

Local tomography and the role of the complex numbers in quantum mechanics

Gerd Niestegge

gerd.niestegge@web.de

Abstract

Various reconstructions of finite-dimensional quantum mechanics result in a formally real Jordan algebra A and a last step remains to conclude that A is the self-adjoint part of a C^* -algebra. Using a quantum logical setting, it is shown that this can be achieved by postulating that there is a locally tomographic model for a composite system consisting of two copies of the same system. Local tomography is a feature of classical probability theory and quantum mechanics; it means that state tomography for a multipartite system can be performed by simultaneous measurements in all subsystems. The quantum logical definition of local tomography is sufficient, but not as strong as the prevalent definition in the literature and involves some subtleties concerning the so-called spin factors.

Key Words: Local tomography; Jordan algebra; quantum logic

PACS: 03.65.Ta, 03.67.-a, 02.30.Sa

MSC: 17C55; 81P10; 81P16; 81R15

1. Introduction

The quantum-mechanical need of the mathematical Hilbert space apparatus including the complex numbers has been a matter of fundamental research since the emergence of this theory one hundred years ago. Over time, quantum logical, algebraic, operational and information theoretic approaches to reconstruct quantum mechanics from a small number of plausible principles have been proposed, using algebraic methods, generalized probabilistic theories or category theory. Particular emphasis has been laid on information theoretic principles in the last two decades.

Several approaches [3, 6, 7, 8, 23, 29, 32, 33, 36, 37, 38] succeed in deriving the need of formally real Jordan algebras. A further approach [25, 27] also results in the formally real Jordan algebras, when it is restricted to the finite-dimensional case. This type of Jordan algebras includes quantum mechanics with the complex numbers, but still other versions with the real numbers, quaternions, octonions and mixtures of these versions.

One feature that distinguishes complex quantum mechanics has been known for some time [39, 40]. It concerns multipartite systems: the state of the multipartite system is completely determined when the states on the subsystems including the correlations are identified. This means that state tomography of the multipartite system can be performed by simultaneous measurements at all the subsystems. Since this feature has become a candidate for a last postulate to perform the final step in the quantum-mechanical reconstruction process and to rule out the non-complex versions, it has been named *local tomography* [6, 9, 10, 11, 13, 14, 21, 22].

In the present paper, local tomography is first defined in a quantum logical setting and is then postulated only for bipartite systems consisting of two copies of the same system. This postulate is sufficient to prove the need of the complex numbers in quantum mechanics, when the Jordan algebraic setting is settled.

The main result can be applied to complete reconstructions of quantum mechanics that first derive the formally real Jordan algebras [3, 6, 7, 8, 23, 29, 32, 33, 36, 37, 38]. In some of these approaches, local tomography has already been used. However, their methods, frameworks and assumptions differ from those of the present paper and, particularly, their local tomography requirement is stronger than the quantum logical definition of local tomography.

Formally real Jordan algebras and probabilistic models of multipartite systems that do not necessarily entail local tomography are studied in [4, 5, 32].

The formally real Jordan algebras include the so-called spin factors which involve some peculiarities in the quantum logical setting. With a spin factor, the maximum number of possible outcomes in a single measurement is two. Quantum logics with this property have a very weak mathematical structure. In this case, the geometric methods of the early quantum logical approaches fail [30, 34], the postulates of [29] are not satisfied, and the Gleason theorem does not hold [12, 16].

This paper is based on the theory of Jordan algebras as presented in the monographs [2, 19]. The results needed here are briefly sketched in section 4.

The quantum logical structure used in this paper is introduced in the next section. In section 3, a reasonable model of a composite system and local tomography are defined in the quantum logical setting. Section 4 provides the brief sketch of the formally real Jordan algebras and some first simple results that will be needed later. The local tomography postulate, a discussion of the spin factors and some further results for later use are presented in section 5. Section 6 prepares some auxiliaries needed to prove the major results in section 7.

