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Optimal dual frames and frame completions for majorization ☆

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ABSTRACT

In this paper we consider two problems in frame theory. On the one hand, given a set of vectors $\mathcal{F}$ we describe the spectral and geometrical structure of optimal completions of $\mathcal{F}$ by a finite family of vectors with prescribed norms, where optimality is measured with respect to majorization. In particular, these optimal completions are the minimizers of a family of convex functionals that include the mean square error and the Benedetto–Fickus' frame potential. On the other hand, given a fixed frame $\mathcal{F}$ we describe explicitly the spectral and geometrical structure of optimal frames $\mathcal{G}$ that are in duality with $\mathcal{F}$ and such that the Frobenius norms of their analysis operators is bounded from below by a fixed constant. In this case, optimality is measured with respect to submajorization of the frames operators. Our approach relies on the description of the spectral and geometrical structure of matrices that minimize submajorization on sets that are naturally associated with the problems above.

1. Introduction

Finite frame theory is a well-established research field that has attracted the attention of many researchers (see [9,16,23] for general references to frame theory). On the one hand, finite frames provide redundant linear encoding–decoding schemes that are useful when dealing with transmission of signals through noisy channels. Indeed, the redundancy of frames allows for reconstruction of a signal, even when some frame coefficients are lost. Moreover, frames have also shown to be robust under erasures of the frame coefficients when a blind reconstruction strategy is considered (see [4,6,5,24,26,28,33]). On the other hand, there are several problems in frame theory that have deep relations with problems in other areas of mathematics (such as matrix analysis, operator theory and operator algebras) which constitute a strong motivation for research. For example, we can mention the relation between the Feichtinger conjecture in frame theory and some major open problems in operator algebra theory such as the Kadison–Singer problem (see [13,14]). Other examples of this phenomenon are the design problem in frame theory, the so-called Paulsen problem in frame theory and frame completion problems [1,8,10–12,17,18,20,21,25,29] which are known to be equivalent to different aspects of the Schur–Horn theorem. Recently, matrix analysis has served as a tool to show some structural properties of minimizers of the Benedetto–Fickus frame potential [2,15] and other convex functionals in the finite setting [30–32].

Following [1,17,29–32], in this paper we explore new connections of problems that arise naturally in frame theory with some results in matrix theory related with the notion of (sub)majorization between vectors and positive matrices. Indeed, one of the main problems in frame theory is the design of frames with some prescribed parameters and such

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that they are optimal in some sense. Optimal frames $\mathcal{F}$ are usually the minimizers of a tracial convex functional i.e., a functional of the form $P_f(\mathcal{F}) = \operatorname{tr}(f(S_{\mathcal{F}}))$ for some convex function $f(x)$, where $S_{\mathcal{F}}$ is the frame operator of $\mathcal{F}$. For example, we mention the Benedetto–Fickus' frame potential (i.e. $f(x) = x^2$) or the mean square error (i.e. $f(x) = x^{-1}$) or the negative of von Neumann's entropy (i.e. $f(x) = x\log(x)$). Thus, in many situations it is natural to ask whether the optimal frames corresponding to different convex potentials coincide: that is, whether optimality with respect to these potentials is an structural property. One powerful tool to deal with this type of problems is the notion of (sub)majorization between positive operators, because of its relation with tracial inequalities with respect to convex functions as above (see Section 2.3). Hence, a (sub)majorization based strategy can reveal structural properties of optimal frames. It is worth pointing out that (sub)majorization is not a total preorder and therefore the task of computing minimizers of this relation within a given set of positive operators – if such minimizers exist – is usually a non-trivial problem.

In this paper we consider the following two optimality problems in frame theory in terms of (sub)majorization (see Section 2 for the notation and terminology). Given a finite sequence of vectors $\mathcal{F}_0 \subseteq \mathcal{H} \cong \mathbb{C}^d$ and a finite sequence of positive numbers $\mathbf{b}$ we are interested in computing optimal frame completions of $\mathcal{F}_0$, denoted by $\mathcal{F}$, obtained by adding vectors with norms prescribed by the entries of $\mathbf{b}$ (see Section 3.1 for the motivation and a detailed description of this problem). In this context we show the existence of minimizers of majorization in the set of frame completions of $\mathcal{F}_0$ with prescribed norms, under certain hypothesis on $\mathbf{b}$; we also compute the spectral and geometrical structure of these optimal completions. Our results can be considered as a further step in the classical frame completion and frame design problems considered in [1,8,10,12,17,18,21,25]. In particular, we solve the frame completion problem recently posed in [21], where optimality is measured with respect to the mean square error of the completed frame.

On the other hand, given a fixed frame $\mathcal{F}$ for a finite-dimensional Hilbert space $\mathcal{H} \cong \mathbb{C}^d$, let $\mathcal{D}(\mathcal{F})$ denote the set of all frames $\mathcal{G}$ that are in duality with $\mathcal{F}$. It is well known that the canonical dual of $\mathcal{F}$, denoted $\mathcal{F}^{\#}$, has some optimality properties among the elements in $\mathcal{D}(\mathcal{F})$. Nevertheless, although optimal in some senses, there might be alternate duals that are more suitable for applications (see [7,22,26,28,33,34]). In order to search for optimal alternative duals for $\mathcal{F}$ we restrict attention the set $\mathcal{D}_t(\mathcal{F})$ which consists of frames $\mathcal{G}$ that are in duality with $\mathcal{F}$ and such that the Frobenius norm of their frame operators is bounded from below by a constant t. Therefore, in this paper we show the existence of minimizers of submajorization in $\mathcal{D}_t(\mathcal{F})$ and we explicitly describe their spectral and geometrical structure (see Section 3.2 for the motivation and a detailed description of this problem).

Both problems above are related with the minimizers of (sub)majorization in certain sets $\mathcal{S}$ of positive semidefinite matrices that arise naturally. We show that these sets $\mathcal{S}$ that we consider have minimal elements with respect to (sub)majorization, a fact that is of independent interest (see Theorems A.12, A.16 and 3.12). Notably, the existence of such minimizers is essentially obtained with insights coming from frame theory.

The paper is organized as follows: In Section 2 we establish the notation and terminology used throughout the paper, and we state some basic facts from frame theory and majorization theory. In Sections 3.1 and 3.2 we give a detailed description of the two main problems of frame theory mentioned above, including motivations, related results and specific notations. Section 3 ends with the definitions and statements of the matrix theory results of the paper, which give a unified matrix model for the frame problems; in order to avoid some technical aspects of these results, their proofs are presented in Appendix A. In Section 4 we apply the previous analysis of the matrix model to obtain the solutions of the frame problems, including algorithmic implementations and several examples. With respect to the problem of optimal completions, we obtain a complete description in several cases, that include the case of uniform norms for the added vectors. With respect to the problem of minimal duals, we completely describe their spectral and geometrical structure. Appendix A, contains the proofs of the matrix theory results of Section 3.3; it is divided in three subsections in which we develop the following steps: the characterization of the set of vectors of eigenvalues of elements in the matrix model, the description of the minimizers for submajorization in this set, and the description of the geometric structure of the matrices which are minimizers for submajorization in the matrix model.

2. Preliminaries

In this section we describe the basic notions that we shall consider throughout the paper. We first establish the general notations and then we recall the basic facts from frame theory that are related with our main results. Finally, we describe submajorization which is a notion from matrix analysis, that will play a major role in this note.

2.1. General notations

Given $m \in \mathbb{N}$ we denote by $\mathbb{I}_m = \{1, \ldots, m\} \subseteq \mathbb{N}$ and $\mathbb{1} = \mathbb{1}_m \in \mathbb{R}^m$ denotes the vector with all its entries equal to 1. For a vector $x \in \mathbb{R}^m$ we denote by $x^{\downarrow}$ the rearrangement of x in decreasing order, and $\mathbb{R}^{m\downarrow} = \{x \in \mathbb{R}^m\colon\ x = x^{\downarrow}\}$ the set of ordered vectors.

Given $\mathcal{H} \cong \mathbb{C}^d$ and $\mathcal{K} \cong \mathbb{C}^n$, we denote by $L(\mathcal{H}, \mathcal{K})$ the space of linear operators $T : \mathcal{H} \to \mathcal{K}$. Given an operator $T \in L(\mathcal{H}, \mathcal{K})$, $R(T) \subseteq \mathcal{K}$ denotes the image of T, $\ker T \subseteq \mathcal{H}$ the null space of T and $T^* \in L(\mathcal{K}, \mathcal{H})$ the adjoint of T. If $d \leqslant n$ we say that $U \in L(\mathcal{H}, \mathcal{K})$ is an isometry if $U^*U = I_{\mathcal{H}}$. In this case, U^* is called a coisometry. If $\mathcal{K} = \mathcal{H}$ we denote by $L(\mathcal{H}) = L(\mathcal{H}, \mathcal{H})$, by $\mathcal{G}l(\mathcal{H})$ the group of all invertible operators in $L(\mathcal{H})$, by $L(\mathcal{H})^+$ the cone of positive operators and by $\mathcal{G}l(\mathcal{H})^+ = \mathcal{G}l(\mathcal{H}) \cap L(\mathcal{H})^+$. If $T \in L(\mathcal{H})$, we denote by $\sigma(T)$ the spectrum of T, by $\operatorname{rk} T = \dim R(T)$ the rank of T, and by

$\operatorname{tr} T$ the trace of T. By fixing an orthonormal basis (onb) of the Hilbert spaces involved, we shall identify operators with matrices, using the following notations:

By $\mathcal{M}_{n,d}(\mathbb{C}) \cong L(\mathbb{C}^d, \mathbb{C}^n)$ we denote the space of complex $n \times d$ matrices. If $n = d$ we write $\mathcal{M}_n(\mathbb{C}) = \mathcal{M}_{n,n}(\mathbb{C})$. $\mathcal{H}(n)$ is the $\mathbb{R}$-subspace of self-adjoint matrices, $\mathcal{G}l(n)$ the group of all invertible elements of $\mathcal{M}_n(\mathbb{C})$, $\mathcal{U}(n)$ the group of unitary matrices, $\mathcal{M}_n(\mathbb{C})^+$ the set of positive semidefinite matrices, and $\mathcal{G}l(n)^+ = \mathcal{M}_n(\mathbb{C})^+ \cap \mathcal{G}l(n)$. If $d \leqslant n$, we denote by $\mathcal{I}(d,n) \subseteq \mathcal{M}_{n,d}(\mathbb{C})$ the set of isometries, i.e. those $U \in \mathcal{M}_{n,d}(\mathbb{C})$ such that $U^*U = I_d$. Given $S \in \mathcal{M}_n(\mathbb{C})^+$, we write $\lambda(S) \in \mathbb{R}_+^{n\downarrow}$ the vector of eigenvalues of S – counting multiplicities – arranged in decreasing order. If $\lambda(S) = \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}_+^{n\downarrow}$, a system $\{h_i\}_{i\in\mathbb{I}_n} \subseteq \mathbb{C}^n$ is a "ONB of eigenvectors for S, λ" if it is an orthonormal basis for $\mathbb{C}^n$ such that $Sh_i = \lambda_i h_i$ for every $i \in \mathbb{I}_n$.

If $W \subseteq \mathcal{H}$ is a subspace we denote by $P_W \in L(\mathcal{H})^+$ the orthogonal projection onto W, i.e. $R(P_W) = W$ and $\ker P_W = W^\perp$. Given $x, y \in \mathcal{H}$ we denote by $x \otimes y \in L(\mathcal{H})$ the rank one operator given by $x \otimes y(z) = \langle z, y\rangle x$ for every $z \in \mathcal{H}$. Note that if $\|x\| = 1$ then $x \otimes x = P_{\operatorname{span}\{x\}}$.

For vectors in $\mathbb{C}^n$ we shall use the Euclidean norm. On the other hand, for $T \in \mathcal{M}_{n,d}(\mathbb{C})$ we shall use both the spectral norm, denoted $\|T\|$, and the Frobenius norm, denoted $\|T\|_2$, given by

$$\|T\| = \max_{\|x\|=1} \|Tx\| \quad \text{and} \quad \|T\|_2 = \left(\operatorname{tr} T^*T\right)^{1/2} = \left(\sum_{i\in\mathbb{I}_n,\, j\in\mathbb{I}_d} |T_{ij}|^2\right)^{1/2}.$$

2.2. Basic framework of finite frames and their dual frames

In what follows we consider (n,d)-frames. See [2,9,16,23,30] for detailed expositions of several aspects of this notion.

Let $d, n \in \mathbb{N}$, with $d \leqslant n$. Fix a Hilbert space $\mathcal{H} \cong \mathbb{C}^d$. A family $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n} \in \mathcal{H}^n$ is an (n,d)-frame for $\mathcal{H}$ if there exist constants $A, B > 0$ such that

$$A\|x\|^2 \leqslant \sum_{i=1}^n \left|\langle x, f_i\rangle\right|^2 \leqslant B\|x\|^2 \quad \text{for every } x \in \mathcal{H}. \tag{1}$$

The **frame bounds**, denoted by $A_\mathcal{F}, B_\mathcal{F}$ are the optimal constants in (1). If $A_\mathcal{F} = B_\mathcal{F}$ we call $\mathcal{F}$ a tight frame. Since $\dim \mathcal{H} < \infty$, a family $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n}$ is an (n,d)-frame if and only if $\operatorname{span}\{f_i\colon i \in \mathbb{I}_n\} = \mathcal{H}$. We shall denote by $\mathbf{F} = \mathbf{F}(n,d)$ the set of all (n,d)-frames for $\mathcal{H}$.

Given $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n} \in \mathcal{H}^n$, the operator $T_\mathcal{F} \in L(\mathcal{H}, \mathbb{C}^n)$ defined by

$$T_\mathcal{F} x = \left(\langle x, f_i\rangle\right)_{i\in\mathbb{I}_n}, \quad \text{for every } x \in \mathcal{H} \tag{2}$$

is the **analysis** operator of $\mathcal{F}$. Its adjoint $T_\mathcal{F}^*$ is called the **synthesis** operator:

$$T_\mathcal{F}^* \in L\left(\mathbb{C}^n, \mathcal{H}\right) \quad \text{given by } T_\mathcal{F}^* v = \sum_{i\in\mathbb{I}_m} v_i f_i \quad \text{for every } v = (v_1, \ldots, v_n) \in \mathbb{C}^n.$$

Finally, we define the **frame operator** of $\mathcal{F}$ as $S_\mathcal{F} = T_\mathcal{F}^* T_\mathcal{F} = \sum_{i\in\mathbb{I}_n} f_i \otimes f_i \in L(\mathcal{H})^+$. Notice that, if $\mathcal{F} \in \mathbf{F}(n,d)$, then $\langle S_\mathcal{F} x, x\rangle = \sum_{i\in\mathbb{I}_n} |\langle x, f_i\rangle|^2$ for every $x \in \mathcal{H}$, so $S_\mathcal{F} \in \mathcal{G}l(\mathcal{H})^+$ and

$$A_\mathcal{F}\|x\|^2 \leqslant \langle S_\mathcal{F} x, x\rangle \leqslant B_\mathcal{F}\|x\|^2 \quad \text{for every } x \in \mathcal{H}. \tag{3}$$

In particular, $A_\mathcal{F} = \lambda_{\min}(S_\mathcal{F}) = \|S_\mathcal{F}^{-1}\|^{-1}$ and $\lambda_{\max}(S_\mathcal{F}) = \|S_\mathcal{F}\| = B_\mathcal{F}$. Moreover, $\mathcal{F}$ is tight if and only if $S_\mathcal{F} = \frac{\tau}{d} I_\mathcal{H}$, where $\tau = \operatorname{tr} S_\mathcal{F} = \sum_{i\in\mathbb{I}_n} \|f_i\|^2$.

The frame operator plays an important role in the reconstruction of a vector x using its frame coefficients $\{\langle x, f_i\rangle\}_{i\in\mathbb{I}_n}$. This leads to the definition of the canonical dual frame associated to $\mathcal{F}$: for every $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n} \in \mathbf{F}(n,d)$, the **canonical dual** frame associated to $\mathcal{F}$ is the sequence $\mathcal{F}^\# \in \mathbf{F}$ defined by

$$\mathcal{F}^\# \stackrel{\text{def}}{=} S_\mathcal{F}^{-1} \cdot \mathcal{F} = \left\{S_\mathcal{F}^{-1} f_i\right\}_{i\in\mathbb{I}_m} \in \mathbf{F}(n,d).$$

Therefore, we obtain the reconstruction formulas

$$x = \sum_{i\in\mathbb{I}_n} \langle x, f_i\rangle S_\mathcal{F}^{-1} f_i = \sum_{i\in\mathbb{I}_n} \left\langle x, S_\mathcal{F}^{-1} f_i\right\rangle f_i \quad \text{for every } x \in \mathcal{H}. \tag{4}$$

Observe that the canonical dual $\mathcal{F}^\#$ satisfies that given $x \in \mathcal{H}$, then

$$T_{\mathcal{F}^\#} x = \left(\left\langle x, S_\mathcal{F}^{-1} f_i\right\rangle\right)_{i\in\mathbb{I}_n} = \left(\left\langle S_\mathcal{F}^{-1} x, f_i\right\rangle\right)_{i\in\mathbb{I}_n} \quad \text{for } x \in \mathcal{H} \quad \Longrightarrow \quad T_{\mathcal{F}^\#} = T_\mathcal{F} S_\mathcal{F}^{-1}. \tag{5}$$

Hence $T_{\mathcal{F}^\#}^* T_\mathcal{F} = I_\mathcal{H}$ and $S_{\mathcal{F}^\#} = S_\mathcal{F}^{-1} T_\mathcal{F}^* T_\mathcal{F} S_\mathcal{F}^{-1} = S_\mathcal{F}^{-1}$.

