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Genuine multipartite entanglement in quantum phase transitions

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We demonstrate that the global-entanglement (GE) measure defined by Meyer and Wallach [J. Math. Phys. **43**, 4273 (2002)] is maximal at the critical point for the Ising chain in a transverse magnetic field. Our analysis is based on the equivalence of GE to the averaged linear entropy, allowing an understanding of multipartite entanglement (ME) features through a generalization of GE for bipartite blocks of qubits. Moreover, in contrast to GE, the proposed ME measure can distinguish three paradigmatic entangled states: GHZ_N (Greenberger-Horne-Zeilinger), W_N, and $\mathrm{EPR}^{\otimes N/2}$. As such the generalized measure can detect a genuine ME and is maximal at the critical point.

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Entanglement is a correlation of exclusively quantum nature present (in principle) in any set of postinteracting quantum systems [1]. As such, multipartite entanglement (ME) is expected to play a key role on quantum-phase transition (QPT) phenomena in the same way that classical (statistical) correlation does on classical phase transitions [2,3]. In ordinary phase transitions, at the critical point, a nonzero order parameter characterizes a long-range correlation (given by the correlation length divergence). In the same way, in QPTs the ME is expected to be maximal at the critical point, in the sense that all the system parties would be entangled with each other [2]. However, this conjecture could not be proved in general either by measures of pairwise entanglement or by the proposed ME measures. Even after a considerable effort a deep understanding of multipartite-entangled states (MES) is lacking. Thus it is still a great challenge to capture the essential features of ME from a conceptual point of view as well as from a quantitative approach, defining a measure that among its other properties is able to distinguish MES [4,5].

Indeed, concerning the legitimate quantum correlations in QPTs it would certainly be important to know exactly what kind of entanglement we should expect to be maximal at the critical point. The great majority of efforts trying to answer this question made use of two kinds of bipartite entanglement measures, both calculated for spin-1/2 lattice models such as the Ising model in a transverse magnetic field [6]. The first measure, namely the pairwise entanglement (concurrence) between two spins in the chain, was studied in Refs. [2,3]. The second measure, the entropy of entanglement between one part of the chain (a block of L spins) and the rest of the chain, was investigated in Refs. [2,7,8]. Some candidates for ME measures were also evaluated in systems exhibiting QPTs [9–11]. Nevertheless, none of the entanglement measures employed in the references cited are maximal at the critical point except the single site entropy for the Ising model [2] in the thermodynamical limit and the localizable entanglement [11] for an Ising chain with 14 spins. Furthermore, in Refs. [2,3] the authors have independently shown that bipartite entanglement vanishes when the distance between the two spins is greater than two lattice sites [12]. This is not expected since long-range quantum correlations should be present at the critical point. It was then suggested that bipartite entanglement at the critical point should be decreased in order to increase ME due to entanglement sharing [2]. In other words, ME only appears at the expense of pairwise entanglement, and at the critical point we should expect a genuine MES.

In this paper we demonstrate that the global entanglement (GE) introduced in Ref. [13] indeed captures the essential point to be maximal at the critical point for the Ising model in a transverse magnetic field in the thermodynamical limit. We also prove that there exists an interesting relation between the GE, the von Neumann entropy, linear entropy (LE), and 2-tangle [14–16], showing that they are all equivalent to detect QPTs. Furthermore, this relation helps us to understand the results obtained in Ref. [2], as outlined in the previous paragraph, and suggests that they are not unique to the Ising model but common to all MES with translational invariance. In addition to this, we generalize the GE and propose a ME measure that is also maximal at the critical point for the Ising model and can detect genuine MES, and in contrast to GE furnishes different values for the entanglement of the GHZ_N, (Greenberger-Horne-Zeilinger), W_N, and $\mathrm{EPR}^{\otimes N/2}$ states, thus being able to distinguish between MES.

For a N qubit system (spin-1/2 chain) it was noted that GE is simply related to the N single qubit purities [16–18] by

$$E_G^{(1)} = 2 - \frac{2}{N}\sum_{j=1}^{N} \mathrm{Tr}(\rho_j^2) = \frac{1}{N}\sum_{j=1}^{N} S_L(\rho_j) = \langle S_L\rangle, \tag{1}$$

where GE is hereafter identified as $E_G^{(1)}$, $\rho_j=\mathrm{Tr}_{\bar{j}}\{\rho\}$ is the jth qubit reduced-density matrix obtained by tracing out the other $\bar{j}$ qubits, and $S_L(\rho_j)=[d/(d-1)][1-\mathrm{Tr}(\rho_j^2)]$ is the standard definition of LE. This relation shows that $E_G^{(1)}$ is just the mean of LE. It was also noted in Refs. [16,19] that