2. Quantum logics

The quantum logic of usual quantum mechanics consists of the observables with the simple spectrum $\{0, 1\}$ (or, equivalently, the self-adjoint projection operators on the Hilbert space or the closed linear subspaces of the Hilbert space) and forms an orthomodular lattice. Originally, therefore, a quantum logic was mostly assumed to be an orthomodular lattice [30, 34]. However, there is no physical motivation for the existence of the lattice operations for propositions that are not compatible, and later a quantum logic was often assumed to be an orthomodular partially ordered set only. This is what we will do here.

In this paper, a *quantum logic* shall be an orthomodular partially ordered set L with order relation \leq , smallest element 0 , largest element $\mathbb{I} \neq 0$ and an orthocomplementation $'$. This means that the following conditions are satisfied by the $p, q \in L$:

- (a) $q \leq p$ implies $p' \leq q'$.
- (b) $(p')' = p$.
- (c) $p \leq q'$ implies that $p \vee q$, the supremum of p and q , exists.
- (d) $p \vee p' = \mathbb{I}$.
- (e) Orthomodular law: $q \leq p$ implies $p = q \vee (p \wedge q')$. Here, $p \wedge q$ denotes the infimum of p and q , which exists iff $p' \vee p'$ exists.

The elements of the quantum logic are called *propositions*. A proposition p is called *minimal*, if there is no proposition q with $q \leq p$ and $0 \neq q \neq p$. The minimal propositions are also called *atoms* in the common literature. Two propositions p and q are *orthogonal*, if $p \leq q'$ or, equivalently, $q \leq p'$; in this case, $p \vee q$ will be noted by $p + q$ in the following. The interpretation of this mathematical terminology is as follows: orthogonal events are exclusive, p' is the negation of p , and $p + q := p \vee q$ is the disjunction of the exclusive events p and q .

The mathematical structure used in classical probability theory is the *Boolean lattice*, and it can be expected that those subsets of the quantum logic L , that are Boolean lattices, behave classically. Therefore, two propositions p_1 and p_2 in L shall be called *compatible*, if there is a subset of L that is a Boolean lattice and that includes both propositions p_1 and p_2 . This is equivalent to the existence of three pairwise orthogonal propositions $q_1, q_2, q_3 \in L$ such that $p_1 = q_1 + q_2$ and $p_2 = q_2 + q_3$.

A *state* allocates probability values to the propositions and is a non-negative function μ on the quantum logic L such that $\mu(\mathbb{I}) = 1$ and $\mu(p+q) = \mu(p) + \mu(q)$ for any two orthogonal propositions p and q in L .

3. Composite systems and local tomography

In quantum mechanics as well as in classical probability theory, the global state of a multipartite system can be determined completely by specifying joint probabilities of outcomes for measurements performed simultaneously on each subsystem. However, this is not possible if a real version of quantum mechanics is considered instead of the usual one using the complex Hilbert space. This quantum-mechanical feature has been known for a long time [39, 40]. When it later got significance in different approaches to reconstruct quantum mechanics, it was called *local tomography* [6, 9, 11, 13, 14, 21, 22]. These approaches often use local tomography as a last postulate to finally segregate usual quantum theory with the complex Hilbert space from the real and other versions. The concept of local tomography shall now be transferred to the quantum logical setting, which requires a reasonable quantum logical model of a composite system.

When the quantum logics L_1 and L_2 are used as models for two single systems and the quantum logic L_{12} for the composite system consisting of these two systems, there shall be a map $\otimes : L_1 \times L_2 \rightarrow L_{12}$ such that the following conditions hold:

- (C1) $\mathbb{I} \otimes \mathbb{I}$ is the largest element of L_{12} .
- (C2) For $p_1 \in L_1$ and $p_2 \in L_2$, $p_1 \otimes p_2 = 0$ holds in L_{12} if and only if $p_1 = 0$ or $p_2 = 0$.
- (C3) If p_1 and q_1 are orthogonal in L_1 or if p_2 and q_2 are orthogonal in L_2 , then $p_1 \otimes p_2$ and $q_1 \otimes q_2$ are orthogonal in L_{12} .
- (C4) If p_1 and q_1 are orthogonal in L_1 , then $(p_1 + q_1) \otimes p_2 = p_1 \otimes p_2 + q_1 \otimes p_2$ for any $p_2 \in L_2$.
If p_2 and q_2 are orthogonal in L_2 , then $p_1 \otimes (p_2 + q_2) = p_1 \otimes p_2 + p_1 \otimes q_2$ for any $p_1 \in L_1$.
- (C5) If μ_1 and μ_2 are states on L_{12} such that $\mu_1(p_1 \otimes p_2) = \mu_2(p_1 \otimes p_2)$ for all $p_1 \in L_1, p_2 \in L_2$, then $\mu_1 = \mu_2$.

The maps $p_1 \rightarrow p_1 \otimes \mathbb{I}$ and $p_2 \rightarrow \mathbb{I} \otimes p_2$ provide embeddings of L_1 and L_2 in L_{12} . For every $p_1 \in L_1$ and every $p_2 \in L_2$, $p_1 \otimes \mathbb{I}$ and $\mathbb{I} \otimes p_2$ are compatible in L_{12} , since $p_1 \otimes \mathbb{I} = p_1 \otimes p_2 + p_1 \otimes p'_2$, $\mathbb{I} \otimes p_2 = p_1 \otimes p_2 + p'_1 \otimes p_2$ and $p_1 \otimes p_2, p'_1 \otimes p_2, p_1 \otimes p'_2$ are pairwise orthogonal in L_{12} .

Lemma 3.1. *If the first four conditions (C1-4) hold and p_k, q_k is a compatible pair in L_k for $k = 1, 2$, then $p_1 \otimes p_2$ and $q_1 \otimes q_2$ are compatible in L_{12} .*

Proof. Suppose that p_k, q_k are compatible in L_k for $k = 1, 2$. Then there are pairwise orthogonal propositions $x_1, x_2, x_3 \in L_1$ and pairwise orthogonal propositions $y_1, y_2, y_3 \in L_2$ such that $p_1 = x_1 + x_2$, $q_1 = x_2 + x_3$, $p_2 = y_1 + y_2$, $q_2 = y_2 + y_3$. Consider $z_1 := x_1 \otimes y_1 + x_2 \otimes y_1 + x_1 \otimes y_2$, $z_2 := x_2 \otimes y_2$, $z_3 := x_2 \otimes y_3 + x_3 \otimes y_2 + x_3 \otimes y_3$. These are pairwise orthogonal propositions in L_{12} with $z_1 + z_2 = p_1 \otimes p_2$ and $z_2 + z_3 = q_1 \otimes q_2$. \square

Lemma 3.2. *If the first four conditions (C1-4) hold and $q_k \leq p_k$ with $p_k, q_k \in L_k$, $k = 1, 2$, then $q_1 \otimes q_2 \leq p_1 \otimes p_2$ in L_{12} .*

Proof. Suppose $q_k \leq p_k$ for $k = 1, 2$. Then $p_k = q_k + p_k \wedge q'_k$ and $p_1 \otimes p_2 = q_1 \otimes q_2 + q_1 \otimes (p_2 \wedge q'_2) + (p_1 \wedge q'_1) \otimes q_2 + (p_1 \wedge q'_1) \otimes (p_2 \wedge q'_2) \geq q_1 \otimes q_2$. \square

(C1-4) are general, purely algebraic postulates for modeling a composite system in the quantum logical setting, and (C5) is basically the local tomography postulate.

The only-if part of (C2) [$p_1 \otimes p_2 = 0 \Rightarrow p_1 = 0$ or $p_2 = 0$] means that the embeddings of L_1 and L_2 in L_{12} are *logically independent*. Logical independence is usually defined for von Neumann subalgebras [18, 31] and becomes a necessary and sufficient condition for the *C*-independence* of two commuting von Neumann subalgebras [31]; C*-independence was introduced by Haag and Kastler in the framework of algebraic quantum field theory [17].

