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On The Three-Receiver Multilevel Broadcast Channel with Random Parameters

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***Abstract*—In this paper we extend the analysis of two-receiver broadcast channels with random parameters to the three-receivers case. Specifically we base our work on Nair and El Gamal's results for the three-receiver discrete memoryless multilevel broadcast channel and assume that state information is available non-causally at the transmitter. We provide an achievable rate region for this setting and acknowledge its importance in the study of multiuser cognitive radio configurations.**

I. Introduction

The broadcast channel in its simplest configuration consists of one transmitter and two receivers. In general the messages intended for each receiver are different although there could be a common message to be decoded by both receivers. The first results for this channel are due to Cover [1], who develops an encoding scheme for this channel: *superposition coding*. Capacity results for the degraded broadcast channel were first proved by Bergmans [2][3] and Gallager [4].

Capacity results for other special cases of broadcast channels: the *less noisy* [5] and *more capable* [6] broadcast channels have been found as well. The best known achievable rate region for the general broadcast channel is due to Marton [7]. The two-receiver broadcast channel with random parameters was analysed by Steinberg *et al.*, first for the degraded case [8] and later for the general case [9]. Gel'fand-Pinsker [10] coding is used in their approach to obtain the achievable rates. More recently, Nair and El Gamal [11] proved that the straightforward extension of Korner and Marton's region [12] to more than two receivers is not optimal in general. The authors showed that this natural extension applied to the three-receiver multilevel broadcast channel is strictly smaller than the capacity region which they found.

In this work we intend to extend Nair and El Gamal's results for the multilevel broadcast channel to the case where state information is available non-causally at the transmitter. The analysis of channels with states known at the transmitter have been motivated by the increasing research in the area of cognitive networks. In cognitive configurations the available information at the cognitive transmitter (primary user's message) is utilised by the encoder to increase the overall system capacity region by applying a precoding against primary user's transmissions [13].

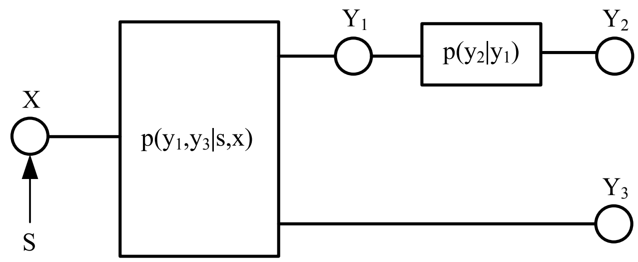

Figure 1. Multilevel broadcast channel with random parameters.

II. Channel model and definitions

First of all we review the encoding scheme of [11]. The three-receiver multilevel broadcast channel is resembled by the channel depicted in Figure 1 omitting the presence of the state s. Nair and El Gamal considered two messages to be transmitted, one common intended to all receivers and one private intended to receiver 1 only. They proved that the capacity region of such a channel consists of the set of rate pairs (R_0, R_1) such that

$$\begin{aligned} R_0 &\leq \min\{I(U;Y_2), I(V;Y_3)\} \\ R_1 &\leq I(X;Y_1|U) \\ R_0 + R_1 &\leq I(V;Y_3) + I(X;Y_1|V) \end{aligned} \tag{1}$$

where U, V and X are random variables (r.v.) used to encode the messages. Their encoding scheme is such that allows *indirect* decoding at receiver 3. The common message m_0 is encoded in U and the private message m_1 is divided into two parts, m_{10} which is encoded in V and m_{11} which is encoded in X. The codes are superimposed using superposition coding. Receiver 2 obtains m_0 by decoding U, receiver 3 obtains m_0 indirectly by decoding V and receiver 1 obtains both messages by decoding the three codes.

The presence of state information at the transmitter in the channel of Figure 1 allows us to combine Gel'fand-Pinsker coding with the encoding scheme previously described as it will be detailed later on.

