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# Two-country Model and Foreign Exchange Dynamics

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**Abstract.** We establish the nature of the dynamics of the exchange rate in a two country model with heterogenous firms *a la* Abadir and Talmain (2002).

We are picking an “off-the-shelf” two country (‘home’ or ‘foreign’) monopolistic competition model from Cook and Devereux (2016), denoted as CD later, and adapt it to our setting. We generalize their model by allowing heterogenous productivities among firms, and heterogeneity in the shares of demand for each commodity. Breaking the homogeneity of firms is required for important reasons. First, it is the reality, and it leads to a very different and nonlinear solution of the model. Second, the “representative firm” assumption has been shown in our previous work to be far from innocent, as it linearizes the dynamics of GDP and the other aggregate variables to a process that contradicts the data (loses the long-memory and turning-point features). We also simplify aspects of the model, as CD were interested in a very different problem: the implication of the zero lower-bound of interest rates on monetary policy. We assume no disutility of work (i.e., inelastic labour supply), no capital mobility, and no price friction. The first two restrictions and the Calvo price adjustment (as assumed in CD) simplify the calculations but do not change the spirit of the model.

The home country [resp. foreign] is inhabited by an infinitely lived dynasty with  $N_t$  [resp.  $N_t^*$ ] agents at time  $t$ . The home dynasty aims to maximize, at time  $t$ , the intertemporal utility:

$$U_t = \mathbb{E}_t \sum_{j=0}^{\infty} \beta^j \left[ u_h^{1-\rho} c_{h,j}^\rho + u_f^{1-\rho} c_{f,j}^\rho \right]^{1/\rho}, \text{ with } \sum_{i=h,f} u_i = 1, 0 < \beta, \rho < 1, \quad (1)$$

where  $c_{h,t}$  and  $c_{f,t}$  are the consumption of home and foreign goods by the home representative agent at time  $t$ ,  $u_h$  and  $u_f$  are coefficients of the time-invariant felicity function for the home agents, the coefficients  $\beta$  and  $\rho$  encapsulate time preferences and the degree of substitutability between the home and foreign commodities, and  $\mathbb{E}_t$  denotes the expectation operator conditional on the information available at time  $t$ . Likewise, the intertemporal utility for the foreign country is

$$U_t^* = \mathbb{E}_t \sum_{j=0}^{\infty} \beta^j \left[ u_h^{*1-\rho} c_{h,j}^{*\rho} + u_f^{*1-\rho} c_{f,j}^{*\rho} \right]^{1/\rho}, \text{ with } \sum_{i=h,f} u_i^* = 1, \quad (2)$$

where  $c_{h,t}^*$  and  $c_{f,t}^*$  are the consumption of their *own* home and foreign commodities by the foreign representative agent, and  $u_h^*$  and  $u_f^*$  are the coefficients of the foreign felicity function. In both countries, each agent supplies inelastically one unit of labour in each period, as leisure is absent from the felicity function.

Symmetric felicities: Home and foreign agents share the same felicity function when

$$u_h = u_f^* \text{ and } u_f = u_h^*, \quad (3)$$

as the home good in the ‘home’ country becomes the foreign good when shipped to the ‘foreign’ country and vice versa.

### 1.1 Composite commodities bundles and production.

Following CD, production in the home and foreign countries uses labour according to

$$Y_t = \theta_t N_t^{employed}, \text{ and } Y_t^* = \theta_t^* N_t^{employed*},$$

where  $\theta_t$  and  $\theta_t^*$  are the technical efficiencies at home and abroad, and  $N_t^{employed}$  and  $N_t^{employed*}$  are the quantity of labour used at time  $t$ . As the supply of labour is inelastically, labour market clearing requires that the demand for labour by the firm equals the number of agents in each country:

