J. Math. Anal. Appl. 402 (2013) 103–118

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Geometry of unitary orbits of pinching operators

Eduardo Chiumiento [a,c,*], María E. Di Iorio y Lucero [b,c]

[a] Departamento de Matemática, FCE-UNLP, Calles 50 y 115, (1900) La Plata, Argentina

[b] Instituto de Ciencias, UNGS, J. M. Gutierrez 1150, (1613) Los Polvorines, Buenos Aires, Argentina

[c] Instituto Argentino de Matemática, "Alberto P. Calderón", CONICET, Saavedra 15 Piso 3, (1083) Buenos Aires, Argentina

ARTICLE INFO

Article history:
Received 16 January 2012
Available online 29 December 2012
Submitted by Lawrence A. Fialkow

Keywords:
Pinching operator
Left representation
Symmetrically-normed ideal
Submanifold
Covering space

ABSTRACT

Let $\mathfrak{I}$ be a symmetrically-normed ideal of the space of bounded operators acting on a Hilbert space $\mathcal{H}$. Let $\{p_i\}_1^w (1 \le w \le \infty)$ be a family of mutually orthogonal projections on $\mathcal{H}$. The pinching operator associated with the former family of projections is given by

$$P : \mathfrak{I} \longrightarrow \mathfrak{I}, \quad P(x) = \sum_{i=1}^{w} p_i x p_i .$$

Let $\mathcal{U}_{\mathfrak{I}}$ denote the Banach–Lie group of the unitary operators whose difference with the identity belongs to $\mathfrak{I}$. We study geometric properties of the orbit

$$\mathcal{U}_{\mathfrak{I}}(P) = \{L_u P L_{u^*} : u \in \mathcal{U}_{\mathfrak{I}}\},$$

where L_u is the left representation of $\mathcal{U}_{\mathfrak{I}}$ on the algebra $\mathcal{B}(\mathfrak{I})$ of bounded operators acting on $\mathfrak{I}$. The results include necessary and sufficient conditions for $\mathcal{U}_{\mathfrak{I}}(P)$ to be a submanifold of $\mathcal{B}(\mathfrak{I})$. Special features arise in the case of the ideal $\mathfrak{K}$ of compact operators. In general, $\mathcal{U}_{\mathfrak{K}}(P)$ turns out to be a non complemented submanifold of $\mathcal{B}(\mathfrak{K})$. We find a necessary and sufficient condition for $\mathcal{U}_{\mathfrak{K}}(P)$ to have complemented tangent spaces in $\mathcal{B}(\mathfrak{K})$. We also show that $\mathcal{U}_{\mathfrak{I}}(P)$ is a covering space of another orbit of pinching operators.

1. Introduction

Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space and $\mathcal{B}(\mathcal{H})$ the space of bounded linear operators acting on $\mathcal{H}$. We denote by $\mathcal{U}$ the group of unitary operators on $\mathcal{H}$. Let Φ be a symmetric norming function and $\mathfrak{I} = \mathfrak{S}_\Phi$ the corresponding symmetrically-normed ideal of $\mathcal{B}(\mathcal{H})$ equipped with the norm $\|\cdot\|_{\mathfrak{I}}$. Let $\mathcal{U}_{\mathfrak{I}}$ denote the group of unitaries which are perturbations of the identity by an operator in $\mathfrak{I}$, i.e.

$$\mathcal{U}_{\mathfrak{I}} = \{u \in \mathcal{U} : u - 1 \in \mathfrak{I}\}.$$

It is a real Banach–Lie group with the topology defined by the metric $d(u_1, u_2) = \|u_1 - u_2\|_{\mathfrak{I}}$, and its Lie algebra equals

$$\mathfrak{I}_{sh} = \{x \in \mathfrak{I} : x^* = -x\},$$

which is the real Banach space of skew-hermitian operators in $\mathfrak{I}$ (see [4]).

Let $\{p_i\}_1^w$ $(1 \le w \le \infty)$ be a family of mutually orthogonal hermitian projections in $\mathcal{B}(\mathcal{H})$. We do not make any assumption on the sum of all the projections of the family, so we could have that the projection $p_0 := 1 - \sum_{i=1}^w p_i$ is

* Corresponding author at: Departamento de Matemática, FCE-UNLP, Calles 50 y 115, (1900) La Plata, Argentina.
E-mail addresses: eduardo@mate.unlp.edu.ar (E. Chiumiento), mdiiorio@ungs.edu.ar (M.E. Di Iorio y Lucero).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2012.12.060

nonzero. The *pinching operator* associated with $\{ p_i \}_1^w$ is defined by

$$P : \mathfrak{I} \longrightarrow \mathfrak{I}, \qquad P(x) = \sum_{i=1}^{w} p_i x p_i,$$

where in case $w = \infty$ the series is convergent in the uniform norm. Let $\mathcal{B}(\mathfrak{I})$ denote the Banach algebra of bounded operators acting on $\mathfrak{I}$. Left multiplication defines the bounded linear operators $L_x : \mathfrak{I} \longrightarrow \mathfrak{I}$, $L_x(y) = xy$, for $x \in \mathcal{B}(\mathcal{H})$ and $y \in \mathfrak{I}$. The left representation of $\mathcal{U}_\mathfrak{I}$ on $\mathcal{B}(\mathfrak{I})$, namely $\mathcal{U}_\mathfrak{I} \longrightarrow \mathcal{B}(\mathfrak{I})$, $u \mapsto L_u$, allows us to introduce the following orbit

$$\mathcal{U}_\mathfrak{I}(P) := \{ L_u P L_{u^*} : u \in \mathcal{U}_\mathfrak{I} \} .$$

The aim of this paper is to study geometric properties of this orbit. Since every pinching operator is a continuous projection, the present work might be regarded as a contribution to the vast literature on the differential geometry of unitary orbits of projections in different settings (see e.g. [1,3,6,12,13,24]). Despite of some usual geometric properties that have already been studied in the afore-mentioned papers and still hold in this special orbit, we will also show some new special features of $\mathcal{U}_\mathfrak{I}(P)$, especially concerning with its submanifold structure (see Theorem 3.7, Theorem 4.7). We also go further into the topological structure of this orbit by proving that $\mathcal{U}_\mathfrak{I}(P)$ is a covering space of an orbit of pinching operators containing P (see Theorem 5.5). This topological result is another motivation for the study of $\mathcal{U}_\mathfrak{I}(P)$, and it has its counterpart in von Neumann algebras with unitary orbits of conditional expectations [3].

Pinching operators generalize the so-called notion of pinching of block matrices developed in matrix analysis (see e.g. [14,15,7]). In the framework of symmetrically-normed ideals, these operators have been studied in [17,23]. If $\mathfrak{I}$ is the trace class ideal, pinching operators arise in quantum mechanics due to a well-known postulate of von Neumann on the measurement of density operators [25]. More recently, they have been shown to be examples of the quantum reduction maps introduced in [21].

2. Preliminaries

Symmetrically-normed ideals. We begin with some basic facts on symmetrically-normed ideals. For a deeper discussion of this subject we refer the reader to [17] or [23].

Let $\mathcal{H}$ be a Hilbert space. No confusion will arise if $\| \cdot \|$ denotes the norm of vectors in $\mathcal{H}$ and the uniform norm in $\mathcal{B}(\mathcal{H})$. For $\xi, \eta \in \mathcal{H}$, let $\xi \otimes \eta$ be the rank one operator defined by $(\xi \otimes \eta)(\zeta) = \langle \zeta, \eta \rangle \xi$, for $\zeta \in \mathcal{H}$. By a *symmetrically-normed ideal* we mean a two-sided ideal $\mathfrak{I}$ of $\mathcal{B}(\mathcal{H})$ endowed with a norm $\| \cdot \|_\mathfrak{I}$ satisfying

- $(\mathfrak{I}, \| \cdot \|_\mathfrak{I})$ is a Banach space.
- $\|xyz\|_\mathfrak{I} \le \|x\| \, \|y\|_\mathfrak{I} \|z\|$, for $x, z \in \mathcal{B}(\mathcal{H})$ and $y \in \mathfrak{I}$.
- $\|\xi \otimes \eta\|_\mathfrak{I} = \|\xi\| \, \|\eta\|$, for $\xi, \eta \in \mathcal{H}$.

A result that goes back to J. Calkin [10] states the inclusions $\mathfrak{F} \subseteq \mathfrak{I} \subseteq \mathfrak{K}$, where $\mathfrak{F}$ is the set of all the finite rank operators, $\mathfrak{I}$ is a two-sided ideal of $\mathcal{B}(\mathcal{H})$ and $\mathfrak{K}$ the ideal of compact operators on $\mathcal{H}$.

Symmetrically-normed ideals are closely related to the following class of norms. Let $\hat{c}$ be the real vector space consisting of all sequences with a finite number of nonzero terms. A *symmetric norming function* is a norm $\Phi : \hat{c} \to \mathbb{R}$ satisfying the following properties:

- $\Phi(1, 0, 0, \ldots) = 1$.
- $\Phi(a_1, a_2, \ldots, a_n, 0, 0, \ldots) = \Phi(|a_{j_1}|, |a_{j_2}|, \ldots, |a_{j_n}|, 0, 0, \ldots)$, where $j_1, \ldots, j_n$ is any permutation of the integers $1, 2, \ldots, n$ and $n \ge 1$.

Any symmetric norming function Φ gives rise to two symmetrically-normed ideals. Indeed, for any compact operator x one may consider the sequence $(s_n(x))_n$ of its singular values arranged in non-increasing order, and thus define

$$\|x\|_\Phi := \sup_{k \ge 1} \Phi(s_1(x), s_2(x), \ldots, s_k(x), 0, 0, \ldots) \in [0, \infty].$$

It turns out that

$$\mathfrak{S}_\Phi := \{ x \in \mathfrak{K} : \|x\|_\Phi < \infty \}$$

and the $\| \cdot \|_\Phi$-closure in $\mathfrak{S}_\Phi$ of the finite rank operators, that is

$$\mathfrak{S}_\Phi^{(0)} := \overline{\mathfrak{F}}^{\| \cdot \|_\Phi},$$

are symmetrically-normed ideals. It is not difficult to show that $\mathfrak{S}_\Phi^{(0)} = \mathfrak{S}_\Phi$ if and only if $\mathfrak{S}_\Phi$ is separable. Moreover, any separable symmetrically-normed ideal coincides with some $\mathfrak{S}_\Phi^{(0)}$ (see [17, p. 89]).

Submanifolds. In the paper we will use different notions of submanifold of a (Banach) manifold. Since the terminology is not uniform in the literature, we need to mention that we follow Bourbaki [9]. To be precise, let M be a manifold and N a topological space contained in M. Recall that a subspace F of a Banach space E is said to be complemented if F is closed and there exists a closed subspace F_1 such that $F \oplus F_1 = E$. We will use the following definitions:

- N is a *submanifold* of M if for each point $x \in N$ there exists a Banach space E and a chart $(\mathcal{W}, \phi)$ at x, $\phi : \mathcal{W} \subseteq M \longrightarrow E$, such that $\phi(\mathcal{W} \cap N)$ is a neighborhood of 0 in a complemented subspace of E.
- N is a *quasi submanifold* of M if for each point $x \in N$ there exists a Banach space E and a chart $(\mathcal{W}, \phi)$ at x, $\phi : \mathcal{W} \subseteq M \longrightarrow E$, such that $\phi(\mathcal{W} \cap N)$ is a neighborhood of 0 in a closed subspace of E.

The following criterion will be useful (see [9]).

Proposition 2.1. *Let M be a manifold, N be a topological space and $N \subseteq M$. Then N is a submanifold (resp. quasi submanifold) of M if and only if the topology of N coincides with the topology inherited from M and the differential map of the inclusion map $N \hookrightarrow M$ has complemented range (resp. closed range) at every $x \in N$.*

Pinching operators. Let Φ a symmetric norming function, and $\mathfrak{I} = \mathfrak{S}_\Phi$. Recall that given a family $\{ p_i \}_1^w$ $(1 \leq w \leq \infty)$ of mutually orthogonal hermitian projections, i.e.

$$p_i = p_i^*, \qquad p_i p_j = \delta_{ij},$$

we define the pinching operator associated with the family by

$$P : \mathfrak{I} \longrightarrow \mathfrak{I}, \qquad P(x) = \sum_{i=1}^{w} p_i x p_i.$$

Notice that we might have $w = \infty$. Since x is compact, the series, which at first converges in the strong operator topology, turns out to be convergent in the uniform norm (see [17, p. 52]). It is also noteworthy that P is well defined in the sense that $P(x) \in \mathfrak{I}$ whenever $x \in \mathfrak{I}$ (see [17, p. 82]).

