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DEPARTMENT OF

AERONAUTICS & FLUID MECHANICS

AN ALGORITHM FOR THE CALCULATION OF THE
POTENTIAL FLOW ABOUT AN ARBITRARY TWO
DIMENSIONAL AEROFOIL

by

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SUMMARY

A vortex panel method has been developed to calculate the potential flow about an arbitrary two dimensional aerofoil or axisymmetric shape at fixed incidence in a steady, uniform, irrotational, incompressible flow.

The procedure replaces the contour by a suitably inscribed polygon, on which surface vorticity varies linearly and continuously along the panel and is piecewise continuous at the panel corners.

The Neumann boundary condition is satisfied at control points situated at the midpoint of each panel and the classical Kutta condition is specified at the trailing edge by setting the net vorticity there equal to zero.
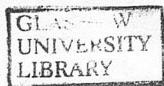
This particular algorithm offers much flexibility in the treatment of a greater range of aerofoil geometries and at higher incidence than other surface singularity methods.

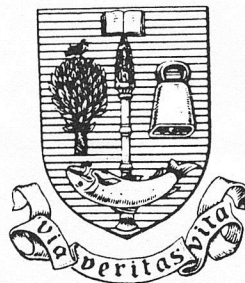Programme flow charts and FORTRAN code listings are given in the User Guide[1].

CONTENTS

NOMENCLATURE

A_{ij}, B_{ij}	elements of the influence coefficients
C_{ij}	influence coefficient
C	the boundary contour
C_p	pressure coefficient
L	length of panel
R	region external to C
S	distance along panel
V	velocity
x,y	co-ordinates
α	angle of incidence
γ	vortex sheet strength

Subscripts:

i	refers to i th panel
j	refers to j th panel
n	normal component
p,c	arbitrary point
∞	conditions in the free stream

1.   Introduction

The prediction of the Newtonian flow about an arbitrary two-dimensional body and, in particular, an aerofoil, still poses great difficulties for the aerodynamicist.   The problem is complicated by the wide variety of viscous phenomena and the indeterminacy of the equations currently used to describe turbulent flows.

Although a general solution, including unsteady effects, is not yet available, many useful procedures with certain limitations and simplifications have been developed.   Originally, viscous effects were ignored and potential flow assumed, as in the method of Theodorsen[2]. This method was, of course, restricted to the treatment of the analytic Joukouski aerofoil profiles.   The extension of the method by Theodorsen and Garrick[3] to the analysis of arbitrary aerofoil shapes became a significant advance.   It was soon realised, however, that due to the neglect of viscous effects, poor predictions of the pressure distribution about the aerofoil at higher angles of incidence were obtained.   Early attempts to account for viscous effects by Pinkerton[4] allowed the relaxation of the Kutta condition, with the specification of the circulation about the aerofoil that followed from the measured lift coefficient at that angle of incidence.   This procedure gave improved comparisons of theoretical and experimental pressure distributions in the vicinity of the leading edge but allowed infinite velocities to exist at the trailing edge.

Preston[5] took a different approach, and showed that the required viscous effects on an unstalled aerofoil may be accounted for by displacing the aerofoil surface an amount equal to the local boundary layer displacement thickness, repeating the potential flow calculation using the new contour and iterating to convergence.

The full treatment of the viscous flow about an aerofoil can, of course, be accomplished using the complete Navier-Stokes equations. The computational effort and storage requirements are at present prohibitive. Generally then, an alternative approach must be made by engineers who require aerodynamic predictions.

The advent of fast digital processors led to the active development and use of "panel methods" which, in effect, replace the aerofoil contour by a suitably proportioned inscribed polygon on which is placed appropriate singularity distributions. The strength of the distribution is chosen to satisfy the condition of flow tangency on the contour and the classic Kutta condition.

Popular methods use some combination of source and vortex singularities distributed in a prescribed manner on the surface of the panels. A method which uses a uniform source distribution along each panel but which varies in strength from panel to panel and a uniform vorticity distribution around the contour is generally credited to A.M.O. Smith and his co-workers [5,6]. This method has been used for a wide variety of problems in both two[6] and three[7] dimensions. One of the short-comings of the method, however, is the manner in which the Kutta condition is specified. This is done by equating tangential velocities at the mid points of the upper and lower panels adjacent to the trailing edge. Hence, the specification depends on the panel distribution and is not applied in a unique manner. In contrast, a vortex panel method which uses some variation in vorticity along the panel, overcomes this non-unique specification by deducing another property that is a direct consequence of finite velocities at the trailing edge. This is done by equating vorticity values, in the opposite sense, on the upper and lower panels at the trailing edge.

It was because of the ability for this method to be extended to multi-element aerofoils and in the modelling of separated trailing edge flows that the authors chose to use this technique and developed their own algorithm. It is hoped that this algorithm will form the basis of a series of computer programmes for predicting many of the observed viscous flow phenomena. The Glasgow University implementation is described herein and shown to be very satisfactory.

2. The algorithm

2.1 Mathematical description of the problem

The problem is to calculate the potential flow in a region R exterior to a contour C.

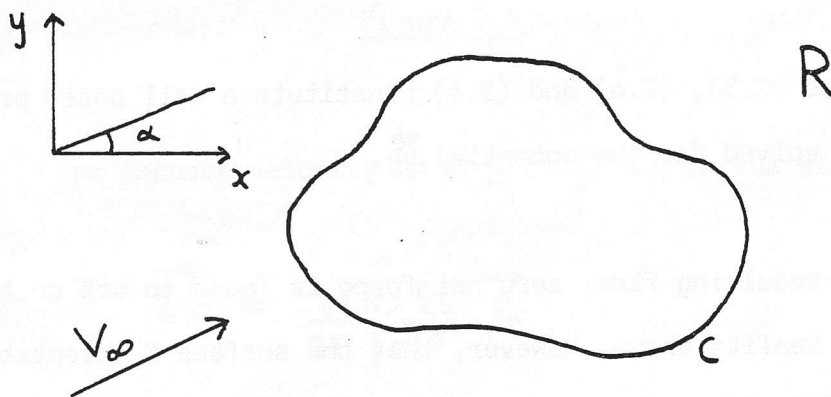Figure 2.1

The fluid velocity at any point is given by

$$\vec{V} = \vec{V}_\infty + \vec{v}$$

(2.1)

where $\vec{V}_\infty$ denotes the velocity of the uniform onset flow, i.e.,

$$\vec{V}_\infty = |V_\infty| \cos \alpha \, \vec{i} + |V_\infty| \sin \alpha \, \vec{j}$$

(2.2)

α is the flow onset angle relative to a fixed axis system.

The vector $\vec{V}$ is the perturbation velocity at that point due to the contour C, which is assumed impermeable to the fluid.

For potential flow in region R, the following equations must hold

$$\vec{V} = \text{grad } \phi \tag{2.3}$$

and

$$\nabla^2 \phi = 0 \qquad \text{(Laplaces Equation)} \tag{2.4}$$

where ϕ is some potential function.

Together with the Neumann boundary condition:

$$\frac{\partial \phi}{\partial n} = \text{grad } \phi \cdot \vec{n} = \vec{V} \cdot \vec{n} = 0 \tag{2.5}$$

on the contour C.

Equations (2.3), (2.4) and (2.5) constitute a well posed problem which can be solved for the potential ϕ.

