Journal of Physics: Conference Series

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Journal of Physics: Conference Series **246** (2010) 012030 doi:10.1088/1742-6596/246/1/012030

Diverging Fano factors

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Abstract. We illustrate, with reference to the so-called Fano factor, that dramatic differences between the two most popular phase-space distributions (Wigner's and Husimi's) become evident at the level of the harmonic oscillator treatment.

1. Introduction

In statistics, the Fano factor [1] is a measure of the dispersion of a probability distribution, defined as

$$F = \frac{\sigma_U^2}{\mu_U}, \tag{1}$$

where σ_U^2 is the variance and μ_U is the mean of a random process in some time-window U. The Fano factor can be viewed as a kind of noise-to-signal ratio, being a measure of the reliability with which the random variable can be estimated from a time window that on average contains several random events. For a Poisson process, the variance in the count equals the mean count, so that $F = 1$. If the time window is chosen to be infinity, the Fano factor is similar to the variance-to-mean ratio (VMR) which in statistics is also known as the "index of dispersion", dispersion index or coefficient of dispersion. This is a normalized measure of the dispersion of a probability distribution. In other words, it tells us whether a set of observed occurrences are clustered or dispersed compared to a standard statistical model. As just stated, the Poisson distribution has $F = 1$, but both the geometric and the negative binomial distributions have $F > 1$, while in the binomial instance we find $F < 1$. A constant "random variable" has $F = 0$. The Fano factor turns out to be a convenient noise-indicator of a non-classical field. In the case of a photon distribution (N photons) it reads [2]

$$F \equiv \sigma = \frac{(\Delta N)^2}{\langle \hat{N} \rangle}, \tag{2}$$

and is in this sense related to the so-called Mandel parameter Q [3]

$$Q = \frac{(\Delta N)^2}{\langle \hat{N} \rangle} - 1 \equiv F - 1, \tag{3}$$

In consonance with the above remarks, for $F < 1$ ($Q \leq 0$), emitted light is referred to as sub-Poissonian since it has photo-count noise smaller than that of coherent (ideal laser) light with the same intensity ($F = 1$; $Q = 0$), whereas for $F > 1$, ($Q > 0$) the light is called super-Poissonian, exhibiting photo-count noise higher than the coherent-light noise. Of course, one wishes to minimize the Fano factor.

For a coherent state the Mandel parameter vanishes, i.e., $Q = 0$ and $F = 1$. It is important to note that a field in a coherent state is considered to be the closest possible one to a classical field, since it saturates the Heisenberg uncertainty relation and has the same uncertainty in each quadrature component. Therefore, $Q = 0$ or $F = 1$ define a boundary between a classical and a quantum field. It is clear then that both Q and F can function as indicators on non-classicality. Indeed, for a thermal state one has $Q > 0$ and $F > 1$, corresponding to a photon distribution broader than the Poissonian. For $Q < 0$, ($F < 1$) the photon distribution becomes narrower than that of a Poisson-PDF and the corresponding state is non-classical. The most elementary examples of non-classical states are number states. Since they are eigenstates of the photon number operator $\hat{N}$ the fluctuations in $\hat{N}$ vanish and the Mandel parameter reads $Q = -1$ ($F = 0$) [4].

