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Finding intersection models: From chordal to Helly circular-arc graphs

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ABSTRACT

Every chordal graph **G** admits a representation as the intersection graph of a family of subtrees of a tree. A classic way of finding such an intersection model is to look for a maximum spanning tree of the valuated clique graph of **G**. Similar techniques have been applied to find intersection models of chordal graph subclasses as interval graphs and path graphs. In this work, we extend those methods to be applied beyond chordal graphs: we prove that a graph **G** can be represented as the intersection of a Helly separating family of graphs belonging to a given class if and only if there exists a spanning subgraph of the clique graph of **G** satisfying a particular condition. Moreover, such a spanning subgraph is characterized by its weight in the valuated clique graph of **G**. The specific case of Helly circular-arc graphs is treated. We show that the canonical intersection models of those graphs correspond to the maximum spanning cycles of the valuated clique graph.

1. Introduction and background

We consider simple, finite, undirected graphs. Without loss of generality, we shall only deal with connected graphs. Given a graph **G**, $V_{\mathbf{G}}$ and $E_{\mathbf{G}}$ denote the vertex set and the edge set of **G**, respectively. A *complete set* of **G** is a subset of pairwise adjacent vertices. A *clique* is a complete set that is not properly contained in any other complete set. The clique family of **G** is denoted by $\mathcal{C}(\mathbf{G})$. For any $v \in V_{\mathbf{G}}$, the set whose elements are the cliques of **G** containing v is denoted by $Q(v)$, that is $Q(v) = \{Q \in \mathcal{C}(\mathbf{G}) / v \in Q\}$.

Let $\mathcal{F}$ be a set family with *members* F_i for $i \in I$. The *intersection* or *total intersection* of $\mathcal{F}$, written $\cap\mathcal{F}$, is the set $\cap_{i\in I} F_i$. The family $\mathcal{F}$ has the *Helly property* or is a *Helly family* if every pairwise intersecting subfamily has a non-empty total intersection. We say that $\mathcal{F}$ is *separating* when for every pair of elements u and v of $\cup_{i\in I} F_i$, there exists a member F_i of the family such that $u \in F_i$ and $v \notin F_i$.

The *intersection graph* of $\mathcal{F}$ is obtained by considering a vertex for each member of the family and an edge between two vertices whenever the corresponding sets have non-empty intersection. The following theorem is a direct consequence of a well-known result demonstrated by Berge in the context of hypergraph theory [2, Chapter 1, Section 8, Proposition 1]. It proves that any graph is the intersection graph of a unique Helly separating set family.

Theorem 1. *Let* **G** *be a graph. The set family* $(Q(v))_{v\in V_{\mathbf{G}}}$ *is Helly and separating. The intersection graph of this family is the same graph* **G**; *moreover, if* **G** *is the intersection graph of a Helly separating set family* $\mathcal{F}$, *then* $\mathcal{F} = (Q(v))_{v\in V_{\mathbf{G}}}$.

Many graph classes such as chordal graphs, interval graphs, circular-arc graphs and line graphs, have been characterized or defined as intersection graphs of set families whose members are vertex sets of certain subgraphs of some particular graphs [4]. In order to write in a simpler way the definition of such classes, we introduce the term intersection graph of a

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graph family, instead of intersection graph of a set family. The *intersection graph of a graph family* $(\mathbf{F}_i)_{i\in I}$ is the intersection graph of the corresponding vertex set family $(V_{\mathbf{F}_i})_{i\in I}$, thus it is obtained by representing each graph $\mathbf{F}_i$ by a vertex i, and connecting two vertices i and j by an edge if and only if the vertex sets of $\mathbf{F}_i$ and $\mathbf{F}_j$ intersect. We say that *a graph family has the Helly property* if the corresponding vertex set family has the Helly property and, in an analogous way, that *a graph family is separating* when the corresponding vertex set family is separating.

Chordal graphs have been widely studied; one of the reasons is that chordal graphs have a natural intersection model. In many applications the intersection representation of a graph is more important than the graph itself. The following standard properties of chordal graphs are discussed in more detail in [1,3,5,6,13] and are also contained in books [4,9,12].

A graph $\mathbf{G}$ is chordal if and only if there exists a tree $\mathbf{T}$ and a family $\mathcal{T}$ of subtrees of $\mathbf{T}$ such that $\mathbf{G}$ is the intersection graph of $\mathcal{T}$. The pair $(\mathbf{T}, \mathcal{T})$ is called a *tree representation* of $\mathbf{G}$ and $\mathbf{T}$ is called the *support* of the representation. In many publications $\mathbf{T}$ is called the *host tree*. It is clear that given a tree representation other tree representations can be obtained, for instance, by just subdividing an edge of the support and subdividing the same edge in the members of the family covering it. A tree representation is called *canonical* when its support has a minimum number of vertices. A chordal graph can admit many distinct canonical representations but the support of each one of them is always a spanning tree of the clique graph of $\mathbf{G}$; moreover, it is a spanning tree that can be recognized by its weight in the valuated clique graph of $\mathbf{G}$. In what follows we discuss these results in detail.

The *clique graph* of $\mathbf{G}$, written $K(\mathbf{G})$, is the intersection graph of $\mathcal{C}(\mathbf{G})$. Notice that for each $v \in V_{\mathbf{G}}$, $Q(v)$ is a subset of vertices of $K(\mathbf{G})$. Given a spanning tree $\mathbf{T}$ of $K(\mathbf{G})$, $\mathbf{T}[Q(v)]$ denotes the subgraph of $\mathbf{T}$ induced by the vertices belonging to $Q(v)$.

Theorem 2. *A graph* $\mathbf{G}$ *admits a tree representation if and only if there exists a spanning tree* $\mathbf{T}$ *of* $K(\mathbf{G})$ *such that, for any vertex* $v \in V_{\mathbf{G}}$, $\mathbf{T}[Q(v)]$ *is a subtree of* $\mathbf{T}$.

Any spanning tree $\mathbf{T}$ of $K(\mathbf{G})$ satisfying the conditions of the previous theorem is called a *clique-tree* of $\mathbf{G}$.

The *weighted clique graph* $K^w(\mathbf{G})$ of $\mathbf{G}$ is the clique graph $K(\mathbf{G})$ with each edge $e = QQ'$ weighted by $w(e) = |Q \cap Q'|$. The weight $w(H)$ of a subgraph H of $K^w(G)$ is $\sum_{e\in E_H} w(e)$.