Below we need to consider the Banach algebra $\mathcal{B}(\mathfrak{I})$ of all bounded operators on $\mathfrak{I}$ with the usual operator norm: for $X \in \mathcal{B}(\mathfrak{I})$,

$$\|X\|_{\mathcal{B}(\mathfrak{I})} = \sup_{\|y\|_{\mathfrak{I}}=1} \|X(y)\|_{\mathfrak{I}}.$$

We will denote by $R(X)$ the range of X. In the next proposition, we collect some basic properties of pinching operators.

Proposition 2.2. *Let Φ a symmetric norming function, and $\mathfrak{I} = \mathfrak{S}_\Phi$. Let P be the pinching operator associated with a family $\{ p_i \}_1^w$. The following assertions hold:*

(i) $P^2 = P$.
(ii) $P(xyz) = xP(y)z$, *where* $x, z \in R(P)$ *and* $y \in \mathfrak{I}$.
(iii) $P(x)^* = P(x^*)$.
(iv) *P is continuous. In fact,* $\|P\|_{\mathcal{B}(\mathfrak{I})} = 1$.

Proof. The proofs of (i)–(iii) are trivial. For a proof of (iv) we refer the reader to [17, p. 82]. □

Now we show that $\mathcal{U}_\mathfrak{I}(P)$ has a smooth manifold structure endowed with the quotient topology.

Lemma 2.3. *Let $x \in \mathcal{B}(\mathcal{H})$. Then $L_x P = P L_x$ if and only if $x = \sum_{i=0}^{w} p_i x p_i$.*

Proof. Suppose that $L_x P = P L_x$, which actually means that

$$\sum_{i=1}^{w} (p_i x - x p_i) y p_i = 0, \tag{1}$$

for all $y \in \mathfrak{I}$. Let $i \geq 0$ and $(e_{i,n})_n$ be a sequence of finite rank projections such that $e_{i,n} \leq p_i$ and $e_{i,n} \nearrow p_i$ in the strong operator topology. We first assume that $i \geq 1$. Replacing y by $e_{i,n}$, we get $p_i x e_{i,n} = x e_{i,n}$ for all $n \geq 1$. This gives $p_i x p_i = x p_i$ for all $i \geq 1$. Thus $p_j x p_i = 0$ for all $i \geq 1$, $j \geq 0$ and $i \neq j$. In the case in which $i = 0$ we replace y by $e_{0,n} x^*$ and multiply on the right Eq. (1) by p_j, where $j \geq 1$. Then we see that $p_j x e_{0,n} x^* p_j = 0$, so that $p_j x p_0 x^* p_j = 0$, and this implies $p_j x p_0 = 0$ for all $j \geq 1$. □

Proposition 2.4. *Let Φ a symmetric norming function, and $\mathfrak{I} = \mathfrak{S}_\Phi$. Then $\mathcal{U}_\mathfrak{I}(P)$ is a real analytic homogeneous space of $\mathcal{U}_\mathfrak{I}$.*

Proof. Note that the isotropy group at P of the natural underlying action of $\mathcal{U}_\mathfrak{I}$ is

$$G = \{ u \in \mathcal{U}_\mathfrak{I} : L_u P = P L_u \}.$$

It is a closed subgroup of $\mathcal{U}_\mathfrak{I}$. Its Lie algebra can be identified with

$$\mathcal{G} = \{ z \in \mathfrak{I}_{sh} : L_z P = P L_z \}.$$

We will prove that G is a Banach–Lie subgroup of $\mathcal{U}_\mathfrak{I}$. Let $u = e^z \in G$, with $z \in \mathfrak{I}_{sh}$ and $\|z\|_\mathfrak{I} < \pi$. By the condition on the norm of z, we have $z = \log(u) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)} (u-1)^{n+1}$. Notice that $L_u P = P L_u$, or $L_{u-1} P = P L_{u-1}$, clearly implies $L_{r(u-1)} P =$

$PL_{r(u-1)}$ for any polynomial $r \in \mathbb{R}[X]$, and by continuity we have $L_zP = PL_z$. Denote by $\exp_{\mathcal{U}_\mathfrak{I}} : \mathfrak{I}_{sh} \longrightarrow \mathcal{U}_\mathfrak{I}$, $\exp_{\mathcal{U}_\mathfrak{I}}(z) = e^z$ the exponential map of the Banach–Lie group $\mathcal{U}_\mathfrak{I}$. Hence we have proved that $\exp_{\mathcal{U}_\mathfrak{I}}(\mathcal{G} \cap V) = G \cap \exp_{\mathcal{U}_\mathfrak{I}}(V)$, for any sufficiently small neighborhood V of the origin in $\mathfrak{I}_{sh}$.

On the other hand, by Lemma 2.3 we can rewrite the Lie algebra as

$$\mathcal{G} = \left\{ \sum_{i=0}^{w} p_i z p_i : z \in \mathfrak{I}_{sh} \right\},$$

which is a real closed subspace of $\mathfrak{I}_{sh}$. Moreover, the following subspace

$$\mathcal{M} = \{ z \in \mathfrak{I}_{sh} : p_i z p_i = 0,\ \forall\, i \geq 0 \} = \left\{ \sum_{i \neq j} p_i z p_j : z \in \mathfrak{I}_{sh} \right\}$$

is a closed supplement for $\mathcal{G}$ in $\mathfrak{I}_{sh}$. Then, G is a Banach–Lie subgroup of $\mathcal{U}_\mathfrak{I}$, and by [24, Theorem 8.19] we conclude that $\mathcal{U}_\mathfrak{I}(P)$ is a real analytic homogeneous space of $\mathcal{U}_\mathfrak{I}$. □

3. Submanifold structure of $\mathcal{U}_\mathfrak{I}(P)$. The case $\mathfrak{I} \neq \mathfrak{K}$

In this section, we discuss the submanifold structure of $\mathcal{U}_\mathfrak{I}(P)$ under the assumption that $\mathfrak{I} \neq \mathfrak{K}$. Recall that given the pinching operator P associated with a family of mutually orthogonal projections $\{ p_i \}_1^w$ $(1 \leq w \leq \infty)$, we may consider the larger family $\{ p_i \}_0^w$, where $p_0 = 1 - \sum_{i=1}^w p_i$. However, the pinching operator P is always associated with the first family $\{ p_i \}_1^w$. The following estimate will be useful.

Lemma 3.1. *Let Φ a symmetric norming function, and $\mathfrak{I} = \mathfrak{S}_\Phi$. Then*

$$\|L_xP - PL_x\|_{\mathcal{B}(\mathfrak{I})} \geq \|p_i x p_j\|,$$

for $x \in \mathfrak{I}$, $i \geq 1$, $j \geq 0$ and $i \neq j$.

Proof. Consider the Schmidt expansion of the compact operator $p_i x p_j$, namely

$$p_i x p_j = \sum_{k=1}^{\infty} s_k\, \xi_k \otimes \eta_k,$$

where s_k are the singular values of $p_i x p_j$ arranged in non increasing order and $(\xi_k)_k$, $(\eta_k)_k$ are orthonormal systems of vectors (see [17, p. 28]). Note that $p_i x p_j \eta_1 = s_1 \xi_1$, where $s_1 = \|p_i x p_j\|$, $\eta_1 \in R(p_j)$ and $\xi_1 \in R(p_i)$. Since $i \neq j$, we have $P(\eta_1 \otimes \xi_1) = 0$. Also note that $PL_x(\eta_1 \otimes \xi_1) = p_i x(\eta_1 \otimes \xi_1)p_i = p_i x(\eta_1 \otimes \xi_1)$ does not vanish because $i \geq 1$. It follows that

$$(L_xP - PL_x)(\eta_1 \otimes \xi_1) = -p_i x(\eta_1 \otimes \xi_1) = -p_i x p_j(\eta_1 \otimes \xi_1) = -s_1(\xi_1 \otimes \xi_1).$$

Hence we get

$$\|L_xP - PL_x\|_{\mathcal{B}(\mathfrak{I})} \geq \|(L_xP - PL_x)(\eta_1 \otimes \xi_1)\|_\mathfrak{I} = s_1 \|\xi_1 \otimes \xi_1\|_\mathfrak{I} = s_1 \|\xi_1\|\, \|\xi_1\| = \|p_i x p_j\|. \quad \square$$

The first obstruction for $\mathcal{U}_\mathfrak{I}(P)$ to be a submanifold of $\mathcal{B}(\mathfrak{I})$ lies in the fact that its tangent spaces may not be closed. The tangent space of $\mathcal{U}_\mathfrak{I}(P)$ at Q (i.e. the derivatives at Q of smooth curves inside $\mathcal{U}_\mathfrak{I}(P)$) is apparently given by

$$(T\mathcal{U}_\mathfrak{I}(P))_Q = \{ L_zQ - QL_z : z \in \mathfrak{I}_{sh} \}.$$

We denote tangent vectors briefly by $[L_z, Q]$. In the next lemma we give a characterization of when tangent spaces of $\mathcal{U}_\mathfrak{I}(P)$ are closed. Similar questions have been addressed and answered in [20] (in particular Chapter VII, Lemma VII.3) in a different setting.

Lemma 3.2. *Assume that $\mathfrak{I} \neq \mathfrak{K}$. Then tangent spaces of $\mathcal{U}_\mathfrak{I}(P)$ are closed in $\mathcal{B}(\mathfrak{I})$ if and only if $w < \infty$ and there is only one infinite rank projection in the family $\{ p_i \}_0^w$.*

Proof. It suffices to prove the statement for the tangent space at P. Indeed, if $Q = L_uPL_{u^*}$ for some $u \in \mathcal{U}_\mathfrak{I}$, then $[L_z, Q] = L_u[L_{u^*zu}, P]L_{u^*}$. Thus $(T\mathcal{U}_\mathfrak{I}(P))_Q$ is closed in $\mathcal{B}(\mathfrak{I})$ if and only if $(T\mathcal{U}_\mathfrak{I}(P))_P$ is closed in $\mathcal{B}(\mathfrak{I})$.

Suppose that $(T\mathcal{U}_\mathfrak{I}(P))_P$ is closed in $\mathcal{B}(\mathfrak{I})$. Let $x \notin \mathfrak{I}$ be a compact operator and $(e_n)_n$ be a sequence of finite rank projections such that $e_n \nearrow 1$ in the strong operator topology. Since x is compact, the sequence of finite rank operators $z_n = e_n x e_n$ satisfies $\|x - z_n\| \to 0$. Taking into account that tangent vectors have the expression $[L_z, P]$, where z is a skew-hermitian operator, we will need to consider in our next computation the real and imaginary parts of an operator. Given $y \in \mathcal{B}(\mathcal{H})$, recall that the real part of y is defined by $\Re e(y) = \frac{1}{2}(y + y^*)$ and the imaginary part by $\Im m(y) = \frac{1}{2i}(y - y^*)$.

Now note that for every $n \geq 1$, the operators $[L_{i\Re e(z_n)}, P]$ and $[L_{i\Im m(z_n)}, P]$ belong to the tangent space at P. Then we see that

$$\begin{aligned} \| [L_{i\Re e(z_n)}, P] - [L_{i\Re e(x)}, P] \|_{\mathcal{B}(\mathfrak{I})} &\leq 2\|L_{i\Re e(z_n)} - L_{i\Re e(x)}\|_{\mathcal{B}(\mathfrak{I})} \\ &= 2\|\Re e(z_n) - \Re e(x)\| \leq 2\|z_n - x\| \to 0. \end{aligned}$$

Since we have made the assumption that $(T\mathcal{U}_{\mathfrak{I}}(P))_P$ is closed, there exists some $z_0 \in \mathfrak{I}_{sh}$ such that $[L_{z_0}, P] = [L_{i\Re e(x)}, P]$. We can proceed analogously with the imaginary part to find another operator $z_1 \in \mathfrak{I}_{sh}$ such that $[L_{z_1}, P] = [L_{i\Im m(x)}, P]$. Hence we obtain $[L_x, P] = [L_z, P]$ for $z = -iz_0 + z_1 \in \mathfrak{I}$. By Lemma 2.3 the latter can be rephrased as

$$x - z = \sum_{i=0}^{w} p_i(x - z)p_i.$$

In particular, we see that

$$x - \sum_{i=0}^{w} p_i x p_i \in \mathfrak{I}. \tag{2}$$

Recall that $\mathfrak{I} = \mathfrak{S}_\Phi$ for some symmetric norming function Φ. Since $\mathfrak{I}$ is different from the compact operators, there exists a sequence of positive numbers $(a_n)_n$ such that $a_n \to 0$ and $\Phi((a_n)_n) = \infty$.