Here we will concern ourselves with "quantal" phase-space distributions and are forced thus to speak of the celebrated Wigner quasi-probability distribution $D_W(x,p)$ (also called the Wigner function or the Wigner-Ville distribution), a special type of quasi-probability distribution. It was introduced by Eugene Wigner in 1932 [5] to study quantum corrections to classical statistical mechanics. In trying to approximate it in some fashion one is prone to fall into the semiclassical domain. Wigner's goal was to supplant the wave-function that appears in Schröedinger's equation with a probability distribution D_W in phase space. This D should function as a generating function for all spatial autocorrelation functions of a given quantum-mechanical wave-function $\psi(x)$. Thus, in the map between real phase-space functions and Hermitian operators introduced by Hermann Weyl in 1927 [6], D_W maps on the quantum density matrix [6]. One speaks of the Weyl-Wigner transform of the density matrix. In 1949 Moyal [7], who had also re-derived it independently, recognized D_W as the quantum moment-generating functional, *i.e.*, as the basis of an elegant encoding of all quantum expectation values, and hence quantum mechanics in phase space (*Weyl quantization*). It has applications in statistical mechanics, quantum chemistry, quantum optics, classical optics and signal analysis in diverse fields such as electrical engineering, seismology, biology, speech processing, and engine design [8]. Thus, Wigner's is the most elaborate phase-space (PS) formulation of quantum mechanics [5, 9, 10].
A rival phase space distribution is the one developed by Husimi (see Sect. 4 below) [11, 12, 13]. Although both the Wigner and the Husimi distributions carry complete information regarding a quantum state, they exhibit different features. The Wigner function displays large oscillations and may adopt negative values which make it a quasi-distribution rather than a classical probability density. On a compact phase space of area A it is able to reveal fine structures on a sub- Planck scale of order $\hbar^2/A$ [14], structures that can be traced to quantum interferences from distant localized objects [14, 15, 16, 17], that in turn enhance the state's sensitivity to perturbations [14, 15, 16, 17]. Instead, the Husimi distribution is known to be a Gaussian smearing of the Wigner function on an area of size $\hbar$ that washes out the negative part and hence it is suitable as a probability density [18]. However, such smoothing may hide significant important attributes or aspects of the Wigner function [19]. Summing up, while the Wigner function exhibits high resolution, it is not free of long range quantum interferences. The Husimi distribution washes out quantum interferences at the price of hiding important semiclassical structures [19]. We will here illustrate Wigner-Husimi differences in dramatic fashion by recourse to the Fano factor for the harmonic oscillator (HO).

In this work we will compare the Wigner and Husimi distributions in the presence of noise as described by the Fano factor and see that remarkable oddities emerge.

2. Fisher information

The last years have viewed a great deal of effort revolving around physical applications of Fisher's information measure (FIM) [20]. FIM is the source of a powerful variational principle, the extreme physical information one, that yields most of the canonical Lagrangians of theoretical physics [20], characterizing also in quite a proper fashion an "arrow of time", alternative to the one associated with Boltzmann's entropy [21, 22]. The classical Fisher information associated with translations of a one-dimensional observable x with corresponding probability density $\rho(x)$ is [23]

$$I_x = \int \mathrm{d}x\, \rho(x) \left(\frac{\partial \ln \rho(x)}{\partial x}\right)^2, \tag{4}$$

which obeys the so-called Cramer-Rao inequality

$$(\Delta x)^2 \geq I_x^{-1} \tag{5}$$

involving the variance of the stochastic variable x [23]

$$(\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2 = \int \mathrm{d}x\, \rho(x)\, x^2 - \left(\int \mathrm{d}x\, \rho(x)\, x\right)^2. \tag{6}$$

3. Thermal Wigner distribution

It is well known that the Wigner distribution for a general density matrix is given by [24]

$$f_w(x,p) = \int_{-\infty}^{\infty} \mathrm{d}s\, e^{ips/\hbar} \left\langle x - \frac{s}{2} \middle| \hat{\rho} \middle| x + \frac{s}{2} \right\rangle, \tag{7}$$

which is normalized according to $\int (\mathrm{d}^2 z/\pi) f_w(x,p) = 1$, with $\mathrm{d}^2 z = \mathrm{d}x\mathrm{d}p/2\pi\hbar$. The pertinent analytic expression for the harmonic-oscillator thermal density is thus [24]

$$f_w(x,p) = 2\tanh(\beta\hbar\omega/2)\, e^{-2\tanh(\frac{\beta\hbar\omega}{2})|z|^2}, \tag{8}$$

where $|z|^2 = x^2/4\sigma_x^2 + p^2/4\sigma_p^2$, with $\sigma_x^2 = \hbar/2m\omega$ and $\sigma_p^2 = \hbar m\omega/2$.
The HO-Fisher Information measure associated to the thermal Wigner distribution is then [13]