For the resulting flow, zero net force is found to act on the surface C. Reality shows, however, that the surface C orientated at some angle of incidence will experience both a lift and drag force. Hence, it can be concluded that both lift and drag forces are ultimately due to viscosity. As the objective is to mathematically model the flow of a real viscous fluid about the contour C using a potential formulation, an auxiliary conditon is specified to fix the value of lift generated by the contour. In the case of an aerofoil shape, the auxiliary condition used is that of a finite velocity at the trailing edge, the so called Kutta condition.

Hence, the solution of equations (2.3) and (2.4) subject to (2.5) and the Kutta condition, is the mathematical problem considered herein.

The method chosen to satisfy the preceeding conditions was to replace the contour C by a vortex sheet of unknown variable strength, as shown in figure (2.2).
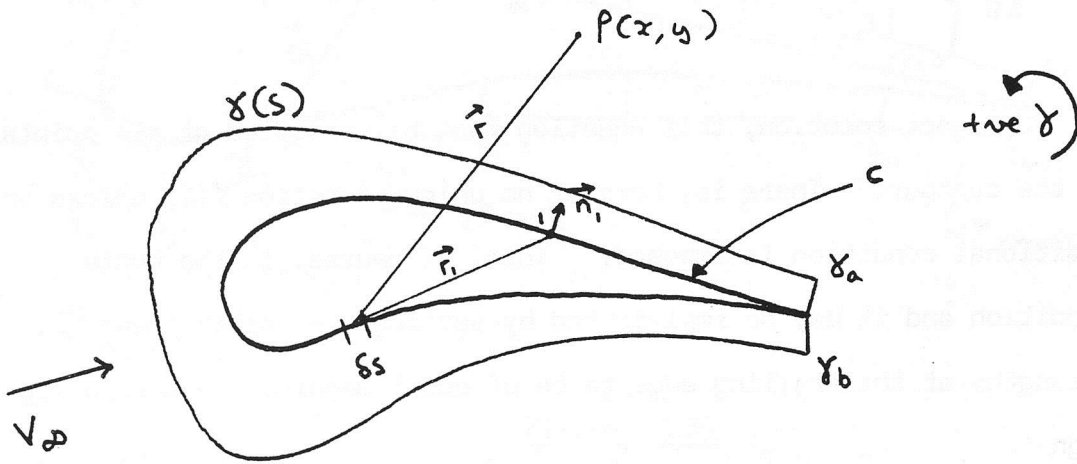

Figure 2.2

The induced velocity $\vec{\delta v}$ at P (x,y) due to the element δs is given by

$$\vec{\delta v} = \frac{\gamma(s) \, \delta s}{2\pi \, |r|^2} \vec{r_n} \tag{2.6}$$

where $\vec{r_n}$ is the unit normal vector to $\vec{r}$ at P. Hence the total induced velocity at P due to the vortex sheet is

$$\vec{v} = \oint_C \frac{\gamma(s)}{2\pi \, |r|^2} \vec{r_n} \, ds \tag{2.7}$$

and the velocity is

$$\vec{V} = \vec{V_\infty} + \frac{1}{2\pi} \oint_C \frac{\gamma(s)}{|r|^2} \vec{r_n} \, ds \tag{2.8}$$

The normal velocity on the contour, at the arbitrary point 1, may then be found from

$$\vec{V_n} = \left[\vec{V_\infty} + \frac{1}{2\pi} \oint_C \frac{\gamma(s)}{|r_1|^2} \vec{r_{n_1}} \, ds \right] \cdot \vec{n_1} \tag{2.9}$$

and by invoking the boundary condition (equation 5) the following integral equation is obtained

$$\frac{1}{2\pi} \oint_C \frac{\gamma(s)}{|\vec{r_1}|^2} \vec{r}_{n_1} \cdot \vec{n_1} \, ds + \vec{V}_\infty \cdot \vec{n_1} = 0 \qquad (2.10)$$

For exact solution, this equation must be satisfied at all points on the contour. There is, however, no unique function $\gamma(s)$ unless an additional condition is imposed. This, of course, is the Kutta condition and it may be implemented by setting the vortex sheet strengths at the trailing edge to be of equal magnitude but opposite sign

$$\text{i.e.,} \quad \gamma_a + \gamma_b = 0 \qquad (2.11)$$

If $\gamma(s)$ is obtainable, then the surface velocity may be simply obtained from

$$V = |\gamma(s)| \qquad (2.12)$$

2.2 Method of solution

Equation 10 is satisfied by a numerical technique where the smooth vortex sheet on C is approximated by a suitable polygon whose sides (panels) consist of vortex sheets with a linear variation of strength, as shown in figure (2.3) below.
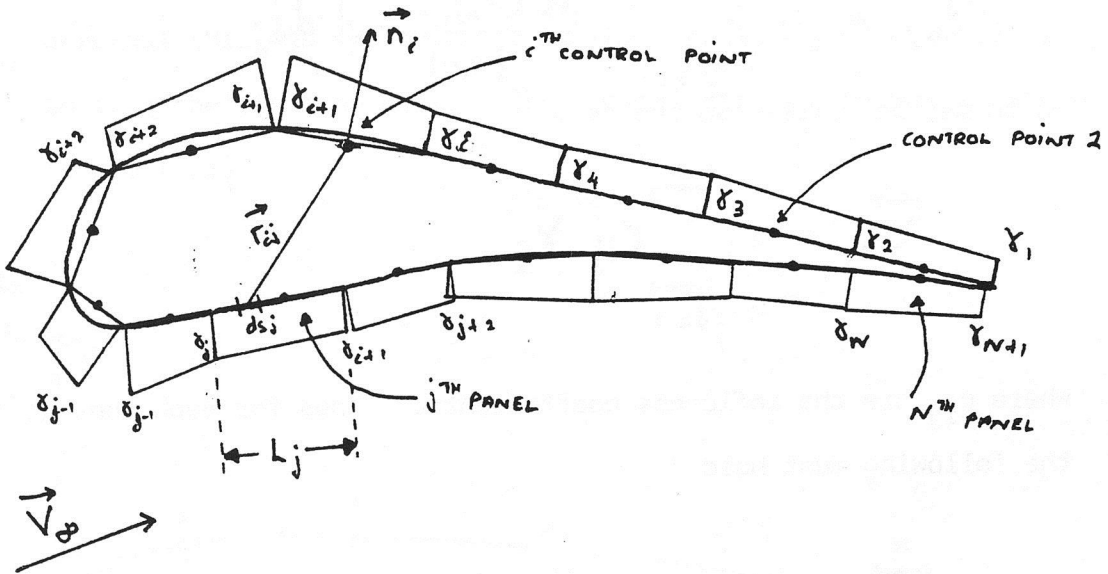
Figure (2.3)

The boundary condition (i.e., equation 2.5) is satisfied only at centrally located control points on each panel. Considering the velocity induced at the i th control point due to the element ds_j on the j th panel, we have

$$\vec{\delta v}_{ij} = \frac{\gamma_s\, ds_j}{2\bar{n}\, |r_{ij}|^2}\, \vec{r}_{n_{ij}} \tag{2.13}$$

and hence

$$\vec{v}_{ij} = \int_0^{L_j} \frac{\gamma_s}{2\bar{n}\, |r_{ij}|^2}\, \vec{r}_{n_{ij}}\, ds_j \tag{2.14}$$