Suppose that the family $\{ p_i \}_0^w$ has two projections $p_i, p_j, i \neq j$, such that both have infinite rank. Let $(\xi_n)_n$ be an orthonormal basis of $R(p_i)$ and $(\eta_n)_n$ be an orthonormal basis of $R(p_j)$. Consider the following compact operator:

$$x = \sum_{n=1}^{\infty} a_n \, \xi_n \otimes \eta_n.$$

From our choice of the sequence $(a_n)_n$ it follows that $x \notin \mathfrak{I}$. Thus we find that $x = p_i x p_j = x - \sum_{i=0}^{w} p_i x p_i \notin \mathfrak{I}$, which contradicts Eq. (2). Hence it is impossible to have two different projections with infinite rank in the family $\{ p_i \}_0^w$.

It remains to prove that $w < \infty$. Suppose that there is an infinite number of projections $p_1, p_2, \ldots$. We can construct an orthonormal system of vectors $(\xi_i)_i$ such that $\xi_i \in R(p_i)$. Then we define the following compact operator:

$$x = \sum_{n=1}^{\infty} a_n \, \xi_{n+1} \otimes \xi_n.$$

It is easily seen that $x = \sum_{n=1}^{\infty} p_{n+1} x p_n = x - \sum_{i=0}^{\infty} p_i x p_i \notin \mathfrak{I}$. We thus get again a contradiction with Eq. (2).

In order to prove the converse we assume that the family $\{ p_i \}_0^w$ satisfies $w < \infty$ and it has only one projection p_{i_0} with infinite rank. Let $(z_k)_k$ be a sequence in $\mathfrak{I}_{sh}$ such that $\| [L_{z_k}, P] - X\|_{\mathcal{B}(\mathfrak{I})} \to 0$, where $X \in \mathcal{B}(\mathfrak{I})$. It is worth noting that by Lemma 2.3 the sequence $(z_k)_k$ can be chosen satisfying $p_i z_k p_i = 0$ for all k and $i = 0, \ldots, w$. Since $([L_{z_k}, P])_k$ is a Cauchy sequence in $\mathcal{B}(\mathfrak{I})$, Lemma 3.1 implies that

$$\|p_i(z_k - z_r)p_j\| \underset{k,r\to\infty}{\longrightarrow} 0$$

for $i = 1, \ldots, w, j = 0, \ldots, w$ and $i \neq j$. Note that the rank of the operators $p_i(z_k - z_r)p_j$ is uniformly bounded on the subscripts k and r by $C := \max\{ \operatorname{rank}(p_j) : j = 0, \ldots, w, \; j \neq i_0 \}$. Then we get

$$\|p_j(z_k - z_r)p_i\|_{\mathfrak{I}} \leq C\|p_j(z_r - z_k)p_i\| \underset{k,r\to\infty}{\longrightarrow} 0.$$

Hence each $(p_j z_k p_i)_k$ converges in the ideal norm to some $z_{ij} \in \mathfrak{I}$. We can construct an operator z by defining its matricial blocks with respect to the projections $p_0, p_1, \ldots, p_w$ as follows:

$$p_i z p_j := \begin{cases} 0 & \text{if } i = j, \\ z_{ij} & \text{if } i \neq j. \end{cases}$$

Then z is a skew-hermitian operator in $\mathfrak{I}$ satisfying

$$\|z - z_k\|_{\mathfrak{I}} \leq \sum_{i\neq j} \|p_j z p_i - p_j z_k p_i\|_{\mathfrak{I}} = \sum_{i\neq j} \|z_{ij} - p_j z_k p_i\|_{\mathfrak{I}} \to 0.$$

Therefore

$$\| [L_{z_k}, P] - [L_z, P] \|_{\mathcal{B}(\mathfrak{I})} \leq 2\|L_{z_k} - L_z\|_{\mathcal{B}(\mathfrak{I})} = 2\|z_k - z\| \leq 2\|z_k - z\|_{\mathfrak{I}} \to 0.$$

Hence we conclude $X = [L_z, P]$, and the lemma is proved. $\square$

We can endow $\mathcal{U}_{\mathfrak{I}}(P)$ with two natural topologies. According to Proposition 2.4 we have that $\mathcal{U}_{\mathfrak{I}}(P) \simeq \mathcal{U}_{\mathfrak{I}}/G$ has a real analytic manifold structure in the quotient topology in such way that the map $\pi : \mathcal{U}_{\mathfrak{I}} \longrightarrow \mathcal{U}_{\mathfrak{I}}(P)$, $\pi(u) = L_u P L_{u^*}$ is a real analytic submersion. On the other hand, we can regard $\mathcal{U}_{\mathfrak{I}}(P)$ as a subset of $\mathcal{B}(\mathfrak{I})$ with the inherited topology. In this case, we denote the projection map by $\tilde{\pi} : \mathcal{U}_{\mathfrak{I}} \longrightarrow \mathcal{U}_{\mathfrak{I}}(P)$, $\tilde{\pi}(u) = L_u P L_{u^*}$. Note that $\tilde{\pi}$ is also continuous, and the following diagram commutes

$$\begin{tikzcd}
\mathcal{U}_{\mathfrak{I}} \arrow[r, "\pi"] \arrow[dr, "\tilde{\pi}"'] & \mathcal{U}_{\mathfrak{I}}(P) \simeq \mathcal{U}_{\mathfrak{I}}/G \arrow[d, "id"] \\
 & \mathcal{U}_{\mathfrak{I}}(P) \subseteq \mathcal{B}(\mathfrak{I}).
\end{tikzcd}$$

Here *id* stands for the identity map. Note that *id* is always continuous, but it may not be a homeomorphism. In fact, we will show that the two topologies defined on $\mathcal{U}_{\mathfrak{I}}(P)$ coincide if and only if tangent spaces are closed. As we will see, the proof of this result depends on the existence of continuous local cross sections for the action.

Remark 3.3. Let P be the pinching operator associated with a family $\{p_i\}_1^w$. We will consider the unitary orbit of each projection p_i, i.e.

$$\mathcal{O}_i := \{ u p_i u^* : u \in \mathcal{U}_{\mathfrak{I}} \}.$$

If $\mathfrak{I}$ is the ideal of Hilbert–Schmidt operators and p_i has infinite-dimensional range, the above defined orbits are usually known as the connected component of p_i in the restricted Grassmannian (see e.g. [22]). Note that $\mathcal{O}_i \subseteq p_i + \mathfrak{I}$, so we may endow each orbit with the subspace topology defined by the metric $(up_iu^*, vp_iv^*) \mapsto \|up_iu^* - vp_iv^*\|_{\mathfrak{I}}$.

Lemma 3.4. *Assume that $w < \infty$ and there is only one infinite rank projection in the family $\{p_i\}_0^w$. Then the map*

$$F_i : \mathcal{U}_{\mathfrak{I}}(P) \longrightarrow \mathcal{O}_i, \qquad F_i(L_u P L_{u^*}) = u p_i u^*$$

is continuous for $i = 0, 1, \ldots, w$, when $\mathcal{U}_{\mathfrak{I}}(P)$ is endowed with the topology inherited from $\mathcal{B}(\mathfrak{I})$.

Proof. We first show that the function F_i is well defined for $i = 0, 1, \ldots, w$. From Lemma 2.3 we know that $L_u P L_{u^*} = L_v P L_{v^*}$ implies $v^*u = \sum_{i=0}^{w} p_i v^* u p_i$. Then we get $v^*up_i = p_iv^*up_i = p_iv^*u$, or equivalently, $up_iu^* = vp_iv^*$.

To prove the continuity of F_i we will actually see that F_i is Lipschitz. Since the underlying actions are isometric, it suffices to estimate the distance from $F_i(L_uPL_{u^*}) = up_iu^*$ to $F_i(P) = p_i$. For $u \in \mathcal{U}_{\mathfrak{I}}$, set $a(u) := \|L_uPL_{u^*} - P\|_{\mathcal{B}(\mathfrak{I})} = \|\,[L_u, P]\,\|_{\mathcal{B}(\mathfrak{I})}$. From Lemma 3.1 it follows that

$$\|p_iup_j\| = \|p_i(u-1)p_j\| \leq a(u),$$

for $j = 0, 1, \ldots, w$, $i = 1, \ldots, w$ and $i \neq j$. The same estimate can be extended for all $i \neq j$. In fact, we have

$$\|p_jup_i\| = \|p_iu^*p_j\| \leq a(u^*) = a(u).$$

Let p_{i_0} be the unique infinite rank projection in the family $\{p_i\}_0^w$. For $u \in \mathcal{U}_{\mathfrak{I}}$, we note that

$$\operatorname{rank}(p_iup_j) \leq \min\{\operatorname{rank}(p_i), \operatorname{rank}(p_j)\},$$

and then we get

$$\max\{\operatorname{rank}(p_jup_i) : i, j = 0, 1, \ldots, w,\ i \neq j\} \leq \max\{\operatorname{rank}(p_j) : j = 0, 1, \ldots, w,\ j \neq i_0\} := C.$$

This implies that $\|p_iup_j\|_{\mathfrak{I}} \leq C\|p_iup_j\|$ for $i \neq j$. Thus we get

$$\begin{aligned}
\|F_i(L_uPL_{u^*}) - F_i(P)\|_{\mathfrak{I}} &= \|up_i - p_iu\|_{\mathfrak{I}} \leq \left\| \sum_{j=0}^{w} p_jup_i - \sum_{k=0}^{w} p_iup_k \right\|_{\mathfrak{I}} \\
&= \left\| \sum_{j\neq i} p_jup_i - \sum_{k\neq i} p_iup_k \right\|_{\mathfrak{I}} \leq \sum_{j:j\neq i} \|p_jup_i\|_{\mathfrak{I}} + \sum_{k:k\neq i} \|p_iup_k\|_{\mathfrak{I}} \\
&\leq C \left(\sum_{j:j\neq i} \|p_jup_i\| + \sum_{k:k\neq i} \|p_iup_k\| \right) \leq 2wC\|L_uPL_{u^*} - P\|_{\mathcal{B}(\mathfrak{I})}, \qquad (3)
\end{aligned}$$

which shows that F is Lipschitz. $\square$

Remark 3.5. Let $\mathcal{M}$ be the supplement of the Lie algebra defined in Proposition 2.4. Suppose that $w = \infty$ or there exist two different infinite rank projections in the family $\{ p_i \}_0^w$. Under the assumption that $\mathfrak{I} \neq \mathfrak{K}$, we will construct a sequence $(z_k)_k$ in $\mathcal{M}$ satisfying $\|z_k\| \to 0$ and $\|z_k\|_{\mathfrak{I}} = 1$. To this end, put

$$a_k := \Phi(\underbrace{1, 1, \ldots, 1}_{k}, 0, 0, \ldots),$$

where Φ is a symmetric norming function such that $\mathfrak{I} = \mathfrak{S}_\Phi$. Since $\mathfrak{I} \neq \mathfrak{K}$, it follows that Φ is not equivalent to the uniform norm of ℓ^∞, so that $a_k \to \infty$ (see [17, p. 76]). In the case in which $w = \infty$, let $(\xi_i)_i$ be an orthonormal system such that $\xi_i \in R(p_i)$ for all $i \geq 1$. It is not difficult to see that the sequence defined by

$$z_k := a_{2k}^{-1} \sum_{i=1}^{k} \xi_{2i-1} \otimes \xi_{2i} - \xi_{2i} \otimes \xi_{2i-1}$$

satisfies the required properties. In the case in which there exist two different infinite rank projections p_i and p_j, let $(\xi_i)_i$ be an orthonormal system such that $\xi_{2k-1} \in R(p_i)$ and $\xi_{2k} \in R(p_j)$ for all $k \geq 1$. Then we can define the sequence $(z_k)_k$ in the same fashion as before.

