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SHAPE SENSITIVITY ANALYSIS FOR STRUCTURAL PROBLEMS WITH NON-LINEAR MATERIAL BEHAVIOUR

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SUMMARY

This paper describes some considerations around the analytical structural shape sensitivity analysis when the structural behaviour is computed using the finite element method with a non-linear constitutive material model. Depending on the type of non-linear behaviour two different approaches are proposed. First, a new direct (non-incremental) formulation is proposed for material models characterized by the fact that the stresses at any time t can be expressed in terms of the strains at the same time t and, in some cases, the strains at a specific past time t^{u} ($t^{\mathrm{u}} < t$). This is the case of elasticity (linear as well as non-linear), perfect plasticity and damage models. Second, a more classical incremental approach is proposed for general plasticity cases. A special strategy is also proposed for material models with strain softening. The quality and reliability of the proposed approaches are assessed through their application in different examples. Copyright © 1999 John Wiley & Sons, Ltd.

KEY WORDS: sensitivity analysis; non-linear material; finite elements

1. INTRODUCTION

Roughly speaking, sensitivities are understood as relations between some control parameters, like the structural response, and some design variables that define the structural shape. From this point of view, if a design variable q is selected, the sensitivity of the structural response can be defined in terms of the displacements $\mathbf{u}$ as the relation $\mathrm{d}\mathbf{u}/\mathrm{d}q$ or in terms of the stresses as $\mathrm{d}\boldsymbol{\sigma}/\mathrm{d}q$.

In optimization problems these sensitivities are normally defined in terms of the objective function and the constraints. In general, these functions depend on the structural response and, due to that, only the sensitivities in terms of the displacements or the stresses have been considered in this work.

There have been different contributions for performing the sensitivity analysis in non-linear structural systems [1–10]. In particular, in the case of non-linear material models, most of these contributions are related to the use of plasticity models, see [11–14]. Due to the fact that using an incremental approach solves most of the non-linear structural equilibrium problems, the

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sensitivity analysis is normally obtained as an addition of increments

$$\frac{\mathrm{d}\mathbf{u}^{t+\Delta t}}{\mathrm{d}q} = \frac{\mathrm{d}\mathbf{u}^{t}}{\mathrm{d}q} + \Delta\left(\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}q}\right)^{\Delta t} \tag{1}$$

The incremental magnitude of the sensitivities is obtained at each step of the incremental approach through differentiation of the incremental integral equilibrium equation Vidal *et al.* [12]

$$\frac{\mathrm{d}}{\mathrm{d}q}\left[\int_V \mathbf{B}^{\mathrm{T}}\,\dot{\boldsymbol{\sigma}}\,\mathrm{dV} - \dot{\mathbf{f}}\right] = 0 \tag{2}$$

where $\mathbf{B}$ is the deformation matrix, $\dot{\boldsymbol{\sigma}}$ is the stress increment vector and $\dot{\mathbf{f}}$ is the external forces increment vector.

Another alternative is to start from the incremental finite element discrete equilibrium equations, [13, 14]

$$\mathbf{K}_{\mathbf{T}}(\mathbf{u}, q)\Delta\mathbf{u}(q) = \mathbf{f}^{t+\Delta t} - \mathbf{r}^{t} \tag{3}$$

where $\mathbf{K}_{\mathbf{T}}$ is the tangent stiffness matrix, $\Delta\mathbf{u}$ is the incremental displacement vector, $\mathbf{f}^{t+\Delta t}$ is the nodal external forces vector and $\mathbf{r}^{t}$ is the nodal internal forces vector corresponding to the last equilibrium configuration.

These strategies involve high computational cost because the sensitivity analysis must be computed after the convergence of each load increment. In addition, its incremental nature favours carrying and accumulation of errors depending on the resolution strategy used to solve the equilibrium problems. Then, the quality of this type of sensitivity analysis can depend on the resolution strategy and the size of the load increments.

Another general approach of the sensitivity analysis with non-linear material problems appears [15], and it is applied to perfect plasticity [16]. In this last reference the differentiation of the global equilibrium equation in its integral form is proposed:

$$\frac{\mathrm{d}}{\mathrm{d}q}\left[\int_V \mathbf{B}^{\mathrm{T}}\,\boldsymbol{\sigma}\,\mathrm{d}V - \mathbf{f}\right] = 0 \tag{4}$$

where $\boldsymbol{\sigma}$ are the stresses and $\mathbf{f}$ is the external nodal forces vector.

The present work follows similar arguments for the development of one of the proposed strategies that can be used for elastic, perfect plasticity and damage models. On the other side, a more 'classical' strategy is proposed for general plasticity models. In addition, in order to solve problems involving constitutive material models with strain softening a specific approach is also proposed.

Next sections present the proposed sensitivity analysis formulations. Two different concepts are developed: The first one is a formulation for the sensitivity analysis of a non-linear constitutive material model without taking into account, yet, the possibility of a strain-softening behaviour. This concept is specific for the cases of damage and plasticity models. The second one is the possibility of considering a constitutive material model with strain softening. The additional considerations to be taken into account are developed for the case of the damage model with strain softening and can be easily generalised for any other constitutive model. One of these additional considerations is the necessity of including a special approach based on an arc-length strategy for a proper use of the sensitivity analysis to project the structural behaviour from an original structure to a modified one. Finally, the use of all the presented formulations is illustrated through some assessment examples.

2. FORMULATION OF THE SENSITIVITY ANALYSIS IN NON-LINEAR STRUCTURAL PROBLEMS

For a typical structural equilibrium problem, it is well known that when the applied external loads are in equilibrium with the internal loads the finite element discrete equilibrium equation at each step of the analysis can be written as

$$\sum_{\text{elem}} \int_V \mathbf{B}^{\mathrm{T}} \boldsymbol{\sigma}^t \, \mathrm{d}V - \mathbf{f}^t = 0 \tag{5}$$

Here we have used the superscript t to indicate the pseudo-time that corresponds to the step of the analysis in which the equilibrium equation has been considered.