Notation and notion of ϵ-typicality, strong typicality of [14] is used throughout the paper. Consider the three-receiver discrete memoryless multilevel broadcast channel with random parameters shown in Figure 1. It consists of an input alphabet

$\mathcal{X}$, state space $\mathcal{S}$, output alphabets $\mathcal{Y}_1$, $\mathcal{Y}_2$, and $\mathcal{Y}_3$, and a probability transition function $p(y_1, y_2, y_3|x, s)$, where the state s is random, taking values in $\mathcal{S}$ according to the probability mass function (PMF) $p(s)$. Due to the memoryless assumption, we have:

$$p^n(y_1^n, y_2^n, y_3^n|s^n, x^n) = \prod_{i=1}^{n} p(y_{1,i}, y_{2,i}, y_{3,i}|s_i, x_i) \quad (2)$$

and

$$p^n(s^n) = \prod_{i=1}^{n} p(s_i). \quad (3)$$

A $(2^{nR_0}, 2^{nR_1}, n)$ two-degraded message set code for a three-receiver broadcast channel consists of a pair of uniformly distributed messages $m_0 \in [1 : 2^{nR_0}]$ and $m_1 \in [1 : 2^{nR_1}]$ and complies with the encoding function

$$f : [1 : 2^{nR_0}] \times [1 : 2^{nR_1}] \times \mathcal{S}^n \longrightarrow \mathcal{X}^n \quad (4)$$

and 3 decoding functions

$$\begin{aligned} g_{y_1} &: \mathcal{Y}_1^n \longrightarrow [1 : 2^{nR_0}] \times [1 : 2^{nR_1}] \\ g_{y_2} &: \mathcal{Y}_2^n \longrightarrow [1 : 2^{nR_0}] \\ g_{y_3} &: \mathcal{Y}_3^n \longrightarrow [1 : 2^{nR_0}] \end{aligned}$$

A rate tuple (R_0, R_1) is said to be achievable if there exists a sequence of $(2^{nR_0}, 2^{nR_1}, n)$ two-degraded message set codes with probability of error bounded ($P_e^n \longrightarrow 0$). Due to the degradedness of the channel and the degraded message sets condition, we have: $p(y_1, y_2, y_3|x, s) = p(y_1, y_3|x, s)p(y_2|y_1)$.

III. MAIN RESULT

Our main result consists of an achievable rate region for the three-receiver multilevel broadcast channel with random parameters and can be stated as a theorem as follows:

Theorem 1. *An achievable rate region of the discrete memoryless three-receiver multilevel broadcast channel with random parameters is the set of rate pairs (R_0,R_1) such that:*

$$\begin{aligned} R_0 &\leq \min\big\{I(U; Y_2), I(V; Y_3) - I(V; S|U)\big\} \\ &\quad - I(U; S) \\ R_1 &\leq I(W; Y_1|U) - I(W; S|V) - I(V; S|U) \\ R_0 + R_1 &\leq I(V; Y_3) + I(W; Y_1|V) \\ &\quad - \big[I(W; S|V) + I(V; S|U) + I(U; S)\big] \end{aligned} \quad (5)$$

for some $p(u)p(v|u)p(w|v)p(xs|wvu)$ where U, V and W are auxiliary random variables with cardinalities satisfying $|\mathcal{U}| \leq |\mathcal{X}||\mathcal{S}| + 8$, $|\mathcal{V}| \leq (|\mathcal{X}||\mathcal{S}| + 8)(|\mathcal{X}||\mathcal{S}| + 4)$, $|\mathcal{W}| \leq (|\mathcal{X}||\mathcal{S}| + 8)(|\mathcal{X}||\mathcal{S}| + 4)(|\mathcal{X}||\mathcal{S}| + 1)$.

Proof: The proof follows standard steps and combines the analysis for channels with states with simultaneous decoding. First the codebook generation is explained followed by the encoding process, the analysis of the probability of error for each receiver and finally a derivation of an upper bound to the cardinality of each auxiliary random variable is presented.

1) Codebook generation: Generate 2^{nL_0} independent codewords $u^n(j_0, m_0)$ of length n and throw them into $M_0 = 2^{nR_0}$ bins uniformly. Set $J_0 = 2^{L_0 - R_0}$. For each $u^n(j_0, m_0)$ generate 2^{nL_1} sequences $v^n(j_1, m_{11})$ of length n and throw them into $M_{11} = 2^{nS_1}$ bins uniformly. Set $J_1 = 2^{L_1 - S_1}$. For each $v^n(j_1, m_{11})$ generate 2^{nL_2} sequences $w^n(j_2, m_{12})$ and throw them into $M_{12} = 2^{nS_2}$ bins uniformly. Set $J_2 = 2^{L_2 - S_2}$. Generate a sequence x^n according to the memoryless distribution defined by the n-product $P_{X|UVWS}$.

