A second-order semi-Lagrangian particle FEM method for the incompressible Navier-Stokes equations

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OUTLINE

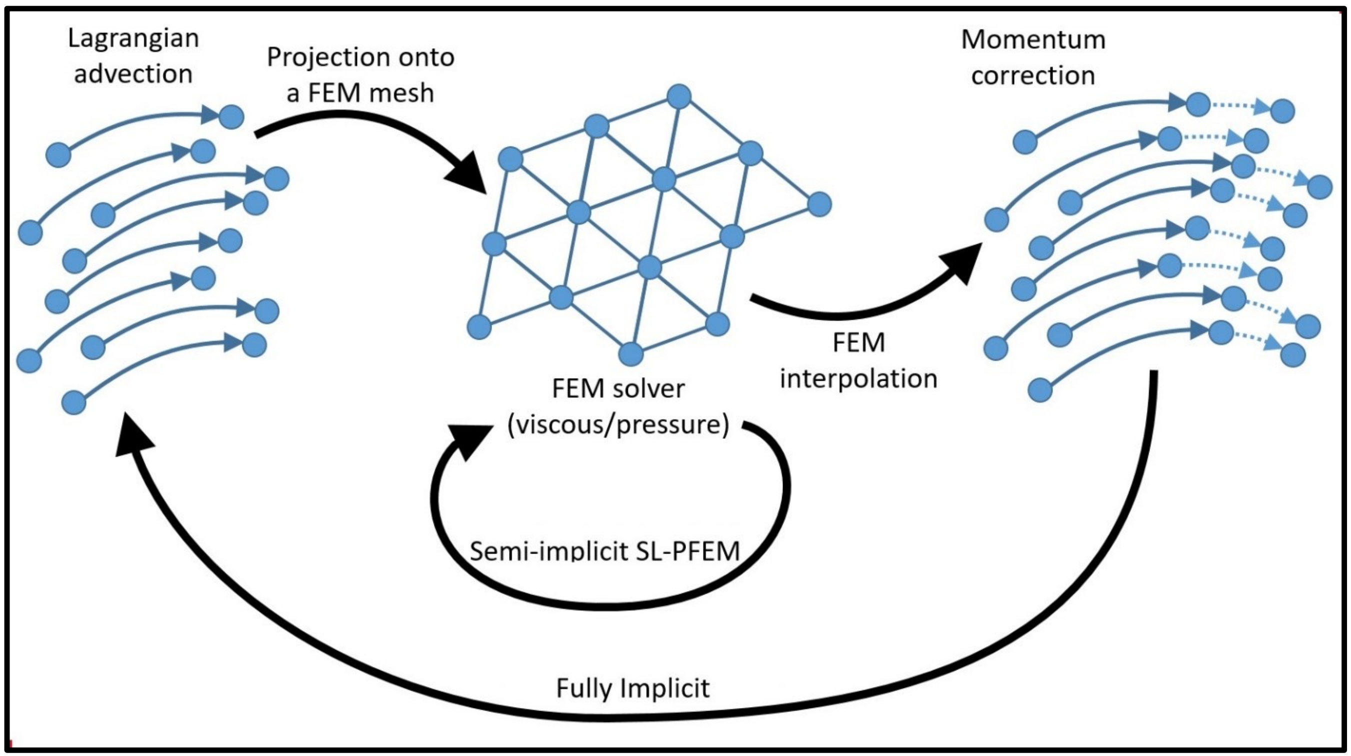
- ✓ Introduction
  - ✓ Semi-Lagrangian approach
  - ✓ SL-PFEM
- ✓ Verification and convergence analyses
  - ✓ Taylor-Green vortex
  - ✓ Lid driven cavity flow
  - ✓ 3D flow past a cylinder
- ✓ Ongoing and future work
- ✓ Acknowledgements

INTRODUCTION

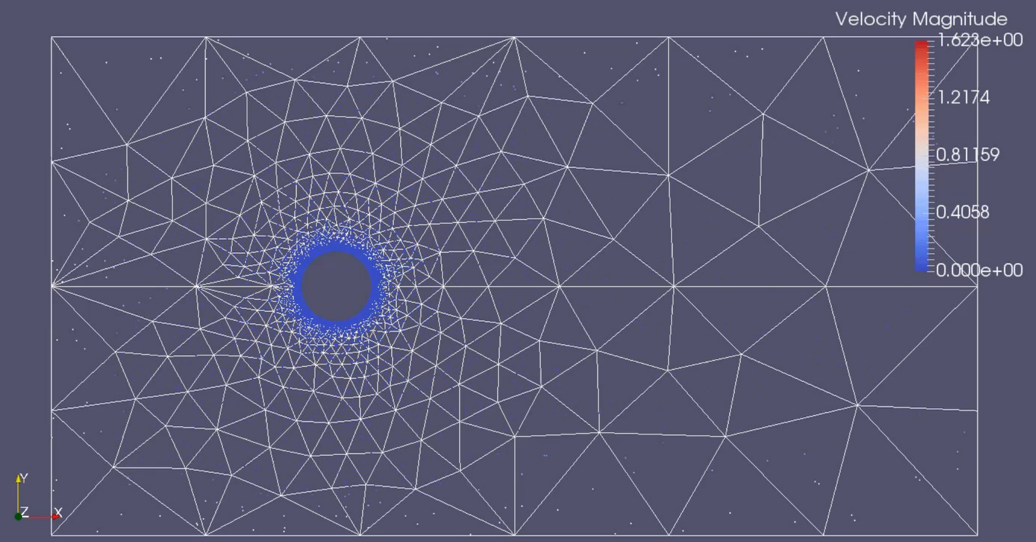
SEMI-LAGRANGIAN APPROACH

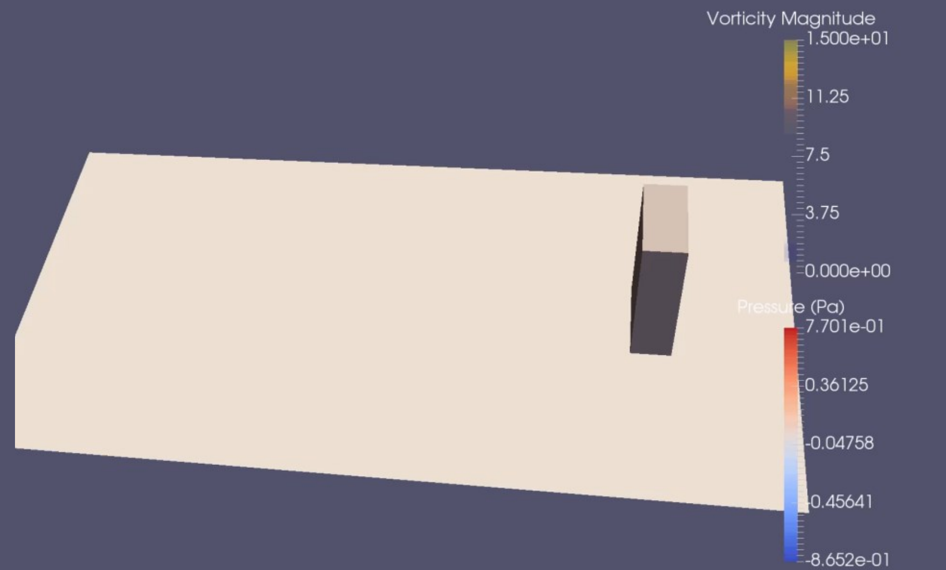
Conceps of the Semi-Lagrangian particle Finite Element Method (SL-PFEM)

First introduced by: S. R. Idelsohn, N. Nigro, A. Limache, E. Oñate: Large time-step explicit integration method for solving problems with dominant convection. Computer Methods in Applied Mechanics and Engineering 217-220, 168–185 (2012). DOI 10.1016/j.cma.2011.12. 008.