Instead of only two, three or more systems could be combined to a single one. The subsystems could be very different or could be copies of the same system. In the following, local tomography will be postulated only for a composite system consisting of two copies of the same system; this means that a quantum logic L shall satisfy the above postulates with $L_1 = L_2 = L$.

4. Jordan algebras

A *Jordan algebra* is a linear space A with a commutative bilinear product \circ that satisfies the identity $(x^2 \circ y) \circ x = x^2 \circ (y \circ x)$ for $x, y \in A$. A Jordan algebra over the real numbers is called *formally real*, if $x_1^2 + \dots + x_m^2 = 0$ implies $x_1 = \dots = x_m = 0$ for any $x_1, \dots, x_m \in A$ and any positive integer m . The formally real Jordan algebras were introduced, studied and classified in [35]. In the finite-dimensional case, they coincide with the so-called JB-algebras and JBW-algebras [2, 19] and with the Euclidean Jordan algebras [5, 15, 37].

Each finite-dimensional formally real Jordan algebra A possesses a multiplicative identity $\mathbb{1}$ and a natural order relation such that $a^2 \geq 0$ holds for all $a \in A$; the system of the idempotent elements

$$L_A := \{p \in A : p^2 = p\}$$

becomes a quantum logic with $p' := \mathbb{1} - p$. In the following, L_A will be called the quantum logic of the Jordan algebra A , and the elements of L_A will be called propositions.

For each finite-dimensional formally real Jordan algebra A , there are two characteristic numbers. The first is the *dimension* of A as real vector space

which will be denoted by n_A in the following. The second is the *rank* of A ; this is the maximal number of pairwise orthogonal minimal propositions in L_A which will be denoted by k_A in the following. Then

$$\mathbb{I} = \sum_{j=1}^{k_A} p_j$$

with k_A pairwise orthogonal minimal propositions p_1, p_2, \dots, p_{k_A} . Moreover, $k_A \leq n_A$; the identity $k_A = n_A$ means

$$A = \sum_{j=1}^{k_A} \mathbb{R}p_j$$

and holds if and only if the Jordan product is associative.

Two elements x and y in a Jordan algebra A are said to *operator-commute*, if $x \circ (y \circ z) = y \circ (x \circ z)$ for all $z \in A$. The *center* of A consists of all elements that operator-commute with every element of A and becomes an associative subalgebra of A . A is simple (irreducible) iff its center is $\mathbb{R}\mathbb{I}$. Moreover, any two operator-commuting elements of A lie in a joint associative subalgebra of A .

In the study of Jordan algebras, an important role is played by the co-called *Jordan triple product* which is defined as $\{x, y, z\} := x \circ (y \circ z) - y \circ (z \circ x) + z \circ (x \circ y)$ for three elements x, y, z in a Jordan algebra A . In the case of the special Jordan product $x \circ y := (xy + yx)/2$ in an associative algebra, the identity $\{x, y, x\} = xyx$ holds.

A Jordan algebra A is said to be the direct sum of the subalgebras A_1 and A_2 ($A = A_1 \oplus A_2$) if A is the direct sum of A_1 and A_2 as linear spaces and if $a_1 \circ a_2 = 0$ holds for all $a_1 \in A_1, a_2 \in A_2$. Every finite-dimensional formally real Jordan algebra decomposes into a direct sum of simple (irreducible) subalgebras; these are either one-dimensional (the real numbers), spin factors (a spin factor is characterized by $k_A = 2$ and $n_A \geq 3$) or can be represented as algebras of the Hermitian $k \times k$ -matrices over the real numbers \mathbb{R} , complex numbers \mathbb{C} , quaternions \mathbb{H} with $k = 3, 4, 5, \dots$ or over the octonions \mathbb{O} with $k = 3$ only [19, 35]. The product of the matrices x, y is given by $x \circ y := (xy + yx)/2$. These Jordan matrix algebras are denoted by $H_k(\mathbb{R}), H_k(\mathbb{C}), H_k(\mathbb{H})$ and $H_3(\mathbb{O})$. Note that the indexes k and 3 coincide with the rank k_A of these algebras.