$$I_w = \frac{1}{4} \int \frac{\mathrm{d}^2 z}{\pi}\, f_w(z) \left(\frac{\partial \ln f_w(z)}{\partial |z|}\right)^2, \tag{9}$$

and integrating over phase space we immediately get [25, 26]

$$I_w = 2\tanh(\beta\hbar\omega/2). \tag{10}$$

(Note that $0 \leq I_w^{sc} \leq 2$.) Thus, we realize that the HO-thermal Wigner function can be expressed entirely in terms of the Fisher measure (10) [13], i.e.,

$$f_w(x,p) = I_w\, e^{-I_w |z|^2}. \tag{11}$$

We know [27] that, in phase space terms, the HO-Hamiltonian reads

$$\hat{H} = \hbar\omega |z|^2, \tag{12}$$

so that the mean energy becomes

$$E_{Wigner} = \hbar\omega \int \frac{\mathrm{d}^2 \mathrm{z}}{\pi} |z|^2 f_w(|z|) = \frac{\hbar\omega}{2\tanh(\beta\hbar\omega/2)} = \frac{\hbar\omega}{I_w}. \tag{13}$$

4. Husimi distribution

Although the oldest and most elaborate phase space (PS) formulation of quantum mechanics is the one discussed above [5, 10], the ensuing distribution function can, as has been noted (regrettably enough), assume negative values so that a probabilistic interpretation becomes questionable. Such limitation was overcome, among others, by Husimi [11].
The Husimi probability distributions $\mu(x, p)$ can be regarded as a "smoothed Wigner distributions" [9]. Indeed, $\mu(x, p)$ is a Wigner-distribution smeared over an $\hbar$ sized region of phase-space [27]. Thus, it is an approximation to the "true" distribution that, specifically, neglects effects of order $\hbar^2$ [9]. As a consequence, it should not be surprising to find out that a Husimi treatment yields results different from the exact ones.
In terms of the concomitant Husimi probability distributions, quantum mechanics can be completely reformulated [12, 28, 29, 30]. This phase space distribution has the appearance

$$\mu(x, p) \equiv \mu(z) = \langle z|\hat{\rho}|z\rangle, \tag{14}$$

where $\hat{\rho}$ is the density operator of the system and $|z\rangle$ are the coherent states (see, for instance, [31] and references therein). The function $\mu(x, p)$ is normalized in the fashion

$$\int \frac{\mathrm{d}^2 z}{\pi}\, \mu(x, p) = 1. \tag{15}$$

For a thermal equilibrium case $\hat{\rho} = Z^{-1} e^{-\beta \hat{H}}$, $Z = \mathrm{Tr}(e^{-\beta\hat{H}})$ is the partition function and $\beta = 1/T$. Specializing things for the HO of frequency ω, with eigenstates $|n\rangle$ associated to the eigenenergies $E_n = \hbar\omega\,(n + 1/2)$, one has [13]

$$\langle z|\hat{\rho}|z\rangle = \frac{1}{Z} \sum_n e^{-\beta\hat{H}} |\langle z|n\rangle|^2, \tag{16}$$

and the normalized Husimi probability distribution is [13]

$$\mu(z) = (1 - e^{-\beta\hbar\omega})\, e^{-(1-e^{-\beta\hbar\omega})|z|^2}. \tag{17}$$