Taking into account that our objective is to compute the sensitivities at the equilibrium point we can differentiate the discrete equilibrium equation with respect to a design variable q

$$\frac{\mathrm{d}}{\mathrm{d}q}\left[\sum_{\text{elem}} \int_V \mathbf{B}^{\mathrm{T}} \boldsymbol{\sigma}^t \, \mathrm{d}V - \mathbf{f}^t\right] = \sum_{\text{elem}} \left[\frac{\mathrm{d}}{\mathrm{d}q} \int_V \mathbf{B}^{\mathrm{T}} \boldsymbol{\sigma}^t \, \mathrm{d}V\right] - \frac{\mathrm{d}\mathbf{f}^t}{\mathrm{d}q} = 0 \tag{6}$$

The last expression can be developed for each finite element of the mesh. Assuming that the iso-parametric mapping can be used for the integration domain of each element the last expression becomes

$$\sum_{\text{elem}} \left[\int_{V_0} \frac{\mathrm{d}}{\mathrm{d}q} (\mathbf{B}^{\mathrm{T}} \boldsymbol{\sigma}^t |\mathbf{J}|) \, \mathrm{d}V_0\right] - \frac{\mathrm{d}\mathbf{f}^t}{\mathrm{d}q} = 0 \tag{7}$$

where $\mathbf{J}$ is the jacobian of the iso-parametric mapping. If now we develop the expression (7) for each element of the finite element mesh we obtain

$$\int_{V_0} \frac{\mathrm{d}}{\mathrm{d}q} (\mathbf{B}^{\mathrm{T}} \boldsymbol{\sigma}^t |\mathbf{J}|) \, \mathrm{d}V_0 = \int_{V_0} \frac{\mathrm{d}\mathbf{B}^{\mathrm{T}}}{\mathrm{d}q} \boldsymbol{\sigma}^t |\mathbf{J}| \, \mathrm{d}V_0 + \int_{V_0} \mathbf{B}^{\mathrm{T}} \frac{\mathrm{d}\boldsymbol{\sigma}^t}{\mathrm{d}q} |\mathbf{J}| \, \mathrm{d}V_0 + \int_{V_0} \mathbf{B}^{\mathrm{T}} \boldsymbol{\sigma}^t \frac{\mathrm{d}|\mathbf{J}|}{\mathrm{d}q} \, \mathrm{d}V_0 \tag{8}$$

where V_0 refers to the integration domain corresponding to one element. Most of the integral terms of the last expression are well known and can be obtained using the well-established strategies for linear problems (see [17]). Nevertheless, the second integral term contains the expression $\mathrm{d}\boldsymbol{\sigma}^t/\mathrm{dq}$ that includes the non-linearity that characterizes the constitutive material model. All the difficulties of the sensitivity analysis for material non-linear problems are concentrated in the computation of this term. Next, we will see how this term can be obtained in a direct way for some particular non-linear material models.

2.1. Damage and perfect plasticity cases

For a general non-linear material model, during the equilibrium structural analysis the stresses at each point for a given pseudo-time t can depend on the entire history of deformation. Due to this, obtaining the term $\mathrm{d}\boldsymbol{\sigma}^t/\mathrm{d}q$ is not an easy task, and normally, it involves an incremental procedure using expressions (1), (2) or (3) as commented above. Nevertheless, there are some cases where the material model allows expression of the stresses at time t in terms of the strains at the same time t and, in other cases, the strains at a past time t^{u} such as

1. Linear and non-linear elastic models: this is an obvious case.
2. Perfect plasticity models: this was presented by Silva *et al.* [16].
3. Damage models: this will be shown in detail in the next subsection.

In a general plasticity case the stresses at each time t can be expressed in terms of the total strains $\boldsymbol{\varepsilon}^t$ at the same time t and the plastic strains $\boldsymbol{\varepsilon}_{\mathrm{p}}^t$ at the same time as follows:

$$\boldsymbol{\sigma}^t = \mathbf{D}(\boldsymbol{\varepsilon}^t - \boldsymbol{\varepsilon}_{\mathrm{p}}^t) \tag{9}$$

where $\mathbf{D}$ is the linear elastic constitutive matrix. For obtaining the plastic strain at time t there are three possible situations:

(i) If the stresses have not reached the yield surface yet, the plastic strains are null and then the expression (9) reduces to $\boldsymbol{\sigma}^t = \mathbf{D}\boldsymbol{\varepsilon}^t$.
(ii) When the stresses lie in the yield surface the plastic strain at time t can be obtained in terms of the total strain $\boldsymbol{\varepsilon}^t$ in a direct way by using the consistency equation and a radial return algorithm. In this case expression (9) holds, and $\boldsymbol{\varepsilon}_{\mathrm{p}}^t$ can be written in terms of $\boldsymbol{\varepsilon}^t$.
(iii) When due to an unloading situation, the stresses go back to the interior of the space limited by the yield surface; the value of the plastic strain $\boldsymbol{\varepsilon}_{\mathrm{p}}^t$ at time t is the same as the plastic strain $\boldsymbol{\varepsilon}_{\mathrm{p}}^u$ at the time where the unloading process started t^u. In this case equation (9) transforms into $\boldsymbol{\sigma}^t = \mathbf{D}(\boldsymbol{\varepsilon}^t - \boldsymbol{\varepsilon}_{\mathrm{p}}^u)$.

As shown now, in the particular case of the perfect plasticity the stresses at time t can be expressed in terms of the strains at time t and a set of internal variables $\boldsymbol{\kappa}^t$ (plastic strains) which, in turn, depend on the strains at time t and, in some situations, the strains at a past time t^u. This can be expressed as follows:

$$\boldsymbol{\sigma}^t = \boldsymbol{\sigma}^t(\boldsymbol{\varepsilon}^t, \boldsymbol{\kappa}^t(\boldsymbol{\varepsilon}^t, \boldsymbol{\varepsilon}^{\mathrm{u}}, l_{\mathrm{c}})) \tag{10}$$

In expression (10) a possible dependence with respect to a characteristic length l_{c} has been considered. This is normally used if the material model contains a strain softening behaviour. The use of the characteristic length l_c is necessary for the regularisation of the solutions obtained from smeared (continuous) models after the analysis of a cracking (discontinuous) phenomena, specially for those with softening (see Oliver [18]).

On the other hand, in the case of a simple damage model, $\boldsymbol{\kappa}^t$ is the damage parameter d^t that characterizes the amount of existing damage and so expression (10) can be written in the following way:

$$\boldsymbol{\sigma}^t = (1 - d^t)\mathbf{D}\boldsymbol{\varepsilon}^t \tag{11}$$

where the obtainment of the damage parameter d^{t} in terms of the total stresses will be shown in a next section.

In all non-linear material models, for every step of the analysis the variation of the strains and stresses at the time t are related through the tangent constitutive matrix $\mathbf{D}_{\mathrm{T}}^t$ in the following way:

$$\mathrm{d}\boldsymbol{\sigma}^t = \mathbf{D}_{\mathrm{T}}^t\, \mathrm{d}\boldsymbol{\varepsilon}^t \tag{12}$$

And taking into account the equation (10) this matrix $\mathbf{D}_T^t$ can be written as follows:

$$\mathbf{D}_T^t = \frac{\partial \boldsymbol{\sigma}^t}{\partial \boldsymbol{\varepsilon}^t} + \frac{\partial \boldsymbol{\sigma}^t}{\partial \kappa^t} \frac{d\kappa^t}{d\boldsymbol{\varepsilon}^t} \tag{13}$$

where all the derivatives have to be evaluated at time t. For instance, for the classical plasticity model of equation (9) it is

$$\mathbf{D}_T^t = \mathbf{D}\left(1 - \frac{\partial \boldsymbol{\varepsilon}_p^t}{\partial \boldsymbol{\varepsilon}^t}\right) \tag{14}$$

and for the damage model of equation (11) it is

$$\mathbf{D}_T^t = \mathbf{D}\left(1 - d^t - \boldsymbol{\varepsilon}^t \frac{\partial d^t}{\partial \boldsymbol{\varepsilon}^t}\right) \tag{15}$$