$$N_t^{employed} = N_t \text{ and } N_t^{employed*} = N_t^*.$$

We will assume that the technical productivities  $\theta_t$  and  $\theta_t^*$  follow the process derived in Abadir and Talmain (2002), henceforth AT, and empirically validated by Abadir et al. (2013). One could endogenize this assumption by assuming, as in AT, that the home and foreign commodities are produced by aggregating, according to a CES function, the output of a large number of monopolistically firms whose log-productivities follow stable autoregressive processes, such as AR(1) with AR coefficients distributed according to a Beta distribution on the unit interval. The profit of the firms,  $\Pi_t$  and  $\Pi_t^*$ , are (in their local currencies)

$$\Pi_t = P_{h,t} Y_t - W_t N_t \text{ and } \Pi_t^* = P_{h,t}^* Y_t^* - W_t^* N_t^*,$$

where  $P_{h,t}$  and  $W_t$  [resp.  $P_{h,t}^*$  and  $W_t^*$ ] are the price of the home good and the wage in the home country [resp. foreign country] in their local currencies. The associated unit costs are

$$\text{unit cost}_t = \frac{W_t}{\theta_t} \text{ and } \text{unit cost}_t^* = \frac{W_t^*}{\theta_t^*}.$$

Home and foreign goods are not storable and can only be used for consumption in the period they are produced, hence market clearance in these markets requires

$$Y_t = C_{h,t} + C_{f,t}^*, \text{ and } Y_t^* = C_{h,t}^* + C_{f,t}, \quad (4)$$

where  $C_{h,t}$  and  $C_{f,t}^*$  are the aggregate consumption at home and abroad of the good produced in the ‘home’ country, and likewise for the foreign produced good:

$$C_{h,t} = N_t c_{h,t}, \quad C_{f,t}^* = N_t^* c_{f,t}^*, \quad C_{f,t} = N_t c_{f,t}, \text{ and } C_{h,t}^* = N_t^* c_{h,t}^*.$$

## 1.2 Demand for each good.

As in CD, we assume that the law of one price holds for each good:

$$P_{f,t} = S_t P_{h,t}^*, \text{ and } P_{f,t}^* = \frac{P_{h,t}}{S_t}, \quad (5)$$

where  $S_t$  is the nominal exchange rate (home price of foreign currency). Free trade in goods is encapsulated in this assumption; however, we assume that factors are immobile. Hence, the home dynasty cannot transfer income across periods, as it would require capital mobility, and must optimize the felicity in (1) or (2) subject to current income in every period. Let the absorptions  $A_t$  and  $A_t^*$  in each country be:

$$A_t = \left[ u_h^{1-\rho} C_{h,t}^\rho + u_f^{1-\rho} C_{f,t}^\rho \right]^{1/\rho}, \text{ and } A_t^* = \left[ u_h^{*1-\rho} C_{h,t}^{*\rho} + u_f^{*1-\rho} C_{f,t}^{*\rho} \right]^{1/\rho}.$$

The aggregate consumptions must satisfy

$$C_{h,t} = \frac{\lambda_t u_h}{P_{h,t}^{1/(1-\rho)}} A_t, \text{ and } C_{f,t}^* = \frac{\lambda_t^* u_f^*}{P_{f,t}^{*1/(1-\rho)}} A_t^*, \quad (6)$$

where  $\lambda_t$  and  $\lambda_t^*$  are Lagrange multipliers which satisfy

$$\frac{1}{\lambda_t^\rho} = \frac{u_h}{P_{h,t}^{\rho/(1-\rho)}} + \frac{u_f}{P_{f,t}^{\rho/(1-\rho)}}, \quad (7)$$

and likewise for  $\lambda_t^*$ . Hence, by the law of one price (5), the aggregate demand for good  $h$  is  $D_t$

$$D_t = C_{h,t} + C_{f,t}^* = \frac{\lambda_t u_h A_t + \lambda_t^* u_f^* A_t^* S_t^{1/(1-\rho)}}{P_{h,t}^{1/(1-\rho)}}, \quad (8)$$

and likewise for the foreign aggregate demand.