Notice that, by definition, any clique-tree of $\mathbf{G}$ is a spanning subgraph of $K(\mathbf{G})$ and thus of $K^w(\mathbf{G})$. The following theorem characterizes the clique-trees of $\mathbf{G}$ by means of their weight in $K^w(\mathbf{G})$.

Theorem 3. *Let* $\mathbf{G}$ *be a graph. A spanning tree* $\mathbf{T}$ *of* $K^w(\mathbf{G})$ *is a clique-tree of* $\mathbf{G}$ *if and only if*

$$w(\mathbf{T}) = \left(\sum_{Q\in\mathcal{C}(\mathbf{G})} |Q|\right) - |V_{\mathbf{G}}|.$$

Moreover, in this case, $\mathbf{T}$ *is a maximum spanning tree of* $K^w(\mathbf{G})$.

In this paper we generalize these results in order to apply them to other graph classes. In Section 2, a generalization of Theorem 2 is supplied by Theorem 7 which states that a graph $\mathbf{G}$ can be represented as intersection of a Helly separating family of graphs belonging to a given class if and only if there exists a spanning subgraph of $K(\mathbf{G})$ satisfying a particular condition. Such a spanning subgraph is the clique-tree in the case of chordal graphs.

In Section 3, the mentioned spanning subgraph of $K(\mathbf{G})$ is characterized by its weight in $K^w(\mathbf{G})$. Theorem 10 generalizes Theorem 3.

In Section 4, the general results of the previous sections are applied to the classes of chordal, interval and Helly circular-arc graphs. In the first and in the second case, already known results are obtained in an easy way. In the case of Helly circular-arc graphs, we get the main Theorem 15 which provides a new unpublished characterization of the canonical representation of Helly circular-arc graphs. A related but distant work has been published in [11].

In Section 5, other examples are offered.

2. Intersection graphs of Helly separating graph families

As we noted in the previous section, for any graph $\mathbf{G}$ and $v \in V_{\mathbf{G}}$, the set $Q(v)$ is a subset of vertices of $K(\mathbf{G})$; in [2], the set family $(Q(v))_{v\in V_{\mathbf{G}}}$ is called the dual of the clique family of $\mathbf{G}$. Since v is a vertex in the intersection of any two cliques belonging to $Q(v)$, $Q(v)$ induces a complete subgraph of $K(\mathbf{G})$ which is denoted by $\mathbf{K}_v$.

The following theorem is a clear consequence of Theorem 1.

Theorem 4. *Any graph is the intersection graph of some Helly separating graph family. Moreover, a graph* $\mathbf{G}$ *is the intersection graph of a Helly separating graph family* $(\mathbf{F}_v)_{v\in V_{\mathbf{G}}}$ *if and only if, for every* $v \in V_{\mathbf{G}}$, $\mathbf{F}_v$ *is a spanning subgraph of* $\mathbf{K}_v$.

Proof. The first assertion is trivial using Theorem 1 and considering every $Q(v)$ as the vertex set of a graph without edges. Let $\mathbf{G}$ be the intersection graph of the Helly separating graph family $(\mathbf{F}_v)_{v\in V_{\mathbf{G}}}$; this means that the corresponding vertex set family is Helly separating and its intersection graph is $\mathbf{G}$. By Theorem 1, the corresponding vertex set family must be $(Q(v))_{v\in V_{\mathbf{G}}}$. Since each $\mathbf{K}_v$ is a complete graph with vertex set $Q(v)$, every member $\mathbf{F}_v$ is a spanning subgraph of $\mathbf{K}_v$.

To prove the converse implication, for every $v \in V_{\mathbf{G}}$, let $\mathbf{F}_v$ be a spanning subgraph of $\mathbf{K}_v$ and consider the graph family $(\mathbf{F}_v)_{v\in V_{\mathbf{G}}}$. The corresponding vertex set family is $(Q(v))_{v\in V_{\mathbf{G}}}$ and hence, by Theorem 1, it is Helly separating and its intersection graph is $\mathbf{G}$. □

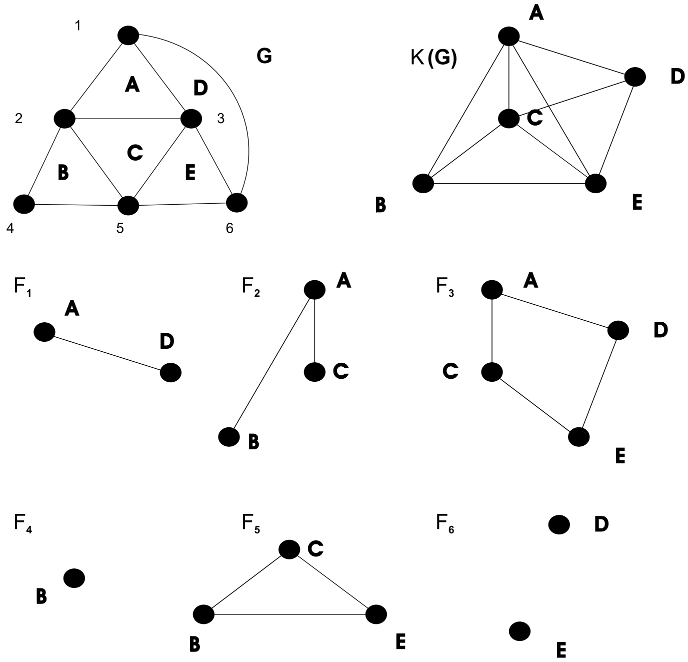

Fig. 1. Graph **G**, its clique graph $K(\mathbf{G})$, and a Helly separating graph family $(\mathbf{F}_v)_{1\le v\le 6}$ whose intersection graph is **G**.

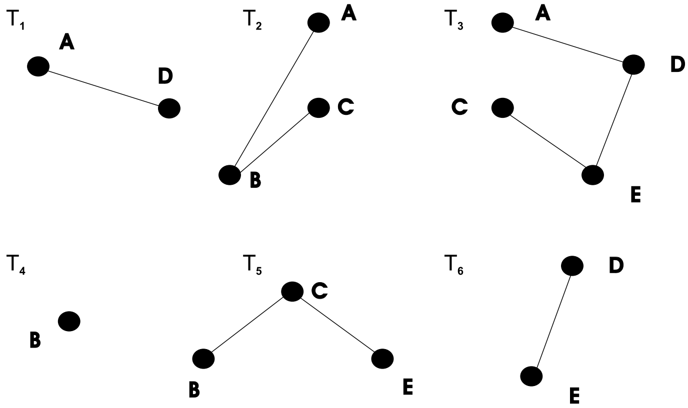

Fig. 2. A Helly separating *Paths*-family whose intersection graph is the graph **G** in Fig. 1.