In [29], four postulates for a quantum logic L have been presented that are satisfied if and only if L is the quantum logic L_A of some finite-dimensional formally real Jordan algebra A the decomposition of which into simple algebras does not include spin factors. This means that, in this case, the mathematical structure of the quantum logic L_A of the algebra A is so rich that the complete algebra A (with its linear and multiplicative structure) can be recovered from L_A (with the order relation \leq and the orthocomplementation $'$).

Let A be a finite-dimensional formally real Jordan algebra. A linear functional $\mu : A \rightarrow \mathbb{R}$ is called *positive*, if $\mu(a) \geq 0$ holds for all $a \in A$ with $a \geq 0$. A positive linear functional μ with $\mu(\mathbb{I}) = \mathbb{I}$ is called a *state* on A ; its restriction to L_A becomes a state on this quantum logic. By Gleason's theorem [16] and its

extension to Jordan algebras [12], each state μ on the quantum logic L_A has a unique linear extension to A , if the decomposition of A into simple subalgebras does not include spin factors. In this case, the states on the quantum logic L_A can be identified with the states on A . When spin factors are included, however, it is necessary to distinguish between the linear states on the algebra A and the states on quantum logic L_A . Each linear functional μ on A can be written as $\mu = t_1\mu_1 - t_2\mu_2$ with non-negative real number t_1, t_2 and states μ_1, μ_2 on A .

The next lemma is a collection of facts from the theory of the formally real Jordan algebras [2, 19, 35] that will be needed in the following sections.

Lemma 4.1. *Let A be a finite-dimensional formally real Jordan algebra and let p and q be propositions in its quantum logic L_A .*

- (i) $0 \leq \{p, q, p\} \leq p$, and $0 \leq \{p, a, p\}$ for any $a \in A$ with $a \geq 0$.
- (ii) p is minimal iff $\{p, A, p\} = \mathbb{R}p$ and $p \neq 0$.
- (iii) p and q are orthogonal iff $p \circ q = 0$ iff $\{p, q, p\} = 0$ iff $\{q, p, q\} = 0$.
Moreover, $p \leq q$ iff $p \circ q = p$ iff $\{p, q, p\} = p$ iff $\{q, p, q\} = p$.
- (iv) Orthogonal propositions in L_A operator-commute in A .
- (v) Spectral theorem: Every $a \in A$ can be written as $a = \sum_j r_j q_j$ with pairwise orthogonal minimal proposition $q_j \in L_A$ and $r_j \in \mathbb{R}$, $j = 1, 2, \dots, m$.
 $\{p, A, p\}$ is a Jordan subalgebra of A and its elements a can be written as $a = \sum_j r_j q_j$ with pairwise orthogonal minimal proposition $q_j \leq p$ and $r_j \in \mathbb{R}$, $j = 1, 2, \dots, m$.
- (vi) If μ is a linear state on A with $\mu(p) = 1$, then $\mu(\{p, a, p\}) = \mu(a)$ for all $a \in A$.

Definition 4.2. *Let p be a minimal proposition in the quantum logic L_A of a finite-dimensional formally real Jordan algebra A . For each $a \in A$ there is $r_a \in \mathbb{R}$ with $\{p, a, p\} = r_a p$ by lemma 4.1 (ii). Thus p induces a linear state $\mathbb{P}_p : A \rightarrow \mathbb{R}$, $a \rightarrow r_a$ with $\mathbb{P}_p(p) = 1$.*