The induced velocity normal to the surface at the i th control point is given by

$$\vec{v}_{n_{ij}} = \frac{1}{2\bar{n}} \int_0^{L_j} \frac{\gamma_s}{|r_{ij}|^2}\, (\vec{r}_{n_{ij}} \cdot \vec{n}_i)\, ds_j \tag{2.15}$$

and the total induced velocity (V_{n_c}) at i is

$$\vec{v}_{n_i} = \frac{1}{2\pi} \sum_{j=1}^{N} \int_0^{L_j} \frac{\gamma_s (\vec{r}_{n_{ci}} \cdot \hat{n}_c)}{|r_{cj}|^2} ds_j \tag{2.16}$$

which may be written in the form

$$\vec{v}_{n_i} = \frac{1}{2\pi} \sum_{j=1}^{N} c_{ij} \gamma_j \tag{2.17}$$

where c_{ij} are the influence coefficients. Thus for each control point the following must hold

$$\frac{1}{2\pi} \sum_{j=1}^{N} c_{ij} \gamma_j + \vec{V}_\infty \cdot \vec{n}_i = 0 \tag{2.18}$$

This represents a set of N linear simultaneous equations containing $N + 1$ unknowns, i.e., $\gamma_1 \rightarrow \gamma_{N} + 1$. The necessary additional equation is equation 2.11 which is the specification of the Kutta condition, i.e.,

$$\gamma_1 + \gamma_{N+1} = 0 \tag{2.19}$$

3. Details of the numerical procedure

3.1 The vortex panel distribution

The efficiency of the algorithm for a given number of panels is independent of whether equal or unequal panel lengths are used. The accuracy of the method is, however, very sensitive to the size of panel since the resulting polygon must be an appropriate representation of the original contour. It is, therefore, necessary to use small panels in regions of large curvature and, for the sake of efficiency, larger panels in regions of low curvature. The maximum size of the panels is, of course, constrained by the appropriateness of the quasi-linear approximation to the continuous vortex sheet strength.

The procedure adopted was to analytically specify the distribution of the panel size along the contour. For the case of an arbitrary aerofoil this was short panels at the leading edge and fairly long ones towards the trailing edge. The typical arrangement was as shown in figure (3.1).
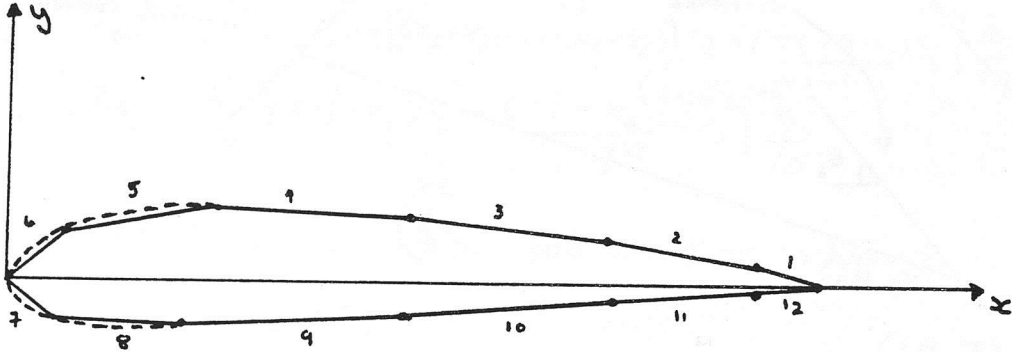

Figure 3.1

3.2 Calculation of the influence coefficients

The influence coefficients C_{ij} contained in equation 2.17 were obtained as follows for $i \neq j$

The vorticity γ_s is given by

$$\gamma_s = \gamma_j + \left(\frac{\gamma_{j+1} - \gamma_j}{L_j} \right) S_j \tag{3.1}$$

or

$$\gamma_s = \frac{\gamma_j}{L_j} \left(L_j - S_j \right) + \gamma_{j+1} \frac{S_j}{L_j} \tag{3.2}$$

Thus from equation 2.15 we have

$$A_{ij} + B_{ij} = \int_0^{L_j} \frac{\gamma_j}{L_j}(L_j - S_j) \frac{(\overrightarrow{r_{n_{ij}}} \cdot \overrightarrow{n_i})}{|r_{ij}|^2} ds_j + \int_0^{L_j} \frac{\gamma_{j+1}}{L_j}(\overrightarrow{r_{n_{ij}}} \cdot \frac{\overrightarrow{n_i}}{|r_{ij}|^2}) ds_j \qquad (3.3)$$

Considering figure 3.2
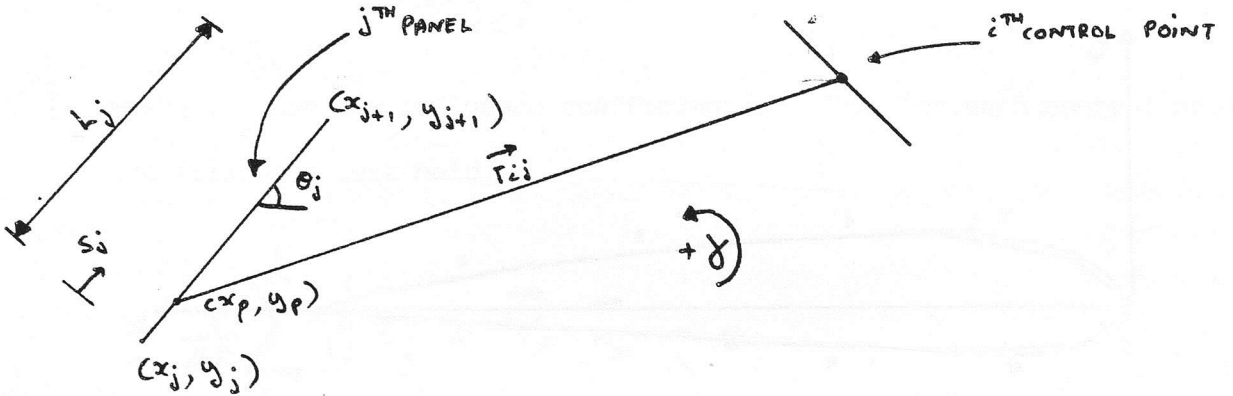
Figure 3.2

$$\overrightarrow{r_{ij}} = \left\{ (x_i - x_p), (y_i - y_p) \right\} \qquad (3.4)$$

$$\overrightarrow{r_{n_{ij}}} = \left\{ (y_p - y_i), (x_i - x_p) \right\} \qquad (3.5)$$

also $$|r_{ij}|^2 = (x_i - x_p)^2 + (y_i - y_p)^2 \qquad (3.6)$$

Furthermore we have

$$x_p = x_j + S_j \cos\Theta_j = x_j + \left(\frac{x_{j+1} - x_j}{L_j} \right) S_j \qquad (3.7)$$

and $$y_p = y_j + S_j \sin\Theta_j = y_j + \left(\frac{y_{j+1} - y_j}{L_j} \right) S_j \qquad (3.8)$$