We show an example in Fig. 1. Since every graph $\mathbf{F}_v$ in that figure is a spanning subgraph of the corresponding complete subgraph $\mathbf{K}_v$ of $K(\mathbf{G})$, the family $(\mathbf{F}_v)_{1\le v\le 6}$ is Helly and separating, and its intersection graph is **G**.

Let $\widetilde{S}$ be a graph class. A $\widetilde{S}$-family is any graph family such that each member belongs to $\widetilde{S}$. A natural question arises: is every graph the intersection graph of some Helly separating $\widetilde{S}$-family? Answering this question, we have established the following theorem.

Theorem 5. *Let $\widetilde{S}$ be a graph class. A graph* **G** *is the intersection graph of some Helly separating $\widetilde{S}$-family if and only if, for every $v \in V_{\mathbf{G}}$, there exists $\mathbf{S} \in \widetilde{S}$ such that $|V_{\mathbf{S}}| = |Q(v)|$.*

Proof. By Theorem 4, **G** is the intersection graph of some Helly separating $\widetilde{S}$-family if and only if, for every vertex $v \in V_{\mathbf{G}}$, a spanning subgraph of $\mathbf{K}_v$ belonging to $\widetilde{S}$ can be chosen. Since every $\mathbf{K}_v$ is a complete graph, this can be done if and only if the class $\widetilde{S}$ contains, for every $v \in V_{\mathbf{G}}$, some graph with $|Q(v)|$ vertices. $\square$

For example, the graph **G** in Fig. 1 cannot be represented as the intersection graph of a Helly separating family of triangles. Observe that for the vertex 1 of **G**, $|Q(1)| = 2$ and there is no triangle with 2 vertices. On the other hand, any graph can be represented as the intersection graph of a Helly separating family of paths because there exist paths of any order. The Helly separating *Paths*-family in Fig. 2 represents graph **G** in Fig. 1.

The following two theorems tell us when a graph is the intersection graph of a Helly separating $\widetilde{S}$-family such that all its members are subgraphs, or induced subgraphs, of some particular graph belonging to a given class.

The *union* of a graph family $\mathcal{F} = (\mathbf{F}_i)_{i\in I}$ is the graph $\mathbf{U}_\mathcal{F}$ whose edge set and vertex set are, respectively, the union of the edge sets and the union of the vertex sets of the members of the family, that is

$$V_{\mathbf{U}_\mathcal{F}} = \bigcup_{i\in I} V_{\mathbf{F}_i} \quad \text{and} \quad E_{\mathbf{U}_\mathcal{F}} = \bigcup_{i\in I} E_{\mathbf{F}_i}.$$

Notice that each member $\mathbf{F}_i$ of $\mathcal{F}$ is a subgraph of the graph $\mathbf{U}_\mathcal{F}$.

Let $\widetilde{S}$ and $\widetilde{H}$ be graph classes. An $\widetilde{H}$–$\widetilde{S}$-family is an $\widetilde{S}$-family whose union belongs to $\widetilde{H}$. When, in addition, each member of the family is an induced subgraph of the union, we say that the family is an *induced* $\widetilde{H}$–$\widetilde{S}$-family.

For instance, the family $(\mathbf{T}_i)_{1\le i\le 6}$ in Fig. 2 is an induced *Cycles-Paths*-family since the union of the six paths members of this family is a cycle, and each path is an induced subgraph of it. For the same reason, this is not a *Trees-Paths*-family.

Theorem 6. *Let $\widetilde{S}$ and $\widetilde{H}$ be graph classes. A graph $\mathbf{G}$ is the intersection graph of some Helly separating $\widetilde{H}$–$\widetilde{S}$-family if and only if, for every $v \in V_\mathbf{G}$, a spanning subgraph $\mathbf{F}_v$ of $\mathbf{K}_v$ belonging to $\widetilde{S}$ can be chosen in such a way that the union graph of $(\mathbf{F}_v)_{v\in V_\mathbf{G}}$ belongs to $\widetilde{H}$.*

Proof. It is a direct consequence of Theorem 4. □

Theorem 7. *Let $\widetilde{S}$ and $\widetilde{H}$ be graph classes. A graph $\mathbf{G}$ is the intersection graph of some induced Helly separating $\widetilde{H}$–$\widetilde{S}$-family if and only if there exists a spanning subgraph $\mathbf{H}$ of $K(\mathbf{G})$ belonging to $\widetilde{H}$ such that, for every $v \in V_\mathbf{G}$, the subgraph $\mathbf{H}[Q(v)]$ induced in $\mathbf{H}$ by $Q(v)$ belongs to $\widetilde{S}$.*

Proof. Assume that $\mathbf{G}$ is the intersection graph of an induced $\widetilde{H}$–$\widetilde{S}$-family $\mathcal{F} = (\mathbf{F}_v)_{v\in V_\mathbf{G}}$. We claim that the necessary condition is satisfied considering $\mathbf{H} = \mathbf{U}_\mathcal{F}$. Indeed,

(i) $\mathbf{U}_\mathcal{F}$ is a spanning subgraph of $K(\mathbf{G})$ since, by Theorem 4, $\mathbf{F}_v$ is a spanning subgraph of $\mathbf{K}_v$ for every $v \in V_\mathbf{G}$.
(ii) $\mathbf{U}_\mathcal{F} \in \widetilde{H}$, by definition of $\widetilde{H}$–$\widetilde{S}$-family.
(iii) By definition of *induced* $\widetilde{H}$–$\widetilde{S}$-family, each $\mathbf{F}_v$ is an induced subgraph of $\mathbf{U}_\mathcal{F}$. The vertex set of $\mathbf{F}_v$ is $Q(v)$, thus the subgraph induced in $\mathbf{U}_\mathcal{F}$ by $Q(v)$ is $\mathbf{F}_v$. By the $\widetilde{H}$–$\widetilde{S}$-family definition, $\mathbf{F}_v \in \widetilde{S}$. It follows that, for every $v \in V_\mathbf{G}$, the subgraph induced in $\mathbf{U}_\mathcal{F}$ by $Q(v)$ belongs to $\widetilde{S}$.

The converse implication is straightforward. □

Notice that the previous theorem generalizes Theorem 2. In analogy with the term *clique-tree* used for a chordal graph, we introduce the following definition.