Pricing. The elasticity of demand  $\nu$  associated with (8) is

$$\nu \equiv \frac{1}{1-\rho}. \quad (9)$$

The associated optimal pricing for each firm are

$$P_{h,t} = \frac{\text{unit cost}_t}{\rho} \text{ and } P_{h,t}^* = \frac{\text{unit cost}_t^*}{\rho}.$$

After substitution for the unit cost, the associated product prices are

$$P_{h,t} = \frac{W_t}{\rho \theta_t} \text{ and } P_{h,t}^* = \frac{W_t^*}{\rho \theta_t^*}, \text{ so } P_{f,t} = \frac{S_t W_t^*}{\rho \theta_t^*}, \quad (10)$$

$$P_{h,t}^* = \frac{W_t^*}{\rho \theta_t^*} \text{ and } P_{f,t}^* = \frac{P_{h,t}}{S_t} = \frac{W_t}{\rho \theta_t S_t}. \quad (11)$$

Aggregate prices. The aggregate price in the home country  $P_t$  is such that the value of absorption is equal to the aggregate value of its constituents, given from (6)

$$P_t A_t = P_{h,t} C_{h,t} + P_{f,t} C_{f,t}.$$

It is related to the Lagrange multiplier from (7)

$$\frac{1}{P_t^{\rho/(1-\rho)}} = \frac{1}{\lambda_t^\rho},$$

which yields the aggregate price:

$$\frac{1}{P_t^{\rho/(1-\rho)}} = \frac{u_h}{P_{h,t}^{\rho/(1-\rho)}} + \frac{u_f}{P_{f,t}^{\rho/(1-\rho)}}.$$

It can be directly linked to the home aggregate technical efficiency  $\theta_{\varpi,t}$  by the substituting for  $P_{h,t}$  and  $P_{f,t}$  from (10):

$$P_t = \frac{W_t}{\rho \theta_{\varpi,t}}, \text{ with } \theta_{\varpi,t}^{\rho/(1-\rho)} \equiv u_h \theta_t^{\rho/(1-\rho)} + u_f \left( \frac{\theta_t^*}{\varpi_t} \right)^{\rho/(1-\rho)}, \quad (12)$$

where the effective wage ratio  $\varpi_t$  is

$$\varpi_t \equiv \frac{S_t W_t^*}{W_t}.$$

Likewise for the foreign country, the aggregate price  $P_t^*$  is related to the foreign aggregate technical efficiency  $\theta_{\varpi,t}^*$

$$P_t^* = \frac{W_t}{\rho \theta_{\varpi,t}^* S_t} = \frac{W_t^*}{\rho \theta_{\varpi,t}^*}, \text{ with } [\theta_{\varpi,t}^*]^{\rho/(1-\rho)} \equiv u_h^* \left( \frac{\theta_t^*}{\varpi_t} \right)^{\rho/(1-\rho)} + u_f^* \theta_t^{\rho/(1-\rho)}, \text{ and } \theta_{\varpi,t}^* \equiv \varpi_t \theta_{\varpi,t}^*. \quad (13)$$

Symmetric felicities: In this case, the home aggregate technical efficiency is equal to the foreign one in equilibrium:

$$u_h^* = u_f \text{ and } u_f^* = u_h \implies \theta_{\varpi,t} = \theta_{\varpi,t}^*.$$

### 1.3 Good markets clearing.

In the home goods market, substituting from (12) and (7) into (6), we obtain

$$C_{h,t} = \left[ \frac{P_t}{P_{h,t}} \right]^{1/(1-\rho)} u_h A_t, \text{ and } C_{f,t}^* = \left[ \frac{S_t P_t^*}{P_{h,t}} \right]^{1/(1-\rho)} u_f^* A_t^*.$$

Substituting for the prices from (10) and (12) and (13), we obtain

$$C_{h,t} = \left[ \frac{\theta_t}{\theta_{\varpi,t}} \right]^{1/(1-\rho)} u_h A_t, \text{ and } C_{f,t}^* = \left[ \frac{\theta_t}{\theta_{\varpi,t}^*} \right]^{1/(1-\rho)} u_f^* A_t^*. \quad (14)$$