Substitution of equations 3.7 and 3.8 into 3.6 leads to

$$|r_{ij}|^2 = 1.0\, S_j^2 + b\, S_j + c \qquad (3.9)$$

where $$b = -\frac{2}{L_j} \left\{ (x_i - x_j)(x_{j+1} - x_j) + (y_i - y_j)(y_{j+1} - y_j) \right\} \qquad (3.10)$$

and $$c = (x_i - x_j)^2 + (y_i - y_j)^2 \qquad (3.11)$$

Further substitution of the above in equation 3.3 leads to

$$A_{ij}\gamma_j + B_{ij}\gamma_{j+1} = \frac{1}{2\pi L_j}\left\{(I_1\gamma_j + I_2\gamma_{j+1})\vec{i} + (I_3\gamma_j + I_4\gamma_{j+1})\vec{j}\right\}\cdot\vec{n_i} \qquad (3.12)$$

where $I_1 = \int_0^{L_j}\left\{L_j(y_i - y_j) - [(y_j - y_{j+1}) + (y_j - y_i)]s_j + \frac{(y_j - y_{j+1})}{L_j}s_j^2\right\}\frac{ds_j}{r_{ij}}$ $\qquad (3.13)$

$$I_2 = \int_0^{L_j}\left\{(y_j - y_i)s_j - \frac{(y_j - y_{j+1})}{L_j}s_j^2\right\}\frac{ds_j}{r_{ij}} \qquad (3.14)$$

$$I_3 = \int_0^{L_j}\left\{L_j(x_i - x_j) - [(x_j - x_{j+1}) + (x_j - x_i)]s_j + \frac{(x_j - x_{j+1})}{L_j}s_j^2\right\}\frac{ds_j}{r_{ij}} \qquad (3.15)$$

$$I_4 = \int_0^{L_j}\left\{(x_j - x_i)s_j - \frac{(x_j - x_{j+1})}{L_j}s_j^2\right\}\frac{ds_j}{r_{ij}} \qquad (3.16)$$

The above integrals are of a standard form given in Appendix I.

The influence coefficient C_{ij} is obtained from

$$C_{ij} = A_{ij} + B_{ij-1} \quad (2 \le j \le N) \quad ; C_{i1} = A_{i1} \; ; \; C_{iN+1} = B_{iN} \qquad (3.17)$$
$$(1 \le i \le N)$$

3.3 Elements of the influence coefficient of a panel which contains the considered control point

When the control point lies on the j th panel, the preceeding analysis cannot be used and the following applies

From equation (2.14)

$$\vec{v}_{jj} = \int_0^{L_j}\frac{\gamma_s}{2\pi|r|}ds_j \qquad (3.18)$$

substituting $r = s_j - s_c$ we have

$$\vec{v}_{jj} = \int_0^{L_j}\frac{\gamma_s}{2\pi(s_j - s_c)}ds_j \qquad (3.19)$$

and from equation 2.20 3.1

$$\vec{v}_{jj} = \frac{1}{2\pi}\int_0^{L_j}\frac{\gamma_j}{s_j - s_c}ds_j - \frac{1}{2\pi}\int_0^{L_j}\frac{\gamma_j s_j\,ds_j}{L_j(s_j - s_c)} + \frac{1}{2\pi}\int_0^{L_j}\frac{\gamma_{j+1}s_j\,ds_j}{L_j(s_j - s_c)} \qquad (3.20)$$

$$= \frac{\gamma_j}{2\pi}\ln\left[\frac{L_j - s_c}{s_c}\right] + \frac{1}{2\pi}\left[\gamma_{j+1} - \gamma_j\right] + \frac{1}{2\pi}\left[s_c\ln\left(\frac{L_j - s_c}{s_c}\right)\right] \qquad (3.21)$$

now at the panel mid points $S_c = L_j / 2$

$$\therefore \quad \vec{v}_{jj} = \frac{1}{2\pi} \left(\gamma_{j+1} - \gamma_j \right) \tag{3.22}$$

$$A_{jj} \, \gamma_j + B_{jj} \, \gamma_{j+1} = \frac{1}{2\pi} \gamma_j - \frac{1}{2\pi} \gamma_{j+1} \tag{3.23}$$

and C_{ij} for $i = j$ follows from equation (3.17).

3.4   Solution of the equations

From the above analysis, the equation set to be solved is

$$\sum_{j=1}^{N} C_{ij} \, \gamma_i + \vec{V}_\rho \cdot \vec{n}_j = 0 \qquad (i = 1, N+1) \tag{3.24}$$
$$\text{plus: equation (2.19)}$$

The $N+1$ unknowns γ_j are obtained, at present, via a Gauss-Jordan elimination technique.   This method, described in ref. (1), yields fast stable solutions for up to 60 panels after which solution time is significantly increased.

It will be seen from equation 3.24 that the influence coefficients C_{ij} are constant for all angles of attack of a given aerofoil section and hence numerous solutions for various α can be efficiently obtained.

The surface velocity is given directly by

$$V = |\gamma| \tag{3.25}$$

and, in particular, at the control points x_i, y_i

$$V_i = \left| \frac{\gamma_i + \gamma_{i+1}}{2} \right| \tag{3.26}$$

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The surface pressure coefficient is then given by

$$C_p = 1 - \left(\frac{V_i}{V_\delta}\right)^2$$

(3.27)

4.   A selection of typical results

As a test case, the algorithm was applied to that of the 2-D flow about a circular cylinder, the analytic result being given by a simple expression.   As presented in Figs. (4.1) and (4.2) excellent agreement is obtained for forty panels.

As another panel method was not available to the authors at the time of writing, no direct comparisons with the present algorithm could be made.   Comparisons were, however, made with data refs. (3) and (9), using the NACA 0012 aerofoil section.   A selection of pressure plots is presented in Figs. (4.3) to (4.7).

5.   Discussion

As a potential flow analysis is usually the first step in an aerodynamic design study, the computer programme in its present format provides an extremely useful working tool for this purpose.   The resulting velocity distributions on the aerofoil surface can be utilised in a boundary layer program and thereby providing useful predictions on the aerodynamic characteristics.   Furthermore, it may be readily extended to the treatment of multi-element aerofoils and in the modelling of separated trailing edge wakes such as in refs. (9) and (10).

One of the most pleasing aspects of the current algorithm is in the treatment of the Kutta condition.   The Kutta condition is used in a potential formulation, such as presented, to fix the value of lift.

For an aerofoil shape, the classic Kutta condition specifies that the stagnation point must lie on the trailing edge in order that infinite velocities do not exist.   In other words, we stipulate that the flow leaves the trailing edge smoothly and there is no static pressure jump at this point.   It has been found that different workers have their own interpretation of this condition in a numerical algorithm. However, it is emphasised here that as the overall flow field is governed by the Kutta condition, its specification must be accurate and unique otherwise deviations would be expected to occur, especially at high angles of incidence.   More often than not, panel methods apply the Kutta condition some small distance away from the trailing edge[6], and hence, the resulting solution will be dependent on the number of panels and, more importantly, their distribution.   In the present algorithm the Kutta condition is specified directly at the trailing edge in a unique manner and there can be no ambiguity in the results.   The algorithm is considered superior in this respect.

The Kutta condition can only be applied to arbitrary aerofoil shapes or axisymmetric bodies.   The algorithm will not work in its present format for an arbitrary body with a blunt/bluff trailing edge.   If this objective is required, it is recommended that the alternative programme version AEROPF2 (see User Guide) be used, in which both the angle of attack and circulation values can be specified by the user.   The FORTRAN programme AEROPF2 has been formulated along the ideas presented by Pinkerton[3] for use in experimental comparisons of aerofoil data.

It may be argued that the positioning of the control points on the mid-points of the panels means, in fact, that the control points do not lie on the contour of the actual aerofoil.   Although this is so, the error incurred in the solution will be negligible so long as a

"reasonable" number of panels is taken.   A value felt as "reasonable" at the time of writing is about 40 to 60 panels.   More elaborate routines could be used to "smooth" the input data to give control points approximately on the contour but it is doubtful as to the gain in accuracy over the computational effort required.