The Fisher information associated to an Husimi PDF has been derived in Ref. [27]. One starts with

$$I_h = \frac{1}{4} \int \frac{\mathrm{d}^2 z}{\pi}\, \mu(z) \left(\frac{\partial \ln \mu(z)}{\partial |z|} \right)^2, \tag{18}$$

so that inserting the $\mu-$expression into (18) we straightforwardly find for the HO-case the analytic form [27]

$$I_h = 1 - e^{-\beta\hbar\omega}, \tag{19}$$

leading to the following limits: for $T \to 0$ one has $I_h = 1$ and for $T \to \infty$ one has $I_h = 0$ as it should be expected (null information at infinite temperature). On the other hand, if we evaluate the mean energy with the Husimi distribution we obtain [27]

$$E_{Husimi} = \hbar\omega \int \frac{\mathrm{d}^2 \mathrm{z}}{\pi} |z|^2 \mu(|z|) = \frac{\hbar\omega}{1 - e^{-\beta\hbar\omega}} = \frac{\hbar\omega}{I_h}. \tag{20}$$

Remember that on a compact phase space of area A the Wigner function reveals fine structures on a sub- Planck scale of order $\hbar^2/A$ [14], while the Husimi distribution is known to be a Gaussian smearing of the Wigner function on an area of size $\hbar$ that washes out the negative Wigner parts and hence it is suitable as a probability density [18]. However, such smoothing may hide significant important attributes or aspects of the Wigner function [19].

5. HO-Canonical ensemble: energy fluctuation

It is of the essence for our present purposes to revisit the HO-canonical ensemble machinery [32], which yields correct statistical mechanical results. For practical purposes we here from use the abbreviated notation

$$\Omega = \hbar\omega; \;\; b = \beta\Omega; \;\; e_n = \Omega(n + 1/2); \;\; \text{and} \;\; I_h^{-1} = \frac{1}{1 - e^{-b}}. \tag{21}$$

Thus, the canonical partition function reads [32]

$$Z = \sum_n \exp\left[-b\,(n + 1/2)\right] = e^{(-b/2)}\, I_h^{-1}, \tag{22}$$

and the mean energy is [32]

$$\overline{E} = -\frac{d}{d\beta}(\ln Z) = \frac{\Omega}{2} + \frac{\Omega}{e^b - 1} = \frac{\Omega(2 - I_h)}{2I_h}. \tag{23}$$

We arrive now at the vital, decisive stage of our communication. The crux of the matter we are concerned with here is that of the energy fluctuation [32],

$$(\Delta H)^2 = \overline{(E^2)} - (\overline{E})^2 = -\frac{\partial \overline{E}}{\partial \beta}, \tag{24}$$

that implies

$$(\Delta H)^2 = \frac{\Omega^2 e^b}{(e^b - 1)^2} = \frac{\Omega^2(1 - I_h)}{I_h^2}. \tag{25}$$

It is now evident that the canonical HO distribution adopts the appearance

$$f(\epsilon_n) = \frac{e^{-b(n+1/2)}}{Z} = I_h\, \exp\left(-b\, n\right), \tag{26}$$

which does not coincide, of course, neither with the Husimi (17) nor with the Wigner (11) distributions. Also, from Eqs. (23) and (25) we see that the correct statistical mechanics' HO energy-dispersion is

$$(\Delta H)^2 = (\overline{E})^2 - \Omega^2/4, \tag{27}$$

while, for the energy mean-value one gets

$$\overline{E} = \frac{\Omega}{2} + \frac{\Omega}{e^b - 1} = \frac{\Omega(e^b + 1)}{2(e^b - 1)} = \frac{\Omega}{2}\,\frac{1 + e^{-b}}{1 - e^{-b}} = \frac{\Omega}{2I_h}\,(1 + e^{-b}) = \frac{\Omega}{I_w}, \tag{28}$$

which, as should be expected, coincides with the, also exact, Wigner energy (13). We thus find the nice (and original, as far as we know) information theory relation:

$$I_w = \frac{2I_h}{1 + e^{-b}}. \tag{29}$$

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Now comes the crucial point, regarding the Fano factor F, intimately linked to the energy fluctuation because the Hamiltonian equals $\Omega(N+1/2)$. We have

$$F_c = \frac{(\Delta N)^2}{\overline{N}} = \frac{(\Delta H)^2}{\Omega\,[\overline{E}\,-\,\Omega/2]}, \tag{30}$$

and, in more detailed fashion

$$F_c = \frac{(\Delta N)^2}{\overline{N}} == \frac{\exp{(b)}}{\exp{(b)}-1} \equiv \frac{1}{1-\exp(-b)} = I_h^{-1}. \tag{31}$$

Accordingly,

$$\lim_{\beta\to\infty} F_c = 1, \text{ coherent state}'\text{s value} \tag{32}$$

is the correct limiting quantum statistical result for the Fano factor.