To obtain the derivative of the stresses at the time t with respect to the design variable q we can consider again expression (10) taking into account all the possible dependencies of the stresses with respect to the design variable q in the following way:

$$\boldsymbol{\sigma}^t = \boldsymbol{\sigma}^t(\boldsymbol{\varepsilon}^t(q), \boldsymbol{\kappa}^t(\boldsymbol{\varepsilon}^t(q), \boldsymbol{\varepsilon}^u(q), l_c(q), q), q) \tag{16}$$

Nevertheless, normally there is no explicit dependence either of the stresses or of the internal variables with respect to the design variable (this would be the case whether, for instance, the Young modulus or the Poisson ratio be a design variable). Due to that, expression (16) can be reduced to

$$\boldsymbol{\sigma}^t = \boldsymbol{\sigma}^t(\boldsymbol{\varepsilon}^t(q), \boldsymbol{\kappa}^t(\boldsymbol{\varepsilon}^t(q), \boldsymbol{\varepsilon}^u(q), l_c(q))) \tag{17}$$

And then, we can differentiate the stresses with respect to the design variable in the following way:

$$\begin{aligned} \frac{d\boldsymbol{\sigma}^t}{dq} &= \frac{\partial \boldsymbol{\sigma}^t}{\partial \boldsymbol{\varepsilon}^t} \frac{d\boldsymbol{\varepsilon}^t}{dq} + \frac{\partial \boldsymbol{\sigma}^t}{\partial \boldsymbol{\kappa}^t}\left(\frac{\partial \boldsymbol{\kappa}^t}{\partial \boldsymbol{\varepsilon}^t} \frac{d\boldsymbol{\varepsilon}^t}{dq} + \frac{\partial \boldsymbol{\kappa}^t}{\partial \boldsymbol{\varepsilon}^u} \frac{d\boldsymbol{\varepsilon}^u}{dq} + \frac{\partial \boldsymbol{\kappa}^t}{\partial l_c} \frac{dl_c}{dq}\right) \\ &= \left(\frac{\partial \boldsymbol{\sigma}^t}{\partial \boldsymbol{\varepsilon}^t} + \frac{\partial \boldsymbol{\sigma}^t}{\partial \boldsymbol{\kappa}^t} \frac{\partial \boldsymbol{\kappa}^t}{\partial \boldsymbol{\varepsilon}^t}\right) \frac{d\boldsymbol{\varepsilon}^t}{dq} + \frac{\partial \boldsymbol{\sigma}^t}{\partial \boldsymbol{\kappa}^t}\left(\frac{\partial \boldsymbol{\kappa}^t}{\partial \boldsymbol{\varepsilon}^u} \frac{d\boldsymbol{\varepsilon}^u}{dq} + \frac{\partial \boldsymbol{\kappa}^t}{\partial l_c} \frac{dl_c}{dq}\right) \end{aligned} \tag{18}$$

If we combine expressions (13) and (18) we can arrive to the next expression

$$\frac{d\boldsymbol{\sigma}^t}{dq} = \mathbf{D}_T^t \frac{d\boldsymbol{\varepsilon}^t}{dq} + \frac{\partial \boldsymbol{\sigma}^t}{\partial \boldsymbol{\kappa}^t}\left(\frac{\partial \boldsymbol{\kappa}^t}{\partial \boldsymbol{\varepsilon}^u} \frac{d\boldsymbol{\varepsilon}^u}{dq} + \frac{\partial \boldsymbol{\kappa}^t}{\partial l_c} \frac{dl_c}{dq}\right) \tag{19}$$

Clearly, if the non-linear material model does not have any dependence with respect to the characteristic length, the last term of expression (19) vanishes. In addition, in the case of elasticity where there is no dependence of the stresses on any internal variable the terms where $\boldsymbol{\kappa}^t$ appears are null. Finally, the terms where $\boldsymbol{\varepsilon}^u$ appears will be null, unless the t corresponds to an unloading situation. In the latter the tangent constitutive matrix $\mathbf{D}_T^t$ also corresponds to the unloading situation.

On the other hand, we have to take into account that if a classical finite element discretized linear relation between the displacement field and the strains is assumed we have

$$\boldsymbol{\varepsilon}^t = \mathbf{B}\mathbf{u}^t \tag{20}$$

where $\mathbf{u}^t$ is the nodal displacements vector at the time t. Differentiating (20) with respect to a design variable gives

$$\frac{\mathrm{d}\boldsymbol{\varepsilon}^t}{\mathrm{d}q} = \frac{\mathrm{d}\mathbf{B}}{\mathrm{d}q}\mathbf{u}^t + \mathbf{B}\frac{\mathrm{d}\mathbf{u}^t}{\mathrm{d}q} \tag{21}$$

Now, if we substitute expressions (19) and (21) into expression (8) and rearrange terms the following expression is obtained:

$$\int_{V_0} \frac{\mathrm{d}}{\mathrm{d}q}(\mathbf{B}^{\mathrm{T}}\boldsymbol{\sigma}^t|\mathbf{J}|)\,\mathrm{d}V_0 = \int_{V_0}\left(\frac{\mathrm{d}\mathbf{B}^{\mathbf{T}}}{\mathrm{d}q}\boldsymbol{\sigma}^t\,|\mathbf{J}| + \mathbf{B}^{\mathrm{T}}\boldsymbol{\sigma}^t\frac{\mathrm{d}|\mathbf{J}|}{\mathrm{d}q}\right)\mathrm{d}V_0 \tag{22}$$
$$+ \int_{V_0}\mathbf{B}^{\mathrm{T}}\left(\mathbf{D}_{\mathrm{T}}^t\left(\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}q}\mathbf{u}^t + \mathbf{B}\frac{\mathrm{d}\mathbf{u}^t}{\mathrm{d}q}\right) + \frac{\partial\boldsymbol{\sigma}^t}{\partial\boldsymbol{\kappa}^t}\left(\frac{\partial\boldsymbol{\kappa}^t}{\partial\boldsymbol{\varepsilon}^{\mathrm{u}}}\frac{\mathrm{d}\boldsymbol{\varepsilon}^{\mathrm{u}}}{\mathrm{d}q} + \frac{\partial\boldsymbol{\kappa}^t}{\partial l_{\mathrm{c}}}\frac{\mathrm{d}l_{\mathrm{c}}}{\mathrm{d}q}\right)\right)|\mathbf{J}|\,\mathrm{d}V_0$$