SEMI-LAGRANGIAN APPROACH





SEMI-LAGRANGIAN APPROACH

Integrating the particle's equation of motion

Let $\boldsymbol{a}(\boldsymbol{x}, t)$ be an acceleration field and let $\{\lambda\}$ be a set of particles each of them identified with a label λ.

Particle's equation of motion:

$$\mathrm{d_t}\boldsymbol{U}_\lambda(t) = \boldsymbol{A}_\lambda(t) = \boldsymbol{a}(\boldsymbol{X}_\lambda(t), t)$$
$$\mathrm{d_t}\boldsymbol{X}_\lambda(t) = \boldsymbol{U}_\lambda(t)$$

Velocity Verlet algorithm:

$$\boldsymbol{X}_\lambda(t^{n+1}) = \boldsymbol{X}_\lambda(t^n) + \Delta t \boldsymbol{U}_\lambda(t^n) + \frac{\Delta t^2}{2}\boldsymbol{A}_\lambda(t^n) + O(\Delta t^3)$$
$$\boldsymbol{U}_\lambda(t^{n+1}) = \boldsymbol{U}_\lambda(t^n) + \frac{\Delta t}{2}\left(\boldsymbol{A}_\lambda(t^n) + \boldsymbol{A}_\lambda(t^{n+1})\right) + O(\Delta t^3)$$

SL-PFEM for the incompressible Navier-Stokes equations

Let $\boldsymbol{u}(\boldsymbol{x}, t)$ be a fluid velocity field and let´s define the acceleration field:

$$\boldsymbol{a} = \mathrm{d}_t \boldsymbol{u} = \partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} = -\boldsymbol{\nabla}\left(\frac{P}{\rho}\right) + \nu \Delta \boldsymbol{u} + \boldsymbol{f}$$

$$\boldsymbol{X}_\lambda^{n+1} = \boldsymbol{X}_\lambda^n + \Delta t \boldsymbol{u}^n(\boldsymbol{X}_\lambda^n) + \frac{\Delta t}{2} \boldsymbol{a}^n(\boldsymbol{X}_\lambda^n)$$

$$\boldsymbol{U}_\lambda^{n+1} = \underbrace{\boldsymbol{U}_\lambda^n + \frac{\Delta t}{2} \boldsymbol{a}^n(\boldsymbol{X}_\lambda^n)}_{\boldsymbol{U}_\lambda^{n+1/2}} + \frac{\Delta t}{2} \underbrace{\boldsymbol{a}^{n+1}(\boldsymbol{X}_\lambda^{n+1})}_{Implicit}$$

$\boldsymbol{u}(\boldsymbol{X}_\lambda), \boldsymbol{a}(\boldsymbol{X}_\lambda)$: Interpolated mesh velocity and acceleration

SL-PFEM for the incompressible Navier-Stokes equations

Projection onto FEM mesh to solve $\boldsymbol{a}^{n+1}$ on Eulerian description such that $\boldsymbol{u}^{\mathrm{n}+1}(\boldsymbol{x}) = \mathcal{P}_{\{\lambda\}}^{n+1}\left[\left\{\boldsymbol{U}_\lambda^{n+1}\right\}\right]$:

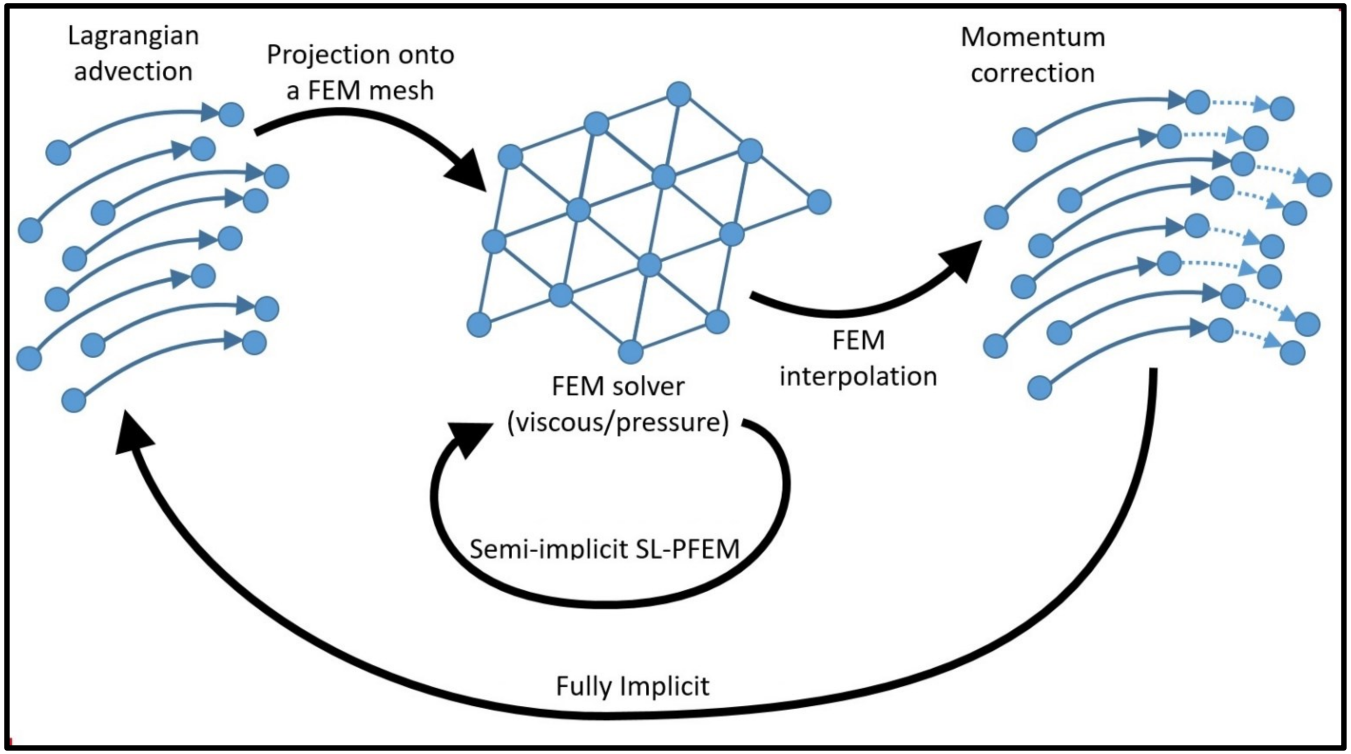
$$\boldsymbol{U}_\lambda^{n+1} = \boldsymbol{U}_\lambda^{n+1/2} + \frac{\Delta t}{2}\boldsymbol{a}^{n+1}\left(\boldsymbol{X}_\lambda^{n+1}\right)$$

$$\boldsymbol{u}^{\mathrm{n}+1}(\boldsymbol{x}) = \underbrace{\mathcal{P}_{\{\lambda\}}^{n+1}\left[\left\{\boldsymbol{U}_\lambda^{n+1/2}\right\}\right]}_{\boldsymbol{u}^{\mathrm{n}+1/2}} + \frac{\Delta t}{2}\mathcal{P}_{\{\lambda\}}^{n+1}\left[\left\{\boldsymbol{a}^{n+1}\left(\boldsymbol{X}_\lambda^{n+1}\right)\right\}\right]$$

Coherence condition: $\boldsymbol{a}^{n+1}(\boldsymbol{x}) = \mathcal{P}_{\{\lambda\}}^{n+1}\left[\left\{\boldsymbol{a}^{n+1}\left(\boldsymbol{X}_\lambda^{n+1}\right)\right\}\right]$

SL-PFEM

SL-PFEM for the incompressible Navier-Stokes equations



Remark: the coherence condition makes it unneccesary to iterate at the outter implicit loop.