In their seminal work [2], Benedetto and Fickus introduced a functional defined (on unit norm frames), the so-called frame potential, given by

$$FP(\{f_i\}_{i\in\mathbb{I}_n}) = \sum_{i,j\in\mathbb{I}_n} |\langle f_i, f_j\rangle|^2.$$

One of their major results shows that tight unit norm frames – which form an important class of frames because of their simple reconstruction formulas – can be characterized as (local) minimizers of this functional among unit norm frames. Since then, there has been interest in (local) minimizers of the frame potential within certain classes of frames, since such minimizers can be considered as natural substitutes of tight frames (see [15,30,31]). Notice that, given $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n} \in \mathcal{H}^n$ then $FP(\mathcal{F}) = \operatorname{tr}(S_{\mathcal{F}}^2) = \sum_{i\in\mathbb{I}_d} \lambda_i(S_{\mathcal{F}})^2$. These remarks have motivated the definition of general convex potentials as follows:

Definition 2.1. Let $f : [0,\infty) \to [0,\infty)$ be a convex function. Following [30] we consider the (generalized) frame potential associated to f, denoted P_f, given by

$$P_f(\mathcal{F}) = \operatorname{tr}(f(S_{\mathcal{F}})) \quad \text{for } \mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n} \in \mathcal{H}^n.$$

Of course, one of the most important generalized potential is the Benedetto–Fickus' (BF) frame potential. As shown in [30, Section 4] these convex functionals (which are related with the so-called entropic measures of frames) share many properties with the BF-frame potential. Indeed, under certain restrictions both the spectral and geometric structures of minimizers of these potentials coincide (see [30]).

2.3. Submajorization

Next we briefly describe submajorization, a notion from matrix analysis theory that will be used throughout the paper. For a detailed exposition of submajorization see [3].

Given $x, y \in \mathbb{R}^d$ we say that x is **submajorized** by y, and write $x \prec_w y$, if

$$\sum_{i=1}^k x_i^\downarrow \leqslant \sum_{i=1}^k y_i^\downarrow \quad \text{for every } k \in \mathbb{I}_d.$$

If $x \prec_w y$ and $\operatorname{tr} x = \sum_{i=1}^d x_i = \sum_{i=1}^d y_i = \operatorname{tr} y$, then we say that x is **majorized** by y, and write $x \prec y$.

On the other hand, we write $x \leqslant y$ if $x_i \leqslant y_i$ for every $i \in \mathbb{I}_d$. It is a standard exercise to show that $x \leqslant y \Rightarrow x^\downarrow \leqslant y^\downarrow \Rightarrow x \prec_w y$. Majorization is usually considered because of its relation with tracial inequalities for convex functions. Indeed, given $x, y \in \mathbb{R}^d$ and $f : I \to \mathbb{R}$ a convex function defined on an interval $I \subseteq \mathbb{R}$ such that $x, y \in I^d$, then (see for example [3]):

1. If one assumes that $x \prec y$, then $\operatorname{tr} f(x) \overset{\text{def}}{=} \sum_{i=1}^d f(x_i) \leqslant \sum_{i=1}^d f(y_i) = \operatorname{tr} f(y)$.
2. If only $x \prec_w y$, but the map f is also increasing, then still $\operatorname{tr} f(x) \leqslant \operatorname{tr} f(y)$.
3. If $x \prec_w y$ and f is an strictly convex function such that $\operatorname{tr}(f(x)) = \operatorname{tr}(f(y))$ then there exists a permutation σ of $\mathbb{I}_d$ such that $y_i = x_{\sigma(i)}$ for $i \in \mathbb{I}_d$.

The notion of submajorization can be extended to the context of self-adjoint matrices as follows: given $S_1, S_2 \in \mathcal{H}(d)$ we say that S_1 is **submajorized** by S_2, denoted $S_1 \prec_w S_2$, if $\lambda(S_1) \prec_w \lambda(S_2)$. If $S_1 \prec_w S_2$ and $\operatorname{tr}(S_1) = \operatorname{tr}(S_2)$ we say that S_1 is **majorized** by S_2 and write $S_1 \prec S_2$. Thus, $S_1 \prec S_2$ if and only if $\lambda(S_1) \prec \lambda(S_2)$. Notice that (sub)majorization is an spectral relation between self-adjoint operators.

3. Description and modeling of the main problems

We begin this section with a detailed description of our two main problems together with their motivations. In both cases we search for optimal frame designs (frame completions and duals), that are of potential interest in applied situations. In order to tackle these problems we obtain (see Sections 3.1 and 3.2) equivalent versions of them in a matrix analysis context. In Section 3.3 we present a unified matrix model and develop some notions and results that allow us to solve the two problems in frame theory (see Section 4).

3.1. Frame completions with prescribed norms

We begin by describing the following frame completion problem posed in [21]. Let $\mathcal{H} \cong \mathbb{C}^d$ and let $\mathcal{F}_0 = \{f_i\}_{i\in\mathbb{I}_{n_o}} \in \mathcal{H}^{n_o}$ be a fixed (finite) sequence of vectors. Let $n > n_o$ be an integer; denote by $k = n - n_o$ and assume that $\operatorname{rk} S_{\mathcal{F}_0} \geqslant d - k$. Consider a sequence $\mathbf{a} = \{\alpha_i\}_{i\in\mathbb{I}_n} \in \mathbb{R}_{>0}^n$ such that $\|f_i\|^2 = \alpha_i$ for every $i \in \mathbb{I}_{n_o}$.

With the fixed data from above, the problem posed in [21] is to find a sequence $\mathcal{F}_1 = \{f_i\}_{i=n_o+1}^n \in \mathcal{H}^k$ with $\|f_i\|^2 = \alpha_i$, for $n_o + 1 \leqslant i \leqslant n$, such that the mean square error of the resulting completed frame $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1) = \{f_i\}_{i\in\mathbb{I}_n} \in \mathbf{F}(n, d)$, namely $\operatorname{tr}(S_{\mathcal{F}}^{-1})$, is minimal among all possible such completions. It is worth pointing out that the mean square error of $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1)$ depends on $\mathcal{F}$ through the eigenvalues $\lambda(S_{\mathcal{F}})$ of its frame operator.

Note that there are other possible ways to measure robustness of the completed frame $\mathcal{F}$ as above. For example, we can consider optimal (minimizing) completions, with prescribed norms, for the Benedetto–Fickus' potential. In this case we search for a frame $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1) = \{f_i\}_{i\in\mathbb{I}_n} \in \mathbf{F}(n,d)$, with $\|f_i\|^2 = \alpha_i$ for $n_o + 1 \leqslant i \leqslant n$, and such that its frame potential $FP(\mathcal{F}) = \operatorname{tr}(S_{\mathcal{F}}^2)$ is minimal among all possible such completions. As before, we point out that the frame potential of the resulting completed frame $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1)$ depends on $\mathcal{F}$ through the eigenvalues $\lambda(S_{\mathcal{F}})$ of the frame operator of $\mathcal{F}$.

Hence, in order to solve both problems above we need to give a step further in the classical frame completion problem (i.e. decide whether $\mathcal{F}_0$ can be completed to a frame $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1)$ with prescribed norms and frame operator $S \in \mathcal{M}_d(\mathbb{C})^+$) and search for *optimal* (e.g. minimizers of the mean square error or Benedetto–Fickus' frame potential) frame completions with prescribed norms.

At this point a natural question arises as whether the minimizers corresponding to the mean square error and to the Benedetto–Fickus' potential, or even more general convex potentials, coincide (see [2,15,30]). As we shall see, the solutions of these problems are independent of the particular choice of convex potential considered. Indeed, we show that under certain hypothesis on the final sequence $\mathbf{b} = \{\alpha_i\}_{i=n_o+1}^{n}$ (which includes the uniform case) we can explicitly compute the completing sequences $\mathcal{F}_1 = \{f_i\}_{i=n_o+1}^{n} \in \mathcal{H}^k$ such that the frame operators of the completed sequences $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1)$ are minimal with respect to majorization (within the set of frame operators of all completions with norms prescribed by the sequence $\mathbf{a}$). In order to do this, we begin by fixing some notations.

Definition 3.1. Let $\mathcal{F}_0 = \{f_i\}_{i\in\mathbb{I}_{n_o}} \in \mathcal{H}^{n_o}$ and $\mathbf{a} = \{\alpha_i\}_{i\in\mathbb{I}_n} \in \mathbb{R}_{>0}^n$ such that $d - \operatorname{rk} S_{\mathcal{F}_0} \leqslant n - n_o$ and $\|f_i\|^2 = \alpha_i$, $i \in \mathbb{I}_{n_o}$. We consider the sets

$$\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0) = \big\{\{f_i\}_{i\in\mathbb{I}_n} \in \mathbf{F}(n,d)\colon \{f_i\}_{i\in\mathbb{I}_{n_o}} = \mathcal{F}_0 \text{ and } \|f_i\|^2 = \alpha_i \text{ for } i \geqslant n_o + 1\big\},$$
$$\mathcal{SC}_{\mathbf{a}}(\mathcal{F}_0) = \big\{S_{\mathcal{F}}\colon \mathcal{F} \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)\big\}.$$

In what follows we shall need the following solution of the classical frame completion problem.

Proposition 3.2. *(See [1,29].) Let $B \in \mathcal{M}_d(\mathbb{C})^+$ with $\lambda(B) \in \mathbb{R}_+^{d\,w\downarrow}$ and let $\mathbf{b} = (\beta_i)_{i\in\mathbb{I}_k} \in \mathbb{R}_{>0}^k$. Then there exists a sequence $\mathcal{G} = \{g_i\}_{i\in\mathbb{I}_k} \in \mathcal{H}^k$ with frame operator $S_{\mathcal{G}} = B$ and such that $\|g_i\|^2 = \beta_i$ for every $i \in \mathbb{I}_k$ if and only if $\mathbf{b} \prec \lambda(B)$ (completing with zeros if $k \neq d$).* □

Since our criteria for optimality of frame completions will be based on majorization, our analysis of the completed frame $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1)$ will depend on $\mathcal{F}$ through $S_{\mathcal{F}}$. Hence, the following description of $\mathcal{SC}_{\mathbf{a}}(\mathcal{F}_0)$ plays a central role in our approach.

Proposition 3.3. *Let $\mathcal{F}_0 = \{f_i\}_{i\in\mathbb{I}_{n_o}} \in \mathcal{H}^{n_o}$ and $\mathbf{a} = \{\alpha_i\}_{i\in\mathbb{I}_n} \in \mathbb{R}_{>0}^n$ such that $\|f_i\|^2 = \alpha_i$, $i \in \mathbb{I}_{n_o}$. Then, we have that*

$$\mathcal{SC}_{\mathbf{a}}(\mathcal{F}_0) = \big\{S \in \mathcal{G}l(d)^+\colon\ S \geqslant S_0 \text{ and } (\alpha_i)_{i=n_o+1}^{n} \prec \lambda(S - S_0)\big\}.$$

In particular, if we let $k = n - n_o$ then we get the inclusion

$$\mathcal{SC}_{\mathbf{a}}(\mathcal{F}_0) \subseteq \left\{S_{\mathcal{F}_0} + B\colon\ B \in \mathcal{M}_d(\mathbb{C})^+,\ \operatorname{rk} B \leqslant k,\ \operatorname{tr}(S_{\mathcal{F}_0} + B) = \sum_{i=1}^{n} \alpha_i\right\}. \tag{6}$$

Proof. Observe that if $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1) \in \mathbf{F}(n,d)$, then $S_{\mathcal{F}} = S_{\mathcal{F}_0} + S_{\mathcal{F}_1}$. Denote by $S_0 = S_{\mathcal{F}_0}$ and $B = S - S_0$, for any $S \in \mathcal{G}l(d)^+$. Applying Proposition 3.2 to the matrix B (which must be non-negative if $S \in \mathcal{SC}_{\mathbf{a}}(\mathcal{F}_0)$), we get the first equality.

The inclusion in Eq. (6) follows using that, if $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1) \in \mathbf{F}(n,d)$, then $\operatorname{rk} B = \operatorname{rk} S_{\mathcal{F}_1} \leqslant k = d - (d-k)$. On the other hand, recall that $\operatorname{tr}(S_{\mathcal{F}}) = \sum_{i=1}^{n} \|f_i\|^2$. □

3.2. Dual frames of a fixed frame with tracial restrictions

Let $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n} \in \mathbf{F}(n,d)$. Then $\mathcal{F}$ induces and encoding–decoding scheme as described in Eq. (4), in terms of the canonical dual $\mathcal{F}^{\#}$. But, in case that $\mathcal{F}$ has nonzero redundancy then we get a family of reconstruction formulas in terms of different frames that play the role of the canonical dual. In what follows we say that $\mathcal{G} = \{g_i\}_{i\in\mathbb{I}_n} \in \mathbf{F}(n,d)$ is a **dual** frame for $\mathcal{F}$ if $T_{\mathcal{G}}^* T_{\mathcal{F}} = I_{\mathcal{H}}$ (and hence $T_{\mathcal{F}}^* T_{\mathcal{G}} = I_{\mathcal{H}}$), or equivalently if the following reconstruction formulas hold:

$$x = \sum_{i\in\mathbb{I}_n} \langle x, f_i\rangle g_i = \sum_{i\in\mathbb{I}_n} \langle x, g_i\rangle f_i \quad \text{for every } x \in \mathcal{H}.$$

We denote by

$$\mathcal{D}(\mathcal{F}) \stackrel{\text{def}}{=} \{\mathcal{G} \in \mathbf{F}(n,d)\colon T_{\mathcal{G}}^* T_{\mathcal{F}} = I_{\mathcal{H}}\}$$

the set of all dual frames for $\mathcal{F}$. Observe that $\mathcal{D}(\mathcal{F}) \neq \varnothing$ since $\mathcal{F}^\# \in \mathcal{D}(\mathcal{F})$.

Notice that the fact that $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n} \in \mathbf{F}(n,d)$ implies that $T_{\mathcal{F}}^*$ is surjective. In this case, a sequence $\mathcal{G} \in \mathcal{D}(\mathcal{F})$ if and only if its synthesis operator $T_{\mathcal{G}}^*$ is a pseudo-inverse of $T_{\mathcal{F}}$. Moreover, the synthesis operator $T_{\mathcal{F}^\#}^*$ of the canonical dual $\mathcal{F}^\#$ corresponds to the Moore–Penrose pseudo-inverse of $T_{\mathcal{F}}$. Indeed, notice that $T_{\mathcal{F}} T_{\mathcal{F}^\#}^* = T_{\mathcal{F}} S_{\mathcal{F}}^{-1} T_{\mathcal{F}}^* \in L(\mathbb{C}^n)^+$, so that it is an orthogonal projection. From this point of view, the canonical dual $\mathcal{F}^\#$ has some optimal properties that come from the theory of pseudo-inverses. Nevertheless, the canonical dual frame might not be the optimal choice for a dual frame from an applied point of view. For example, it is well known that there are classes of structured frames that admit alternate duals that share this structure but for which their canonical duals are not structured [7,34]; in the theory of signal transmission through noisy channels, it is well known that there are alternate duals that perform better than $\mathcal{F}^\#$ [26,28,33] when we assume that the frame coefficients can be corrupted by the noise in the channel. There are other cases in which $\mathcal{F}^\#$ may be ill-conditioned or simply too difficult to compute: for example, it is known (see [22]) that under certain hypothesis we can find Parseval dual frames $\mathcal{G} \in \mathcal{D}(\mathcal{F})$ (i.e. such that $S_{\mathcal{G}} = I_{\mathcal{H}}$), which lead to more stable reconstruction formulas for vectors in $\mathcal{H}$.

In the general case, we can measure the stability of the reconstruction formula induced by a dual frame $\mathcal{G} \in \mathcal{D}(\mathcal{F})$ in terms of the spread of the eigenvalues of the frame operator $S_{\mathcal{G}}$; this can be seen if we consider, as it is usual in applied situations, the condition number of $S_{\mathcal{G}}$ as a measure of stability of linear processes that depend on $S_{\mathcal{G}}$. There are finer measures of the dispersion which take into account all the eigenvalues of $S_{\mathcal{G}}$, if one restricts to the case of fixed trace. As an example of such a measure we can mention the Benedetto–Fickus' potential. Our approach based on majorization – which is the structural measure of the spread of eigenvalues for matrices with a fixed trace – allows us to show that minimizers with respect to a large class of convex potentials coincide. The main advantages of considering the partition of $\mathcal{D}(\mathcal{F})$ into slices determined by the trace condition $\operatorname{tr}(S_{\mathcal{G}}) = t$ are:

- There exists a unique vector $\nu(t)$ of eigenvalues which is minimal for majorization among the vectors $\lambda(S_{\mathcal{G}})$, for dual frames $\mathcal{G} \in \mathcal{D}(\mathcal{F})$ with $\operatorname{tr} S_{\mathcal{G}} = t$.
- Moreover, the vector $\nu(t)$ is also submajorized by the vectors $\lambda(S_{\mathcal{G}})$ for every $\mathcal{G} \in \mathcal{D}(\mathcal{F})$ with $\operatorname{tr} S_{\mathcal{G}} \geqslant t$.
- The map $t \mapsto \nu(t)$ is increasing (in each entry) and continuous.
- Continuous sections $t \mapsto \mathcal{G}_t \in \mathcal{D}(\mathcal{F})$ such that $\lambda(\mathcal{G}_t) = \nu(t)$ can be computed.
- In addition, the condition number of $\nu(t)$ decreases when t grows until a critical point (which is easy to compute).

We point out that both the vector $\nu(t)$ and the duals $\mathcal{G}_t$ can be computed explicitly in terms of implementable algorithms. In order to obtain a convenient formulation of the problem we consider the following notions and simple facts.