Now, if we substitute expression (22) into expression (7) we obtain the following matrix expression:

$$\mathbf{K}_{\mathrm{T}}^t\frac{\mathrm{d}\mathbf{u}^{\mathbf{t}}}{\mathrm{d}q} = \mathbf{f}^{t*} \tag{23}$$

where $\mathbf{K}_{\mathbf{T}}$ is the tangent stiffness matrix and $\mathbf{f}^{t*}$ is a pseudo-load vector, both evaluated at the time t. Their expressions are:

$$\mathbf{K}_{\mathrm{T}}^t = \sum_{\text{elem}}\left[\int_{V_0}\mathbf{B}^t\mathbf{D}_{\mathrm{T}}^t\mathbf{B}|\mathbf{J}|\,\mathrm{d}V_0\right] \tag{24}$$

$$\mathbf{f}^{t*} = \frac{\mathrm{d}\mathbf{f}^t}{\mathrm{d}q} - \sum_{\text{elem}}\left[\int_{V_0}\left(\frac{\mathrm{d}\mathbf{B}^{\mathrm{T}}}{\mathrm{d}q}\boldsymbol{\sigma}^t|\mathbf{J}| + \mathbf{B}^{\mathrm{T}}\left(\boldsymbol{\sigma}^t\frac{\mathrm{d}|\mathbf{J}|}{\mathrm{d}q} + \left(\mathbf{D}_{\mathrm{T}}^t\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}q}\mathbf{u}^t + \frac{\partial\boldsymbol{\sigma}^t}{\partial\boldsymbol{\kappa}^t}\left(\frac{\partial\boldsymbol{\kappa}^t}{\partial\boldsymbol{\varepsilon}^{\mathrm{u}}}\frac{\mathrm{d}\boldsymbol{\varepsilon}^{\mathrm{u}}}{\mathrm{d}q} + \frac{\partial\boldsymbol{\kappa}^t}{\partial l_{\mathrm{c}}}\frac{\mathrm{d}l_{\mathrm{c}}}{\mathrm{d}q}\right)\right)|\mathbf{J}|\right)\right)\mathrm{d}V_0\right] \tag{25}$$

Equations (23)–(25) form a linear system of equations whose solution provides the sensitivities of the displacement field with respect to a design variable. The coefficient matrix of this system is the same tangent stiffness matrix normally used for the solution of the structural equilibrium problem. Note that, often, this matrix is already factorized during the solution process. The pseudo-load vector, with the exception of the last integral term, can be computed using the same techniques as in the case of linear structural problems.

It has to be emphasized that the displacements and the tangent stiffness matrix that appear in equations (23)–(25) are obtained during the solution of the structural equilibrium problem in the usual incremental way. This means that they incorporate all the dependencies with respect to the strains and stresses assigned by the non-linear material model.

Aa a consequence, despite the fact that traditionally the structural sensitivity analysis for non linear problems is performed in an incremental way, the above expressions show that, in the mentioned particular cases, this analysis can be carried out in a direct way after the solution of the structural equilibrium problem is obtained.

2.2. Detailed expressions for a specific damage model

As mentioned above, continuous damage material models are based on an internal variable d^t that controls the mechanical behaviour of the material. The constitutive equation of the simplest damage model depends on a single parameter d^t in the way expressed in (11)

$$\boldsymbol{\sigma}^t = (1 - d^t)\mathbf{D}\boldsymbol{\varepsilon}^t = (1 - d^t)\mathbf{DBu}^t, \quad 0 \leqslant d^t \leqslant 1 \tag{26}$$

The damage parameter evolves in terms of the strain state. In this work the following evolution equation has been used (see [19]):

$$d^t = 1 - \frac{\boldsymbol{\tau}^*}{r^t}\,\mathbf{e}^{A(1 - r^t/\tau^*)}, \quad r^t = \max_{s \in [0,\, t]} (\boldsymbol{\tau}^*, \boldsymbol{\tau}^s) \tag{27}$$

where τ^t is a norm of the stress state given by

$$\tau^t = [1 + v(n - 1)]\sqrt{(\boldsymbol{\sigma}_1^{\mathrm{e}})^2 + (\boldsymbol{\sigma}_2^{\mathrm{e}})^2 + (\boldsymbol{\sigma}_3^{\mathrm{e}})^2} \tag{28}$$

with

$$v = \sum_{i=1}^{3} \frac{\langle \boldsymbol{\sigma}_i^e \rangle}{|\boldsymbol{\sigma}_i^e|}, \quad \langle \boldsymbol{\sigma}_i^e \rangle = \frac{1}{2}[\boldsymbol{\sigma}_i^e + |\boldsymbol{\sigma}_i^e|], \quad n = \frac{f_{\mathrm{c}}}{f_t} \tag{29}$$

where $\boldsymbol{\sigma}^{\mathrm{e}} = \mathbf{D}\boldsymbol{\varepsilon}$ are the elastic stresses, $\boldsymbol{\sigma}_i^{\mathrm{e}}$ are the principal elastic stresses and n is the ratio between the maximum allowable stresses in compression (f_{c}) and in tension (f_t). Finally, τ^* is the threshold value of this norm (a material property) above which the material starts to damage, and A is a softening parameter which depends on the characteristic length of the elements l_{c} (a measure of the element size), fracture energy G_f, maximum traction stress f_t and Young modulus E as follows (see [19]):

$$A = \frac{2 l_{\mathrm{c}} f_t^2}{2 G_{\mathrm{f}} E - 1} \tag{30}$$

The dependence of A on the mesh size l_{c} accounts for the proper numerical structural response with respect to the size of the finite elements. As mentioned earlier, this dependence is necessary when a continuous displacement field (smeared model) is used for the analysis of cracking phenomena, where the displacements are known to become discontinuous (see [18]). This means that, in practice, the constitutive equation at each element depends on its size. In case of shape sensitivity analysis this effect must be taken into account because a change on the structural shape can affect the element sizes and, as a consequence, also the stress values.

Expression (27) indicates the following:

(a) The behaviour of the material is initially elastic ($d^t = 1$) until the stress norm reaches the threshold value τ^*.
(b) Once the stress norm becomes higher than the threshold value, an increase of the strains results in an increase of the stress norm. Due to this, in this situation the value of r^t corresponds to the value of the stress norm τ^t, and the value of the damage parameter at time t can be obtained in terms of the strains at time t.

(c) In case of an unloading situation of a damaged material the value of r^t corresponds to the value of the stress norm ($r^t = \tau^{\mathrm{u}}$), obtained at the previous time t^{u} when the unloading process started. This norm is obtained in terms of the stresses at that time $\boldsymbol{\varepsilon}^{\mathrm{u}}$. In this case, the value of the damage parameter at time t can be obtained in terms of the strains at time t^{u}.

To our knowledge, the above comments are valid for any damage material model. As can be seen from these comments, expression (19) is valid for this damage model because the stresses at time t can be obtained in terms of the strains at time t, and time t^{u}, as well as a characteristic length. Due to this, the structural sensitivity analysis for this material model can be obtained using expressions (23)–(25).

