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Essentially commuting projections

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ABSTRACT

Let $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ be a fixed orthogonal decomposition of a Hilbert space, with both subspaces of infinite dimension, and let E_+, E_- be the projections onto $\mathcal{H}_+$ and $\mathcal{H}_-$. We study the set $\mathcal{P}_{cc}$ of orthogonal projections P in $\mathcal{H}$ which *essentially commute* with E_+ (or equivalently with E_-), i.e.

$$[P, E_+] = PE_+ - E_+P \quad \text{is compact.}$$

By means of the projection π onto the Calkin algebra, one sees that these projections $P \in \mathcal{P}_{cc}$ fall into nine classes. Four *discrete* classes, which correspond to $\pi(P)$ being 0, 1, $\pi(E_+)$ or $\pi(E_-)$, and five *essential* classes which we describe below. The discrete classes are, respectively, the finite rank projections, finite co-rank projections, the Sato Grassmannian of $\mathcal{H}_+$ and the Sato Grassmannian of $\mathcal{H}_-$. Thus the connected components of each of these classes are parametrized by the integers (via de rank, the co-rank or the Fredholm index, respectively). The essential classes are shown to be connected. We are interested in the geometric structure of $\mathcal{P}_{cc}$, being the set of selfadjoint projections of the C^*-algebra $\mathcal{B}_{cc}$ of operators in $\mathcal{B}(\mathcal{H})$ which essentially commute with E_+. In particular, we study the problem of existence of minimal geodesics joining two given projections in the same component. We show that

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the Hopf–Rinow Theorem holds in the discrete classes, but not in the essential classes. Conditions for the existence and uniqueness of geodesics in these latter classes are found.

1. Introduction

Let $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ be a fixed decomposition of a separable Hilbert space, with both $\mathcal{H}_+, \mathcal{H}_-$ infinite dimensional. Denote by E_+ and E_- the orthogonal projections onto $\mathcal{H}_+$ and $\mathcal{H}_-$, respectively. We shall study the unitary group $\mathcal{U}_{cc}$ and the set of projections $\mathcal{P}_{cc}$ of the C^*-algebra $\mathcal{B}_{cc} = \mathcal{B}_{cc}(\mathcal{H}; \mathcal{H}_+, \mathcal{H}_-)$ given by

$$\mathcal{B}_{cc} = \{T \in \mathcal{B}(\mathcal{H}) : [T, E_+] \text{ is compact}\}.$$

Here [,] denotes the commutator. Note that this condition is equivalent to $[T, E_-]$ compact. If we denote by J the symmetry which is the identity in $\mathcal{H}_+$ and minus the identity in $\mathcal{H}_-$ (i.e. $J = 2E_+ - 1 = 1 - 2E_-$), this condition is equivalent to $[T, J]$ compact. If one writes operators in $\mathcal{H}$ as two by two matrices in terms of the given decomposition, elements in $\mathcal{B}_{cc}$ have compact off-diagonal entries (with this matricial characterization, it is straightforward to verify that $\mathcal{B}_{cc}$ is a C^*-algebra). If we denote by

$$\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{C}(\mathcal{H}) = \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$$

the homomorphism onto the Calkin algebra, and $e_+ = \pi(E_+)$, then

$$\mathcal{B}_{cc} = \pi^{-1}(\{e_+\}'),$$

where $\{e_+\}'$ denotes the set of elements in $\mathcal{C}(\mathcal{H})$ that commute with e_+.

The set $\mathcal{P}_{cc}$ relates to the so called restricted or Sato Grassmannian (see e.g. [13,14], or [5,12] for a version using Hilbert–Schmidt operators instead of compact operators). In fact, $\mathcal{P}_{cc}$ is disconnected, and several of its components form the restricted Grassmannian of $\mathcal{H}_+$ (as well as the restricted Grassmannian of $\mathcal{H}_-$). Thus this framework enables one to regard the restricted Grassmannian as (certain components of) the set of projections of a C^*-algebra. Again, by means of the homomorphism π, one sees that $\mathcal{P}_{cc}$ decomposes into nine classes. If $P \in \mathcal{P}_{cc}$, then $\pi(P)$ is one of the following (written as 2×2 matrices in terms of $e_+, e_- = 1 - e_+$):

$$0,\ 1,\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ \begin{pmatrix} p_+ & 0 \\ 0 & 0 \end{pmatrix},\ \begin{pmatrix} p_+ & 0 \\ 0 & 1 \end{pmatrix},\ \begin{pmatrix} 0 & 0 \\ 0 & p_- \end{pmatrix},$$
$$\begin{pmatrix} 1 & 0 \\ 0 & p_- \end{pmatrix} \text{ and } \begin{pmatrix} p_+ & 0 \\ 0 & p_- \end{pmatrix}.$$

The first four classes are called here *discrete*, and denoted by $\mathbb{D}_i$ $(1 \leq i \leq 4)$. The latter five classes are called *essential*, and denoted by $\mathbb{E}_j$ $(1 \leq j \leq 5)$.

The main results of this paper are the following:

- The classes $\mathbb{E}_j$ are connected (Theorem 5.3)
- The Hopf–Rinow Theorem (two projections in the same connected component can be joined by a minimal geodesic) holds in the classes $\mathbb{D}_i$ (Theorems 6.4 and 6.6)
- The Hopf–Rinow Theorem does not hold in the classes $\mathbb{E}_j$ (Corollary 6.8)

The overall contents of the paper are the following. In Section 2 we establish basic facts on the structure of essentially commuting projections. In Section 3 we describe the discrete classes $\mathbb{D}_i$. In particular we show that their connected components are parametrized by the integers, by means of the rank, the co-rank, or the Fredholm index, depending on the class. In Section 4 we study properties of the action of $\mathcal{U}_{cc}$ on $\mathcal{P}_{cc}$. In Section 5 we study the structure of the essential classes $\mathbb{E}_j$. In Section 6 we recall basic facts on the differential geometry of the set of projections [9], and on the index of pairs of projections [1,4], and study the geodesic structure of $\mathcal{P}_{cc}$.

2. Structure of $\mathcal{P}_{cc}$

We begin this section with some basic facts on the unitary group $\mathcal{U}_{cc}$.

Remark 2.1. The group $\mathcal{U}_{cc}$ is not connected. It is known as the restricted unitary group in the literature [14]. Elementary computations show that if $U \in \mathcal{U}_{cc}$ is written in matrix form in terms of the given decomposition of $\mathcal{H}$,

$$U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix},$$

then u_{ii} are Fredholm operators in their respective spaces, and $ind(u_{11}) = -ind(u_{22})$. We shall denote by $ind(U) = ind(u_{11})$. The connected components of $\mathcal{U}_{cc}$ are parametrized by this index: two unitaries lie in the same connected component if and only if they have the same index. These facts on the group $\mathcal{U}_{cc}$ can be found in [8,7]. A similar argument holds for the invertible group of $\mathcal{B}_{cc}$: its connected components are parametrized by the index of the $1,1$ entry.

From the fact that $\mathcal{U}_{cc}$ is the unitary group of a C^*-algebra, it follows $\mathcal{U}_{cc}$ is a Banach Lie group. Its Banach Lie algebra is given by

$$\mathfrak{u}_{cc} = \{A \in \mathcal{B}_{cc} : A^* = -A\}.$$

The group $\mathcal{U}_{cc}$ acts on $\mathcal{P}_{cc}$ with the usual coadjoint action:

$$U \cdot P = UPU^*.$$

Now we focus on the set of projections $\mathcal{P}_{cc}$ of the $\mathcal{B}_{cc}$. Since $\mathcal{P}_{cc}$ is the set of projections of a C^*-algebra, using the facts proved by Corach, Porta and Recht in [9], it is a C^∞ submanifold of $\mathcal{B}_{cc}$. Moreover, the action of the unitary group induces a homogeneous reductive structure on $\mathcal{P}_{cc}$. For any fixed $P_0 \in \mathcal{P}_{cc}$ the map

$$\pi_{P_0} : \mathcal{U}_{cc} \to \mathcal{P}_{cc}, \qquad \pi_{P_0}(U) = UP_0U^*$$

is a C^∞ submersion. The range of this map is the orbit of P_0, which is not the whole $\mathcal{P}_{cc}$. We shall examine these orbits in the next section. Let us focus here in the structure of elements in $\mathcal{P}_{cc}$. Pick $P \in \mathcal{P}_{cc}$. Written as a matrix in terms of $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, we have

$$P = \begin{pmatrix} x & a \\ a^* & y \end{pmatrix}.$$

The fact that P is a selfadjoint projection implies that $0 \le x, y \le 1$, $\|a\| \le 1$, and the relations

$$x - x^2 = aa^*, \qquad y - y^2 = a^*a \quad \text{and} \quad xa + ay = a. \tag{1}$$

Since a is compact, the first two relations imply that $x - x^2$ and $y - y^2$ are compact operators. Therefore the spectra of x and y, which lie in the unit interval, are discrete sets which may accumulate at roots of the polynomial $t - t^2$, i.e. 0 or 1. It follows that x and y can be diagonalized using the eigenspaces of aa^* and a^*a, respectively, and in particular, all spectral values of x and y, with the possible exception of 0 and 1, are eigenvalues of finite multiplicity. As we will see in the next lemma, there is also a symmetry between these eigenvalues. Given an operator T, we denote by $N(T)$ and $R(T)$, the nullspace and the range of T, respectively.