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$$E_G^{(1)} = \frac{1}{N}\sum_{j=1}^{N} \tau_{j,rest} = \langle \tau \rangle, \tag{2}$$

where $\tau_{j,rest}=C^2$ is the 2-tangle [14,16] (the square of concurrence C) [20]. Both LE and the 2-tangle can thus be used to quantify the entanglement between any block bipartition of a system of N qubits. (They quantify the entanglement between one qubit j and the *rest* of the $N-1$ qubits of the chain [16].) The proof of Eq. (2) is based on the Schmidt decomposition [21] that also allows us to use for pure systems the reduced von Neumann entropy, $S_V(\rho_{j(\bar{j})})=-\mathrm{Tr}_{j(\bar{j})} \times[\rho_{j(\bar{j})}\log_d(\rho_{j(\bar{j})})]$ as a good bipartite-entanglement measure [22]. Here $d=\min\{\dim \mathcal{H}_j, \dim \mathcal{H}_{\bar{j}}\}$ and $\dim \mathcal{H}_{j(\bar{j})}$ is the Hilbert-space dimension of subsystem $j(\bar{j})$. Recalling that S_V is bounded from below by S_L and employing Eqs. (1) and (2) we obtain the following important relation:

$$E_G^{(1)} = \langle \tau \rangle = \langle S_L \rangle \leqslant \langle S_V \rangle, \tag{3}$$

which states that the GE is nothing but the mean LE of single qubits with the rest of the chain. Furthermore, the GE is also equal to the mean 2-tangle and a lower bound for the mean von Neumann entropy. An immediate consequence of this result shows up when we deal with linear chains with translational invariance. This implies that $\langle S_L\rangle=S_L(\rho_j)$ and that $\langle S_V\rangle=S_V(\rho_j)$. Hence, Eq. (3) becomes $E_G^{(1)}=S_L(\rho_j)\leqslant S_V(\rho_j)$. Since $S_L(\rho_j)$ and $S_V(\rho_j)$ have the same concavity and both entropies attain their maximal value for a maximally mixed state this last relation shows that $E_G^{(1)}$ is as efficient as the linear and the von Neumann entropies to detect QPTs. In Ref. [2] the authors used S_V and in Ref. [9] $E_G^{(1)}$ was employed to detect QPTs in the Ising model. Needless to say, both works arrived at the same results for a given range of parameters, notwithstanding their use of different entanglement measures, which by that time were thought to be unrelated.

Despite its success to detect the Greenberger-Horne-Zeilinger (GHZ) state [19,23], $E_G^{(1)}$ sometimes fails to distinguish different multipartite states. This is best understood if we study $E_G^{(1)}$ for three paradigmatic multipartite states. The first is $|\mathrm{GHZ}_N\rangle=(1/\sqrt{2})(|0\rangle^{\otimes N}+|1\rangle^{\otimes N})$, where $|0\rangle^{\otimes N}$ and $|1\rangle^{\otimes N}$ represent N tensor products of $|0\rangle$ and $|1\rangle$ respectively. The second is a tensor product of $N/2$ Bell states [18], $|\mathrm{EPR}_N\rangle=|\Phi^+\rangle^{\otimes N/2}$, where $|\Phi^+\rangle=(1/\sqrt{2})(|00\rangle+|11\rangle)$. This state is obviously not a MES. Only the pairs of qubits $(2j–1,2j)$, where $j=1,2,\ldots,N$, are entangled. Nevertheless, for both states $E_G^{(1)}=1$. The last is the W state [4], $|W_N\rangle =(1/\sqrt{N})\Sigma_{j=1}^{N}|00\cdots 1_j \cdots 00\rangle$. The state $|00\cdots 1_j\cdots 00\rangle$ represents N qubits in which the jth is $|1\rangle$ and the others are $|0\rangle$. As shown in Ref. [13], $E_G^{(1)}(W_N)=4(N-1)/N^2$.

We now present a generalization of GE. There are three main features of this approach. First, it becomes clear that we have different classes of ME measures, where $E_G^{(1)}$ is the first one. Second, the first nontrivial class, $E_G^{(2)}$, furnishes different values for the three states considered above. Third, it gives new insights into the study of QPT and ME.

In order to define $E_G^{(2)}$ we need the following function:

TABLE I. Comparison among the three paradigmatic states.