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SL-PFEM

Projection

Minimization of the least square error (LSE).

$\{\Psi_\lambda\}$: set of particles' values $\xrightarrow{Projection} \mathcal{P}[\{\Psi_\lambda\}] = \{\psi_c^*\}$ projected nodal values

Interpolated-projected values on particles: $\psi_h(\boldsymbol{X}_\lambda) = \sum_c N^c(\boldsymbol{X}_\lambda)\psi_c^*$

Square error: $\epsilon_\psi = \sum_\lambda (\psi_h(\boldsymbol{X}_\lambda) - \Psi_\lambda)^2$

LSE: $\frac{\partial \epsilon_\psi}{\partial \psi_b^*} = 0 \rightarrow \sum_\lambda \left(\sum_c N^b(\boldsymbol{X}_\lambda) N^c(\boldsymbol{X}_\lambda)\psi_c^*\right) = \sum_\lambda N^b(\boldsymbol{X}_\lambda)\Psi_\lambda$

Fulfils the coherence condition naturally

$$\Psi_\lambda = \psi_h(\boldsymbol{X}_\lambda) \rightarrow \epsilon_\psi = 0$$

SL-PFEM

Semi-Lagrangian approach for the incompressible Navier-Stokes equations

Equations in the Eulerian description:

$$\frac{\boldsymbol{u}^{\mathrm{n+1}} - \boldsymbol{u}^{\mathrm{n+1/2}}}{\Delta t} = \frac{1}{2}\underbrace{\left(-\boldsymbol{\nabla}\left(\frac{P^{n+1}}{\rho}\right) + \nu\Delta\boldsymbol{u}^{n+1} + \boldsymbol{f}^{n+1}\right)}_{\boldsymbol{a}^{n+1}}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u}^{\mathrm{n+1}}=0$$

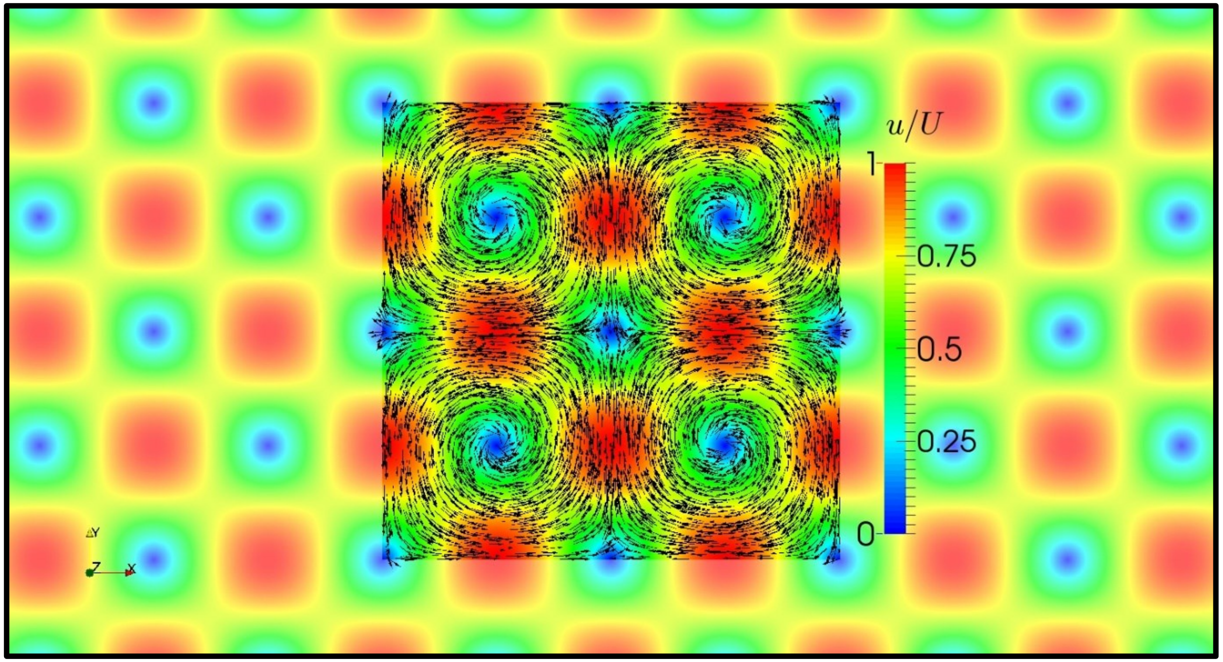
Solved using FEM implicit scheme inspired in the fractional step method.

VERIFICATION AND CONVERGENCE ANALYSES

TAYLOR-GREEN VORTEX

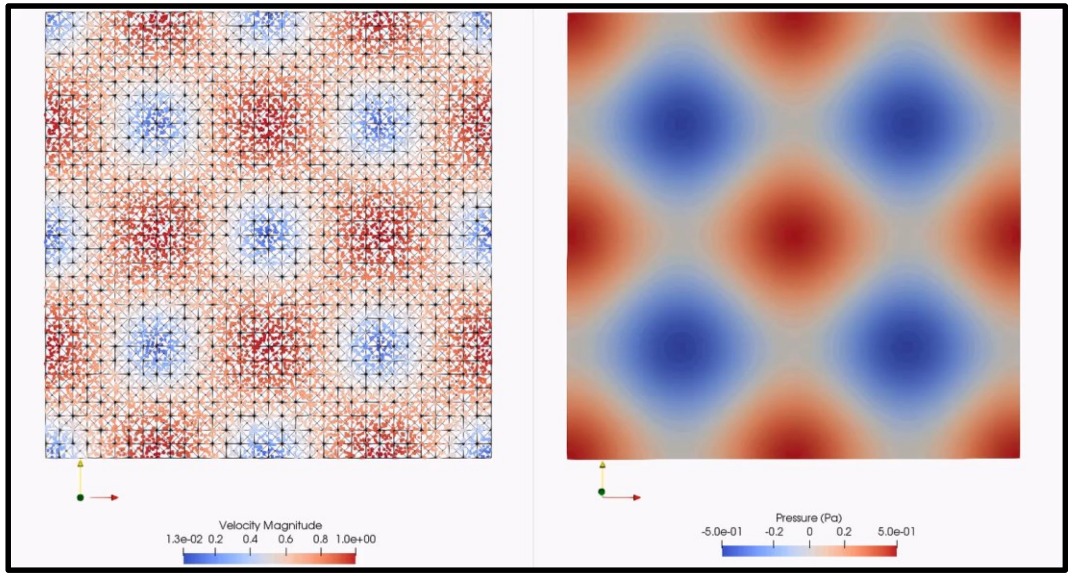
Taylor-Green vortex solution

$$u_x(x,y,t) = -\sin(x)\cos(y)\,e^{-2\nu t}$$
$$u_y(x,y,t) = +\cos(x)\sin(y)\,e^{-2\nu t}$$
$$P(x,y,t) = \frac{1}{4}[\cos(2x) + \cos(2y)]e^{-4\nu t}$$