Definition 3.4. Let $\mathcal{F} \in \mathbf{F}(n,d)$. We denote by

$$\mathcal{SD}(\mathcal{F}) = \{S_{\mathcal{G}}\colon \mathcal{G} \in \mathcal{D}(\mathcal{F})\}$$

the set of frame operators of all dual frames for $\mathcal{F}$.

Proposition 3.5. *Let $\mathcal{F} \in \mathbf{F}(n,d)$. Then*

$$\mathcal{SD}(\mathcal{F}) = \{S_{\mathcal{F}^\#} + B\colon B \in \mathcal{M}_d(\mathbb{C})^+ \text{ and } \operatorname{rk} B \leqslant n-d\}. \tag{7}$$

Proof. Given $\mathcal{G} \in \mathbf{F}(n,d)$, then $\mathcal{G} \in \mathcal{D}(\mathcal{F}) \Leftrightarrow Z = T_{\mathcal{G}} - T_{\mathcal{F}^\#} \in L(\mathcal{H}, \mathbb{C}^n)$ satisfies $Z^* T_{\mathcal{F}} = 0$. In this case, by Eq. (5), we know that $T_{\mathcal{F}^\#} = T_{\mathcal{F}} S_{\mathcal{F}}^{-1} \Rightarrow Z^* T_{\mathcal{F}^\#} = 0$, and

$$S_{\mathcal{G}} = (T_{\mathcal{F}^\#} + Z)^*(T_{\mathcal{F}^\#} + Z) = S_{\mathcal{F}^\#} + B = S_{\mathcal{F}}^{-1} + B, \quad \text{where } B = Z^*Z \in \mathcal{M}_d(\mathbb{C})^+.$$

Moreover, $S_{\mathcal{F}} = T_{\mathcal{F}}^* T_{\mathcal{F}} \in \mathcal{G}l(d)^+ \Rightarrow \operatorname{rk} T_{\mathcal{F}} = d$, and the equation $T_{\mathcal{F}}^* Z = 0$ implies that

$$R(Z) \subseteq \ker T_{\mathcal{F}}^* = R(T_{\mathcal{F}})^\perp \quad \Longrightarrow \quad \operatorname{rk} B = \operatorname{rk}(Z^*Z) = \operatorname{rk} Z \leqslant n-d.$$

Since any $B \in \mathcal{M}_d(\mathbb{C})^+$ with $\operatorname{rk} B \leqslant n-d$ can be represented as $B = Z^*Z$ for some $Z \in L(\mathcal{H}, R(T_{\mathcal{F}})^\perp)$, we have proved Eq. (7). □

Fix a system $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n} \in \mathbf{F}(n,d)$. Notice that Proposition 3.5 shows that if $\mathcal{G} \in \mathcal{D}(\mathcal{F})$ then $S_{\mathcal{F}^\#} \leqslant S_{\mathcal{G}}$, which is a strong minimality property of the frame operator of the canonical dual $\mathcal{F}^\#$. As we said before, we are interested in considering alternate duals that are more stable than $\mathcal{F}^\#$. In order to do this, we consider the set $\mathcal{D}_t(\mathcal{F})$ of dual frames $\mathcal{G} \in \mathcal{D}(\mathcal{F})$ with a further restriction, namely that $\operatorname{tr}(S_{\mathcal{G}}) \geqslant t$ for some $t \geqslant \operatorname{tr}(S_{\mathcal{F}^\#})$. Therefore, the problem we focus in is to find dual frames $\mathcal{G}_t \in \mathcal{D}_t(\mathcal{F})$ such that their frame operators $S_{\mathcal{G}_t}$ are minimal with respect to submajorization within the set

$$\mathcal{SD}_t(\mathcal{F}) \stackrel{\text{def}}{=} \{S_{\mathcal{G}}:\ \mathcal{G} \in \mathcal{D}_t(\mathcal{F})\}. \tag{8}$$

Notice that as an immediate consequence of Proposition 3.5 we get the identity

$$\mathcal{SD}_t(\mathcal{F}) = \{S_{\mathcal{F}^\#} + B:\ B \in \mathcal{M}_d(\mathbb{C})^+,\ \operatorname{rk} B \leqslant n-d,\ \operatorname{tr}(S_{\mathcal{F}^\#} + B) \geqslant t\}. \tag{9}$$

As we shall see, these optimal duals $\mathcal{G}_t$ decrease the condition number and, in some cases are even tight frames. Moreover, because of the relation between submajorization and increasing convex functions, our optimal dual frames $\mathcal{G}_t \in \mathcal{D}_t(\mathcal{F})$ are also minimizers of a family of convex frame potentials (see Definition 2.1 below) that include the Benedetto–Fickus' frame potential.

3.3. A unified matrix model for the frame problems and submajorization

In this section we introduce and develop some aspects of a set $U_t(S_0, m) \subseteq \mathcal{M}_d(\mathbb{C})^+$ that will play an essential role in our approach to the frame problems described above (see Remark 3.7). Our main results related with $U_t(S_0, m)$ are Theorem 3.12 and Proposition 3.14. In order to avoid some technicalities, we postpone their proofs to Appendix A.

Definition 3.6. Let $S_0 \in \mathcal{M}_d(\mathbb{C})^+$ with $\lambda(S_0) = \lambda \in \mathbb{R}_+^{d\downarrow}$, $t_0 = \operatorname{tr} S_0$, and $t \geqslant t_0$. For any integer $m < d$ we consider the following subset of $\mathcal{M}_d(\mathbb{C})^+$:

$$U_t(S_0, m) = \{S_0 + B:\ B \in \mathcal{M}_d(\mathbb{C})^+,\ \operatorname{rk} B \leqslant d-m,\ \operatorname{tr}(S_0 + B) \geqslant t\}. \tag{10}$$

Observe that if $m \leqslant 0$ then $U_t(S_0, m) = \{S \in \mathcal{M}_d(\mathbb{C})^+:\ S \geqslant S_0, \operatorname{tr}(S) \geqslant t\}$. △

Remark 3.7. As a consequence of Eqs. (6) and (9) we see that the two main problems are intimately related with the structure of the set $U_t(S_0, m)$ for suitable choices of the parameters $S_0 \in \mathcal{M}_d(\mathbb{C})^+$, $m < d$ and $t \geqslant \operatorname{tr} S_0$:

1. Note that Eq. (6) shows that $\mathcal{SC}_{\mathbf{a}}(\mathcal{F}_0) \subseteq U_t(S_{\mathcal{F}_0}, m)$, where $t = \operatorname{tr} \mathbf{a}$ and $m = d - n + n_0$.
2. Similarly, Eq. (9) shows that identity $\mathcal{SD}_t(\mathcal{F}) = U_t(S_{\mathcal{F}^\#}, m)$ where $m = 2d - n$.

Remark 3.8. Given $\lambda = \lambda(S_0) \in \mathbb{R}_+^{d\downarrow}$ and $m < d$ we look for a $\prec_w$-minimizer on the set

$$\Lambda(U_t(S_0, m)) \stackrel{\text{def}}{=} \{\lambda(S):\ S \in U_t(S_0, m)\} \subseteq \mathbb{R}_+^{d\downarrow}. \tag{11}$$

Heuristic computations suggest that in some cases such a minimizer should have the form

$$\nu = (\lambda_1, \dots, \lambda_r, c, \dots, c) \in \mathbb{R}_{>0}^{d\downarrow} \quad \text{with } \operatorname{tr} \nu = t \text{ for some } r \in \mathbb{I}_{d-1} \text{and } c \in \mathbb{R}_{>0}.$$

Observe that if $\nu \in \Lambda(U_t(S_0, m))$ then $\lambda \leqslant \nu = \nu^\downarrow$. Hence we need that

$$c = \frac{t - \sum_{j=1}^r \lambda_j}{d-r} \quad \text{and} \quad \lambda_{r+1} \leqslant c \leqslant \lambda_r.$$

These restrictions on the numbers r and c suggest the following definitions:

Definition 3.9. Let $\lambda \in \mathbb{R}_+^{d\downarrow}$ and $t \in \mathbb{R}$ such that $\operatorname{tr} \lambda \leqslant t < d\lambda_1$. Consider the set

$$A_\lambda(t) \stackrel{\text{def}}{=} \left\{r \in \mathbb{I}_{d-1}:\ p_\lambda(r, t) \stackrel{\text{def}}{=} \frac{t - \sum_{j=1}^r \lambda_j}{d-r} \geqslant \lambda_{r+1}\right\}.$$

Observe that $t \geqslant \operatorname{tr} \lambda \Longrightarrow t - \sum_{j=1}^{d-1} \lambda_j \geqslant \lambda_d$, so that $d-1 \in A_\lambda(t) \neq \varnothing$. The t-irregularity of the ordered vector λ, denoted $r_\lambda(t)$, is defined by

$$r_\lambda(t) \stackrel{\text{def}}{=} \min A_\lambda(t) = \min\{r \in \mathbb{I}_{d-1}:\ p_\lambda(r, t) \geqslant \lambda_{r+1}\}. \tag{12}$$

If $t \geqslant d\lambda_1$, we set $r_\lambda(t) \stackrel{\text{def}}{=} 0$ and $p_\lambda(0, t) = t/d$.

For example, if $t_0 = \operatorname{tr}\lambda$ then for every $r \in \mathbb{I}_{d-1}$ we have that

$$p_\lambda(r, t_0) = \frac{t_0 - \sum_{j=1}^{r} \lambda_j}{d-r} = \frac{\sum_{j=r+1}^{d} \lambda_j}{d-r} \geqslant \lambda_{r+1} \quad \Longleftrightarrow \quad \lambda_{r+1} = \lambda_d.$$

Therefore in this case

- If $\lambda = c\mathbb{1}_d$ for some $c \in \mathbb{R}_{>0}$, then $r_\lambda(t_0) = 0$.
- If $\lambda_1 > \lambda_d$, then

$$r_\lambda(t_0) + 1 = \min\{i \in \mathbb{I}_d \colon \lambda_i = \lambda_d\} \quad \text{and} \quad r_\lambda(t_0) = \max\{r \in \mathbb{I}_{d-1} \colon \lambda_r > \lambda_d\}. \tag{13}$$

Definition 3.10. Let $\lambda \in \mathbb{R}_+^{d\downarrow}$ and $t_0 = \operatorname{tr}\lambda$. We define the functions

$$r_\lambda : [t_0, +\infty) \to \{0, \dots, d-1\} \quad \text{given by } r_\lambda(s) \overset{(12)}{=} \text{the } s\text{-irregularity of } \lambda, \tag{14}$$

$$c_\lambda : [t_0, +\infty) \to \mathbb{R}_{\geqslant 0} \quad \text{given by } c_\lambda(s) = p_\lambda\big(r_\lambda(s), s\big) = \frac{s - \sum_{i=1}^{r_\lambda(s)} \lambda_i}{d - r_\lambda(s)}, \tag{15}$$

for every $s \in [t_0, +\infty)$, where we set $\sum_{i=1}^{0} \lambda_i = 0$.

Fix $\lambda \in \mathbb{R}_+^{d\downarrow}$. As we shall show in Lemma A.8, the vector $\nu = (\lambda_1, \dots, \lambda_{r_\lambda(t)}, c_\lambda(t)\mathbb{1}_{d-r_\lambda(t)}) \in \mathbb{R}_{>0}^{d\downarrow}$ for every $t \geqslant t_0$, and the map c_λ is piece-wise linear, strictly increasing and continuous. This last claim allows us to introduce the following parameter: given $m \in \mathbb{I}_{d-1}$ we denote by

$$s^* = s^*(\lambda, m) \overset{\text{def}}{=} c_\lambda^{-1}(\lambda_m) = \sum_{i=1}^{m} \lambda_i + (d-m)\lambda_m \tag{16}$$

that is, the unique $s \in [t_0, +\infty)$ such that $c_\lambda(s) = \lambda_m$. These facts and other results of Appendix A give consistency to the following definitions:

Definition 3.11. Let $\lambda \in \mathbb{R}_+^{d\downarrow}$, $t_0 = \operatorname{tr}\lambda$. Take an integer $m < d$. If $m > 0$ and $t \in [t_0, +\infty)$ let

$$c_{\lambda,m}(t) \overset{\text{def}}{=} \begin{cases} c_\lambda(t) & \text{if } t \leqslant s^* \\ \lambda_m + \frac{t-s^*}{d-m} & \text{if } t > s^* \end{cases} \quad \text{and}$$

$$r_{\lambda,m}(t) \overset{\text{def}}{=} \min\big\{r \in \mathbb{I}_{d-1} \cup \{0\} \colon c_{\lambda,m}(t) \geqslant \lambda_{r+1}\big\}.$$

If $m \leqslant 0$ and $t \in [t_0, +\infty)$ we define $c_{\lambda,m}(t) = c_\lambda(t)$ and $r_{\lambda,m}(t) = r_\lambda(t)$.

Note that, by Eq. (31) of Lemma A.8, $r_{\lambda,m}(t) = r_\lambda(t)$ for every $t \leqslant s^*$. The following results will be used throughout Section 4; see Appendix A for their proofs.

Theorem 3.12. *Let $S_0 \in \mathcal{M}_d(\mathbb{C})^+$ with $\lambda = \lambda(S_0)$ and $m < d$ be an integer. For $t \geqslant \operatorname{tr} S_0$, let us denote by $r' = \max\{r_{\lambda,m}(t), m\}$ and $c = c_{\lambda,m}(t)$. Then, there exists $\nu \in \Lambda(U_t(S_0, m))$ such that*

1. *The vector ν is $\prec_w$-minimal in $\Lambda(U_t(S_0, m))$, i.e. $\nu \prec_w \mu$ for every $\mu \in \Lambda(U_t(S_0, m))$.*
2. *For every matrix $S \in U_t(S_0, m)$ the following conditions are equivalent:*
   (a) *$\lambda(S) = \nu$ (i.e. S is $\prec_w$-minimal in $U_t(S_0, m)$).*
   2.1. *There exists $\{v_i\}_{i\in\mathbb{I}_d}$, an ONB of eigenvectors for S_0, λ such that*

$$B = S - S_0 = \sum_{i=1}^{d-r'} (c - \lambda_{r'+i}) v_{r'+i} \otimes v_{r'+i}. \tag{17}$$

3. *If we further assume any of the following conditions:*
   - *$m \leqslant 0$,*
   - *$m \geqslant 1$ and $\lambda_m > \lambda_{m+1}$, or*
   - *$m \geqslant 1$ and $\lambda_m = \lambda_{m+1}$ but $t \leqslant s^*(\lambda, m)$ (see Eq. 16),*

   then B and S are unique. Moreover, in these cases Eq. (17) holds for any ONB of eigenvectors of S_0 as above. □

Remark 3.13. Suppose that $m \geqslant 1$. In this case the map $c_{\lambda,m}(\cdot)$ is continuous and strictly increasing. Indeed, by Lemma A.8 we know that $s^* = \sum_{i=1}^m \lambda_i + (d-m)\lambda_m$. Hence

$$c_{\lambda,m}(t) = \lambda_m + \frac{t-s^*}{d-m} = \frac{t - \sum_{j=1}^m \lambda_j}{d-m} \quad \text{for every } t > s^*. \tag{18}$$

The fact that the map c_λ is continuous and strictly increasing will be also proved in Lemma A.8. Let us abbreviate by $r = r_{\lambda,m}(t)$ for any fixed $t > s^*$. Then, if $r > 0$ we have that

$$r < m \quad \text{and} \quad \lambda_r \geqslant c_{\lambda,m}(t) = \lambda_m + \frac{t-s^*}{d-m} \geqslant \lambda_{r+1}. \tag{19}$$

Finally, notice that the previous remarks allow to define

$$s^{**} = c_{\lambda,m}^{-1}(\lambda_1) \overset{(18)}{=} (d-m)\lambda_1 + \sum_{j=1}^m \lambda_j \geqslant s^* \quad (\text{with equality} \Leftrightarrow \lambda_1 = \lambda_m). \tag{20}$$

Then $c_{\lambda,m}(t) \geqslant \lambda_1$ and $r = r_{\lambda,m}(t) = 0$ for every $t > s^{**}$ (by Definition 3.11). These remarks are necessary to characterize the vector ν of Theorem 3.12:

Proposition 3.14. *Let $S_0 \in \mathcal{M}_d(\mathbb{C})^+$ with $\lambda = \lambda(S_0)$, $t_0 = \operatorname{tr}(S_0)$ and $m \in \mathbb{Z}$ such that $m < d$. Fix $t \in [t_0, +\infty)$ and denote by $r = r_{\lambda,m}(t)$. Then, the minimal vector $\nu = \nu(\lambda, m, t) \in \mathbb{R}_+^{d\downarrow}$ of Theorem 3.12 has $\operatorname{tr} \nu = t$ and it is given by the following rule:*

- *If $m \leqslant 0$ then $\nu = (\lambda_1, \ldots, \lambda_r, c_{\lambda,m}(t)\mathbb{1}_{d-r}) = (\lambda_1, \ldots, \lambda_r, c_\lambda(t)\mathbb{1}_{d-r})$.*

If $m \geqslant 1$ we have that

- *$\nu = (\lambda_1, \ldots, \lambda_r, c_{\lambda,m}(t)\mathbb{1}_{d-r})$ for $t \leqslant s^*$ (so that $r \geqslant m$ and $c_{\lambda,m}(t) \leqslant \lambda_m$).*
- *$\nu = (\lambda_1, \ldots, \lambda_r, c_{\lambda,m}(t)\mathbb{1}_{d-m}, \lambda_{r+1}, \ldots, \lambda_m)$ for $t \in (s^*, s^{**})$, and*
- *$\nu = (c_{\lambda,m}(t)\mathbb{1}_{d-m}, \lambda_1, \ldots, \lambda_m)$ for $t \geqslant s^{**}$.*

If $\lambda_1 = \lambda_m$, the second case above disappears. □

4. Solutions of the main problems

In this section we present the solutions of the problems in frame theory described in Section 3. Our strategy is to apply Theorem 3.12 and Proposition 3.14 to the matrix-theoretic reformulations of these problems obtained in Sections 3.1 and 3.2. We point out that our arguments are not only constructive but also algorithmically implementable. This last fact together with recent progress in algorithmic constructions of solutions to the classical frame design problem allow us to effectively compute the optimal frames from Theorems 4.3 and 4.12 below (for optimal completions see Remarks 4.5, 4.6 and Section 4.2; for optimal duals see Remark 4.14 and Example 4.15).