	$E_G^{(1)}$	$G(2,1)$	$E_G^{(2)}$
GHZ_N	1	2/3	2/3
EPR_N	1	$\frac{N-2}{2(N-1)}$	$\frac{(2N-1)(N-2)}{2(N-1)^2}$
W_N	$\frac{4(N-1)}{N^2}$	$\frac{16(N-2)}{3N^2}$	$\frac{16(N-2)}{3N^2}$

$$G(2,l) \equiv \frac{4}{3}\left(1-\frac{1}{N-l}\sum_{j=1}^{N-l}\mathrm{Tr}(\rho_{j,j+l}^2)\right), \tag{4}$$

where $\rho_{j,j+l}$ is the density matrix of qubits j and $j+l$ obtained by tracing out the other $N-2$ qubits. The index $0<l<N$ is the distance in the chain of two qubits and $4/3$ is a normalization constant assuring that $G(2,l)\leqslant 1$. Of interest here are two quantities that can be considered ME measures in the same sense that $E_G^{(1)}$ is:

$$G(2,1) \equiv \frac{4}{3}\left(1-\frac{1}{N-1}\sum_{j=1}^{N-1}\mathrm{Tr}(\rho_{j,j+1}^2)\right), \tag{5}$$

and

$$E_G^{(2)} = \frac{1}{N-1}\sum_{l=1}^{N-1} G(2,l). \tag{6}$$

We can interpret $G(2,1)$ as the mean LE of all two-qubit nearest neighbors with the rest of the chain. Similar interpretations are valid for the others $G(2,l)$. $E_G^{(2)}$ is the mean of all $G(2,l)$ and it gives the mean LE of all two qubits, independent of their distance, with the rest of the chain [24]. To define $E_G^{(3)}$ we need the function $G(3,l_1,l_2)$ with one more parameter, since now we can have different distances between the three qubits of the reduced state. A complete analysis of this ME measure and its usefulness to detect MES is discussed elsewhere [25].

Table I shows the quantities given by Eqs. (5) and (6) for $\mathrm{GHZ}_N,\mathrm{EPR}_N$, and W_N. We note that due to translational symmetry, $G(2,1)$ and $E_G^{(2)}$ are identical for GHZ_N and W_N. It is worthy of mention that depending on the value of N the states are classified differently by $G(2,1)$. A similar behavior is observed for $E_G^{(2)}$ [24]. In this case, however, EPR_N is the most entangled state for long chains. The reason for that lies in the definition of $E_G^{(2)}$. For EPR_N, $G(2,l)=1$ for any $l\geqslant 2$. Thus, since $E_G^{(2)}$ is the average of all $G(2,l)$, for long chains $G(2,1)$ does not contribute significantly and $E_G^{(2)}\rightarrow 1$.

It is worth noting that even at the thermodynamical limit, $N\rightarrow\infty$, $E_G^{(2)}$, and $G(2,1)$ still distinguish the three states. However, the ordering of the states is different. As already explained, this is due to the contribution of $G(2,l)$, $l\geqslant 2$, in the calculation of $E_G^{(2)}(\mathrm{EPR}_N)$.

Now we specify to the one-dimensional Ising model in a transverse magnetic field, which is given by the following Hamiltonian:

$$H = \lambda \sum_{i=1}^{N} \sigma_i^x \sigma_{i+1}^x + \sum_{i=1}^{N} \sigma_i^z, \tag{7}$$

where i represents the ith qubit, λ is a free parameter related to the inverse strength of the magnetic field, and we work in the thermodynamical limit. We assume periodic boundary conditions: $\sigma_{N+1}=\sigma_1$. As we have shown, for a system with translational symmetry GE is nothing but the LE of one spin with the rest of the chain. We only need, then, the LE to obtain the GE. For that purpose we must calculate the single-qubit (or single-site) reduced-density matrix that is obtained from the two-qubits (two-sites) reduced-density matrix. It is a 4×4 matrix and can be written as

$$\rho_{ij} = \mathrm{Tr}_{\overline{ij}}[\rho] = \frac{1}{4}\sum_{\alpha,\beta} p_{\alpha\beta}\sigma_i^\alpha \otimes \sigma_j^\beta, \tag{8}$$