2) Encoding: Given the state sequence s^n the encoding proceeds as follows:
To send (m_0, m_1), the encoder splits m_1 into m_{11} and m_{12}. Let $j_0(m_0, s^n)$ be the smallest integer such that the codeword in the bin m_0, $u^n(j_0, m_0)$ is jointly typical with s^n. If such j_0 does not exist, set $j_0(m_0, s^n) = J_0$ and an encoding error is declared.
Similarly let $j_1(m_0, m_{11}, s^n)$ be the smallest j_1 in bin m_{11} such that $v^n(j_1, m_{11})$ is jointly typical with s^n given $u^n(j_0, m_0)$. If such j_1 does not exist, set $j_1(m_{11}, s^n) = J_1$ and an encoding error is declared.
Finally, let $j_2(m_0, m_{11}, m_{12}, s^n)$ be the smallest j_2 in bin m_{12} such that $w^n(j_2, m_{12})$ is jointly typical with s^n given $v^n(j_1, m_{11})$. If such j_2 does not exist, set $j_2(m_0, m_{11}, m_{12}, s^n) = J_2$ and an encoding error is declared.
Generate x^n according to $\prod_{t=1}^{n} P(x^{(t)}|u^{(t)}, v^{(t)}, w^{(t)}, s^{(t)})$.

3) Decoding: Receiver 2: A $u^n(j_0, m_0)$ jointly typical with y_2^n is sought. If all the elements found have the same message index, this will be regarded as the message sent. Otherwise M_0 is assumed to be sent and a decoding error is declared.
Receiver 1: Receiver 1 declares that (m_0, m_{11}, m_{12}) is sent if it is the unique triple such that $u^n(m_0)$, $v^n(m_0, m_{11})$, $w^n(m_0, m_{11}, m_{12})$ and y_1^n are jointly typical.
Receiver 3: Receiver 3 declares that m_0 is sent if it is the unique index such that $u^n(m_0)$, $v^n(m_0, m_{11})$ and y_3^n are jointly typical for some $m_{11} \in [1 : 2^{nS_1}]$.

4) Analysis of the probability of error: Receiver 2: Define the error event sets:

$$\begin{aligned} A_1(m_0, s^n) &= \big\{\nexists\, j_0 \in \{1, 2, \ldots, J_0\} : \\ &\qquad \big(u^n(j_0, m_0), s^n\big) \in T_{US}\big\} \\ A_2(m_{11}, s^n|u^n) &= \big\{\nexists\, j_1 \in \{1, 2, \ldots, J_1\} : \\ &\qquad \big(v^n(j_1, m_{11}), s^n\big) \in T_{VS|U}\big\} \\ A_3(m_{12}, s^n|v^n) &= \big\{\nexists\, j_2 \in \{1, 2, \ldots, J_2\} : \\ &\qquad \big(w^n(j_2, m_{12}), s^n\big) \in T_{WS|V}\big\} \\ A_4(m_0, s^n) &= \big\{\big(u^n(j_0, m_0), y_2^n\big) \notin T_{UY_2}\big\} \\ A_5(m_0, s^n) &= \big\{\exists\, u^n(j_0', m_0') : m_0' \neq m_0 \text{ and} \\ &\qquad u^n(j_0', m_0') \in T_{UY_2}\big\} \end{aligned}$$

where T_{US} is the set of jointly typical sequences u^n and s^n. The probability of error can be bounded as in (6).

$$
\begin{aligned}
P_e = \frac{1}{M_0 M_{11} M_{12}} & \sum_{m_0,m_{11},m_{12}} \sum_{s\in T_S} p^n(s^n)\big[P(A_1) \\
& + P(A_2|A_1^c) + P(A_3|A_1^c A_2^c) \\
& + P(A_4|A_1^c A_2^c A_3^c) + P(A_5|A_1^c A_2^c A_3^c A_4^c)\big] \\
& + P_S(T_S^c) \qquad (6)
\end{aligned}
$$