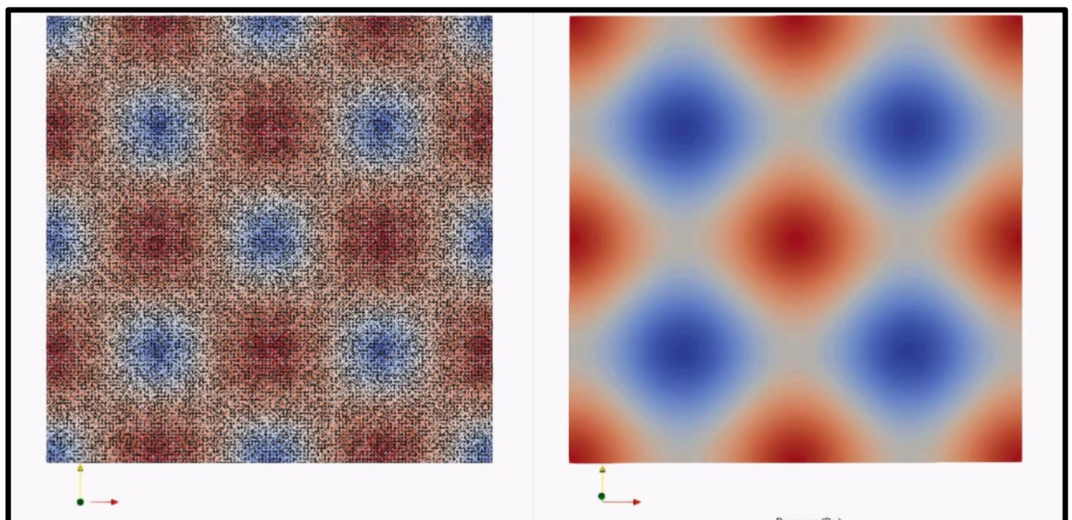


TAYLOR-GREEN VORTEX

Case 32	
Number of elements	4096
Number of Nodes	2113
Number of Particles	12288
Courant number	2.54
Reynolds number	2000
Number of time steps	500
Simulation time	250 s

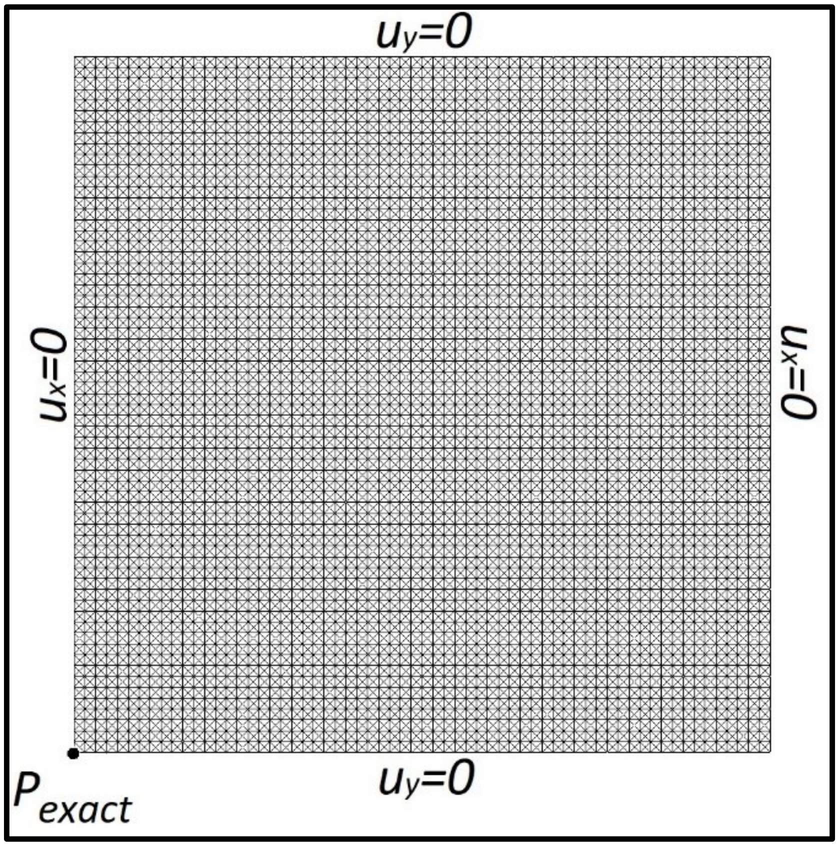


Case 128	
Number of elements	65536
Number of Nodes	33025
Number of Particles	196608
Courant number	10.1
Reynolds number	2000
Number of time steps	500
Simulation time	250 s

