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Generalized Schur complements and P-complementable operators

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Abstract

Let A be a selfadjoint operator and P be an orthogonal projection both operating on a Hilbert space $\mathscr{H}$. We say that A is P-complementable if $A - \mu P \geqslant 0$ holds for some $\mu \in \mathbb{R}$. In this case we define $I_P(A) = \max\{\mu \in \mathbb{R} : A - \mu P \geqslant 0\}$. As a tool for computing $I_P(A)$ we introduce a natural generalization of the Schur complement or shorted operator of A to $\mathscr{S} = R(P)$, denoted by $\Sigma(A, P)$. We give expressions and a characterization for $I_P(A)$ that generalize some known results for particular choices of P. We also study some aspects of the shorted operator $\Sigma(A, P)$ for P-complementable A, under the hypothesis of *compatibility* of the pair $(A, \mathscr{S})$. We give some applications in the finite dimensional context.

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1. Introduction

Let $\mathscr{H}$ be a Hilbert space and $L(\mathscr{H})$ the algebra of bounded linear operators on $\mathscr{H}$. Given a closed subspace $\mathscr{S} \subseteq \mathscr{H}$ and $P = P_{\mathscr{S}} \in L(\mathscr{H})$ the orthogonal

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projection onto $\mathscr{S}$, we study the following two problems: for any selfadjoint operator $A \in L(\mathscr{H})$,

1. determine whether there exists some $\mu \in \mathbb{R}$ such that
$$A - \mu P \geqslant 0; \tag{1}$$
2. in case Eq. (1) holds for some μ, compute the optimum number
$$I_P(A) = \max\{\mu \in \mathbb{R} : A - \mu P \geqslant 0\}. \tag{2}$$

The general solution of problem (1) is well known, see for example [14] or Proposition 3.3. Also, if $A \geqslant 0$, problem (2) has a known answer (see, for example, [7]). We state this case in the preliminary Section 2 (Corollary 2.2). Therefore our main interest is to study problem (2) in the non-positive case. It should be mentioned that the general case seems not to be easily reduced to the positive case (see Remark 5.1).

If condition (1) is satisfied by A, we shall say that A is *P-complementable*, because in this case there exists the *shorted operator* (or Schur complement, see [1]) defined as follows:
$$\Sigma(A, P) = \max\{D \in L(\mathscr{H}) : D = D^*, D \leqslant A, D(\mathscr{H}) \subseteq \mathscr{S}\}.$$
Using the identity $\Sigma(A - \mu P, P) = \Sigma(A, P) - \mu P$, in Section 3 we extend several known properties of shorted operators of positive operators to our case. On the other hand, in Section 5 we show that, if $A \ngeqslant 0$ but it is P-complementable, then
$$I_P(A) = \lambda_{\min}(\Sigma(A, P)),$$
where $\lambda_{\min}(C)$ denotes the minimum of the spectrum $\sigma(C)$ of $C \in L(\mathscr{H})$.

Although most applications of the problems mentioned above appear in matrix theory, i.e., when $\dim \mathscr{H} < \infty$, an additional hypothesis of the operator A allows to extend all finite dimensional results to our setting. This hypothesis is the so called *compatibility* of the pair $(A, \mathscr{S})$. This notion, defined by Corach, Maestripieri and the second author in [4–6], is the following: The pair $(A, \mathscr{S})$ is compatible if there exists a A-selfadjoint projection onto $\mathscr{S}^\perp$, i.e., $Q \in L(\mathscr{H})$ such that $Q^2 = Q$, $AQ = Q^*A$ and $R(Q) = \mathscr{S}^\perp$.

There are several characterization of the compatibility of $(A, \mathscr{S})$ and general properties of such pairs in the case $A \geqslant 0$ (see, for example, [4]); some of them are stated in Section 4 of this paper, where we also extend these properties to the non-positive case. We say that compatibility is an additional condition because, if $(1 - P)A(1 - P) \geqslant 0$ and $(A, \mathscr{S})$ is compatible , then A is P-complementable. The reverse implication is false in general, but it is true if $\dim \mathscr{S}^\perp < \infty$, in particular in the finite dimensional case.

Section 5 is devoted to the computation of the number $I_P(A)$ for A selfadjoint, not necessarily positive. We first obtain the formula
$$I_P(A) = \inf\{\langle A\xi, \xi\rangle : \xi \in \mathscr{H}, \|P\xi\| = 1\}$$
so that, if $\mathscr{S}$ is the subspace generated by the unit vector $\xi \in \mathscr{H}$, then
$$I_P(A) = \inf\{\langle A\eta, \eta\rangle : \eta \in \mathscr{H}, \langle \eta, \xi\rangle = 1\}.$$

If $(A, \mathscr{S})$ is compatible, we show that the computation of $I_P(A)$ can be reduced to the case in which $\mathscr{S} \subseteq \overline{R(A)}$, by replacing $\mathscr{S}$ by $\overline{\mathscr{S} \cap R(A)}$. We state the results of the rest of this section in the following theorem:

Theorem. *Let $A = A^* \in L(\mathscr{H})$, $A \not\geqslant 0$, and $P = P^* = P^2 \in L(\mathscr{H})$ with $R(P) = \mathscr{S}$. Suppose that $(1 - P)A(1 - P) \geqslant 0$ and $(A, \mathscr{S})$ is compatible. Then*

1. $R(\Sigma(A, P)) = \overline{\mathscr{S} \cap R(A)} \neq \{0\}$.
2. *If $\mathscr{T} = \overline{\mathscr{S} \cap R(A)}$ and $Q = P_{\mathscr{T}}$, then the pair $(A, \mathscr{T})$ is compatible, $\Sigma(A, P) = \Sigma(A, Q)$ and $I_P(A) = I_Q(A)$.*
3. *If $R(A)$ is closed, then $\mathscr{T} = \mathscr{S} \cap R(A)$ and*
$$I_P(A) = I_Q(A) = \lambda_{\min}\big((QA^\dagger Q)^\dagger\big), \tag{3}$$
where $C^\dagger$ denotes the Moore–Penrose pseudoinverse of a closed range operator C.

Formula (3) is the natural generalization of $I_P(A) = \|PA^\dagger P\|^{-1}$, which holds if A is positive (semidefinite) with closed range (see Corollary 2.2). In Section 6 we study some applications of the mentioned results, particularly to problems posed by Fiedler–Markham [9] and Reams [15]. Given a completely positive map $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$, we also compute the number

$$I(\Phi) = \max\{\mu \in \mathbb{R} : \Phi - \mu \cdot \mathrm{Id} \text{ is completely positive}\},$$

which can be considered as a notion of index for such maps.

2. Preliminary results

In this paper $\mathscr{H}$ denotes a Hilbert space, $L(\mathscr{H})$ is the algebra of all linear bounded operators on $\mathscr{H}$, $\mathrm{Gl}(\mathscr{H})$ is the group of invertible operators in $L(\mathscr{H})$ and $L(\mathscr{H})^+$ is the subset of $L(\mathscr{H})$ of all positive (semidefinite) operators. If $\dim \mathscr{H} = n < \infty$ we shall identify $\mathscr{H}$ with $\mathbb{C}^n$ and $L(\mathscr{H})$ with the space of $n \times n$ complex matrices $M_n(\mathbb{C})$. The elements of $\mathbb{C}^n$ are considered as column vectors. For simplicity we sometimes describe a column vector $\xi \in \mathbb{C}^n$ as $\xi = (\xi_1, \ldots, \xi_n)$.

For every $C \in L(\mathscr{H})$ its range is denoted by $R(C)$, $\sigma(C)$ denotes the spectrum of C and $\rho(C)$ the spectral radius of C. If $C^* = C$, we denote $\lambda_{\min}(C)$ the minimum of $\sigma(C)$. If $R(C)$ is closed, then $C^\dagger$ denotes the Moore–Penrose pseudoinverse of C. The orthogonal projection onto a closed subspace $\mathscr{S}$ is denoted by $P_{\mathscr{S}}$. We use the notations $\mathbb{Q} = \{Q \in L(\mathscr{H}) : Q^2 = Q\}$ for the set of idempotents and $\mathbb{P} = \{P \in \mathbb{Q} : P = P^*\}$ for the set of orthogonal projections. For every $P \in \mathbb{P}$, the decomposition $\mathscr{H} = R(1 - P) \oplus R(P)$ induces a 2×2 representation of $A \in L(\mathscr{H})$:

$$A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix},$$

which we call the matrix representation induced by P.

