets of newspapers according to this new set of preferences. This new reference set where alternatives are sequentially eliminated, does not represent the absolute idea of freedom proposed in the first section of this paper. Instead, it is a compromised idea of freedom that can only be justified when the set of reasonable preferences cannot discriminate among the sets of alternatives.

Two different lexicographic versions are studied in this section. The first one eliminates the most preferred elements in a reasonable person's preferences until the preferences are able to discriminate between the two sets. A second lexicographic ranking proposes a stronger criterion. This second criterion declares A to be weakly preferred to B only if the intersection of A with the set of relevant alternatives generated from the sequential elimination of the most preferred elements in the set of reasonable persons' preferences is always equal or greater than that of B.

Let R_j^1 be the preference where the first element has been removed. Denote the set of alternatives in X that are in R_j^1 as $\widetilde{R_j^1}$. In general

$$R_j^i \equiv \{R_j^{i-1} - \{\Theta\} \mid \forall y, x \in \widetilde{R_j^{i-1}}, x \in \Theta \leftrightarrow xR_j^{i-1}y\}$$

where $R_j^i - \{\Theta\}$ denotes the preference R_j^i where the alternatives in Θ have been removed. I denote by $\wp^i = (R_1^i, \ldots, R_n^i)$ the set of preference orderings over X where the best elements have been removed i times from the set of preferences that a reasonable person may have. Thus $\max^i(A)$ is the set of all alternatives x in A such that x is a best alternative in X for some ordering in $\wp^i$. Let $\mathscr{P}(\wp^i)$ be the set of alternatives in $\max^i(X)$.

For all $A, B \in Z$, $[A \succsim^i B]$ will be interpreted as "A offers at least as much freedom as B according to $\wp^i$". For all $A, B \in Z$, $[A \succ^i B$ iff $A \succsim^i B$ and $\neg(B \succsim^i A)]$ and $[A \sim^i B$ iff $A \succsim^i B$ and $B \succsim^i A]$.

Now I consider a number of properties of the binary relation $\succsim$ over Z.

Definition 2. *A binary relation* $\succsim$ *over* Z *satisfies:*

1. *simple non-dominance by levels* (SND^i) *iff, for all* $x, y \in X$, [*if* $\#\max^i(x) = \#\max^i(y)$, *then* $\{x\} \sim^i \{y\}$];
2. *inclusion monotonicity by levels* ($IMON^i$) *iff, for all* $A, B \in Z$ *if* $A \supseteq B$ *and* $\max^i(A \backslash B) \neq \emptyset$, *then* $A \succ^i B$. *If* $A \supseteq B$ *and* $\max^i(A \backslash B) = \emptyset$, *then* $A \sim^i B$;
3. *property 1 (P-1) iff, for all* $A, B \in Z$ *if* $\#\max^i(A) = \#\max^i(B)$ *for all* i, *then* $A \sim B$;
4. *property 2 (P-2) iff, for all* $A, B \in Z$ *if* $\#\max^i(A) > \#\max^i(B)$ *and* $\#\max^j(A) = \#\max^j(B)$ *for all* $j < i$, *then* $A \succ B$;
5. *property 3 (P-3) iff, for all* $A, B \in Z$ *if there is a level* i *such that* $\#\max^i(A) > \#\max^i(B)$, *then* $\neg(B \succsim A)$.

The first two axioms are an adaptation of the SND and IMON defined in Sect. 1. They apply to the different levels in which the set of relevant alternatives may be defined. The axioms P-1, P-2 and P-3 are three preference-dependent axioms. Property 1 stipulates that, if two sets have the same number of relevant alternatives for any $\wp^i$, then they will be considered indifferent in terms of freedom. Axiom P-2 requires that A be preferred to B in terms of freedom if the number of relevant alternatives in A is bigger than that in B according to $\mathscr{P}(\wp^i)$, and no other $\mathscr{P}(\wp^j)$, $j < i$, gives a different number of relevant alternatives for one of the sets. Axiom P-3 implies that if A has more relevant elements than B according to some $\mathscr{P}(\wp^i)$, then B is not going to be preferred to A.

The binary relation $\succsim^{\text{l}}$ that represents a lexicographic version of $\succsim^*$ is such that,

$$\text{for all } A, B \in Z, \left[A \succsim^{\text{l}} B \text{ iff } \begin{cases} \#\max^j(A) = \#\max^j(B) \text{ for all } j < i \\ \exists i \text{ such that } \#\max^i(A) > \#\max^i(B). \end{cases} \right] \tag{12}$$

Proposition 2. $\succsim$ *satisfies* SND^i, *COM, P-1, P-2 and* $IMON^i$ *if and only if* $\succsim = \succsim^{\text{l}}$.