Definition 1. Let $\widetilde{S}$ and $\widetilde{H}$ be graph classes. A *clique-$\widetilde{H}$–$\widetilde{S}$-support* of a graph $\mathbf{G}$ is any spanning subgraph $\mathbf{H} \in \widetilde{H}$ of $K(\mathbf{G})$ satisfying that for all $v \in V_\mathbf{G}$ the subgraph induced in $\mathbf{H}$ by $Q(v)$ belongs to $\widetilde{S}$.

Using this definition, Theorem 7 can be formulated as follows:

> A graph $\mathbf{G}$ is the intersection graph of an induced Helly separating $\widetilde{H}$–$\widetilde{S}$-family if and only if there exists a clique-$\widetilde{H}$–$\widetilde{S}$-support of $\mathbf{G}$.

Moreover, Theorem 4 shows that the clique-$\widetilde{H}$–$\widetilde{S}$-supports provide the only representations of $\mathbf{G}$ as intersection of induced Helly separating $\widetilde{H}$–$\widetilde{S}$-families.

3. Characterization of clique-$\widetilde{H}$–$\widetilde{S}$-supports

It is clear that, in general, determining if there exists a clique-$\widetilde{H}$–$\widetilde{S}$-support of $\mathbf{G}$ could be a hard problem. In order to help in the solution of this problem, in this section, we present partial and total characterizations of clique-$\widetilde{H}$–$\widetilde{S}$-supports for graph classes $\widetilde{S}$ and $\widetilde{H}$ satisfying certain conditions. The characterizations are based on the weight of the support and generalize Theorem 3: the problem of finding a clique-$\widetilde{H}$–$\widetilde{S}$-support of $\mathbf{G}$ is reduced to the problem of finding a spanning subgraph $\mathbf{H} \in \widetilde{H}$ of $K^w(\mathbf{G})$ with a specific weight which, in addition, must be maximum.

The following result is used in the proofs of the main theorems.

Lemma 1. *Let $\mathbf{G}$ be a graph. If $\mathbf{H}$ is a spanning subgraph of $K^w(\mathbf{G})$, then*

$$w(\mathbf{H}) = \sum_{v\in V_\mathbf{G}} |E_{\mathbf{H}[Q(v)]}|.$$

Proof. Notice that the weight $w(e) = |Q \cap Q'|$ of an edge $e = QQ'$ of $K(\mathbf{G})$ can be alternatively formulated as

$$w(e) = |\{v \in V_\mathbf{G} / e \in E_{\mathbf{K}_v}\}|.$$

Table 1
For any class $\widetilde{H}$, class $\widetilde{S}$ satisfies property ***A*** with $\widetilde{H}$. This means that if $\mathbf{H}' \in \widetilde{S}$ is an induced subgraph with n vertices of a graph $\mathbf{H}$ with k vertices, then $|E_{\mathbf{H}'}| = f(n,k)$.

$\widetilde{S}$	f	
Trees	$f(n,k) = n-1$	for $n \geq 1$
Paths	$f(n,k) = n-1$	for $n \geq 1$
Unicycles	$f(n,k) = n$	for $n \geq 3$
Cycles	$f(n,k) = n$	for $n \geq 3$
Complete graphs	$f(n,k) = \binom{n}{2}$	for $n \geq 1$
Maximal planar graphs	$f(n,k) = \begin{cases} n-1 \\ 3n-6 \end{cases}$	for $1 \leq n \leq 2$ for $n \geq 3$

Table 2
Class $\widetilde{S}$ satisfies property ***C*** with class $\widetilde{H}$.

$\widetilde{H}$	$\widetilde{S}$	f	
Paths	*Paths*	$f(n,k) = n-1$	for $n \geq 1$
Trees	*Trees*	$f(n,k) = n-1$	for $n \geq 1$
Cycles	*Arcs* = *Cycles* ∪ *Paths*	$f(n,k) = \begin{cases} n-1 \\ n \end{cases}$	for $1 \leq n \leq k-1$ for $n = k$
h-Unicycles	*h-Unicycles* ∪ $(h-1)$*-Trees*	$f(n,k) = \begin{cases} n-1 \\ n \end{cases}$	for $1 \leq n \leq h-1$ for $n \geq h$
Planar graphs	*Maximal planar graphs*	$f(n,k) = \begin{cases} n-1 \\ 3n-6 \end{cases}$	for $1 \leq n \leq 2$ for $n \geq 3$

Let $\mathbf{H}$ be a spanning subgraph of $K(\mathbf{G})$; we have that

$$w(\mathbf{H}) = \sum_{e \in E_{\mathbf{H}}} w(e) = \sum_{e \in E_{\mathbf{H}}} |\{v \in V_{\mathbf{G}} / e \in E_{\mathbf{K}_v}\}|$$
$$= \sum_{v \in V_{\mathbf{G}}} |\{e \in E_{\mathbf{H}} / e \in E_{\mathbf{K}_v}\}| = \sum_{v \in V_{\mathbf{G}}} |E_{\mathbf{H}[Q(v)]}|. \quad \square$$

Next, we propose properties to be satisfied by the classes $\widetilde{S}$ and $\widetilde{H}$ in order to obtain different characterizations of the clique-$\widetilde{H}$–$\widetilde{S}$-supports. Afterwards, we give several examples of classes satisfying the properties.

We say that the class $\widetilde{S}$ satisfies *Property* ***A*** with the class $\widetilde{H}$ when there exists a function f such that

$$\mathbf{H} \in \widetilde{H}, \mathbf{H}' \text{ is an induced subgraph of } \mathbf{H}, \mathbf{H}' \in \widetilde{S} \Rightarrow |E_{\mathbf{H}'}| = f(|V_{\mathbf{H}'}|, |V_{\mathbf{H}}|).$$

We say that the class $\widetilde{S}$ satisfies *Property* ***B*** with the class $\widetilde{H}$ when $\widetilde{S}$ satisfies property ***A*** with $\widetilde{H}$ and, in addition,

$$\mathbf{H} \in \widetilde{H}, \mathbf{H}' \text{ is an induced subgraph of } \mathbf{H} \Rightarrow |E_{\mathbf{H}'}| \leq f(|V_{\mathbf{H}'}|, |V_{\mathbf{H}}|).$$

We say that the class $\widetilde{S}$ satisfies *Property* ***C*** with the class $\widetilde{H}$ when $\widetilde{S}$ satisfies property ***A*** with $\widetilde{H}$ and, in addition,

$$\mathbf{H} \in \widetilde{H}, \mathbf{H}' \text{ is an induced subgraph of } \mathbf{H}, \mathbf{H}' \notin \widetilde{S} \Rightarrow |E_{\mathbf{H}'}| < f(|V_{\mathbf{H}'}|, |V_{\mathbf{H}}|).$$

It is clear that property ***C*** implies property ***B*** which in turn implies property ***A***.