Now we state the well known criterion due to Douglas [8] (see also [10]) about ranges and factorization of operators:

Theorem 2.1. *Let $A, B \in L(\mathscr{H})$. Then the following conditions are equivalent*:

1. $R(B) \subseteq R(A)$.
2. *There exists a positive number μ such that $BB^* \leqslant \mu AA^*$.*
3. *There exists $D \in L(\mathscr{H})$ such that $B = AD$.*

Moreover, in this case there exists a unique solution D of the equation $AX = B$ such that $R(D) \subseteq \overline{R(A)}$. The operator D is called the reduced solution of the equation $AX = B$ and $\|D\|^2 = \min\{\mu : BB^* \leqslant \mu AA^*\}$. If $R(A)$ is closed, then $D = A^\dagger B$.

Corollary 2.2. *Let $A, B \in L(\mathscr{H})^+$. Then there exists $\mu > 0$ such that $A - \mu B \geqslant 0$ if and only if $R(B^{1/2}) \subseteq R(A^{1/2})$. In this case, if $B \neq 0$ and D is the reduced solution of the equation $A^{1/2}X = B^{1/2}$, we have*

$$\max\{\mu \geqslant 0 : A - \mu B \geqslant 0\} = \|D\|^{-2}.$$

If $R(A)$ is closed, this number coincides with $\rho(A^\dagger B)^{-1} = \|B^{1/2}A^\dagger B^{1/2}\|^{-1}$.

Proof. If $R(A)$ is closed, then $R(A) = R(A^{1/2})$ and $D = (A^{1/2})^\dagger B^{1/2}$. Hence

$$\|D\|^2 = \|D^*D\| = \|B^{1/2}A^\dagger B^{1/2}\| = \rho(B^{1/2}A^\dagger B^{1/2}) = \rho(A^\dagger B). \qquad \square$$

Corollary 2.3. *Let $A \in L(\mathscr{H})^+$ and $\xi \in \mathscr{H}$ with $\|\xi\| = 1$. Consider the rank one projection $P = \xi \otimes \xi = P_\xi$ onto the subspace generated by ξ. If*

$$I_\xi(A) = \max\{\mu \geqslant 0 : A - \mu P \geqslant 0\},$$

then $I_\xi(A) \neq 0 \iff \xi \in R(A^{1/2})$. In this case, if $\eta \in \ker A^\perp$ satisfies $A^{1/2}\eta = \xi$, we get $I_\xi(A) = \|\eta\|^{-2}$. If $\xi \in R(A)$, then for every $\zeta \in \mathscr{H}$ such that $A\zeta = \xi$ it holds $I_\xi(A) = \langle A\zeta, \zeta\rangle^{-1}$. If $R(A)$ is closed, then $I_\xi(A) = \langle A^\dagger\xi, \xi\rangle^{-1}$.

Proof. The first part follows from Corollary 2.2. Let $\eta \in \ker A^\perp$ such that $A^{1/2}\eta = \xi$. Then the reduced solution of the equation $A^{1/2}X = P$ is $\eta \otimes \xi$, the one rank operator defined by

$$\eta \otimes \xi(\gamma) = \langle \gamma, \xi\rangle\eta, \quad \gamma \in \mathscr{H}.$$

It is easy to see that $\|\eta \otimes \xi\| = \|\xi\|\,\|\eta\| = \|\eta\|$. If there exists $\zeta \in \mathscr{H}$ such that $A\zeta = \xi$, then $A^{1/2}\zeta = \eta$ and $\langle A\zeta, \zeta\rangle = \|\eta\|^2$. If $R(A)$ is closed, then $\eta = (A^{1/2})^\dagger\xi$, so that $\|\eta\|^2 = \langle \eta, \eta\rangle = \langle A^\dagger\xi, \xi\rangle$. $\square$

2.4. Suppose that $\dim \mathscr{H} = n < \infty$. We identify $L(\mathscr{H})$ with $M_n(\mathbb{C})$, the algebra of $n \times n$ complex matrices. Let $A \in L(\mathscr{H})^+$. In [16] the notion of minimal index for A was defined as

$$\begin{aligned} I(A) &= \max\left\{\mu \geqslant 0 : A \circ B \geqslant \mu \cdot B \quad \forall\, B \in L(\mathscr{H})^+\right\} \\ &= \max\left\{\mu \geqslant 0 : \Phi_A - \mu \cdot \mathrm{Id} \geqslant 0 \quad \text{on } L(\mathscr{H})^+\right\} \\ &= \max\left\{\mu \geqslant 0 : A - \mu \cdot ee^t \geqslant 0\right\}, \end{aligned} \tag{4}$$

where $e = (1, \ldots, 1)$, the symbol $\circ$ denotes the Hadamard product of matrices and $\Phi_A(C) = A \circ C$, $C \in L(\mathscr{H})$. The last equality follows from the fact that for $C \in L(\mathscr{H})$, $\Phi_C \geqslant 0 \Leftrightarrow C \geqslant 0$ (see [13]).

Note that, if $\xi = n^{-1/2}e$, then $\|\xi\| = 1$ and $I(A) = n^{-1} I_\xi(A)$. By Corollary 2.3, $I(A) > 0$ if and only if e belongs to the range of A. In [7,16] it is shown that, in this case, for any vector y such that $A(y) = e$,

$$I(A) = \langle Ay, y\rangle^{-1} = \langle A^\dagger e, e\rangle^{-1} = \min\{\langle Az, z\rangle : \langle z, e\rangle = 1\}. \tag{5}$$

Note that the first two equalities are particular cases of Corollary 2.3.

3. The shorted operator for selfadjoint operators

Let $A = A^* \in L(\mathscr{H})$ and $P \in \mathbb{P}$. We first need a characterization of those pairs (A, P) such that, for some $\mu \in \mathbb{R}$, it holds

$$A - \mu P \geqslant 0. \tag{6}$$

The solution of this problem is well known, see for example [14]. We shall give a brief survey of the characterization of pairs (A, P) satisfying Eq. (6), for the sake of completeness.

Note that if $Px = 0$ then $\langle (A - \mu P)x, x\rangle = \langle Ax, x\rangle$. Thus, a necessary condition for A and P to satisfy condition (6) is that $(1 - P)A(1 - P) \geqslant 0$.

Definition 3.1. Let $A \in L(\mathscr{H})$ such that $A = A^*$ and let $P \in \mathbb{P}$. We shall say that A is P-positive if $(1 - P)A(1 - P) \geqslant 0$.

Remark 3.2. Let $e = (1, \ldots, 1) \in \mathbb{C}^n$ and let $P_e \in M_n(\mathbb{C})$ denote the orthogonal projection onto the subspace generated by e. A real symmetric matrix $A \in M_n(\mathbb{R})$ is called *almost positive* if $\langle A\xi, \xi\rangle \geqslant 0$ for all $\xi \in \mathbb{R}^n$ such that $\langle \xi, e\rangle = 0$. Therefore a real selfadjoint matrix $A \in M_n(\mathbb{C})$ is almost positive if and only if it is P_e-positive.

Proposition 3.3. *Let $P \in \mathbb{P}$ with $R(P) = \mathscr{S}$ and $A \in L(\mathscr{H})$ be hermitian and P-positive. Let $A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$ be the representation induced by P. Then the following conditions are equivalent*:

1. *There exists $\mu \in \mathbb{R}$ such that $A - \mu P \geqslant 0$.*
2. *The partial matrix $\begin{pmatrix} a & b \\ b^* & ? \end{pmatrix}$ admits a positive completion.*

3. *The set* $M(A,\mathscr{S})=\{D\in L(\mathscr{H}):D=D^*,\,D\leqslant A,\,R(D)\subseteq\mathscr{S}\}$ *is not empty.*
4. *There exists* $x\in L(\mathscr{S},\mathscr{S}^\perp)$ *such that* $b=a^{1/2}x$.
5. $R(b)\subseteq R(a^{1/2})$.
   and, if $R(a)$ *is closed, also*
6. $\ker a=\ker A\cap\mathscr{S}^\perp$.