4.1. Optimal completions with prescribed norms

Next we show how our previous results and techniques allow us to partially solve the frame completion problem described in Section 3.1 (which includes the problem posed in [21]). We begin by extracting the relevant data for the problem:

Definition 4.1. In what follows, we fix the following data: A space $\mathcal{H} \cong \mathbb{C}^d$.

D1. A sequence of vectors $\mathcal{F}_0 = \{f_i\}_{i\in\mathbb{I}_{n_0}} \in \mathcal{H}^{n_0}$.
D2. An integer $n > n_0$. We denote by $k = n - n_0$. We assume that $\operatorname{rk} S_{\mathcal{F}_0} \geqslant d - k$.
D3. A sequence $\mathbf{a} = \{\alpha_i\}_{i\in\mathbb{I}_n} \in \mathbb{R}_{>0}^n$ such that $\|f_i\|^2 = \alpha_i$ for every $i \in \mathbb{I}_{n_0}$.
D4. We shall denote by $t = \operatorname{tr}\mathbf{a}$ and by $\mathbf{b} = \{\alpha_i\}_{i=n_0+1}^n \in \mathbb{R}_{>0}^k$.
D5. The vector $\lambda = \lambda(S_{\mathcal{F}_0}) \in \mathbb{R}_+^{d\downarrow}$.
D6. The integer $m = d - k = (d + n_0) - n$. Observe that $d - m = k = n - n_0$.

In order to apply the results of Section 3.3 to this problem, we need to recall and restate some objects and notations:

Definition 4.2. Fix the data $\mathcal{F}_0 = \{f_i\}_{i\in\mathbb{I}_{n_0}}$ and $\mathbf{a} = \{\alpha_i\}_{i\in\mathbb{I}_n}$ as in 4.1. Recall that $t = \operatorname{tr}\mathbf{a}$, $\lambda = \lambda(S_{\mathcal{F}_0})$ and $m = d - k$. We rename some notions of previous sections:

1. The vector $\nu(\mathcal{F}_0, \mathbf{a}) = \nu \in \mathbb{R}^d_{\geqslant 0}$ of Theorem 3.12 (see also Proposition 3.14).
2. The number $c = c(\mathcal{F}_0, \mathbf{a}) \stackrel{\text{def}}{=} c_{\lambda,m}(t)$ (see Definition 3.11).
3. The integer $r = r(\mathcal{F}_0, \mathbf{a}) \stackrel{\text{def}}{=} \max\{r_{\lambda,m}(t), m\}$ (see Definition 3.11). Note that $d - r \leqslant k$.
4. Now we consider the vector

$$\mu = \mu(\mathcal{F}_0, \mathbf{a}) \stackrel{\text{def}}{=} \big(c(\mathcal{F}_0, \mathbf{a}) - \lambda_{r+j}\big)_{j \in \mathbb{I}_{d-r}} \in \big(\mathbb{R}^{d-r}_{\geqslant 0}\big)^{\uparrow}.$$

Observe that $\operatorname{tr} \mu = \operatorname{tr} \nu - \operatorname{tr} \lambda = t - \sum_{i \in \mathbb{I}_{n_0}} \|f_i\|^2 = \operatorname{tr} \mathbf{a} - \sum_{i \in \mathbb{I}_{n_0}} \alpha_i = \operatorname{tr} \mathbf{b}$.

Throughout the rest of this section we shall denote by $S_0 = S_{\mathcal{F}_0}$ the frame operator of $\mathcal{F}_0$. Recall the following notations of Section 3.1:

$$\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0) = \big\{\{f_i\}_{i \in \mathbb{I}_n} \in \mathbf{F}(n, d)\colon \{f_i\}_{i \in \mathbb{I}_{n_0}} = \mathcal{F}_0 \text{ and } \|f_i\|^2 = \alpha_i \text{ for } i \geqslant n_0 + 1\big\},$$

$$\mathcal{SC}_{\mathbf{a}}(\mathcal{F}_0) = \big\{S_{\mathcal{F}}\colon \mathcal{F} \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)\big\} \quad \text{and} \quad \Lambda_{\mathbf{a}}(\mathcal{F}_0) \stackrel{\text{def}}{=} \big\{\lambda(S)\colon S \in \mathcal{SC}_{\mathbf{a}}(\mathcal{F}_0)\big\}.$$

Theorem 4.3. *Fix the data of 4.1 and 4.2. If we assume that $\mathbf{b} \prec \mu(\mathcal{F}_0, \mathbf{a})$ then*

1. *The vector $\nu = \nu(\mathcal{F}_0, \mathbf{a}) \in \Lambda_{\mathbf{a}}(\mathcal{F}_0)$.*
2. *We have that $\nu \prec \beta$ for every other $\beta \in \Lambda_{\mathbf{a}}(\mathcal{F}_0)$.*
3. *Let $r = r(\mathcal{F}_0, \mathbf{a})$. Given $\mathcal{F}_1 = \{f_i\}_{i=n_0+1}^{n} \in \mathcal{H}^k$ such that $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1) \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$, the following conditions are equivalent:*
   (a) *$\lambda(S_{\mathcal{F}}) = \nu$ (i.e. $S_{\mathcal{F}}$ is $\prec$-minimal in $\mathcal{SC}_{\mathbf{a}}(\mathcal{F}_0)$).*
   (b) *There exists $\{h_i\}_{i \in \mathbb{I}_d}$, an ONB of eigenvectors for $S_0, \lambda(S_0)$ such that*

$$S_{\mathcal{F}_1} = B = \sum_{i=1}^{d-r} \mu_i h_{r+i} \otimes h_{r+i}. \tag{21}$$

   Since, by the hypothesis, $\mathbf{b} \prec \mu = \mu(\mathcal{F}_0, \mathbf{a})$ then such an $\mathcal{F}_1$ exists.
4. *Moreover, if any of the conditions in item 3 of Theorem 3.12 holds, then*
   (a) *Any ONB of eigenvectors for S_0, λ produces the same operator B via (21).*
   (b) *Any $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1) \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$ satisfies that $\lambda(S_{\mathcal{F}}) = \nu(\mathcal{F}_0, \mathbf{a}) \Leftrightarrow S_{\mathcal{F}_1} = B$.*

Proof. Since the elements of $\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$ must be frames, we have first to show that $\nu(\mathcal{F}_0, \mathbf{a}) > 0$. By the description of $\nu = \nu(\mathcal{F}_0, \mathbf{a}) \in \mathbb{R}^{d\downarrow}_{+}$ given in Proposition 3.14, there are two possibilities: In one case $\nu_d = \lambda_m$ which is positive because we know from the data given in 4.1 that $\operatorname{rk} S_0 \geqslant m$. Otherwise $t \leqslant s^*$ so that $\nu_d = c(\mathcal{F}_0, \mathbf{a}) = c_\lambda(t)$ by Proposition 3.14 and Definition 3.11. But $c_\lambda(t) > 0$ because $\mathbf{b} > 0 \Rightarrow t > \operatorname{tr} S_0$ (see Lemma 3.13 and Definition 3.10).

By Proposition 3.2, we know that the hypothesis $\mathbf{b} \prec \mu = \mu(\mathcal{F}_0, \mathbf{a})$ assures that there exists a sequence $\mathcal{F}_1 = \{f_i\}_{i=n_0+1}^{n} \in \mathcal{H}^k$ such that $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1) \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$ and $S_{\mathcal{F}_1} = B$. Then

$$\lambda(S_{\mathcal{F}}) = \lambda(S_{\mathcal{F}_0} + S_{\mathcal{F}_1}) = \lambda(S_{\mathcal{F}_0} + B) = \nu(\mathcal{F}_0, \mathbf{a}) \in \Lambda_{\mathbf{a}}(\mathcal{F}_0),$$

by Theorem 3.12. Observe that $\Lambda_{\mathbf{a}}(\mathcal{F}_0) \subseteq \Lambda(U_t(S_{\mathcal{F}_0}, m))$, by Remark 3.7. Hence the majorization of item 2, the equivalence of item 3 and the uniqueness results of item 4 follow from Theorem 3.12. Note that all the vectors of $\Lambda_{\mathbf{a}}(\mathcal{F}_0)$ have the same trace. So we have $\prec$ instead of $\prec_w$. $\square$

Theorem 4.4. *Fix the data of 4.1 and 4.2. If we assume that $\mathbf{b} \prec \mu(\mathcal{F}_0, \mathbf{a})$ then*

1. *Any $\mathcal{F} \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$ such that $\lambda(S_{\mathcal{F}}) = \nu(\mathcal{F}_0, \mathbf{a})$ satisfies that*

$$\sum_{i \in \mathbb{I}_d} f\big(\nu(\mathcal{F}_0, \mathbf{a})_i\big) = P_f(\mathcal{F}) \leqslant P_f(\mathcal{G}) \quad \textit{for every } \mathcal{G} \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0),$$

   and every (not necessarily increasing) convex function $f : [0, \infty) \to [0, \infty)$.
2. *If f is strictly convex then, for every global minimizer $\mathcal{F}'$ of $P_f(\cdot)$ on $\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$ we get that $\lambda(S_{\mathcal{F}'}) = \nu(\mathcal{F}_0, \mathbf{a})$.*

In particular the previous items holds for the Benedetto–Fickus' potential and the mean square error.

Proof. It follows from Theorem 3.12 and the majorization facts described in Section 2.3. $\square$

Fix the data $\mathcal{F}_0$ and $\mathbf{b}$ of 4.1 and 4.2. We shall say that "the completion problem is **feasible**" if the condition $\mathbf{b} \prec \mu(\mathcal{F}_0, \mathbf{a})$ of Theorem 4.3 is satisfied.

Remark 4.5. The data $\nu(\mathcal{F}_0, \mathbf{a})$, $r(\mathcal{F}_0, \mathbf{a})$, $c(\mathcal{F}_0, \mathbf{a})$ and $\mu(\mathcal{F}_0, \mathbf{a})$ are essential for Theorem 4.3, both for checking the feasibility hypothesis $\mathbf{b} \prec \mu(\mathcal{F}_0, \mathbf{a})$ and for the construction of the matrix B of (21), which is the frame operator of the optimal extensions of $\mathcal{F}_0$. Notice that the vector $\mu(\mathcal{F}_0, \mathbf{a})$ measures how restrictive is the feasibility condition. Fortunately, this condition can be easily computed according to the following algorithm:

1. The numbers $t = \operatorname{tr} \mathbf{a}$ and $m = d - k$ are included in the data 4.1.
2. The main point is to compute the irregularity $r = r(\mathcal{F}_0, \mathbf{a}) = \max\{r_{\lambda,m}(t), m\}$. If $m \leqslant 0$ then (12) allows us to compute $r_\lambda(t)$. If $m \geqslant 1$, the number $s^* = s^*(\lambda, m)$ of Eq. (16) allows us to compute r: If $t > s^*$ then $r = m$ by Eq. (19), and if $t \leqslant s^*$ then $r = r_\lambda(t)$ by the remark which follows Definition 3.11.
3. Once r is obtained, we can see that the wideness of the allowed weights $\mathbf{b}$ depends on the dispersion of the eigenvalues $(\lambda_{r+1}, \dots, \lambda_d)$ of $S_{\mathcal{F}_0}$.
4. Indeed, the number $t_1 \overset{\text{def}}{=} \operatorname{tr} \mathbf{b} = t - \operatorname{tr} S_{\mathcal{F}_0}$ is known data. Also $\operatorname{tr} \mu(\mathcal{F}_0, \mathbf{a}) = t_1$. Hence $c(\mathcal{F}_0, \mathbf{a})$ and $\mu(\mathcal{F}_0, \mathbf{a})$ can be directly computed: Let $s = \sum_{i=r+1}^d \lambda_i$. Then
$$t_1 = \operatorname{tr} \mu = (d-r)c(\mathcal{F}_0, \mathbf{a}) - s \quad \Longrightarrow \quad c(\mathcal{F}_0, \mathbf{a}) = \frac{t_1 + s}{d-r} = \frac{\operatorname{tr} \mathbf{b} + \sum_{i=r+1}^d \lambda_i}{d-r}.$$
And we have the vector $\mu = \mu(\mathcal{F}_0, \mathbf{a}) = (c(\mathcal{F}_0, \mathbf{a}) - \lambda_{r+j})_{j \in \mathbb{I}_{d-r}} \in (\mathbb{R}_{\geqslant 0}^{d-r})^\uparrow$. Then
$$\mathbf{b} \prec \mu \quad \Longleftrightarrow \quad \sum_{i=1}^p \mathbf{b}_i^\downarrow + \lambda_{d-i+1} \leqslant \frac{p}{d-r}\left(\operatorname{tr} \mathbf{b} + \sum_{i=r+1}^d \lambda_i\right) \quad \text{for } 1 \leqslant p < d-r,$$
since the last inequalities $s + \sum_{i=1}^p \mathbf{b}_i^\downarrow \leqslant s + \operatorname{tr} \mathbf{b}$ (for $d - r \leqslant p \leqslant k$) clearly hold.

It is interesting to note that the closer $\mathcal{F}_0$ is to be tight (at least in the last r entries of λ), the more restrictive Theorem 4.3 becomes; but in this case $\mathcal{F}_0$ and $\mathcal{F}_0^\#$ are already "good".

On the other hand, if $\mathcal{F}_0$ is far from being tight then the sequence $(\lambda_{r+1}, \dots, \lambda_d)$ has more dispersion and the feasibility condition $\mathbf{b} \prec \mu(\mathcal{F}_0, \mathbf{a})$ becomes less restrictive. It is worth mentioning that in the uniform case $\mathbf{b} = b\mathbb{1}_k$ is always feasible and Theorem 4.3 can be applied.

Observe that as the number k of vectors increases (or as the weights α_i increase) the trace t grows and the numbers r and m become smaller, taking into account more entries λ_i of $\lambda(\mathcal{F}_0)$. This fact offers a criterion for choosing a convenient data k and $\mathbf{b}$ for the completing process. We remark that the vector μ (and therefore the feasibility) only depends on λ, k and **the trace** of $\mathbf{b}$, so the feasibility can also be obtained by changing $\mathbf{b}$ maintaining its length (size) and its trace.

The above algorithm (which tests the feasibility of our method for fixed data $\mathcal{F}_0$ and $\mathbf{b}$) can be easily implemented in MATLAB with low complexity (see Section 4.2).

Remark 4.6 *(Construction of optimal completions for the mean square error).* Consider the data in 4.1. Apply the algorithm described in Remark 4.5 and assume that $\mathbf{b} \prec \mu(\mathcal{F}_0, \mathbf{a})$. Then construct B as in Eq. (21). In order to obtain an optimal completion of $\mathcal{F}_0$ with prescribed norms we have to construct a sequence $\mathcal{F}_1 \in \mathcal{H}^k$ with frame operator B and norms given by the sequence $\mathbf{b}$ (which is minimal for the mean square error by Theorem 4.4). But once we know B and the weights $\mathbf{b}$ we can apply the results in [8] in order to concretely construct the sequence $\mathcal{F}_1$. In fact, in [8] they give a MATLAB implementation which works fast, with low complexity.

We also implemented a MATLAB program which compute the matrix B as in Eq. (21). This process is direct, but it is more complex because it depends on finding a ONB of eigenvectors for the matrix $S_{\mathcal{F}_0}$. In Section 4.2 we shall present several examples which use these programs for computing explicit solutions.

Consider the data in 4.1 and 4.2. Let $f : [0, \infty) \to [0, \infty)$ be an strictly convex function. By Remark 3.7, Theorem 3.12 and the remarks in Section 2.3, in general we have that
$$\sum_{i \in \mathbb{I}_d} f\big(\nu(\mathcal{F}_0, \mathbf{a})_i\big) \leqslant P_f(\mathcal{F}) \quad \text{for every } \mathcal{F} \in \mathcal{C}_\mathbf{a}(\mathcal{F}_0). \tag{22}$$
Notice that although the left-hand side of Eq. (22) can be effectively computed, the inequality might not be sharp. Indeed, Eq. (22) is sharp if and only if the completion problem is feasible and, in this case, the lower bound is attained if and only if $\lambda(S_\mathcal{F}) = \nu(\mathcal{F}_0, \mathbf{a})$. Nevertheless, Eq. (22) provides a general lower bound that can be of interest for optimization problems in $\mathcal{C}_\mathbf{a}(\mathcal{F}_0)$.

4.2. Examples of optimal completions with prescribed norms

In this section we show several examples obtained by implementing the algorithms described in Remarks 4.5 and 4.6 in MATLAB, for different choices of $\mathcal{F}_0 = \{f_i\}_{i \in \mathbb{I}_{n_0}}$ and $\mathbf{a} = \{\alpha_i\}_{i \in \mathbb{I}_n}$ (as in 4.1). Indeed, we have implemented the computation of r, c, μ and ν by a fast algorithm using $\mathbf{b} = \{\alpha_i\}_{i=n_0+1}^{n} \in \mathbb{R}_{>0}^k$ and the vector $\lambda = \lambda(S_{\mathcal{F}_0})$ as data. Then, after computing the eigenvectors of $S_{\mathcal{F}_0}$ with the function 'eig' in MATLAB we computed the matrix B, and we apply the one-sided Bendel–Mickey algorithm (see [17] for details) to construct the vectors of $\mathcal{F}_1$ satisfying the desired properties. The corresponding M-files that compute all the previous objects are freely distributed by the authors.