Let $\widetilde{H}$ be any graph class. Table 1 shows examples of classes $\widetilde{S}$ satisfying property ***A*** with $\widetilde{H}$. The corresponding function f is showed in the second column. A *unicycle*, or *unicyclic graph*, is any connected graph with a unique cycle, this is a tree plus one edge. Intersection graphs of families of subtrees of a unicycle have been studied in [8].

Notice that, in these examples, f does not depend on k. A case in which f depends on k is shown in the third row of Table 2.

On the other hand, given any function $f : \mathbb{N} \to \mathbb{N}$, a graph class $\widetilde{S}$ satisfying property ***A*** with $\widetilde{H}$ can be defined as

$$\widetilde{S} = \{\mathbf{S} / |V_{\mathbf{S}}| = n \text{ and } |E_{\mathbf{S}}| = f(n)\}.$$

The class *Paths* satisfies property ***B*** with the class *Trees*, but not property ***C***. Examples of classes satisfying property ***C*** and the corresponding functions are showed in Table 2; some of them are explained immediately after the table. An *h-unicycle* is any unicycle with a cycle of size h and an $(h-1)$*-tree* is a tree with at most $h-1$ vertices.

$\widetilde{H} =$ *Trees*, $\widetilde{S} =$ *Trees*: let $\mathbf{H}$ be a tree with k vertices, $k \geq 1$, and $\mathbf{H}'$ an induced subgraph with n vertices. If $\mathbf{H}' \in \widetilde{S}$, $\mathbf{H}'$ is also a tree; then the number of edges of $\mathbf{H}'$ is $n-1$. If $\mathbf{H}'$ is not a tree, then it is not connected and so the number of edges of $\mathbf{H}'$ is less than $n-1$.

$\widetilde{H} =$ *Cycles*, $\widetilde{S} =$ *Arcs*: let $\mathbf{H}$ be a cycle with k vertices, $k \geq 3$, and $\mathbf{H}'$ an induced subgraph with n vertices. If $\mathbf{H}' \in \widetilde{S}$ and $n < k$, then $\mathbf{H}'$ must be a path with n vertices and thus the number of edges of $\mathbf{H}'$ is $n-1$. If $\mathbf{H}' \in \widetilde{S}$ and $n = k$, then $\mathbf{H}'$ must be the whole cycle and thus the number of edges of $\mathbf{H}'$ is n. If $\mathbf{H}' \notin \widetilde{S}$, then it is not connected. It follows that the number of edges of $\mathbf{H}'$ is less than $n-1$.

$\widetilde{H} = h$-*Unicycles*, $\widetilde{S} = h$-*Unicycles* $\cup$ $(h-1)$-*Trees*: for any $h \geq 3$, let $\mathbf{H}$ be an h-unicycle with k vertices, $k \geq h$, and $\mathbf{H}'$ an induced subgraph with n vertices. If $\mathbf{H}' \in \widetilde{S}$ and $n \leq (h-1)$, then $\mathbf{H}'$ must be an $(h-1)$-tree. It follows that the number of edges of $\mathbf{H}'$ is $n-1$. If $\mathbf{H}' \in \widetilde{S}$ and $n \geq h$, then $\mathbf{H}'$ must be an h-unicycle. It follows that the number of edges of $\mathbf{H}'$ is n. If $\mathbf{H}'$ is not in $\widetilde{S}$, it is clear that the number of edges of $\mathbf{H}'$ is less than n.

In what follows, assume that $\mathbf{G}$ is an arbitrary fixed graph. Let c_v and $c_{\mathbf{G}}$ be the number of cliques containing the vertex v and the number of cliques of $\mathbf{G}$, respectively. In other words, $c_v = |Q(v)|$ and $c_{\mathbf{G}} = |V_{K(\mathbf{G})}|$.

Theorem 8. *Let $\widetilde{S}$ be a graph class satisfying property **A** with a class $\widetilde{H}$ and a function f. If $\mathbf{H}$ is a clique-$\widetilde{H}$–$\widetilde{S}$-support of $\mathbf{G}$, then*

$$w(\mathbf{H}) = \sum_{v \in V_{\mathbf{G}}} f(c_v, c_{\mathbf{G}}).$$

Proof. Let $\mathbf{H}$ be a clique-$\widetilde{H}$–$\widetilde{S}$-support of $\mathbf{G}$, this means that $\mathbf{H}$ is a spanning subgraph of $K(\mathbf{G})$ satisfying $\mathbf{H} \in \widetilde{H}$ and $\mathbf{H}[Q(v)] \in \widetilde{S}$, for all $v \in V_{\mathbf{G}}$.

By property ***A*** and since $\mathbf{H}[Q(v)]$ is an induced subgraph of $\mathbf{H}$,

$$|E_{\mathbf{H}[Q(v)]}| = f(|V_{\mathbf{H}[Q(v)]}|, |V_{\mathbf{H}}|) = f(|Q(v)|, |V_{K(\mathbf{G})}|) = f(c_v, c_{\mathbf{G}}).$$

On the other hand, since $\mathbf{H}$ is a spanning subgraph of $K^w(\mathbf{G})$, by Lemma 1,

$$w(\mathbf{H}) = \sum_{v \in V_{\mathbf{G}}} |E_{\mathbf{H}[Q(v)]}| = \sum_{v \in V_{\mathbf{G}}} f(c_v, c_{\mathbf{G}}). \quad \square$$

For a graph class $\widetilde{H}$, a *maximum spanning $\widetilde{H}$-subgraph* of $K^w(\mathbf{G})$ is a spanning subgraph $\mathbf{H}$ of $K^w(\mathbf{G})$ such that $\mathbf{H} \in \widetilde{H}$ and $w(\mathbf{H}) \geq w(\mathbf{H}_0)$ for any other $\mathbf{H}_0$ spanning subgraph of $K^w(\mathbf{G})$ belonging to $\widetilde{\mathbf{H}}$.

Theorem 9. *Let $\widetilde{S}$ be a graph class satisfying property **B** with a class $\widetilde{H}$ and a function f. If $\mathbf{H}$ is a clique-$\widetilde{H}$–$\widetilde{S}$-support of $\mathbf{G}$, then $\mathbf{H}$ is a maximum spanning $\widetilde{H}$-subgraph of $K^w(\mathbf{G})$.*

Proof. Let $\mathbf{H}$ be a clique-$\widetilde{H}$–$\widetilde{S}$-support of $\mathbf{G}$, thus $\mathbf{H} \in \widetilde{H}$, $\mathbf{H}$ is a spanning subgraph of $K(\mathbf{G})$, and $\mathbf{H}[Q(v)] \in \widetilde{S}$, for all $v \in V_{\mathbf{G}}$.