If we compute the value of the different derivatives that appear in the last term of equation (25) we have

$$\frac{\partial \boldsymbol{\sigma}^t}{\partial \boldsymbol{\kappa}^t} = \frac{\partial \boldsymbol{\sigma}^t}{\partial d^t} = -\mathbf{D}\boldsymbol{\varepsilon}^t = -\mathbf{D}\mathbf{B}\mathbf{u}^t \tag{31}$$

$$\frac{\partial \boldsymbol{\kappa}^t}{\partial \boldsymbol{\varepsilon}^{\mathrm{u}}} = \frac{\partial d^t}{\partial \boldsymbol{\varepsilon}^{\mathrm{u}}} = \frac{\partial d^t}{\partial \tau^{\mathrm{u}}}\frac{\partial \tau^{\mathrm{u}}}{\partial \boldsymbol{\varepsilon}^{\mathrm{u}}} \tag{32}$$

The derivatives of last expression can be obtained through the direct derivation of expressions (26) and (27). Anyway, these must be taken into account only in unloading situations and in already damaged material. If this is the case, the term $\mathrm{d}\boldsymbol{\varepsilon}^{\mathrm{u}}/\mathrm{d}q$ must be computed at the time t^{u} when the unloading process starts. This term can be obtained from the sensitivity of the displacements at time t^{u} using

$$\frac{\mathrm{d}\boldsymbol{\varepsilon}^{\mathrm{u}}}{\mathrm{d}q} = \frac{\mathrm{d}\mathbf{B}}{\mathrm{d}q}\mathbf{u}^{\mathrm{u}} + \mathbf{B}\frac{\mathrm{d}\mathbf{u}^{\mathrm{u}}}{\mathrm{d}q} \tag{33}$$

Another necessary term for (25) is obtained from (27) and (30) as

$$\frac{\partial \boldsymbol{\kappa}^t}{\partial l_{\mathrm{c}}} = \frac{\partial d^t}{\partial l_{\mathrm{c}}} = -\frac{\tau^*}{\tau^t}\left(1 - \frac{\tau^{\mathrm{t}}}{\tau^*}\right)\mathrm{e}^{A(1-\tau^t/\tau^*)}\frac{2f_{\mathrm{t}}^2}{2G_{\mathrm{f}}E - 1} \tag{34}$$

Finally, if the characteristic length is taken in the following way:

$$l_{\mathrm{c}} = (V_{\mathrm{element}})^{1/n} \tag{35}$$

where n takes the values 1, 2 or 3 for one, two or three-dimensional problems, respectively, then after some algebraic manipulation we have

$$\frac{\mathrm{d}l_{\mathrm{c}}}{\mathrm{d}q} = \frac{1}{n l_{\mathrm{c}}^{n-1}}\frac{\mathrm{d}|\mathbf{J}|}{\mathrm{d}q} \tag{36}$$

Finally, if we substitute (31)–(36) in expression (25) we obtain the final expression for the pseudo-load vector corresponding to this type of material model.

2.3. General plasticity case

In the case of a general plasticity model the dependence of the stresses with respect to the strains cannot be expressed using expression (10). The reason is that the strains at time t can depend not only on the strains at some specific time (as it was the previous case) but also on the complete deformations history and in a continuous way. If this is the case, the structural sensitivity analysis must be computed using an incremental approach. Typically, this is done using expression (1), where the sensitivities of the displacements are obtained as an addition of increments. In this work, the approach proposed by Kleiber *et al.* [10, 13, 14] has been used. Instead of obtaining the $\mathrm{d}\boldsymbol{\sigma}^t/\mathrm{d}q$ term of expression (8), this formulation is based on the obtainment of the sensitivity of the incremental displacements through the derivation of expression (3) with respect to the design variable. If the equilibrium equation (5) is developed using a first-order expansion around an equilibrium point we obtain

$$\mathbf{K}_{\mathrm{T}}^{t+\Delta t}(\mathbf{u}^{t+\Delta t})\,\Delta\mathbf{u}^{t+\Delta t} = \mathbf{f}^{\,t+\Delta t} - \mathbf{r}^t \tag{37}$$

where $\mathbf{K}_{\mathrm{T}}^t$ is the consistent tangent stiffness matrix at time t and $\mathbf{r}^t$ is the internal forces vector obtained after the last load increment. Their corresponding expressions are

$$\mathbf{K}_{\mathrm{T}}^t = \int_V \mathbf{B}^{\mathrm{T}}\mathbf{D}_{\mathrm{T}}^t\mathbf{B}\,\mathrm{d}V \tag{38}$$

$$\mathbf{r}^t = \int_V \mathbf{B}^{\mathrm{T}}\boldsymbol{\sigma}^t\,\mathrm{d}V \tag{39}$$

If now we derive expression (37) with respect to q and rearrange terms we obtain the following:

$$\mathbf{K}_{\mathrm{T}}^t\,\Delta\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}q} = \frac{\mathrm{d}\mathbf{f}^{\,t+\Delta t}}{\mathrm{d}q} - \frac{\mathrm{d}\mathbf{r}^t}{\mathrm{d}q} - \frac{\mathrm{d}\mathbf{K}_{\mathrm{T}}^t}{\mathrm{d}q}\,\Delta\mathbf{u} \tag{40}$$

Expression (40) is a linear system of equations whose solution provides the sensitivity of the incremental displacements. The sensitivity of the total displacements can be obtained using (1).

The reader is referred to Kleiber *et al.* [10, 13, 14] and Gil [20] to see the details of the computation of the different terms of expression (40). They are not detailed here since these expressions are quite complex and have already been presented in the context of other works.

3. DISCUSSION AND STRATEGY FOR THE SENSITIVITY ANALYSIS OF STRAIN-SOFTENING PROBLEMS

The solution of (23)–(25), as well as that of any other method based on a traditional incremental approach, provides the sensitivities of the structural behaviour assuming that when the structural shape is perturbed the loads remain constant. These sensitivities are the derivatives of the structural behaviour with respect to the design variables, and are obtained assuming an infinitesimal perturbation of these design variables. Nevertheless, the sensitivity analysis is often used for the first-order prediction of the non-linear behaviour of a modified structure that has been obtained by the application of a finite perturbation. By using this approach, one can estimate the answer of the new structure under the same load level rather than the original structure. This means that the behaviour of a structure that we would obtain after a finite perturbation of the design variable q can be approximated by the following expression:

$$\mathbf{u}(q+\Delta q) \approx \mathbf{u}(q) + \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}q}\Delta q \tag{41}$$

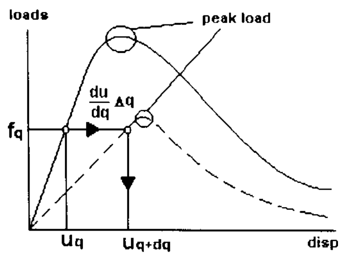

Figure 1. First-order projection of the structural response using the sensitivity analysis

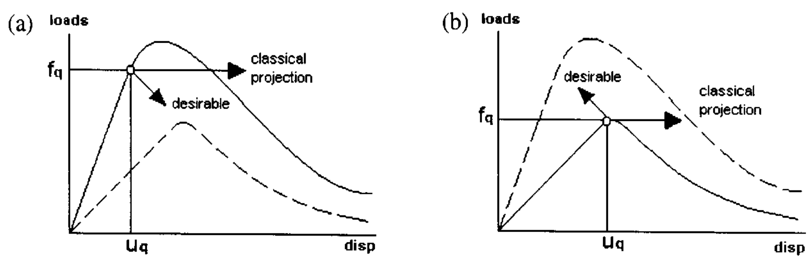

Figure 2. Non feasible projections of the structural response using a constant load sensitivity analysis, (a) and (b)

Figure 1 shows the meaning of this type of projection assuming that we have the full load–displacement curve for the original and the perturbed structures. In this figure, the continuous line shows the equilibrium response curve (force vs. displacement) of a given structure under a specific loading history, and the dashed line shows the estimation of the corresponding equilibrium response curve for a perturbed structure under the same loading history.