Lemma 2.2. *If $\lambda \neq 0, 1$ is an eigenvalue of y, then $1 - \lambda$ is an eigenvalue of x, and the operator $a|_{N(y - \lambda 1_{\mathcal{H}_-})}$ maps $N(y - \lambda 1_{\mathcal{H}_-})$ isomorphically onto $N(x - (1 - \lambda)1_{\mathcal{H}_+})$. Thus in particular, these eigenvalues have the same multiplicity. Moreover,*

$$aP_{N(y - \lambda 1_{\mathcal{H}_-})} = P_{N(x - (1 - \lambda)1_{\mathcal{H}_+})}a.$$

Proof. Let $\xi \in \mathcal{H}$, $\xi \neq 0$, such that $y\xi = \lambda\xi$ (with $\lambda \neq 0, 1$). Then by the third relation in (1) one has

$$a\xi = xa\xi + ay\xi = xa\xi + \lambda a\xi, \quad \text{i.e. } xa\xi = (1 - \lambda)a\xi.$$

Also note that

$$N(a) = N\big(a^*a\big) = N\big(y - y^2\big) = N(y) \oplus N(y - 1_{\mathcal{H}_-}),$$

and thus $a\xi \neq 0$ is an eigenvector for x, with eigenvalue $1 - \lambda$, and the map $a|_{N(y - \lambda 1_{\mathcal{H}_-})}$ is injective from $N(y - \lambda 1_{\mathcal{H}_-})$ to $N(x - (1 - \lambda)1_{\mathcal{H}_+})$. Therefore

$$\dim\big(N(y-\lambda 1_{\mathcal{H}_-})\big) \le \dim\big(N\big(x-(1-\lambda)1_{\mathcal{H}_+}\big)\big).$$

By a symmetric argument, using a^* (and the relation $ya^* + a^*x = a^*$), one obtains equality.

Pick now an arbitrary $\xi \in \mathcal{H}_-$, $\xi = \xi_1 + \xi_2$, with $\xi_1 \in N(y-\lambda 1_{\mathcal{H}_-})$ and $\xi_2 \perp N(y-\lambda 1_{\mathcal{H}_-})$. Then

$$aP_{N(y-\lambda 1_{\mathcal{H}_-})}\xi = a\xi_1.$$

On the other hand

$$P_{N(x-(1-\lambda)1_{\mathcal{H}_+})}a\xi_1 = a\xi_1,$$

by the fact proven above. Let us see that $P_{N(x-(1-\lambda)1_{\mathcal{H}_+})}a\xi_2 = 0$, which would prove our claim. Since $\xi_2 \perp N(y-\lambda 1_{\mathcal{H}_-})$, $\xi_2 = \sum_{l\ge 2}\eta_l + \eta_0 + \eta_1$, where η_l, $l \ge 2$, are eigenvectors of y corresponding to eigenvalues λ_l different from 0, 1 and λ, $\eta_0 \in N(y)$, $\eta_1 \in N(y-1_{\mathcal{H}_-})$ (where these two latter may be trivial). Note then that $\eta_0, \eta_1 \in N(a)$, and thus

$$a\xi_2 = \sum_{l\ge 2} a\eta_l,$$

where the (non-nil) vectors $a\eta_l$ are eigenvectors of x corresponding to eigenvalues $1-\lambda_l$, different from $0, 1$ and $1-\lambda$. Thus $P_{N(x-(1-\lambda)1_{\mathcal{H}_+})}a\xi_2 = 0$. □

Remark 2.3. One obtains nine types of projections in $\mathcal{P}_{cc}$, by means of the homomorphism onto the Calkin algebra $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{C}(\mathcal{H}) = \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. Denote by $e_+ = \pi(E_+)$ and $e_- = \pi(E_-)$. Note that since both $\mathcal{H}_+$ and $\mathcal{H}_-$ are infinite dimensional, these projections are non-trivial. If $P \in \mathcal{P}_{cc}$, then $p = \pi(P)$ is one of the following projections in the Calkin algebra (written as 2×2 matrices in terms of e_+, e_-):

$$0,\ 1,\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} p_+ & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} p_+ & 0 \\ 0 & 1 \end{pmatrix},$$
$$\begin{pmatrix} 0 & 0 \\ 0 & p_- \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & p_- \end{pmatrix} \text{ and } \begin{pmatrix} p_+ & 0 \\ 0 & p_- \end{pmatrix},$$

where p_+ and p_- are proper projections in $\mathcal{C}(\mathcal{H}_+)$ and $\mathcal{C}(\mathcal{H}_-)$. It is known that proper projections are unitarily equivalent in the Calkin algebra, even more, that they are homotopic. Thus these nine types are nine different classes in the set of projections of $\mathcal{C}(\mathcal{H})$, modulo the action of the unitaries in $\mathcal{C}(\mathcal{H})$ that commute with e_+ (and e_-), i.e. modulo diagonal unitaries of $\mathcal{C}(\mathcal{H})$.

In particular, this fact implies that these different types cannot be unitarily equivalent in $\mathcal{B}_{cc}$. Projections of the first four types will be called *discrete*, and their classes referred as (respectively) $\mathbb{D}_i$, $i = 1,2,3,4$. Projections of the latter five classes will be called *essential*, and their classes $\mathbb{E}_j$, $j = 1,2,3,4,5$.

These classes can be also characterized by means of the spectra of x and y. For instance, projections in $\mathbb{E}_1$ satisfy that the spectrum of x (and thus also the spectrum of y) is finite, and that 1 has finite multiplicity (possibly zero). We shall not pursue this description here, though the spectral properties of the different classes will be discussed later.

3. Discrete projections

In this section we give a characterization of the connected components of the discrete classes.

Proposition 3.1. *Projections in $\mathbb{D}_1$ have finite rank. The connected components of $\mathbb{D}_1$ are parametrized by the rank: two projections of finite rank lie in the same component if and only if they have the same rank.*

Proof. Recall that $P \in \mathbb{D}_1$ if $\pi(P) = 0$, then P is compact, and therefore of finite rank. The assertion on the components is well known. □

Next we examine discrete projections of the second type:

Proposition 3.2. *Projections in $\mathbb{D}_2$ have finite co-rank. The connected components of $\mathbb{D}_2$ are parametrized by the co-rank.*

Proof. If $P \in \mathbb{D}_2$, then $\pi(P) = 1$. It follows that P is a Fredholm operator. Thus it has finite dimensional nullspace, i.e. finite co-rank. □

Note that $P \in \mathcal{P}_{cc}$ if and only if $P^\perp = 1-P \in \mathcal{P}_{cc}$. Taking the orthogonal complement gives a diffeomorphism between $\mathbb{D}_1$ and $\mathbb{D}_2$.

Projections of the third type belong to the restricted Grassmannian. Let us recall its definition [14]:

A projection P belongs to the restricted Grassmannian $\mathcal{P}_{res}(\mathcal{H}_+)$ with respect to the decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, or more precisely, with respect to subspace $\mathcal{H}_+$ (which is the name that we shall adopt here, since the roles of $\mathcal{H}_+$ and $\mathcal{H}_-$ are not interchangeable) if and only if

1. $$E_+P|_{R(P)} : R(P) \to \mathcal{H}_+ \in \mathcal{B}\big(R(P), \mathcal{H}_+\big)$$

   is a Fredholm operator in $\mathcal{B}_p(R(P), \mathcal{H}_+)$, and

2. $$E_-P|_{R(P)} : R(P) \to \mathcal{H}_- \in \mathcal{B}\big(R(P), \mathcal{H}_-\big)$$

   is compact.

The index of the first operator characterizes the connected components of $\mathcal{P}_{res}(\mathcal{H}_+)$. The following result is elementary:

Lemma 3.3. *Let $P \in \mathcal{P}$ with matrix (in terms of $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$)*

$$P = \begin{pmatrix} x & a \\ a^* & y \end{pmatrix}.$$

Then $P \in \mathcal{P}_{res}(\mathcal{H}_+)$ if and only if x is Fredholm in $\mathcal{B}(\mathcal{H}_+)$, and y and a are compact.

Proof. Suppose first that $P \in \mathcal{P}_{res}(\mathcal{H}_+)$. Then $E_+P \in \mathcal{B}(R(P), \mathcal{H}_+)$ is Fredholm, and thus

$$E_+P(E_+P)^*|_{\mathcal{H}_+} = E_+PE_+|_{\mathcal{H}_+} = x$$

is Fredholm in $\mathcal{H}_+$. Also $E_-P \in \mathcal{B}(R(P), \mathcal{H}_-)$ is compact, and thus

$$E_-P(E_-P)^*|_{\mathcal{H}_-} = E_-PE_-|_{\mathcal{H}_-} = y$$

is compact in $\mathcal{H}_-$. The fact that P is a projection, implies the relation $y - y^2 = a^*a$, and thus a is compact.

Conversely, if a and y are compact, then $E_-P \in \mathcal{B}(R(P), \mathcal{H}_-)$ is compact.