Proof. $1\to2$: Take $d=c-\mu P$. Then $\begin{pmatrix}a & b\\ b^* & d\end{pmatrix}=A-\mu P\geqslant 0$.

$2\to3$: Let $d\in L(\mathscr{S})$ such that $\begin{pmatrix}a & b\\ b^* & d\end{pmatrix}\geqslant 0$. Then $D=\begin{pmatrix}0 & 0\\ 0 & c-d\end{pmatrix}\in M(A,\mathscr{S})$.

$3\to1$: If $D\in M(A,\mathscr{S})$, take $\mu\in\mathbb{R}$ such that $-D\leqslant-\mu P$.

$4\leftrightarrow5$: It is a consequence of Douglas Theorem 2.1.

$2\leftrightarrow5$: It is well known (see [1] or [14]). For example, if $b=a^{1/2}x$ with $x\in L(\mathscr{S},\mathscr{S}^\perp)$, then

$$\begin{pmatrix}a & b\\ b^* & x^*x\end{pmatrix}=\begin{pmatrix}a^{1/2} & 0\\ x^* & 0\end{pmatrix}\begin{pmatrix}a^{1/2} & x\\ 0 & 0\end{pmatrix}\geqslant 0.$$

If $R(a)$ is closed, then $R(a^{1/2})=R(a)=(\ker a)^\perp$. In this case

$$\begin{aligned}R(b)\subseteq R(a)&\Leftrightarrow \ker a\subseteq\ker b^*\\ &\Leftrightarrow(\forall\,\xi\in\mathscr{S}^\perp,\ a\xi=0\Rightarrow a\xi+b^*\xi=A\xi=0),\end{aligned}$$

i.e., condition 5 is equivalent to $\ker a\subseteq\ker A\cap\mathscr{S}^\perp$. Note that the reverse inclusion always holds. □

Remark 3.4. With the notations of Proposition 3.3, if $R(a)$ is not closed, then conditions 1–5 still imply, with the same proof, that $\ker a=\ker A\cap\mathscr{S}^\perp$.

Definition 3.5. Let $A\in L(\mathscr{H})$ be hermitian and $P\in\mathbb{P}$ such that A is P-positive.

1. A is called P-*complementable* if any of the conditions of Proposition 3.3 holds.
2. In this case we define: $I_P(A)=\max\{\mu\in\mathbb{R}:A-\mu P\geqslant 0\}$.

If A is P-complementable, the shorted operator can be defined for the pair (A,P), and several results for shorted operators of positive operators (see [1]) remain true in this case. We show these properties in the rest of this section.

Definition 3.6. Let $P\in\mathbb{P}$ with $R(P)=\mathscr{S}$ and let $A=\begin{pmatrix}a & b\\ b^* & c\end{pmatrix}\in L(\mathscr{H})$ be hermitian P-complementable. Let $d\in L(\mathscr{S},\mathscr{S}^\perp)$ be the reduced solution of the equation $b=a^{1/2}x$. Then we define the Schur complement (or shorted operator) of A with respect to $\mathscr{S}$ as

$$\Sigma(A, P) = \begin{pmatrix} 0 & 0 \\ 0 & c - d^*d \end{pmatrix}.$$

Proposition 3.7. *Let $P \in \mathbb{P}$ with $R(P) = \mathscr{S}$ and let A be P-complementable.*

1. *If $A \geqslant 0$, then $\Sigma(A, P)$ is the usual shorted operator for A and $\mathscr{S}$.*
2. *Let $\mu \in \mathbb{R}$. Then $\Sigma(A - \mu P, P) = \Sigma(A, P) - \mu P$.*
3. $\Sigma(A, P) = \max\{D \in L(\mathscr{H}) : D = D^*, D \leqslant A, R(D) \subseteq \mathscr{S}\}$.
4. $\Sigma(A, P) = \inf\{QAQ^* : Q = Q^2, R(Q) = \mathscr{S}\}$.
5. *Let $\xi \in \mathscr{S}$. Then*

$$\langle \Sigma(A, P)\xi, \xi\rangle = \inf\left\{\langle A(\xi + \eta), \xi + \eta\rangle,\ \eta \in \mathscr{S}^\perp\right\}.$$

6. *If $a = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$ and $R(a)$ is closed, then*

$$\Sigma(A, P) = \begin{pmatrix} 0 & 0 \\ 0 & c - b^*a^\dagger b \end{pmatrix}.$$

where $a^\dagger$ is the Moore–Penrose pseudoinverse of a in $L(\mathscr{S}^\perp)$.

Proof
1. It is shown in [1].
2. It is clear by definition.
3. If $A \geqslant 0$, then $\Sigma(A, P) = \max\{D \in L(\mathscr{H}) : D \geqslant 0,\ D \leqslant A \text{ and } R(D) \subseteq \mathscr{S}\}$ (see [1]). The general case can be easily deduced from the positive case using item 2.
4. If $A \geqslant 0$, then $\Sigma(A, P) = \inf\{QAQ^* : Q = Q^2, R(Q) = \mathscr{S}\}$ (see [1]). The general case can be easily deduced from the positive case using item 2 and the fact that, if $Q \in \mathbb{Q}$ has $R(Q) = \mathscr{S}$, then $QPQ^* = P$.
5. The positive was shown in [1]. If $A \not\geqslant 0$, denote by $B = A - I_P(A)P \geqslant 0$. By item 2, $\Sigma(A, P) = \Sigma(B, P) + I_P(A)P$. Thus,

$$\begin{aligned}\langle \Sigma(B, P)\xi, \xi\rangle &= \inf\{\langle B(\xi + \eta), \xi + \eta\rangle, \eta \in \mathscr{S}^\perp\} \\ &= \inf\{\langle A(\xi + \eta), \xi + \eta\rangle, \eta \in \mathscr{S}^\perp\} - I_P(A)\|\xi\|^2.\end{aligned}$$

6. If $R(a)$ is closed, then $R(a^{1/2})$ is also closed, $(a^{1/2})^\dagger = (a^\dagger)^{1/2}$ and $d = (a^{1/2})^\dagger b$ is the reduced solution of the equation $a^{1/2}x = b$. $\square$

Remark 3.8. The following properties are easy consequences of Proposition 3.7 and the corresponding results for the positive case (see [1,11]):

1. Let $P, Q \in \mathbb{P}$ such that $P \leqslant Q$, let $A \in L(\mathscr{H})$ P-complementable and $B \in L(\mathscr{H})$ such that $A \leqslant B$. Then B is P-complementable, $\Sigma(A, P) \leqslant \Sigma(B, P)$, A is Q-complementable and $\Sigma(A, P) \leqslant \Sigma(A, Q)$.

2. Let $\{E_n\} \in L(H)$ be a monotone decreasing sequence of positive operators strongly convergent to 0 and let $A \in L(H)$ be P-complementable. Then $\Sigma(A + E_n, P)$ converges strongly to $\Sigma(A, P)$.
3. Let $A \in L(H)$ be an invertible P-complementable operator. Then $\|\Sigma(A+\epsilon, P) - \Sigma(A, P)\| \to 0$ as $\epsilon \to 0^+$.
4. Let $A \in L(H)$ be P-complementable. Then there exist unique operators F and G such that $A = F + G$ with $R(F) \subseteq \mathscr{S}$, $G \geqslant 0$ and $R(G^{1/2}) \cap \mathscr{S} = \{0\}$.
5. Let $A \in L(\mathscr{H})$ and $P \in \mathbb{P}$ such that A is P-complementable. Let f be a operator monotone map defined on $\sigma(A) \cup \sigma(\Sigma(A, P))$ such that $f(0) \geqslant 0$. Then $\Sigma(f(A), P) \geqslant f(\Sigma(A, P))$.
6. Let $\{P_n\} \in \mathbb{P}$ be a decreasing sequence of projections such that $P_n \xrightarrow{\text{S.O.T}} P$ and let $A \in L(\mathscr{H})$ be P-complementable. Then $\{\Sigma(A, P_n)\}$ decreases to $\Sigma(A, P)$ (see [14] or [2]).