Lemma 3.6. *Assume that $\mathfrak{I} \neq \mathfrak{K}$. Then the following conditions are equivalent:*

(i) *The quotient topology of $\mathcal{U}_{\mathfrak{I}}(P)$ coincides with the topology inherited from $\mathcal{B}(\mathfrak{I})$.*
(ii) *$w < \infty$ and there is only one infinite rank projection in the family $\{ p_i \}_0^w$.*

Proof. Suppose that the quotient topology of $\mathcal{U}_{\mathfrak{I}}(P) \simeq \mathcal{U}_{\mathfrak{I}}/G$ coincides with the topology inherited form $\mathcal{B}(\mathfrak{I})$. Let $\mathcal{M}$ be the supplement of the Lie algebra of G defined in Proposition 2.4. Recall that a real analytic atlas of $\mathcal{U}_{\mathfrak{I}}(P)$ compatible with the quotient topology can be constructed by translation of the homeomorphism

$$\psi : \mathcal{W} \subseteq \mathcal{M} \longrightarrow \psi(\mathcal{W}), \qquad \psi(z) = (\pi \circ \exp_{\mathcal{U}_{\mathfrak{I}}})(z) = L_{e^z} P L_{e^{-z}},$$

where $\mathcal{W}$ is an open neighborhood of $0 \in \mathcal{M}$ and $\psi(\mathcal{W})$ an open neighborhood of P (see for instance [4, Theorem 4.19]). Assume that the family $\{ p_i \}_0^w$ does not satisfy the claimed properties. This leads us to consider two cases, namely $w = \infty$ or there exist two different infinite rank projections in $\{ p_i \}_0^w$. In any case we can find a sequence $(z_k)_k$ in $\mathcal{M}$ such that $\|z_k\| \to 0$ and $\|z_k\|_{\mathfrak{I}} = 1$ according to Remark 3.5. Then note that

$$\|L_{e^{z_k}} P L_{e^{-z_k}} - P\|_{\mathcal{B}(\mathfrak{I})} = \| [L_{e^{z_k}-1}, P] \|_{\mathcal{B}(\mathfrak{I})} \leq 2\|e^{z_k} - 1\| \to 0,$$

and using that the quotient topology of $\mathcal{U}_{\mathfrak{I}}(P)$ coincides with the subspace topology, we arrive at a contradiction: $\|z_k\|_{\mathfrak{I}} = \|\psi^{-1}(L_{e^{z_k}} P L_{e^{-z_k}})\|_{\mathfrak{I}} \to 0$.

To prove the converse, assume that $w < \infty$ and there is only one infinite rank projection in the family $\{ p_i \}_0^w$. Clearly, our assertion about the topology of $\mathcal{U}_{\mathfrak{I}}(P)$ will follow if we show that the projection map

$$\tilde{\pi} : \mathcal{U}_{\mathfrak{I}} \longrightarrow \mathcal{U}_{\mathfrak{I}}(P), \qquad \tilde{\pi}(u) = L_u P L_{u^*}$$

have continuous local cross sections, when $\mathcal{U}_{\mathfrak{I}}(P)$ is considered with the relative topology of $\mathcal{B}(\mathfrak{I})$. To this end, for $i = 0, 1, \ldots, w$, we need to consider the orbits

$$\mathcal{O}_i := \{ u p_i u^* : u \in \mathcal{U}_{\mathfrak{I}} \}.$$

In [1, Proposition 2.2] the authors showed that the maps

$$\pi_i : \mathcal{U}_{\mathfrak{I}} \longrightarrow \mathcal{O}_i, \qquad \pi_i(u) = u p_i u^*,$$

have continuous local cross sections, when $\mathfrak{I}$ is the ideal of Hilbert–Schmidt operators. Actually, the same proof works out for any symmetrically-normed ideal $\mathfrak{I}$, so we have that there exist continuous maps

$$\psi_i : \{ q \in \mathcal{O}_i : \|q - p_i\|_{\mathfrak{I}} < 1 \} \subseteq p_i + \mathfrak{I} \longrightarrow \mathcal{U}_{\mathfrak{I}}$$

such that $\psi_i(u p_i u^*) p_i \psi_i(u p_i u^*)^* = u p_i u^*$ for any $u \in \mathcal{U}_{\mathfrak{I}}$ such that $\|u p_i u^* - p_i\|_{\mathfrak{I}} < 1$.

Now we can explicitly give the required section for $\tilde{\pi}$, namely

$$\sigma : \left\{ Q \in \mathcal{U}_{\mathfrak{I}}(P) : \|Q - P\|_{\mathcal{B}(\mathfrak{I})} < 1/2wC \right\} \longrightarrow \mathcal{U}_{\mathfrak{I}}, \qquad \sigma(L_u P L_{u^*}) = \sum_{i=0}^{w} \psi_i(u p_i u^*) p_i.$$

If $Q = L_u P L_{u^*}$ lies in the domain of σ, then by the estimate (3) in Lemma 3.4, the operators $u p_i u^*$ do lie in the domain of each ψ_i. Our next task is to show that $\sigma = \sigma(L_u P L_{u^*}) \in \mathcal{U}_{\mathfrak{I}}$. In fact, we see that

$$\sigma\sigma^* = \left(\sum_{i=0}^{w} \psi_i(u p_i u^*) p_i \right) \left(\sum_{i=0}^{w} p_i \psi_i(u p_i u^*)^* \right) = \sum_{i=0}^{w} \psi_i(u p_i u^*) p_i \psi(u p_i u^*)^* = \sum_{i=0}^{w} u p_i u^* = 1.$$

Note that $p_j\psi_j(up_ju^*)^*\psi_i(up_iu^*)p_i = \psi_j(up_ju^*)^*up_jp_iu^*\psi_i(up_iu^*) = \delta_{ij}$, then

$$\sigma^*\sigma = \left(\sum_{i=0}^{w} p_i\psi_i(up_iu^*)^*\right)\left(\sum_{i=0}^{w}\psi_i(up_iu^*)p_i\right) = \sum_{i=0}^{w} p_i = 1.$$

Also we see that

$$\sigma - 1 = \sum_{i=0}^{w}(\psi_i(up_iu^*) - 1)p_i \in \mathfrak{I}.$$

On the other hand, the map σ is actually a section for π: for any $y \in \mathfrak{I}$,

$$L_{\sigma(L_uPL_{u^*})}PL_{\sigma(L_uPL_{u^*})^*}(y) = \sum_{i=0}^{w}\sigma(L_uPL_{u^*})p_i\sigma(L_uPL_{u^*})^*yp_i = \sum_{i=0}^{w} up_iu^*yp_i = L_uPL_{u^*}(y).$$

Finally, to show the continuity of σ, it is enough to remark that

$$\sigma(L_uPL_{u^*}) = \sum_{i=0}^{w}\psi_i(F_i(L_uPL_{u^*}))p_i$$

and use the continuity of each F_i, which has already been proved in Lemma 3.4. $\square$

Now our main result on the differential structure of $\mathcal{U}_{\mathfrak{I}}(P)$ follows.

Theorem 3.7. *Let Φ a symmetric norming function, and $\mathfrak{I} = \mathfrak{S}_\Phi$. Assume that $\mathfrak{I} \neq \mathfrak{K}$. Let P be the pinching operator associated with a family $\{p_i\}_1^w$ $(1 \leq w \leq \infty)$. Then the following assertions are equivalent:*

(i) *The quotient topology on $\mathcal{U}_{\mathfrak{I}}(P)$ coincides with topology inherited from $\mathcal{B}(\mathfrak{I})$.*
(ii) *Tangent spaces of $\mathcal{U}_{\mathfrak{I}}(P)$ are closed in $\mathcal{B}(\mathfrak{I})$.*
(iii) *$w < \infty$ and there is only one infinite rank projection in the family $\{p_i\}_0^w$.*
(iv) *$\mathcal{U}_{\mathfrak{I}}(P)$ is a submanifold of $\mathcal{B}(\mathfrak{I})$.*

Proof. Suppose that $\mathcal{U}_{\mathfrak{I}}(P)$ is a submanifold of $\mathcal{B}(\mathfrak{I})$. By Proposition 2.1, tangent spaces of $\mathcal{U}_{\mathfrak{I}}(P)$ has to be closed in $\mathcal{B}(\mathfrak{I})$. From Lemma 3.2 it follows that the family $\{p_i\}_0^w$ satisfies the stated properties.

Now we assume that $w < \infty$ and there is only one infinite rank projection in the family $\{p_i\}_0^w$. According to Lemmas 3.2 and 3.6, what is left to prove is that tangent spaces are complemented in $\mathcal{B}(\mathfrak{I})$. Clearly, it suffices to show that $(T\mathcal{U}_{\mathfrak{I}}(P))_P$ is complemented in $\mathcal{B}(\mathfrak{I})$.

We will divide the proof into two cases according to whether the rank of p_0 is infinite or finite. Let us first assume that $\mathrm{rank}(p_0) = \infty$, so that $\mathrm{rank}(p_i) < \infty$ for all $i = 1, \ldots, w$. Then $X(p_i)$ is well defined for any $X \in \mathcal{B}(\mathfrak{I})$, $i = 1, \ldots, w$, and we can set

$$\hat{z} : \mathcal{B}(\mathfrak{I}) \longrightarrow \mathfrak{I}_{sh}, \qquad \hat{z}(X) = 2i\Im m\left(\sum_{i=1}^{w}\sum_{j=0}^{i-1} p_jX(p_i)\right).$$

Clearly $\hat{z}$ is a continuous linear operator. Then we define a bounded linear projection onto the tangent space by

$$E : \mathcal{B}(\mathfrak{I}) \longrightarrow (T\mathcal{U}_{\mathfrak{I}}(P))_P, \qquad E(X) = [L_{\hat{z}(X)}, P].$$

In order to show that E actually defines a projection we pick $X = [L_z, P]$ for some $z \in \mathfrak{I}_{sh}$. Notice that $X(p_i) = (1 - p_i)zp_i$, for all $i = 1, \ldots, w$, then we get that

$$\hat{z}(X) = 2i\Im m\left(\sum_{i=1}^{w}\sum_{j=0}^{i-1} p_jzp_i\right) = z - \sum_{i=0}^{w} p_izp_i.$$

From Lemma 2.3 we deduce that $E(X) = [L_{\hat{z}(X)}, P] = X$, which proves that E is a projection. Finally, the continuity of $\hat{z}$ easily implies that of E.

Now we consider the case in which the infinite rank projection is not p_0. Without loss of generality we may assume that $\mathrm{rank}(p_1) = \infty$. Let us point out that the above definition of the operator $\hat{z}(X)$ does not work in this case for two different reasons: on one hand, since $p_1 \notin \mathfrak{I}$ we cannot evaluate any $X \in \mathcal{B}(\mathfrak{I})$ at p_1, and on the other hand, every tangent vector $[L_z, P]$ vanishes at p_0.

In order to solve this case we need to modify the definition of the operator $\hat{z}$. Recall that $\mathrm{rank}(p_0) < \infty$ since $\mathrm{rank}(p_1) = \infty$. Let $\eta_1, \ldots, \eta_m$ be an orthonormal basis of $R(p_0)$. Let $\xi \in R(p_1)$ be a unit vector. Then we define

$$\hat{z} : \mathcal{B}(\mathfrak{I}) \longrightarrow \mathfrak{I}_{sh}, \qquad \hat{z}(X) = 2i\Im m\left(\sum_{i=2}^{w}\sum_{j=0}^{i-1} p_jX(p_i) - \sum_{k=1}^{m} X(\eta_k \otimes \xi)\xi \otimes \eta_k\right),$$

and the projection onto the tangent space is

$$E : \mathcal{B}(\mathfrak{I}) \longrightarrow (T\mathcal{U}_{\mathfrak{I}}(P))_P, \qquad E(X) = [L_{\hat{z}(X)}, P].$$

It is apparent that E is continuous, so we are left with the task of proving that E is a projection. To this end, let $X = [L_z, P]$ for some $z \in \mathfrak{I}_{sh}$. Note that

$$X(\eta_k \otimes \xi) = \sum_{i=1}^{w}(zp_i - p_i z)(\eta_k \otimes \xi)p_i = (zp_1 - p_1 z)(\eta_k \otimes \xi)p_1 = -p_1 z(\eta_k \otimes \xi),$$

and then

$$\sum_{k=1}^{m} X(\eta_k \otimes \xi)\xi \otimes \eta_k = -p_1 z p_0.$$

Thus we get

$$\hat{z}(X) = 2i\Im m\left(\sum_{i=2}^{w}\sum_{j=0}^{i-1} p_j z p_i + p_1 z p_0\right) = z - \sum_{i=0}^{w} p_i z p_i.$$

Hence we conclude that $E([L_z, P]) = [L_z, P]$, and the proof is complete. □

4. Submanifold structure of $\mathcal{U}_{\mathfrak{K}}(P)$

In this section we turn to the case $\mathfrak{I} = \mathfrak{K}$. The following estimate is a somewhat improved version of Lemma 3.1.