Since property ***B*** implies property ***A***, by Theorem 8,

$$w(\mathbf{H}) = \sum_{v \in V_{\mathbf{G}}} f(c_v, c_{\mathbf{G}}).$$

We claim that $\mathbf{H}$ is maximum spanning $\widetilde{H}$-subgraph. Indeed, let $\mathbf{H}_0 \in \widetilde{H}$ be a spanning subgraph of $K^w(\mathbf{G})$. By Lemma 1,

$$w(\mathbf{H}_0) = \sum_{v \in V_{\mathbf{G}}} |E_{\mathbf{H}_0[Q(v)]}|.$$

By property ***B***, since $\mathbf{H}_0 \in \widetilde{H}$ and $\mathbf{H}_0[Q(v)]$ is an induced subgraph of $\mathbf{H}_0$, we obtain

$$|E_{\mathbf{H}_0[Q(v)]}| \leq f(|V_{\mathbf{H}_0[Q(v)]}|, |V_{\mathbf{H}_0}|) = f(c_v, c_{\mathbf{G}}), \quad \text{for every } v \in V_{\mathbf{G}}.$$

Hence,

$$w(\mathbf{H}_0) = \sum_{v \in V_{\mathbf{G}}} |E_{\mathbf{H}_0[Q(v)]}| \leq \sum_{v \in V_{\mathbf{G}}} f(c_v, c_{\mathbf{G}}) = w(\mathbf{H}).$$

It follows that $\mathbf{H}$ is a maximum spanning $\widetilde{H}$-subgraph of $K^w(\mathbf{G})$. $\square$

Theorem 10. *Let $\widetilde{S}$ be a graph class satisfying property **C** with a class $\widetilde{H}$ and a function f. Consider a spanning subgraph $\mathbf{H} \in \widetilde{H}$ of $K(\mathbf{G})$. If*

$$w(\mathbf{H}) = \sum_{v \in V_{\mathbf{G}}} f(c_v, c_{\mathbf{G}}),$$

then $\mathbf{H}$ is a clique-$\widetilde{H}$–$\widetilde{S}$-support of $\mathbf{G}$. $\square$

Proof. Assume $\mathbf{H} \in \widetilde{H}$ is a spanning subgraph of $K^w(\mathbf{G})$ with weight

$$w(\mathbf{H}) = \sum_{v \in V_{\mathbf{G}}} f(c_v, c_{\mathbf{G}}).$$

By Lemma 1,

$$w(\mathbf{H}) = \sum_{v \in V_{\mathbf{G}}} |E_{\mathbf{H}[Q(v)]}|;$$

thus

$$\sum_{v \in V_{\mathbf{G}}} |E_{\mathbf{H}[Q(v)]}| = \sum_{v \in V_{\mathbf{G}}} f(c_v, c_{\mathbf{G}}). \quad (1)$$

Let v be any vertex of **G**. By property ***B***, since $\mathbf{H} \in \widetilde{H}$ and $\mathbf{H}[Q(v)]$ is an induced subgraph of **H**, we have

$$|E_{\mathbf{H}[Q(v)]}| \leq f(|V_{\mathbf{H}[Q(v)]}|, |V_{\mathbf{H}}|) = f(c_v, c_{\mathbf{G}}).$$

It follows from Eq. (I) that $|E_{\mathbf{H}[Q(v)]}| = f(|V_{\mathbf{H}[Q(v)]}|, |V_{\mathbf{H}}|)$ and thus, by property ***C***, $\mathbf{H}[Q(v)] \in \widetilde{S}$. We have proved that **H** is a clique-$\widetilde{H}$–$\widetilde{S}$-support of **G**. □

Corollary 1. *Let $\widetilde{S}$ be a graph class satisfying property **C** with a class $\widetilde{H}$. If a graph **G** admits a clique-$\widetilde{H}$–$\widetilde{S}$-support, then every maximum spanning $\widetilde{H}$-subgraph of $K^w(\mathbf{G})$ is a clique-$\widetilde{H}$–$\widetilde{S}$-support.*

4. Chordal, interval and Helly circular-arc graphs

A connected induced subgraph of a cycle is called an *arc* of the cycle. A graph **G** is a *circular-arc graph* if there exists a cycle **C** and a family $\mathcal{A}$ of arcs of **C** such that **G** is the intersection graph of $\mathcal{A}$. The pair $(\mathbf{C}, \mathcal{A})$ is called a *circular-arc representation* of **G**, and **C** is the support of the representation. When an edge of **C** is covered by no arc of the family, **C** can be reduced to a path and the arcs become intervals of the path; in this case, we speak of *interval graphs* and *interval representations*.

Any family of subtrees of a tree has the Helly property, but not every family of arcs of a cycle has the Helly property. Graphs admitting a *Helly circular-arc representation*, i.e. a circular-arc representation such that the family of arcs satisfies the Helly property, are the graphs belonging to the well known class of *Helly circular-arc graphs* [4]. Interval graphs are included in the intersection of chordal graphs and Helly circular-arc graphs. For a general review of the class of circular-arc graphs and its subclasses, we refer the reader to [10].

In accordance with the terminology of the previous sections and [7,6], chordal, interval and Helly circular-arc graphs can be defined as follows.

- A graph is a chordal graph if and only if it is the intersection graph of an induced Helly separating *Trees*–*Trees*-family.
- A graph is an interval graph if and only if it is the intersection graph of an induced Helly separating *Paths*–*Paths*-family.
- A non interval graph is a Helly circular-arc graph if and only if it is the intersection graph of an induced Helly separating *Cycles*-*Arcs*-family.

Applying Theorem 7 to these classes we obtain respectively Theorem 2 and the following two already known theorems [7].

Theorem 11. *A graph **G** is an interval graph if and only if there exists a spanning path **P** of $K(\mathbf{G})$ such that, for every $v \in V_{\mathbf{G}}$, the subgraph induced in **P** by $Q(v)$ is a subpath of **P**.*

Theorem 12. *A non interval graph **G** is a Helly circular-arc graph if and only if there exists a spanning cycle **C** of $K(\mathbf{G})$ such that, for every $v \in V_{\mathbf{G}}$, the subgraph induced in **C** by $Q(v)$ is an arc of **C**.*

Notice that, in accordance with the terminology of Definition 1, the spanning tree in Theorem 2, the spanning path in Theorem 11 and the spanning cycle in Theorem 12 are a clique-*Trees*–*Trees*-support, a clique-*Paths*–*Paths*-support and a clique-*Cycles*-*Arcs*-support of **G**, respectively.