where ρ is the broken-symmetry ground state in the thermodynamical limit and $p_{\alpha\beta}=\mathrm{Tr}[\sigma_i^\alpha\sigma_j^\beta\rho_{ij}]=\langle\sigma_i^\alpha\sigma_j^\beta\rangle$. $\mathrm{Tr}_{\overline{ij}}$ is the partial trace over all degrees of freedom except the spins at sites i and j, σ_i^α is the Pauli matrix acting on the site $i, \alpha, \beta = 0, x, y, z$ where σ^0 is the identity matrix, and $p_{\alpha\beta}$ is real. Therefore, all we need are the ground state two-point correlation functions (CFs). Using symmetry arguments for the ground state [2] the only nonzero CFs are p_{00}, p_{xx}, p_{yy}, p_{zz}, $p_{0x}=p_{x0}$, $p_{0z}=p_{z0}$, and $p_{xz}=p_{zx}$. Due to normalization $p_{00}=1$ and a direct calculation gives $p_{xz}=p_{zx}=0$ for $\lambda\leqslant 1$. On the other hand, the Schwartz inequality necessarily gives $0\leqslant|p_{xz}|\leqslant|\langle\sigma_i^x\rangle\langle\sigma_i^z\rangle|$, thus allowing that the lower and upper bounds for entanglement be calculated for $\lambda>1$. We plot the upper bound for entanglement by taking $p_{xz}=0$. By continuity the true value for entanglement must show a similar behavior.

Those CFs have already been calculated [6], and we just highlight the main results. The two-point CFs and the mean values of σ^x and σ^z are

$$\langle\sigma_1^x\sigma_l^x\rangle = \begin{vmatrix} g(-1) & g(-2) & \cdots & g(-l) \\ g(0) & g(-1) & \cdots & g(-l+1) \\ \vdots & \vdots & \ddots & \vdots \\ g(l-2) & g(l-3) & \cdots & g(-1) \end{vmatrix}, \tag{9}$$

$$\langle\sigma_1^y\sigma_l^y\rangle = \begin{vmatrix} g(1) & g(0) & \cdots & g(-l+2) \\ g(2) & g(1) & \cdots & g(-l+3) \\ \vdots & \vdots & \ddots & \vdots \\ g(l) & g(l-1) & \cdots & g(1) \end{vmatrix}, \tag{10}$$

where $\langle\sigma_1^z\sigma_l^z\rangle=\langle\sigma_1^z\rangle^2-g(l)g(-l)$, $\langle\sigma_1^z\rangle=g(0)$, and $\langle\sigma_1^x\rangle=0$ for $\lambda\leqslant 1$ or $\langle\sigma_1^x\rangle=(1-\lambda^{-2})^{1/8}$ for $\lambda>1$. Here $g(l)=\mathcal{L}(l)+\lambda\mathcal{L}(l+1)$, $\mathcal{L}(l)=(1/\pi)\int_0^\pi dk\cos(kl)/[1+\lambda^2+2\lambda\cos(k)]$, and $l\geqslant 1$ is the lattice-site distance between two qubits. By tracing out one of the qubits we obtain the single-qubit density matrix, which allows us to obtain $E_G^{(1)}$ as a function of λ. This is shown in Fig. 1. As a matter of fact $E_G^{(1)}$ is maximal (with a singular derivative) at the critical point $\lambda=1$. For comparison, in Fig. 1 we plot $S_V(\rho_j)$, which was already shown to be also maximal at the critical point for the broken-symmetry state [2]. We emphasize that these measures quantify entanglement in the global system by measuring how mixed the subsystems are. The physical meaning behind studying "mixedness" lies in the fact that the more entangled two subsystems are the more mixed their reduced-density matrix should be [9,18]. However, in a many-body system there are many ways in which one could divide the global system into subsystems. The first nontrivial generalization is to study the LE of two sites with the rest of the chain. Using ρ_{ij} we can calculate $G(2,l)$ for the Ising model (Fig. 2). It has a similar behavior to $E_G^{(1)}$, since it is also maximal (with a singular derivative) at the critical point. This feature demonstrates that both a pair of nearest-neighbors sites and the sites themselves are maximally entangled with the rest of the chain at the critical point. But this is not unique to nearest neighbors as shown in Fig. 3, where $G(2,1)$, $G(2,15)$, and $E_G^{(2)}=(1/15)\Sigma_{i=1}^{15}G(2,i)$ is plotted. $G(2,15)$ is also maximal at the critical point indicating that in a QPT entanglement sharing at the critical point is favored by an increase of all types of MEs. Moreover, Fig. 3 shows that $G(2,15)$ is only slightly different from $E_G^{(2)}=\frac{1}{15}\Sigma_{i=1}^{15}G(2,i)$. This is due to the rapid convergence of $G(2,l)$ as l is increased. At the critical point $\lim_{l\to\infty}G(2,l)$ is 0.675, and thus higher than the values for

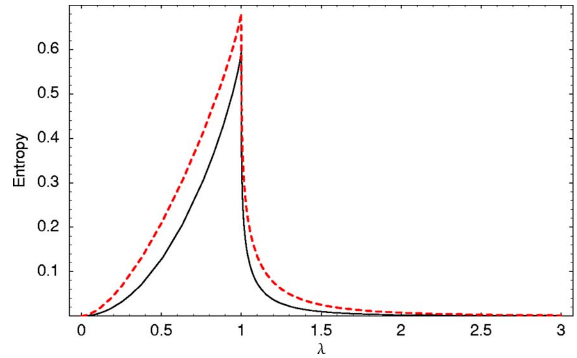

FIG. 1. (Color online) von Neumann entropy (dashed curve) and GE/LE (solid curve) as a function of λ.

