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Non-commutative supersymmetric quantum mechanics

Ashok Das [a,b], H. Falomir [c], J. Gamboa [d,*], F. Méndez [d]

[a] Departament of Physics and Astronomy, University of Rochester, USA
[b] Saha Institute of Nuclear Physics, 1/AF Bidhannagar, Calcutta-700 064, India
[c] IFLP/CONICET - Departamento de Física, Facultad de Ciencias Exactas, Universidad Nacional de la Plata, C.C. 67, (1900) La Plata, Argentina
[d] Departamento de Física, Universidad de Santiago de Chile, Casilla 307, Santiago, Chile

A R T I C L E I N F O

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A B S T R A C T

General non-commutative supersymmetric quantum mechanics models in two and three dimensions are constructed and some two- and three-dimensional examples are explicitly studied. The structure of the theory studied suggest other possible applications in physical systems with potentials involving spin and non-local interactions.

1. Introduction

The non-commutativity of spacetime is an old idea [1] and in physics, the first example of non-commutativity was probably discussed by Landau in 1930 [2]. It has been revived in recent years within the context of string theory [3] and since then, non-commutative field theories have attracted much attention in various fields such as mathematical physics [4,5] and phenomenology [6]. Normally, in discussing a quantum field theory on a non-commutative manifold, we assume that only spatial coordinates do not commute in order to avoid possible conflict with unitarity of the theory. On a manifold where spatial coordinates do not commute, the product of functions generalizes to the Moyal product (also conventionally known as the $\star$-product) which is an associative and non-commutative product defined as

$$(A \star B)(\mathbf{x}) = e^{\frac{i}{2}\theta^{ij}\partial_i^{(1)}\partial_j^{(2)}} A(\mathbf{x}_1)B(\mathbf{x}_2)|_{\mathbf{x}_1=\mathbf{x}_2=\mathbf{x}}, \tag{1}$$

where A, B denote two arbitrary functions of coordinates and $\theta^{ij} = -\theta^{ji}$ denotes the constant parameter of non-commutativity with $i, j = 1, 2, \ldots, n$ taking values over only the spatial indices. It follows from (1) that

$$\left[x^i, x^j\right]_\star := x^i \star x^j - x^j \star x^i = i\theta^{ij}, \tag{2}$$

which reflects the basic spatial non-commutativity of the manifold.

A particularly interesting example of non-commutative theories is non-commutative quantum mechanics (NCQM) [7] which contains most of the distinguishing properties of a non-commutative quantum field theory and yet is simple enough to carry out explicit calculations. At the quantum mechanical level, the space non-commutativity (2) reflects in the Schrödinger equation with the change of the usual products of functions by the $\star$-product (1). Namely, the time dependent Schrödinger equation in a non-commutative space takes the form

$$i\frac{\partial\phi(\mathbf{x},t)}{\partial t} = \left[-\frac{1}{2}\nabla^2 + V(\mathbf{x})\right] \star \phi(\mathbf{x},t) = -\frac{1}{2}\nabla^2\phi(\mathbf{x},t) + V(\mathbf{x}) \star \phi(\mathbf{x},t), \tag{3}$$

where $\phi(\mathbf{x},t)$ denotes the quantum mechanical wave function (and we have set $\hbar = 1 = m$). We note from (1) that the Moyal product definition does not affect the derivatives and the only change appears in the potential term which can be simplified using (1) to have the form (we are suppressing the time dependence of the wave function for simplicity)

$$V(\mathbf{x}) \star \phi(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}} \int d^n k\, \tilde{V}(\mathbf{k}) e^{ik_i[x^i - \frac{\theta^{ij}p_j}{2}]}\phi(\mathbf{x}) =: V(\mathbf{x} - \tilde{\mathbf{p}}/2)\phi(\mathbf{x}), \tag{4}$$

* Corresponding author.
E-mail addresses: das@pas.rochester.edu (A. Das), falomir@fisica.unlp.edu.ar (H. Falomir), jgamboa55@gmail.com (J. Gamboa), feritox@gmail.com (F. Méndez).

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where $\tilde{V}(\mathbf{k})$ denotes the Fourier transform of the potential, $p_i := -\imath\partial/\partial x^i$ represents the momentum operator and we have identified $\tilde{p}^i := \theta^{ij}p_j$ for simplicity. The simple replacement of x^i by $x^i - \tilde{p}^i/2$ in (4) in going from a non-commutative star product to an ordinary product is known as the Bopp shift. Notice that there is no ordering ambiguity in this definition since

$$[\mathbf{k}\cdot\mathbf{x}, \mathbf{k}\cdot\tilde{\mathbf{p}}] = k_i k_j \theta^{jk}[x^i, p_k] = k_i k_j \theta^{jk}\imath\delta^i_k = -\imath k_i k_j \theta^{ij} = 0, \tag{5}$$

where we have used $[p_i, x^j] = -\imath(\partial_i x^j) = -\imath\delta_i^j$. The procedure sketched here was employed in [8] to illustrate some of the properties of non-commutative quantum field theories using a general realization of NCQM. In particular, several interesting properties of the central potentials within the context of NCQM were derived using this formalism.

Another class of quantum mechanical theories that have been studied from various points of view is known as supersymmetric quantum mechanics. In this connection much work has been done within the context of one-dimensional supersymmetric quantum mechanical theories with some isolated attempts at supersymmetric quantum mechanics in higher dimensions. The goal of the present work is to construct general (higher dimensional) non-commutative supersymmetric quantum mechanical systems and, in particular, to study three-dimensional non-commutative problems. Although the supersymmetric extensions of non-commutative quantum mechanical systems have been partly discussed in the literature (see, for example, [9] and [10] for the harmonic oscillator and the Landau problems), a general formalism to study such systems is still lacking, and we wish to systematically develop it in this work.

Our paper is organized as follows. In Section 2 we give a general discussion of Witten's supersymmetric quantum mechanics (in ordinary space) and discuss in detail several subtle points related to its extension to higher (three) dimensional systems. In Section 3 we generalize this to non-commutative space and construct general supersymmetric theories in higher dimensional non-commutative spaces. We study, as examples, the non-commutative supersymmetric harmonic oscillator, the non-commutative supersymmetric Landau problem as well as the three-dimensional harmonic oscillator. We present our conclusions and discussions in Section 4 and collect some technical results in two appendices.