Now we need to bound only the partial probabilities of error. Starting with the first error event set

$$
\begin{aligned}
P\big(A_1(m_0,s^n)\big) &= P\Big(\bigcap_{j_0=1}^{J_0}\big\{\big(u^n(j_0,m_0),s^n\big)\notin T_{US}\big\}\Big) \\
&= \big[P\big((u^n(1,m_0),s^n)\notin T_{US}\big)\big]^{J_0} \\
&= \big[1-P\big((u^n(1,m_0),s^n)\in T_{US}\big)\big]^{J_0} \\
&\leq \big[1-2^{-n(I(U;S)+3\epsilon)}\big]^{J_0} \\
&\leq e^{-2^{-n(I(U;S)+3\epsilon-L_0+R_0)}} \qquad (7)
\end{aligned}
$$

where the last inequality follows the form of [14, p. 323] and ϵ is an arbitrarily small positive number. The probability of error will decay to 0 as $n\to\infty$ as long as

$$
L_0 - R_0 > I(U;S) + 3\epsilon. \qquad (8)
$$

Similarly for $A_2|A_1^c$ and for $A_3|A_1^c A_2^c$ the two following inequalities allow the probability of error to be bounded:

$$
L_1 - S_1 > I(V;S|U) + 3\epsilon \qquad (9)
$$
$$
L_2 - S_2 > I(W;S|V) + 3\epsilon \qquad (10)
$$

Conditioning on A_1^c,A_2^c and A_3^c, $(u^n,v^n,w^n,s^n)\in T_{UVWS}$ which implies that

$$
\begin{aligned}
P\big(\{(u^n,v^n,w^n,s^n,y_2^n)\in T_{UVWSY_2}\}|A_1^c,A_2^c,A_3^c\big) &\longrightarrow 1, \\
\text{as } n &\longrightarrow \infty. \qquad (11)
\end{aligned}
$$

Moreover this fact implies that $(u^n,y_2^n)\in T_{UY_2}$ (it is the true distribution) and therefore

$$
\begin{aligned}
P\big(A_4|A_1^c,A_2^c,A_3^c\big) &\longrightarrow 0, \\
\text{as } n &\longrightarrow \infty. \qquad (12)
\end{aligned}
$$

For $P(A_5(m_0,s^n)|A_1^c,A_2^c,A_3^c,A_4^c)$, it is not difficult to see that conditioned on A_1^c,A_2^c,A_3^c,A_4^c, Y_2^n is typical and for $m_0'\neq m_0$, $u^n(j_0',m_0')$ and y_2^n are independent.

$$
\begin{aligned}
P(A_5(m_0,s^n)&|A_1^c,A_2^c,A_3^c,A_4^c) = \\
&= P\Bigg(\bigcup_{j_0',m_0':m_0'\neq m_0}\big\{u^n(j_0',m_0'),y_2^n\in T_{UY_2}\big\}\Bigg) \\
&\leq J_0 2^{nR_0} P\big(u^n(j_0',m_0'),y_2^n\in T_{UY_2}\big) \\
&= J_0 2^{nR_0} \sum_{u^n:(u^n,y_2^n)\in T_{UY_2}} P_u(u) \\
&\leq J_0 2^{nR_0} 2^{-n(I(U;Y_2)-\epsilon)} \\
&= 2^{n(L_0-R_0+R_0-I(U;Y_2)+\epsilon)} \qquad (13)
\end{aligned}
$$

then for

$$
L_0 < I(U;Y_2) - \epsilon, \qquad (14)
$$

the probability of error is bounded. Combining (8) and (14) we obtain

$$
R_0 < I(U;Y_2) - I(U;S) - 4\epsilon. \qquad (15)
$$

Hence we have bounded the encoding error and the decoding error for receiver 2.