Proof. The necessity part of the proposition is straightforward; I prove the sufficiency part. Let $\succsim$ satisfy SND^i, COM, P-1, P-2 and IMON^i. First, I show:

$$\text{for all } A, B \in Z, \text{ if } \#\max^i(A) = \#\max^i(B) \text{ for all } i, \text{ then } A \sim B. \tag{13}$$

Suppose $A, B \in Z$ and $\#\max^i(A) = \#\max^i(B) = g$. Let $\max^i(A) = \{a_1, \ldots, a_g\}$ and $\max^i(B) = \{b_1, \ldots, b_g\}$. By SND^i,

$$\{a_1\} \sim^i \{b_1\} \tag{14}$$

and

$$\{a_2\} \sim^i \{b_2\}. \tag{15}$$

$\{a_1\} \cap \{b_1\} = \{a_2\} \cap \{b_2\} = \varnothing$ and, further $\max^i(\{a_1\}) = \{a_1\}, \max^i(\{a_2\}) = \{a_2\}$ and $\max^i(\{b_1\}) = \{b_1\}, \max^i(\{b_2\}) = \{b_2\}$ (since $a_1, a_2 \in \max^i(A)$ and $b_1, b_2 \in \max^i(B)$). Hence by (14), (15) and COM, I have

$$\{a_1, a_2\} \sim^i \{b_1, b_2\}. \tag{16}$$

By SND^i, again,

$$\{a_3\} \sim^i \{b_3\}. \tag{17}$$

By (16), (17) and COM,

$$\{a_1, a_2, a_3\} \sim^i \{b_1, b_2, b_3\}. \tag{18}$$

Proceeding in this way, I finally have $\{a_1, \ldots, a_g\} \sim^i \{b_1, \ldots, b_g\}$, that is, $\max^i(A) \sim^i \max^i(B)$. If $A = \max^i(A)$, then $A \sim^i \max^i(B)$. Suppose $\{A \backslash \max^i(A)\} \neq \varnothing$. Let $\{A \backslash \max^i(A)\} = \{\bar{a}_1, \ldots, \bar{a}_m\} \neq \varnothing$. It is clear that $T_1 = \max^i(A) \cup \{\bar{a}_1\}$ is such that $T_1 \subseteq A$ and $\max^i(A \backslash T_1) = \varnothing$. Then, by IMON^i and (18), I have $A \sim^i \max^i(B)$ in this case. Thus, in all cases,

$$A \sim^i \max^i(B). \tag{19}$$

Similarly, using IMON^i in (19), $A \sim^i B$ for all i. By P-1, $A \sim B$, which proves (13).

Next, I show:

$$\text{for all } A, B \in Z, \text{ if } \begin{cases} \#\max^j(A) = \#\max^j(B) \text{ for all } j < i \\ \exists i \text{ such that } \#\max^i(A) > \#\max^i(B) \end{cases} \text{ then } A \succ B. \tag{20}$$

Suppose $A, B \in Z$ and $\#\max^i(A) > \#\max^i(B)$. Let $\#\max^i(B) = g$ and $\#\max^i(A) = g + t$ (where $t > 0$). Further, let $\max^i(B) = \{b_1, \ldots, b_g\}$ and $\max^i(A) = \{a_1, \ldots, a_g, \ldots, a_{g+t}\}$. Note that $\max^i\{a_1, \ldots, a_g\} = \{a_1, \ldots, a_g\}$. Hence, by (13),

$$\{a_1, \ldots, a_g\} \sim^i B \tag{21}$$

since $\max^i(A) = \{a_1, \ldots, a_g, \ldots, a_{g+t}\}$, it is clear that $T_{g+1} = \max^i\{a_1, \ldots, a_g\} \cup \{a_{g+1}\}$ is such that $\{a_1, \ldots, a_g\} \subseteq T_{g+1}$ and $\max^i(T_{g+1} \backslash \{a_1, \ldots, a_g\}) \neq \varnothing$. Then, by IMON^i and (21), it follows that

$$T_{g+1} \succ^i \{a_1, \ldots, a_g\}$$

and by (21)

$$T_{g+1} \succ^i \max^i(B). \tag{22}$$

Taking (22), adding $a_{g+2}, \ldots, a_{g+t}$ on the left hand side, and using IMON^i, I have

$$\{a_1, \ldots, a_{g+t}\} \succ^i B. \tag{23}$$

Taking (23) and using an argument similar to the one used to prove (19), by IMON^i, I have $A \succ^i B$. It is also known that A $\sim^j B$ by (13). Then, by P-2,

$A \succ B$ which proves (20). (13) and (20) together establish the sufficiency part of the proposition. ■

I have already mentioned the differences between the rule just characterized and the relation $\succsim^*$. There are situations where one may be interested in strengthening the requirements for declaring a set preferred to another in terms of freedom. In the previous example about free press we were dealing with an issue of fundamental rights. Once any reasonable person agrees that no available journal represents a reasonable choice, we may need to impose a stronger criterion to discriminate between the two sets. One way of doing this is to use full lexicographic domination.