Example 4.7. Consider the frame $\mathcal{F}_0 \in \mathbf{F}(7,5)$ whose analysis operator is

$$T_{\mathcal{F}_0}^* = \begin{bmatrix} 0.9202 & -0.7476 & -0.4674 & 0.9164 & 0.1621 & 0.3172 & -0.5815 \\ 0.4556 & 0.0164 & 0.0636 & 1.0372 & -1.6172 & 0.3688 & 0.2559 \\ -0.0885 & -0.3495 & -0.9103 & 0.3672 & -0.6706 & -0.9252 & 0.6281 \\ 0.1380 & -0.4672 & -0.6228 & -0.1660 & 0.9419 & 1.0760 & 1.1687 \\ 0.7082 & 0.2412 & -0.1579 & -1.8922 & -0.4026 & 0.1040 & 1.6648 \end{bmatrix}.$$

The spectrum of it frame operator is $\lambda = \lambda(S_{\mathcal{F}_0}) = (9,5,4,2,1)$ and $t_0 = \operatorname{tr} S_{\mathcal{F}_0} = 21$. As in 4.1, fix the data $k = 2$ and $\mathbf{b} = \{\alpha_i\}_{i=8}^{9} = (3, 2.5) \in \mathbb{R}_{>0}^2$, so that $m = d - k = 3$. We compute:

1. The number $r_{\lambda,m}(26.5) = 2$ and the vector $\mu = (2.25, 3.25)$. Notice that, in this case, $\mathbf{b} = (3, 2.5) \prec (2.25, 3.25) = \mu$. Therefore the completion problem is feasible.
2. The optimal spectrum is $\nu_{\lambda,m}(26.5) = (9, 5, 4.25, 4.25, 4)$.
3. An optimal completion $\mathcal{F}_1$ of $\mathcal{F}_0$, with squared norms given by $\mathbf{b}$ is given by:

$$T_{\mathcal{F}_1}^* = \begin{bmatrix} -0.6120 & -1.1534 \\ 0.9087 & 0.1097 \\ -1.0680 & 0.7154 \\ 0.3735 & 0.7676 \\ -0.1404 & -0.7462 \end{bmatrix}.$$

4. If we take $\mathbf{b} = (3.5, 2)$ then the number $t = t_0 + \operatorname{tr} \mathbf{b}$ (and so also r and μ) are the same as before but the problem is not feasible, because in this case $\mathbf{b} \not\prec \mu$.

Example 4.8. We want to complete frame $\mathcal{F}_0$ of Example 4.7 with 4 vectors in $\mathbb{R}^4$, whose norms are given by $\mathbf{b} = \{\alpha_i\}_{i=8}^{11} = (1, 1, \frac{1}{2}, \frac{1}{4}) \in \mathbb{R}_{>0}^4$. We can compute that

1. $m = d - k = 1$ and $r = r_{\lambda,m}(23.75) = 3$.
2. The vector $\mu = (0.875, 1.875)$, so that $\mathbf{b} \prec \mu$ and the problem is feasible.
3. The optimal spectrum is $\nu = (9, 5, 4, 2.875, 2.875)$.
4. An example of optimal completion $\mathcal{F}_1$ of $\mathcal{F}_0$, with squared norms given by $\mathbf{b}$ is given by:

$$T_{\mathcal{F}_1}^* = \begin{bmatrix} -0.7086 & -0.3232 & -0.5011 & -0.3543 \\ 0.1730 & 0.5746 & 0.1224 & 0.0865 \\ 0.2597 & -0.7252 & 0.1836 & 0.1299 \\ 0.4674 & 0.1935 & 0.3305 & 0.2337 \\ -0.4267 & -0.0457 & -0.3017 & -0.2133 \end{bmatrix}.$$

5. If we take $\mathbf{b} = (2, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}) \in \mathbb{R}_{>0}^4$ the problem becomes not feasible.

Example 4.9. Suppose now that $\mathcal{H} = \mathbb{C}^5$ and that our original set of vectors $\mathcal{F}_0 = \{f_i\}_{i \in \mathbb{I}_6} \in \mathcal{H}^6$ is such that the spectrum of $S_{\mathcal{F}_0}$ is given by $\lambda = (7,4,4,3,1)$. Thus $t_0 = \operatorname{tr} S_{\mathcal{F}_0} = 19$. Let $\mathbf{b} = (2,2,1)$. Then, $k = 3$, $m = d - k = 2$ and $t = 24$.

With these initial data, we obtain the values $r_{\lambda,m}(24) = 1$ and $c_{\lambda,m}(24) = 4.33$. The spectrum of the completion B is $\mu = (0.33, 1.33, 3.33)$ (notice that $\mathbf{b} \prec \mu$) and the optimal spectrum is $\nu_{\lambda,m}(24) = (7, 4.33, 4.33, 4.33, 4)$. However the frame operator B of the optimal completion is not unique, since $\lambda_m = \lambda_{m+1}$ and $t = t_0 + \operatorname{tr} \mathbf{b} = 24 > 23 = s^*$ (see Theorem 3.12).

4.3. Minimizing potentials in $\mathcal{D}_t(\mathcal{F})$

In this section we show how our previous results and techniques allow us to solve the problem of computing optimal duals in $\mathcal{D}_t(\mathcal{F}) = \{\mathcal{G} \in \mathbf{F}(n,d)\colon T_{\mathcal{G}}^* T_{\mathcal{F}} = I \text{ and } \operatorname{tr} S_{\mathcal{G}} \geqslant t\}$ for a given frame $\mathcal{F}$, described in Section 3.2. In order to state our main results we introduce the set $\Lambda_t(\mathcal{D}(\mathcal{F}))$, called the spectral picture of the set $\mathcal{SD}(\mathcal{F})$ (see Eq. (8)), given by

$$\Lambda_t\big(\mathcal{D}(\mathcal{F})\big) = \big\{\lambda(S_{\mathcal{G}})\colon \mathcal{G} \in \mathcal{D}_t(\mathcal{F})\big\}.$$

Remark 4.10. Recall from Remark 3.7 that if $\mathcal{F} \in \mathbf{F}(n,d)$ with $\lambda = \lambda(S_{\mathcal{F}}^{-1})$, $m = 2d - n$ and $t \geqslant \operatorname{tr}\lambda$, then $\mathcal{SD}_t(\mathcal{F}) = U_t(S_{\mathcal{F}^\#}, m)$. Hence, by Theorem 3.12, there exists a unique $\nu \in \Lambda_t(\mathcal{D}(\mathcal{F}))$ that is $\prec_w$-minimizer on this set. Moreover, recall that such vector ν is explicitly described in Proposition 3.14.

Theorem 4.11 *(Spectral structure of Global minima in $\mathcal{D}_t(\mathcal{F})$). Let $\mathcal{F}=\{f_i\}_{i\in\mathbb{I}_n}\in\mathbf{F}(n,d)$ with $\lambda=\lambda(S_{\mathcal{F}}^{-1})$, $m=2d-n$ and $t\geqslant\operatorname{tr}\lambda$. Let $\nu=\nu(\lambda,m,t)\in\mathbb{R}_+^{d\downarrow}$ be as in Proposition 3.14. Then, $\nu\in\Lambda_t(\mathcal{D}(\mathcal{F}))$ and we have that:*

1. *If $\mathcal{G}_t\in\mathcal{D}_t(\mathcal{F})$ is such that $\lambda(S_{\mathcal{G}_t})=\nu$ then*
$$\sum_{i\in\mathbb{I}_d} f(\nu_i)=P_f(\mathcal{G}_t)\leqslant P_f(\mathcal{G})\quad \text{for every } \mathcal{G}\in\mathcal{D}_t(\mathcal{F}),$$
and every increasing convex function $f:[0,\infty)\to[0,\infty)$.
2. *If we assume further that f is strictly convex then, for every global minimizer $\mathcal{G}_t'$ of $P_f(\cdot)$ on $\mathcal{D}_t(\mathcal{F})$ we get that $\lambda(\mathcal{G}_t')=\nu$.*

Proof. As explained in Remark 4.10 we see that $\nu\in\Lambda_t(\mathcal{D}(\mathcal{F}))$ is such that $\nu\prec_w\mu$ for every $\mu\in\Lambda_t(\mathcal{D}(\mathcal{F}))$. By the remarks in Section 2.3 we conclude that, if $\mathcal{G}_t$ is as above and $\mathcal{G}\in\mathcal{D}_t(\mathcal{F})$ then
$$P_f(\mathcal{G}_t)=\operatorname{tr}\big(f(S_{\mathcal{G}_t})\big)=\operatorname{tr} f(\nu)\leqslant\operatorname{tr} f\big(\lambda(S_{\mathcal{G}})\big)=P_f(\mathcal{G}),$$
since $\lambda(S_{\mathcal{G}})\in\Lambda_t(\mathcal{D}(\mathcal{F}))$. Assume further that f is strictly convex and let $\mathcal{G}_t'$ be a global minimizer of $P_f(\cdot)$ on $\mathcal{D}_t(\mathcal{F})$. Then, we have that
$$\nu\prec_w\lambda(S_{\mathcal{G}_t'})\quad\text{but}\quad\operatorname{tr} f\big(\lambda(S_{\mathcal{G}_t'})\big)=P_f\big(\mathcal{G}_t'\big)\leqslant P_f(\mathcal{G}_t)=\operatorname{tr} f(\nu).$$
These last facts imply (see Section 2.3) that $\lambda(S_{\mathcal{G}_t'})=\nu$ as desired. □

Next we describe the geometric structure of the global minimizers of the (generalized) frame potential $P_f(\cdot)$ in $\mathcal{D}_t(\mathcal{F})$, in terms of their frame operators.

Theorem 4.12 *(Geometric Structure of global minima in $\mathcal{D}_t(\mathcal{F})$). Let $\mathcal{F}\in\mathbf{F}(n,d)$, $m=2d-n$, let $t\geqslant\operatorname{tr} S_{\mathcal{F}}^{-1}$ and denote by $\lambda=\lambda(S_{\mathcal{F}}^{-1})$. Let $f:[0,\infty)\to[0,\infty)$ an increasing and strictly convex function.*

1. *If $\mathcal{G}\in\mathcal{D}_t(\mathcal{F})$ is a global minimum of P_f in $\mathcal{D}_t(\mathcal{F})$ then there exists $\{h_i\}_{i\in\mathbb{I}_d}$, an ONB of eigenvectors for $S_{\mathcal{F}}^{-1}$, λ such that*
$$S_{\mathcal{G}}=S_{\mathcal{F}}^{-1}+\sum_{i=1}^{d-r'}\big(c_{\lambda,m}(t)-\lambda_{r'+i}\big)h_{r'+i}\otimes h_{r'+i},$$
where $r'=\max\{r_{\lambda,m}(t),m\}$.
2. *If we further assume any of the conditions of item 3 of Theorem 3.12, there exists a unique $S_t\in\mathcal{SD}_t(\mathcal{F})$ such that if $\mathcal{G}$ is a global minimum of P_f in $\mathcal{D}_t(\mathcal{F})$ then $S_{\mathcal{G}}=S_t$.*

Proof. It is a consequence of Theorems 3.12 and 4.11 together with Proposition 3.14. □

Remark 4.13. Fix $\mathcal{F}=\{f_i\}_{i\in\mathbb{I}_n}\in\mathbf{F}(n,d)$ and $m=2d-n$. Denote by $\lambda=\lambda(S_{\mathcal{F}}^{-1})$. If $m>0$ then there exist $t\in\mathbb{R}_{>0}$ and a constant vector
$$c\mathbb{1}_d\in\Lambda_t\big(\mathcal{D}(\mathcal{F})\big)\quad\Longleftrightarrow\quad\lambda_1=\lambda_m. \tag{23}$$
In this case $c=\lambda_1$ and $t=d\lambda_1$. The proof uses the characterization of $\Lambda_t(\mathcal{D}(\mathcal{F}))$ given in Remark 3.7 and Corollary A.7 (see also Definition A.6). Indeed, if $\nu=c\mathbb{1}_d\in\Lambda_t(\mathcal{D}(\mathcal{F}))$ then, by Eq. (29),
$$c=\nu_d\leqslant\lambda_m\leqslant\lambda_1\leqslant\nu_1=c\quad\Longrightarrow\quad\lambda_1=\lambda_m=c.$$
Conversely, if $\lambda_1=\lambda_m$ and $t=d\lambda_1$, then by Corollary A.7 it is easy to see that the vector $\lambda_1\mathbb{1}_d\in\Lambda_t(\mathcal{D}(\mathcal{F}))$. Therefore the frame $\mathcal{F}$ has a dual frame which is tight if and only if

- $m=2d-n\leqslant 0$. Recall that in this case $\nu_{\lambda,m}(t)=\frac{t}{d}\cdot\mathbb{1}_d$ for every $t\geqslant d\lambda_1$.
- $m\in\mathbb{I}_{d-1}$ and $\lambda_{d-m+1}(S_{\mathcal{F}})=\lambda_d(S_{\mathcal{F}})$ i.e., the multiplicity of the smaller eigenvalue $\lambda_d(S_{\mathcal{F}})$ of $S_{\mathcal{F}}$ is greater or equal than m. This is a consequence of Eq. (23).

In particular, if $m\in\mathbb{I}_{d-1}$ then there is a Parseval dual frame for $\mathcal{F}$ if and only if
$$\lambda_{d-m+1}(S_{\mathcal{F}})=\lambda_d(S_{\mathcal{F}})=1\quad\Longleftrightarrow\quad S_{\mathcal{F}}\geqslant I_d\quad\text{and}\quad\operatorname{rk}(I_d-S_{\mathcal{F}})\leqslant d-m=\dim\ker T_{\mathcal{F}}^*.$$
Observe that the equivalence also holds if $2d\leqslant n$. In this case there is a Parseval dual frame for $\mathcal{F}\Longleftrightarrow S_{\mathcal{F}}\geqslant I_d$, because the restriction $\dim\ker T_{\mathcal{F}}^*=n-d\geqslant d\geqslant\operatorname{rk}(I_d-S_{\mathcal{F}})$ is irrelevant. This characterization was already proved by Han in [22], even for the infinite-dimensional case.

Remark 4.14. Using the characterization of $\mathcal{D}(\mathcal{F})$ given in the proof of Proposition 3.5 every optimal dual frame $\mathcal{G} = \{g_i\}_{i\in\mathbb{I}_n}$ is constructed from the canonical dual of $\mathcal{F}$: each $g_i = S_{\mathcal{F}}^{-1} f_i + h_i$, for a family $\mathcal{F}_1 = \{h_i\}_{i\in\mathbb{I}_n}$ which satisfies $T_{\mathcal{F}_1}^* T_{\mathcal{F}_1} = B$ and $T_{\mathcal{F}_1}^* T_{\mathcal{F}} = 0$.

As it was done with the completion problem, the previous results can be implemented in Matlab in order to construct optimal dual frames for a given one when a tracial condition is imposed.

It turns out that in this case, once we have calculated the optimal B, we must improve a different type of factorization of B. Now $B = X^*X$ should satisfy $R(T_{\mathcal{F}}) \subseteq \ker X^*$. In the algorithm developed, $X^* = B^{1/2}W^*$ where $B^{1/2}$ has no cost of construction since we already have the eigenvectors of $S_{\mathcal{F}}^{-1}$. In addition W is constructed using the first $d-r$ vectors of the ONB of $\ker T_{\mathcal{F}}^*$ (computed with the 'null' function) and adding r zero vectors in order to obtain an $n\times d$ partial isometry.

Example 4.15. The frame operator of the following frame $\mathcal{F} \in \mathbf{F}(8,5)$ has eigenvalues listed by $\lambda = (\frac{5}{2}, 2, \frac{2}{3}, \frac{1}{3}, \frac{1}{4})$:

$$T_{\mathcal{F}}^* = \begin{bmatrix} -0.5124 & 0.5695 & 0.4542 & -0.3527 & -0.2452 & 0.1260 & 0.0558 & -0.3513 \\ -0.4965 & 0.0478 & 0.1579 & -0.2299 & -0.9348 & -0.6935 & -0.0836 & 0.7641 \\ 0.2777 & 0.2875 & -0.4974 & 0.0086 & 0.1893 & -0.0916 & 0.2501 & -0.0722 \\ -0.3793 & -0.7849 & -0.4783 & -0.2566 & 0.3450 & -0.0749 & -0.2939 & 0.3785 \\ 0.0725 & -0.0803 & -0.2075 & -0.2967 & -0.1518 & 0.2077 & -0.2050 & 0.4226 \end{bmatrix}.$$

Therefore, $\lambda = \lambda(S_{\mathcal{F}}^{-1}) = (4, 3, \frac{3}{2}, \frac{1}{2}, \frac{2}{5})$ and $\operatorname{tr} S_{\mathcal{F}}^{-1} = 9.4$. We also have that $m = 2d - n = 2$. Consider $t = 16.5$, then an optimal dual $\mathcal{G} \in \mathcal{D}_t(\mathcal{F})$ for $\mathcal{F}$ is given by

$$T_{\mathcal{G}}^* = \begin{bmatrix} -1.0236 & 0.2319 & -0.0181 & -0.5802 & 0.0438 & 0.7316 & 1.0846 & -0.0143 \\ -0.2583 & 0.6148 & 0.5219 & 0.1585 & 0.6493 & -0.7116 & 0.2109 & 1.2138 \\ 0.3080 & 0.6525 & -1.0323 & -0.8031 & 0.2306 & -0.8740 & 0.2856 & -0.3488 \\ -1.1868 & 0.1198 & -0.8331 & 0.4816 & 0.4222 & -0.0495 & -0.8551 & -0.0836 \\ 0.5506 & 0.5428 & -0.2035 & -0.5871 & -0.2309 & 1.1268 & -0.7891 & 0.8432 \end{bmatrix}.$$

Here, the optimal spectrum $\nu = \nu_{\lambda,m}(16.5)$ is given by $\nu = (4, 3.166, 3.166, 3.166, 3)$.