4. *A*-selfadjoint projections

Given $P \in \mathbb{P}$ with $R(P) = \mathscr{S}$ and $A \in L(\mathscr{H}) P$-positive, we shall consider a condition stronger than being P-complementable which is the existence of A-selfadjoint projections onto $\mathscr{S}^{\perp}$, i.e., $Q \in \mathbb{Q}$ such that $AQ = Q^*A$ and $R(Q) = \mathscr{S}^{\perp}$.

Definition 4.1. Let $A = A^* \in L(\mathscr{H})$ and $\mathscr{S} \subseteq \mathscr{H}$ a closed subspace. We denote by

$$\mathscr{P}(A, \mathscr{S}) = \left\{Q \in \mathbb{Q} : R(Q) = \mathscr{S}^{\perp}, AQ = Q^*A\right\}.$$

The pair $(A, \mathscr{S})$ is said to be *compatible* if $\mathscr{P}(A, \mathscr{S})$ is not empty.

The notion of a compatible pair was introduced in [4], where a characterization of compatible pairs $(A, \mathscr{S})$ in terms of the Schur complements $\Sigma(A, P)$ is given, in case that $A \geqslant 0$. The following two results are taken from [4]:

Lemma 4.2. *Let $A = A^* \in L(\mathscr{H})$ and $Q \in \mathbb{Q}$. Then the following conditions are equivalent*:

1. *Q satisfies that $AQ = Q^*A$, i.e., Q is A-selfadjoint.*
2. $\ker Q \subseteq A^{-1}(R(Q)^{\perp})$.
   and, if $A \geqslant 0$,
3. $Q^*AQ \leqslant A$.

Proposition 4.3. *Given $A = A^* \in L(\mathscr{H})$ and $P \in \mathbb{P}$ with $R(P) = \mathscr{S}$, the following conditions are equivalent*:

1. *The pair $(A, \mathscr{S})$ is compatible (i.e., $\mathscr{P}(A, \mathscr{S})$ is not empty).*

2. *If* $A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$ *then* $R(b) \subseteq R(a)$.
3. $\mathscr{S}^\perp + A^{-1}(\mathscr{S}) = \mathscr{H}$.

In this case, for every $E \in \mathscr{P}(A, \mathscr{S})$, $\ker E \subseteq A^{-1}(\mathscr{S})$.

Corollary 4.4. *If* $(A, \mathscr{S})$ *is compatible and* A *is* P*-positive, then* A *is* P*-complementable.*

Proof. Just note that, if $a = (1 - P)A(1 - P) \geqslant 0$, then $R(a) \subseteq R(a^{1/2})$. $\square$

Remark 4.5. Let $A \in L(\mathscr{H})$ be hermitian and $P \in \mathbb{P}$ with $R(P) = \mathscr{S}$ such that A is P-positive and suppose that $R((1 - P)A(1 - P))$ is closed. Then $(A, \mathscr{S})$ is compatible if and only if A is P-complementable. This last condition holds whenever $\dim \mathscr{S}^\perp < \infty$. Therefore if $\mathscr{H}$ is a finite dimensional space and A is P-positive, the conditions $(A, \mathscr{S})$ is compatible and A is P-complementable are equivalent.

Proposition 4.6. *Let* $A = A^* \in L(\mathscr{H})$ *such that* A *is* P*-positive and the pair* $(A, \mathscr{S})$ *is compatible. Let* $E \in \mathscr{P}(A, \mathscr{S})$ *and* $Q = I - E$. *Then*

1. $\Sigma(A, P) = AQ = Q^*A = Q^*AQ$.
2. $\Sigma(A, P) = \min\{FAF^* : F \in \mathbb{Q}, R(F) = \mathscr{S}\}$.
3. $R(\Sigma(A, P)) \subseteq R(A) \cap S$.

Proof. The case $A \geqslant 0$ was shown in [4] (with equality in item 3). The general case follows from the fact that if $F \in \mathbb{Q}$ and $R(F) = \mathscr{S}$, then $FP = PF^* = FPF^* = P$. Recall that if $B = A - I_P(A)P$, then $\Sigma(A, P) = \Sigma(B, P) + I_P(A)P$; and $R((I - E)^*) = \ker(I - E)^\perp = \mathscr{S}$. Item 3 is clear because $R(AQ) \subseteq R(A)$. $\square$

Lemma 4.7. *Let* $A = A^* \in L(\mathscr{H})$ *and* $P \in \mathbb{P}$ *with* $R(P) = \mathscr{S}$. *Suppose that* A *is* P*-positive and* $(A, \mathscr{S})$ *is compatible. Let* $E \in \mathscr{P}(A, \mathscr{S})$ *and* $Q = 1 - E$. *Consider the operator* $T = (1 - P) + Q$. *Then*

1. $T \in \mathrm{Gl}(\mathscr{H})$ *with* $T^{-1} = E + P$.
2. *If* $A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$ *in terms of* P, *then*

$$T^*AT = \begin{pmatrix} a & 0 \\ 0 & \Sigma(A, P) \end{pmatrix}. \tag{7}$$

3. *If* $A \in \mathrm{Gl}(\mathscr{H})$ *then* $a \in \mathrm{Gl}(\mathscr{S}^\perp)$ *and* $\Sigma(A, P) \in \mathrm{Gl}(\mathscr{S})$. *Moreover, if we view* $\Sigma(A, P) \in L(\mathscr{S})$, *then* $\Sigma(A, P)^{-1} = PA^{-1}P$ *or, in other words,*

$$\Sigma(A, P) = (PA^{-1}P)^\dagger. \tag{8}$$

Proof

1. Since $R(1-P) = R(E) = \ker P = \ker Q = \mathscr{S}^\perp$, then $(1-P)E = E$ and $QP = Q$. Thus $T(E+P) = E+Q = 1$. The other case is similar.
2. The fact that $R(Q) = \ker E \subseteq A^{-1}(\mathscr{S})$ implies that $Q^*A(1-P) = (1-P)AQ = 0$. By Proposition 4.6, $Q^*AQ = \Sigma(A,P)$.
3. Note that $(T^*AT)^{-1} = T^{-1}A^{-1}(T^*)^{-1} = (E+P)A^{-1}(E^*+P)$. But $PE = E^*P = 0$, so that $\Sigma(A,P)^{-1} = P(T^*AT)^{-1}P = PA^{-1}P$. $\square$

Proposition 4.8. *Let $A = A^* \in L(\mathscr{H})$ and $P \in \mathbb{P}$ with $R(P) = \mathscr{S}$. Suppose that A is P-positive and $(A, \mathscr{S})$ is compatible. Then $R(\Sigma(A,P)) = R(A) \cap \mathscr{S}$.*

Proof. We use the notations of Lemma 4.7. By formula (7), $R(T^*AT) \cap \mathscr{S} = R(\Sigma(A,P))$. On the other hand, if $\xi \in \mathscr{S}$, then $T^*\xi = Q^*\xi = \xi$, because $R(Q^*) = \ker Q^\perp = \mathscr{S}$ and $Q^* \in \mathbb{Q}$. Hence $R(A) \cap \mathscr{S} = R(AT) \cap \mathscr{S} \subseteq R(T^*AT) \cap \mathscr{S} = R(\Sigma(A,P))$. The reverse inclusion was shown in Proposition 4.6. $\square$

5. Computation of $I_P(A)$

Let $P \in \mathbb{P}$ and $A = A^* \in L(\mathscr{H})$. Recall that, if A is P-complementable, we have defined

$$I_P(A) = \max\{\mu \in \mathbb{R} : A - \mu P \geqslant 0\}.$$

Remark 5.1. If $A \geqslant 0$ then, by Corollary 2.2, $I_P(A) \neq 0$ if and only if $R(P) \subseteq R(A^{1/2})$ and, in this case, $I_P(A) = \|D\|^{-2}$, where D is the reduced solution of the equation $A^{1/2}X = P$. Thus, if $R(A)$ is closed, then $I_P(A) = \rho(A^\dagger P)$.