6. The Husimi and Wigner fluctuation-treatments

We see from Eqs. (11) and (17) that we can recast the general distribution in a common form [33]

$$f_{h,w}(z) = \lambda_{h,w}\, e^{-\lambda_{h,w}|z|^2}, \tag{33}$$

with $\lambda_h = I_h$ for the Husimi case or $\lambda_w = I_w$ for the Wigner instance. Accordingly, the mean energy is

$$\overline{E}_{h,w} = \frac{\Omega}{\lambda_{h,w}}, \tag{34}$$

and

$$\overline{E^2}_{h,w} = 2\frac{\Omega^2}{\lambda_{h,w}^2}. \tag{35}$$

Therefore, the energy fluctuation adopts the appearance

$$(\Delta H)^2 = \overline{E^2}_{h,w} - (\overline{E}_{h,w})^2 = \frac{\Omega^2}{\lambda_{h,w}^2}, \tag{36}$$

and we ascertain that this result disagrees with the correct canonical one of the preceding Section. Then, the Fano factor becomes now

$$F_{h,w} = \frac{\lambda_{h,w}^{-1}}{1-\frac{\lambda_{h,w}}{2}}. \tag{37}$$

We illustrate in Figure 1 the thermal Fano factor as a function of T.
Since $I_w = 2$ for $T = 0$, we see that the HO-Wigner Fano factor diverges at this temperature (infinite noise), which is of course wrong. We obtain three different Fano factor-forms, with the canonical the correct one. That the Wigner instance exhibits a non-physical divergence is our main result.

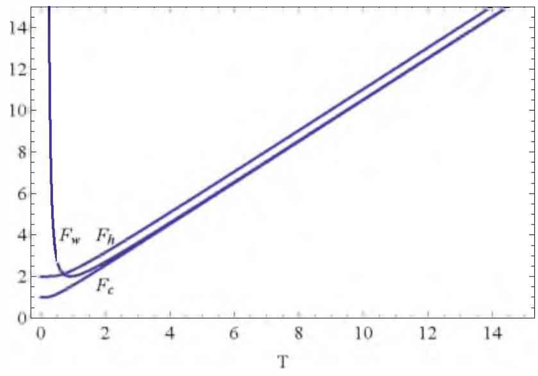

Figure 1. Fano factor as a function of T for canonical, Husimi and Wigner treatments.

7. Conclusions

It is known that there are some cases for which the Weyl-Wigner procedure, that maps Hermitian operators $\hat{A}$ to phase-space functions $A_W(x,p)$, does not give the the correct correspondence between classical and quantum operators [34]. Among these cases one encounters the square of the Hamiltonian [34], an instance in which the Wigner function yields wrong results [34]. Thus, a priori one could have anticipated problems with the Wigner-expectation value of this operator. What is new here is the dramatic nature of the errors in connection with the Fano factor.
Indeed, we have vividly illustrated this effect for the case of the HO-Fano factor, which diverges when evaluated à la Wigner. Instead, the Husimi treatment (HT) is no affected by the Weyl-failure. Since the HT consists in a smearing of the Wigner function over areas of the order of $\hbar$, we gather, as a further result, that the Wigner defect is a sub-$\hbar$ (sub-Planck) effect, of order $\hbar^2$, that disappears after smearing procedures and becomes weaker and weaker with increasing temperatures. Of course, the canonical treatment of statistical mechanics gives always the correct result.
Note also the important role played by the Fisher information measure in these proceedings.

Acknowledgments

F. Pennini would like to thank for partial financial support FONDECYT, grant 1080487.

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