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Appendix A. Proof of Theorem 3.12

In this section we obtain the proofs of Theorem 3.12 and Proposition 3.14 stated in Section 3.3 in a series of steps. In the first step we introduce the set $U(S_0, m) := U_{\operatorname{tr}(S_0)}(S_0, m)$ and characterize its spectral picture $\Lambda(U(S_0, m))$ – i.e. the subset of $\mathbb{R}_+^{d\downarrow}$ of eigenvalues $\lambda(S)$, for $S \in U(S_0, m)$ – in terms of the so-called Fan–Pall inequalities. In the second step we show the existence of a $\prec_w$-minimizer within the set $\Lambda(U_t(S_0, m))$ and give an explicit (algorithmic) expression for this vector. Finally, in the third step we characterize the geometrical structure of the positive operators $S \in U_t(S_0, m)$ such that $\lambda(S)$ are $\prec_w$-minimizers within the set $\Lambda(U_t(S_0, m))$, in terms of the relation between the eigenspaces of S and the eigenspaces of S_0. It is worth pointing out that the arguments in this section are constructive, and lead to algorithms that allow to effectively compute all the parameters involved.

A.1. Step 1: spectral picture of $U(S_0, m)$

Recall that $\mathbb{R}_+^{d\downarrow}$ is the set of vectors $\mu \in \mathbb{R}_+^d$ with non-negative and decreasing entries (i.e. $\mu \in \mathbb{R}_+^d$ with $\mu^\downarrow = \mu$); also, given $S \in \mathcal{M}_d(\mathbb{C})^+$, $\lambda(S) \in \mathbb{R}_+^{d\downarrow}$ denotes the vector of eigenvalues of S – counting multiplicities – and arranged in decreasing order.

Given $S_0 \in \mathcal{M}_d(\mathbb{C})^+$, $m < d$ and integer and $t \geqslant \operatorname{tr}(S_0)$ then in Eq. (10) we introduced $U_t(S_0, m) = \{S_0 + B\colon\ B \in \mathcal{M}_d(\mathbb{C})^+,\ \operatorname{rk} B \leqslant d - m,\ \operatorname{tr}(S_0 + B) \geqslant t\}$. In this section we consider

$$U(S_0, m) := U_{\operatorname{tr}(S_0)}(S_0, m) = \{S_0 + B\colon\ B \in \mathcal{M}_d(\mathbb{C})^+, \operatorname{rk} B \leqslant d - m\}$$

together with its spectral picture $\Lambda(U(S_0, m)) := \Lambda(U_{\operatorname{tr}(S_0)}(S_0, m))$ (see Eq. (11) in Remark 3.8). We shall also use the following notations:

1. Given $x \in \mathbb{C}^d$ then $D(x) \in \mathcal{M}_d(\mathbb{C})$ denotes the diagonal matrix with main diagonal x.
2. If $d \leqslant n$ and $y \in \mathbb{C}^d$, we write $(y, 0_{n-d}) \in \mathbb{C}^n$, where 0_{n-d} is the zero vector of $\mathbb{C}^{n-d}$. In this case, we denote by $D_n(y) = D((y, 0_{n-d})) \in \mathcal{M}_n(\mathbb{C})$.

Theorem A.1. *Let $S_0 \in \mathcal{M}_d(\mathbb{C})^+$, $m < d$ be an integer and $\mu \in \mathbb{R}_+^{d\downarrow}$. Then the following conditions are equivalent*:

1. *There exists $S \in U(S_0, m)$ such that $\lambda(S) = \mu$.*
2. *There exists an orthogonal projection $P \in \mathcal{M}_{2d-m}(\mathbb{C})$ such that* $\operatorname{rk} P = d$ *and*

$$\lambda\big(P D_{2d-m}(\mu) P\big) = \big(\lambda(S_0), 0_{d-m}\big). \tag{24}$$

Proof.

$1 \Rightarrow 2$. Let $B \in \mathcal{M}_d(\mathbb{C})^+$ be such that $\operatorname{rk}(B) \leqslant d - m$ and $\lambda(S_0 + B) = \mu$. Thus, B can be factorized as $B = V^* V$ for some $V \in \mathcal{M}_{d-m,d}(\mathbb{C})$. If

$$T = \begin{pmatrix} S_0^{1/2} \\ V \end{pmatrix} \in \mathcal{M}_{2d-m,d}(\mathbb{C}) \quad \Longrightarrow \quad T^*T = S_0 + B \quad \text{and} \quad TT^* = \begin{pmatrix} S_0 & S_0^{1/2} V^* \\ V S_0^{1/2} & VV^* \end{pmatrix}. \tag{25}$$

Let $U \in \mathcal{U}(2d-m)$ be such that $U(TT^*)U^* = D(\lambda(TT^*)) = D_{2d-m}(\mu)$ and let $P \in \mathcal{M}_{2d-m}(\mathbb{C})$ be given by $P = UP_1U^*$, where $P_1 = I_d \oplus 0_{d-m}$. Notice that, by construction, P is an orthogonal projection with $\operatorname{rk} P = d$ and, by the previous facts,

$$P D_{2d-m}(\mu) P = U P_1 \big(TT^*\big) P_1 U^* = U \begin{pmatrix} S_0 & 0 \\ 0 & 0 \end{pmatrix} U^*,$$

which shows that Eq. (24) holds in this case.

$2 \Rightarrow 1$. Let $P \in \mathcal{M}_{2d-m}(\mathbb{C})$ be a projection as in item 2. Then, there exists $U \in \mathcal{U}(2d-m)$ such that $U^*PU = P_1$, where $P_1 = I_d \oplus 0_{d-m}$ as before. Hence, we get that

$$\lambda\big(P_1 \big(U^* D_{2d-m}(\mu) U\big) P_1\big) = \lambda(S_0, 0_{d-m}). \tag{26}$$

Since $\operatorname{rk}(U^* D_{2d-m}(\mu) U) \leqslant d$ then we see that there exist $T \in \mathcal{M}_{2d-m,d}(\mathbb{C})$ such that $U^* D_{2d-m}(\mu) U = TT^*$. Let $T_1 \in \mathcal{M}_d(\mathbb{C})^+$ and $T_2 \in \mathcal{M}_{d-m,d}(\mathbb{C})$ such that

$$T = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} \quad \Longrightarrow \quad U^* D_{2d-m}(\mu) U = TT^* = \begin{pmatrix} T_1T_1^* & T_1T_2^* \\ T_2T_1^* & T_2T_2^* \end{pmatrix}.$$

Then $\lambda(T_1T_1^*) = \lambda(S_0)$ by Eq. (25). On the other hand, notice that $\lambda(T^*T) = \mu$ and

$$T^*T = T_1^*T_1 + T_2^*T_2 \stackrel{\text{def}}{=} S_1 + B_1 \quad \text{with } \lambda(S_1) = \lambda\big(T_1T_1^*\big) = \lambda(S_0) \text{ and } \operatorname{rk}(B_1) \leqslant d - m.$$

Let $W \in \mathcal{U}(d)$ such that $W^*S_1W = S_0$. Then $S \stackrel{\text{def}}{=} W^*(T^*T)W = S_0 + B$ satisfies that $\lambda(S) = \mu$ and $\operatorname{rk}(B) = \operatorname{rk}(W^*B_1W) \leqslant d - m$. Then $\mu = \lambda(T^*T) = \lambda(S) \in \Lambda(U(S_0, m))$. $\square$

Remark A.2. Let $S_0 \in \mathcal{M}_d(\mathbb{C})^+$, $m < d$ be an integer and $\mu \in \mathbb{R}_+^{d\downarrow}$ as in Theorem A.1. It turns out that condition (24) can be characterized in terms of interlacing inequalities.

More explicitly, given $\mu \in \mathbb{R}_+^{d\downarrow}$, by the Fan–Pall inequalities (see [19,27]), the existence of a projection $P \in \mathcal{M}_{2d-m}(\mathbb{C})$ satisfying (24) for μ is equivalent to the following inequalities:

1. $\mu \geqslant \lambda(S_0)$, i.e. $\mu_i \geqslant \lambda_i(S_0)$ for every $i \in \mathbb{I}_d$.
2. If $m \geqslant 1$ then μ also satisfies

   $$\mu_{d-m+i} \leqslant \lambda_i(S_0) \quad \text{for every } i \in \mathbb{I}_m,$$

   where the last inequalities compare the first m entries of $\lambda(S_0)$ with the last m of μ.

These facts together with Theorem A.1 give a complete description of the spectral picture of the set $U(S_0, m)$, which we write as follows.

Corollary A.3. *Let $S_0 \in \mathcal{M}_d(\mathbb{C})^+$ and $m < d$ be an integer. Then, the set $\Lambda(U(S_0, m))$ can be characterized as follows*:

1. *If $m \leqslant 0$, we have that*

$$\mu \in \Lambda\big(U(S_0, m)\big) \quad \Longleftrightarrow \quad \mu \geqslant \lambda(S_0). \tag{27}$$

2. *If $m \geqslant 1$, then*

$$\mu \in \Lambda\big(U(S_0, m)\big) \quad \Longleftrightarrow \quad \mu \geqslant \lambda(S_0) \quad \textit{and} \quad \mu_{d-m+i} \leqslant \lambda_i(S_0) \quad \textit{for } i \in \mathbb{I}_m. \tag{28}$$

Proof. It follows from Theorem A.1 and the Fan–Pall inequalities of Remark A.2. □

Corollary A.4. *Let $S_0 \in \mathcal{M}_d(\mathbb{C})^+$ and $m < d$ be an integer. Then $\Lambda(U(S_0, m))$ is convex.*

Proof. It is clear that the inequalities given in Eqs. (27) and (28) are preserved by convex combinations. Observe that also the set $\mathbb{R}_+^{d\downarrow}$ is convex. □

Remark A.5. Let $S_0 \in \mathcal{M}_d(\mathbb{C})^+$, $m < d$ be an integer and $S \in \mathcal{M}_d(\mathbb{C})^+$. The reader should note that the fact that $\lambda(S) \in \Lambda(U(S_0, m))$ does not imply that $S \in U(S_0, m)$. Indeed, it is fairly easy to produce examples of this phenomenon. Therefore, the spectral picture of $\Lambda(U(S_0, m))$ does not determine the set $U(S_0, m)$. This last assertion is a consequence of the fact that $U(S_0, m)$ is not saturated by unitary equivalence. Nevertheless, $\Lambda(U(S_0, m))$ allows to compute minimizers of submajorization in $U(S_0, m)$, since submajorization is an spectral preorder.

A.2. Step 2: minimizers for submajorization in $\Lambda(U_t(S_0, m))$

The spectral picture of $U(S_0, m)$ studied in the previous section motivates the definition of the following sets.

Definition A.6. Let $\lambda \in \mathbb{R}_+^{d\downarrow}$ and take an integer $m < d$. We consider the set

$$\Lambda(\lambda, m) = \begin{cases} \{\mu \in \mathbb{R}_+^{d\downarrow}\colon\ \mu \succcurlyeq \lambda\} & \text{if } m \leqslant 0, \\ \{\mu \in \Lambda(\lambda, 0)\colon\ \mu_{d-m+i} \leqslant \lambda_i \text{ for every } i \in \mathbb{I}_m\} & \text{if } m \geqslant 1. \end{cases} \tag{29}$$

Denote by $t_0 = \operatorname{tr}\lambda$. For $t \geqslant t_0$, we also consider the set

$$\Lambda_t(\lambda, m) = \{\mu \in \Lambda(\lambda, m)\colon\ \operatorname{tr}\mu \geqslant t\}.$$

Now Corollary A.3 can be rewritten as

Corollary A.7. *Let $S_0 \in \mathcal{M}_d(\mathbb{C})^+$ with $\lambda(S_0) = \lambda$, $m < d$ be an integer and $t \geqslant \operatorname{tr}(\lambda)$. Then we have the identities $\Lambda(U(S_0, m)) = \Lambda(\lambda, m)$ and $\Lambda(U_t(S_0, m)) = \Lambda_t(\lambda, m)$.* □

In this section, as a second step towards the proof of Theorem 3.12, we show that the sets $\Lambda_t(\lambda, m)$ have minimal elements with respect to submajorization and we describe explicitly these elements.

Let $\lambda \in \mathbb{R}_+^{d\downarrow}$ and $t_0 = \operatorname{tr}\lambda$. We recall the maps $r_\lambda(\cdot)$ and $c_\lambda(\cdot)$ introduced in 3.3. Fix $t \geqslant t_0$. Then

1. Given $r \in \mathbb{I}_{d-1} \cup \{0\}$ we denote by $p_\lambda(r, t) = \frac{t - \sum_{j=1}^r \lambda_j}{d-r}$, where we set $\sum_{j=1}^0 \lambda_j = 0$.
2. The maps $r_\lambda : [t_0, +\infty) \to \mathbb{I}_{d-1} \cup \{0\}$ and $c_\lambda : [t_0, +\infty) \to \mathbb{R}_{\geqslant 0}$ given by

$$r_\lambda(t) = \min\{r \in \mathbb{I}_{d-1} \cup \{0\}\colon\ p_\lambda(r, t) \geqslant \lambda_{r+1}\} \quad \text{and} \quad c_\lambda(t) = \frac{t - \sum_{i=1}^{r_\lambda(t)} \lambda_i}{d - r_\lambda(t)}. \tag{30}$$

In the following lemma we state several properties of these maps, which we shall use below. The proofs are technical but elementary, so that we only sketch the essential arguments.

Lemma A.8. *Let $\lambda \in \mathbb{R}_+^{d\downarrow}$ and $t_0 = \operatorname{tr}\lambda$.*

1. *The function r_λ is non-increasing and right-continuous, with $\lambda_{r_\lambda(t_0)+1} = \lambda_d$.*
2. *The image of r_λ is the set $\mathcal{B} = \{k \in \mathbb{I}_{d-1}\colon\ \lambda_k > \lambda_{k+1}\} \cup \{0\}$.*
3. *The map c_λ is piece-wise linear, strictly increasing and continuous.*
4. *We have that $c_\lambda(t_0) = \lambda_d$ and $c_\lambda(t) = t/d$ for $t \geqslant d\lambda_1$.*
5. *For every $t \in [t_0, d\lambda_1)$, if $r = r_\lambda(t)$ then $\lambda_{r+1} \leqslant c_\lambda(t) < \lambda_r$. In other words*

$$r_\lambda(t) = \min\{r \in \mathbb{I}_{d-1} \cup \{0\}\colon\ \lambda_{r+1} \leqslant c_\lambda(t)\}. \tag{31}$$

6. *For any $k \in \mathcal{B}$ let $s_k = \sum_{i=1}^k \lambda_i + (d-k)\lambda_{k+1}$. Then $r_\lambda(s_k) = k$ and $c_\lambda(s_k) = \lambda_{k+1}$. Moreover, the set $\mathcal{A}$ of discontinuity points of r_λ satisfies that*

$$\mathcal{A} = \{t \in (t_0, +\infty)\colon\ c_\lambda(t) = \lambda_{r_\lambda(t)+1}\} = c_\lambda^{-1}\{\lambda_i\colon\ \lambda_i \neq \lambda_d\} = \{s_k\colon\ k \in \mathcal{B}\}.$$

7. *Given $t \in [t_0, +\infty)$, such that $c_\lambda(t) = \lambda_m$ (even if $m \notin \mathcal{B}$), then*

- $t \in \mathcal{A} \Leftrightarrow \lambda_m \neq \lambda_d$.
- $r_\lambda(t) = 0 \Leftrightarrow c_\lambda(t) = \lambda_1 \Leftrightarrow t = d\lambda_1$.
- *If* $\lambda_m \neq \lambda_1$, *then* $r_\lambda(t) = \max\{j \in \mathbb{I}_d\colon \lambda_j > \lambda_m\}$ *and*

$$t = \sum_{i=1}^{m} \lambda_i + (d-m)\lambda_m = \sum_{i=1}^{r_\lambda(t)} \lambda_i + \big(d - r_\lambda(t)\big)\lambda_m. \tag{32}$$

Proof. Given $t \in [t_0, d\lambda_1)$ and $1 \leqslant r \leqslant d-1$, then $r = r_\lambda(t)$ if and only if

$$c_\lambda(t) = p_\lambda(r,t) \geqslant \lambda_{r+1} \quad \text{and} \quad p_\lambda(r-1,t) < \lambda_r. \tag{33}$$

On the other hand, the map $t \mapsto p_\lambda(r,t)$ is linear, continuous and increasing for any r fixed. From these facts one easily deduces the right continuity of the map r_λ, and that the map c_λ is continuous at the points where r_λ is. We can also deduce that if $c_\lambda(t) \neq \lambda_{r_\lambda(t)+1}$ then r_λ is continuous (i.e. constant) near the point t. Observe that, if $r = r_\lambda(t)$, then

$$\lambda_r \overset{(33)}{>} p_\lambda(r-1,t) = \frac{(d-r)p_\lambda(r,t) + \lambda_r}{d-r+1} \implies \lambda_r > p_\lambda(r,t) \geqslant \lambda_{r+1} \implies r \in \mathcal{B}. \tag{34}$$

Using that $r_\lambda(t) = 0$ for $t \geqslant d\lambda_1$, that $c_\lambda(t_0) = \lambda_d$, and the right continuity of the map r_λ, we have that $\mathcal{A} = \{t \in (t_0, +\infty)\colon c_\lambda(t) = \lambda_{r_\lambda(t)+1}\} = c_\lambda^{-1}\{\lambda_i\colon \lambda_i \neq \lambda_d\}$.