We showed in Table 2 that the class *Paths* satisfies property **C** with the class *Paths*, and also that the class *Arcs* satisfies property ***C*** with the class *Cycles*. It follows that Theorems 8–10 can be applied to chordal, interval and Helly circular-arc graphs in order to obtain the support of any canonical representation by its weight in the valuated clique graph. For this, it is necessary to evaluate the functions given in Table 2.

For a chordal or an interval graph **G**, the function to be used is the same function given in the first and in the second row of Table 2. So, for chordal and interval graphs,

$$\sum_{v \in V_{\mathbf{G}}} f(c_v, c_{\mathbf{G}}) = \sum_{v \in V_{\mathbf{G}}} (c_v - 1) = \left(\sum_{v \in V_{\mathbf{G}}} c_v\right) - \left(\sum_{v \in V_{\mathbf{G}}} 1\right).$$

An easy exercise shows that

$$\sum_{v \in V_{\mathbf{G}}} c_v = \sum_{Q \in \mathcal{C}(\mathbf{G})} |Q|.$$

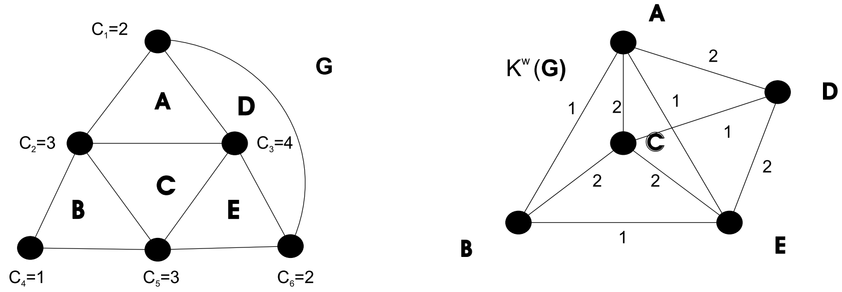

Fig. 3. Graph **G** and its valuated clique graph $K^w(\mathbf{G})$.

As a consequence, we obtain the following theorems.

Theorem 13 (Theorem 3). *Let* **G** *be a graph. A spanning tree* **T** *of* $K^w(\mathbf{G})$ *is a clique* — Trees–Trees—*support of* **G** *if and only if*

$$w(\mathbf{T}) = \left(\sum_{Q\in\mathcal{C}(\mathbf{G})} |Q|\right) - |V_{\mathbf{G}}|.$$

Moreover, in this case, **T** *is a maximum spanning tree of* $K^w(\mathbf{G})$.

Theorem 14. *Let* **G** *be a graph. A spanning path* **P** *of* $K^w(\mathbf{G})$ *is a clique*—Paths–Paths—*support of* **G** *if and only if*

$$w(\mathbf{P}) = \left(\sum_{Q\in\mathcal{C}(\mathbf{G})} |Q|\right) - |V_{\mathbf{G}}|.$$

Moreover, in this case, **P** *is a maximum spanning path of* $K^w(\mathbf{G})$.

For a Helly circular-arc graph **G**, the function to be used is the one given in the third row of Table 2. To evaluate this function we partition the vertex set of **G** into the set $U_{\mathbf{G}}$ of all universal vertices of **G** and its complement $\overline{U}_{\mathbf{G}}$.

If $v \in U_{\mathbf{G}}$, then $c_v = c_{\mathbf{G}}$ and so $f(c_v, c_{\mathbf{G}}) = c_v$. If $v \in \overline{U}_{\mathbf{G}}$, then $c_v < c_{\mathbf{G}}$ and so $f(c_v, c_{\mathbf{G}}) = c_v - 1$. It follows that

$$\begin{aligned}\sum_{v\in V_{\mathbf{G}}} f(c_v, c_{\mathbf{G}}) &= \sum_{v\in U_{\mathbf{G}}} f(c_v, c_{\mathbf{G}}) + \sum_{v\in \overline{U}_{\mathbf{G}}} f(c_v, c_{\mathbf{G}}) \\ &= \sum_{v\in U_{\mathbf{G}}} c_v + \sum_{v\in \overline{U}_{\mathbf{G}}} (c_v - 1) = \sum_{v\in V_{\mathbf{G}}} c_v - \sum_{v\in \overline{U}_{\mathbf{G}}} 1 \\ &= \left(\sum_{Q\in\mathcal{C}(\mathbf{G})} |Q|\right) - |\overline{U}_{\mathbf{G}}|.\end{aligned}$$

We obtain the following theorem,

Theorem 15. *Let* **G** *be a graph. A spanning cycle* **C** *of* $K^w(\mathbf{G})$ *is a clique*—Cycles — Arcs—*support of* **G** *if and only if*

$$w(\mathbf{C}) = \left(\sum_{Q\in\mathcal{C}(\mathbf{G})} |Q|\right) - |\overline{U}_{\mathbf{G}}|.$$

Moreover, in this case, **C** *is a maximum spanning cycle of* $K^w(\mathbf{G})$.

The following theorem is proved in an analogous way.

Theorem 16. *Let* **G** *be a graph and* h *an integer larger than* 2. *A spanning* h*-unicycle* **A** *of* $K^w(\mathbf{G})$ *is a clique-*[h*-Unicycles*] — [h*-Unicycles* $\cup$ $(h-1)$*-Trees*]*-support of* **G** *if and only if*

$$w(\mathbf{A}) = \left(\sum_{Q\in\mathcal{C}(\mathbf{G})} |Q|\right) - |V_{h-1}|,$$

where V_{h-1} *is the set of vertices* $v \in \mathbf{G}$ *with* $c_v \le h-1$.

Moreover, in this case, **A** *is a maximum spanning* h*-unicycle of* $K^w(\mathbf{G})$.

In the following section we apply these theorems in a particular example.