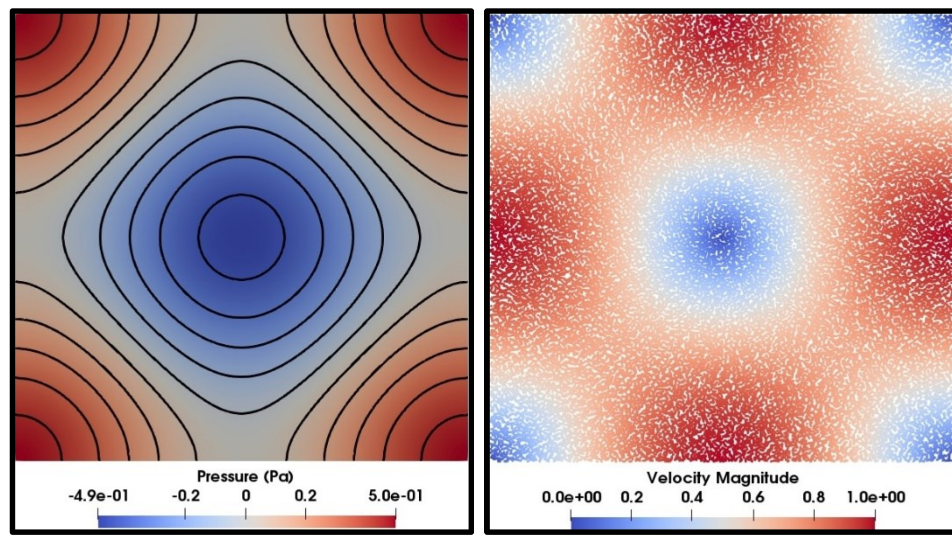


TAYLOR-GREEN VORTEX

Taylor-Green vortex decay

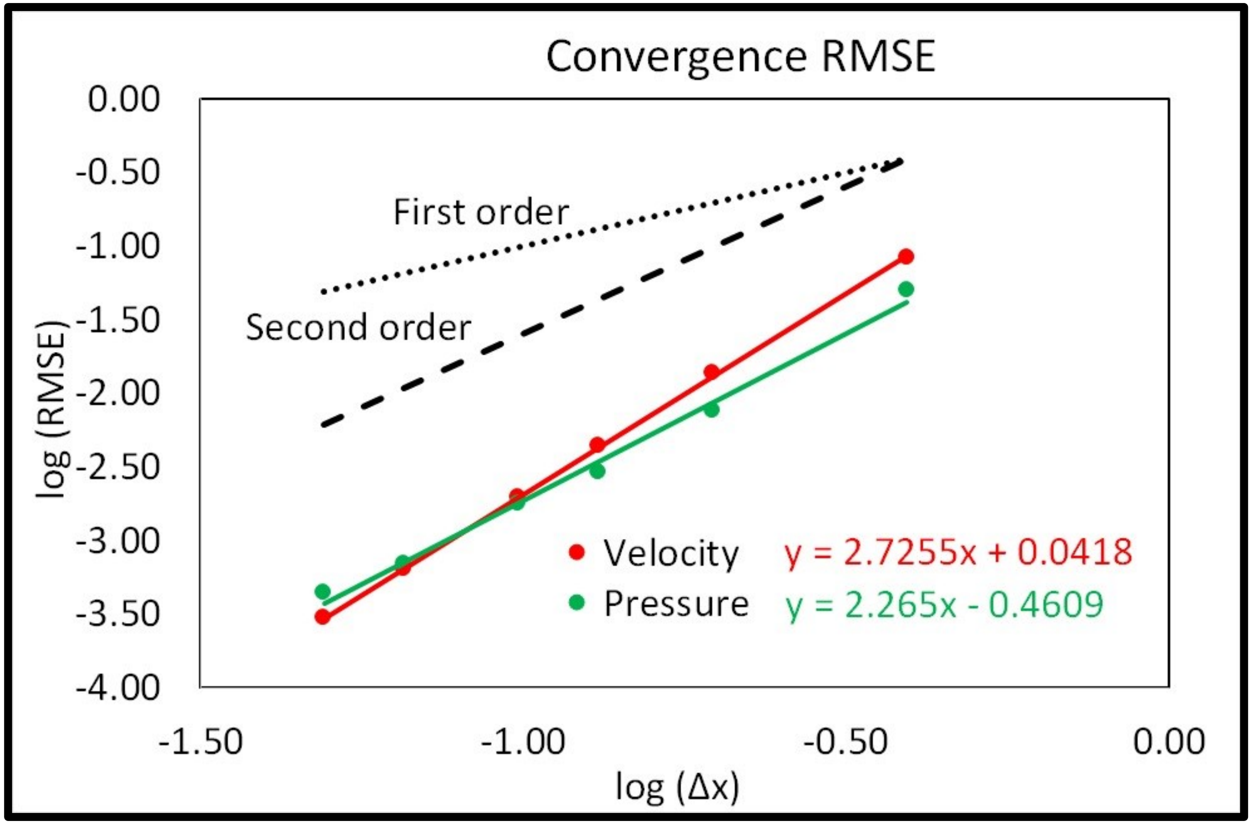


Particulars	
Maximum velocity (m/s)	1
Reynolds number	**3140**
Domain size (mxm)	$(0, \Pi)$ x $(0, \Pi)$
Courant number	**0.5**
Simulation time (s)	10



CONVERGENCE ANALYSIS

Taylor-Green vortex decay



Case	Δx(m)	Δt(s)	N Δt
8x8	$\pi/8$	0.2	50
16x16	$\pi/16$	0.1	100
24x24	$\pi/24$	0.0667	150
32x32	$\pi/32$	0.05	200
48x48	$\pi/48$	0.0333	300
64x64	$\pi/64$	0.025	400

CONVERGENCE ANALYSIS

Steady-state Taylor Green Vortex

Mass forces:

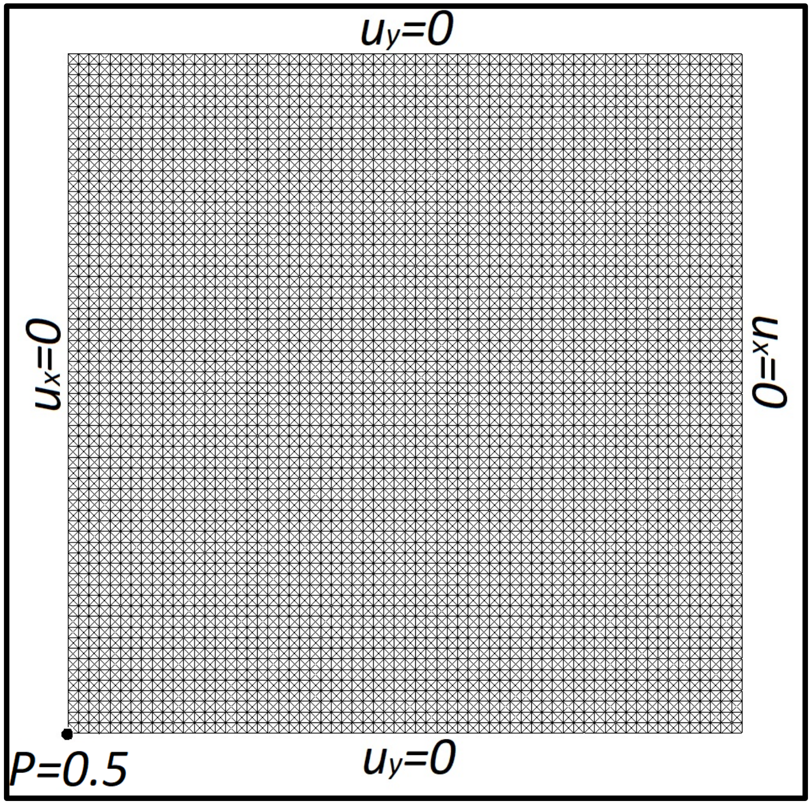
$$f_x(x,y) = -2\nu \sin(x)\cos(y)$$

$$f_y(x,y) = 2\nu \cos(x)\sin(y)$$

Analytical solution:

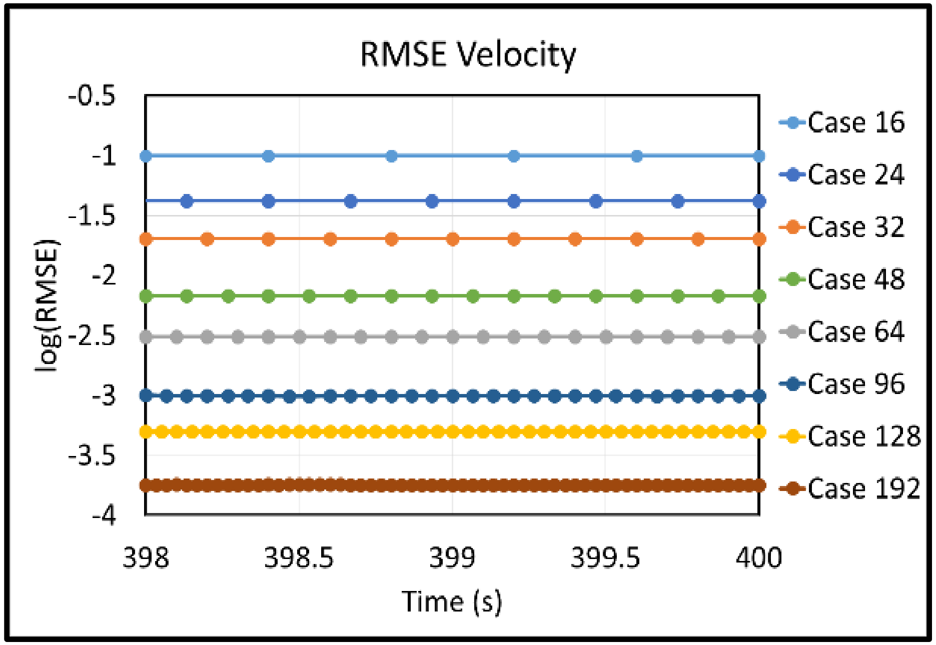
$$u_x(x,y) = -\sin(x)\cos(y)$$
$$u_y(x,y) = \cos(x)\sin(y)$$
$$P(x,y) = 0.25[\cos(2x) + \cos(2y)]$$