2. Supersymmetric quantum mechanics

One-dimensional supersymmetric quantum mechanics (SUSYQM) has been studied extensively in the past couple of decades (for a review, see [11]). While in the beginning it drew attention as an interesting toy model for studying the mechanism of supersymmetry breaking, since then it has found many interesting applications. Basically, SUSYQM in one dimension is described by the simple closed superalgebra (graded algebra with curly brackets denoting anticommutators)

$$\{Q, Q^\dagger\} = 2H, \qquad [Q, H] = 0, \qquad [Q^\dagger, H] = 0, \qquad \{Q, Q\} = 0, \quad \{Q^\dagger, Q^\dagger\} = 0, \tag{6}$$

where the supercharges $Q, Q^\dagger$ define the Grassmann odd generators of the algebra. In one dimension the supercharges can be given the coordinate representation

$$Q := A\psi = \left(\frac{d}{dx} + W(x)\right)\psi, \qquad Q^\dagger := A^\dagger\psi^\dagger = \left(-\frac{d}{dx} + W(x)\right)\psi^\dagger, \tag{7}$$

where $W(x)$ is chosen to be real and is known as the superpotential and ψ and $\psi^\dagger$ are nilpotent Grassmann odd elements satisfying the algebra

$$\{\psi, \psi^\dagger\} = 1, \qquad \{\psi, \psi\} = 0 = \{\psi^\dagger, \psi^\dagger\}. \tag{8}$$

In one dimension the algebra (8) can be explicitly realized through the identification

$$\psi \to \sigma_+, \qquad \psi^\dagger \to \sigma_-, \tag{9}$$

where $\sigma_\pm = (\sigma_1 \pm i\sigma_2)/2$ with σ_k, $k = 1, 2, 3$, denoting the Pauli matrices. The supersymmetric Hamiltonian can now be constructed from (6) to have the form

$$2H = -\frac{d^2}{dx^2} + W^2(x) + W'(x)\sigma_3, \tag{10}$$

with the supersymmetric partner Hamiltonians of the form

$$2H_\pm = -\frac{d^2}{dx^2} + W^2(x) \pm W'(x), \tag{11}$$

where "prime" denotes differentiation with respect to x. It is worth emphasizing here that the nilpotency of the supercharges in the explicit realization (7) is guaranteed by the nilpotency $\psi^2 = 0 = (\psi^\dagger)^2$ of the Grassmann odd elements.

We note here that although our discussion here has assumed that the superpotential is a real function as is conventional, this can be relaxed. If $W(x)$ is a complex superpotential, then we can write

$$Q = \left(\frac{d}{dx} + W(x)\right)\sigma_+, \qquad Q^\dagger = \left(-\frac{d}{dx} + W^*(x)\right)\sigma_-, \tag{12}$$

which leads to the supersymmetric Hamiltonian

$$\begin{aligned}2H = \{Q, Q^\dagger\} &= -\frac{d^2}{dx^2} - 2i\,\mathrm{Im}\,W(x)\frac{d}{dx} - i\,\mathrm{Im}\,W'(x) + WW^* + \mathrm{Re}\,W'(x)\sigma_3\\ &= -\left(\frac{d}{dx} + i\,\mathrm{Im}\,W(x)\right)^2 + \left(\mathrm{Re}\,W(x)\right)^2 + \mathrm{Re}\,W'(x)\sigma_3.\end{aligned} \tag{13}$$

with the supersymmetric partner Hamiltonians of the form

$$2H_{\pm} = -\left(\frac{d}{dx} + i\,\mathrm{Im}\,W(x)\right)^2 + \left(\mathrm{Re}\,W(x)\right)^2 \pm \mathrm{Re}\,W'(x). \tag{14}$$

The Hamiltonian is by construction Hermitian even though the superpotential is complex and (13) shows that the imaginary part of the superpotential behaves like a gauge potential, $\mathcal{A} = -\mathrm{Im}\,W$, and when the superpotential is real, (13) reduces to (10). Furthermore, the zero-energy ground state of the theory, satisfying $Q\phi_0(x) = 0$ or $Q^\dagger\phi_0(x) = 0$ with the nonzero element proportional to (note that ϕ_0 is a two component wave function)

$$\phi_0(x) \sim e^{\int^x dx' (\mp \mathrm{Re}\,W(x') + i\,\mathrm{Im}\,W(x'))}, \tag{15}$$

is normalizable only if $\mathrm{Re}\,W(x)$ is an odd function of x which does not vanish too fast for $|x| \to \infty$, without any restriction on the imaginary part of the superpotential. Simple examples of unbroken supersymmetry are given by the superpotentials $W(x) = x + ia$, $W(x) = x^3 + iax^2$ and so on.

This construction can be generalized to two dimensions quite easily. Let us label the two-dimensional coordinates as $x^k, k = 1, 2$, and introduce two complex superpotentials W_k. Defining $A_k := \partial_k + W_k(x)$ and identifying $\psi_k = (-1)^{k+1}\frac{\sigma_k}{2}$, we can write the manifestly rotation invariant supercharges as ($k, l = 1, 2$ and repeated indices are summed)

$$\begin{aligned} Q &= A_k(\delta_{kl} - i\epsilon_{kl})\psi_l = (\partial_k + W_k)(\delta_{kl} - i\epsilon_{kl})(-1)^{l+1}\frac{\sigma_l}{2} = (A_1 + iA_2)\sigma_+ = \left(\partial_1 + i\partial_2 + (W_1 + iW_2)\right)\sigma_+, \\ Q^\dagger &= (A_1^\dagger - iA_2^\dagger)\sigma_- = \left(-\partial_1 + i\partial_2 + (W_1^* - iW_2^*)\right)\sigma_-, \end{aligned} \tag{16}$$

where $\epsilon_{12} = 1 = -\epsilon_{21}$ and $\epsilon_{11} = 0 = \epsilon_{22}$.

