NUMERICAL SOLUTIONS FOR DAEs USING LEGENDRE WAVELETS

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Abstract: In this paper, a method for solving singular differential algebraic equations combining Legendre wavelets with a collocation technique is presented. Numerical examples show that the method is easy to implement and yields very accurate results.

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1 INTRODUCTION

Differential algebraic equations (DAEs) play an important role in many branches of science and engineering. DAEs arise in a wide variety of systems, for instances, circuit analysis, computer-aided design, simulation of mechanical systems and optimal control problems.

DAEs are a type of differential equation where one or more derivatives of dependent variables are not present in the equations. Variables that appear in the equations without their derivative are called algebraic, and the presence of algebraic variables means that you cannot write down the equations in the explicit form $\mathbf{y}' = \mathbf{f}(t, \mathbf{y})$. They are characterized by their differential index, which is a measure of their singularity. By differentiating equations you can eliminate algebraic variables, and if this is done enough times, then the equations take the form of a system of explicit ODEs. The differential index of DAEs is the smallest number of derivatives you must take to express the system as an equivalent system of explicit ODEs. In particular, ODEs have a differential index of 0.

In recent years, DAEs have attracted the attention of numerical analysists. One of the reasons is that when the index is greater than 1 they can be difficult to solve numerically. Vanani and Aminataei (see [1]) obtain numerical solutions using a multiquadric approximation scheme, using as basis functions a type of spline approximation with good regularity properties. Other authors, as Hosseini ([3]), have proposed reducing index methods for semi-explicit DAEs.

The aim of this work is to present the Legendre wavelets approximation method (LWM) for DAEs (see [2]) and to make a comparison with multiquadric scheme and ODE solvers in case they can be used.

This paper is organized as follows. In Section 2 some notations and definitions of the DAE forms are given. Section 3 describes basic concepts of the Legendre wavelet method and how they are applied on DAEs. In Section 4 two illustrative numerical experiments are presented. Finally, Section 5 consists of some obtained conclusions.

2 DIFFERENTIAL ALGEBRAIC EQUATIONS. DAEs

Let us consider a system of differential equations of the form:

$$\mathbf{M}(t, \mathbf{y})\mathbf{y}' = \mathbf{f}(t, \mathbf{y}) \tag{1}$$

where the mass matrix $\mathbf{M}(t, \mathbf{y})$ is singular. That means that you have a Differential Algebraic Equation (DAE) rather than an ODE system. And it is important to note that solving DAEs is more complicated because, as it is demonstrated in ([4]), DAEs has a solution only if the initial conditions are consistent .

Rewriting Eq.(1) in semi-explicit form,

$$\begin{aligned} \mathbf{u}' &= f_1(t, \mathbf{u}, \mathbf{v}) \\ 0 &= f_2(t, \mathbf{u}, \mathbf{v}) \end{aligned} \tag{2}$$

where $\mathbf{y} = (\mathbf{u}, \mathbf{v})$, the system is of index 1 if the matrix of partial derivatives of the algebraic system f_2 with respect to the algebraic variables (i.e. $df_2/d\mathbf{v}$) is non-singular. And in mass matrix form (Eq.(1)), the index 1 condition is satisfied when the matrix $(\mathbf{M} + \lambda * df_2/d\mathbf{v})$ is nonsingular, for all non-zero λ.

In Matlab, $ode15s$ and $ode23t$ solvers can be used to find numerical solutions for DAEs of index 1. In that case it is demonstrated that there is any distinction between solving a DAE and an ODE system (see [4] for details). If the index of the equations is 2 or higher, then it is necessary to rewrite the equations as an equivalent system of index-1 DAEs.

3 Properties of Legendre wavelets. Collocation method

Wavelet theory is a relative new and emerging area in mathematical research ([5], [6], [7]). Different wavelets have found their way into many fields of science and engineering, as they constitute a family of functions generated from dilation and translation of a single function called the *mother wavelet*. When the dilation parameter a and the translation parameter b vary continuosly, the following family of continuos wavelets is constructed:

$$\psi_{a,b}(t) = |a|^{-1/2}\psi(\frac{t-b}{a}), \qquad a, b \in R, a \neq 0 \tag{3}$$

If the parameters a and b are restricted to discret values as $a = a_0^{-k}$ and $b = nb_0a_0^{-k}$ where $a_0 > 1$, $b_0 > 0$, and n and k are positive integers, the family of discrete wavelets,

$$\psi_{k,n}(t) = |a_0|^{-1/2}\psi(a_0^k t - nb_0), \tag{4}$$

is a wavelet basis for $L^2(R)$. When $a_0 = 2$ and $b_0 = 1$ they form an orthonormal basis.