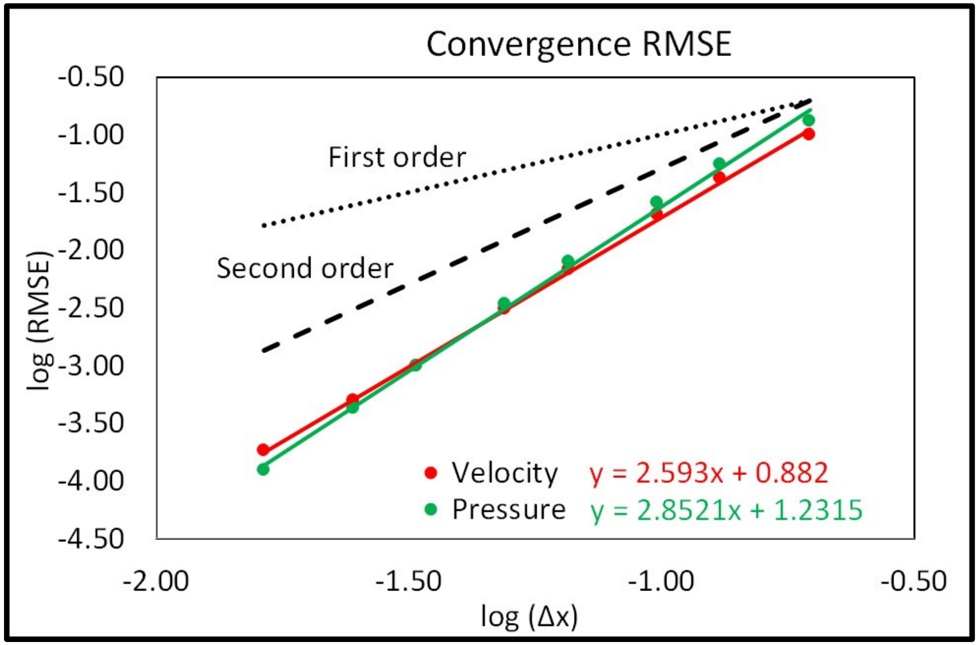
Particulars	
Maximum velocity (m/s)	1
Reynolds number	**314**
Domain size (mxm)	$(0,\Pi)$ x$(0,\Pi)$
Courant number	**2.04**
Simulation time (s)	400



CONVERGENCE ANALYSIS

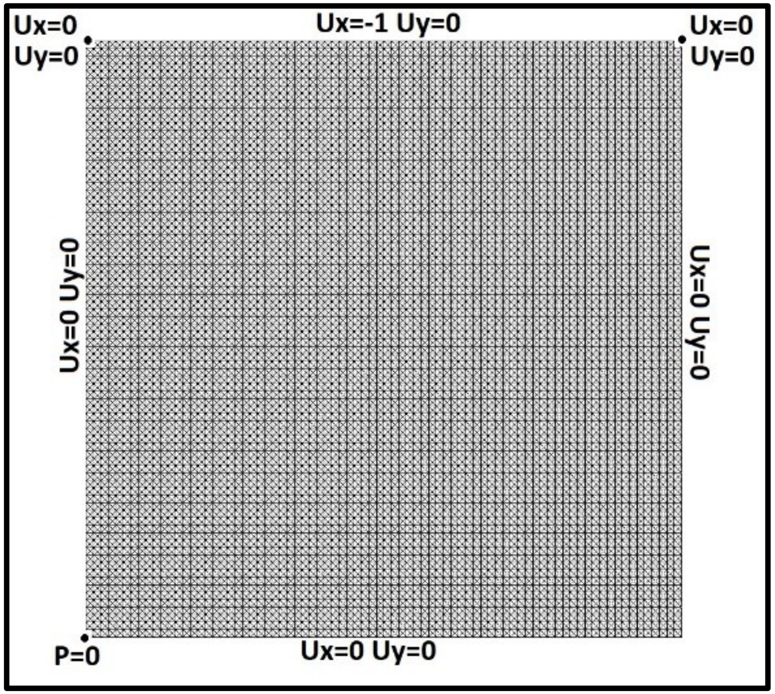
Case	Δx(m)	Δt(s)	N Δt
16x16	$\pi/16$	0.4	1000
24x24	$\pi/24$	0.2667	1500
32x32	$\pi/32$	0.2	2000
48x48	$\pi/48$	0.1333	3000
64x64	$\pi/64$	0.1	4000
96x96	$\pi/96$	0.0667	6000
128x128	$\pi/128$	0.05	8000
192x192	$\pi/192$	0.0333	12000





LID DRIVEN CAVITY FLOW

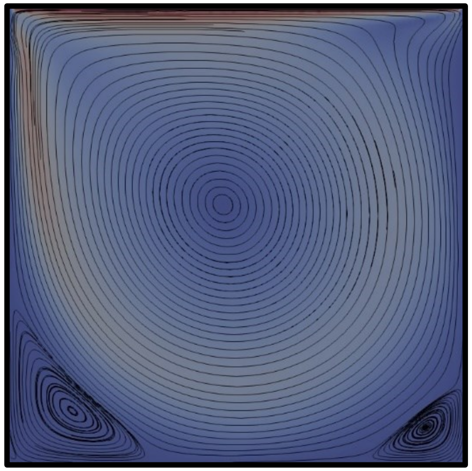
Lid velocity V	$-1m/s$
Reynolds number Re	1000
Domain size	$1m\ x\ 1m$
Domain discretization	$80x80$
Number of elements	25600
Number of nodes	12961
Particles per element	3
Mesh size $\Delta x = \Delta y$	0.0125
Time step Δt	0.1
Courant number	**8**
Simulation time	$10^5 \Delta t$
Sampling time	$10^2 \Delta t$