Since $E_+P(E_+P)^*|_{\mathcal{H}_+} = x$ is Fredholm, it follows that E_+P in $\mathcal{B}(R(P), \mathcal{H}_+)$ has closed range (equal to the range of x) with finite codimension. Let us prove that its nullspace is finite dimensional. Let $\xi = \xi_+ + \xi_- = P\xi$ such that $E_+\xi = 0$ ($\xi_+ \in \mathcal{H}_+$, $\xi_- \in \mathcal{H}_-$). These imply

$$0 = x\xi_+ + a\xi_- \quad \text{and} \quad \xi_- = y\xi_-,$$

i.e. ξ_- lies in the 1-eigenspace of the compact operator y. Thus ξ_- lies in a finite dimensional space. It follows that $N(E_+P|_{R(P)})$ is finite dimensional. □

From this result it is apparent that:

Proposition 3.4. *The set $\mathbb{D}_3$ of discrete projections of the third type coincides the restricted Grassmannian $\mathcal{P}_{res}(\mathcal{H}_+)$. The connected components are parametrized (in the integers) by the index of the operator $E_+P|_{R(P)} \in \mathcal{B}(R(P), \mathcal{H}_+)$.*

With a similar argument, or taking orthogonal complements ($\perp$ maps $\mathbb{D}_3$ onto $\mathbb{D}_4$), one proves:

Proposition 3.5. *The set $\mathbb{D}_4$ of discrete projections of the fourth type coincides the restricted Grassmannian $\mathcal{P}_{res}(\mathcal{H}_-)$. The connected components are parametrized by the index of the operator $E_-P|_{R(P)} \in \mathcal{B}(R(P), \mathcal{H}_-)$.*

In order to study the classes of essential projections, it will be useful to establish first certain facts concerning the action of $\mathcal{U}_{cc}$ on $\mathcal{P}_{cc}$.

4. Unitary action

Apparently, projections of the nine different types cannot be unitary equivalent (with a unitary in $\mathcal{U}_{cc}$). Projections in $\mathbb{D}_3$ and $\mathbb{D}_4$ (i.e. in the restricted Grassmannian of $\mathcal{H}_+$ and $\mathcal{H}_-$) have infinite rank and corank, therefore they are unitarily equivalent in $\mathcal{B}(\mathcal{H})$, but not with a unitary in $\mathcal{U}_{cc}$: $\mathcal{U}_{cc}$ (or $\mathcal{U}_{res}$) acts in the restricted Grassmannian [12].

Also it is apparent that, if P has finite rank, the unitary orbit

$$\{UPU^* : U \in \mathcal{U}_{cc}\} = \{Q \in \mathcal{P}_{cc} : rank(Q) = rank(P)\},$$

which is also the full unitary orbit of P (with unitaries in $\mathcal{B}(\mathcal{H})$). Indeed, it is easy to verify that two projection with equal (finite) rank are conjugate with a unitary which is a finite rank perturbation of the identity (thus in $\mathcal{U}_{cc}$).

An analogous result follows in the case when P has finite corank, or arguing by means of the symmetry $P \mapsto P^{\perp}$.

In [14] it was remarked that the action of the invertible group of $\mathcal{B}_{cc}$ is transitive in the restricted Grassmannian of $\mathcal{H}_+$. The action is given by $G \cdot L = G(L)$, if L belongs to the restricted Grassmannian and G is invertible in $\mathcal{B}_{cc}$. Let us prove that the unitary coadjoint actions on projections (of the third and fourth type) is also transitive. To this effect, we need the following lemmas (the first result is well known [9]):

Lemma 4.1. *Let P and Q be orthogonal projections such that there exists an invertible element G in $\mathcal{B}_{cc}$ such that $GPG^{-1} = Q$. Then there exists a unitary element $U \in \mathcal{U}_{cc}$ such that $UPU^* = Q$.*

Proof. $GP = QG$ implies that $PG^* = G^*Q$, so that $G^*GP = PG^*G$, i.e. G^*G commutes with P. Then $|G| = (G^*G)^{1/2}$ also commutes with P. Let $G = U|G|$ be the polar decomposition of G. Then

$$UPU^* = UPU^{-1} = G|G|^{-1}P|G|G^{-1} = GPG^{-1} = Q. \qquad \square$$

Lemma 4.2. *Let L be a subspace of $\mathcal{H}$ such that $P_L \in \mathcal{P}_{cc}$, and let G be invertible in $\mathcal{B}_{cc}$. Then there exists an invertible operator T in $\mathcal{B}_{cc}$ such that*

$$TP_LT^{-1} = P_{G(L)}.$$

Proof. Note that the idempotents $P_{G(L)}$ and $Q = GP_LG^{-1}$ have the same range, namely $G(L)$. Put $L_0 = G(L)$. Consider

$$T_0 = QP_{L_0} + (1-Q)(1-P_{L_0}) = Q + (1-Q)(1-P_{L_0}).$$

First note that $T_0P_{L_0} = QP_{L_0} = QT_0$. Also T_0 is invertible. Indeed, $T_0|_{L_0} = T_0|_{R(Q)} = 1_{L_0}$, and $T_0|_{N(Q)} = (1 - P_{L_0})|_{N(Q)}$. Since $N(Q)$ is a supplement for $N(1 - P_{L_0}) = L_0 = R(Q)$, this latter operator is an isomorphism between $N(Q)$ and $R(1-P_{L_0}) = L_0^\perp$. Finally, note that T_0 belongs to $\mathcal{B}_{cc}$. This is clear since $Q, P_{L_0} \in \mathcal{B}_{cc}$. Thus $T_0P_{G(L)}T_0^{-1} = GP_LG^{-1}$. □

Let us say that two idempotents P and Q are similar in $\mathcal{B}_{cc}$ if there exists an invertible operator G in $\mathcal{B}_{cc}$ such that $GPG^{-1} = Q$. This is clearly an equivalence relation.

Proposition 4.3. *Let L be a subspace of $\mathcal{H}$ such that $P_L \in \mathcal{P}_{cc}$, and let G be invertible in $\mathcal{B}_{cc}$. Then there exists a unitary $U \in \mathcal{U}_{cc}$ such that*

$$UP_LU^* = P_{G(L)}.$$

Proof. By Lemma 4.2, P_L and $P_{G(L)}$ are similar in $\mathcal{B}_{cc}$. By Lemma 4.1, this implies that they are unitary equivalent, with a unitary in $\mathcal{U}_{cc}$. □

If we specialize to projections in $\mathbb{D}_3$ and $\mathbb{D}_4$, we have the following:

Corollary 4.4. *The action of $\mathcal{U}_{cc}$ on $\mathbb{D}_3$ (the restricted Grassmannian of $\mathcal{H}_+$) is transitive. The same statement holds for $\mathbb{D}_4$ (the restricted Grassmannian of $\mathcal{H}_-$).*

Proof. In [12] it was proved that the action $G\cdot L = G(L)$ (G invertible in $\mathcal{B}_{cc}$) is transitive in the restricted Grassmannian of $\mathcal{H}_+$: given L_1 and L_2 such that P_{L_i} are projections in $\mathbb{D}_3$, then there exists an invertible operator G in $\mathcal{B}_{cc}$ such that $P_{L_2} = P_{G(L_1)}$. By the above proposition, there exists a unitary $U \in \mathcal{U}_{cc}$ such that $P_{L_2} = P_{G(L_1)} = UP_{L_1}U^*$. The same argument works in $\mathbb{D}_4$. □

Note that U in the proof above satisfies $U(L_1) = L_2$. An easy consequence of the transitivity of the unitary action is the following fact:

Remark 4.5. Two projections P, Q in the same component of the restricted Grassmannian verify that the difference $P-Q$ is compact. Indeed, it is known in the zero index connected component ([6,5]): the unitary Fredholm group acts transitively on this component, thus $Q = UPU^*$ with $U = 1 + K$, K compact. Then $P - Q = P - (1 + K)P(1 + K^*) = -KP - PK^* - KPK^*$ is compact. In any other component, since the action is transitive, $P = WE_+W^*$ for some $W \in \mathcal{U}_{cc}$. Thus W^*QW lies in the same component as $W^*PW = E_+$. It follows that $P - Q = W(E_+ - W^*QW)W^*$ is compact.

5. Essential projections

Since the unitary group $\mathcal{U}_{cc}$ acts on $\mathcal{P}_{cc}$, and leaves invariant the discrete classes, it leaves invariant also the set of essential projections. It will be useful to describe the effect of the two fundamental symmetries of $\mathcal{P}_{cc}$ in the class of essential projections. The first symmetry is $\perp$: $\mathcal{P}_{cc} \to \mathcal{P}_{cc}$, $\perp(P) = 1 - P$. Note that this symmetry leaves $\mathcal{P}_{cc}$ invariant. The second symmetry depends on the choice of orthonormal bases for $\mathcal{H}_+$ and $\mathcal{H}_-$. Or equivalently, using a model $\mathcal{H} = \mathcal{L} \times \mathcal{L}$, let

$$J = \begin{pmatrix} 0 & 1_{\mathcal{L}} \\ 1_{\mathcal{L}} & 0 \end{pmatrix}.$$

Apparently, J maps $\mathcal{H}_+ = \mathcal{L} \times 0$ onto $\mathcal{H}_- = 0 \times \mathcal{L}$ (and vice versa). Consider the inner automorphism

$$Ad_J : \mathcal{B}_{cc} \to \mathcal{B}_{cc}, \qquad Ad_J(X) = JXJ.$$

It is clear that Ad_J maps $\mathcal{B}_{cc}$ onto itself and that $Ad_J \circ Ad_J = id_{\mathcal{B}_{cc}}$. Then

- $\perp$ maps $\mathbb{E}_1$ onto $\mathbb{E}_2$, and $\mathbb{E}_3$ onto $\mathbb{E}_4$.
- Ad_J maps $\mathbb{E}_1$ onto $\mathbb{E}_3$ and $\mathbb{E}_2$ onto $\mathbb{E}_4$.
- both symmetries leave $\mathbb{E}_5$ fixed.