The treatment of aerofoils with cusped or excessively thin trailing edge regions has been found problematic, as under these circumstances, the influence coefficient matrix tends to become singular.   Generally, however, these aerofoils are encountered infrequently in practice, and the user will be guided to the "non-applicability" of the aerofoil by the erroneous results.

Aerofoils with a finite thickness at the trailing edge, such as the NACA 0012, are treated by representing the contour by an open polygon, the "open" part being at the trailing edge.   The algorithm is applied as before and no difficulties have been found with this approach.

CONCLUSIONS AND RECOMMENDATIONS

(a) A potential flow algorithm for calculating the velocity (or pressure) distribution on a steady two-dimensional aerofoil or axisymmetric shape has been developed. The algorithm replaces the aerofoil contour by an appropriately inscribed polygen on each side of which is placed a linear variation of surface vorticity.

(b) Satisfactory agreement has been obtained with results available at the time of writing.

(c) The Kutta condition has been accurately satisfied in a unique manner at the trailing edge by setting the net vorticity there equal to zero.

(d) It is recommended that approximately 40 to 60 panels be used. Very often, however, the user will be left to rely on existing published aerofoil coordinates. (Refer to User Guide).

(e) The present algorithm is inappropriate to the analysis of "flat plate" aerofoils and those with cusped or long, thin trailing edge regions.

(f) The velocity (or pressure) can be directly calculated at any point on the contour when the surface vorticity is known.

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FIGURES

C_p

-3

-2

-1

0

1

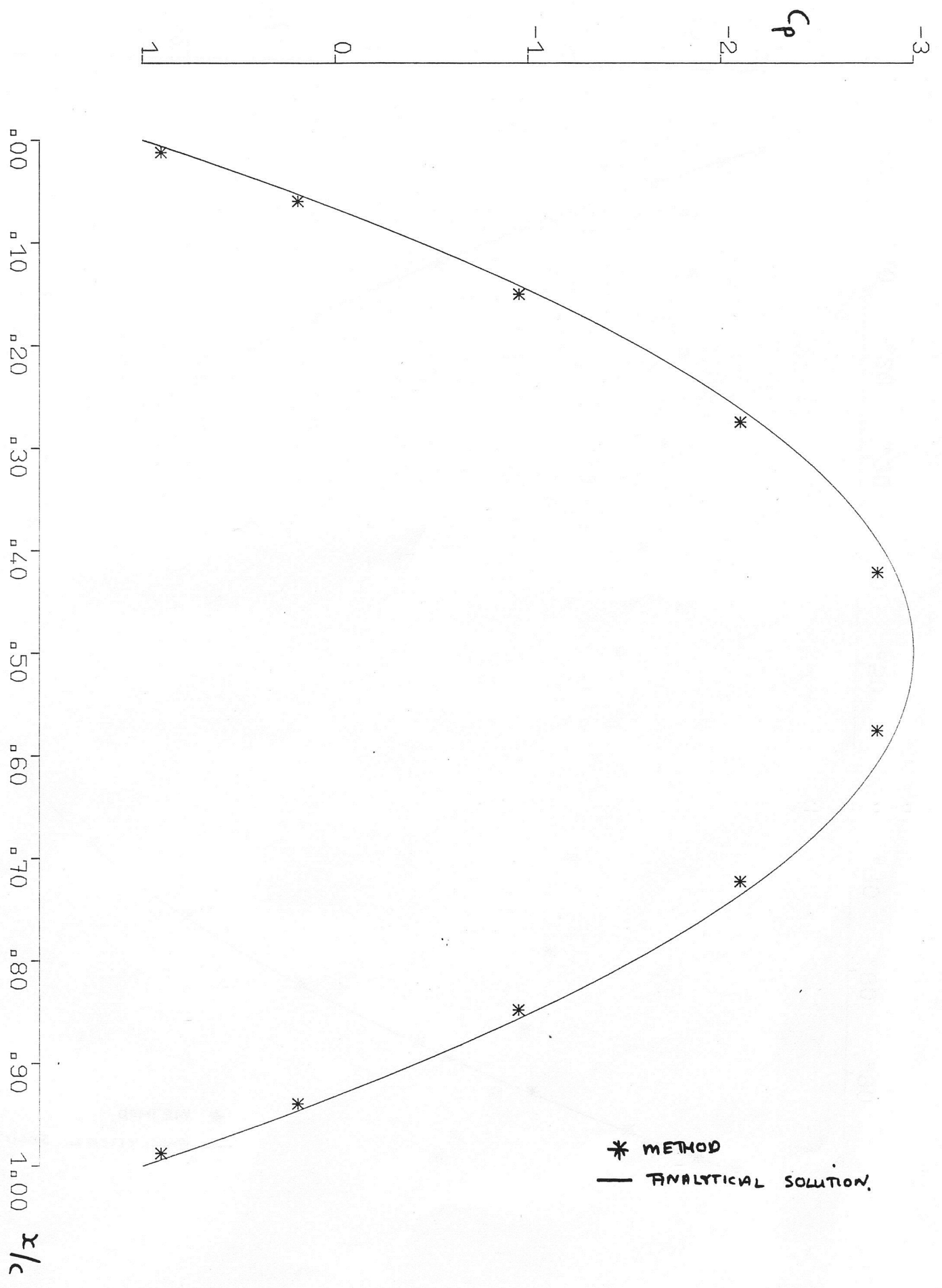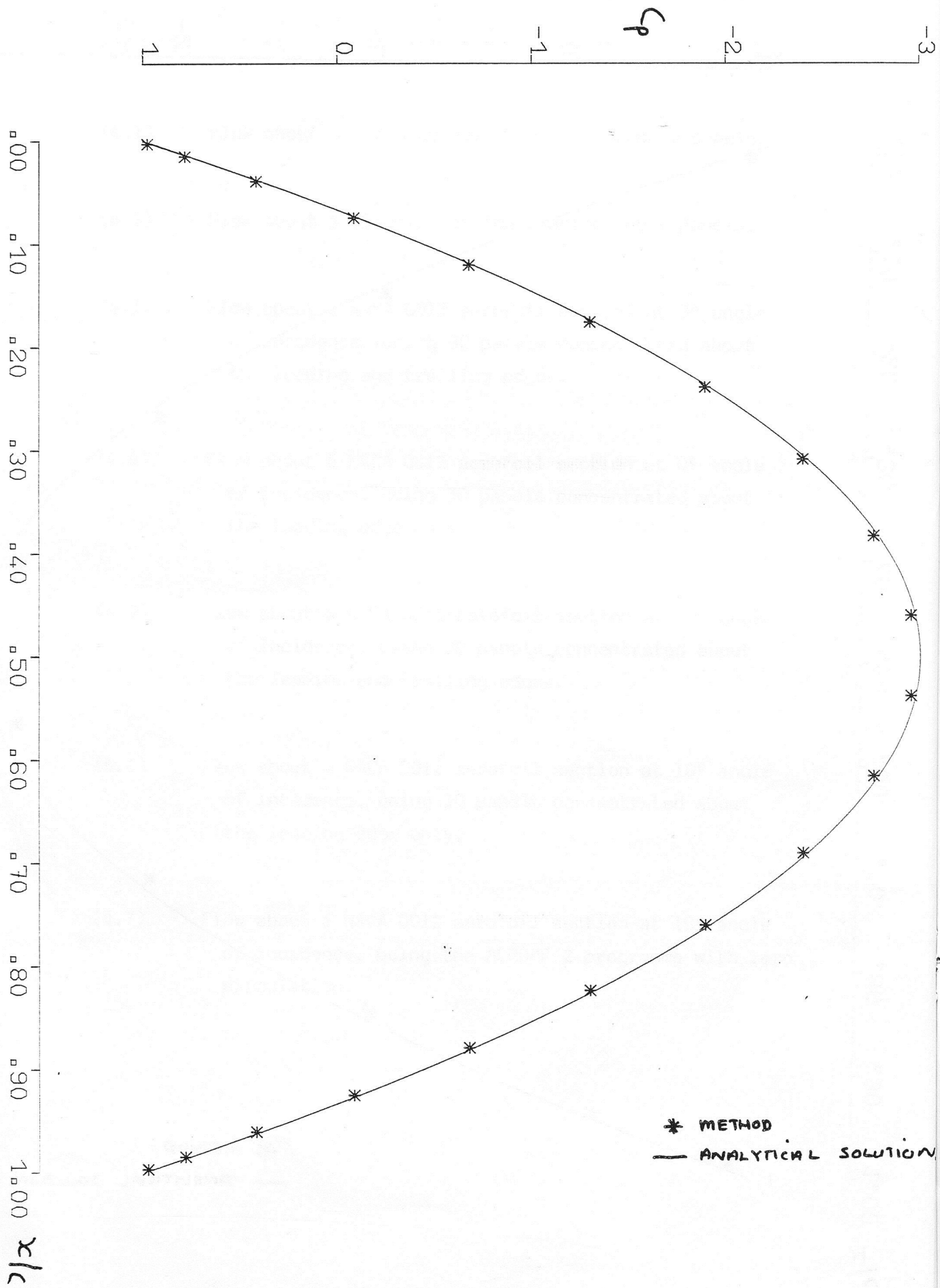.00   .10   .20   .30   .40   .50   .60   .70   .80   .90   1.00
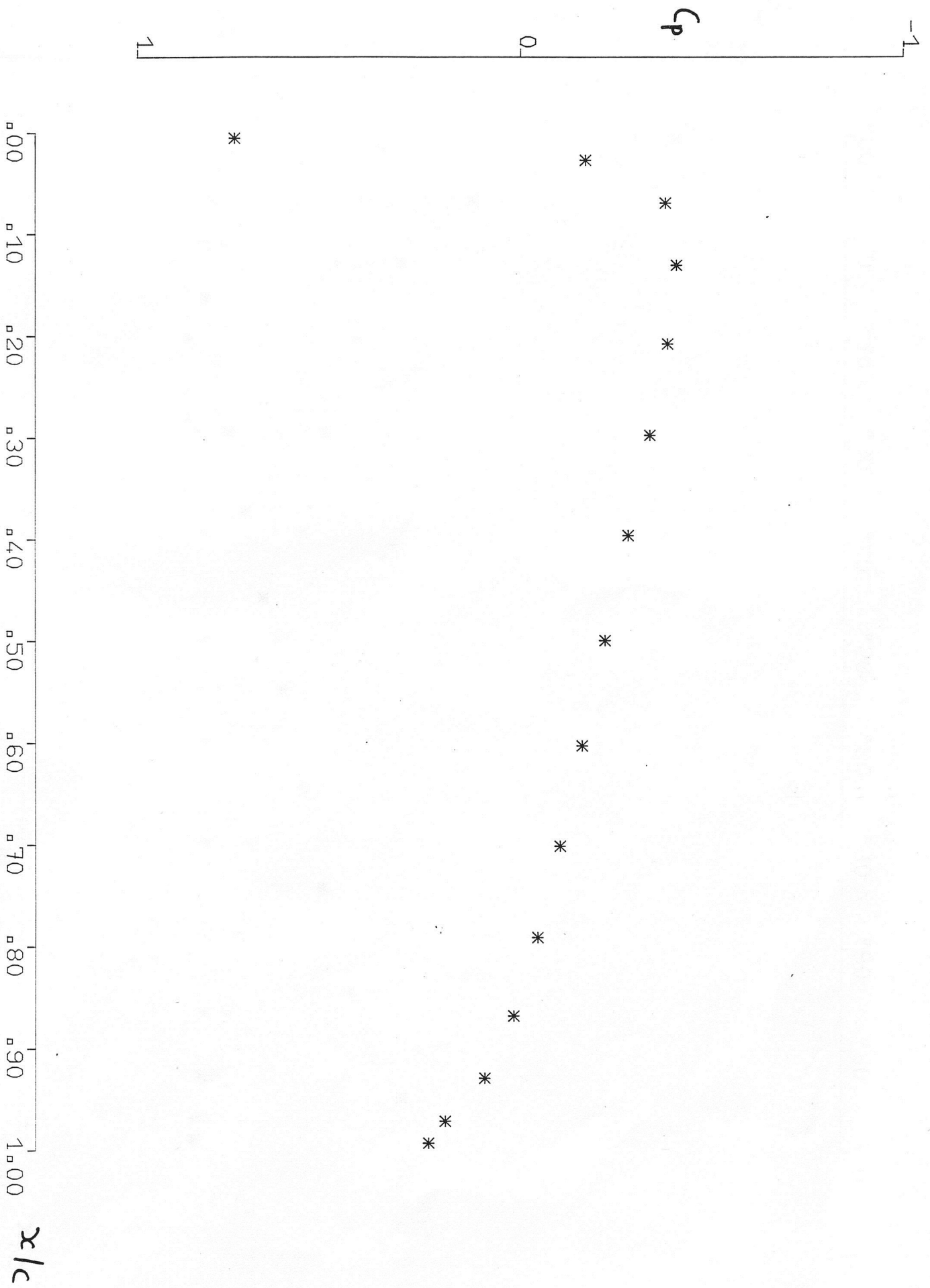
x/c