Lemma 4.1. *Let $x \in \mathfrak{K}$ such that $p_i x p_i = 0$ for all $i \geq 1$. Then*

$$\|L_x P - P L_x\|_{\mathcal{B}(\mathfrak{K})} \geq \|x(1 - p_0)\|,$$

where $p_0 = 1 - \sum_{i=1}^{w} p_i$.

Proof. To estimate the norm of $L_x P - P L_x$ as an operator acting on $\mathfrak{K}$ we need to consider the following projections: if $\operatorname{rank}(p_i) = \infty$, let $(p_{i,k})_k$ be a sequence of finite rank projections satisfying $p_{i,k} \leq p_i$ and $p_{i,k} \nearrow p_i$, and if $\operatorname{rank}(p_i) < \infty$, we set $p_{i,k} = p_i$ for all $k \geq 1$. Now assume that the pinching operator P is associated with a family $\{p_i\}_1^w$ such that $w < \infty$. Then the projections given by $e_k = \sum_{i=1}^{n} p_{i,k}$ have finite rank. We thus get

$$\|L_x P - P L_x\|_{\mathcal{B}(\mathfrak{K})} \geq \|(L_x P - P L_x)(e_k)\| = \left\|\sum_{i=1}^{w}(1 - p_i) x p_{i,k}\right\| = \left\|x \sum_{i=1}^{w} p_{i,k}\right\|,$$

where in the last equality we use that $p_i x p_i = 0$. Using that $x \in \mathfrak{K}$ and $p_{i,k} \nearrow p_i$, we find that

$$\|L_x P - P L_x\|_{\mathcal{B}(\mathfrak{K})} \geq \|x(1 - p_0)\|.$$

In the case where $w = \infty$, we set $e_{n,k} = \sum_{i=1}^{n} p_{i,k}$. In the same fashion as above we find that

$$\|L_x P - P L_x\|_{\mathcal{B}(\mathfrak{K})} \geq \left\|x \sum_{i=1}^{n} p_{i,k}\right\|.$$

Letting $k \to \infty$, we have

$$\|L_x P - P L_x\|_{\mathcal{B}(\mathfrak{K})} \geq \left\|x \sum_{i=1}^{n} p_i\right\|,$$

for all $n \geq 1$. Now letting $n \to \infty$, we get the estimate in this case. □

Proposition 4.2. *Tangent spaces of $\mathcal{U}_{\mathfrak{K}}(P)$ are closed in $\mathcal{B}(\mathfrak{K})$.*

Proof. By the remark at the beginning of the proof of Lemma 3.2, we may restrict, without loss of generality, to verify the statement for the tangent space at P. Let $(z_k)_k$ be a sequence in $\mathfrak{K}_{sh}$ such that $p_i z_k p_i = 0$ for all $i \geq 0$ and $k \geq 1$. Suppose that $\|\,[L_{z_k}, P] - X\,\|_{\mathcal{B}(\mathfrak{K})} \to 0$ for some $X \in \mathcal{B}(\mathfrak{K})$. According to Lemma 4.1,

$$\|(z_k - z_r)(1 - p_0)\| \leq \|\,[L_{z_k - z_r}, P]\,\|_{\mathcal{B}(\mathfrak{K})}.$$

Also note that

$$\|(z_k - z_r)p_0\| = \|p_0(z_k - z_r)\| = \|p_0(z_k - z_r)(1 - p_0)\| \leq \|\,[L_{z_k - z_r}, P]\,\|_{\mathcal{B}(\mathfrak{K})}.$$

Therefore $(z_k)_k$ is a Cauchy sequence and thus has a limit $z_0 \in \mathfrak{K}_{sh}$. Then we see that

$$\| [L_{z_k}, P] - [L_{z_0}, P] \| \le 2\|z_k - z_0\| \to 0.$$

Thus we conclude that $X = [L_{z_0}, P]$. □

Now we turn to the study of the topology of $\mathcal{U}_{\mathfrak{K}}(P)$. We will find that the quotient topology and the topology inherited from $\mathcal{B}(\mathfrak{K})$ coincide regardless of the number or the ranks of the projections in the family $\{ p_i \}_0^w$.

Remark 4.3. Let P be the pinching operator associated with a family $\{ p_i \}_1^w$ $(1 \le i \le w)$. In this subsection we need to consider again the unitary orbit of the projections, which we denote by

$$\mathcal{O}_i = \{ up_iu^* : u \in \mathcal{U}_{\mathfrak{K}} \}$$

for $i = 0, \dots, w$. We claim that the map

$$F_0 : \mathcal{U}_{\mathfrak{J}}(P) \longrightarrow \mathcal{O}_0, \qquad F_0(L_uPL_{u^*}) = up_0u^*$$

is Lipschitz. In fact, according to Lemma 4.1 applied with $x = u - 1 - \sum_{i=0}^w p_i(u-1)p_i = u - \sum_{i=0}^w p_iup_i$ we have that

$$\|p_0u(1-p_0)\| = \left\| p_0 \left(u - \sum_{i=0}^w p_iup_i \right)(1-p_0) \right\| \le \left\| \left(u - \sum_{i=0}^w p_iup_i \right)(1-p_0) \right\| \le \|L_uPL_{u^*} - P\|_{\mathcal{B}(\mathfrak{K})}.$$

Replacing u by u^* we find that

$$\|(1-p_0)up_0\| = \|p_0u^*(1-p_0)\| \le \|L_uPL_{u^*} - P\|_{\mathcal{B}(\mathfrak{K})}.$$

Thus we get

$$\begin{aligned} \|F_0(L_uPL_{u^*}) - F_0(P)\| &= \|up_0u^* - p_0\| \\ &\le \|(1-p_0)up_0\| + \|p_0u(1-p_0)\| \le 2\|L_uPL_{u^*} - P\|, \end{aligned}$$

which proves our claim.

Lemma 4.4. *Let $u, v \in \mathcal{U}_{\mathfrak{K}}$. Then*

$$\left\| \sum_{i=0}^w up_iu^*p_i - vp_iv^*p_i \right\| \le 3\|L_uPL_{u^*} - L_vPL_{v^*}\|_{\mathcal{B}(\mathfrak{K})},$$

where in the case in which $w = \infty$ the series on the left side is convergent in the uniform norm.

Proof. For each $i \ge 1$, let $(p_{i,k})_k$ be a sequence of finite rank projections such that $p_{i,k} \le p_i$ and $p_{i,k} \nearrow p_i$. In case p_i has finite rank, we set $p_{i,k} = p_i$ for all k. We will use the orthogonal projections defined by $e_k = \sum_{i=1}^n p_{i,k}$. Put $a(u, v) := \|L_uPL_{u^*} - L_vPL_{v^*}\|_{\mathcal{B}(\mathfrak{K})}$. Then

$$\left\| \sum_{i=1}^w (up_iu^* - vp_iv^*)p_{i,k} \right\| = \|(L_uPL_{u^*} - L_vPL_{v^*})(e_k)\| \le a(u, v).$$

Note that for each $i \ge 1$, the operator $up_iu^* - vp_iv^*$ is compact. Letting $k \to \infty$, we get that

$$\left\| \sum_{i=1}^w (up_iu^* - vp_iv^*)p_i \right\| \le a(u, v).$$

Combining this with the Remark 4.3 it gives that

$$\left\| \sum_{i=0}^w (up_iu^* - vp_iv^*)p_i \right\| \le 3a(u, v). \tag{4}$$

This finishes the proof for the case $w < \infty$. If $w = \infty$, we note that

$$\sum_{i=0}^\infty up_iu^*p_i - vp_iv^*p_i = \sum_{i=0}^\infty up_i(u^*-1)p_i - vp_i(v^*-1)p_i + (u-v)p_i.$$

Since the operators $u^* - 1$, $v^* - 1$ and $u - v$ are compact, this series converges in the uniform norm. Letting $w \to \infty$ in (4), the desired inequality follows. □

In the following proposition we extend the technique developed in [1] to construct continuous local cross sections.

Proposition 4.5. *The map*

$$\pi : \mathcal{U}_{\mathfrak{K}} \longrightarrow \mathcal{U}_{\mathfrak{K}}(P) \subseteq \mathcal{B}(\mathfrak{K}), \qquad \pi(u) = L_u P L_{u^*},$$

has continuous local cross sections, when $\mathcal{U}_{\mathfrak{K}}(P)$ is considered with the topology inherited from $\mathcal{B}(\mathfrak{K})$.

Proof. Let P be the pinching operator associated with a family $\{p_i\}_1^w$ $(1 \le w \le \infty)$. Since the action of $\mathcal{U}_{\mathfrak{K}}$ is isometric it will be enough to find a continuous section σ in a neighborhood of P. Also we will restrict ourselves to prove the case $w = \infty$. The case $w < \infty$ needs less care, and it can be handled in much the same fashion.

We consider the following neighborhood of P to define the cross section,

$$\mathcal{V} := \left\{ Q \in \mathcal{U}_{\mathfrak{K}}(P) : \|Q - P\|_{\mathcal{B}(\mathfrak{K})} < 1/3 \right\}.$$

Given $Q = L_u P L_{u^*} \in \mathcal{V}$, where $u \in \mathcal{U}_{\mathfrak{K}}$, let $q_i = F_i(Q) = u p_i u^*$ for $i \ge 0$. According to the proof of Lemma 3.4 the function F_i is well defined. Then, we set

$$s = s(Q) := \sum_{i=0}^{\infty} q_i p_i.$$

This series is convergent in the strong operator topology. In fact, we can rewrite the series as

$$\sum_{i=0}^{\infty} q_i p_i = \sum_{i=0}^{\infty} u p_i (u^* - 1) p_i + (u - 1) p_i + p_i,$$

where the first and second summand on the right are convergent in the uniform norm, while the third is convergent in the strong operator topology. On the other hand, note that by Lemma 4.4, we get

$$\|s - 1\| \le 3 \|Q - P\|_{\mathcal{B}(\mathfrak{K})} < 1.$$

Then we get that s is invertible. Moreover, it follows that

$$s - 1 = u \left(\sum_{i=0}^{\infty} p_i (u^* - 1) p_i + 1 \right) - 1 = u \sum_{i=0}^{\infty} p_i (u^* - 1) p_i + u - 1 \in \mathfrak{K},$$

which is due to the fact that $\sum_{i=0}^{\infty} p_i (u^* - 1) p_i \in \mathfrak{K}$. Now we will show that

$$\sigma = \sigma(Q) := s|s|^{-1}$$

is a continuous local cross section for π. To this end, note that $sp_i = q_i p_i = q_i s$, so that $p_i |s|^2 = s^* q_i s = |s|^2 p_i$, which implies

$$\sigma p_i \sigma^* = s|s|^{-1} p_i |s|^{-1} s^* = s p_i |s|^{-2} s^* = s p_i s^{-1} = q_i.$$

This allows us to prove that σ is a section: for any $y \in \mathfrak{K}$, we have

$$L_\sigma P L_{\sigma^*}(y) = \sum_{i=1}^{\infty} \sigma p_i \sigma^* y p_i = \sum_{i=1}^{\infty} q_i y p_i = Q(y).$$

On the other hand, we have $|s|^2 - 1 \in \mathfrak{K}$, and consequently, $|s| - 1 = (|s|^2 - 1)(|s| + 1)^{-1} \in \mathfrak{K}$. Therefore we can conclude

$$\sigma - 1 = s|s|^{-1} - 1 = (s - |s|)|s|^{-1} = (s - 1)|s|^{-1} + (1 - |s|)|s|^{-1} \in \mathfrak{K}.$$

Hence $\sigma \in \mathcal{U}_{\mathfrak{K}}$. Let $Gl(\mathcal{H})$ denote the group of invertible operators on $\mathcal{H}$. In order to prove the continuity of σ we consider the subgroup of $Gl(\mathcal{H})$ given by

$$Gl_{\mathfrak{K}} = \{ g \in Gl(\mathcal{H}) : g - 1 \in \mathfrak{K} \}.$$

It is a Banach–Lie group endowed with the topology defined by $(g_1, g_2) \mapsto \|g_1 - g_2\|$ (see [4]). From Lemma 4.4 the map $s : \mathcal{V} \longrightarrow Gl_{\mathfrak{K}}$ is continuous. Also note that the map $Gl_{\mathfrak{K}} \longrightarrow \mathcal{U}_{\mathfrak{K}}, s \mapsto s|s|^{-1}$, is real analytic by the regularity properties of the Riesz functional calculus. Thus σ is continuous, being the composition of continuous maps. $\square$

Our next task in the study of the submanifold structure of $\mathcal{U}_{\mathfrak{K}}(P)$ is to ask about the existence of a supplement for $(T\mathcal{U}_{\mathfrak{I}}(P))_P$ in $\mathcal{B}(\mathfrak{K})$. The existence of such supplement is closely related to the fact that for an infinite dimensional Hilbert space $\mathcal{H}$ the compact operators are not complemented in $\mathcal{B}(\mathcal{H})$. A proof of this result can be found, for instance, in [11]. It is based on the following well known result: c_0 (sequences which converges to zero) is not complemented in ℓ^∞ (bounded sequences). The reader can find a proof of this latter fact in [26].