Suppose now that $A \ngeqslant 0$. It is easy to see that if $B = A + \mu P$, then $I_P(B) = I_P(A) + \mu$. Therefore a way to compute $I_P(A)$ would be to find a lower bound $\mu \leqslant I_P(A)$ in order to compute firstly $I_P(B)$ for $B = A - \mu P \geqslant 0$, reducing the general case to the positive case. Nevertheless this way seems to be not applicable. For example, it is easy to get, for any $M > 0$, selfadjoint matrices $A \in M_2(\mathbb{C})$ with $\|A\| \leqslant 2$ such that $I_P(A) < -M$, where P is a fixed projection of rank one. Indeed, take $P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $A = \begin{pmatrix} \varepsilon & 1 \\ 1 & 0 \end{pmatrix}$, for $\varepsilon < M^{-1}$.

We first show the key relation between $I_P(A)$ and the shorted operator $\Sigma(A,P)$:

Proposition 5.2. *Let $A \in L(\mathscr{H})$ be hermitian, $A \ngeqslant 0$, and $P \in \mathbb{P}$ with $R(P) = \mathscr{S}$ such that A is P-complementable. Then*

$$I_P(A) = \lambda_{\min}(\Sigma(A,P)) = \min\{\langle \Sigma(A,P)\xi, \xi\rangle : \xi \in \mathscr{S}, \|\xi\| = 1\}. \tag{9}$$

Proof. Denote by $\mu = \lambda_{\min}(\Sigma(A, P))$. Since $A \ngeqslant 0$, it is easy to see that $\mu < 0$. In particular this shows the last equality in Eq. (9). Note that $\mu P \leqslant \Sigma(A, P)$, so that

$$A - \mu P \geqslant A - \Sigma(A, P) \geqslant 0 \quad \text{and} \quad \mu \leqslant I_P(A).$$

On the other hand, since $A - I_P(A)P \geqslant 0$, then $I_P(A)P \in M(A, \mathscr{S})$ and $I_P(A)P \leqslant \Sigma(A, P)$ (see Propositions 3.3 and 3.7), which implies that $I_P(A) \leqslant \mu$. $\square$

Remark 5.3. With the notations of Proposition 5.2, if $A \geqslant 0$, then the identity $I_P(A) = \min\{\langle\Sigma(A, P)\xi, \xi\rangle : \xi \in \mathscr{S}, \|\xi\| = 1\}$ remains true; and this number coincides with $\lambda_{\min}(\Sigma(A, P))$ if we consider the spectrum of $\Sigma(A, P)$ as an operator of $L(\mathscr{S})$ (in order to remove the number 0 if necessary).

The following properties of $I_P(A)$ follow immediately from Remark 3.8 and Proposition 5.2.

Corollary 5.4. *Let $A \in L(\mathscr{H})$ be hermitian and $P \in \mathbb{P}$ such that A is P-complementable*:

1. *Let $Q \in \mathbb{P}$ such that $P \leqslant Q$ and suppose that $A \ngeqslant 0$. Then $I_P(A) \leqslant I_Q(A)$. If $A \geqslant 0$ this property may fail because of the fact observed in Remark* 5.3.
2. *Let $B \in L(\mathscr{H})$ such that $A \leqslant B$. Then $I_P(A) \leqslant I_P(B)$.*
3. *Let $\{E_n\} \in L(\mathscr{H})$ be a monotone* (*not necessary strictly*) *decreasing sequence of positive operators strongly convergent to* 0. *Then the sequence $\{I_P(A + E_n)\}$ decreases to $I_P(A)$.*
4. *Let $\{A_n\} \in L(\mathscr{H})$ be a sequence of P-complementable operators which is norm convergent to an invertible P-complementable operator A. Then $\{I_P(A_n)\}$ converges to $I_P(A)$.*
5. *Let f be a operator monotone map defined on $\sigma(A) \cup \sigma(\Sigma(A, P))$ such that $f(0) \geqslant 0$. Then $I_P(f(A)) \geqslant f(I_P(A))$.*
6. *Let $\{P_n\}$ be a decreasing sequence of orthogonal projections such that $P_n \xrightarrow{\text{S.O.T}} P$. Then $\{I_{P_n}(A)\}$ decreases to $I_P(A)$.*

Remark 5.5. It was pointed out in [16] that the hypothesis in item 3 can not be relaxed, i.e the map $A \mapsto I_P(A)$ is not norm continuous in general, as we see in the following example.

Example 5.6. Let $a \neq 1$ and $\{b_n\} \subseteq \mathbb{R}_{>a}$ such that $\lim_{n\to\infty} b_n = a$. Then the sequence of positive matrices

$$A_n = \begin{pmatrix} a^2 + a^{-2} & ab_n + (ab_n)^{-1} \\ ab_n + (ab_n)^{-1} & b_n^2 + b_n^{-2} \end{pmatrix}$$

converges in norm to $A = (a^2 + a^{-2})ee^t$, where $e = (1, 1)$. Let $x_n = (a, b_n)$ and $y_n = (a^{-1}, b_n^{-1})$. Note that $A_n = x_n x_n^* + y_n y_n^*$ and $e = \lambda_n x_n + \mu_n y_n$, with $\lambda_n = (a + b_n)^{-1}$ and $\mu_n = ab_n(a + b_n)^{-1}$. If a vector z satisfies that $A_n z = e$, then

$$e = A_n z = (x_n x_n^* + y_n y_n^*)z = \langle z, x_n\rangle x_n + \langle z, y_n\rangle y_n$$

and $\langle z, e\rangle^{-1} = (\langle z, x_n\rangle^2 + \langle z, y_n\rangle^2)^{-1} = \frac{(a+b_n)^2}{1+a^2b_n^2}$. Since $I_{P_e}(A_n) = 2\langle A_n^{-1}e, e\rangle^{-1} = \frac{2(a+b_n)^2}{1+a^2b_n^2}$, we get

$$\lim_{n\to\infty} I_{P_e}(A_n) = \lim_{n\to\infty} \frac{2(a + b_n)^2}{a^2b_n^2 + 1} = \frac{8}{a^2 + a^{-2}} \neq 2(a^2 + a^{-2}) = I_{P_e}(A).$$

The following results are the natural generalizations of formula (5) to our setting.

Corollary 5.7. *Let $A \in L(\mathscr{H})$ be hermitian and $P \in \mathbb{P}$ with $R(P) = \mathscr{S}$ such that A is P-complementable. Then*

$$I_P(A) = \inf\{\langle A\xi, \xi\rangle : \xi \in \mathscr{H}, \|P\xi\| = 1\}. \tag{10}$$

Proof. It is a consequence of Eq. (9) in Proposition 5.2 (or Remark 5.3 in case that $A \geqslant 0$) and item 5 of Proposition 3.7. $\square$

Corollary 5.8. *Let A and P be as above and suppose that $P = \xi \otimes \xi$ for some unit vector $\xi \in \mathscr{H}$. Then*

$$I_P(A) = \inf\{\langle A\eta, \eta\rangle : \eta \in \mathscr{H}, \langle \eta, \xi\rangle = 1\}. \tag{11}$$

Proof. Note that $P\eta = \langle \eta, \xi\rangle\xi$ and $\|P\eta\| = |\langle \eta, \xi\rangle|$. Also, if $\omega \in \mathbb{C}$ has $|\omega| = 1$, then $\langle A\omega\eta, \omega\eta\rangle = \langle A\eta, \eta\rangle$. $\square$

Throughout, we shall consider $P \in \mathbb{P}$ with $R(P) = \mathscr{S}$ and $A \in L(\mathscr{H})$ P-positive such that $(A, \mathscr{S})$ is compatible. In this case almost all results which can be shown for matrices can be extended to the infinite dimensional case.

Remark 5.9. Let $A \in L(\mathscr{H})$ be hermitian and $P \in \mathbb{P}$ with $R(P) = \mathscr{S}$ such that $(A, \mathscr{S})$ is compatible. Suppose that $I_P(A) \neq 0$. Then

$$R(A) \cap \mathscr{S} = R(\Sigma(A, P)) \neq \{0\}.$$

Indeed, since $(A, \mathscr{S})$ is compatible, $R(\Sigma(A, P)) = R(A) \cap \mathscr{S}$ by Proposition 4.8. On the other hand, $0 \neq I_P(A) = \lambda_{\min}(\Sigma(A, P))$, by Proposition 5.2. Hence $\Sigma(A, P) \neq 0$.