Consider $P \in \mathcal{P}_{cc}$ (not necessarily an essential projection). Without loss of generality we may suppose $\mathcal{H} = \mathcal{L} \times \mathcal{L}$, with $\mathcal{H}_+ = \mathcal{L} \times 0$ and $\mathcal{H}_- = 0 \times \mathcal{L}$. By the result in Lemma 2.2, writing as before

$$P = \begin{pmatrix} x & a \\ a^* & y \end{pmatrix},$$

we know that the spectra of x and y are related (and can be described) in the following fashion:

- The spectrum of $0 \leq x \leq 1$ (as an operator in $\mathcal{L}$) consists of two strictly positive (disjoint, eventually finite) sequences α_n, β_m, such that $\frac{1}{2} > \alpha_n \to 0$, $\frac{1}{2} \leq \beta_m < 1$ and $\beta_m \to 1$, plus 0 and 1, which may or may not be eigenvalues.
- The spectrum of y consists of the sequences $1 - \alpha_n$, $1 - \beta_m$, plus 0 and 1.
- The multiplicity of α_n (resp. β_m) in x equals the multiplicity of $1 - \alpha_n$ (resp. $1 - \beta_n$) in y. These multiplicities are finite.

With these facts, and using the relations (1), we can describe the entries x, y and a of $P \in \mathcal{P}_{cc}$:

$$x = \sum_n \alpha_n P_n + \sum_m \beta_m Q_m + E_x,$$

$$y = \sum_n (1 - \alpha_n) P'_n + \sum_m (1 - \beta_m) Q'_m + E_y,$$

and

$$a = \sum_n \lambda_n \xi_n \otimes \xi'_n + \sum_m \mu_m \eta_m \otimes \eta'_m,$$

with $rank(P_n) = rank(P'_n)$, $rank(Q_m) = rank(Q'_m)$, $\lambda_n = \sqrt{\alpha_n - \alpha_n^2}$ and $\mu_n = \sqrt{\beta_m - \beta_m^2}$. Here E_x, E_y denote the spectral projections corresponding to the spectral value 1 (which may be trivial or have rank of any dimension) of x and y, respectively. As it is usual, the rank one operators $\xi \otimes \eta$ are defined by $\langle \cdot, \eta \rangle \xi$. The orthonormal sets

$$\{\xi_k : k \geq 1\}, \qquad \{\xi'_k : k \geq 1\}, \qquad \{\eta_l : l \geq 1\} \quad \text{and} \quad \{\eta'_l : l \geq 1\}$$

span, respectively, the subspaces

$$\bigoplus_{n \geq 1} R(P_n), \qquad \bigoplus_{n \geq 1} R(P'_n), \qquad \bigoplus_{m \geq 1} R(Q_m) \quad \text{and} \quad \bigoplus_{m \geq 1} R(Q'_m),$$

and are eigenvectors of x and y:

$$x\xi_k = \alpha_{n(k)} \xi_k, \qquad x\eta_l = \beta_{m(l)} \eta_l, \qquad y\xi'_k = (1 - \alpha_{n(k)})\xi'_k, \qquad y\eta'_l = (1 - \beta_{m(l)})\eta'_l.$$

Denote by $N_x = P_{N(x)}$ and $N_y = P_{N(y)}$ (which are also unconditioned). Consider the following projection (related to P above):

$$P_d = \begin{pmatrix} \sum_m Q_m + E_x & 0 \\ 0 & \sum_n P'_n + E_y \end{pmatrix}.$$

Consider also the operator $B \in \mathcal{B}_{cc}$,

$$B = P + P_d - 1.$$

Lemma 5.1. *B is invertible in $\mathcal{B}_{cc}$, and has zero index.*

Proof. By direct computation, writing the identity operator as

$$\begin{pmatrix} \sum_n P_n + \sum_m Q_m + E_x + N_x & 0 \\ 0 & \sum_n P'_n + \sum_m Q'_m + E_y + N_y \end{pmatrix},$$

one obtains that B is

$$\begin{pmatrix} \sum_n (\alpha_n - 1) P_n + \sum_m \beta_m Q_m + E_x - N_x & a \\ a^* & \sum_n (1 - \alpha_n) P'_n + \sum_m (-\beta_m) Q'_m + E_y - N_y \end{pmatrix}.$$

Note that the diagonal entries are invertible operators, whereas the co-diagonal entries are compact. It follows that B is of the form invertible plus compact, and thus it is a

Fredholm operator. In particular, it has closed range. Let us prove that $N(B) = \{0\}$. Since B is selfadjoint, this would imply that B is invertible (apparently, it belongs to $\mathcal{B}_{cc}$). Note that since $B = P - (1 - P_d)$ is a difference of orthogonal projections, then

$$\begin{aligned} N(B) &= \big(N(P) \cap N(1 - P_d)\big) \oplus \big(R(P) \cap R(1 - P_d)\big) \\ &= \big(N(P) \cap R(P_d)\big) \oplus \big(R(P) \cap N(P_d)\big). \end{aligned}$$

Let us first see that $N(P) \cap R(P_d) = \{0\}$. If $(\xi, \eta) \in N(P) \cap R(P_d)$,

$$\sum_m Q_m \xi + E_x \xi = \xi, \qquad \sum_n P'_n \eta + E_y \eta = \eta.$$

These imply that $P_n \xi = 0$ for all n, $N_x \xi = 0$, $Q'_m \eta = 0$ for all m, and $N_y \eta = 0$.

Also one has

$$\sum_n \alpha_n P_n \xi + \sum_m \beta_m Q_m \xi + E_x \xi + a\eta = 0. \tag{2}$$

Then applying Q_{m_0}, one obtains

$$\beta_{m_0} Q_{m_0} \xi + Q_{m_0} a\eta = 0.$$

Note that

$$Q_{m_0} a\eta = \sum_k \mu_k Q_{m_0} \eta_k \otimes \eta'_k(\eta) = \sum_k \mu_k \langle \eta, \eta'_k \rangle Q_{m_0} \eta_k.$$

Since $\eta'_k \in R(Q'_{m(k)})$ for some $m(k)$, this sum equals

$$\sum_k \mu_k \langle \eta, Q'_{m(k)} \eta'_k \rangle Q_{m_0} \eta_k = \sum_k \mu_k \langle Q'_{m(k)} \eta, \eta'_k \rangle Q_{m_0} \eta_k = 0,$$

because $Q'_m \eta = 0$ for all m. Thus

$$\beta_{m_0} Q_{m_0} \xi = 0,$$

which implies that $Q_m \xi = 0$ for all m ($\beta_m > 0$). Similarly, $P'_n \eta = 0$. Note then that the fact that $(\xi, \eta) \in R(P_d)$ now implies

$$E_x \xi = \xi, \qquad E_y \eta = \eta.$$

On the other hand, Eq. (2) above now means

$$E_x \xi + a\eta = 0,$$

since these operators have orthogonal ranges, $E_x\xi = 0$. Using the relation analogous to (2), also $E_y\eta = 0$. It follows that $N(P) \cap R(P_d) = \{0\}$. One proves that $R(P) \cap N(P_d) = \{0\}$ in a similar fashion.

Finally, the $1,1$ entry of B above is invertible, therefore B has zero index. □

Lemma 5.2. *Let F, G be two essential projections which are diagonal (with respect to $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$). Then they are homotopic if and only if they belong to the same class $\mathbb{E}_i$. In that case, they are unitarily equivalent with a unitary in $\mathcal{U}_{cc}$ of index zero.*

Proof. If F and G are homotopic, they clearly belong to the same class $\mathbb{E}_i$. Let us first suppose that $F, G \in \mathbb{E}_1$. The projections F and G are of the form

$$F = \begin{pmatrix} P_+ & 0 \\ 0 & F_- \end{pmatrix}, \qquad G = \begin{pmatrix} P_+ & 0 \\ 0 & G_- \end{pmatrix},$$

where P_+ is a projection with infinite rank and co-rank, and F_-, G_- have finite rank. It suffices to show that any of these two is homotopic (and unitarily equivalent with a unitary in $\mathcal{U}_{cc}$ of index zero) to a projection of the form

$$E = \begin{pmatrix} P_+ & 0 \\ 0 & 0 \end{pmatrix}.$$

Let $\{e_n : 1 \le n < \infty\}$ be an orthonormal basis for $R(P_+)$, $\{e'_l : 1 \le l < \infty\}$ an orthonormal basis for $\mathcal{H}_+ \ominus R(P_+)$ and $\{f_k : 1 \le k < \infty\}$ an orthonormal basis for $\mathcal{H}_-$, such that $f_1, \ldots, f_M$ generate $R(F_-)$ ($M < \infty$). Consider the unitary operator U given by

$$\begin{cases} U(e_n) = f_n & \text{for } 1 \le n \le M \\ U(e_n) = e_{n-M} & \text{for } n \ge M+1 \\ U(e'_l) = e'_{l+M} & \text{for } 1 \le l < \infty \\ U(f_n) = e'_n & \text{for } 1 \le n \le M \\ U(f_n) = f_n & \text{for } n \ge M+1. \end{cases}$$

Clearly $U(R(E)) = R(F)$ and $U(R(E)^\perp) = R(F)^\perp$, so that $UEU^* = F$. Also it is clear that $E_+UE_- = 0$ and that E_-UE_+ is a partial isometry with range generated by $\{f_1, \ldots, f_M\}$. Thus $U \in \mathcal{U}_{cc}$. Moreover, it is not difficult to see that $ind(U) = 0$. This proves our claim for that case when $F, G \in \mathbb{E}_1$. Using the symmetry $\perp$, the result holds for the class $\mathbb{E}_2$. Using then the symmetry Ad_J in these two classes, the result holds also for the classes $\mathbb{E}_3$ and $\mathbb{E}_4$.