Remark 4.6. We will need a slightly modified version of the afore-mentioned result. We first note that $\mathfrak{K}_{sh}$ is not complemented in $\mathcal{B}(\mathcal{H})_{sh}$. Otherwise we would have a real bounded projection $E : \mathcal{B}(\mathcal{H})_{sh} \longrightarrow \mathfrak{K}_{sh}$, then we can define a bounded projection $\tilde{E} : \mathcal{B}(\mathcal{H}) \longrightarrow \mathfrak{K}$, $\tilde{E}(x) = -iE(i\Re e(x)) + iE(i\Im m(x))$, a contradiction.

Let q_1, q_2 two infinite rank orthogonal projections on $\mathcal{H}$. We claim that $q_1\mathfrak{K}_{sh}q_2$ is not complemented in $q_1\mathcal{B}(\mathcal{H})_{sh}q_2$. In fact, suppose that there exists a real bounded projection $E : q_1\mathcal{B}(\mathcal{H})_{sh}q_2 \longrightarrow q_1\mathfrak{K}_{sh}q_2$. Let v a partial isometry on $\mathcal{H}$ such that $v^*v = q_1$ and $vv^* = q_2$. Then we have that $L_vEL_{v^*} : \mathcal{B}(q_2(\mathcal{H}))_{sh} \longrightarrow q_2\mathfrak{K}_{sh}q_2$ is a bounded projection, which is impossible by the previous paragraph.

In the following result we collect the above proved properties of $\mathcal{U}_{\mathfrak{K}}(P)$ and we give a complete characterization of the submanifold structure.

Theorem 4.7. *Let P be the pinching operator associated with a family $\{p_i\}_1^w$ $(1 \le w \le \infty)$. Then $\mathcal{U}_{\mathfrak{K}}(P)$ is a quasi submanifold of $\mathcal{B}(\mathfrak{K})$. Furthermore, $\mathcal{U}_{\mathfrak{K}}(P)$ is a submanifold of $\mathcal{B}(\mathfrak{K})$ if and only if $w < \infty$ and there is only one infinite rank projection in the family $\{ p_i \}_0^w$.*

Proof. The first statement about the quasi submanifold structure of $\mathcal{U}_{\mathfrak{K}}(P)$ has already been proved in Propositions 4.2 and 4.5. Assume that $w < \infty$ and there is only one infinite rank projection in the family $\{ p_i \}_0^w$. The same proof of Theorem 3.7 can be carried out to show that $(T\mathcal{U}_{\mathfrak{K}}(P))_P$ is complemented in $\mathcal{B}(\mathfrak{K})$.

Suppose now that $\mathcal{U}_{\mathfrak{K}}(P)$ is a submanifold of $\mathcal{B}(\mathfrak{K})$. According to Proposition 2.1, there is a bounded linear projection $E : \mathcal{B}(\mathfrak{K}) \longrightarrow (T\mathcal{U}_{\mathfrak{K}}(P))_P$. Two cases should be considered: first, that there are two infinite rank projections in the family $\{ p_i \}_0^w$, and second, that $w = \infty$. In the first case, let $q_1 \in \{ p_0, p_1, \dots, p_w \}$ be an infinite rank projection and $q_2 \in \{ p_1, \dots, p_w \} \setminus \{ q_1 \}$ be other infinite rank projection. In the second case, we set $q_1 = \sum_{k=0}^{\infty} p_{2k}$ and $q_2 = \sum_{k=0}^{\infty} p_{2k+1}$. In any case we define the following bounded linear map

$$\tilde{E} : q_1\mathcal{B}(\mathcal{H})_{sh}q_2 \longrightarrow q_1\mathfrak{K}_{sh}q_2, \qquad \tilde{E}(q_1xq_2) = (L_{q_1}E)(\,[L_{q_1xq_2+q_2xq_1}, P]\,)(q_2).$$

We claim that $\tilde{E}$ is a projection onto $q_1\mathfrak{K}_{sh}q_2$. In fact, notice that for each $x \in \mathcal{B}(\mathcal{H})_{sh}$ there is $z \in \mathfrak{K}_{sh}$ such that $E(\,[L_{q_1xq_2+q_2xq_1}, P]\,) = [L_z, P]$. In the case in which there are two infinite rank projections, note that

$$\tilde{E}(q_1xq_2) = q_1 \sum_{i=1}^{w} (zp_i - p_iz)q_2p_i = q_1(zq_2 - q_2z)q_2 = q_1zq_2.$$

On the other hand, when $w = \infty$,

$$\tilde{E}(q_1xq_2) = q_1 \sum_{i=1}^{\infty} (zp_i - p_iz)q_2p_i = q_1 \sum_{k=0}^{\infty} (zp_{2k+1} - p_{2k+1}z)p_{2k+1} = q_1z \sum_{k=0}^{\infty} p_{2k+1} = q_1zq_2.$$

This proves that the range of $\tilde{E}$ is contained in $p_1\mathfrak{K}_{sh}p_2$. Moreover, let $x \in \mathfrak{K}_{sh}$, then we have that $E(\,[L_{q_1xq_2+q_2xq_1}, P]\,) = [L_{q_1xq_2+q_2xq_1}, P]$. We thus get that

$$\tilde{E}(q_1xq_2) = q_1(q_1xq_2 + q_2xq_1)q_2 = q_1xq_2.$$

Hence $\tilde{E}$ is a continuous linear projection onto $q_1\mathfrak{K}_{sh}q_2$. In other words, $q_1\mathfrak{K}_{sh}q_2$ is complemented in $q_1\mathcal{B}(\mathcal{H})_{sh}q_2$, but this contradicts Remark 4.6. $\square$

As an application of the previous results on the topology of $\mathcal{U}_{\mathfrak{I}}(P)$, we include a study of the topology of $\mathcal{U}_{\mathfrak{I}}$-unitary orbits of a compact normal operator. Let a be a compact normal operator. The $\mathcal{U}_{\mathfrak{I}}$-unitary orbit of a is given by

$$\mathcal{U}_{\mathfrak{I}}(a) = \{ uau^* : u \in \mathcal{U}_{\mathfrak{I}} \}.$$

These type of unitary orbits may be endowed with the quotient topology, though there is another quite natural topology, the one defined by the norm of the ideal $\mathfrak{I}$. We show that both topologies coincide if and only if the compact operator has finite rank. This result is related with several works [2,5,8,19], where for many different ideals $\mathfrak{I}$, the finite rank condition appears as sufficient to the statement on the topologies. On the other hand, it is worth pointing out that this problem was completely solved in [16] for the usual unitary orbits.

Remark 4.8. The main idea to link unitary orbits of pinching operators with the $\mathcal{U}_{\mathfrak{I}}$-unitary orbit of a compact operator is the following. By the spectral theorem we may rewrite the compact normal operator a as a uniform norm convergent series, namely

$$a = \sum_{i=1}^{w} \lambda_i p_i, \tag{5}$$

where $1 \le w \le \infty$, λ_i are the nonzero distinct eigenvalues of a and $\{ p_i \}_1^w$ is a family of mutually orthogonal finite rank projections. Indeed, p_i is the orthogonal projection onto $\ker(a - \lambda_i)$. Then we take P to be the pinching operator associated with $\{ p_i \}_1^w$.

Let $u \in \mathcal{U}_{\mathfrak{I}}$ such that $ua = au$. If we use the spectral decomposition of a, we see that u must be block diagonal with respect to the family $\{ p_i \}_0^w$. This says that the isotropy group at a coincides with the isotropy group at P, i.e.

$$\{ u \in \mathcal{U}_{\mathfrak{I}} : ua = au \} = \{ u \in \mathcal{U}_{\mathfrak{I}} : L_u P = PL_u \} = G.$$

Hence it turns out that the quotient topology on $\mathcal{U}_{\mathfrak{I}}(a) \simeq \mathcal{U}_{\mathfrak{I}}/G$ is equal to the quotient topology on $\mathcal{U}_{\mathfrak{I}}(P)$.

Corollary 4.9. *Let Φ a symmetric norming function, and $\mathfrak{I} = \mathfrak{S}_{\Phi}$. Let a be a compact normal operator. Then the quotient topology on $\mathcal{U}_{\mathfrak{I}}(a)$ coincides with the topology inherited from $a + \mathfrak{I}$ if and only if* $\operatorname{rank}(a) < \infty$.

Proof. Suppose that $\operatorname{rank}(a) < \infty$. This is equivalent to state that $w < \infty$ in the spectral decomposition of a given by Eq. (5). Under this assumption the family $\{ p_i \}_0^w$ has only one projection of infinite rank, namely $p_0 = 1 - \sum_{i=1}^w p_i$. Indeed, note that p_0 is the orthogonal projection onto $\ker(a)$. According to Lemma 3.6 when $\mathfrak{I} \neq \mathfrak{K}$, or Proposition 4.5 when $\mathfrak{I} = \mathfrak{K}$, the quotient topology coincides with the topology inherited from $\mathcal{B}(\mathfrak{I})$ on $\mathcal{U}_{\mathfrak{I}}(P)$.

Since the quotient topology on $\mathcal{U}_{\mathfrak{I}}(a)$ is always stronger than the topology inherited from $a + \mathfrak{I}$, it remains to prove that any sequence $(u_n)_n$ in $\mathcal{U}_{\mathfrak{I}}$ satisfying $\|u_n a u_n^* - a\|_{\mathfrak{I}} \to 0$ has to be convergent to a in the quotient topology. To this end, note that $p_i(u_n a - a u_n)p_j = (\lambda_i - \lambda_j)p_i u_n p_j$, and then

$$\|p_i u_n p_j\|_{\mathfrak{I}} \leq |\lambda_i - \lambda_j|^{-1} \|u_n a - a u_n\|_{\mathfrak{I}} \to 0,$$

for all $i, j \geq 0$ and $i \neq j$ (where we set $\lambda_0 = 0$). Now let $x \in \mathfrak{I}$ such that $\|x\|_{\mathfrak{I}} = 1$. Since

$$\left\| \sum_{i=1}^{w} (u_n p_i - p_i u_n) x p_i \right\|_{\mathfrak{I}} \leq \sum_{i=1}^{w} \|u_n p_i - p_i u_n\|_{\mathfrak{I}} \leq 2 \sum_{i \neq j} \|p_j u_n p_i\|_{\mathfrak{I}},$$

we see that

$$\|L_{u_n} P L_{u_n^*} - P\|_{\mathcal{B}(\mathfrak{I})} = \|L_{u_n} P - P L_{u_n}\|_{\mathcal{B}(\mathfrak{I})} \leq 2 \sum_{i \neq j} \|p_j u_n p_i\|_{\mathfrak{I}} \to 0.$$

By the remarks in the first paragraph of this proof and Remark 4.8, the latter is equivalent to say that $u_n a u_n^* \to a$ in the quotient topology.