Legendre wavelets (see [2]) $\psi_{nm}(t) = \psi(k, \hat{n}, m, t)$ have four arguments $\hat{n} = 2n-1, n = 1, 2, \cdots, 2^{k-1}$, k is any positive integer, m is the order of the Legendre polynomials and t is the normalized time (they are defined on the interval $[0, 1)$). These polynomials are orthogonal with respect to the weight function 1 on the interval $[-1, 1]$ and they can be constructed easily by a recursive formula (see [2]).

Using the above family, a function $f(t)$ defined over $[0, 1)$ may be expanded as

$$f(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{nm}\psi_{nm}(t), \tag{5}$$

where $c_{nm} = \langle f(t), \psi_{nm}(t)\rangle$, and the following approximation can be obtained if the series is truncated:

$$f(t) \approx \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{nm}\psi_{nm}(t) \tag{6}$$

In this way, the approximate solution for the function $\mathbf{y}$ in Eq.(1) can be expanded as follows :

$$\mathbf{y}(t) = C^T\Psi(t) \tag{7}$$

where C and $\Psi(t)$ are $2^{k-1}M \times 1$ matrices with the unknown coefficients and Legendre wavelets, respectively (see [2] for details).

Thus for the solution of above DAEs Eq.(2) it is sufficient to suppose that

$$\mathbf{u}(t) = C_1^T \Psi(t), \qquad \mathbf{v}(t) = C_2^T \Psi(t) \tag{8}$$

and therefore,

$$\begin{aligned} C_1^T \Psi'(t) &= f_1(t, C_1^T \Psi, C_2^T \Psi) \\ 0 &= f_2(t, C_1^T \Psi, C_2^T \Psi) \end{aligned} \tag{9}$$

Collocation points t_i are required in order to apply LWM . They are selected in $[0,1]$. Thus, the following algebraic system is obtained,

$$\begin{aligned} C_1^T \Psi'(t_i) &= f_1(t_i, C_1^T \Psi, C_2^T \Psi) \\ 0 &= f_2(t_i, C_1^T \Psi, C_2^T \Psi) \end{aligned} \tag{10}$$

and adding initial conditions, it can be solved and the unknown coefficients are found.

4 NUMERICAL EXAMPLES

In this section we present two numerical examples concerning a problem of the form

$$\mathbf{A}(t)\mathbf{y}'(t) + \mathbf{B}(t)\mathbf{y} = 0 \qquad 0 \le t \le 1 \tag{11}$$

where,

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and,

$$\mathbf{B}_1 = \begin{pmatrix} 1 & 0 & -t & 1 \\ -1 & 1 & -t^2 & t \\ -t^3 & t^2 & -1 & 0 \\ t & -1 & t & -1 \end{pmatrix}, \qquad \mathbf{B}_2 = \begin{pmatrix} 1 & -t^2 & 1 & t \\ 0 & t & 0 & -3 \\ -(1-t^2) & -t^3 & 0 & t^2 \\ t & t & -1 & -1 \end{pmatrix}$$

correspond to two different problems, of index 1 (see [2]).

They were solved using a computer program in MATLAB, with $k = 1$ and $M = 2^j + 1$ collocation points, $j = 3, 4, 5$.

Exact solutions in each case are: $\mathbf{y}_1(t) = (e^{-t}, te^{-t}, e^t, te^t)^T$, $\mathbf{y}_2(t) = (e^{-t^2}, e^{t^2}, te^{-t^2}, te^{t^2})^T$. Using both, the multiquadric approximation ([1]) and the Legendre wavelet method, numerical results were obtained. They were similar in accuracy (relative errors $\approx 10^{-6}$ for $j = 4$) but with the advantage that the matrices in LWM method are better conditioned. In addition, they were solved using $ode15s$ and a comparison of the behaviour was made.

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5 Conclusions

The results reveal that the technique introduced here is effective and convenient in solving DAEs and it can be extended for solving nonlinear problems. Areas of future work include performing further validations of the method to solve high-index semi-explicit DAEs and understanding the behaviour of singular solutions, in particular, those arising in modelling electric circuits.

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