5. Examples

On the left side of Fig. 3, we have the graph **G** of Fig. 1. Each vertex v is labeled with c_v, the number of cliques containing v; and the five cliques are indicated with capital letters. On the right, the valuated clique graph of **G** is presented, the vertices

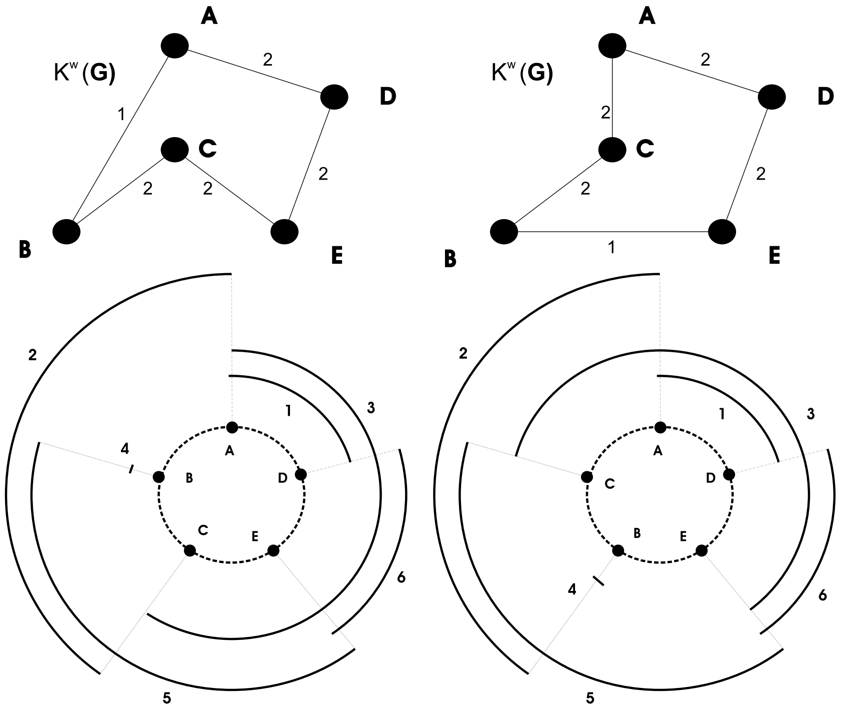

Fig. 4. The maximum spanning cycles of $K^w(\mathbf{G})$ and the corresponding representations.

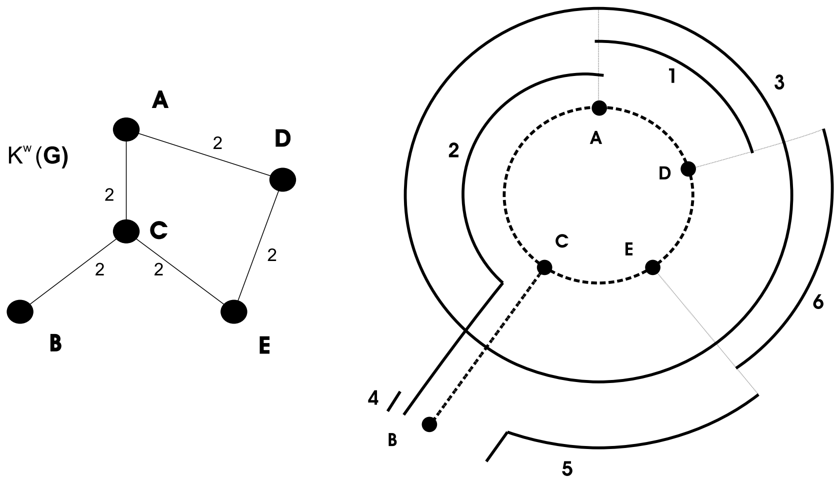

Fig. 5. The only maximum 4-unicycle of $K^w(\mathbf{G})$ and the corresponding representation of **G**.

are labeled as the corresponding cliques, and the edges with the number of vertices in the intersection that each of them represents. The following values are easily calculated. Recall that $V_k = \{v \in V_{\mathbf{G}}/c_v \leq k\}$.

$$|V_{\mathbf{G}}| = 6 \qquad |\overline{U}_{\mathbf{G}}| = 6 \qquad \sum_{Q \in \mathcal{C}(\mathbf{G})} |Q| = 15$$

$$|V_2| = 3 \qquad |V_3| = 5 \qquad |V_4| = 6.$$

In agreement with Theorem 13, to represent **G** as the intersection of a Helly separating family of trees of a tree, we need a maximum spanning tree **T** of $K^w(\mathbf{G})$ with weight equal to $w(\mathbf{T}) = \sum_{Q \in \mathcal{C}(\mathbf{G})} |Q| - |V_{\mathbf{G}}| = 15 - 6 = 9$. Any maximum spanning tree has weight 8, thus **G** is not a chordal graph; indeed vertices 1, 2, 5 and 6 induce a cycle.

Using Theorem 15, to represent **G** as the intersection of a Helly separating family of arcs of a cycle, we need a maximum spanning cycle **C** of $K^w(\mathbf{G})$ with weight equal to $w(\mathbf{C}) = \sum_{Q \in \mathcal{C}(\mathbf{G})} |Q| - |\overline{U}_{\mathbf{G}}| = 15 - 6 = 9$. There are exactly two such cycles. They are shown in Fig. 4. Each one gives a representation of G. No other canonical representation is possible.

From Theorem 16, to represent **G** as intersection of a Helly separating family of induced subgraphs of a 3-unicycle, the subgraphs being 3-unicycles or trees with less than 3 vertices, we need a spanning 3-unicycle **A** of $K^w(\mathbf{G})$ with maximum weight equal to $w(\mathbf{A}) = (\sum_{Q\in\mathcal{C}(\mathbf{G})} |Q|) - |V_{3-1}| = 15 - 3 = 12$. Since any spanning unicycle is a spanning tree plus an edge, and the maximum spanning tree has weight 8 and any edge has weight less than or equal to 2, there is no such 3-unicycle, thus the required representation is not possible.

To represent **G** as intersection of a Helly separating family of induced subgraphs of a 4-unicycle, the subgraphs being 4-unicycles or trees with less than 4 vertices, we need a spanning 4-unicycle **A** of $K^w(\mathbf{G})$ with maximum weight equal to $w(\mathbf{A}) = (\sum_{Q\in\mathcal{C}(\mathbf{G})} |Q|) - |V_{4-1}| = 15 - 5 = 10$. Since any spanning unicycle is a spanning tree plus and edge and the maximum spanning tree has weight 8 and any edge has weight less than or equal to 2, if such a 4-unicycle exists, then it must be obtained using the 5 edges of $K^w(\mathbf{G})$ with weight 2; these edges actually form a 4-unicycle. Fig. 5 shows the maximum 4-unicycle of $K^w(\mathbf{G})$ and the only way of representing **G** as intersection of a Helly separating family of induced subgraphs of a 4-unicycle which can be 4-unicycles or trees with at most 3 vertices.

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