When the structural behaviour includes strain softening the use of this type of projection for finite perturbations can become meaningless. In principle, we do not know if a certain finite modification of a design variable will increase or decrease the peak load of the structural response. In particular, if the peak load decreases, it makes no sense to use a standard sensitivity analysis to project the structural behaviour to the perturbed one. A clear example of this situation is shown in Figure 2(a). Note that, in this case, a projection from the highest loaded equilibrium points of the original structure is driven to situations where the perturbed structure cannot be equilibrated.

On the other side, in the cases where the perturbed structure has a higher peak load than the original one, it would not be possible to predict its value. This case is shown in Figure 2(b) where we cannot estimate any of the highest loaded equilibrium points of the perturbed structure by a horizontal projection from the original one.

It seems clear that in the two mentioned pathological situations the projections from the original structure to the perturbed one should involve a variation not only of the displacement field but also of the load level. This type of projections is named as desirable projections in Figures 2(a) and 2(b). In these cases it is not enough to know the sensitivities of the unknown variables of the equilibrium equations. It is also necessary to know the sensitivities of the load forces.

It should be emphasized that this type of problems appears because the sensitivity analysis that has been performed assuming an infinitesimal perturbation of a design variable is used to predict the behaviour of a new structure that is defined through a finite perturbation of a design variable. In fact, when infinitesimal perturbations are applied there are no finite variations of the peak load.

From the mechanical point of view structural problems with strain softening present a reduction of the structural resistance after the peak load equilibrium point. After this situation it is not possible to increase the magnitude of the loads applied to the structure. Consequently, a classical analysis strategy based on an incremental application of the loads does not allow studying the structural behaviour after the peak load point.

However, the displacement field can always be increased producing new equilibrium states in the structural response curve, even if the structural behaviour contains strain softening. This allows the use of arc-length method for the study of the structural behaviour of strain-softening problems after the peak load. This type of method is based on the simultaneous accomplishment of the equilibrium equations and some conditions about the displacement field [21]. Both conditions ensure to obtain new equilibrium points in the displacements–loads curve.

Taking into account that the solution strategies of this type of problems are based on the use of a fixed arc-length for each equilibrium point instead of a fixed load level, it seems logical to use this type of condition to predict the behaviour of a perturbed structure. In this case, the projections will be made assuming that the arc-length condition, and not the load level, will remain constant. This approach has the advantage that no assumption is made on the peak load of the perturbed structure; in addition, it is completely consistent with the equilibrium equations.

Arc-length methods are based on the simultaneous accomplishment of the two following equations:

$$\sum_{\text{elem}} \int_V \mathbf{B}^t \boldsymbol{\sigma}\, \mathrm{d}V - \lambda \mathbf{f} = 0 \tag{42}$$

$$g(\mathbf{u}, \lambda) = 0 \tag{43}$$

where λ is a load factor parameter that controls the magnitude of the load applied at each equilibrium point and (43) is a condition on the values of the displacement field at the same point. Traditionally, this condition is applied on the value of the displacement of a single node or on the value of a norm of the nodal displacement vector. Compared with a classical incremental approach the arc-length methods introduce the additional unknown λ and the additional equation (43).

The sensitivity analysis of the arc-length equations involves the simultaneous differentiation of (42) and (43). This process leads to the following system of equations:

$$\frac{\mathrm{d}}{\mathrm{d}q}\left[\sum_{\text{elem}} \int_V \mathbf{B}^t \boldsymbol{\sigma}\, \mathrm{d}V\right] - \frac{\mathrm{d}\lambda}{\mathrm{d}q}\mathbf{f} - \lambda \frac{\mathrm{d}\mathbf{f}}{\mathrm{d}q} = 0 \tag{44}$$

$$\frac{\mathrm{d}g}{\mathrm{d}q} = \frac{\partial g}{\partial q} + \frac{\partial g}{\partial \mathbf{u}}\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}q} + \frac{\partial g}{\partial \lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}q} \tag{45}$$

Equation (44) is similar to (6) with the addition of some terms to the pseudo-load vector. On the other side, the different terms of (45) are readily obtained by differentiation of the arc-length condition.

The simultaneous solution of equations (44) and (45) can be grouped in the following, more compact, matrix form:

$$\begin{bmatrix} \mathbf{K}_\mathrm{T} & -\mathbf{f} \\ \partial g/\partial \mathbf{u} & \partial g/\partial \lambda \end{bmatrix} \begin{Bmatrix} \mathrm{d}\mathbf{u}/\mathrm{d}q \\ \mathrm{d}\lambda/\mathrm{d}q \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}^{t^*} \\ 0 \end{Bmatrix} \tag{46}$$

The solution of (46) provides the sensitivities of the displacements and the load factor parameter. If these sensitivities are used to project the structural behaviour by a perturbation of a design variable, it will produce a first-order estimation of the displacements and the load level of the new structure assuming the accomplishment of the same arc-length condition than the original one. This estimation will be obtained by using the following expressions:

$$\mathbf{u}(q + \Delta q) \approx \mathbf{u}(q) + \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}q}\Delta q \tag{47}$$

$$\lambda(q + \Delta q) \approx \lambda(q) + \frac{\mathrm{d}\lambda}{\mathrm{d}q}\Delta q \tag{48}$$

On the other side, it should be mentioned that the addition of the arc-length condition to the equilibrium equation leads to the system of equations (46) that is no longer symmetric. Fortunately, due to the fact that only the last row and the last column are non-symmetric it is possible to use cheap iterative strategies that take full advantage of the factorisation of the original tangent stiffness matrix, see Gil [20] for more details.

In addition, it should be mentioned that the presented strategy could be applied to any type of non-linear structural problem, even if it does not have any strain-softening effect.

It should be noted that this procedure provides the design sensitivity analysis of the structural response, and this allows to predict the complete structural response path (or part of it) which can contain the new peak load. Nevertheless, it does not provide the design sensitivity analysis of the peak load. The obtainment of this last sensitivity is a different problem that has not been treated here.

To our knowledge, a similar approach was first proposed in [14] and it was applied to the sensitivity analysis of geometrically non-linear problems. In the present work, this technique has been applied to structural problems where the non-linearity and the softening are due to the constitutive material model.

4. ASSESSMENT EXAMPLES

The quality and the reliability of the formulations proposed in this paper are assessed here through the resolution of four different test cases:

(i) The first test case uses the damage model with strain softening for the analysis of a two-dimensional beam under a bending moment.
(ii) The second test case is based on the problem similar to that of the first one but for using a Von Mises plasticity model with linear hardening.
(iii) The third test case uses the damage material model for the analysis of a short beam with a variable cross section.