Notice that the two complex supercharges are nilpotent by construction, and they can also be written in the form

$$Q = \left((\partial_1 - i\mathcal{A}_1) + i(\partial_2 - i\mathcal{A}_2)\right)\sigma_+, \qquad Q^\dagger = \left(-(\partial_1 - i\mathcal{A}_1) + i(\partial_2 - i\mathcal{A}_2)\right)\sigma_-, \tag{17}$$

where we have introduced the equivalent gauge fields

$$\mathcal{A}_k = -(\mathrm{Im}\,W_k + \epsilon_{kl}\,\mathrm{Re}\,W_l). \tag{18}$$

The supersymmetric Hamiltonian can now be easily obtained from the superalgebra

$$2H = \{Q, Q^\dagger\} = -(\partial_1 - i\mathcal{A}_1)^2 - (\partial_2 - i\mathcal{A}_2)^2 + \mathcal{F}_{12}\sigma_3 = -\mathbf{D}\cdot\mathbf{D} + \mathcal{F}_{12}\sigma_3, \tag{19}$$

where we have identified the field strength

$$\mathcal{F}_{12} = \partial_1\mathcal{A}_2 - \partial_2\mathcal{A}_1. \tag{20}$$

Therefore, this supersymmetric system behaves like a minimally gauge coupled system with a magnetic dipole interaction. The planar Landau model is a typical example of such a system. Notice that only the combinations of real and imaginary parts of the superpotentials in Eq. (18) enter in the expression of the Hamiltonian (as well as the supercharges).

Although more realistic physical systems with supersymmetry would correspond to three-dimensional quantum mechanical systems [12–14], surprisingly there is only a few isolated discussions about higher dimensional SUSYQM. A construction similar to the discussions above can be generalized to three dimensions in ordinary space using suitable generalizations of (8) and (9). In this case, the space of the Grassmann elements needs to be enlarged. Indeed, in three dimensions we can proceed as in [15]. Introducing three A_k's with $k = 1, 2, 3$ corresponding to the three coordinates, we can write the supercharges as

$$Q = \sum_{k=1}^{3} A_k\psi_k, \qquad Q^\dagger = \sum_{k=1}^{3} A_k^\dagger\psi_k^\dagger, \tag{21}$$

where as in (7) we have assumed that the superpotential is complex and have identified

$$A_k := \left(\partial_k + W_k(\mathbf{x})\right), \qquad A_k^\dagger := \left(-\partial_k + W_k^*(\mathbf{x})\right), \quad k = 1, 2, 3. \tag{22}$$

Clearly, in this case, we have three distinct complex superpotentials $W_k(\mathbf{x})$ and six Grassmann odd elements $\psi_k, \psi_k^\dagger$.

The nilpotency of the supercharges in (21) requires that

$$\psi_k\psi_l = 0 = \psi_k^\dagger\psi_l^\dagger, \tag{23}$$

for any pair of $k, l = 1, 2, 3$, which in turn implies the standard Grassmann property

$$\{\psi_k, \psi_l\} = 0 = \{\psi_k^\dagger, \psi_l^\dagger\}. \tag{24}$$

However, it is worth noting that (23) is a stronger condition than (24) and this seems not to have been appreciated in the past. A realization of the Grassmann odd elements satisfying (23) leads to

$$\psi_k = \sigma_k \otimes \sigma_+, \qquad \psi_k^\dagger = \sigma_k \otimes \sigma_-, \quad k = 1, 2, 3, \tag{25}$$

and the supersymmetric Hamiltonian H can now be obtained from the algebra (6) to correspond to

$$2H := \{Q, Q^\dagger\} = \frac{1}{2}\{A_k, A_l^\dagger\}[\psi_k, \psi_l^\dagger] + \frac{1}{2}[A_k, A_l^\dagger][\psi_k, \psi_l^\dagger]. \tag{26}$$

Using the explicit realization of the Grassmann elements in (25) we now obtain

$$\{\psi_k,\psi_l^\dagger\}=\delta_{kl}\mathbf{1}_2\otimes\mathbf{1}_2+\imath\epsilon_{klm}\sigma_m\otimes\sigma_3,\qquad [\psi_k,\psi_l^\dagger]=\delta_{kl}\,\mathbf{1}_2\otimes\sigma_3+\imath\epsilon_{klm}\sigma_m\otimes\mathbf{1}_2,\tag{27}$$

and substituting these into (26) the supersymmetric Hamiltonian for the system takes the general form

$$\begin{aligned}2H&=\big(-(\partial_k+i\,\mathrm{Im}\,W_k)^2+(\mathrm{Re}\,W_k)^2\big)\mathbf{1}_2\otimes\mathbf{1}_2+(\partial_k\,\mathrm{Re}\,W_k)\mathbf{1}_2\otimes\sigma_3\\&\quad+\epsilon_{klm}\big(i\big((\partial_k\,\mathrm{Re}\,W_l)+2\,\mathrm{Re}\,W_l\partial_k-2i\,\mathrm{Im}\,W_k\,\mathrm{Re}\,W_l\big)\sigma_m\otimes\sigma_3+(\partial_k\,\mathrm{Im}\,W_l)\sigma_m\otimes\mathbf{1}_2\big)\\&=\big(-\mathbf{D}^2+(\mathrm{Re}\,\mathbf{W})^2\big)\mathbf{1}_2\otimes\mathbf{1}_2+(\nabla\cdot\mathrm{Re}\,\mathbf{W})\mathbf{1}_2\otimes\sigma_3-i(\mathrm{Re}\,\mathbf{W}\times\mathbf{D}-\mathbf{D}\times\mathrm{Re}\,\mathbf{W})\cdot\boldsymbol{\sigma}\otimes\sigma_3-\frac{1}{2}\epsilon_{klm}\mathcal{F}_{kl}\sigma_m\otimes\mathbf{1}_2,\end{aligned}\tag{28}$$

where we have identified

$$\mathbf{D}=\nabla+i\,\mathrm{Im}\,\mathbf{W}=\nabla-i\boldsymbol{\mathcal{A}},\qquad \mathcal{F}_{ij}=\partial_i\mathcal{A}_j-\partial_j\mathcal{A}_i.\tag{29}$$

The supersymmetric partner Hamiltonians can now be obtained from (28) to correspond to

$$2H_\pm=\big(-\mathbf{D}^2+(\mathrm{Re}\,\mathbf{W})^2\pm(\nabla\cdot\mathrm{Re}\,\mathbf{W})\big)\mathbf{1}_2-\frac{1}{2}\epsilon_{klm}\mathcal{F}_{kl}\sigma_m\mp i(\mathrm{Re}\,\mathbf{W}\times\mathbf{D}-\mathbf{D}\times\mathrm{Re}\,\mathbf{W})\cdot\boldsymbol{\sigma}.\tag{30}$$

We note here that a sufficient condition for the invariance of the Hamiltonian under supersymmetry transformations is the nilpotency of the supercharges, which is ensured by (23). Indeed, from (26) it follows that under supersymmetry transformations with Grassmann parameters ε and $\bar{\varepsilon}$, the change in the Hamiltonian is given by

$$[\varepsilon Q+\bar{\varepsilon}Q^\dagger,H]=\varepsilon\big[Q^2,Q^\dagger\big]+\bar{\varepsilon}\big[(Q^\dagger)^2,Q\big],\tag{31}$$

which clearly vanishes if $Q^2=0=(Q^\dagger)^2$. We will show this in detail in Appendix A and simply note here that this continues to be true in the non-commutative case as well.