To finish the proof, note that any pair of diagonal projections in $\mathbb{E}_5$ is unitarily equivalent with a unitary in $\mathcal{U}_{cc}$ of index zero. Indeed, pick F, G two such projections,

$$F = \begin{pmatrix} F_+ & 0 \\ 0 & F_- \end{pmatrix}, \qquad G = \begin{pmatrix} G_+ & 0 \\ 0 & G_- \end{pmatrix}$$

with $F_\pm, G_\pm$ of infinite rank and co-rank. Apparently, there exist unitary operators V_+ and V_- in $\mathcal{H}_+$ and $\mathcal{H}_-$, respectively, such that

$$V_+F_+V_+^* = G_+, \qquad V_-F_-V_-^* = G_-.$$

Then $V = V_+ \oplus V_-$ lies in $\mathcal{U}_{cc}$ of index zero, and implements the equivalence between F and G. □

This fact implies that the classes $\mathbb{E}_i$ are connected:

Theorem 5.3. *The sets $\mathbb{E}_i$ ($1 \le i \le 5$) are connected. The action of $(\mathcal{U}_{cc})_0$, the component of the identity in $\mathcal{U}_{cc}$, is transitive in each $\mathbb{E}_i$.*

Proof. Consider the invertible operator $B = P + P_d - 1$ of Lemma 5.1. Clearly,

$$BP_d = PP_d = PB.$$

Thus P and P_d are similar in $\mathcal{B}_{cc}$. Thus by Lemma 4.1 they are unitary equivalent in $\mathcal{B}_{cc}$. Moreover, as in the proof of Lemma 4.1, a unitary operator U implementing this equivalence is the unitary part of S in its polar decomposition:

$$U = B\big(B^2\big)^{-1/2} = sgn(B),$$

which is in fact a symmetry (sgn is the sign function). Since B lies in the connected component of the identity in the invertible group of $\mathcal{B}_{cc}$ (it has zero index), there exists a continuous path $B(t)$ of invertible elements in $\mathcal{B}_{cc}$ such that $B(0) = 1$ and $B(1) = B$. Then

$$U(t) = B(t)\big(B^*(t)B(t)\big)^{-1/2}$$

is a continuous path in $\mathcal{U}_{cc}$, joining 1 and U. Thus, we obtain that $U \in (\mathcal{U}_{cc})_0$ and $P_d = U^*PU$. The same argument can be carried out to find a unitary $V \in (\mathcal{U}_{cc})_0$ such that $Q_d = V^*QV$, where Q is another projection in the same class of P and Q_d has the obvious meaning. According to Lemma 5.2 the diagonal projections P_d and Q_d are unitarily equivalent with a unitary in $(\mathcal{U}_{cc})_0$. Hence P and Q are also unitarily equivalent with a unitary in $(\mathcal{U}_{cc})_0$. □

6. Geodesics

In [9] a natural reductive structure was introduced to the homogeneous space of selfadjoint projections $\mathcal{P}(\mathcal{A})$ of an abstract C^*-algebra $\mathcal{A}$. In particular, geodesics were characterized. In [11] it was proved that these geodesics have minimal length, if the manifold of projections is endowed with the Finsler metric obtained by considering the usual norm of the algebra. For the special case of $\mathcal{A} = \mathcal{B}(\mathcal{H})$, more recently [3] (see also [2]), necessary and sufficient conditions were given for the existence of a geodesic (which additionally has minimal length for the Finsler metric considered) joining two given projections. Let us summarize this information in the following remark.

Remark 6.1.

1. $\mathcal{P}(\mathcal{A})$ is a complemented submanifold of $\mathcal{A}$. Its tangent space $(T\mathcal{P}(\mathcal{A}))_P$ at P is given by

$$\big(T\mathcal{P}(\mathcal{A})\big)_P = \big\{Y = iXP - iPX : X \in \mathcal{A}, X^* = X\big\},$$

which consists of selfadjoint operators Y which are co-diagonal with respect to P (i.e. $PYP = (1-P)Y(1-P) = 0$). Denote by $\mathcal{A}_h$ the space of selfadjoint elements of $\mathcal{A}$. A natural projection $E_P : \mathcal{A}_h \to (T\mathcal{P}(\mathcal{A}))_P$ is given by

$$E_P(X) = \text{co-diagonal part of } X = PX(1-P) + (1-P)XP.$$

This map induces a linear connection in $\mathcal{P}(\mathcal{A})$: if $X(t)$ is a tangent field along a curve $\gamma(t) \in \mathcal{P}$,

$$\frac{DX}{dt} = E_\gamma(X).$$

2. For any $P \in \mathcal{P}(\mathcal{A})$, the map

$$\pi_P : \mathcal{U}(\mathcal{A}) \to \mathcal{P}, \qquad \pi_P(U) = UPU^*,$$

($\mathcal{U}(\mathcal{A})$ = unitary group of $\mathcal{A}$) whose range is the unitary orbit of P in $\mathcal{P}(\mathcal{A})$, is a C^∞ submersion. In particular, it has C^∞ local cross sections.
3. If $P_0, P_1 \in \mathcal{P}(\mathcal{A})$ with $\|P_0 - P_1\| < 1$, there exists a unique selfadjoint element $Z \in \mathcal{A}$, with $\|Z\| < \pi/2$, which is co-diagonal with respect to P_0, such that

$$P_1 = e^{iZ} P_0 e^{-iZ}.$$

The curve $\delta(t) = e^{itZ} P_0 e^{-itZ}$ is the unique geodesic of $\mathcal{P}(\mathcal{A})$ joining P_0 and P_1 (up to reparametrization).
4. If one defines a Finsler metric in $\mathcal{P}(\mathcal{A})$, by endowing each tangent space with the usual norm in $\mathcal{A}$, then the above geodesic has minimal length, among all rectifiable curves in $\mathcal{P}(\mathcal{A})$ joining the same endpoints. We point that this Finsler metric is non-smooth, nor regular.
5. In the special case $\mathcal{A} = \mathcal{B}(\mathcal{H})$, if $\|P_1 - P_2\| = 1$, there exists a geodesic (equivalently, a minimal geodesic) if and only if

$$\dim\big(R(P_1) \cap N(P_2)\big) = \dim\big(R(P_2) \cap N(P_1)\big).$$

If this dimension is zero, the geodesic is unique (note that there may exist a unique geodesic even if $\|P_1 - P_2\| = 1$). If it is non-zero, there are infinitely many geodesics. Below we describe with more precision how these occur.

Let us recall the notion of Fredholm pairs of projections [4,1]:
Let P, Q be projections. The pair (P, Q) is a Fredholm pair if

$$QP|_{R(P)} : R(P) \to R(Q)$$

is a Fredholm operator. The index $i(P, Q)$ is defined as the index of this operator. The index coincides with the integer

$$i(P, Q) = \dim\big(R(P) \cap N(Q)\big) - \dim\big(R(Q) \cap N(P)\big).$$

In other words, the index of the pair can be interpreted as the obstruction for the existence of a geodesic joining the projections of the pair.

Note that if P lies in the restricted Grassmannian of $\mathcal{H}_+$, (P, E_+) is a Fredholm pair, with $i(P, E_+)$ equal to the index of P.

Let us transcribe from [4] and [1] other properties of this index, which will be useful.

Remark 6.2. Let (P, Q) be a Fredholm pair.

1. (Q, P) is also a Fredholm pair, with

$$i(Q, P) = -i(P, Q).$$

2. If U is a unitary operator, (UPU^*, UQU^*) is also a Fredholm pair, and

$$i\big(UPU^*, UQU^*\big) = i(P, Q).$$

3. If (Q, R) is another Fredhlom pair, and either $Q - R$ or $P - Q$ is compact, then (P, R) is a Fredholm pair and

$$i(P, R) = i(P, Q) + i(Q, R).$$

4. If P, Q are such that $P - Q$ is of trace class, then the index of the Fredholm pair (P, Q) equals

$$i(P, Q) = Tr(P - Q),$$

where Tr is the usual trace.

Let us consider now our main problem in this section, namely the existence of (minimal) geodesics of $\mathcal{P}_{cc}$ joining given projections. Clearly the projections must lie in the same component, so we proceed this analysis considering the different types of projections in $\mathcal{P}_{cc}$.

Remark 6.3. The symmetries $\perp$ and Ad_J are isometries for the Finsler structure and preserve geodesics. Ad_J is the restriction of an (inner) automorphism, so it is clearly isometric, and preserves algebraic data: Z is P co-diagonal if and only if $Ad_J(Z)$ is $Ad_J(P)$ co-diagonal, etc.

$$\perp : \mathcal{P}_{cc} \to \mathcal{P}_{cc}, \qquad \perp (P) = P^\perp = 1 - P$$

is an isometry, and preserves geodesics. Indeed, pick a smooth curve $P(t)$ of projections with $P(0) = P$ and $\dot{P}(0) = X$. Then $\frac{d}{dt}P^\perp(t)|_{t=0} = -X$, thus the differential of $\perp$ at P is

$$(d \perp)_P(X) = -X,$$

which is isometric. A geodesic $\delta(t) = e^{itZ}Pe^{itZ}$ (Z is selfadjoint and P-codiagonal) is transformed to $\delta^\perp(t) = 1 - e^{itZ}Pe^{itZ} = e^{itZ}P^\perp e^{itZ}$, which is a geodesic because P-codiagonal is the same as $P^\perp$-codiagonal.