Receiver 3: Let us define the event sets:

$$
\begin{aligned}
B_1(m_0,m_{11}) &= \big\{(u^n(j_0,m_0),v^n(j_1,m_{11}), \\
&\qquad y_3^n)\notin T_{UVY_3}\big\} \\
B_2(m_0,m_{11}) &= \big\{m_0'\neq m_0 \text{ and any } m_{11}: \\
&\qquad (u^n(j_0',m_0'),v^n(j_1',m_{11}),y_3^n)\in T_{UVY_3}\big\}.
\end{aligned}
$$

Similarly as for receiver 2 the following inequalities can be obtained:

$$
L_0 + L_1 < I(U,V;Y_3) \qquad (16)
$$
$$
L_0 + L_1 < I(V;Y_3), \qquad (17)
$$

where the last step is due to markovity ($U\to V\to W$) and for simplicity the dependency on epsilon has been omitted. Combining (9) and (17) the following inequality is obtained for this receiver:

$$
R_0 + S_1 < I(V;Y_3) - I(V;S|U) - I(U;S). \qquad (18)
$$

Receiver 1: For receiver 1 we have the following error event sets:

$$
\begin{aligned}
D_1(m_0,m_{11},m_{12}) &= \big\{(u^n(j_0,m_0),v^n(j_1,m_{11}), \\
&\qquad w^n(j_2,m_{12}),y_1^n)\notin T_{UVWY_1}\big\} \\
D_2(m_0,m_{11},m_{12}) &= \big\{\exists\, w^n(j_2',m_{12}'): \\
&\qquad m_{12}'\neq m_{12} \text{ and } (u^n(j_0,m_0),v^n(j_1,m_{11}), \\
&\qquad w^n(j_2',m_{12}'),y_1^n)\in T_{UVWY_1}\big\} \\
D_3(m_0,m_{11},m_{12}) &= \big\{\exists\, v^n(j_1',m_{11}') \text{ and } w^n(j_2',m_{12}'): \\
&\qquad m_{11}'\neq m_{11} \text{ and } m_{12}'\neq m_{12} \text{ and} \\
&\qquad (u^n(j_0,m_0),v^n(j_1',m_{11}'), \\
&\qquad w^n(j_2',m_{12}'),y_1^n)\in T_{UVWY_1}\big\} \\
D_4(m_0,m_{11},m_{12}) &= \big\{\exists\, u^n(j_0',m_0'),v^n(j_1',m_{11}') \text{ and} \\
&\qquad w^n(j_2',m_{12}'): m_0'\neq m_0, m_{11}'\neq m_{11} \text{ and} \\
&\qquad m_{12}'\neq m_{12} \text{ and } (u^n(j_0',m_0'),v^n(j_1',m_{11}'), \\
&\qquad w^n(j_2',m_{12}'),y_1^n)\in T_{UVWY_1}\big\}.
\end{aligned}
$$

Similar analysis of the probability of error events leads to the following set of inequalities, where the dependencies on epsilon have been omitted again for simplicity:

$$
L_0 + L_1 + L_2 < I(W;Y_1) \qquad (19)
$$
$$
L_1 + L_2 < I(W;Y_1|U) \qquad (20)
$$
$$
L_2 < I(W;Y_1|V) \qquad (21)
$$

and using (8), (9) and (10) along with (15) and (18) produce the set of inequalities:

$$R_0 < I(U;Y_2) - I(U;S) \tag{22}$$
$$R_0 + S_1 < I(V;Y_3) - I(V;S|U) - I(U;S) \tag{23}$$
$$\begin{aligned} R_0 + S_1 + S_2 < I(W;Y_1) - I(W;S|V) - \\ - I(V;S|U) - I(U;S) \end{aligned} \tag{24}$$
$$\begin{aligned} S_1 + S_2 < I(W;Y_1|U) - I(W;S|V) - \\ - I(V;S|U) \end{aligned} \tag{25}$$
$$S_2 < I(W;Y_1|V) - I(W;S|V). \tag{26}$$

Applying the Fourier-Motzkin [15] elimination procedure to the preceding set of inequalities to eliminate S_1 and S_2 produces the set of the theorem.