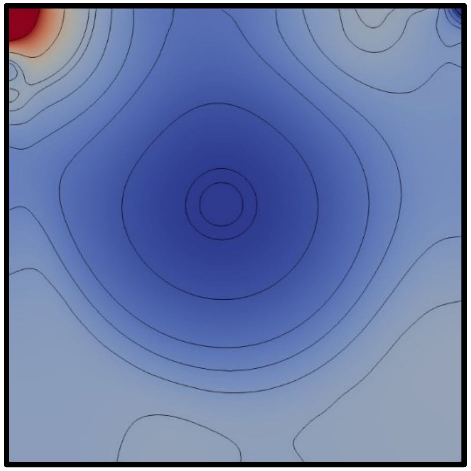


LID DRIVEN CAVITY FLOW

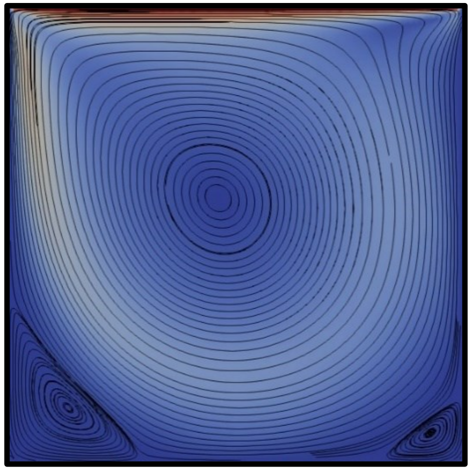
Velocity-streamlines | Pressure

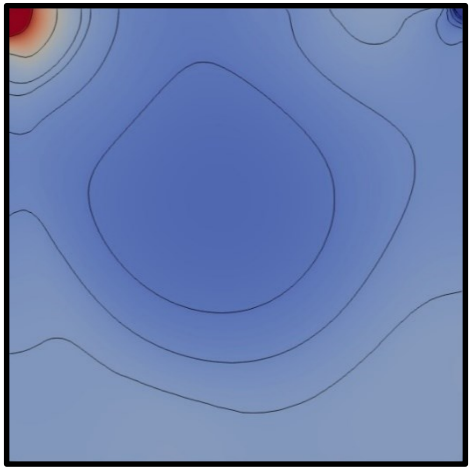
Second order Verlet
Verlet SL-PFEM





First order in time
Euler SL-PFEM

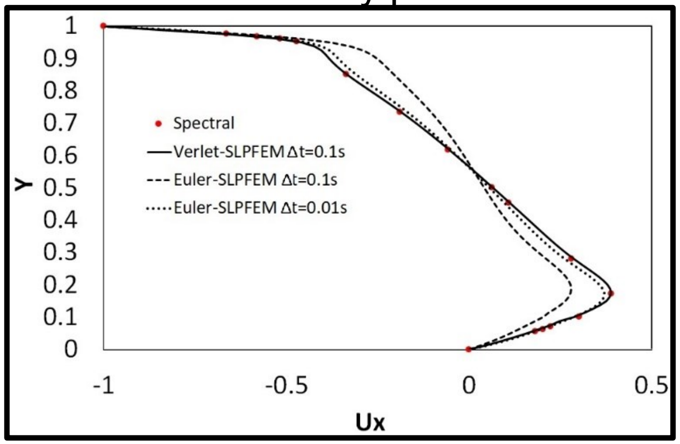




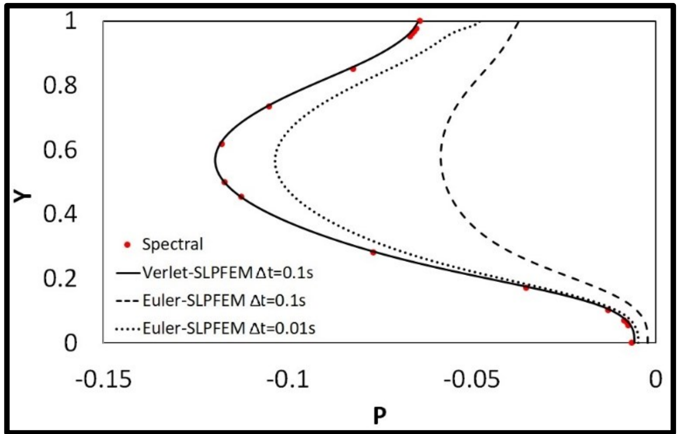
LID DRIVEN CAVITY FLOW

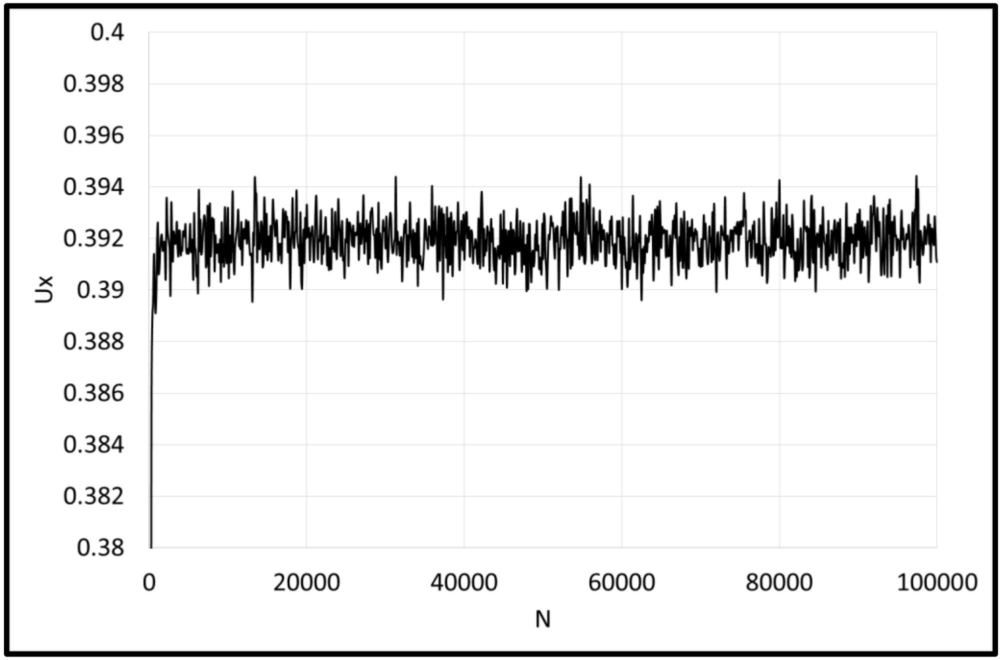
Horizontal velocity and pressure profiles at the mid-section x=0.5. Solid line: Second Order Verlet-SLPFEM. Dash line: First Order in time Euler-SLPFEM. Red dots: spectral method (O. Botella and R. Peyret. Benchmark spectral results on the lid-driven cavity flow. Computers & Fluids Vol. 27, No. 4, pp. 421-433, 1998)

Velocity profile



Pressure profile



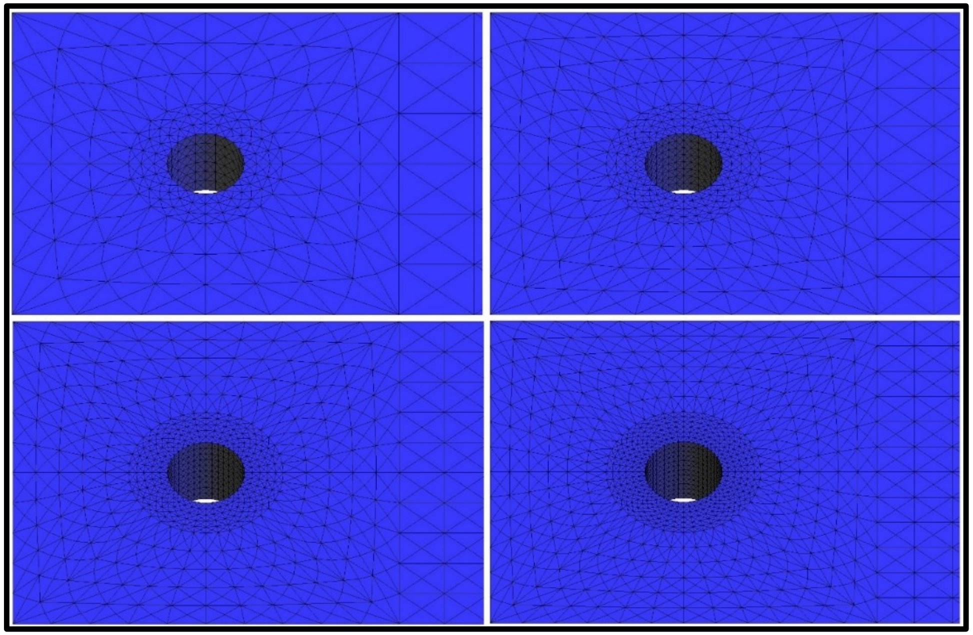


Horizontal velocity evolution at (x,y)=(0.5,0.175) for the second-order Verlet -SLPFEM