Theorem 6.4. *Let $P_1, P_2 \in \mathbb{D}_1$ or $\mathbb{D}_2$, in the same connected component (i.e. with the same rank if they are of the first type, or the same corank if they are of the second type). Then there exists a minimal geodesic joining P_1 and P_2. It is unique if and only $R(P_1) \cap N(P_2) = R(P_2) \cap N(P_1) = \{0\}$.*

Proof. Consider first the case when $rank(P_1) = rank(P_2) = n < \infty$. Let $\mathcal{L}$ be the (finite dimensional) subspace generated by $R(P_1)$ and $R(P_2)$. Clearly $\mathcal{L}$ reduces both P_1 and P_2, and both projections act trivially in $\mathcal{L}^\perp$. Also it is clear that $Tr(P_i|_\mathcal{L}) = Tr(P_i) = n$, $i = 1, 2$. So that by the above remarks, there exists a $P_1|_\mathcal{L}$-codiagonal selfadjoint operator Z' acting in $\mathcal{L}$, with $\|Z'\| \le \pi/2$ such that

$$e^{iZ'}P_1|_\mathcal{L}e^{-iZ'} = P_2|_\mathcal{L}.$$

Consider $Z \in \mathcal{B}(\mathcal{H})$, $Z = Z' \oplus 0$ in $\mathcal{L} \oplus \mathcal{L}^\perp$. Z is selfadjoint, has finite rank (is compact), satisfies $\|Z\| = \|Z'\| \le \pi/2$ and

$$e^{iZ}P_1e^{-iZ} = P_2,$$

and is P_1-codiagonal, which means that P_1 and P_2 are joined by a minimal geodesic in $\mathcal{P}_{cc}$.

If P_1 and P_2 have finite (and equal) corank, by what was proved above, there exists a compact selfadjoint operator Z as above (which is $P_1^\perp$-codiagonal), such that $e^{iZ}P_1^\perp e^{-iZ} = P_2^\perp$. $P_1^\perp$-codiagonal is the same as P_1-codiagonal, and this equality trivially implies $e^{iZ}P_1e^{-iZ} = P_2$. □

In order to study the next types of projections, the following result will we useful.

Proposition 6.5. *Let $P_1, P_2 \in \mathcal{P}_{cc}$ such that $P_1 - P_2$ is compact and $i(P_1, P_2) = 0$. Then there exists a minimal geodesic of $\mathcal{P}_{cc}$ joining them.*

Proof. Also in this case we proceed by decomposing $\mathcal{H}$ in two orthogonal subspaces reducing P_1 and P_2. Denote by $A = P_1 - P_2$. Elementary computations (see [3] for instance) show that

$$\mathcal{H}_1 = N\big(A^2 - 1\big) = N(A - 1) \oplus N(A + 1) = \big(R(P_1) \cap N(P_2)\big) \oplus \big(R(P_2) \cap N(P_1)\big).$$

Put $\mathcal{H}_0 = \mathcal{H}_1^\perp$. Clearly these subspaces reduce P_1 and P_2. Since A is compact, $\mathcal{H}_1$ is finite dimensional.

Also $i(P_1, P_2) = 0$, and thus

$$\dim\big(R(P_1) \cap N(P_2)\big) = \dim\big(R(P_2) \cap N(P_1)\big).$$

This implies the existence of geodesics of $\mathcal{P}$ joining P_1 and P_2. They can be constructed explicitly. Pick any isometric isomorphism (between finite dimensional spaces)

$$V : R(P_1) \cap N(P_2) \to R(P_2) \cap N(P_1).$$

Put $U' : \mathcal{H}_1 \to \mathcal{H}_1$

$$U'(\xi, \eta) = \big(V^*\eta, -V\xi\big), \qquad (\xi, \eta) \in \big(R(P_1) \cap N(P_2)\big) \oplus \big(R(P_2) \cap N(P_1)\big) = \mathcal{H}_1.$$

Apparently

$$U' P_1|_{\mathcal{H}_1} U'^* = P_2|_{\mathcal{H}_1}.$$

Moreover, putting $Z' = -i\frac{\pi}{2}U'$, one sees that Z' is selfadjoint, P_1-codiagonal, $\|Z'\| = \pi/2$ and satisfies

$$e^{iZ'} = U'.$$

That is, the geodesic $\delta'(t) = e^{itZ'}P_{\mathcal{H}_1}e^{-itZ'}$ joins $P_1|_{\mathcal{H}_1}$ with $P_2|_{\mathcal{H}_1}$ (inside $\mathcal{H}_1$).

In $\mathcal{H}_0 = (N(A-1) \oplus N(A+1))^\perp$, since A is a compact contraction (and we are erasing the eigenspaces 1 and -1), one has that

$$\|P_1|_{\mathcal{H}_0} - P_2|_{\mathcal{H}_0}\| = \|A|_{\mathcal{H}_0}\| < 1.$$

By the above remarks, there exists a unique $Z_0 = Z_0^*$, which is $P_1|_{\mathcal{H}_0}$-codiagonal, $\|Z_0\| < \pi/2$, and

$$e^{iZ_0} P_1|_{\mathcal{H}_0} e^{-iZ_0} = P_2|_{\mathcal{H}_0}.$$

Let us prove that Z_0 is compact in $\mathcal{B}(\mathcal{H}_0)$. That Z_0 is $P_1|_{\mathcal{H}_0}$-codiagonal, implies that it anticommutes with the symmetry (selfadjoint unitary) $\epsilon_1 = 2P_1|_{\mathcal{H}_0} - 1_{\mathcal{H}_0}$. Denote by $\epsilon_2 = 2P_2|_{\mathcal{H}_0} - 1_{\mathcal{H}_0}$ the other symmetry. Thus the equation above implies

$$\epsilon_2 = e^{iZ_0}(2P_1|_{\mathcal{H}_0} - 1)e^{-iZ_0} = e^{iZ_0}\epsilon_1 e^{-iZ_0} = e^{2iZ_0}\epsilon_1.$$

Therefore $e^{2iZ_0} = \epsilon_2\epsilon_1$. Note that

$$\|\epsilon_2\epsilon_1 - 1\| = \|\epsilon_2 - \epsilon_1\| = 2\|P_1|_{\mathcal{H}_0} - P_2|_{\mathcal{H}_0}\| < 2,$$

and that

$$\epsilon_2\epsilon_1 - 1 = \epsilon_2(\epsilon_1 - \epsilon_2) = 2\epsilon_2(P_1|_{\mathcal{H}_0} - P_2|_{\mathcal{H}_0}) = 2\epsilon_2\big((P_1 - P_2)|_{\mathcal{H}_0}\big)$$

is a compact operator. It is a known fact (which can be proved using elementary spectral theory), that the exponential map $X \mapsto e^{iX}$ is a diffeomorphism between

$$\big\{X \in \mathcal{B}(\mathcal{H}_0) : X \text{ compact, } X^* = X, \|X\| < \pi\big\}$$

and

$$\big\{U \in \mathcal{U}(\mathcal{H}_0) : U - 1 \text{ is compact, and } \|U - 1\| < 2\big\}.$$

Note that $\|2Z_0\| < \pi$. It follows that Z_0 is compact.

Therefore, the operator $Z = Z_0 \oplus Z_1$ acting in $\mathcal{H}_0 \oplus \mathcal{H}_1 = \mathcal{H}$, is selfadjoint, compact, P_1-codiagonal, satisfies $\|Z\| \leq \pi/2$, and

$$e^{iZ} P_1 e^{-iZ} = P_2. \qquad \square$$

We analyze now the case of the restricted Grassmannians.

Theorem 6.6. *Let $P_1, P_2 \in \mathbb{D}_3$ or $\mathbb{D}_4$, in the same connected component (i.e. in the restricted Grassmannian of $\mathcal{H}_+$ or $\mathcal{H}_-$, and having the same index). Then there exists a minimal geodesic joining P_1 and P_2. It is unique if and only $R(P_1) \cap N(P_2) = R(P_2) \cap N(P_1) = \{0\}$.*

Proof. Let us suppose that P_1, P_2 lie in the restricted Grassmannian of $\mathcal{H}_+$ (the other case is similar or follows using the symmetry $\perp$). Note that, by Remark 4.5, $A = P_1 - P_2$ is compact. Moreover, by the properties of the index of pairs listed in the above remark,

$$i(P_1, E_+) = i(P_1, P_2) + i(P_2, E_+).$$

The fact that P_i lie in the same connected component implies that $i(P_1, E_+) = i(P_2, E_-)$. Thus $i(P_1, P_2) = 0$, and the proof follows using the above proposition. □

In other words, the Hopf–Rinow theorem is valid in the discrete classes $\mathbb{D}_i$ of $\mathcal{P}_{cc}$. We examine now the case of the essential projections.

Examples 6.7.