5) Cardinality bounds: The proof of the cardinality bounds of the auxiliary random variables is based on the support lemma [16, p.310] and standard arguments by Ahlswede and Korner [17]. Without loss of generality let us assume $m = |\mathcal{X}||\mathcal{S}|$. Given $(U,V,W,S,X) \sim p(u)p(v|u)p(w|v)p(sx|wvu)$ consider the following $m+8$ continuous functions of $p(v|u)$:

$g_j(p_{v|u}(v|u))$	j
$p_{xs|u}(j|u)$	$1,...,m-1$
$H(Y_1|U=u)$	m
$H(Y_2|U=u)$	$m+1$
$H(Y_3|U=u)$	$m+2$
$H(Y_1|V,U=u)$	$m+3$
$H(Y_3|V,U=u)$	$m+4$
$H(Y_1|W,V,U=u)$	$m+5$
$H(S|U=u)$	$m+6$
$H(S|V,U=u)$	$m+7$
$H(S|W,V,U=u)$	$m+8$

Now due to the support lemma there exists a random variable U_1 restricted to the cardinality $m+8$ such that:

$$\begin{aligned} I(U_1;Y_2) &= I(U;Y_2) \\ I(U_1;Y_3) &= I(U;Y_3) \\ I(V';Y_1|U_1) &= I(V;Y_1|U) \\ I(W';Y_1|V',U_1) &= I(W;Y_1|V,U) \\ I(V';S|U_1) &= I(V;S|U) \\ I(U_1;S) &= I(U;S) \\ I(W';Y_1|U_1) &= I(W;Y_1|U) \\ I(Y_3;V',U_1) &= I(Y_3;V,U) \end{aligned}$$

The distributions of V and W are not necessarily preserved, and the resulting random variables are denoted by V' and W'. For each $U_1 = u_1$ consider the following continuous functions of $p(w'|v',u_1)$:

$g_j(p_{w'|v'u_1}(w'|v'u_1))$	j
$p_{xs|v'u_1}(j|v'u_1)$	$1,...,m-1$
$H(Y_1|V'=v',U_1=u_1)$	m
$H(Y_1|W',V'=v',U_1=u_1)$	$m+1$
$H(Y_3|V'=v',U_1=u_1)$	$m+2$
$H(S|V'=v',U_1=u_1)$	$m+3$
$H(S|W',V'=v',U_1=u_1)$	$m+4$

Again due to the support lemma there is a $V(u_1)$ with cardinality $m+4$ such that:

$$\begin{aligned} I(V(u_1);Y_3|U_1) &= I(V';Y_3|U_1) = I(V;Y_3|U) \\ I(V(u_1);Y_1|U_1) &= I(V';Y_1|U_1) = I(V;Y_1|U) \\ I(V(u_1);S|U_1) &= I(V';S|U_1) = I(V;S|U) \\ I(W'';Y_1|V(u_1),U_1) &= I(W';Y_1|V',U_1) = I(W;Y_1|V,U). \end{aligned}$$

Similarly the distribution of W' is not necessarily preserved and the resulted r.v. is denoted by W''. Furthermore the Markov chain is not necessarily preserved neither, however for the selection $V_1 = (V(u_1),U_1)$ we have:

$$\begin{aligned} I(V_1;Y_3) &= I(V(u_1),U_1;Y_3) \\ &= I(U_1;Y_3) + I(V(u_1);Y_3|U_1) \\ &= I(U_1;Y_3) + I(V';Y_3|U_1) \\ &= I(U;Y_3) + I(V;Y_3|U) \\ &= I(V;Y_3) \end{aligned} \tag{27}$$

and

$$\begin{aligned} I(V_1;S|U_1) &= I(V(u_1),U_1;S|U_1) \\ &= I(V(u_1);S|U_1) \\ &= I(V';S|U_1) \\ &= I(V;S|U) \end{aligned}$$

preserving markovity ($U_1 \to V_1 \to W''$). Finally, for each $U_1 = u_1$ and $V_1 = v_1$, consider the following continuous functions of $p(xs|w'',v_1,u_1)$:

$g_j(p_{xs|w''v_1u_1}(xs|w''v_1u_1))$	j
$p_{xs|w''v_1u_1}(j|wv_1u_1)$	$1,...,m-1$
$H(Y_1|W''=w'',V_1=v_1,U_1=u_1)$	m
$H(S|W''=w'',V_1=v_1,U_1=u_1)$	$m+1$