The binary relation $\succsim^1$ that represents a lexicographic version of $\succsim^*$ is such that,

$$\text{for all } A, B \in Z, \ [A \succsim^1 B \text{ iff } \#\max^i(A) \geq \#\max^i(B) \text{ for all } i] \tag{24}$$

Proposition 3. *$\succsim$ satisfies SND^i, COM, P-1, P-3 and IMON^i if and only if $\succsim = \succsim^1$.*

Proof. The necessity part of the proposition is straightforward; I prove only the sufficiency part. Let $\succsim$ satisfy SND^i, COM, P-1, P-3 and IMON^i. First, I show:

$$\text{for all } A, B \in Z, \ \text{if } \#\max^i(A) = \#\max^i(B) \text{ for all } i, \text{ then } A \sim B. \tag{25}$$

Suppose $A, B \in Z$ and $\#\max^i(A) = \#\max^i(B) = g$. Let $\max^i(A) = \{a_1, \ldots, a_g\}$ and $\max^i(B) = \{b_1, \ldots, b_g\}$. By SND^i,

$$\{a_1\} \sim^i \{b_1\} \tag{26}$$

and

$$\{a_2\} \sim^i \{b_2\}. \tag{27}$$

$\{a_1\} \cap \{b_1\} = \{a_2\} \cap \{b_2\} = \varnothing$ and, further $\max^i(\{a_1\}) = \{a_1\}, \max^i(\{a_2\}) = \{a_2\}$ and $\max^i(\{b_1\}) = \{b_1\}, \max^i(\{b_2\}) = \{b_2\}$ (since $a_1, a_2 \in \max^i(A)$ and $b_1, b_2 \in \max^i(B)$). Hence by (26), (27) and COM, I have

$$\{a_1, a_2\} \sim^i \{b_1, b_2\}. \tag{28}$$

By SND^i, again,

$$\{a_3\} \sim^i \{b_3\}. \tag{29}$$

By (28), (29) and COM,

$$\{a_1, a_2, a_3\} \sim^i \{b_1, b_2, b_3\}. \tag{30}$$

Proceeding in this way, finally I have $\{a_1, \ldots, a_g\} \sim^i \{b_1, \ldots, b_g\}$, that is $\max^i(A) \sim^i \max^i(B)$. If $A = \max^i(A)$, then $A \sim^i \max^i(B)$. Suppose $\{A \backslash \max^i(A)\} \neq \varnothing$. Let $\{A \backslash \max^i(A)\} = \{\bar{a}_1, \ldots, \bar{a}_m\} \neq \varnothing$. It is clear that $T_1 = \max^i(A) \cup \{\bar{a}_1\}$ is such that $T_1 \subseteq A$ and $\max^i(A \backslash T_1) = \varnothing$. Then by IMON^i and (30), I have $A \sim^i \max^i(B)$ in this case. Thus, in all cases,

$$A \sim^i \max^i(B). \tag{31}$$

Similarly, using IMON^i in (31), $A \sim^i B$ for all i. By P-1, $A \sim B$ which proves (25).

Next, I show:

$$\text{for all } A, B \in Z, \text{ if } \#\max^i(A) > \#\max^i(B) \text{ for all } i, \text{ then } A \succ B. \qquad (32)$$

Suppose $A, B \in Z$ and $\#\max^i(A) > \#\max^i(B)$. Let $\#\max^i(B) = g$ and $\#\max^i(A) = g + t$ (where $t > 0$). Further, let $\max^i(B) = \{b_1, \ldots, b_g\}$ and $\max^i(A) = \{a_1, \ldots, a_g, \ldots, a_{g+t}\}$. Note that $\max^i\{a_1, \ldots, a_g\} = \{a_1, \ldots, a_g\}$. Hence, by (25),

$$\{a_1, \ldots, a_g\} \sim^i B. \qquad (33)$$

Since $\max^i(A) = \{a_1, \ldots, a_g, \ldots, a_{g+t}\}$, it is clear that $T_{g+1} = \max^i\{a_1, \ldots, a_g\} \cup \{a_{g+1}\}$ is such that $\{a_1, \ldots, a_g\} \subseteq T_{g+1}$ and $\max^i(T_{g+1} \backslash \{a_1, \ldots, a_g\}) \neq \varnothing$. Then by IMON^i and (33), it follows that