3. Non-commutative supersymmetric quantum mechanics

In this section we will extend the construction of general SUSYQM of the last section to non-commutative space. We will take the supersymmetry algebra to correspond to the graded algebra of (6) except that the brackets would now involve $\star$-product defined in (1). Thus, for example, we have

$$2H^{(\mathrm{NC})}:=\{Q,Q^\dagger\}_\star=Q\star Q^\dagger+Q^\dagger\star Q,\qquad \{Q,Q\}_\star=0=\{Q^\dagger,Q^\dagger\}_\star.\tag{32}$$

We note that the non-commutative spaces we are interested in are of spatial dimension two or higher (since we are considering non-commutativity only for space coordinates).

3.1. Two-dimensional case

Let us consider a simple non-commutative manifold with two spatial dimensions. In such a case, the parameter of non-commutativity can be characterized by a single parameter θ through the identification

$$\theta^{ij}=\theta\epsilon^{ij},\quad i,j=1,2,\tag{33}$$

so that the basic commutator of coordinates (2) reduces to

$$\big[x^i,x^j\big]_\star=\imath\theta\epsilon^{ij}.\tag{34}$$

In such a case, the Bopp shift (4) takes the simple form

$$x^i\to x^i-\frac{\theta}{2}\epsilon^{ij}p_j,\tag{35}$$

where the momentum operator can be identified in the coordinate representation with $p_i=-i\partial_i$.

To construct the general non-commutative SUSYQM model in two dimensions, we take the supersymmetric charges already constructed in (16) with two general complex superpotentials. The Hamiltonian now can be obtained in a straightforward manner to be

$$2H^{(\mathrm{NC})}=\{Q,Q^\dagger\}_\star=\big(-(\partial_1-i\mathcal{A}_1)_\star^2-(\partial_2-i\mathcal{A}_2)_\star^2\big)\mathbf{1}_2+\big(\partial_1\mathcal{A}_2-\partial_2\mathcal{A}_1-i[\mathcal{A}_1,\mathcal{A}_2]_\star\big)\sigma_3=-(\mathbf{D}\cdot\mathbf{D})_\star\mathbf{1}_2+\mathcal{F}_{12}\sigma_3,\tag{36}$$

with the supersymmetric partner Hamiltonians

$$2H_\pm^{(\mathrm{NC})}=-(\mathbf{D}\cdot\mathbf{D})_\star\pm\mathcal{F}_{12},\tag{37}$$

where $\mathcal{A}_k$, $k=1,2$, are defined in (20) and we have identified

$$\mathcal{F}_{12}=\partial_1\mathcal{A}_2-\partial_2\mathcal{A}_1-i[\mathcal{A}_1,\mathcal{A}_2]_\star,\tag{38}$$

which is the generalization of the field strength tensor associated with the Abelian gauge field in a non-commutative theory. The Moyal products can be evaluated using the Bopp shift (35)

$$W_k(\mathbf{x})\to W_k\left(x^1-\frac{\theta}{2}p_2,x^2+\frac{\theta}{2}p_1\right).\tag{39}$$

so that (36) gives the most general (depending on the choice of $W_k(\mathbf{x})$) two-dimensional supersymmetric Hamiltonian in the non-commutative space.

We note here that one of the key ingredients in solving exactly the spectrum of the one-dimensional problems in the conventional case is the connection between the ground state wave function and the superpotential. In the conventional two-dimensional problems, something similar can also be done if one finds normalizable ground state solutions satisfying $Q\phi_0(x) = 0$ or $Q^\dagger\phi_0(x) = 0$. In the non-commutative case, however, the ground state equations involve Moyal products which cannot be easily integrated since (contrary to the conventional case) they contain arbitrarily many derivatives because of the Bopp shift.

As an example of the general two-dimensional theory, let us consider the SUSY Landau problem in the non-commutative space. The Hamiltonian describes the motion of a charged particle on a plane moving under the influence of a constant magnetic field. Therefore, in the commutative space we can choose the vector potentials in the symmetric gauge,

$$\mathcal{A}_1 = -\frac{B}{2}x^2, \qquad \mathcal{A}_2 = \frac{B}{2}x^1, \tag{40}$$

where B represents the constant magnetic field. (We assume $B > 0$ for simplicity.) This can be achieved (see (16), (19)) by choosing, for example, $\operatorname{Re} W_i = 0, \mathcal{A}_i = -\operatorname{Im} W_i$. In this case, we have

$$\begin{aligned} D_i &= \partial_i + \frac{i}{2}B\epsilon_{ij}x^j = i\left(p_i + \frac{B}{2}\epsilon_{ij}x^j\right), \qquad Q = i\left(p_1 + ip_2 - \frac{iB}{2}(x^1 + ix^2)\right)\sigma_+, \\ Q^\dagger &= -i\left(p_1 - ip_2 + \frac{iB}{2}(x^1 - ix^2)\right)\sigma_-. \end{aligned} \tag{41}$$

With these, the Hamiltonian for the Landau problem in commutative space can be obtained to be (see (19))

$$2H = \left(\mathbf{p}^2 + \left(\frac{B}{2}\right)^2\mathbf{x}^2 - BL\right)\mathbf{1}_2 + B\sigma_3, \tag{42}$$

where L denotes the third component of the orbital angular momentum defined to be $L = (x^1p_2 - x^2p_1)$.

Let us note [16,17] that the operator within the parenthesis in (42)

$$\mathcal{H} := \mathbf{p}^2 + \left(\frac{B}{2}\right)^2\mathbf{x}^2 - BL = H_0 - BL, \tag{43}$$

can be formally identified with the Hamiltonian of a two-dimensional isotropic harmonic oscillator of mass $\frac{1}{2}$ and frequency B subjected to an interaction involving the (orbital) angular momentum L. Defining the number operators $N_\pm = a_\pm^\dagger a_\pm$ with $a_\pm = (a_y \pm ia_x)/\sqrt{2}$, we can write the angular momentum and the Hamiltonian operators as $L = N_+ - N_-$ and $H_0 = B(N_+ + N_- + 1)$. As a result, these operators are diagonal in the basis of the number operators and lead to [16,17]

$$H_0|n_+, n_-\rangle = B(n_+ + n_- + 1)|n_+, n_-\rangle, \qquad L|n_+, n_-\rangle = (n_+ - n_-)|n_+, n_-\rangle, \qquad \mathcal{H}|n_+, n_-\rangle = B(2n_- + 1)|n_+, n_-\rangle, \tag{44}$$

with $n_\pm = 0, 1, 2, \ldots$. We note that the spectrum of $\mathcal{H}$ does not depend on the label n_+ and, correspondingly, its eigenvectors are infinitely degenerate.