Theorem 5.10. *Let $A \in L(\mathscr{H})$ be hermitian and $P \in \mathbb{P}$ with $R(P) = \mathscr{S}$, such that A is P-positive and $(A, \mathscr{S})$ is compatible with $I_P(A) \neq 0$. Denote by $\mathscr{T} = \overline{\mathscr{S} \cap R(A)}$ and $Q = P_{\mathscr{T}}$. Then*

1. *A is Q-complementable. Moreover, the pair $(A, \mathscr{T})$ is compatible.*
2. $\Sigma(A, P) = \Sigma(A, Q)$.
3. $I_P(A) = I_Q(A)$.

Proof. If $A \geqslant 0$, by Remark 5.3, we know that $\Sigma(A, P)$ is invertible in $L(\mathscr{S})$. On the other hand, since $(A, \mathscr{S})$ is compatible, $\mathscr{S} = R(\Sigma(A, P)) = R(A) \cap \mathscr{S} \subseteq R(A)$.

Suppose now that $A \ngeqslant 0$. By Remark 5.9, $R(\Sigma(A, P)) = R(A) \cap \mathscr{S} \subseteq \mathscr{T}$. Hence

$$\Sigma(A, P) \in M(A, \mathscr{T}) = \{D \in L(\mathscr{H}) : D = D^*, D \leqslant A, R(D) \subseteq \mathscr{T}\} \neq \emptyset.$$

Therefore, by Proposition 3.3, A is Q-complementable and, by Proposition 3.7, $\Sigma(A, P) \leqslant \Sigma(A, Q)$. The inequality $\Sigma(A, Q) \leqslant \Sigma(A, P)$ follows by Remark 3.8. Then,

$$I_P(A) = \lambda_{\min}(\Sigma(A, P)) = \lambda_{\min}(\Sigma(A, Q)) = I_Q(A).$$

Using Proposition 4.3 item 3, in order to show that the pair $(A, \mathscr{T})$ is compatible, it suffices to verify that $\mathscr{T}^\perp + A^{-1}(\mathscr{T}) = \mathscr{H}$, which follows from the following facts: $\mathscr{S}^\perp + A^{-1}(\mathscr{S}) = \mathscr{H}$ (since $(A, \mathscr{S})$ is compatible), $\mathscr{S}^\perp \subseteq \mathscr{T}^\perp$ and $A^{-1}(\mathscr{S}) = A^{-1}(\mathscr{S} \cap R(A)) \subseteq A^{-1}(\mathscr{T})$. $\square$

Remark 5.11. When $\dim \mathscr{S} = 1$, if A is P-positive and P-compatible we can deduce that $\mathscr{S} \subseteq R(A)$. More generally, if $\dim \mathscr{S} < \infty$, A is injective and $(A, \mathscr{S})$ is compatible , then $\mathscr{S} \subseteq R(A)$. Indeed, note that $\dim A^{-1}(\mathscr{S}) = \dim \mathscr{S} \cap R(A)$, and $A^{-1}(\mathscr{S})$ must be a supplement of $\mathscr{S}^\perp$. Nevertheless, if we remove the condition $(A, \mathscr{S})$ is compatible , this is not true, even if $\dim \mathscr{S} = 1$ and A is injective and P-complementable, as the following example shows.

Example 5.12. Let $A \in L(\mathscr{H})^+$ be injective non-invertible. Let $\xi \in \mathscr{H} \setminus R(A)$ be a unit vector. Denote by $\mathscr{S}$ the subspace generated by ξ, $P = P_{\mathscr{S}}$. If

$$A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$$

in terms of P and $A\xi = \lambda\xi + \eta$ with $\eta \in \mathscr{S}^\perp$, then $\lambda = \langle A\xi, \xi\rangle \neq 0$ and $\eta \neq 0$ (otherwise $\xi \in R(A)$). Therefore $c = \lambda P$ and $b(\mu\xi) = \mu\eta$, $\mu \in \mathbb{C}$.

Suppose that $\eta \in R(a)$, i.e., there exists $\nu \in \mathscr{S}^\perp$ which verifies $a\nu = b\xi$. Then $(1 - P)A(\nu - \xi) = a\nu - b\xi = 0$, so $A(\nu - \xi)$ is a multiple of ξ, which must be 0 ($\xi \notin R(A)$). So $\nu = \xi$, a contradiction. Therefore $R(b) \nsubseteq R(a)$ and the pair $(A, \mathscr{S})$ is incompatible.

Now consider $B = A + \mu P$, for any $\mu \in \mathbb{R}$. It is clear that B must be P-complementable ($B - \mu P = A \geqslant 0$). But the facts that A is injective and $\xi \notin R(A)$, clearly imply that B is injective and $\xi \notin R(B)$.

5.13. Fix $E \in \mathbb{P}$ with range $\mathscr{M}$. Denote by $L(\mathscr{H})_{\mathscr{M}} = \{C \in L(\mathscr{H}) : ECE = C\}$. For $C \in L(\mathscr{H})_{\mathscr{M}}$, denote by $C_0 \in L(\mathscr{M})$ the compression of C to $\mathscr{M}$. With respect to the matrix representation induced by E

$$C = \begin{pmatrix} 0 & 0 \\ 0 & C_0 \end{pmatrix} \begin{matrix} \mathscr{M}^{\perp} \\ \mathscr{M} \end{matrix}.$$

The following properties of this compression are easy to see:

1. The map $L(\mathscr{H})_{\mathscr{M}} \ni C \mapsto C_0 \in L(\mathscr{M})$ is a $*$-isomorphism of C^*-algebras, i.e., it is isometric and compatible with sums, products and adjoints.
2. If $C = C^* \in L(\mathscr{H})_{\mathscr{M}}$ and $R(C) = \mathscr{M}$, then $C_0 \in \mathrm{Gl}(\mathscr{M})$ and $(C_0)^{-1} = (C^{\dagger})_0$. If $R(C)$ is closed, then $(C_0)^{\dagger} = (C^{\dagger})_0$.

Theorem 5.14. *Let $A \in L(\mathscr{H})$ be hermitian, $A \ngeq 0$, and $P \in \mathbb{P}$ with $R(P) = \mathscr{S}$ such that A is P-positive and $(A, \mathscr{S})$ is compatible. Suppose that $R(A)$ is closed. Denote by $\mathscr{T} = \mathscr{S} \cap R(A)$ and $Q = P_{\mathscr{T}}$. Then*

$$I_P(A) = I_Q(A) = \lambda_{\min}(QA^{\dagger}Q)^{\dagger}. \tag{12}$$

Proof. Since we only need to prove the equality $I_Q(A) = \lambda_{\min}(QA^{\dagger}Q)^{\dagger}$, we shall directly suppose that $R(P) \subseteq R(A)$. Denote $\mathscr{M} = R(A)$ and $E = P_{\mathscr{M}}$. Using the notations of 5.13, we have that A, P and $\Sigma(A, P) \in L(\mathscr{H})_{\mathscr{M}}$. It is clear that $\Sigma(A, P)_0 = \Sigma(A_0, P_0)$, $I_P(A) = I_{P_0}(A_0)$ and A_0 is invertible. Therefore, by Lemma 4.7,

$$\Sigma(A_0, P_0) = (P_0(A_0)^{-1}P_0)^{\dagger} = (PA^{\dagger}P)_0{}^{\dagger} = ((PA^{\dagger}P)^{\dagger})_0$$

and

$$\begin{aligned} I_P(A) = I_{P_0}(A_0) &= \lambda_{\min}\Sigma(A_0, P_0) = \lambda_{\min}((PA^{\dagger}P)^{\dagger})_0 \\ &= \lambda_{\min}(PA^{\dagger}P)^{\dagger}. \qquad \square \end{aligned}$$

6. Some applications

The problem of calculating $I_P(A)$ of a P-complementable operator A with respect to a projection P has already been considered for certain projections P, mainly in the finite dimensional case. Reams [15] showed that if $A \in M_n(\mathbb{R})$ is invertible and almost positive (see Remark 3.2), then A is P_e-complementable and $I_{P_e}(A) = n \cdot \langle A^{-1}e, e\rangle^{-1}$, where $e = (1, \ldots, 1) \in \mathbb{C}^n$ and P_e denotes the orthogonal projection onto the subspace generated by e. We obtain a generalization of this result in the non-positive case. The general positive case was already considered in [7] and Corollary 2.3 (for every unit vector $\xi \in \mathbb{C}^n$).