Again due to the support lemma $W(u_1,v_1)$ can be found with cardinality $m+1$ such that:

$$\begin{aligned} I(W(u_1,v_1);Y_1|V_1,U_1) &= I(W'';Y_1|V_1,U_1) \\ &= I(W'';Y_1|V(u_1),U_1) \\ &= I(W';Y_1|V',U_1) \\ &= I(W;Y_1|V,U) \\ &= I(W;Y_1|V) \end{aligned}$$

and

$$\begin{aligned} I(W(u_1,v_1);S|V_1,U_1) &= I(W'';S|V_1,U_1) \\ &= I(W'';S|V(u_1),U_1) \\ &= I(W';S|V',U_1) \\ &= I(W;S|V,U) \\ &= I(W;S|V). \end{aligned}$$

Again the Markov chain is not necessarily preserved, however for the selection $W_1 = (W(u_1, v_1), V_1)$ we have:

$$\begin{aligned} I(W_1; Y_1|U_1) &= I(V_1; Y_1|U_1) \\ &\quad + I(W(u_1, v_1); Y_1|V_1, U_1) \\ &= I(V; Y_1|U) + I(W; Y_1|V) \\ &= I(VW; Y_1|U) \\ &= I(W; Y_1|U) \end{aligned} \tag{28}$$

and

$$\begin{aligned} I(W_1; Y_1|V_1) &= I(V_1; Y_1|V_1) + I(W(u_1, v_1); Y_1|V_1) \\ &= I(W; Y_1|V) \end{aligned}$$

and finally

$$\begin{aligned} I(W_1; S|V_1) &= I(V_1; S|V_1) + I(W(u_1, v_1); S|V_1) \\ &= I(W; S|V) \end{aligned}$$

preserving the Markov chain $U_1 \to V_1 \to W_1$ and completing the proof. ■

Furthermore, it can be pointed out that a more compact representation of the achievable rate region can be obtained by making $V = (U, V)$ and $W = (U, V, W)$ in Eq. 5, resulting in:

$$\begin{aligned} R_0 &\le \min\{I(U; Y_2) - I(U; S), \\ &\qquad I(V, U; Y_3) - I(V, U; S)\} \\ R_1 &\le I(W, V; Y_1|U) - I(W, V; S|U) \\ R_0 + R_1 &\le I(W; Y_1|V, U) + I(V, U; Y_3) - I(W, V, U; S). \end{aligned} \tag{29}$$

The preceding representation will be studied in more detail in a future publication. In the next section we compute our region for a particular Gaussian example.

IV. Gaussian Example

We evaluate the rate region of Theorem 1 for an example that assumes that the receivers are less exposed to noise in the order of the Markov chain $Y_3 \to Y_1 \to Y_2$. For simplicity of exposition we rewrite the rate region as follows:

$$\begin{aligned} R_0 &\le \underbrace{I(U; Y_2) - I(U; S)}_{R_{01}} \\ R_0 &\le \underbrace{I(V; Y_3) - I(V; S|U) - I(U; S)}_{R_{02}} \\ R_1 &\le \underbrace{I(W; Y_1|V) - I(W; S|V)}_{R_{11}} \\ &\quad + \underbrace{I(V; Y_1|U) - I(V; S|U)}_{R_{12}} \\ R_0 + R_1 &\le I(V; Y_3) - I(V; S|U) - I(U; S) \\ &\quad + I(W; Y_1|V) - I(W; S|V). \end{aligned} \tag{30}$$