In the non-commutative space the Hamiltonian for the SUSY Landau problem in the symmetric gauge (40) is obtained from (36) to be

$$2H^{(\mathrm{NC})} = \left(\mathbf{p}^2 + \left(\frac{B}{2}\right)^2\mathbf{x}^2 - BL\right)\mathbf{1}_2 + \mathcal{F}_{12}\sigma_3. \tag{45}$$

We note here from (38) that the symmetric gauge in (40) corresponds to a non-commutative magnetic field of strength

$$\mathcal{F}_{12} = \mathcal{B} = B\left(1 + \frac{\theta B}{4}\right). \tag{46}$$

Namely, the magnetic field in the non-commutative case is scaled by a factor of $(1 + \frac{\theta B}{4})$ compared to the commutative case. Furthermore, the Hamiltonian operator in (45) acts on the wave function through the Moyal product. Alternatively, we can use the Bopp shift (35), to write an equivalent Hamiltonian which would act on the wave functions through ordinary products. Note that under the Bopp shift (35), we have

$$\mathbf{p}^2 \star \phi(\mathbf{x}) = \mathbf{p}^2\phi(\mathbf{x}), \qquad \mathbf{x}^2 \star \phi(\mathbf{x}) = \left(\mathbf{x}^2 + \frac{\theta^2}{4}\mathbf{p}^2 - \theta L\right)\phi(\mathbf{x}), \qquad L \star \phi(\mathbf{x}) = \left(L - \frac{\theta}{2}\mathbf{p}^2\right)\phi(\mathbf{x}), \tag{47}$$

so that the equivalent Hamiltonian with an ordinary product (see (45)) can be written as

$$2H^{(\mathrm{NC})} = \left(\mathbf{p}^2\left(1 + \frac{\theta B}{4}\right)^2 + \left(\frac{B}{2}\right)^2\mathbf{x}^2 - B\left(1 + \frac{\theta B}{4}\right)L\right)\mathbf{1}_2 + \mathcal{B}\sigma_3 = \left(\bar{\mathbf{p}}^2 + \left(\frac{\mathcal{B}}{2}\right)^2\bar{\mathbf{x}}^2 - \mathcal{B}L\right)\mathbf{1}_2 + \mathcal{B}\sigma_3, \tag{48}$$

where we have identified the scaled canonical variables

$$\bar{\mathbf{p}} = \mathbf{p}\left(1 + \frac{\theta B}{4}\right), \qquad \bar{\mathbf{x}} = \frac{\mathbf{x}}{(1 + \frac{\theta B}{4})}. \tag{49}$$

Thus, in the variables (49), the non-commutative SUSY Landau problem is equivalent to the standard SUSY Landau problem with the magnetic field scaled as in (46). The energy spectrum and the energy eigenvectors can be easily computed in terms of those of an isotropic Harmonic oscillator of mass 1/2 and frequency $\omega = \mathcal{B}$, as in (44). For the supersymmetric partner Hamiltonians we get

$$H_\pm^{(\mathrm{NC})}|n_+, n_-, \pm\rangle = E_{n_-, n_+, \pm}|n_+, n_-, \pm\rangle, \tag{50}$$

with (see (44))

$$E_{n_-,n_+,\pm} = \frac{\mathcal{B}}{2}(2n_- + 1) \pm \frac{\mathcal{B}}{2}, \tag{51}$$

which does not depend on n_+, corresponding to an infinite degeneracy of states.

A second example is a particle moving in a linear superpotential

$$W_i = \frac{\alpha}{2} x_i, \tag{52}$$

which in the commutative case leads to the SUSY isotropic two-dimensional harmonic oscillator. In this case, we take $\mathrm{Im}\, W_i = 0$ and we identify $\mathcal{A}_i = -\epsilon_{ij} \mathrm{Re}\, W^j$, so that the supercharges can be written as

$$Q = i\left(p_1 + ip_2 - \frac{i\alpha}{2}(x^1 + ix^2)\right)\sigma_+, \qquad Q^\dagger = -i\left(p_1 - ip_2 + \frac{i\alpha}{2}(x^1 - ix^2)\right)\sigma_-. \tag{53}$$

Comparing these supercharges with (41), we see that this problem can be mapped to the SUSY Landau problem with the identification $\alpha = B$ so that the earlier analysis can be carried over in a straightforward manner.

3.2. Three-dimensional case

To construct a general three-dimensional quantum mechanical theory on a non-commutative space which would provide a realization of the algebra (32) we follow the method discussed in the previous section. For example, we introduce three arbitrary superpotentials and define the supercharges as in (21) and (22). The nilpotency of the supercharges

$$Q \star Q = (A_k \star A_l)\psi_k\psi_l = 0 = Q^\dagger \star Q^\dagger = (A_k^\dagger \star A_l^\dagger)\psi_k^\dagger\psi_l^\dagger, \tag{54}$$

requires as before that

$$\psi_k\psi_l = 0 = \psi_k^\dagger\psi_l^\dagger. \tag{55}$$

As a result, (25) continues to provide a realization for the Grassmann odd elements $\psi_k, \psi_k^\dagger, k = 1, 2, 3$. As before, we can consider a complex superpotential so that we can write

$$A_k = \partial_k + W_k, \qquad A_k^\dagger = -\partial_k + W_k^*. \tag{56}$$

With these the Hamiltonian for the supersymmetric theory, in the present case, can be obtained to be