1. Let P, Q be the following projections in $\mathbb{E}_1$,

$$P = \begin{pmatrix} P_+ & 0 \\ 0 & 0 \end{pmatrix}, \qquad Q = \begin{pmatrix} Q_+ & 0 \\ 0 & 0 \end{pmatrix},$$

where P_+, Q_+ are projections of infinite rank and co-rank in $\mathcal{H}_+$. Apparently,

$$R(P) \cap N(Q) = R(P_+) \cap N(Q_+) \oplus 0 \quad \text{and} \quad R(Q) \cap N(P) = R(Q_+) \cap N(P_+) \oplus 0.$$

Thus, if $\dim(R(P_+) \cap N(Q_+)) \neq \dim(R(Q_+) \cap N(P_+))$, there cannot be a geodesic of $\mathcal{P}$, much less in $\mathcal{P}_{cc}$, joining P and Q. One can obtain easy examples of this situation, for instance if $R(Q_+) \subset R(P_+)$. Then

$$\dim\big(R(Q_+) \cap N(P_+)\big) = 0 \quad \text{and} \quad \dim\big(R(P_+) \cap N(Q_+)\big) = \dim\big(R(P_+) \ominus R(Q_+)\big),$$

and this can be any number in $\mathbb{N} \cup \{\infty\}$.
On the other hand if $\dim(R(P_+) \cap N(Q_+)) = \dim(R(Q_+) \cap N(P_+))$, by Remark 6.1, there exists a selfadjoint compact operator Z_+ in $\mathcal{H}_+$, $\|Z_+\| \leq \pi/2$, which is P_+-codiagonal and such that $e^{iZ_+} P_+ e^{-iZ_+} = Q_+$. Put $Z = Z_+ \oplus 0$. Then Z is selfadjoint, P-codiagonal, $\|Z\| = \|Z_+\| \leq \pi/2$ and

$$e^{iZ} P e^{-iZ} = Q,$$

i.e. P and Q are joined by a minimal geodesic (which is non-unique if $\dim(R(P_+) \cap N(Q_+)) \neq 0$).

2. A similar example can be constructed in the class $\mathbb{E}_5$. Put P and Q

$$P = \begin{pmatrix} P_+ & 0 \\ 0 & P_- \end{pmatrix}, \qquad Q = \begin{pmatrix} Q_+ & 0 \\ 0 & Q_- \end{pmatrix},$$

with $P_\pm, Q_\pm$ of infinite rank and corank. If $\dim(R(P_+) \cap N(Q_+)) = \dim(R(Q_+) \cap N(P_+))$ and $\dim(R(P_-) \cap N(Q_-)) = \dim(R(Q_-) \cap N(P_-))$ one can construct a diagonal geodesic $\delta(t) = e^{itZ} P e^{-itZ}$, with $Z = Z_+ \oplus Z_-$. Note that one can choose P and Q such that $\|P - Q\| = 1$. On the other hand, since

$$R(P) \cap N(Q) = \big(R(P_+) \cap N(Q_+)\big) \oplus \big(R(P_-) \cap N(Q_-)\big),$$

with a similar expression for $N(P) \cap R(Q)$, it follows that one can choose P and Q in order that

$$\dim\big(R(P) \cap N(Q)\big) = \dim\big(R(P_+) \cap N(Q_+)\big) + \dim\big(R(P_-) \cap N(Q_-)\big)$$

is different from $\dim(N(P) \cap R(Q))$. Therefore in this case there exists no geodesic joining P and Q in $\mathcal{P}_{cc}$.

3. One can further choose the pairs P_+, Q_+ and P_-, Q_- above to be in generic position. This implies that P and Q are in generic position, and therefore there exists a unique geodesic in $\mathcal{P}$ (which lies inside $\mathcal{P}_{cc}$) joining P and Q.

These examples imply the following:

Corollary 6.8. *The Hopf–Rinow Theorem is non-valid in $\mathbb{E}_i$, $1 \le i \le 5$. That is, there exist pairs of projections in $\mathbb{E}_i$ ($1 \le i \le 5$) which cannot be joined by a geodesic.*

Proof. The first example shows that this is the case for $\mathbb{E}_1$. By means of the symmetries $\perp$ and Ad_J, one obtains examples in $\mathbb{E}_i$ for $i = 2, 3, 4$. The second example gives the result for the class $\mathbb{E}_5$. □

Remark 6.9. One of the statements in the Hopf–Rinow Theorem for finite dimensional Riemnnian or Finsler manifolds asserts that completeness with the rectifiable metric implies that any pair of points can be joined by a minimal geodesic. It is not difficult to see that the essential classes are complete with the rectifiable metric. Recall that this metric is defined by

$$d_g(P, Q) = \inf\big\{\ell(\gamma) : \gamma \text{ joins } P \text{ and } Q \text{ in } \mathcal{P}_{cc}\big\},$$

where $\ell(\gamma)$ denotes the length of γ, that is, $\ell(\gamma) = \int_a^b \|\dot\gamma(t)\| dt$. Pick a Cauchy sequence (P_n) with respect to the metric d_g. Since straight lines are curves of minimal length in any vector space, it follows that $\|P_n - P_m\| \le d_g(P_n, P_m) \to 0$. Therefore there exists a projection $P \in \mathcal{P}_{cc}$ such that $\|P_n - P\| \to 0$. By the second point in Remark 6.1, there exist unitaries $U_n \in \mathcal{U}_{cc}$ such that $U_n P U_n^* = P_n$, whenever n is large enough. Moreover, $\|U_n - 1\| \to 0$, since the section mentioned in that remark is continuous. In an open ball centered at the origin, where the exponential map of the Banach–Lie group $\mathcal{U}_{cc}$ is a diffeomorphism, it holds that $U_n = e^{iZ_n}$, where Z_n are selfadjoint operators in $\mathcal{B}_{cc}$ and $\|Z_n\| \to 0$. Now consider the curves $\delta_n(t) = e^{itZ_n}Pe^{-itZ_n}$. Then note that

$$\|\dot{\delta_n}(t)\| = \|e^{itZ_n}Z_nPe^{-itZ_n} - e^{itZ_n}PZ_ne^{-itZ_n}\| \leq 2\|Z_nP\| \leq 2\|Z_n\|.$$

Thus, we find that

$$d_g(P_n, P) \leq \ell(\delta_n) \leq 2\|Z_n\| \to 0.$$

This proves that $\mathbb{E}_i$ is complete with the rectifiable distance.

6.1. Unique geodesics

In [10], Chandler Davis proved a result characterizing operators which are the difference of two projections. Let $A = A^*$, with $\|A\| \leq 1$. Consider

$$\mathcal{H}' = N\big(A^2 - 1\big)^{\perp},$$

and $A' = A|_{\mathcal{H}'}$.

Davis proved the following (Theorem 6 [10]):

A selfadjoint contraction A is the difference of two projections if and only if there exists a symmetry V in $\mathcal{H}'$ such that $VA' = -A'V$.

Moreover, he proved that to each symmetry V which anti-commutes with A', there corresponds a unique decomposition $A = P_V - Q_V$, with explicit formulas for P_V and Q_V. The symmetry V is given by

$$V = \big(1 - \big(A'\big)^2\big)^{-1/2}(P + Q - 1).$$

Note that $(1 - (A')^2)^{1/2}$ has trivial nullspace, thus its (eventually unbounded) inverse is defined.

Remark 6.10. This result was used in [3] to prove that if two projections P and Q are in generic position, that is $N(A) = N(A^2 - 1) = 0$, then there exists a unique geodesic in $\mathcal{P}$ joining them. Moreover, there is a relation between the exponential $e^{iZ}Pe^{-iZ} = Q$ (with $\|Z\| \leq \pi/2$) and the symmetry V (characterized by $P = P_V$, $Q = Q_V$). Namely: $V = e^{iZ}(2P - 1)$. Let us prove that this formula still holds if $N(A^2 - 1) = 0$, where $A = P - Q$, which is slightly weaker than asking that P and Q be in generic position. Indeed, note that if $A = P - Q$,

$$N(A) = \big(R(P) \cap R(Q)\big) \oplus \big(N(P) \cap N(Q)\big),$$

and in particular $N(A)$ reduces P and Q simultaneously. Also it is clear that Davis' symmetry $V = (1 - A^2)^{-1/2}(P + Q - 1)$ equals 1 in $R(P) \cap R(Q)$ and -1 in $N(P) \cap N(Q)$. On the other hand,

$$N\big(A^2-1\big)=\big(R(P)\cap N(Q)\big)\oplus\big(N(P)\cap R(Q)\big)=0,$$

means that there is a unique exponent Z, $Z^*=Z$, $\|Z\|\le\pi/2$, such that $e^{iZ}Pe^{-iZ}=Q$. If we denote by P_0 and Q_0 the restrictions of P and Q to their generic part, equal in this case ($N(A^2-1)=0$) to $\mathcal{H}_0=\mathcal{H}'\ominus N(A)$, by the same results on existence and uniqueness of geodesics (now in $\mathcal{H}_0$), there exists a unique Z_0, $Z_0^*=Z_0$, $\|Z_0\|\le\pi/2$, such that $e^{iZ_0}P_0e^{-iZ_0}=Q_0$. Note that P and Q coincide in $N(A)$, therefore the unitary operator $e^{i(Z_0\oplus 0)}$ satisfies also

$$e^{i(Z_0\oplus 0)}Pe^{-i(Z_0\oplus 0)}=Q.$$

Since $\|Z_0\oplus 0\|=\|Z_0\|\le\pi/2$, by uniqueness of geodesics joining P and Q, it follows that

$$Z=Z_0\oplus 0.$$

It follows that $e^{iZ}(2P-1)$ equals 1 in $R(P)\cap R(Q)$ and -1 in $N(P)\cap N(Q)$, which implies that the equality

$$e^{iZ}(2P-1)=V$$

holds in $\mathcal{H}$ (under the hypothesis $N(A^2-1)=0$).