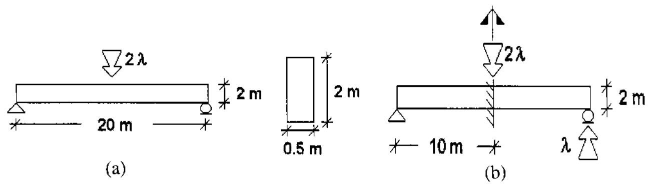

Figure 3. Geometrical definition of test cases 1 and 2, (a) and (b)

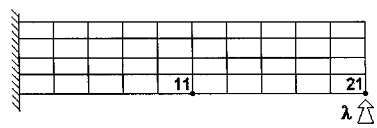

Figure 4. Finite element mesh for test case 1 and 2

(iv) The fourth test case checks the proposed formulation for the case of a three-dimensional short beam using again a Von Mises plasticity with hardening model.

Test case 1: *two-dimensional beam under a bending moment*. *Damage model*. This test case studies the quality of the proposed formulation in the case of a concrete iso-static beam with a bending behaviour. The selected design variable is the thickness of the beam. This test case shows the application of the presented formulation to predict the behaviour of a beam that, which due to construction errors, has a smaller thickness than the designed and analysed one. This test case is analysed with a damage model with strain softening and the main aim is to predict the modification of the peak load produced by the aforementioned error.

The geometry and the applied load of this test case are described in Figures 3(a) and 3(b). Figure 3(b) also describes the symmetry approach used for the analysis of the structural problem.

The different data used for the structural analysis are the following:

(a) A plane stress model with a depth of 50 cm has been assumed.
(b) Arc-length method controlling the displacements of node 11 (see Figure 4) with increments of $\Delta l = 0{\cdot}006$ cm at each step of the solution process.
(c) The convergence criteria for the solution of the equilibrium problem has been defined in terms of the ratio between the norm of the residual forces and the norm of the external forces. This ratio has been limited to a 1 per cent. The finite element mesh shown in Figure 4 contains 40 quadratic 8-noded elements.
(d) The design variable is the thickness of the beam.
(e) The material properties are shown in Table I.

Two different analysis have been performed:

(i) The first analysis corresponds to the original structure. For this, the sensitivities of the displacements with respect to the design variable have been computed at each equilibrium point.

Table I. Material properties

Young modulus E	$2{\cdot}1 \times 10^6$	kN/m^2
Poisson ratio	0·2	
Maximum compressive stress	$2{\cdot}0 \times 10^2$	kN/m^2
Maximum tensile stress	500·0	kN/m^2
Fracture energy	200·0	J/m^2

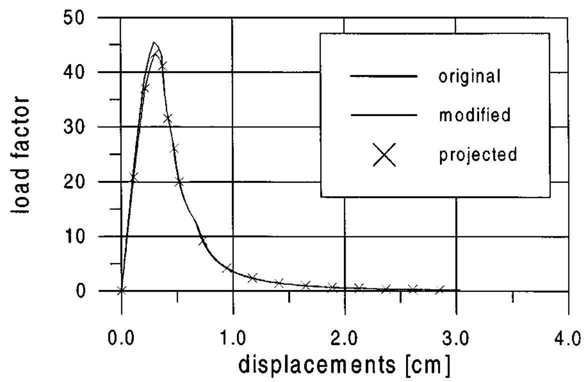

Figure 5. Superposition of the displacement–load curves corresponding to the original, the modified and the projected structures for test case 1

(ii) The second analysis corresponds to a modified (perturbed) structure that has been obtained applying a reduction of 2·5 per cent on the design variable. This means that the thickness of the modified beam is 5 cm smaller than the original one.

In order to check the quality of the proposed formulation the following curves have been compared:

1. Displacement–load curve corresponding to node 21 of the original structure.
2. Displacement–load curve corresponding to the same node of the modified structure obtained by a direct analysis.
3. Displacement–load curve corresponding to the same node of the modified structure obtained by a first-order projection (see expressions (47) and (48)) using the results of the original one and its sensitivities.

The comparison of the response curves between the original, modified and projected structural behaviour is shown in Figure 5. It should be noted that the curves projected and modified superpose quite well. This reveals a good behaviour of the proposed formulation for this test case. In particular, the projected curve allows a very good engineering estimation of the ultimate load of the modified structure (see Figure 6).

Test case 2: *two-dimensional beam under a bending moment. Plasticity with linear hardening*. This problem is similar to the previous one, but in this case the design variable is the span of the beam. This test case is analysed with a Von Mises plasticity material model with linear isotropic hardening. The geometry and the applied load are described in Figures 3(a) and 3(b).

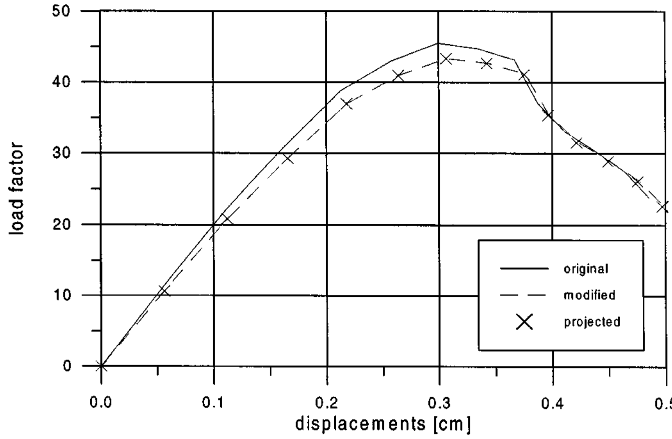

Figure 6. Detail of the zone with the maximum applied load

Table II. Material properties

Young modulus E	$2{\cdot}1 \times 10^8$	kN/m^2
Poisson ratio	0·2	
Yield stress	$2{\cdot}5 \times 10^5$	kN/m^2
Hardening parameter	$2{\cdot}0 \times 10^7$	kN/m^2

The different data used for the structural analysis are the following:

1. Plane stress with beam thickness of 1 m.
2. The convergence criteria for the solution of the equilibrium problem have been defined in terms of the ratio between the norm of the residual forces and the norm of the external forces. This ratio has been limited to a 1 per cent. The total load was been applied using 300 steps, each one of 160 kN.
3. The finite element mesh shown in Figure 4 contains 40 quadratic 8-noded elements.
4. The design variable is the length of the beam.
5. The material properties are shown in Table II.

Two different analysis have been made:

(i) The first analysis corresponds to the original structure. For this structure the sensitivities of the displacements with respect to the design variable have been computed at each equilibrium point.
(ii) The second analysis corresponds to a modified (perturbed) structure that has been obtained by applying an increase of 2 per cent to the design variable. It means that the modified beam is 20 cm longer than the original one.

In order to check the quality of the proposed formulation the following curves have been compared:

(1) Displacement–load curve corresponding to node 21 of the original structure.