$$\begin{aligned} 2H^{(\mathrm{NC})} &= (A_k \star A_l^\dagger)\psi_k\psi_l^\dagger + (A_l^\dagger \star A_k)\psi_l^\dagger\psi_k = \frac{1}{2}\{A_k, A_l^\dagger\}_\star\{\psi_k, \psi_l^\dagger\} + \frac{1}{2}[A_k, A_l^\dagger]_\star[\psi_k, \psi_l^\dagger] \\ &= \left[(\mathbf{p} - \boldsymbol{\mathcal{A}})_\star^2 + (\mathrm{Re}\,\mathbf{W})_\star^2\right]\mathbf{1}_2 \otimes \mathbf{1}_2 + (\mathbf{D}\cdot \mathrm{Re}\,\mathbf{W})_\star \mathbf{1}_2 \otimes \sigma_3 + \epsilon_{klm}\left(-\frac{1}{2}\mathcal{F}_{kl} + i(\mathrm{Re}\,W_k)\star(\mathrm{Re}\,W_l)\right)\sigma_m \otimes \mathbf{1}_2 \\ &\quad + \left[\mathrm{Re}\,\mathbf{W} \times (\mathbf{p} - \boldsymbol{\mathcal{A}}) - (\mathbf{p} - \boldsymbol{\mathcal{A}}) \times \mathrm{Re}\,\mathbf{W}\right]_\star \cdot \boldsymbol{\sigma} \otimes \sigma_3, \end{aligned} \tag{57}$$

where we have identified

$$\mathbf{p} = -i\nabla, \qquad \boldsymbol{\mathcal{A}} = -\mathrm{Im}\,\mathbf{W}, \qquad \mathbf{D} = \nabla - i\boldsymbol{\mathcal{A}}, \qquad \mathcal{F}_{kl} = \partial_k\mathcal{A}_l - \partial_l\mathcal{A}_k - i[\mathcal{A}_k, \mathcal{A}_l]_\star. \tag{58}$$

For any arbitrary complex (three-dimensional) vector superpotential $\mathbf{W}(\mathbf{x})$, (57) represents the Hamiltonian invariant under the supersymmetry transformation generated by the Grassmann odd generators $Q, Q^\dagger$ (through the $\star$-operation). Indeed, since the Moyal product is associative, from (32) we have

$$[\varepsilon Q + \bar{\varepsilon} Q^\dagger, H]_\star = \varepsilon\left[(Q)_\star^2, Q^\dagger\right]_\star + \bar{\varepsilon}\left[(Q^\dagger)_\star^2, Q\right]_\star = 0, \tag{59}$$

for arbitrary Grassmann parameters ε and $\bar{\varepsilon}$ which follows from the nilpotency of the supercharges guaranteed by the realization (23). (The general SUSY transformations in the non-commutative space are discussed in more detail in Appendix B.) Let us note that the terms in (57) can be rearranged to identify the supersymmetric partner Hamiltonians in the present case to be

$$\begin{aligned} 2H_\pm^{(\mathrm{NC})} &= \left((\mathbf{p} - \boldsymbol{\mathcal{A}})_\star^2 + (\mathrm{Re}\,\mathbf{W})_\star^2 \pm (\mathbf{D}\cdot\mathrm{Re}\,\mathbf{W})_\star\right)\mathbf{1}_2 + \epsilon_{klm}\left(-\frac{1}{2}\mathcal{F}_{kl} + i(\mathrm{Re}\,W_k)\star(\mathrm{Re}\,W_l)\right)\sigma_m \\ &\quad \pm \left(\mathrm{Re}\,\mathbf{W} \times (\mathbf{p} - \boldsymbol{\mathcal{A}}) - (\mathbf{p} - \boldsymbol{\mathcal{A}}) \times \mathrm{Re}\,\mathbf{W}\right)_\star \cdot \boldsymbol{\sigma}. \end{aligned} \tag{60}$$

For completeness, we note here that the action of the generators of supersymmetry on functions of the position through the $\star$-operation can be obtained by using the Bopp shift defined in (4) as

$$\begin{aligned} Q \star \phi(\mathbf{x}) &= A_k\psi_k \star \phi(\mathbf{x}) = \left(A_k \star \phi(\mathbf{x})\right)\psi_k = \left(\partial_k + W_k(\mathbf{x} - \bar{\mathbf{p}}/2)\right)\phi(\mathbf{x})\sigma_k \otimes \sigma_+, \\ Q^\dagger \star \phi(\mathbf{x}) &= A_k^\dagger\psi_k^\dagger \star \phi(\mathbf{x}) = \left(A_k^\dagger \star \phi(\mathbf{x})\right)\psi_k^\dagger = \left(-\partial_k + W_k^*(\mathbf{x} - \bar{\mathbf{p}}/2)\right)\phi(\mathbf{x})\sigma_k \otimes \sigma_- \end{aligned} \tag{61}$$

and similarly for the $\star$-action of other operators on functions ($\bar{\mathbf{p}}$ is defined following (4)).

With this construction of the general three-dimensional SUSYQM Hamiltonian in the non-commutative space, we will now discuss one simple example. Although the three-dimensional case in principle is straightforward, finding soluble examples is more difficult than in two-dimensions. We will consider the isotropic three-dimensional harmonic oscillator, for which the superpotential is given by (see (52))

$$\mathbf{W}(\mathbf{x}) = \frac{\alpha}{2}\mathbf{x}, \tag{62}$$

with $\alpha \in \mathbb{R}$. In this case, therefore, the superpotential is real and we have $\mathcal{A} = -\operatorname{Im}\mathbf{W} = 0$. As a result, the Hamiltonian in (57) takes the simpler form

$$2H^{(\mathrm{NC})} = \left(\mathbf{p}^2 + (\mathbf{W})^2_\star\right)\mathbf{1}_2 \otimes \mathbf{1}_2 + (\nabla \cdot \mathbf{W})\mathbf{1}_2 \otimes \sigma_3 + i(\mathbf{W} \times \mathbf{W})_\star \cdot \boldsymbol{\sigma} \otimes \mathbf{1}_2 + (\mathbf{W} \times \mathbf{p} - \mathbf{p} \times \mathbf{W}) \cdot \boldsymbol{\sigma} \otimes \sigma_3. \tag{63}$$

The Moyal (star) products can be evaluated through the Bopp shift. In fact, noting that in three dimensions, the parameter of non-commutativity can be identified with a vector as

$$\theta^{ij} = \epsilon^{ijk}\theta_k, \tag{64}$$

under a Bopp shift we can write (see (4))

$$f(\mathbf{x}) \to f\left(\mathbf{x} + \frac{1}{2}\boldsymbol{\theta} \times \mathbf{p}\right). \tag{65}$$