Corollary 6.1. *Let $\xi \in \mathbb{C}^n$ be a unit vector. Let $A \in M_n(\mathbb{C})$ be non-positive but P_ξ-positive. Then A is P_ξ-complementable if and only if*

$$\forall \eta \in \mathbb{C}^n,\ \langle \eta, \xi\rangle = 0 \quad \textit{and} \quad \langle A\eta, \eta\rangle = 0 \Rightarrow A\eta = 0. \tag{13}$$

In this case $\xi \in R(A)$ and

$$I_{P_\xi}(A) = \langle A^\dagger \xi, \xi\rangle^{-1} = \min\{\langle Az, z\rangle : \langle z, \xi\rangle = 1\}. \tag{14}$$

Proof. Condition (13) is equivalent to $\mathrm{Ker}((1-P_\xi)A(1-P_\xi)) \cap \{\xi\}^\perp = \mathrm{Ker}(A) \cap \{\xi\}^\perp$. By Proposition 3.3, this is equivalent to the fact that A is P_ξ-complementable, since $R(A)$ is closed. Note that $I_{P_\xi}(A) < 0$, since $A \not\geqslant 0$. By Remarks 4.5 and 5.9 we get $R(A) \cap R(P_\xi) \neq \{0\}$. Therefore $\xi \in R(A)$ and $\langle A^\dagger\xi, \xi\rangle \neq 0$. By Eq. (12) in Theorem 5.14,

$$I_{P_\xi}(A) = \lambda_{\min}(P_\xi A^\dagger P_\xi)^\dagger = \lambda_{\min}(\langle A^\dagger \xi, \xi\rangle P_\xi)^\dagger = \langle A^\dagger\xi, \xi\rangle^{-1}.$$

In order to prove Eq. (14), it only remains to show that the infimum in Eq. (11) is actually a minimun. Let $\zeta = A^\dagger \xi$ and $\eta = \langle A^\dagger\xi, \xi\rangle^{-1}\zeta$. Then

$$\langle A\eta, \eta\rangle = \langle A^\dagger \xi, \xi\rangle^{-2}\langle A\zeta, \zeta\rangle = \langle A^\dagger\xi, \xi\rangle^{-2}\langle \xi, A^\dagger \xi\rangle = \langle A^\dagger \xi, \xi\rangle^{-1},$$

and the minimum is attained at η. □

It was also noted in [15] that the problem of calculating $I_P(A)$ with respect to $P = P_e$ is equivalent to a problem posed by Fiedler and Markham in [9], that is to calculate

$$\max\{\lambda_{\min}((A \circ C)C^{-1}), C > 0\}$$

for a positive matrix $A \in M_n(\mathbb{C})$, where $A \circ B$ denotes the Hadamard product of A and B. The corollary above complements the results obtained in [9] in the non-positive, non-invertible case.

Recall that given a positive matrix $A \in M_n(\mathbb{C})$, the minimal index was introduced in [16] as

$$I_A = \max\{\mu \geqslant 0 : A \circ B \geqslant \mu B, B \geqslant 0\}.$$

Given $P \in M_n(\mathbb{C})$ an orthogonal projection and a P-complementable matrix A, there is a relation between $I_P(A)$ and the Schur multiplier induced by A.

Corollary 6.2. *Let $M = \{x_1, \ldots, x_k\} \subseteq \mathbb{C}^n$ be an orthonormal set and let P be the orthogonal projection onto the subspace spanned by M. Suppose that $A \in M_n(\mathbb{C})$ is P-complementable. Then*

$$I_P(A) = \max\left\{\mu \in \mathbb{R} : A \circ B \geqslant \mu \sum_{i=1}^{k} D_{x_i} B D_{x_i}^*,\ B \geqslant 0\right\}, \tag{15}$$

where D_x denotes the diagonal matrix with main diagonal $x \in \mathbb{C}^n$.

Proof. First note that $P = \sum_{i=1}^k x_i x_i^*$. Thus $A - \mu P \geqslant 0$ if and only if every $B \geqslant 0$ satisfies $(A - \mu \sum_{i=1}^k x_i x_i^*) \circ B \geqslant 0$, which is equivalent to $A \circ B \geqslant \mu \sum_{i=1}^k D_{x_i}$

$BD^*_{x_i}$, since a simple calculation shows that $C \circ xx^* = D_x C D^*_x$ for every $C \in M_n(\mathbb{C})$ and $x \in \mathbb{C}^n$. This shows formula (15). $\square$

6.1. Completely positive maps on $M_n(\mathbb{C})$

Definition 6.3. Let $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a linear map. Φ is positive if $\Phi(A) \geqslant 0$ whenever $A \geqslant 0$. Φ is selfadjoint if $\Phi(A^*) = \Phi(A)^*$ or equivalently if $\Phi(A)$ is selfadjoint whenever A is selfadjoint.

Let $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a linear map. If $m \in N$, we denote $\Phi^{(m)}$: $M_m(M_n(\mathbb{C})) \to M_m(M_n(\mathbb{C}))$ the map given by

$$\Phi^{(m)}((a_{ij})_{ij}) = (\Phi(a_{ij}))_{ij}, \quad (a_{ij})_{ij} \in M_m(M_n(\mathbb{C})),$$

and call it the *inflation* of order m of Φ.

Definition 6.4. The linear map $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is called *completely positive* if $\Phi^{(m)}$ is positive for every $m \in N$.

In the following, $\{e_{ij}\} \subseteq M_n(\mathbb{C})$ denotes the canonical basis for $M_n(\mathbb{C})$. Now we state a result due to Choi [3].

Theorem 6.5. *Let $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a linear map. Then Φ is completely positive if and only if $\Phi^{(n)}((e_{ij})_{ij}) = (\Phi(e_{ij}))_{ij} \in M_n(M_n(\mathbb{C}))$ is positive.*

Remark 6.6. Note that the matrix $E = ((e_{ij})_{ij}) \in M_n(M_n(\mathbb{C})) \simeq M_{n^2}(\mathbb{C})$ is a scalar multiple of a rank one projection. Indeed, if $\{e_i\}$ denotes the canonical basis of $\mathbb{C}^n$ and $v \in \mathbb{C}^{n^2}$ is the vector $v = (e_1, \ldots, e_n)$, then $(e_{ij})_{ij} = vv^*$. Thus $E = \frac{1}{n} P_v$, where P_v is the projection onto the subspace generated by v.

Remark 6.7. Let $A \in M_n(A)$. Then the linear map $\Phi_A : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ given by $\Phi_A(B) = A \circ B$ is selfadjoint (resp. positive) if and only if A is selfadjoint (resp. positive). Moreover, if $A \geqslant 0$, then Φ_A is completely positive, since the inflated matrix $A^{(n)} \geqslant 0$ and $\Phi_A^{(n)} = \Phi_{A^{(n)}}$ (see [13]). Therefore $\Phi_A - \mu\mathrm{Id}$ is completely positive if and only if $A - \mu ee^* \geqslant 0$, where $e \in \mathbb{C}^n$ is given by $e = (1, \ldots, 1)$. Note that $ee^* = nP_e$, since $\|e\| = n^{1/2}$. Therefore we conclude that for every P_e-complementable matrix A,

$$I(A) = \max\{\mu \in \mathbb{R} : \Phi_A - \mu\mathrm{Id} \text{ is completely positive}\} = \frac{1}{n}\, I_{P_e}(A),$$

where $I(A)$ is the minimal index of A defined in 2.4 (in fact, its natural generalization for A not necesarily positive, but P_e-complementable).

Definition 6.8. Let $\Phi : M_n(C) \to M_n(C)$ be a selfadjoint map. We say that Φ is complementable if there exists $\mu \in \mathbb{R}$ such that $\Phi - \mu \mathrm{Id}$ is completely positive. In this case we define:

$$I(\Phi) = \max\{\mu \in \mathbb{R} : \Phi - \mu\mathrm{Id} \text{ is completely positive}\}.$$

Note that all completely positive maps Φ are complementable and $I(\Phi) \geqslant 0$. But in general not all selfadjoint maps are complementable. For example, if $A \in M_n(\mathbb{C})$ is selfadjoint, then Φ_A is complementable if and only if A is P_e-complementable.