Clearing in the home product market requires demand from (8) equals supply from (4)

$$\begin{aligned} D_t &= Y_t \\ \Rightarrow \theta_t N_t &= \left[ \frac{\theta_t}{\theta_{\varpi,t}} \right]^{1/(1-\rho)} u_h A_t + \left[ \frac{\theta_t}{\theta_{\varpi,t}^*} \right]^{1/(1-\rho)} u_f^* A_t^* \end{aligned}$$

hence market clearing requires

$$1 = \theta_t^{\rho/(1-\rho)} \left[ \frac{u_h a_t}{\theta_{\varpi,t}^{1/(1-\rho)}} + \frac{u_f^* a_t^* n_t^*}{[\theta_{\varpi,t}^*]^{1/(1-\rho)}} \right], \quad (15)$$

where

$$a_t = \frac{A_t}{N_t}, \quad a_t^* = \frac{A_t^*}{N_t^*} \text{ and } n_t^* = \frac{N_t^*}{N_t}.$$

Likewise for the foreign good,

$$C_{h,t}^* = \left[ \frac{P_t^*}{P_{h,t}^*} \right]^{1/(1-\rho)} u_h^* A_t^*, \text{ and } C_{f,t} = \left[ \frac{P_t}{P_{f,t}} \right]^{1/(1-\rho)} u_f A_t,$$

which implies

$$C_{h,t}^* = \left[ \frac{\theta_t^*}{\theta_{\varpi,t}^* \varpi_t} \right]^{1/(1-\rho)} u_h^* A_t^*, \text{ and } C_{f,t} = \left[ \frac{\theta_t}{\theta_{\varpi,t} \varpi_t} \right]^{1/(1-\rho)} u_f A_t. \quad (16)$$

Clearing in the foreign product market requires

$$1 = \frac{[\theta_t^*]^{\rho/(1-\rho)}}{\varpi_t^{1/(1-\rho)}} \left[ \frac{u_h^* a_t^*}{(\theta_{\varpi,t}^*)^{1/(1-\rho)}} + \frac{u_f a_t / n_t^*}{\theta_{\varpi,t}^{1/(1-\rho)}} \right]. \quad (17)$$

#### 1.4 Equilibrium per capita world-absorption under symmetric felicities.

In this case the home product market clearing condition (15) becomes

$$1 = \frac{\theta_t^{\rho/(1-\rho)} u_h [a_t + a_t^* n_t^*]}{\theta_{\varpi,t}^{1/(1-\rho)}},$$

which reduces, using the elasticity of demand  $\nu$  from (9), to

$$\left[ \frac{\theta_{\varpi,t}}{\theta_t} \right]^\nu = \frac{u_h [a_t + a_t^* n_t^*]}{\theta_t}, \quad (18)$$



which link per capita world-absorption,  $a_t + a_t^* n_t^*$ , to the parameters of the felicities and technical efficiencies and the exchange rate.

Likewise, (17) becomes

$$1 = \frac{[\theta_t^*]^{\nu-1}}{\varpi_t^\nu \theta_{\varpi,t}^\nu} [u_f a_t^* + u_f a_t / n_t^*],$$

hence

$$\left[ \frac{\theta_{\varpi,t}}{\theta_t^*} \right]^\nu = \frac{u_f [a_t + a_t^* n_t^*]}{\varpi_t^\nu \theta_t^* n_t^*}. \quad (19)$$

Dividing (18) by (19), we derive the equilibrium effective wage ratio  $\varpi_t$  in terms of the ratio of technical efficiencies  $\eta_t$  and of the ratio of felicity preferences  $\psi_t$ :

$$\varpi_t = \eta_t^\rho \psi_t^{1-\rho}, \quad (20)$$

where

$$\eta_t \equiv \frac{\theta_t^*}{\theta_t} \text{ and } \psi_t \equiv \frac{u_f}{u_h n_t^*}. \quad (21)$$