Using this the Hamiltonian in (63) can be written in the simple form

$$\begin{aligned} 2H^{(\mathrm{NC})} = {} & \left(\left(1 + \left(\frac{\alpha}{4}\right)^2 \theta^2\right)\mathbf{p}_\perp^2 + \mathbf{p}_\parallel^2 + \frac{\alpha^2}{4}\mathbf{x}^2 - \frac{\alpha^2}{4}\boldsymbol{\theta} \cdot \mathbf{L}\right)\mathbf{1}_2 \otimes \mathbf{1}_2 + \frac{3\alpha}{2}\mathbf{1}_2 \otimes \sigma_3 \\ & - \frac{\alpha^2}{4}\boldsymbol{\theta} \cdot \boldsymbol{\sigma} \otimes \mathbf{1}_2 + \frac{\alpha}{2}\left(2\mathbf{L} - \mathbf{p}^2\boldsymbol{\theta} + (\boldsymbol{\theta} \cdot \mathbf{p})\mathbf{p}\right) \cdot \boldsymbol{\sigma} \otimes \sigma_3. \end{aligned} \tag{66}$$

Here we have denoted the parallel component of $\mathbf{p}$ along the direction $\boldsymbol{\theta}$ by $\mathbf{p}_\parallel = \frac{\boldsymbol{\theta} \cdot \mathbf{p}}{\theta^2}\boldsymbol{\theta}$ while the perpendicular component is given by $\mathbf{p}_\perp = \mathbf{p} - \mathbf{p}_\parallel$ and the orbital momentum is defined as usual as $\mathbf{L} = \mathbf{x} \times \mathbf{p}$. The supersymmetric partner Hamiltonians (see (60)) are now determined to be

$$2H_\pm^{(\mathrm{NC})} = \left(\left(1 + \left(\frac{\alpha}{4}\right)^2 \theta^2\right)\mathbf{p}_\perp^2 + \mathbf{p}_\parallel^2 + \frac{\alpha^2}{4}\mathbf{x}^2 - \frac{\alpha^2}{4}\boldsymbol{\theta} \cdot \mathbf{L} \pm \frac{3\alpha}{2}\right)\mathbf{1}_2 + \left(-\frac{\alpha^2}{4}\boldsymbol{\theta} \pm \frac{\alpha}{2}\left(2\mathbf{L} - \mathbf{p}^2\boldsymbol{\theta} + (\boldsymbol{\theta} \cdot \mathbf{p})\mathbf{p}\right)\right) \cdot \boldsymbol{\sigma}. \tag{67}$$

Even in this simple case, the energy spectrum and the eigenfunctions for the dynamical system cannot be obtained in closed form in a straightforward manner as in the two-dimensional case. A perturbative treatment, however, seems possible.

4. Conclusions

In this paper, we have constructed systematically general supersymmetric quantum mechanical models in higher dimensional non-commutative space. The general models require a restricted nilpotency condition for the Grassmann odd elements and allow for complex superpotentials. This is consistent with the explicit realizations chosen in earlier studies where the nature of the algebra was not fully appreciated. Non-commutativity of spatial coordinates is introduced through the (Moyal) $\star$- product or through the Bopp shift. As we go to higher dimensions, we find that it is not easy to find solvable models. However, all these models have the characteristic feature that they introduce interactions involving angular momentum. This can, therefore, be considered as a starting point for several possible applications in diverse fields where the mixture between non-commutativity and supersymmetry could be of interest. For example nuclear, optics and atomic physics are systems where many potentials involving spin and non-local interactions appear and the general framework developed in this paper would possibly prove useful.

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Appendix A. SUSY invariance and the nilpotency of ψ_k and $\psi_k^\dagger$

The aim of the present appendix is to explicitly show that the nilpotency of ψ_k and $\psi_k^\dagger$ is sufficient to ensure the SUSY invariance of the Hamiltonian, even though these elements do not necessarily satisfy a Clifford algebra (see, for example, the explicit realization in (25)). For simplicity, we will describe the proof in the standard (commutative) case with a real superpotential, the proof for the complex superpotential or the non-commutative case can be shown in a parallel manner.

With the supercharges written as in (21) and (22) and assuming the nilpotency of the Grassmann odd elements as in (23) (without using the explicit realization (25)),

$$\psi_k \psi_l = 0, \qquad \psi_k^\dagger \psi_l^\dagger = 0, \tag{A.1}$$

we obtain the SUSY transformations for the basic elements to be

$$\delta x_i = [\varepsilon Q + \bar{\varepsilon} Q^\dagger, x_i] = \varepsilon \psi_i - \bar{\varepsilon} \psi_i^\dagger, \tag{A.2}$$

$$\delta p_i = [\varepsilon Q + \bar{\varepsilon} Q^\dagger, p_i] = i W_{j,i}(\varepsilon \psi_j + \bar{\varepsilon} \psi_j^\dagger), \tag{A.3}$$

$$\delta \psi_i = [\varepsilon Q + \bar{\varepsilon} Q^\dagger, \psi_i] = A_k^\dagger \bar{\varepsilon} \{\psi_k^\dagger, \psi_i\}, \tag{A.4}$$

$$\delta \psi_i^\dagger = [\varepsilon Q + \bar{\varepsilon} Q^\dagger, \psi_i^\dagger] = A_k \varepsilon \{\psi_k, \psi_i^\dagger\}. \tag{A.5}$$

Here we have assumed that Q generates supersymmetry transformations with the Grassmann parameter ε while $Q^\dagger$ generates those with parameter $\bar{\varepsilon}$.

For simplicity, we put $\bar{\varepsilon}=0$ and consider only the change in the Hamiltonian in (26) coming from transformations generated by Q (proportional to ε, the neglected terms can be obtained by taking the conjugates of the result obtained in the following),

$$4\delta H = \big(\big\{(\partial_i W_k - \partial_k W_i), \partial_j - W_j\big\} + \big\{(\partial_j W_k + \partial_k W_j), \partial_i + W_i\big\}\big)\varepsilon\psi_k\{\psi_i, \psi_j^\dagger\}$$
$$+ \big(\big[(\partial_i W_k - \partial_k W_i), \partial_j - W_j\big] - \big[(\partial_j W_k + \partial_k W_j), \partial_i + W_i\big]\big)\varepsilon\psi_k[\psi_i, \psi_j^\dagger]$$
$$+ \big(\{\partial_i + W_i, -\partial_j + W_j\}\big\{\psi_i, [\varepsilon\psi_k, \psi_j^\dagger]\big\} + [\partial_i + W_i, -\partial_j + W_j]\big[\psi_i, [\varepsilon\psi_k, \psi_j^\dagger]\big]\big)(\partial_k + W_k). \tag{A.6}$$