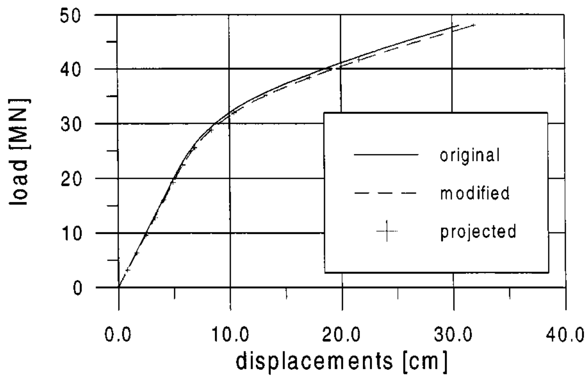

Figure 7. Superposition of the displacement–load curves corresponding to the original, the modified and the projected structures for test case 1

Table III. Material properties

Young modulus E	$2{\cdot}1 \times 10^6$	kN/m^2
Poisson ratio	0·2	
Maximum compression stress	$2{\cdot}0 \times 10^4$	kN/m^2
Maximum traction stress	500·0	kN/m^2
Fracture energy	200·0	J/m^2

(2) Displacement–load curve corresponding to the same node of the modified structure obtained by a direct analysis.
(3) Displacement–load curve corresponding to the same node of the modified structure obtained by a first order projection (see expression (30)) using the results of the original one and its sensitivities.

Looking at the Figure 7, it should be noted that the curves projected and modified superpose quite well. This reveals a good behaviour of the formulation for this test case.

Test case 3: *two-dimensional short cantilever beam. Damage model.* In this test case the structural problem consists of a small cantilever beam with a variable cross section.

The different data used for the structural analysis are the following:

(i) Damage constitutive material model as described in Section 3 (see Table III).
(ii) Plane stress model equations assuming a thickness of 1 cm.
(iii) Arc-length controlling the displacement of node 34 (see Figure 8) allowing an increment in its displacement norm of $\Delta l = 3{\cdot}5 \times 10^{-4}$ cm at each step of the solution process.
(iv) The same convergence criteria as previous test cases.
(v) The design variable is the cross section of the structure.
(vi) A mesh of 12 eight node quadratic elements (see Figure 9).

In a similar way than in test case 1 two different analyses have been performed:

1. The first analysis and sensibilities correspond to the original structure.

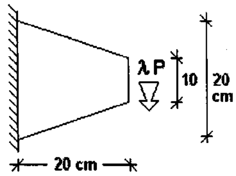

Figure 8. Geometrical description of test case 3

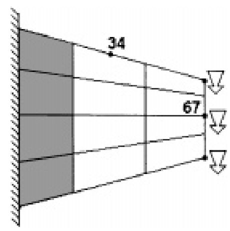

Figure 9. Mesh and structural model, the coloured elements have a lower yield stress in order to localize the damage

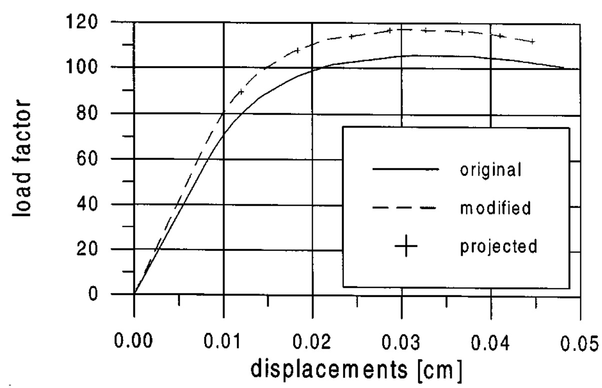

Figure 10. Superposition of the displacement–load curves corresponding to the original, the modified and the projected structures for test case 3

2. The second analysis corresponds to a modified (perturbed) structure that has been obtained by applying a uniform increment of the cross section. It means that the cross section of the modified structure is 6·6 per cent larger than the original one.

Drawing the same type of curves as in previous test cases, Figure 10 shows that the projected curve reproduces very well the behaviour of the perturbed structure. In fact, the coincidence of curves projected and modified is almost perfect.

Test case 4: *short three-dimensional beam. Plasticity model.* This test checks the behaviour of the formulation in a three-dimensional plasticity problem as shown in Figure 11. It consists of a bending short loaded beam. Its material follows a plasticity constitutive law with strain hardening. The details of the parameters are shown in Table IV.

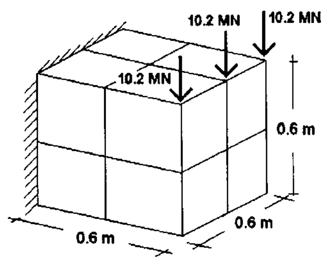

Figure 11. Geometry and structural problem of test case 4

Table IV. Material properties

Young modulus E	$2{\cdot}1 \times 10^8$	kN/m^2
Poisson ratio	0	
Yield stress	$2{\cdot}0 \times 10^5$	kN/m^2
Hardening	$1{\cdot}0 \times 10^6$	kN/m^2

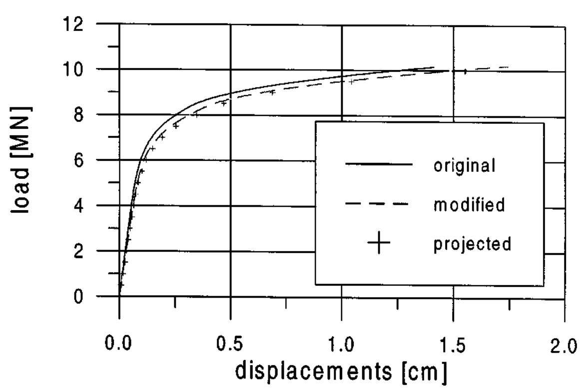

Figure 12. Superposition of the displacement–load curves corresponding to the original, the modified and the projected structures for test case 4

(a) A tridimensional mesh of 8 node bilinear hexahedral elements is used.
(b) The same convergence criteria described in previous cases. The total load has been applied using 200 steps, each one of 51 kN.
(c) The design variable is the length of the beam.

The projected curve, in Figure 12, coincides quite well with the modified one and allows the engineer to predict the structural behaviour of the modified structure.

5. CONCLUSIONS

The differentiation of the discretized global equilibrium equation allows a very satisfactory evaluation of the sensitivities of the structural behaviour when damage or plasticity models are used. In the case of the damage models, the present approach is a new one based on the use of the tangent constitutive matrix and it does not make use of any incremental approach. For the plasticity case the present approach is a ‘classical’ incremental one.

The simultaneous differentiation of the equilibrium equations and the arc-length condition leads to a new strategy for the evaluation of the structural sensitivities that solves the projection problems when the structural behaviour presents strain softening.

The quality of the formulation proposed here for the sensitivity analysis of structures containing damage models, together with the inclusion of the arc-length condition has been assessed through the resolution of different test cases. The results shown in all the test cases are very satisfactory.

In particular, some examples show the good possibilities of the proposed sensitivity analysis for the study of situations where, due to a pathology, the finally built structure is not coincident to the originally designed one.

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