Proposition 6.11. *Let $P,Q\in\mathcal{P}_{cc}$ such that $N(A^2-1)=\{0\}$ ($A=P-Q$). With the notations of the above remark, the following are equivalent:*

1. *$Z\in\mathcal{B}_{cc}$.*
2. *$V\in\mathcal{B}_{cc}$.*
3. *Let $B=P+Q-1$. The polar decomposition of $B=W|B|$ occurs in $\mathcal{B}_{cc}$. i.e. $W\in\mathcal{B}_{cc}$.*

Proof. If $Z\in\mathcal{B}_{cc}$, then apparently $V=e^{iZ}(2P-1)\in\mathcal{B}_{cc}$.

Suppose that $V\in\mathcal{B}_{cc}$. Let $A=P-Q$. Note that

$$(P+Q)^2+(P-Q)^2=2P+2Q,$$

so that $1-A^2=B^2$. Thus $(1-A^2)^{1/2}=|B|$. If $S=W|B|=|B|W$ is the polar decomposition, one has

$$W=|B|^{-1}S=\big(1-A^2\big)^{-1/2}(P+Q-1)=V\in\mathcal{B}_{cc}.\qquad\square$$

The following question arises. Suppose that $P,Q\in\mathcal{P}_{cc}$ satisfy that

$$R(P)\cap N(Q)=N(P)\cap R(Q)=0.$$

This means that there exists a unique minimal geodesic in $\mathcal{P}$ joining P and Q. Does this geodesic lie in $\mathcal{P}_{cc}$? Equivalently, does Z belong to $\mathcal{B}_{cc}$? The second part of the above example settles this question negatively.

If $\|P - Q\| < 1$, B above is invertible in $\mathcal{B}_{cc}$, and thus its polar decomposition occurs in $\mathcal{B}_{cc}$. The first part of the following example shows a case where B is non-invertible (in fact it is compact), but nevertheless its polar decomposition lies in $\mathcal{B}_{cc}$.

Examples 6.12.

1. Let P_n, Q_n, $1 \le n \le \infty$, be mutually orthogonal projections in $\mathcal{L}$ ($\mathcal{H} = \mathcal{L} \times \mathcal{L}$) of rank one such that $1_{\mathcal{L}} = \sum_n P_n + Q_n$. Let (α_n), (β_n) be strictly monotonous sequences such that $0 < \alpha_n < 1/2 < \beta_n < 1$, $\alpha_n \to 0$ and $\beta_n \to 1$. Put $\lambda_n = \sqrt{\alpha_n - \alpha_n^2}$ and $\mu_n = \sqrt{\beta_n - \beta_n^2}$. Consider

$$P = \begin{pmatrix} \sum_n \alpha_n P_n + \beta_n Q_n & \sum_n \lambda_n P_n + \mu_n Q_n \\ \sum_n \lambda_n P_n + \mu_n Q_n & \sum_n (1-\alpha_n) P_n + (1-\beta_n) Q_n \end{pmatrix}$$

and

$$Q = \begin{pmatrix} \sum_n P_n & 0 \\ 0 & \sum_n Q_n \end{pmatrix}.$$

Apparently, $P, Q \in \mathbb{E}_5$. Straightforward computations show that $B = P + Q - 1$ equals

$$B = \begin{pmatrix} \sum_n \alpha_n P_n + (\beta_n - 1) Q_n & \sum_n \lambda_n P_n + \mu_n Q_n \\ \sum_n \lambda_n P_n + \mu_n Q_n & \sum_n (-\alpha_n) P_n + (1-\beta_n) Q_n \end{pmatrix}.$$

Note that all four entries of B are compact in $\mathcal{L}$, and thus B is compact. Also a straightforward computation shows that $N(B) = \{0\}$. Since B is selfadjoint with trivial nullspace, the partial isometry V of the polar decomposition $B = V|B| = |B|V$, is in fact a symmetry. In order to prove that $V \in \mathcal{B}_{cc}$, it suffices to show that the orthogonal projection onto the spectral subspace corresponding to the positive part of B (the sum of the eigenspaces corresponding to positive eigenvalues) belongs to $\mathcal{B}_{cc}$. A simple computation shows that vectors $\xi \in \bigoplus_n R(P_n)$, $\eta \in \bigoplus_n R(Q_n)$, such that $(\xi, \eta) \ne 0$, verify that

$$\langle B(\xi,\eta), (\xi,\eta) \rangle > 0 \quad \text{and} \quad \langle B(\eta,\xi), (\eta,\xi) \rangle < 0.$$

It follows that the spectral subspace corresponding to the positive part of B is $\bigoplus_n R(P_n) \times \bigoplus_n R(Q_n)$. The projection onto this subspace is Q, which belongs to $\mathcal{B}_{cc}$.

2. Let P be as above, and

$$Q = \begin{pmatrix} \sum_n (1-\alpha_n)P_n + (1-\beta_n)Q_n & \sum_n \lambda_n P_n + \mu_n Q_n \\ \sum_n \lambda_n P_n + \mu_n Q_n & \sum_n \alpha_n P_n + \beta_n Q_n \end{pmatrix}.$$

Clearly $P, Q \in \mathbb{E}_5$. Apparently

$$B = P + Q - 1 = \begin{pmatrix} 0 & 2a \\ 2a & 0 \end{pmatrix},$$

where $a = \sum_n \lambda_n P_n + \mu_n Q_n \geq 0$. Then

$$|B| = \begin{pmatrix} 2a & 0 \\ 0 & 2a \end{pmatrix},$$

and the polar decomposition of B is

$$B = V|B| = \begin{pmatrix} 0 & 1_{\mathcal{L}} \\ 1_{\mathcal{L}} & 0 \end{pmatrix} \begin{pmatrix} 2a & 0 \\ 0 & 2a \end{pmatrix}, \quad \text{i.e. } V = \begin{pmatrix} 0 & 1_{\mathcal{L}} \\ 1_{\mathcal{L}} & 0 \end{pmatrix},$$

and thus V does not belong to $\mathcal{B}_{cc}$.

6.2. Geodesics to diagonal projections

Let $P \in \mathcal{P}_{cc}$. Recall from Section 5 that (if $\mathcal{H} = \mathcal{L} \times \mathcal{L}$, $\mathcal{H}_+ = \mathcal{L} \times 0$) P is of the form

$$P = \begin{pmatrix} x & a \\ a^* & y \end{pmatrix},$$

with

$$x = \sum_n \alpha_n P_n + \sum_m \beta_m Q_m + E_x, \qquad y = \sum_n (1-\alpha_n)P'_n + (1-\beta_m)Q'_m + E_y$$

and

$$a = \sum_k \lambda_k \xi_k \otimes \xi'_k + \sum_l \mu_l \eta_l \otimes \eta'_l,$$

where α_n, β_m are strictly monotonous (finite or infinite) sequences $1/2 > \alpha_n \to 0$, $1/2 \leq \beta_m \to 1$, E_x, E_y are the projections onto the eigenspaces of x and y corresponding to the eigenvalue 1, and N_x, N_y are the projections onto the nullspaces of x and y. Also recall the diagonal projection

$$P_d = \begin{pmatrix} \sum_{m \geq 1} Q_m + E_x & 0 \\ 0 & \sum_{n \geq 1} P'_n + E_y \end{pmatrix}.$$

Remark 6.13. P and P_d satisfy $\|P - P_d\| < 1$, and the isometry V corresponding to this pair belongs to $\mathcal{B}_{cc}$. Indeed, the fact that $B = P + P_d - 1$ is invertible, implies that also

$$B^2 = 1 - P - P_d + PP_d + P_dP = 1 - (P - P_d)^2 = 1 - A^2$$

is invertible. It follows that $\pm 1 \notin \sigma(A)$, since A is a contraction, this implies that

$$\|A\| = \|P - P_d\| < 1.$$

The following result is a direct consequence.

Corollary 6.14. *Let $P \in \mathcal{P}_{cc}$, and denote by $\mathcal{DP}$ be the set of diagonal projections in $\mathcal{P}$ (note that $\mathcal{DP} \subset \mathcal{P}_{cc}$). Then*

$$d(P, \mathcal{DP}) = \inf\big\{\|P - Q\| : Q \in \mathcal{DP}\big\} < 1.$$

Corollary 6.15. *Let $P \in \mathcal{P}_{cc}$. Then there exists an element $P_d \in \mathcal{DP}$ which can be joined to P with a minimal geodesic of $\mathcal{P}_{cc}$ of length strictly less than $\pi/2$. In particular,*

$$d_g(P, \mathcal{DP}) = \inf\big\{d_g(P, Q) : Q \in \mathcal{DP}\big\} < \pi/2.$$

Proof. Given $P \in \mathcal{P}_{cc}$, we have shown that $\|P - P_d\| < 1$. By Remark 6.1, there is a selfadjoint operator $Z \in \mathcal{B}_{cc}$ such that $P = e^{iZ} P_d e^{-iZ}$, Z is codiagonal with respect to P_d and $\|Z\| < \pi/2$. Moreover, the curve $\delta(t) = e^{itZ} P_d^{-itZ}$ is a geodesic joining P_d and P satisfying $\ell(\delta) = \|Z\| < \pi/2$. $\square$

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