In order to prove the converse we assume that the quotient topology on $\mathcal{U}_{\mathfrak{I}}(a)$ coincides with the topology inherited from $a + \mathfrak{I}$. We need to consider two cases. In the first case we suppose that $\mathfrak{I} \neq \mathfrak{K}$. Let $\mathcal{M}$ be the supplement of the Lie algebra of G defined in Proposition 2.4. If $\operatorname{rank}(a) = \infty$, we can construct a sequence $(z_k)_k$ in $\mathcal{M}$ such that $\|z_k\| \to 0$ and $\|z_k\|_{\mathfrak{I}} = 1$ (see Remark 3.5).

Given $\epsilon > 0$, let $M \geq 1$ such that $\| \sum_{i=M+1}^{w} \lambda_i p_i \| \leq \epsilon$. Then it follows that

$$\begin{aligned} \|e^{z_k} a e^{-z_k} - a\|_{\mathfrak{I}} &= \|(e^{z_k} - 1)a - a(e^{z_k} - 1)\|_{\mathfrak{I}} \\ &\leq 2 \left(\|e^{z_k} - 1\| \left\| \sum_{i=1}^{M} \lambda_i p_i \right\|_{\mathfrak{I}} + \|e^{z_k} - 1\|_{\mathfrak{I}} \left\| \sum_{i=M+1}^{w} p_i \right\| \right) \\ &\leq 2 \left(\|e^{z_k} - 1\| \left\| \sum_{i=1}^{M} \lambda_i p_i \right\|_{\mathfrak{I}} + e\,\epsilon \right). \end{aligned}$$

Letting $k \to \infty$, we find that $e^{z_k} a e^{-z_k} \to a$ in the norm $\| \cdot \|_{\mathfrak{I}}$, or equivalently, in the quotient topology. By the same argument used at the beginning of Lemma 3.6 we can arrive at $\|z_k\|_{\mathfrak{I}} \to 0$, a contradiction with our previous choice of $(z_k)_k$.

Now we turn to the case where $\mathfrak{I} = \mathfrak{K}$. Under the assumption that both topologies coincide on $\mathcal{U}_{\mathfrak{K}}(a)$ we claim that the map

$$\Lambda : \mathcal{U}_{\mathfrak{K}}(a) \longrightarrow \mathcal{U}_{\mathfrak{K}}(P), \qquad \Lambda(uau^*) = L_u P L_{u^*},$$

is continuous, when one endows $\mathcal{U}_{\mathfrak{K}}(a)$ with the topology inherited from $\mathfrak{K}$ and $\mathcal{U}_{\mathfrak{K}}(P)$ with the topology inherited from $\mathcal{B}(\mathfrak{K})$. In fact, by Proposition 4.5 the quotient and the inherited topologies always coincide on $\mathcal{U}_{\mathfrak{K}}(P)$. Then the map Λ turns out to be the identity map of $\mathcal{U}_{\mathfrak{K}}/G$, and thus our claim follows.

Again we suppose that $\operatorname{rank}(a) = \infty$. We will find a contradiction with the fact that Λ is continuous. Note that there must be an infinite number of finite rank projections in the family $\{ p_i \}_1^w$ and the eigenvalues of a satisfy $\lambda_i \to 0$. Let $(\xi_{i,j(i)})$ be an orthonormal basis of $\mathcal{H}$ such that $(\xi_{i,j(i)})_{j(i)=1,\dots,\operatorname{rank}(p_i)}$ is a basis of $R(p_i)$ for all $i \geq 1$. Then take the following sequence of unitary operators:

$$u_n = \xi_{n+2,1} \otimes \xi_{n+1,1} + \xi_{n+1,1} \otimes \xi_{n+2,1} + e_n,$$

where e_n is the orthogonal projection onto $\{ \xi_{n+1,1}, \xi_{n+2,1} \}^{\perp}$. Note that $u_n - 1$ has finite rank, then $u_n \in \mathcal{U}_{\mathfrak{K}}$. Thus we get

$$\begin{aligned} \|u_n a u_n^* - a\| &= \|u_n a - a u_n\| \\ &= \|(\lambda_{n+1} - \lambda_{n+2})\,(\xi_{n+2,1} \otimes \xi_{n+1,1}) - (\lambda_{n+2} - \lambda_{n+1})(\xi_{n+1,1} \otimes \xi_{n+2,1})\| \\ &\leq 2|\lambda_{n+1} - \lambda_{n+2}| \to 0. \end{aligned}$$

On the other hand, note that

$$\begin{aligned}\|L_{u_n}PL_{u_n^*}-P\|_{\mathcal{B}(\mathfrak{K})} &\geq \|(L_{u_n}PL_{u_n^*}-P)(\xi_{n+1,1}\otimes\xi_{n+1,1})\|\\ &= \left\|u_np_{n+1}u_n^*(\xi_{n+1,1}\otimes\xi_{n+1,1})-\xi_{n+1,1}\otimes\xi_{n+1,1}\right\| = \|\xi_{n+1,1}\otimes\xi_{n+1,1}\| = 1,\end{aligned}$$

where we have used that $u_np_{n+1}u_n^*(\xi_{n+1,1}\otimes\xi_{n+1,1}) = 0$. This contradicts the continuity of Λ. Hence a must have finite rank, and the theorem is proved. □

5. Covering map

For $u \in \mathcal{U}_{\mathfrak{I}}$, consider the inner automorphism given by $Ad_u : \mathfrak{I} \longrightarrow \mathfrak{I}$, $Ad_u(x) = uxu^*$. Given a pinching operator P associated with a family $\{p_i\}_1^w$, there is another orbit of P defined by

$$\mathcal{O}_{\mathfrak{I}}(P) := \{Ad_uPAd_{u^*} : u \in \mathcal{U}_{\mathfrak{I}}\}.$$

Note that all the operators in $\mathcal{O}_{\mathfrak{I}}(P)$ are pinching operators while P is the only pinching operator in $\mathcal{U}_{\mathfrak{I}}(P)$. The isotropy group of the coadjoint action is given by

$$H = \{u \in \mathcal{U}_{\mathfrak{I}} : Ad_uPAd_{u^*} = P\}. \tag{6}$$

In order to find a characterization of the operators in H we need the following lemma. We make the convention $\{0, 1, \ldots, \infty\} = \mathbb{N}_0$.

Lemma 5.1. *Let P be the pinching operator associated with a family $\{p_i\}_1^w$ and Q be the pinching operator associated with another family $\{q_i\}_1^v$. Then $P = Q$ if and only if $w = v$ and $p_i = q_{\sigma(i)}$ for some permutation σ of $\{0, \ldots, w\}$ such that $\sigma(0) = 0$.*

Proof. We first suppose that $P = Q$. This is equivalent to

$$\sum_{i=1}^{w} p_ixp_i = \sum_{j=1}^{v} q_jxq_j, \tag{7}$$

for all $x \in \mathfrak{I}$. If $\operatorname{rank}(p_i) < \infty$, $i \geq 1$, we set $x = p_i$ to get $\sum_{j=1}^{v} q_jp_iq_j = p_i$. Then it follows that $q_jp_i = q_jp_iq_j = p_iq_j$ for all $j \geq 1$. If $\operatorname{rank}(p_i) = \infty$, we use the same idea with a sequence of projections $(e_n)_n$ such that $e_n \leq p_i$, $e_n \nearrow p_i$, to find that $q_je_n = e_nq_j$, which implies that $q_jp_i = p_iq_j$. Since $p_0 = 1 - \sum_{i=1}^{w} p_i$ and $q_0 = 1 - \sum_{i=1}^{v} q_j$, we can conclude that $q_jp_i = p_iq_j$ for all $i, j \geq 0$.

Now we claim that for each $i \geq 0$, we can find a unique $\sigma(i)$ such that $p_i = q_{\sigma(i)}$. To this end, let $\xi \in R(p_i)$, $\xi \neq 0$, and note that $p_i\xi = \xi = \sum_{j=0}^{v} q_j\xi$. This implies that there is some $j := \sigma(i)$ such that $q_j\xi \neq 0$. Then we see that $q_j\xi = q_jp_i\xi = p_iq_j\xi$. Now let $\eta \in R(p_i)$ and insert $x = \eta \otimes q_j\xi$ in Eq. (7). In case $i > 0$ we find that $\eta \otimes q_j\xi = (q_j\eta) \otimes q_j\xi$. If $j = 0$, then $\eta \otimes q_j\xi = 0$. In particular, if we take $\eta = q_j\xi \neq 0$, we obtain a contradiction. Hence we must have $j > 0$, so the equation $\eta \otimes q_j\xi = (q_j\eta) \otimes q_j\xi$ implies that $q_j\eta = \eta$. Since η is arbitrary, we have $R(p_i) \subseteq R(q_j)$. In a similar way, we may choose $\eta \in R(p_j)$ to obtain that $R(q_j) \subseteq R(p_i)$. Thus $p_i = q_j$.

In case $i = 0$, we need to show that $p_0 = q_0$. Suppose that there exists some $j > 0$ such that $q_j\xi \neq 0$. By the preceding paragraph we know that $q_j\xi \in R(p_0)$. Then we insert $x = (q_j\xi) \otimes q_j\xi$ in Eq. (7) to find that $0 = (q_j\xi) \otimes q_j\xi$, and hence $q_j\xi = 0$, a contradiction. Thus we obtain that $\xi = \sum_{j=0}^{v} q_j\xi = q_0\xi$, and consequently, $R(p_0) \subseteq R(q_0)$. Interchanging p_0 and q_0, we can conclude that $p_0 = q_0$. Since $\{q_j\}_0^v$ is a mutually orthogonal family, $\sigma(i)$ is unique and our claim is proved.

In other words, we have proved the existence of a map $\sigma : \{0, \ldots, w\} \to \{0, \ldots, v\}$ satisfying $p_i = q_{\sigma(i)}$ and $\sigma(0) = 0$. Repeating the previous argument with q_j in place of p_i, we can construct another map $\psi : \{0, \ldots, v\} \to \{0, \ldots, w\}$ such that $q_j = p_{\psi(j)}$ and $\psi(0) = 0$. But $p_i = q_{\sigma(i)} = p_{(\psi\sigma)(i)}$ and $q_j = p_{\psi(j)} = q_{(\sigma\psi)(j)}$, so we have that $\sigma\psi = \psi\sigma = 1$. Hence, σ is a permutation and $w = v$.

In order to prove the converse, let σ a permutation of $\{0, \ldots, w\}$, P be the pinching operator associated with a family $\{p_i\}_1^w$ and Q be the pinching operator associated with $\{p_{\sigma(i)}\}_1^w$. Since the case $w < \infty$ is trivial, we suppose $w = \infty$. Set $e_k = \sum_{i=0}^{k} p_i$. For each $x \in \mathfrak{I}$, since x is compact, we find that $\|(1 - e_k)x\| \to 0$. Note that for $k \geq 1$,

$$\sum_{i=1}^{\infty} p_{\sigma(i)}e_kxp_{\sigma(i)} = \sum_{i=1}^{k} p_ixp_i = \sum_{i=1}^{\infty} p_ie_kxp_i.$$

Then we get

$$\left\|\sum_{i=1}^{\infty} p_{\sigma(i)}xp_{\sigma(i)} - \sum_{i=1}^{\infty} p_ixp_i\right\| = \left\|\sum_{i=1}^{\infty} p_{\sigma(i)}(1-e_k)xp_{\sigma(i)} - \sum_{i=1}^{\infty} p_i(1-e_k)xp_i\right\| \leq 2\|(1-e_k)x\| \to 0,$$

which proves that $P = Q$. □

Let P be the pinching operator associated with a family $\{ p_i \}_1^w$ $(1 \leq w \leq \infty)$. Let F be the set of all the permutations σ of $\{ 0, \ldots, w \}$ such that $\sigma(i) = i$ for all but finitely many $i \geq 0$. Note that the definition of the set F becomes unnecessary if $w < \infty$. We will need to consider permutations of a finite number of finite dimensional blocks with the same dimension which fix zero, i.e.

$$\mathcal{F} := \{ \sigma \in F : \sigma(0) = 0, \ \mathrm{rank}(p_i) = \mathrm{rank}(p_{\sigma(i)}) < \infty \text{if } \sigma(i) \neq i \}.$$

Let $(\xi_{i,j(i)})$ be an orthonormal basis of $\mathcal{H}$ such that $(\xi_{i,j(i)})_{j(i)=1,\ldots,\mathrm{rank}(p_i)}$ is a basis of $R(p_i)$, where $i = 0, \ldots, w$. For each $\sigma \in \mathcal{F}$, we define the following permutation block operator matrix:

$$r_\sigma(\xi_{i,j(i)}) := \xi_{\sigma(i),j(\sigma(i))}, \quad i = 0, \ldots, w, \ j(i) = 1, \ldots, \mathrm{rank}(p_i).$$

Note that $\mathrm{rank}(r_\sigma - 1) < \infty$, since $\sigma \in \mathcal{F}$. Hence, it follows that $r_\sigma \in \mathcal{U}_\mathfrak{I}$ for any symmetrically-normed ideal $\mathfrak{I}$.