Also, replacing  $\varpi_t$  from (20) into (12)

$$\left[ \frac{\theta_{\varpi,t}}{\theta_t} \right]^{\rho/(1-\rho)} = u_h + u_f \left( \frac{\eta_t}{\varpi_t} \right)^{\rho/(1-\rho)} = u_h + u_f \left( \frac{\eta_t}{\psi_t} \right)^\rho. \quad (22)$$

Replacing for  $(\theta_{\varpi,t}/\theta_t)$  from (22) into (18), we obtain the equilibrium value of the per capita world-absorption

$$a_t + a_t^* n_t^* = \theta_t u_h^{(1-\rho)/\rho} \left[ 1 + n_t^* \eta_t^\rho \psi_t^{1-\rho} \right]^{1/\rho}. \quad (23)$$

### 1.5 Equilibrium per capita absorption under symmetric felicities.

GDP and CA (current account): at an equilibrium, production is equal to aggregate demand. From (8)

$$\begin{aligned} \text{GDP}_t &= P_t D_t = P_{h,t} C_{h,t} + P_{f,t}^* C_{f,t}^* = P_{h,t} (C_{h,t} + C_{f,t}^*), \\ \text{GDP}_t^* &= P_{f,t} (C_{h,t}^* + C_{f,t}), \end{aligned}$$

by the law of one price. The current account  $\text{CA}_t$  is

$$\begin{aligned} \text{CA}_t &= \text{GDP}_t - P_t A_t \\ &= P_{h,t} C_{f,t}^* - P_{f,t} C_{f,t}. \end{aligned}$$

In the absence of capital mobility, the current account must be balanced in every period:

$$P_{h,t} C_{f,t}^* = P_{f,t} C_{f,t}.$$

Per capita absorption: Substituting for prices and consumptions from (10), (11), (16) and (14) we obtain

$$\begin{aligned} \frac{W_t}{\rho\theta_t} \left[ \frac{\theta_t}{\theta_{\varpi,t}} \right]^\nu u_h A_t^* &= \frac{S_t W_t^*}{\rho\theta_t^*} \left[ \frac{\theta_t^*}{\theta_{\varpi,t}} \right]^\nu u_f A_t \\ \implies a_t^* &= [\zeta_t]^{\frac{2-\rho}{1-\rho}} a_t, \text{ where } \zeta_t \equiv \eta_t^\rho \psi_t^{1-\rho}. \end{aligned} \quad (24)$$

Replacing  $a^*$  in (23),

$$\begin{aligned} \left[ 1 + \left[ \eta_t^{\rho/(1-\rho)} \psi_t \right]^{2-\rho} n_t^* \right] a_t &= \theta_t u_h^{(1-\rho)/\rho} \left[ 1 + \frac{u_f}{u_h} \left( \frac{\eta_t}{\psi_t} \right)^\rho \right]^{1/\rho} \\ \implies a_t &= \frac{\theta_t u_h^{(1-\rho)/\rho} [1 + n_t^* \zeta_t]^{1/\rho}}{1 + n_t^* [\zeta_t]^{(2-\rho)/(1-\rho)}}. \end{aligned} \quad (25)$$

1.6 National Income identity and equilibrium exchange rate under symmetric felicities.

National Income in the home country is the sum of wages and firms' profits. The per-unit profit is, from (10),

$$P_{h,t} - \frac{W_t}{\theta_t} = \frac{1-\rho}{\rho} \frac{W_t}{\theta_t},$$

hence the profit margin is  $(1-\rho)/\rho$ . Since aggregate wages are  $N_t W_t$ , the aggregate profits are

$$\Pi_t = \frac{1-\rho}{\rho} N_t W_t,$$

and the National Income,  $NI_t$ , is

$$NI_t = N_t W_t + \Pi_t = \frac{N_t W_t}{\rho}.$$

Since the current account is always balanced, absorption is equal to national income

$$P_t A_t = \frac{N_t W_t}{\rho} \implies \frac{W_t}{P_t} = \rho a_t. \quad (26)$$