Theorem 6.9. *Let $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a selfadjoint map. Then, with the notations of Remark* 6.6,

1. *Suppose that Φ is not completely positive. In this case Φ is complementable if and only if for all $\eta_1, \ldots, \eta_n \in \mathbb{C}^n$*

$$\sum_{i=1}^{n} (\eta_i)_i = 0 \Rightarrow \sum_{i,\, j=1}^{n} \langle \Phi(e_{ij})\eta_j, \eta_i\rangle \geqslant 0 \tag{16}$$

and

$$\sum_{i=1}^{n} (\eta_i)_i = 0 \quad \textit{and} \quad \sum_{i,\, j=1}^{n} \langle \Phi(e_{ij})\eta_j, \eta_i\rangle = 0 \Rightarrow \sum_{j=1}^{n} \Phi(e_{ij})\eta_j = 0, \quad i = 1, \ldots, n \tag{17}$$

or, equivalently, if $A_\Phi = \Phi^{(n)} E = (\Phi(e_{ij}))_{ij} \in M_{n^2}(\mathbb{C})$ is P_v-complementable.

2. *In this case $I(\Phi) = n \cdot I_{P_v}(A_\Phi)$ and we have*

$$I(\Phi) = \min \left\{ \sum_{i,j=1}^{n} \langle \Phi(e_{ij})\eta_j, \eta_i\rangle : \eta_1, \ldots, \eta_n \in \mathbb{C}^n \ \textit{ and } \sum_{i=1}^{n} (\eta_i)_i = 1 \right\}. \tag{18}$$

3. *If conditions* (16) *and* (17) *hold, there exist $\eta_1, \ldots, \eta_n \in \mathbb{C}^n$ such that*

$$\sum_{j=1}^{n} \Phi(e_{ij})\eta_j = e_i, \quad i = 1, \ldots, n \tag{19}$$

where $\{e_1, \ldots, e_n\}$ is the canonical basis of $\mathbb{C}^n$. For any such vectors,

$$I(\Phi)^{-1} = \sum_{i,\, j=1}^{n} \langle \Phi(e_{ij})\eta_j, \eta_i\rangle. \tag{20}$$

4. *If Φ is completely positive then it is complementable and $I(\Phi) \geqslant 0$. Moreover, $I(\Phi) > 0$ if and only if there exist $\eta_1, \ldots, \eta_n \in \mathbb{C}^n$ such that Eq.* (19) *holds. For any such vectors*, *Eq.* (20) *holds. Also Eq.* (18) *is true in this case.*

Proof. From Theorem 6.5 we conclude that the map Φ is complementable if and only if the matrix $A_\Phi = (\Phi(e_{ij}))_{ij}$ is P_v complementable. It is easy to see that in fact $I(\Phi) = n \cdot I_{P_v}(A_\Phi)$. Thus we can apply Corollary 6.1 to the matrix $A_\Phi \in M_{n^2}(\mathbb{C})$ and the projection P_v. Note that Eq. (16) holds if and only if A_Φ is P_v-positive and condition (13) is equivalent to condition (17). Indeed, if $\eta = (\eta_1, \dots, \eta_n) \in \mathbb{C}^{n^2}$ with $\eta_i \in \mathbb{C}^n$ $(i = 1, \dots, n)$, then $\langle \eta, v\rangle = \sum_{i=1}^n (\eta_i)_i$ and $\langle A_\Phi \eta, \eta\rangle = \sum_{i,\, j=1}^n \langle \Phi(e_{ij})\eta_j, \eta_i\rangle$.

Note that condition (19) is equivalent to the fact that $v \in \mathrm{R}(A_\Phi)$, so this condition and Eq. (18) follow from Eq. (14). Similarly, $I(\Phi) = n \cdot I_{P_v}(A_\Phi) = \langle v, A_\Phi^\dagger v\rangle$.

Let $\zeta = (\zeta_1, \dots, \zeta_n) = A_\Phi^\dagger v$. If $\eta = (\eta_1, \dots, \eta_n) \in \mathbb{C}^{n^2}$ satisfy condition (19) (i.e., $A_\Phi \eta = v$), then $P_{R(A_\Phi)}\eta = \zeta$. Therefore

$$\sum_{i,\, j=1}^n \langle \Phi(e_{ij})\eta_j, \eta_i\rangle = \langle A_\Phi \eta, \eta\rangle = \langle A_\Phi \zeta, \zeta\rangle = \langle v, A_\Phi^\dagger v\rangle = I(\Phi)^{-1}.$$

Suppose now that Φ is completely positive. It is clear that Φ is complementable. By Corollary 2.3, it follows that $I(\Phi) = n \cdot I_{P_v}(A_\Phi) > 0$ if and only if $v \in R(A_\Phi)$, since $R(A_\Phi)$ is closed. This is equivalent to condition (19), and using Corollary 2.3, we can also deduce Eqs. (20) and (18) in this case. □

Example 6.10. Consider the map $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ given by

$$T(A) = \frac{1}{n} Tr(A) I_n = \frac{1}{n} \sum_{i,j=i}^n e_{ij}^* A e_{ij},$$

where $Tr(A) = \sum A_{ii}$ is the usual trace. Then T is completely positive; morever it is a conditional expectation. Note that the matrix

$$A_T = (T(e_{ij}))_{ij} = \frac{1}{n} I_{n^2}.$$

Then $I(T) > 0$, since $A_T(n\, e_i) = e_i$, $1 \leqslant i \leqslant n$, and T satisfies condition (19). Therefore, since $T(e_{ij}) = 0$ if $i \neq j$ and $T(e_{ii}) = \frac{1}{n} I_n$, using Eq. (20),

$$I(T)^{-1} = \sum_{i=1}^n \langle T(e_{ii}) n e_i, n e_i\rangle = \sum_{i=1}^n n = n^2.$$

This result is actually known in index theory of condidional expectations (using that $T^{(n)}(P_v) = n^{-1} A_T = n^{-2} I_{n^2}$, see [12]). Note that the number

$$\begin{aligned} J(T) &= \max\{\lambda \in \mathbb{R} : T - \lambda \text{ Id is positive (not completely)}\} \\ &= n^{-1} \neq n^{-2} = I(T). \end{aligned}$$

Indeed, it is easy to see that $A \geqslant 0$ implies that $Tr(A) \geqslant \rho(A) = \|A\|$, so that

$$T(A) = \frac{1}{n} Tr(A) I_n \geqslant \frac{1}{n} A.$$

Taking $A = e_{11}$ we get $T(A) \ngeqslant \lambda A$ if $\lambda > \frac{1}{n}$; so that $J(T) = n^{-1}$.

Remark 6.11. Let $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a selfadjoint map. The formulation of Theorem 6.9 intends to characterize complementability and to compute $I(\Phi)$ in terms of Φ itself instead of doing it in terms of the "inflated" matrix A_Φ. Another way would be to recall the identity $I(\Phi) = n \cdot I_{P_v}(A_\Phi)$ and use all the previous results of the paper. For example, let $U_1, \ldots, U_m \in M_n(\mathbb{C})$, and suppose that Φ is given by

$$\Phi(A) = \sum_{k=1}^{m} U_k^* A U_k, \quad A \in M_n(\mathbb{C}),$$

a prototypical completely positive map (see [3]). Denote by $V_k \in M_{n^2}(\mathbb{C})$ the block diagonal matrix with copies of U_k in its diagonal. Denote by $v = (e_1, \ldots, e_n) \in \mathbb{C}^{n^2}$ and $E = (e_{ij})_{ij} = vv^*$. Note that $\|V_k v\| = \|U_k\|_2$ and $V_k^* E V_k = (V_k v)(V_k v)^*$. Therefore

$$A_\Phi = (\Phi(e_{ij}))_{ij} = \sum_{k=1}^{m} V_k^* E V_k = \sum_{k=1}^{m} \|U_k\|_2^2 P_{V_k v}.$$

Thus $I(\Phi)$ can be computed using this expression and Corollaries 6.1 and 2.3.

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