* METHOD
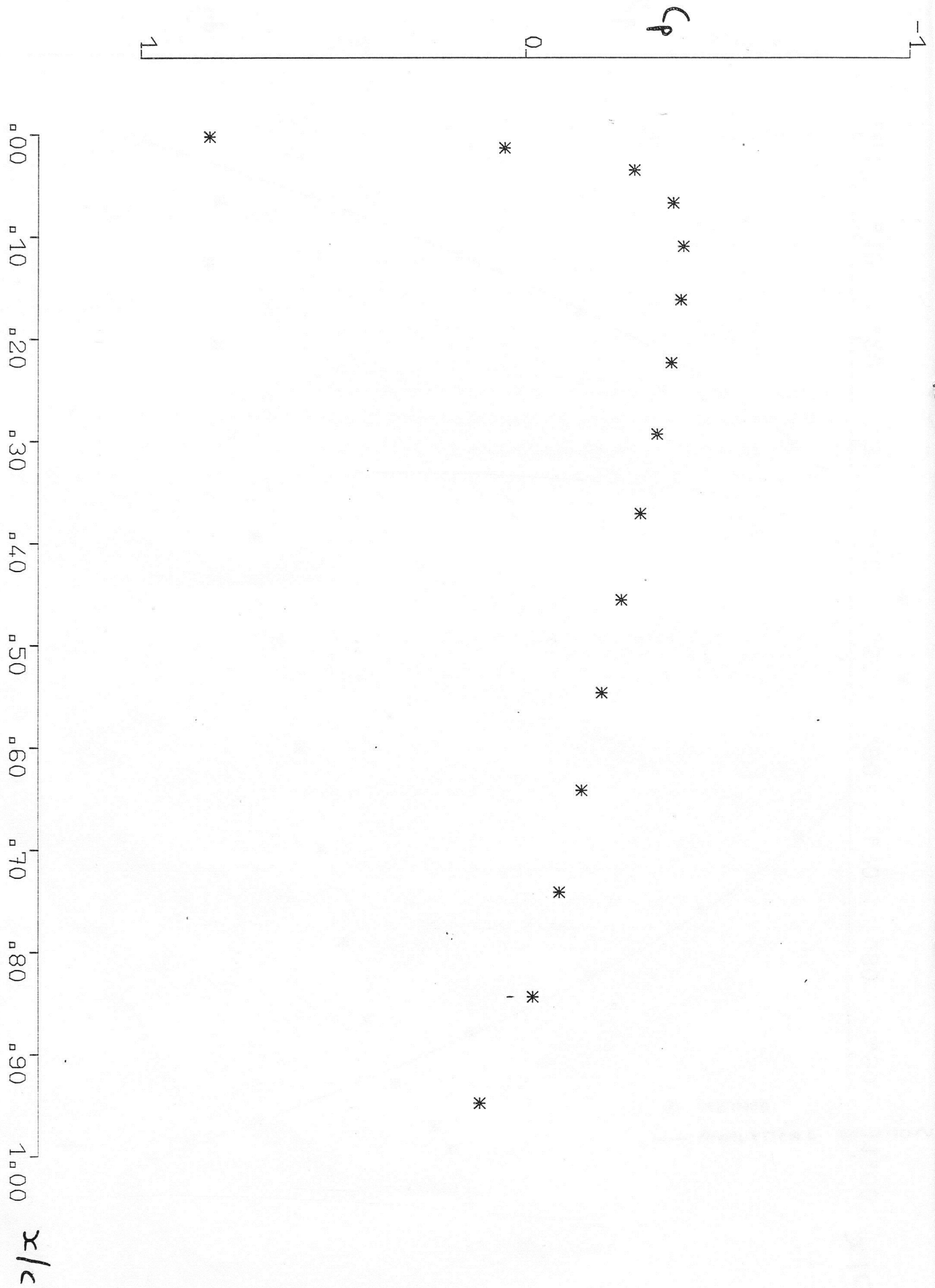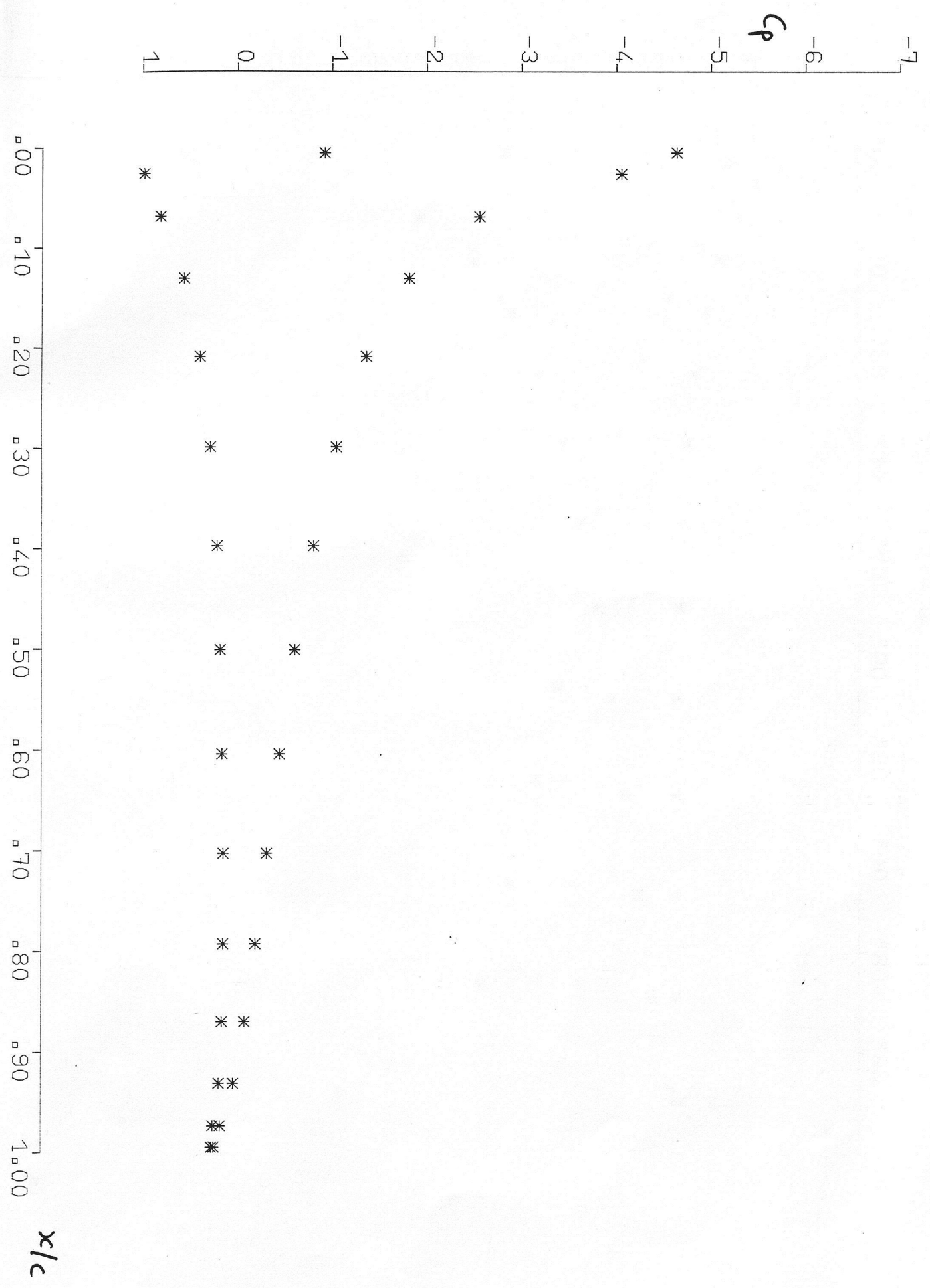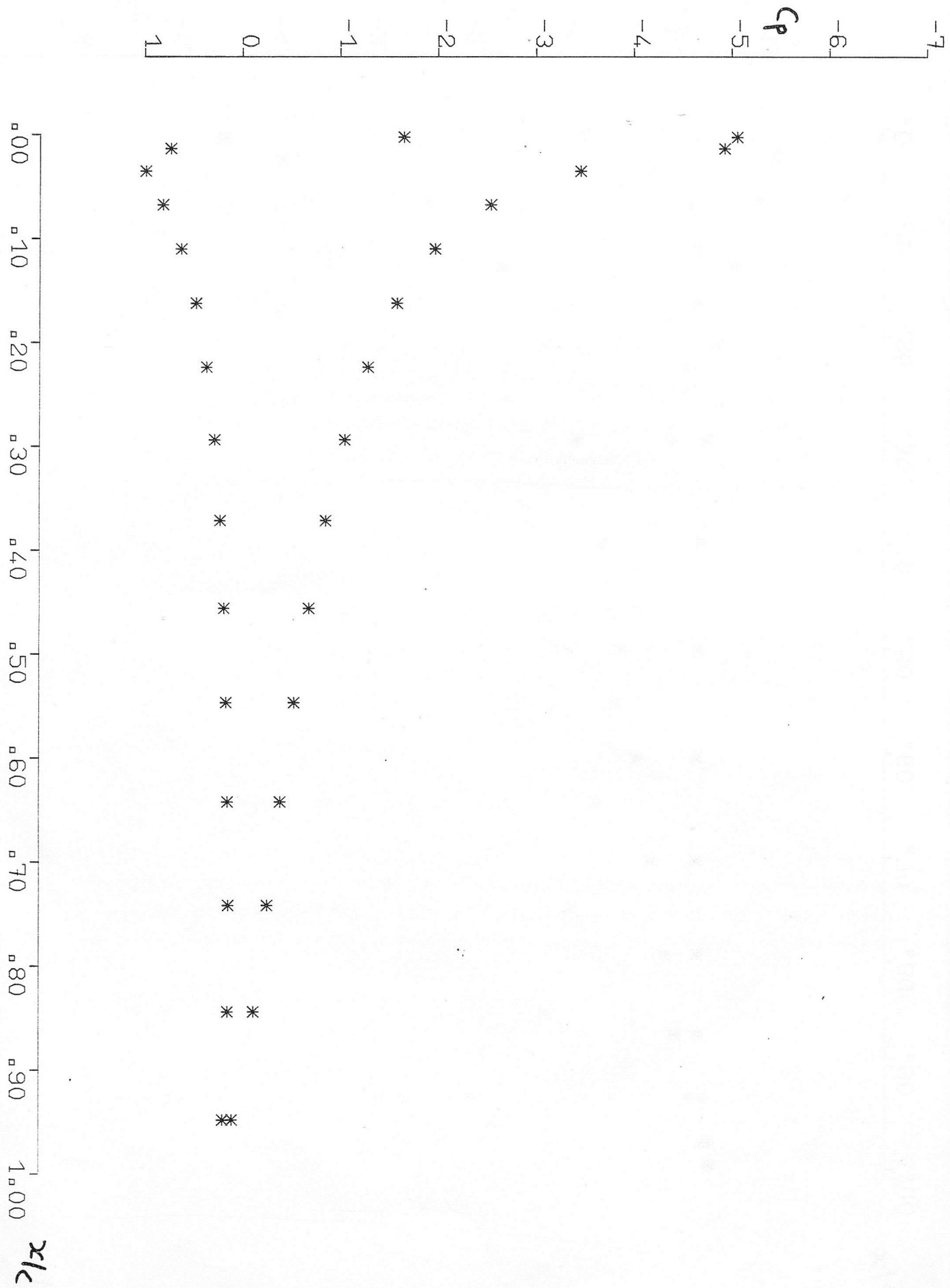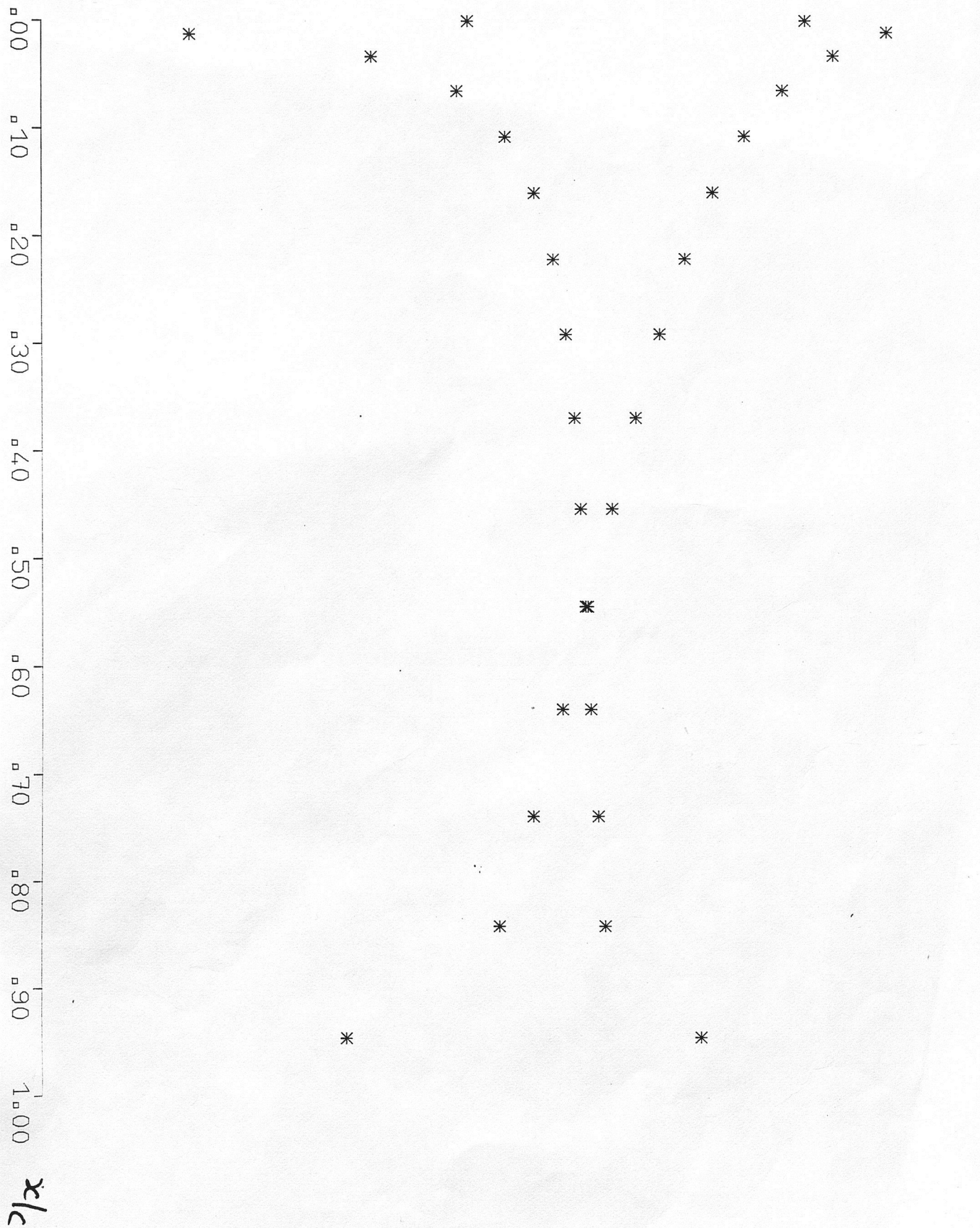— ANALYTICAL SOLUTION.

FIG (4.1)

Fig (4.2)

FIG (4.3).

Fig (4.4)

Fig (4.5)

Fig (4.6)

Fig (4.7)

APPENDIX

Evaluation of relevant integrals

The evaluation of three standard integrals are required

1) $\dfrac{dx}{x^2 + bx + c} = I_1$

2) $\dfrac{xdx}{x^2 + bx + c} = I_2$

3) $\dfrac{x^2\, dx}{x^2 + bx + c} = I_3$

From ref. 11.

$$I_1 = \frac{1}{\sqrt{b^2-4c}}\; \ell n \; \frac{2x+b-\sqrt{b^2-4c}}{2x+b+\sqrt{b^2-4c}}$$

Note: $b^2 - 4c$ will always be > 0

$$I_2 = \frac{1}{2}\, \ell n\, (x^2 + bx + c) - \frac{b}{2}\, I_1$$

$$I_3 = x - \frac{b}{2}\, \ell n\, (x^2 + bx + c) + \frac{b^2 - 2c}{2}\; I_1$$