In this expression the Grassmann odd elements appear in combinations of products of the type $\psi_k\psi_i\psi_j^\dagger$, $\psi_j^\dagger\psi_i\psi_k$ and $\psi_k\psi_j^\dagger\psi_i$. The first two types of terms vanish by virtue of (A.1), while the third leads to

$$2\delta H = \big((\partial_j - W_j)(\partial_i W_k - \partial_k W_i) + (\partial_j W_k + \partial_k W_j)(\partial_i + W_i)$$
$$+ (-\partial_j + W_j)(\partial_i + W_i)(\partial_k + W_k) - (\partial_k + W_k)(-\partial_j + W_j)(\partial_i + W_i)\big)\varepsilon\psi_k\psi_j^\dagger\psi_i.$$

Noting that

$$[\partial_k \mp W_k, \partial_j + W_j] = (\partial_k W_j \pm \partial_j W_k), \tag{A.7}$$

we find

$$2\delta H = \big([\partial_k - W_k, \partial_j + W_j] + [\partial_k + W_k, \partial_j - W_j]\big)(\partial_i + W_i)\varepsilon\psi_k\psi_j^\dagger\psi_i = 0. \tag{A.8}$$

Therefore, the nilpotency assumed in (A.1) is a sufficient condition for the SUSY invariance of H. Let us emphasize that without (A.1) variations proportional to the products of the type $\psi_k\psi_i\psi_j^\dagger$ and $\psi_j^\dagger\psi_i\psi_k$ do not cancel by themselves and a possibility for their cancellation would require the anticommutator $\{\psi_i, \psi_j^\dagger\}$ to be carefully chosen in a nontrivial manner.

Appendix B. SUSY transformation in non-commutative space

As discussed in Appendix A, the nilpotency of the matrices ψ_k and $\psi_k^\dagger$ in (23) is a sufficient condition for the SUSY-invariance of the Hamiltonian in both, the standard and the non-commutative three-dimensional case.

Let us now consider a general SUSY transformation in the three-dimensional non-commutative space, realized unitarily in terms of the generators of transformation as

$$\big(e^{\varepsilon Q + \bar{\varepsilon} Q^\dagger}\big)_\star := 1 + \varepsilon Q + \bar{\varepsilon} Q^\dagger + \frac{1}{2}\{\varepsilon Q, \bar{\varepsilon} Q^\dagger\}_\star, \tag{B.1}$$

where ε and $\bar{\varepsilon} := \varepsilon^\dagger$ are constant Grassmann odd parameters which anticommute with the supercharges. (The forms of the supercharges are described in Sections 2 and 3.) It can be checked easily that

$$\big(e^{\varepsilon Q + \bar{\varepsilon} Q^\dagger}\big)_\star \star \big(e^{\varepsilon Q + \bar{\varepsilon} Q^\dagger}\big)_\star^\dagger = \left\{1 + \varepsilon Q + \bar{\varepsilon} Q^\dagger + \frac{1}{2}\{\varepsilon Q, \bar{\varepsilon} Q^\dagger\}_\star\right\} \star \left\{1 - \bar{\varepsilon} Q^\dagger - \varepsilon Q + \frac{1}{2}\{\varepsilon Q, \bar{\varepsilon} Q^\dagger\}_\star\right\} = 1. \tag{B.2}$$

The supersymmetric transformation of functions depending on the dynamical variables x^k, ψ_k and $\psi_k^\dagger$ can be obtained from

$$\big(e^{\varepsilon Q + \bar{\varepsilon} Q^\dagger}\big)_\star \star F(x_k, p_k, \psi_k, \psi_k^\dagger) \star \big(e^{\varepsilon Q + \bar{\varepsilon} Q^\dagger}\big)_\star^\dagger = F + [\varepsilon Q + \bar{\varepsilon} Q^\dagger, F]_\star + \frac{1}{2}\big[\varepsilon Q + \bar{\varepsilon} Q^\dagger, [\varepsilon Q + \bar{\varepsilon} Q^\dagger, F]_\star\big]_\star \simeq F + \delta F, \tag{B.3}$$

where we have identified the first order changes in the Grassmann parameters with

$$\delta F = [\varepsilon Q + \bar{\varepsilon} Q^\dagger, F]_\star. \tag{B.4}$$

Explicitly, the SUSY transformations for the coordinates can be obtained from

$$\delta x_i = [\varepsilon Q + \bar{\varepsilon} Q^\dagger, x_i]_\star = (\varepsilon\psi_i - \bar{\varepsilon}\psi_i{}^\dagger) - \imath\theta_{ij}\big(W_{k,j}(\mathbf{x})\varepsilon\psi_k + W^*_{k,j}(\mathbf{x})\bar{\varepsilon}\psi_k{}^\dagger\big), \tag{B.5}$$

where we have denoted

$$W_{k,j}(x) = \partial_j W_k(\mathbf{x}). \tag{B.6}$$

For the derivative operator we obtain

$$\delta\partial_i = [\varepsilon Q + \bar{\varepsilon} Q^\dagger, \partial_i]_\star = -\big(W_{k,i}(\mathbf{x})\varepsilon\psi_k + W^*_{k,i}(\mathbf{x})\bar{\varepsilon}\psi_k{}^\dagger\big). \tag{B.7}$$

For the Grassmann odd elements in (9), relation (B.3) yields for ψ_i

$$\delta\psi_i = [\varepsilon Q + \bar{\varepsilon} Q^\dagger, \psi_i]_\star = \big(-\partial_k + W_k^*(\mathbf{x})\big)\bar{\varepsilon}\{\psi_k{}^\dagger, \psi_i\}, \tag{B.8}$$

while for $\psi_i^\dagger$ we have

$$\delta\psi_i{}^\dagger = [\varepsilon Q + \bar{\varepsilon} Q^\dagger, \psi_i{}^\dagger]_\star = \big(\partial_k + W_k(x)\big)\varepsilon\{\psi_k, \psi_i{}^\dagger\}, \tag{B.9}$$

where the anticommutator $\{\psi_k, \psi_i{}^\dagger\}$ is given in (27). Let us note that in the limit $\theta_{ij} \to 0$, these expressions reduce to the transformations for the standard (commutative) space. Finally, we note that since the $\star$-product is associative, we have

$$\delta(F \star G) = [\varepsilon Q + \bar{\varepsilon} Q^\dagger, F \star G]_\star = \delta F \star G + F \star \delta G. \tag{B.10}$$

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