$$T_{g+1} \succ^i \{a_1, \ldots, a_g\}$$

and by (33)

$$T_{g+1} \succ^i \max^i(B). \qquad (34)$$

Taking (34), adding $a_{g+2}, \ldots, a_{g+t}$ on the left hand side, and using IMON^i, I have

$$\{a_1, \ldots, a_{g+t}\} \succ^i B. \qquad (35)$$

Taking (35) and using an argument similar to the one used to prove (31), by IMON^i, I have $A \succ^i B$, and this for all i. By P-3 $\neg(B \succsim A)$ which proves (32).

Next, I show:

$$\text{for all } A, B \in Z, \text{ if } \begin{cases} \#\max^i(A) > \#\max^i(B) \\ \#\max^j(A) < \#\max^j(B) \end{cases}$$

$$\text{then } A \text{ is non comparable with } B. \qquad (36)$$

Using the previous argument, we have $A \succ^i B$, and, by P-3, $\neg(B \succsim A)$, $B \succ^j A$ and $\neg(A \succsim B)$. Then A and B are non comparable.

(25), (32) and (36), together, establish the sufficiency part of the proposition. ■

Both lexicographic rules turn out to be very similar in terms of the axioms that characterized them. They share in their characterizations the axioms SND^i, COM, IMON^i (that link them with $\succsim^*$) and P-1. The difference between $\succsim^{\mathrm{i}}$ and $\succsim^1$ is due to axioms P-2 and P-3. While axiom P-2 gives decisive power over the ranking of A and B to the first $\mathscr{P}(\wp^i)$ that discriminates between both sets, P-3 guarantees that once a decisive set of preferences $\mathscr{P}(\wp^i)$ is found, they will not be contradicted by the ranking of the sets in terms of freedom.

An example may clarify the differences between $\succsim^{\mathrm{i}}$ and $\succsim^1$. Let $A = \{x, z, u\}$ and $B = \{w, u\}$ be two sets of alternatives. Suppose that the preferences that any reasonable person may have in a society are as listed below in columns:

u	z	u	z	w
x	u	y	w	z
v	v	z	x	u
w	x	x	v	x
z	y	w	y	v
y	w	v	u	y

In each ranking shown above, the elements are arranged in strictly descending order of preference. We can easily see that $A \sim^0 B$, $A \succ^1 B$, $A \succ^2 B$, $A \sim^3 B$, $A \sim^4 B$, $A \prec^5 B$. That means that according to the rules defined, $A \succ^l B$. However, A and B are declared non comparable by $\succsim^1$.

5 Concluding remarks

This paper analyses the role of preferences in assessing an agent's freedom of choice. Several authors have argued in favor of introducing preferences as the basis for the evaluation of opportunity sets in terms of freedom (Sen 1991, 1993, Foster 1992, and Puppe 1996 among others). I incorporate the role of preferences in the evaluation of opportunity sets using an approach based on the notion of preference orderings that a reasonable person may have. The main idea is that sets of alternatives offer freedom only if they provide meaningful choices to persons in a society. I argue that only the alternatives chosen by a reasonable person from the set of all possible ones provide a meaningful choice. This paper thus presents an axiomatization of the cardinality rule in this context.

There could be cases where the aspirations of a reasonable person cannot be fulfilled by the alternatives in a feasible set. For example, imagine a situation where the feasible set is composed of alternative ways in which one is to be executed. Even in this case, it may be possible to distinguish between different feasible sets in terms of freedom. For instance, in the context of our example, to be able to choose between being beheaded and being killed in the electric chair may be of some value. This situation can be considered using a 'restricted' idea of freedom. This idea extends the previous approach by sequentially removing the first element in all reasonable persons' preferences and comparing the available sets of alternatives according to these new preferences. Two different rules, a weak and a strong lexicographic version of the cardinality rule, are characterized using this restricted idea of freedom.

In the analysis, alternatives outside an opportunity set are relevant for assessing the freedom that this particular set offers. Such a specification of the set of relevant alternatives takes into account the fact that expectations play a role in analyzing freedom. For example, consider the possibility of going on a business trip from Europe to America in a sailboat. The existence of modern ships and planes makes this alternative irrelevant even if one can not afford a plane ticket. Thus, the perception that a particular agent has of her freedom depends not only on the feasible alternatives but also on infeasible alternatives since the latter contribute towards shaping her aspirations and focusing her achievements.

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