3D FLOW PAST A CYLINDER

Mesh	Δx(m)	Δt(s)	N tetras	N Nodes	N Δt
1	0.1333	0.0667	62208	14487	1500
2	0.1	0.05	147456	33412	2000
3	0.08	0.04	288000	64185	2500
4	0.0667	0.0333	497664	109686	3000

Particulars	
Cylinder diameter (m)	1
Cylinder height (m)	1
Inlet velocity (m/s)	1
Reynolds number	200
Courant number	2



3D FLOW PAST A CYLINDER

Case	St	Error St	C_L
1	0.133	0.0630	0.177
2	0.162	0.0339	0.255
3	0.178	0.0184	0.246
4	0.186	0.0101	0.258
Ref. [*]	0.196		

<table>
<tr><th colspan="3">CPU time in seconds per time step (mesh 2)</th><th colspan="2">SL-PFEM</th><th>FEM</th></tr>
<tr><td rowspan="3">Particles</td><td colspan="2">Move</td><td>0.092</td><td rowspan="3">0.602</td><td rowspan="3"></td></tr>
<tr><td rowspan="2">Projector</td><td>Assembly</td><td>0.363</td></tr>
<tr><td>Solver</td><td>0.147</td></tr>
<tr><td colspan="3">Velocity solver</td><td colspan="2">0.125</td><td>1.532</td></tr>
<tr><td colspan="3">Pressure solver</td><td colspan="2">0.339</td><td>0.320</td></tr>
<tr><td colspan="3">Total</td><td colspan="2">1.066</td><td>1.852</td></tr>
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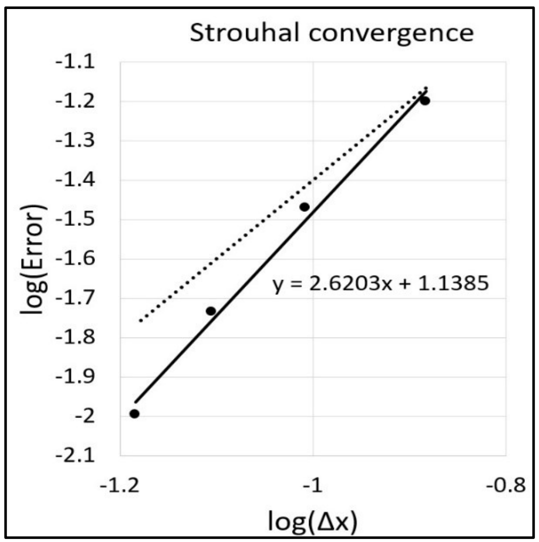
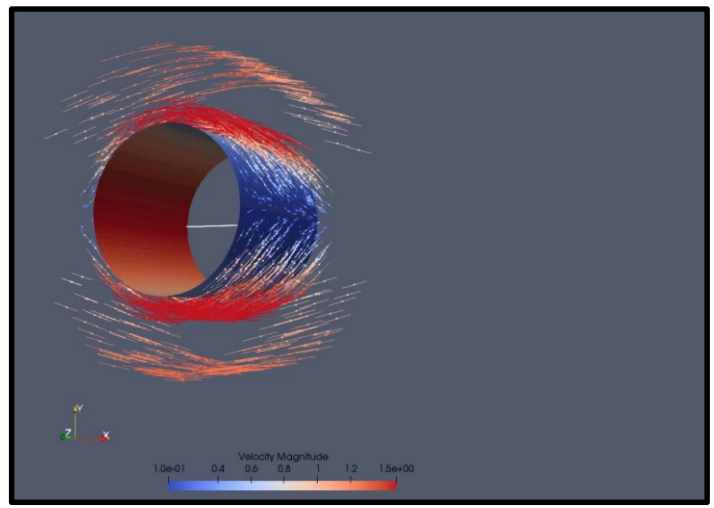




* H. Jiang and L. Cheng. Strouhal Reynolds number relationship for flow past a circular cylinder. J. Fluid Mech. (2017), vol. 832, pp. 170-188

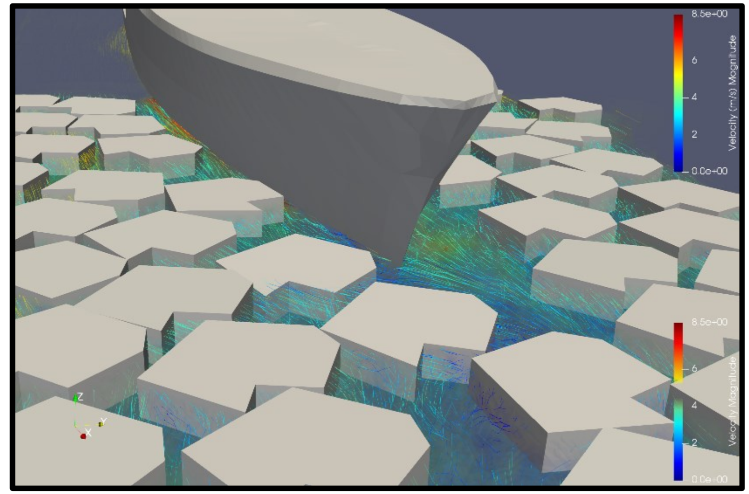
ONGOING AND FUTURE WORK

ONGOING AND FUTURE WORK

Ongoing Work

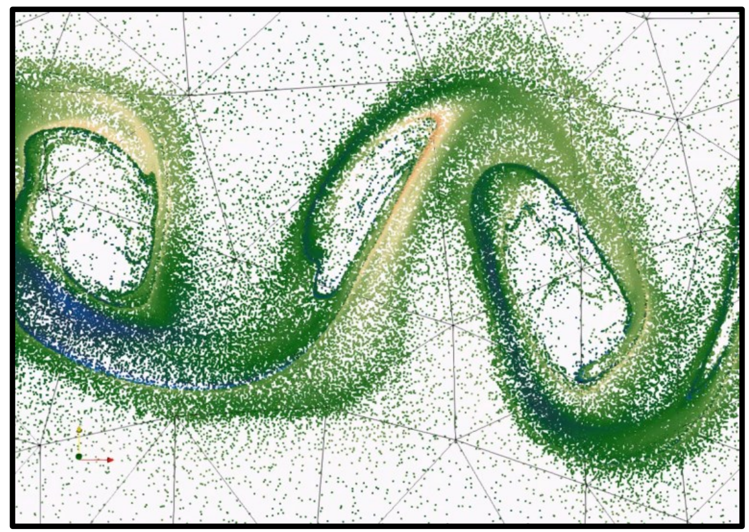
- ✓ FEM Enrichment for fluid interfaces
  - ✓ Fluid solid interaction:

    Advances in the simulation of ship navigation in ice (IS-Ships in ice, Room: Runan @ 16:00-17:40h)
  - ✓ Free surface problems.



Future Work

- ✓ Solving submesh scales
  - ✓ Solve particle scale vorticity using a particle based vortex method.
- ✓ Turbulence modelling



ACKNOWLEDGEMENTS

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