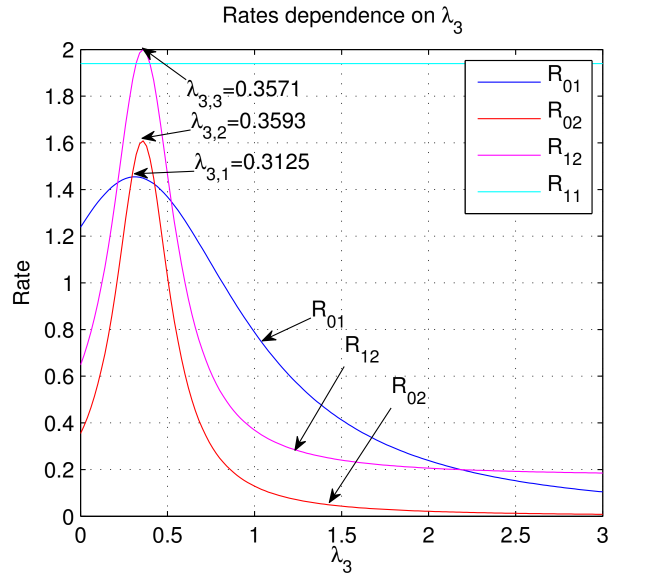

Figure 2. Three-receiver MBC with random parameters rate dependence on λ_3 for $P = 1$, $\alpha = 0.5$, $\alpha_1 = 0.03$, $N_1 = 0.4$ and $N_2 = N_3 = 0.1$.

The Gaussian channel can be described mathematically as follows:

$$\begin{aligned} Y_3 &= X + S + Z_1 \\ Y_1 &= X + S + Z_1 + Z_2 \\ Y_2 &= X + S + Z_1 + Z_2 + Z_3 \end{aligned} \tag{31}$$

where, for simplicity, the state S is assumed to be the same at all receivers in terms of power and correlation, and Z_i for $i = 1, 2, 3$ are additive Gaussian noise at the receivers. Assuming Gaussian inputs the transmission random variables are as follows:

$$\begin{aligned} X &= \tilde{U} + \tilde{V} + \tilde{W} \\ V &= \tilde{U} + \tilde{V} + \lambda_1 S \\ W &= \tilde{U} + \tilde{V} + \tilde{W} + \lambda_2 S \\ U &= \tilde{U} + \lambda_3 S \end{aligned} \tag{32}$$

where $\tilde{U} \sim \mathcal{N}(0, \alpha P)$, $\tilde{V} \sim \mathcal{N}(0, \alpha_1 P)$, $\tilde{W} \sim \mathcal{N}(0, (1 - \alpha - \alpha_1)P)$, $S \sim \mathcal{N}(0, Q)$ and $Z_i \sim \mathcal{N}(0, N_i)$ with $\alpha + \alpha_1 \in [0, 1]$. These random variables are used to compute the region in Eq. 30 utilising standard procedures. The explicit set of inequalities is not shown here due to lack of space, we instead show the optimisation parameters that yield the set. These parameters λ_i obtained during the optimisation process are for R_{01}

$$\lambda_{3,1} = \frac{\alpha P}{P + N_1 + N_2 + N_3}, \tag{33}$$

for R_{02}

$$\lambda_1 = \frac{\alpha P}{P + N_1 + N_2 + N_3}, \tag{34}$$

$$\lambda_{3,2} = \frac{\alpha P}{P + N_1}, \tag{35}$$

for R_{11}, λ_1 is utilised to find

$$\lambda_2 = \frac{P\Big((1-\alpha-\alpha_1)P + (\alpha+\alpha_1)N_2 + N_1\Big)}{(P+N_1)\Big((1-\alpha-\alpha_1)P + N_2 + N_1\Big)}, \quad (36)$$

and for R_{12}, a λ_1 function of λ_3 is optimal. This λ_1 is chosen to be equal to the λ_1 of R_{02} and the following value for λ_3 was found

$$\lambda_{3,3} = \frac{P\Big(\alpha\big((1-\alpha-\alpha_1)P + N_1 + N_2\big) + \alpha_1 N_2\Big)}{(P+N_1)\big((1-\alpha-\alpha_1)P + N_1 + N_2\big)}. \quad (37)$$

Utilising the above values of λ_i the rate region obtained is capacity achieving. It is important to note though that the rate region is λ_3 dependent. For a chosen λ_3 either R_0 or R_1 will be optimal. Figure 2 depicts this dependence.

V. Conclusion

We have demonstrated an achievable rate region for the discrete memoryless three-receiver multilevel broadcast channel with random parameters. This is a step forward toward finding capacity for more general networks with states known at the transmitter. We also presented a Gaussian example and described the dependence of the rate region on an optimisation parameter λ_3. The Gaussian rate region that is obtained utilising the optimisation parameters is the capacity region. In a cognitive network where the cognitive transmitter has knowledge of the primary message, an encoding scheme as the one presented in this work can be utilised in order to boost the cognitive system rates and/or help to boost the primary system transmission rate.

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