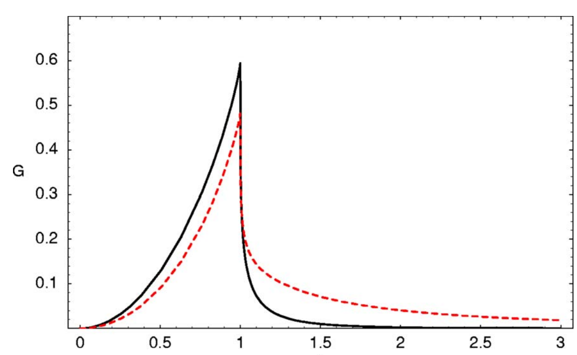

FIG. 2. (Color online) $E_G^{(1)}$ (solid curve) and $G(2,1)$ (dashed curve) as a function of λ. Both quantities are maximal at the critical point $\lambda=1$.

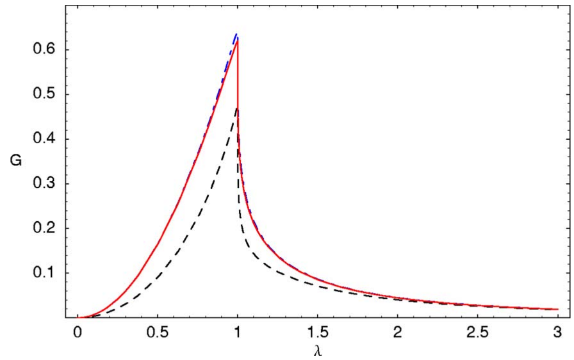

FIG. 3. (Color online) $G(2,1)$ (dashed/black), $G(2,15)$ (solid/red), and $E_G^{(2)}$ (dotted-dashed/blue) as a function of λ. We see that $E_G^{(2)}$ is slightly different from $G(2,15)$ showing that $G(2,l)$ saturates as $l \to \infty$.

$\mathrm{GHZ}_N, \mathrm{EPR}_N$, and W_N, we obtained in the thermodynamical limit, indicating thus a genuine MES. We also note that in addition to $E_G^{(1)}, G(2,l)$, and $E_G^{(2)}$ all being maximal at the critical point, $E_G^{(1)} < E_G^{(2)}$ for every value of λ. However, an interesting change of ordering for $E_G^{(1)}$ and $G(2,1)$ occurs around the critical point. For $\lambda \leqslant 1, E_G^{(1)} > G(2,1)$, but for $\lambda > 1, E_G^{(1)} < G(2,1)$. Thus one type of ME is favored in detriment to the others, depending on the system phase. Also, the fact that at the critical point both $E_G^{(1)}$ and $E_G^{(2)}$ are maximal indicates entanglement sharing, such that all the sites of the chain are strongly (quantum) correlated. Of course this statement is true only if $E_G^{(m)}$ is also shown to be maximal for any $2 < m \leqslant N-1$ (all possible partitions). Furthermore, the fact that $G(2,l)$ always increases as $l \to \infty$ at the critical point suggests a kind of diverging entanglement length. However, its precise definition demands a careful calculation of the scaling of entanglement such as in Refs. [7,9]. These points are left for further investigation [25].

In conclusion we have demonstrated that for an infinite Ising chain both $E_G^{(1)}$ and its generalization, $E_G^{(2)}$, are maximal at the critical point. Furthermore, $E_G^{(2)}$ as defined here is able to detect a genuine ME. We note that the behavior of the ME measures presented here for an infinite chain is in agreement with the localizable entanglement calculated for a finite ($N=14$) Ising chain for the broken-symmetry state [11]. Yet our results were obtained in a relatively simpler fashion and could be used to infer a genuine ME for systems where the localizable entanglement has failed to detect the QPT [26]. Finally, our results reinforced the conjecture of Osborne and Nielsen [5] that at the critical point ME should be high due to entanglement sharing, to the detriment of bipartite entanglement.

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