Example 5.2. A simple example takes place when $\mathcal{H} = \mathbb{C}^n$, $\mathrm{rank}(p_i) = 1$ and $\sum_{i=1}^n p_i = 1$. Here the set of all the matrices of the form r_σ, $\sigma \in \mathcal{F}$, reduces to all the $n \times n$ permutation matrices. According to our next result, H has exactly $n!$ connected components in this example.

Recall that from the proof of Proposition 2.4 we know that the isotropy group G at P corresponding to the action given by the left representation can be characterized as block diagonal unitary operators, i.e.

$$G = \left\{ u \in \mathcal{U}_\mathfrak{I} : \sum_{i=0}^{w} p_i u p_i = u \right\},$$

where P is the pinching operator associated with a family $\{ p_i \}_1^w$.

Lemma 5.3. *Let H be the isotropy group defined in (6). Then,*

$$H = \bigcup_{\sigma \in \mathcal{F}} r_\sigma G,$$

where each set in the union is a connected component of H.

Proof. Let $u \in \mathcal{U}_\mathfrak{I}$ such that $Ad_u P Ad_{u^*} = P$. According to Lemma 5.1 it follows that $u p_i u^* = p_{\sigma(i)}$ for some σ permutation of $\{ 0, \ldots, w \}$ such that $\sigma(0) = 0$. In particular, note that $p_j u p_i = \delta_{j,\sigma(i)} \, p_{\sigma(i)} u$, which actually says that u has only one nonzero block in each row. Since $u - 1 \in \mathfrak{I}$, we get that $\sigma \in \mathcal{F}$. Hence we can write $u = r_\sigma r_{\sigma^{-1}} u$, where $r_{\sigma^{-1}} u \in G$.

To prove the other inclusion it suffices to note that $r_\sigma u p_i u^* r_{\sigma^{-1}} = r_\sigma p_i r_{\sigma^{-1}} = p_{\sigma(i)}$ for any $u \in G$. Then we apply again Lemma 5.1 to obtain that $Ad_u P Ad_{u^*} = P$.

In order to establish the last assertion about the connected components of H, we remark that

$$\|r_\sigma u - r_{\sigma'} v\|_\mathfrak{I} \geq \|r_\sigma u - r_{\sigma'} v\| \geq 1,$$

whenever $\sigma \neq \sigma'$ and $u, v \in G$. This implies that the distance between any pair of sets that appear in the union is greater than one. On the other hand, it is a well known fact that $\mathcal{U}_\mathfrak{I}$ is connected, then so does $r_\sigma G$. Hence the lemma is proved. □

Remark 5.4. As a consequence of Lemma 5.3, H is a Banach–Lie subgroup of $\mathcal{U}_\mathfrak{I}$. Indeed, the connected components of H are diffeomorphic to the Banach–Lie subgroup G of $\mathcal{U}_\mathfrak{I}$. Hence it follows that $\mathcal{O}_\mathfrak{I}(P) \simeq \mathcal{U}_\mathfrak{I}/H$ has a manifold structure endowed with the quotient topology.

Theorem 5.5. *Let Φ a symmetric norming function, and $\mathfrak{I} = \mathfrak{S}_\Phi$. Let P be the pinching operator associated with a family $\{p_i\}_1^w$. If $\mathfrak{I} \neq \mathfrak{K}$ assume in addition that $w < \infty$ and there is only one infinite rank projection in the family $\{ p_i \}_0^w$. Then the map*

$$\Pi : \mathcal{U}_\mathfrak{I}(P) \longrightarrow \mathcal{O}_\mathfrak{I}(P), \qquad \Pi(L_u P L_{u^*}) = Ad_u P Ad_{u^*},$$

is a covering map, when $\mathcal{U}_\mathfrak{I}(P)$ is considered with the topology inherited from $\mathcal{B}(\mathfrak{I})$ and $\mathcal{O}_\mathfrak{I}(P)$ with the quotient topology.

Proof. In the case where $\mathfrak{I} \neq \mathfrak{K}$, under the above hypothesis on the family $\{ p_i \}_1^w$, it was proved in Lemma 3.6 that the quotient topology coincides with the subspace topology on $\mathcal{U}_\mathfrak{I}(P)$. In case $\mathfrak{I} = \mathfrak{K}$ both topologies coincide without additional hypothesis by Proposition 4.5. On the other hand, by Lemma 5.3 the quotient H/G is discrete, then H/G is homomorphic to $\mathcal{F}$. We define an action of $\mathcal{F}$ on $\mathcal{U}_\mathfrak{I}(P)$ given by $\sigma \cdot L_u P L_{u^*} = L_{u r_\sigma} P L_{r_{\sigma^{-1}} u^*}$. Therefore we can make the following identifications:

$$\mathcal{U}_\mathfrak{I}(P)/\mathcal{F} \simeq \mathcal{U}_\mathfrak{I}(P)/(H/G) \simeq (\mathcal{U}_\mathfrak{I}/G)/(H/G) \simeq \mathcal{U}_\mathfrak{I}/H \simeq \mathcal{O}_\mathfrak{I}(P).$$

Thus we may think of Π as the quotient map $\mathcal{U}_\mathfrak{I}(P) \longrightarrow \mathcal{U}_\mathfrak{I}(P)/\mathcal{F}$. Hence to prove that Π is a covering map, it suffices to show that $\mathcal{F}$ acts properly discontinuous on $\mathcal{U}_\mathfrak{I}(P)$ (see [18]). This means that for any $Q \in \mathcal{U}_\mathfrak{I}(P)$, there is an open neighborhood $\mathcal{W}$ of Q such that $\mathcal{W} \cap \sigma \cdot \mathcal{W} = \emptyset$ for all $\sigma \neq 1$. Clearly, there is no loss of generality if we prove this fact for $Q = P$. To this end, define the open neighborhood by

$$\mathcal{W} := \{ Q \in \mathcal{U}_\mathfrak{I}(P) : \|Q - P\|_{\mathcal{B}(\mathfrak{I})} < 1/2 \}.$$

Suppose that $\mathcal{W} \cap \sigma \cdot \mathcal{W} \neq \emptyset$ for some $\sigma \neq 1$. Then there are $Q, \tilde{Q} \in \mathcal{W}$ such that $\tilde{Q} = \sigma \cdot Q$. If $Q = L_u P L_{u^*}$, then we have that $\tilde{Q} = L_{ur_\sigma} P L_{r_{\sigma^{-1}} u^*}$. The distance between Q and $\tilde{Q}$ can be estimated as follows

$$\|Q - \tilde{Q}\|_{\mathcal{B}(\mathfrak{I})} = \|P - L_{r_\sigma} P L_{r_{\sigma^{-1}}}\|_{\mathcal{B}(\mathfrak{I})} \geq \left\| \sum_{i=1}^{w} (p_i - p_{\sigma(i)})(\xi \otimes \xi) p_i \right\|_{\mathfrak{I}} = \|\xi \otimes \xi\|_{\mathfrak{I}} = 1,$$

where $\xi \in R(p_i)$ is such that $\|\xi\| = 1$ and $\sigma(i) \neq i$. But since $Q, \tilde{Q} \in \mathcal{W}$, it follows that $\|Q - \tilde{Q}\|_{\mathcal{B}(\mathfrak{I})} < 1$, a contradiction. Hence the action is properly discontinuous, and the proof is complete. □

Acknowledgment

The authors would like to thank the anonymous referee for constructive comments. They were partially supported by Instituto Argentino de Matemática and CONICET.

References

[1] E. Andruchow, G. Larotonda, Hopf–Rinow theorem in the Sato Grassmanian, J. Funct. Anal. 255 (7) (2008) 1692–1712.
[2] E. Andruchow, G. Larotonda, L. Recht, Finsler geometry and actions of the p-Schatten unitary groups, Trans. Amer. Math. Soc. 362 (2010) 319–344.
[3] E. Andruchow, D. Stojanoff, Geometry of conditional expectations and finite index, Internat. J. Math. 5 (2) (1994) 169–178.
[4] D. Beltiţă, Smooth Homogeneous Structures in Operator Theory, in: Monographs and Surveys in Pure and Applied Mathematics, vol. 137, Chapman and Hall/CRC, 2006.
[5] D. Beltiţă, T. Ratiu, Symplectic leaves in real Banach Lie–Poisson spaces, Geom. Funct. Anal. 15 (4) (2005) 753–779.
[6] D. Beltiţă, T. Ratiu, A. Tumpach, The restricted Grassmannian, Banach Lie–Poisson spaces and coadjoint orbits, J. Funct. Anal. 247 (1) (2007) 138–168.
[7] R. Bhatia, Pinching, trimming, truncating, and averaging of matrices, Amer. Math. Monthly 107 (2000) 602–608.
[8] P. Bóna, Some considerations on topologies of infinite dimensional unitary coadjoint orbits, J. Geom. Phys. 51 (2) (2004) 256–268.
[9] N. Bourbaki, Variétés différentielles et analytiques. Fascicule de résultats. Paragraphes 1 à 7, Hermann, Paris, 1967.
[10] J. Calkin, Two-sided ideals and congruences in the ring of bounded operators in Hilbert space, Ann. of Math. 42 (2) (1941) 839–873.
[11] J.B. Conway, The compact operators are not complemented in $\mathcal{B}(\mathcal{H})$, Proc. Amer. Math. Soc. 32 (1972) 549–550.
[12] G. Corach, H. Porta, L. Recht, Differential geometry of systems of projections in Banach algebras, Pacific J. Math. 143 (1990) 209–228.
[13] G. Corach, H. Porta, L. Recht, The geometry of spaces of projections in C^*-algebras, Adv. Math. 101 (1) (1993) 59–77.
[14] C. Davis, Compressions to finite-dimensional subspaces, Proc. Amer. Math. Soc. 9 (1958) 356–359.
[15] C. Davis, Various averaging operations onto subalgebras, Illinois J. Math. 3 (1959) 538–553.
[16] L.A. Fialkow, A note on unitary cross sections for operators, Canad. J. Math. 30 (1978) 1215–1227.
[17] I.C. Gohberg, M.G. Krein, Introduction to the Theory of Linear Non-Self-Adjoint Operators, Amer. Math. Soc., Providence, RI, 1960.
[18] M. Greenberg, Lectures on Algebraic Topology, Benjamin, New York, 1967.
[19] G. Larotonda, Unitary orbits in a full matrix algebra, Integral Equations Operator Theory 54 (4) (2006) 511–523.
[20] K.-H. Neeb, Infinite-dimensional Lie groups and their representations, in: J.P. Anker, B. Ørsted (Eds.), Lie Theory: Lie Algebras and Representations, in: Progress in Math., vol. 228, Birkhäuser Verlag, 2004.
[21] A. Odzijewicz, T. Ratiu, Banach Lie–Poisson spaces and reduction, Comm. Math. Phys. 243 (1) (2003) 1–54.
[22] A. Pressley, G. Segal, Loop Groups, in: Oxford Mathematical Monographs, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1986.
[23] B. Simon, Trace Ideals and their Applications, in: London Mathematical Society Lecture Note Series, vol. 35, Cambridge University Press, Cambridge, 1979.
[24] H. Upmeier, Symmetric Banach Manifolds and Jordan C^*-Algebras, in: North-Holland Math. Stud., vol. 104, North-Holland, Amsterdam, 1985, Notas de Matemática, vol. 96.
[25] J. von Neumann, Mathematical Foundations of Quantum Mechanics, Princeton Univ. Press, Princeton, NJ, 1955.
[26] R. Whitley, Projecting m onto c_0, Amer. Math. Monthly 73 (1966) 285–286.