Hence, in order to check the continuity of c_λ we have to verify the continuity of c_λ from the left at the points $t > t_0$ for which $c_\lambda(t) = \lambda_{r_\lambda(t)+1}$. Note that, if $r = r_\lambda(t)$, then $r \in \mathcal{B}$ and

$$c_\lambda(t) = p_\lambda(r,t) = \frac{t - \sum_{j=1}^{r}\lambda_j}{d-r} = \lambda_{r+1} \implies t = \sum_{j=1}^{r}\lambda_j + (d-r)\lambda_{r+1}. \tag{35}$$

If $c_\lambda(t) = \lambda_d$ then $t = t_0$ and there is nothing to prove. Assume that $c_\lambda(t) = \lambda_{r_\lambda(t)+1} > \lambda_d$. Then $\hat{r} = \max\{j \in \mathbb{I}_{d-1}\colon \lambda_j = \lambda_{r+1}\}$ is the first element of $\mathcal{B}$ after r. Note that $\lambda_{\hat{r}+1} < \lambda_{\hat{r}} = \lambda_{r+1}$. We shall see that if $s < t$ near t, then $r_\lambda(s) = \hat{r}$. Indeed, as in Eq. (35),

$$p_\lambda(\hat{r}, t+x) = \frac{(d-r)\lambda_{r+1} - \sum_{j=r+1}^{\hat{r}}\lambda_j + x}{d-\hat{r}} = \lambda_{r+1} + \frac{x}{d-\hat{r}} > \lambda_{\hat{r}+1} \quad \text{and}$$

$$p_\lambda(\hat{r}-1, t+x) = \frac{(d-r)\lambda_{r+1} - \sum_{j=r+1}^{\hat{r}-1}\lambda_j + x}{d-\hat{r}+1} = \lambda_{r+1} + \frac{x}{d-\hat{r}+1} < \lambda_{r+1} = \lambda_{\hat{r}}.$$

for $x \in (-\varepsilon, 0]$ if $\varepsilon > 0$ sufficiently small. By Eq. (33) we deduce that $r_\lambda(t+x) = \hat{r} \neq r_\lambda(t)$ for such an x, so that $t \in \mathcal{A}$ (r_λ is discontinuous at t). On the other hand,

$$c_\lambda(t+x) = p_\lambda(\hat{r}, t+x) = \lambda_{r_\lambda(t)+1} + \frac{x}{d-\hat{r}} \implies \lim_{x\to 0^-} c_\lambda(t+x) = \lambda_{r_\lambda(t)+1} = c_\lambda(t).$$

This last fact implies that c_λ is continuous and, since r_λ is right-continuous, that c_λ is a piece-wise linear and strictly increasing function. With the previous remarks, the proof of all other statements of the lemma becomes now straightforward. □

Definition A.9. Fix $\lambda \in \mathbb{R}_+^{d\downarrow}$. Take an integer $m < d$. Recall that if $m > 0$ we denote by

$$s^* = s^*(\lambda, m) = c_\lambda^{-1}(\lambda_m) = \sum_{i=1}^{m}\lambda_i + (d-m)\lambda_m.$$

Now we rewrite the definition of the maps $r_{\lambda,m}$ and $c_{\lambda,m}$: If $m > 0$ and $t \in [t_0, +\infty)$ let

$$c_{\lambda,m}(t) \overset{\text{def}}{=} \begin{cases} c_\lambda(t) & \text{if } t \leqslant s^* \\ \lambda_m + \frac{t-s^*}{d-m} & \text{if } t > s^* \end{cases} \quad \text{and}$$

$$r_{\lambda,m}(t) \overset{\text{def}}{=} \min\big\{r \in \mathbb{I}_{d-1} \cup \{0\}\colon c_{\lambda,m}(t) \geqslant \lambda_{r+1}\big\}.$$

If $m \leqslant 0$ and $t \in [t_0, +\infty)$ we define $c_{\lambda,m}(t) = c_\lambda(t)$ and $r_{\lambda,m}(t) = r_\lambda(t)$. Note that, by Eq. (31), $r_{\lambda,m}(t) = r_\lambda(t)$ for every $t \leqslant s^*$.

Corollary A.10. *Let $\lambda \in \mathbb{R}_+^{d\downarrow}$ and fix an integer $m < d$. Then the map $r_{\lambda,m}$ is not increasing and right continuous and the map $c_{\lambda,m}$ is strictly increasing and continuous on* $[\operatorname{tr}\lambda, +\infty)$.

Proof. The mentioned properties of the map $c_{\lambda,m}$ were proved in Remark 3.13 (whose proof uses Lemma A.8). With respect to the map $r_{\lambda,m}$, the statement follows from Lemma A.8 and A.9. □

A.2.1. Minimizers for submajorization in $\Lambda_t(\lambda, m)$ for $m \leqslant 0$

The following lemma is a standard fact in majorization theory. We include a short proof of it for the sake of completeness.

Lemma A.11. *Let $\alpha, \gamma \in \mathbb{R}^p$, $\beta \in \mathbb{R}^q$ and $x \in \mathbb{R}$ such that $x \leqslant \min_{k \in \mathbb{I}_p} \gamma_k$. Then,*

$$\operatorname{tr}(\gamma, b\mathbb{1}_q) \leqslant \operatorname{tr}(\alpha, \beta) \quad \textit{and} \quad \gamma \prec_w \alpha \quad \Longrightarrow \quad (\gamma, x\mathbb{1}_q) \prec_w (\alpha, \beta).$$

Observe that we are not assuming that $(\alpha, \beta) = (\alpha, \beta)^\downarrow$.

Proof. Let $h = \operatorname{tr} \beta$ and $\rho = \frac{h}{q}\mathbb{1}_q$. Then it is easy to see that

$$\sum_{i \in \mathbb{I}_k} (\gamma^\downarrow, x\mathbb{1}_q)_i \leqslant \sum_{i \in \mathbb{I}_k} (\alpha^\downarrow, \rho)_i \leqslant \sum_{i \in \mathbb{I}_k} (\alpha^\downarrow, \beta^\downarrow)_i \quad \text{for every } k \in \mathbb{I}_{p+q}.$$

Since $(\gamma^\downarrow, x\mathbb{1}_q) = (\gamma, x\mathbb{1}_q)^\downarrow$, we can conclude that $(\gamma, x\mathbb{1}_q) \prec_w (\alpha, \beta)$. □

In the following statement we shall use the maps r_λ and c_λ defined in Eq. (30) (or Definition 3.10).

Theorem A.12. *Fix $m \leqslant 0$. Let $\lambda \in \mathbb{R}_+^{d\downarrow}$, $t_0 = \operatorname{tr} \lambda$ and $t \in [t_0, +\infty)$. Consider the vector*

$$\nu = \nu_\lambda(t) \stackrel{\text{def}}{=} \left(\lambda_1, \ldots, \lambda_{r_\lambda(t)}, c_\lambda(t), \ldots, c_\lambda(t)\right) \quad \textit{if } r_\lambda(t) > 0, \tag{36}$$

or $\nu = \frac{t}{d}\mathbb{1}_d = c_t(\lambda)\mathbb{1}_d \in \Lambda_t(\lambda, m)$ if $r_\lambda(t) = 0$. Then ν satisfies that

$$\nu \in \Lambda_t(\lambda, m), \quad \operatorname{tr} \nu = t \quad \textit{and} \quad \nu \prec_w \mu \quad \textit{for every } \mu \in \Lambda_t(\lambda, m). \tag{37}$$

Proof. Given $t \in [t_0, +\infty)$, we denote by $r = r_\lambda(t)$. If $r = 0$ then,

$$t \geqslant d\lambda_1 \quad \text{and} \quad \lambda = \lambda^\downarrow \quad \Longrightarrow \quad c_\lambda(t) = \frac{t}{d} \geqslant \lambda_1 \quad \Longrightarrow \quad \nu = c\mathbb{1}_d \in \Lambda_t(\lambda, m).$$

It is clear that such a vector must satisfy that $\nu \prec_w \mu$ for every $\mu \in \Lambda_t(\lambda, m)$.

Suppose now that $r \geqslant 1$, so that $t < d\lambda_1$. Recall from Lemma A.8 that in this case we have that $\lambda_{r+1} \leqslant c_\lambda(t) < \lambda_r$. Hence $\nu \geqslant \lambda$ and $\nu = \nu^\downarrow$. It is clear from Eq. (15) that $\operatorname{tr}(\nu) = t$. From these facts we can conclude that $\nu \in \Lambda_t(\lambda, m)$ as claimed.

Now let $\mu \in \Lambda_t(\lambda, m)$ and notice that, since $\mu \geqslant \lambda$, we get that

$$\sum_{i=1}^{k} \mu_i \geqslant \sum_{i=1}^{k} \lambda_i = \sum_{i=1}^{k} \nu_i \quad \text{for every } 1 \leqslant k \leqslant r_\lambda(t).$$

Now we can apply Lemma A.11 (with $p = r_\lambda(t)$ and $x = c_\lambda(t)$) and deduce that $\nu \prec_w \mu$. □

A.2.2. Minimizers for submajorization in $\Lambda_t(\lambda, m)$. The general case

Recall that $\Lambda_t(\lambda, m) = \{\mu \in \mathbb{R}_+^{d\downarrow}\colon \ \mu \geqslant \lambda, \operatorname{tr} \mu \geqslant t$ and $\mu_{d-m+i} \leqslant \lambda_i$ for every $i \in \mathbb{I}_m\}$, for each $m \in \mathbb{I}_{d-1}$. In what follows we shall compute a minimal element in $\Lambda_t(\lambda, m)$ with respect to submajorization in terms of the number $s^* = s^*(\lambda, m) \stackrel{\text{def}}{=} c_\lambda^{-1}(\lambda_m)$ and the maps $r_{\lambda,m}$ and $c_{\lambda,m}$ described in Definition 3.10 (see also A.9).

Proposition A.13. *Let $\lambda \in \mathbb{R}_+^{d\downarrow}$, $t_0 = \operatorname{tr} \lambda$, $m \in \mathbb{I}_d$. If $t \in [t_0, s^*(\lambda, m)]$, then the vector $\nu = (\lambda_1, \ldots, \lambda_{r_\lambda(t)}, c_\lambda(t), \ldots, c_\lambda(t))$ of Eq. (36) satisfies that $\nu \in \Lambda_t(\lambda, m)$. Hence*

$$\operatorname{tr} \nu = t, \quad \nu_d = c_\lambda(t) \quad \textit{and} \quad \nu \prec_w \mu \quad \textit{for every } \mu \in \Lambda_t(\lambda, m).$$

Proof. We already know by Theorem A.12 that $\nu \in \Lambda_t(\lambda, 0)$ and $\operatorname{tr} \nu = t$. Using the inequality $c_\lambda(t) \leqslant c_\lambda(s^*) = \lambda_m$, the verification of the fact that $\nu \in \Lambda_t(\lambda, m)$ is direct. By Theorem A.12, we conclude that $\nu \prec_w \mu$ for every $\mu \in \Lambda_t(\lambda, m) \subseteq \Lambda_t(\lambda, 0)$. □

Recall the number $s^{**} = c_{\lambda,m}^{-1}(\lambda_1) = (d-m)\lambda_1 + \sum_{j=1}^{m} \lambda_j \geqslant s^*$ (with equality $\Leftrightarrow \lambda_1 = \lambda_m$) defined in Eq. (20) (see also Remark 3.13).

Definition A.14. Let $\lambda \in \mathbb{R}_+^{d\downarrow}$, $t_0 = \operatorname{tr}\lambda$ and $m \in \mathbb{Z}$ such that $m < d$. Fix $t \in [t_0, +\infty)$ and denote by $r = r_{\lambda,m}(t)$. Consider the vector $\nu_{\lambda,m}(t) \in \mathbb{R}_+^d$ given by the following rule:

- If $m \leqslant 0$ then $\nu_{\lambda,m}(t) = \nu_\lambda(t) \overset{(36)}{=} (\lambda_1, \dots, \lambda_r, c_{\lambda,m}(t)\mathbb{1}_{d-r})$.

If $m \geqslant 1$ we define

- $\nu_{\lambda,m}(t) = (\lambda_1, \dots, \lambda_r, c_{\lambda,m}(t)\mathbb{1}_{d-r})$ for $t \leqslant s^*$ (so that $r \geqslant m$ and $c_{\lambda,m}(t) \leqslant \lambda_m$).
- $\nu_{\lambda,m}(t) = (\lambda_1, \dots, \lambda_r, c_{\lambda,m}(t)\mathbb{1}_{d-m}, \lambda_{r+1}, \dots, \lambda_m)$ for $t \in (s^*, s^{**})$, and
- $\nu_{\lambda,m}(t) = (c_{\lambda,m}(t)\mathbb{1}_{d-m}, \lambda_1, \dots, \lambda_m)$ for $t \geqslant s^{**}$.

If $\lambda_1 = \lambda_m$, the second case of the definition of $\nu_{\lambda,m}(t)$ disappears.

In the following Lemma we state several properties of the map $\nu_{\lambda,m}(\cdot)$, which are easy to see:

Lemma A.15. *Let $\lambda \in \mathbb{R}_+^{d\downarrow}$, and $m \in \mathbb{Z}$ such that $m < d$. The map $\nu_{\lambda,m}(\cdot)$ of Definition A.14 has the following properties:*

1. *By Remark 3.13 the vector $\nu_{\lambda,m}(t) \in \mathbb{R}_+^{d\downarrow}$ (i.e. it is decreasing) for every t.*
2. *The map $\nu_{\lambda,m}(\cdot)$ is continuous.*
3. *It is increasing in the sense that $t_1 < t_2 \Rightarrow \nu_{\lambda,m}(t_1) \leqslant \nu_{\lambda,m}(t_2)$.*
4. *More precisely, for any fixed $k \in \mathbb{I}_d$, the k-th entry $\nu_{\lambda,m}^{(k)}(t)$ of $\nu_{\lambda,m}(t)$ is given by*

$$\nu_{\lambda,m}^{(k)}(t) = \begin{cases} \max\{\lambda_k, c_{\lambda,m}(t)\} & \text{if } k \leqslant d-m, \\ \min\{\max\{\lambda_k, c_{\lambda,m}(t)\}, \lambda_i\} & \text{if } k = d-m+i,\ i \in \mathbb{I}_m. \end{cases}$$

5. *The vector $\nu_{\lambda,m}(t) \in \Lambda_t(\lambda, m)$ and $\operatorname{tr}\nu_{\lambda,m}(t) = t$ for every $t \in [t_0, +\infty)$.* □

We can now state the main result of this section.

Theorem A.16. *Let $\lambda \in \mathbb{R}_+^{d\downarrow}$, $t_0 = \operatorname{tr}\lambda$ and $t \in [t_0, +\infty)$. Fix $m \in \mathbb{Z}$ such that $m < d$. Then the vector $\nu_{\lambda,m}(t)$ defined in A.14 is the unique element of $\Lambda_t(\lambda, m)$ such that*

$$\nu_{\lambda,m}(t) \prec_w \mu \quad \text{for every } \mu \in \Lambda_t(\lambda, m). \tag{38}$$

Proof. If $m \leqslant 0$ the result follows from Theorem A.12. Suppose now that $m \geqslant 1$. By Lemma A.15, the vector $\nu_{\lambda,m}(t) \in \Lambda_t(\lambda, m)$ and $\operatorname{tr}\nu_{\lambda,m}(t) = t$ for $t \in [t_0, +\infty)$. In Proposition A.13 we have shown that $\nu_{\lambda,m}(t)$ satisfies (38) for every $t \in [t_0, s^*(\lambda, m)]$. Hence we check the other two cases:

Case $t \in (s^*, s^{**})$: fix $\mu \in \Lambda_t(\lambda, m)$ such that $\operatorname{tr}\mu = t$. Let us denote by $r = r_{\lambda,m}(t)$,

$$\alpha = (\mu_1, \dots, \mu_r), \qquad \beta = (\mu_{r+1}, \dots, \mu_{r+d-m}), \qquad \gamma = (\mu_{r+d-m+1}, \dots, \mu_d),$$

$\rho = (\lambda_1, \dots, \lambda_r)$ and $\omega = (\lambda_{r+1}, \dots, \lambda_m)$. Then

$$\mu = (\alpha, \beta, \gamma) \quad \text{and} \quad \nu_{\lambda,m}(t) = \big(\rho, c_{\lambda,m}(t)\mathbb{1}_{d-m}, \omega\big).$$

Since $\mu \in \Lambda_t(\lambda, m)$ and $\operatorname{tr}\nu_{\lambda,m}(t) = \operatorname{tr}\mu = t$, then

$$\rho \leqslant \alpha, \quad \gamma \leqslant \omega \quad \text{and} \quad \operatorname{tr}(\alpha, \beta) \geqslant \operatorname{tr}\big(\rho, c_{\lambda,m}(t)\mathbb{1}_{d-m}\big).$$

Then we can apply Lemma A.11 to deduce that $(\rho, c_{\lambda,m}(t)\mathbb{1}_{d-m}) \prec_w (\alpha, \beta)$. Using this fact jointly with $\gamma \leqslant \omega$ one easily deduces that $\nu_{\lambda,m}(t) \prec \mu$ (because $\operatorname{tr}\mu = \operatorname{tr}\nu_{\lambda,m}(t) = t$).

The case $t \geqslant s^{**}$ for vectors $\mu \in \Lambda_t(\lambda, m)$ such that $\operatorname{tr}\mu = t$ follows similarly.