Likewise in the foreign country

$$\frac{W_t^*}{P_t^*} = \rho a_t^*. \quad (27)$$

Therefore, recalling (24),

$$\frac{W_t^* P_t}{P_t^* W_t} = \frac{a_t^*}{a_t} = [\zeta_t]^{\frac{2-\rho}{1-\rho}}.$$

Recalling from (20) the value of the effective wage ratio:

$$\varpi_t = \frac{W_t^* S_t}{W_t} = \zeta_t,$$

Hence

$$\frac{W_t^* P_t}{P_t^* W_t} = \frac{P_t}{S_t P_t^*} \zeta_t = [\zeta_t]^{\frac{2-\rho}{1-\rho}} \implies \frac{S_t P_t^*}{P_t} = \zeta_t^{-1/(1-\rho)} = \left[ \frac{\theta_t^*}{\theta_t} \right]^{-\rho/(1-\rho)} \frac{u_h n_t^*}{u_f} \quad (28)$$

by (24) and (21).

### 2.1 Implication for the dynamics of the exchange rate.

Taking logs,

$$s_t = \log P_t - \log P_t^* + \frac{\rho}{1-\rho} (\log \theta_t - \log \theta_t^*) + \log n_t^* + \log \frac{u_h}{u_f}. \quad (29)$$

The last term is a constant, and  $\log n_t^* = \log(N_t^*/N_t)$  is a linear time trend if the two populations grow at different rates but constant otherwise. The auto-correlation function (ACF) is invariant to such trends and constants when defined with time-varying means, as in AT and in Abadir et al. (2013),

$$\rho_\tau \equiv \text{corr}(z_t, z_{t-\tau}) \equiv \frac{\text{cov}(z_t, z_{t-\tau})}{\sqrt{\text{var}(z_t) \text{var}(z_{t-\tau})}}, \quad (30)$$

where  $\text{cov}(z_t, z_{t-\tau}) \equiv E[(z_t - E z_t)(z_{t-\tau} - E z_{t-\tau})]$ . Hence, the ACF of  $s_t$  is the same as the ACF of

$$\log P_t - \log P_t^* + \frac{\rho}{1-\rho} (\log \theta_t - \log \theta_t^*); \quad (31)$$

and we now consider what the ACFs of  $(\log \theta_t - \log \theta_t^*)$  and  $(\log P_t - \log P_t^*)$  are, and how they combine.

The ACF of  $\log \theta_t$  (and similarly for  $\log \theta_t^*$ ) has a series expansion. Its leading term was derived on p.766 of AT and its full representation was obtained in Abadir et al. (2013) as

$$\rho_\tau \approx \frac{1 - a[1 - \cos(\omega\tau)]}{1 + (b\tau)^c} \quad (32)$$

whose Fourier inversion produces a spectrum  $f(\lambda)$  of magnitude proportional to  $|\lambda - \omega|^{c-1}$ . If the average length  $\omega$  of the business cycle in the two countries is approximately the same, then the two spectra combine as in Section 4 of Granger (1981) to give the same family of spectra and therefore the ACF of  $(\log \theta_t - \log \theta_t^*)$  is (32) again. Finally, if  $(\log P_t - \log P_t^*)$  has short memory dynamics, then the ACF of the log exchange rate  $s_t$  is dominated by the productivities' term  $(\log \theta_t - \log \theta_t^*)$  and has the form (32). A sufficient (but not necessary) condition for these dynamics to take place is if  $(\log P_t - \log P_t^*)$  is a trend-stationary process: the price ratio is allowed to diverge along some possibly nonlinear trend, but not stray uncontrollably away from it. A special case of this arises if (as is the case in developed countries) the Central Banks try to synchronize the expansion or contraction phases of their monetary policies.

#### Additional references

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