If we have that $\mu \in \Lambda_t(\lambda, m)$ with $\operatorname{tr}\mu = a > t$, then

$$\mu \in \Lambda_a(\lambda, m) \quad \Longrightarrow \quad \nu_{\lambda,m}(t) \leqslant \nu_{\lambda,m}(a) \prec \mu \quad \Longrightarrow \quad \nu_{\lambda,m}(t) \prec_w \mu,$$

where the first inequality follows from Lemma A.15. □

A.3. Step 3: minimizers for submajorization in $U_t(S_0, m)$

Let $S_0 \in \mathcal{M}_d(\mathbb{C})^+$ and let $t \geqslant t_0 = \operatorname{tr}(S_0)$. Notice that Corollary A.7 together with Theorem A.16 show that the sets $U_t(S_0, m)$ have minimal elements with respect to submajorization. We shall describe the geometrical structure of minimal elements in $U_t(S_0, m)$ with respect to submajorization for any $m < d$ in terms of the geometry of S_0. We shall see that, under some mild assumptions, there exists a unique $S_t \in U_t(S_0, m)$ such that $\lambda(S_t) = \nu_{\lambda,m}(t)$ (the vector of Theorem A.16 defined in A.14). In order to do this we recall a series of preliminary results and we fix some notations.

Interlacing inequalities. Let $A \in \mathcal{H}(d)$ with $\lambda(A) \in \mathbb{R}^{d\downarrow}$ and let $P = P^2 = P^* \in \mathcal{M}_d(\mathbb{C})^+$ be a projection with $\operatorname{rk} P = k$. The interlacing inequalities (see [3]) relate the eigenvalues of A with the eigenvalues of $PAP \in \mathcal{H}(d)$ as follows:

$$\lambda_{d-k+i}(A) \leqslant \lambda_i(PAP) \leqslant \lambda_i(A) \quad \text{for every } i \in \mathbb{I}_k. \tag{39}$$

On the other hand, if we have the equalities

$$\lambda_i(PAP) = \lambda_i(A) \quad \text{for every } i \in \mathbb{I}_k \text{ then } PA = AP, \tag{40}$$

and that $R(P)$ has an ONB $\{h_i\}_{i\in\mathbb{I}_k}$ such that $Ah_i = \lambda_i h_i$ for every $i \in \mathbb{I}_k$. Indeed, if $Q = I - P$, then $\operatorname{tr} QAQ = \sum_{i=k+1}^{d} \lambda_i(A)$. The interlacing inequalities applied to QAQ imply that

$$\lambda_{k+j}(A) \leqslant \lambda_j(QAQ) \quad \text{for } j \in \mathbb{I}_{d-k} \quad \Longrightarrow \quad \lambda_j(QAQ) = \lambda_{k+j}(A) \quad \text{for } j \in \mathbb{I}_{d-k}.$$

Taking Frobenius norms, we get that

$$\|A\|_2^2 = \sum_{i=1}^{d} \lambda_i(A)^2 = \|PAP\|_2^2 + \|QAQ\|_2^2 \quad \Longrightarrow \quad PAQ = QAP = 0,$$

so that $A = PAP + QAQ$. The Ky–Fan inequalities (see [3]) assure that

$$\sum_{i=1}^{k} \lambda_i(A) = \max\bigl\{\operatorname{tr} PAP\colon P \in \mathcal{M}_d(\mathbb{C})^+,\ P = P^2 = P^* \text{ and } \operatorname{rk} P = k\bigr\}. \tag{41}$$

As before, given an orthogonal projection P with $\operatorname{rk} P = k$ such that

$$\operatorname{tr} PAP = \sum_{i=1}^{k} \lambda_i(A) \quad \overset{(39)}{\Longrightarrow} \quad \lambda_i(PAP) = \lambda_i(A) \quad \text{for } i \in \mathbb{I}_k \quad \overset{(40)}{\Longrightarrow} \quad PA = AP, \tag{42}$$

and $R(P)$ has an ONB of eigenvectors for A associated to $\lambda_1(A), \dots, \lambda_k(A)$. If we further assume that $\lambda_k(A) > \lambda_{k+1}(A)$ then in both cases (40) and (42) the projection P is unique, since the eigenvectors associated to the first k eigenvalues of A generate a unique subspace of $\mathbb{C}^d$.

Notations. We fix a matrix $S \in \mathcal{M}_d(\mathbb{C})^+$ with $\lambda(S) = \lambda = (\lambda_1, \dots, \lambda_d) \in \mathbb{R}_+^{d\downarrow}$. We shall also fix an orthonormal basis $\{h_i\}_{i\in\mathbb{I}_d}$ of $\mathbb{C}^d$ such that

$$Sh_i = \lambda_i h_i \quad \text{for every } i \in \mathbb{I}_d.$$

Any other such basis will be denoted as a "ONB of eigenvectors for S, λ".

Lemma A.17. *Let $B \in \mathcal{M}_d(\mathbb{C})^+$ and $r \in \mathbb{I}_{d-1}$ such that $\lambda(S + B) = (\lambda_1, \dots, \lambda_r, \alpha)$, for some $\alpha \in \mathbb{R}_+^{d-r\downarrow}$ such that $\alpha_1 \leqslant \lambda_r$. Let $\mathcal{M}_r \overset{\text{def}}{=} \operatorname{span}\{h_i\colon i \in \mathbb{I}_r\}$ and $P = P_{\mathcal{M}_r}$. Then*

$$PB = BP = PBP = 0.$$

Proof. Since $\operatorname{rk} P = r$ and $\operatorname{tr}(PSP) = \sum_{i=1}^{r} \lambda_i$, then the Ky–Fan theorem (41) assures that

$$0 \leqslant \operatorname{tr}(PBP) = \operatorname{tr}\bigl(P(S + B)P\bigr) - \operatorname{tr}(PSP) \leqslant \sum_{i=1}^{r} \lambda_i(S + B) - \sum_{i=1}^{r} \lambda_i = 0.$$

Since $B \geqslant 0$, we have that $\operatorname{tr}(PBP) = 0 \Longrightarrow PBP = 0 \Longrightarrow BP = PB = 0$. $\square$

Proposition A.18. *Let $r \in \mathbb{I}_{d-1}$, then for each $c \in [\lambda_{r+1}, \lambda_r]$ there is a unique $B \in \mathcal{M}_d(\mathbb{C})^+$ such that $\lambda(S+B) = (\lambda_1, \dots, \lambda_r, c\mathbb{1}_{d-r})$. Moreover, it is given by*

$$B = \sum_{i=1}^{d-r} (c - \lambda_{r+i}) h_{r+i} \otimes h_{r+i} \quad \text{and} \quad S + B = \sum_{i=1}^{r} \lambda_i \cdot h_i \otimes h_i + c \cdot \sum_{i=r+1}^{d} h_i \otimes h_i. \tag{43}$$

Proof. Let $\mathcal{M}_r \stackrel{\text{def}}{=} \operatorname{span}\{h_i\colon i \in \mathbb{I}_r\}$ and $P = P_{\mathcal{M}_r}$. Suppose that $B \in \mathcal{M}_d(\mathbb{C})^+$ is such that $\lambda(S+B) = (\lambda_1, \dots, \lambda_r, c\mathbb{1}_{d-r})$. Then, by Lemma A.17, $BP = PB = 0$. Hence

$$P(S+B)P = (S+B)P = SP = \sum_{i=1}^{r} \lambda_i h_i \otimes h_i \quad \overset{\text{Eq. (42)}}{\Longrightarrow} \quad (S+B)Q = cQ,$$

where $Q = I - P$. Hence $B = BQ = cQ - SQ = \sum_{i=1}^{d-r} (c - \lambda_{r+i}) h_{r+i} \otimes h_{r+i}$. $\square$

Remark A.19. In Lemma A.17, we allow the case where $\lambda_r = \lambda_{r+1} = \alpha_1$. In this case we could change h_r by h_{r+1} (or any other eigenvector for λ_r) as a generator for $\mathcal{M}_r$. The proof of the lemma assures that we get another projector P' which also satisfies that $BP' = 0$.

Similarly, in Proposition A.18 we allow the case where $\lambda_r = \lambda_{r+1} = c$. By the previous comments, the projection P in the proof of Proposition A.18 is not unique. Nevertheless, in this case the positive perturbation B is unique, because we have that $\operatorname{rk} B < d - m$ (this follows from the fact that $(c - \lambda_{r+1}) h_{r+1} \otimes h_{r+1} = 0$). In fact $B = cQ - SQ$, where Q is the orthogonal projector onto the sum of the eigenspaces of S for the eigenvalues $\lambda_i < c$.

Lemma A.20. *Let $m \in \mathbb{I}_{d-1}$ and $B \in \mathcal{M}_d(\mathbb{C})^+$ with $\operatorname{rk} B \leqslant d - m$. Assume that*

$$\lambda(S+B) = (c\mathbb{1}_{d-m}, \lambda_1, \dots, \lambda_m),$$

for some $c \geqslant \lambda_1$. Then there exists an ONB $\{v_i\}_{i \in \mathbb{I}_d}$ of eigenvectors for S, λ such that

$$B = \sum_{i=1}^{d-m} (c - \lambda_{m+i}) v_{m+i} \otimes v_{m+i} \quad \text{so that } S + B = \sum_{i=1}^{m} \lambda_i \cdot v_i \otimes v_i + c \cdot \sum_{i=m+1}^{d} v_i \otimes v_i. \tag{44}$$

If we assume further that $\lambda_m > \lambda_{m+1}$ then B is unique, and Eq. (44) holds for any ONB of eigenvectors for S, λ.

Proof. Note that, since $\operatorname{rk} B \leqslant d - m$, then

$$\sum_{i=1}^{d-m} \lambda_i(B) = \operatorname{tr} B = \operatorname{tr}(B+S) - \operatorname{tr} S = c(d-m) - \sum_{j=m+1}^{d} \lambda_j. \tag{45}$$

Take a subspace $\mathcal{M} \subseteq \mathbb{C}^n$ such that $R(B) \subseteq \mathcal{M}$ and $\dim \mathcal{M} = d - m$. Denote by $Q = P_{\mathcal{M}}$. Then $QBQ = B$, and the Ky–Fan inequalities (41) for $S + B$ assure that

$$\begin{aligned} \operatorname{tr}(QSQ) &= \operatorname{tr}\bigl(Q(S+B)Q\bigr) - \operatorname{tr} B \\ &\leqslant \sum_{i=1}^{d-m} \lambda_i(S+B) - \operatorname{tr} B = c(d-m) - \operatorname{tr} B \overset{(45)}{=} \sum_{j=m+1}^{d} \lambda_j. \end{aligned}$$

The equality in Ky–Fan inequalities (for $-S$) force that $\mathcal{M} = \operatorname{span}\{v_{m+1}, \dots, v_d\}$, for some ONB $\{v_i\}_{i \in \mathbb{I}_d}$ of eigenvectors for S, λ (see the remark following Eq. (42)). Thus, we get that $QS = SQ = \sum_{i=1}^{d-m} \lambda_{m+i} v_{m+i} \otimes v_{m+i}$. Since $R(B) \subseteq \mathcal{M}$ then $P \stackrel{\text{def}}{=} I - Q \leqslant P_{\ker B}$, and

$$BP = 0 \quad \Longrightarrow \quad P(S+B)P = SP = \sum_{i=1}^{m} \lambda_i v_i \otimes v_i \quad \overset{\text{Eq. (42)}}{\Longrightarrow} \quad (S+B)Q = cQ.$$

Therefore we can now compute

$$B = BQ = (S+B)Q - SQ = \sum_{i=1}^{d-m} (c - \lambda_{m+i}) v_{m+i} \otimes v_{m+i}. \tag{46}$$

Finally, if we further assume that $\lambda_m > \lambda_{m+1}$ then the subspace $\mathcal{M} = \operatorname{span}\{v_{m+1}, \dots, v_d\}$ is independent of the choice of the ONB of eigenvectors for S, λ. Thus, in this case B is uniquely determined by (46). $\square$

Proposition A.21. *Let $m \in \mathbb{I}_{d-1}$ and $B \in \mathcal{M}_d(\mathbb{C})^+$ with $\operatorname{rk} B \leqslant d-m$. Let $c \in \mathbb{R}$ such that $\lambda_{r+1} \leqslant c < \lambda_r$, for some $r < m$. Assume that*

$$\lambda(S+B) = (\lambda_1, \dots, \lambda_r, c\mathbb{1}_{d-m}, \lambda_{r+1}, \dots, \lambda_m).$$

Then there exists an ONB $\{v_i\}_{i\in\mathbb{I}_d}$ of eigenvectors for S, λ such that

$$B = \sum_{i=1}^{d-m} (c - \lambda_{m+i}) v_{m+i} \otimes v_{m+i} \quad \text{so that} \quad S + B = \sum_{i=1}^{m} \lambda_i \cdot v_i \otimes v_i + c \cdot \sum_{i=m+1}^{d} v_i \otimes v_i.$$

If we further assume that $\lambda_m > \lambda_{m+1}$ then B is unique.

Proof. Consider the subspace $\mathcal{M}_r = \operatorname{span}\{h_1, \dots, h_r\}$ and $P = P_{\mathcal{M}_r}$. By Lemma A.17, we know that $PB = BP = 0$. Let $S_1 = S|_{\mathcal{M}_r^\perp}$ and $B_1 = B|_{\mathcal{M}_r^\perp} (= B)$ considered as operators in $L(\mathcal{M}_r^\perp)$. Then S_1 and B_1 are in the conditions of Lemma A.20, so that there exists an ONB $\{w_i\}_{i\in\mathbb{I}_{d-r}}$ of $\mathcal{M}_r^\perp$ of eigenvectors for S_1, $(\lambda_{r+1}, \dots, \lambda_d)$ such that

$$B = B_1 = \sum_{i=1}^{d-m} (c - \lambda_{m+i}) w_{m+i} \otimes w_{m+i}.$$

Finally, let $\{v_i\}_{i\in\mathbb{I}_d}$ be given by $v_i = h_i$ for $1 \leqslant i \leqslant r$ and $v_{r+i} = w_i$ for $r+1 \leqslant i \leqslant d$. Then $\{v_i\}_{i\in\mathbb{I}_d}$ has the desired properties. Notice that if we further assume that $\lambda_m > \lambda_{m+1}$ then Lemma A.20 implies that B_1 is unique and therefore B is unique, too. □

Remark A.22. With the notations of Lemma A.20 assume that $\lambda_m = \lambda_{m+1}$. In this case B is not uniquely determined. Next we obtain a parametrization of the set of all operators $B \in \mathcal{M}_d(\mathbb{C})^+$ such that $\lambda(S+B) = (c\mathbb{1}_{d-m}, \lambda_1, \dots, \lambda_m)$. Consider $p = (d-m) - \#\{i\colon\ \lambda_i < \lambda_{m+1}\}$ and notice that in this case we have that $1 \leqslant p < \#\{i\colon\ \lambda_i = \lambda_{m+1}\} = \dim\ker(S - \lambda_{m+1} I)$. Then, for every $B \in \mathcal{M}_d(\mathbb{C})^+$ as above there corresponds a subspace $\mathcal{N} = \operatorname{span}\{h_i\colon\ m+1 \leqslant i \leqslant m+p\} \subset \ker(S - \lambda_m I)$ with $\dim \mathcal{N} = p$ such that

$$B = (c - \lambda_m) P_{\mathcal{N}} + \sum_{i=p+1}^{d-m} (c - \lambda_{m+i}) h_{m+i} \otimes h_{m+i}. \tag{47}$$

Conversely, for every subspace $\mathcal{N} \subset \ker(S - \lambda_m I)$ with $\dim \mathcal{N} = p$ then the operator $B \in \mathcal{M}_d(\mathbb{C})^+$ given by (47) satisfies that $\lambda(S+B) = (c\mathbb{1}_{d-m}, \lambda_1, \dots, \lambda_m)$. Since the previous map $B \mapsto P_{\mathcal{N}}$ is bijective, we see that the set of all such operators B is parametrized by the set of projections $P_{\mathcal{N}}$ such that $\mathcal{N} \subset \ker(S - \lambda_m I)$ is a p-dimensional subspace. Moreover, this map is actually an homeomorphism between these sets, with their usual metric structures.

Finally, if we let $k = \#\{i\colon\ \lambda_i > \lambda_m\}$ then the set of operators $S+B$ such that $B \in \mathcal{M}_d(\mathbb{C})^+$ with $\operatorname{rk} B \leqslant m-d$ and such that $\lambda(S+B) = (c\mathbb{1}_{d-m}, \lambda_1, \dots, \lambda_m)$ is given by

$$S + B = \sum_{i=1}^{k} \lambda_i \cdot h_i \otimes h_i + \lambda_m \cdot P_{\mathcal{N}'} + c \cdot \left(P_{\mathcal{N}} + \sum_{i=p+1}^{d-m} h_i \otimes h_i \right),$$

where $\mathcal{N} \subset \ker(S - \lambda_m I)$ is a subspace with $\dim \mathcal{N} = p$ and $\mathcal{N}' = \ker(S - \lambda_{m+1} I) \cap \mathcal{N}^\perp$.

As a consequence of the proof of Proposition A.21, we have a similar description of the operators B of its statement.

A.3.1. Proofs of the main results

Proof of Theorem 3.12. It is a consequence of Corollary A.7, Theorem A.16, and the results of this section (Lemma A.20 and Propositions A.18, A.21). The arrow (b) ⇒ (a) in Item 2 follows by Definition A.14 and the fact that both matrices S_0 and B are diagonal on the same basis (as, for example, in Eq. (43)). □

Proof of Proposition 3.14. It is a consequence of Corollary A.7, Definition A.14, Lemma A.15 and Theorem A.16. □

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