The positivity of \mathbb{P}_p follows from lemma 4.1 (i). For a minimal proposition p and any other proposition q in the quantum logic L_A of a finite-dimensional formally real Jordan algebra A , lemma 4.1 (iii) immediately yields:

$$\mathbb{P}_p(q) = 0 \text{ iff } p \text{ and } q \text{ are orthogonal, and}$$

$$\mathbb{P}_p(q) = 1 \text{ iff } p \leq q.$$

Lemma 4.3. *Let A be a finite-dimensional formally real Jordan algebra and $a \in A$ with $\mathbb{P}_p(a) = 0$ for all minimal minimal propositions $p \in L_A$. Then $a = 0$.*

Proof. By lemma 4.1 (v), $a = \sum_j r_j q_j$ with pairwise orthogonal minimal proposition $q_j \in L_A$, $r_j \in \mathbb{R}$, $j = 1, 2, \dots, m$, and $0 = \mathbb{P}_{q_j}(a) = r_j$ for each j implies $a = 0$. \square

Lemma 4.4. *Let A be a finite-dimensional formally real Jordan algebra and let p and q be propositions in its quantum logic L_A . Then p and q operator-commute in the Jordan algebra A if and only if p and q are compatible in the quantum logic L_A .*

Proof. First suppose that p and q operator-commute in A , which means that both lie in a joint associative subalgebra of A . Then $p \circ q = (p \circ q)^2$ and $x_1 := p \circ q$, $x_2 := p - x_1$, $x_3 := q - x_1$ are pairwise orthogonal propositions in L_A . This means that p and q are compatible in L_A .

Now suppose that p and q are compatible in L_A . This means that there are pairwise orthogonal propositions $x_1, x_2, x_3 \in L_A$ with $p = x_1 + x_2$ and $q = x_2 + x_3$. By lemma 4.1 (iv), x_1, x_2, x_3 pairwise operator-commute and, therefore, p and q operator-commute in A . \square

Two orthogonal propositions p, q in the quantum logic L_A of a Jordan algebra A are said to be *strongly connected*, if there is an element $x \in \{p, A, q\}$ such that $x^2 = p + q$. If A is formally real, finite-dimensional and simple, each pair of orthogonal minimal propositions is strongly connected [19].

Lemma 4.5. *If p_1 and p_2 are any minimal propositions in the quantum logic L_A of a simple finite-dimensional formally real Jordan algebra A , then there is a further minimal proposition $q \in A$ that is neither orthogonal to p_1 nor to p_2 .*

Proof. If p_1 and p_2 are not orthogonal, choose $q := p_1$ or $q := p_2$. Now assume that p_1 and p_2 are orthogonal. Since A is simple, p_1 and p_2 are strongly connected and hence $\{p_1, A, p_2\} \neq \{0\}$. A is the linear span of its minimal propositions and, therefore, $\{p_1, q, p_2\} \neq 0$ must hold for at least one minimal proposition q . Then $0 \neq p_1 \circ (q \circ p_2) - q \circ (p_2 \circ p_1) + p_2 \circ (p_1 \circ q) = p_1 \circ (q \circ p_2) + p_2 \circ (p_1 \circ q)$. Since p_1 and p_2 operator-commute, the last two summands are identical; therefore each one cannot vanish and thus $p_1 \circ q \neq 0 \neq p_2 \circ q$. This means that q is neither orthogonal to p_1 nor to p_2 . \square

5. The local tomography postulate

The local tomography postulate for the quantum logic L_A of a finite-dimensional formally real Jordan algebra A can now be presented:

(LT) *There shall be another finite-dimensional formally real Jordan algebra A^2 and a map $\otimes : L_A \times L_A \rightarrow L_{A^2}$ such that the four conditions (C1-5) in section 3 hold with $L_1 = L_2 = L_A$ and $L_{12} = L_{A^2}$.*

This quantum logical version of local tomography is not as strong as the prevalent version in other papers, where the map \otimes is immediately postulated to exist as a bilinear map on $A \times A$ or as a biconvex map on $[0, \mathbb{I}] \times [0, \mathbb{I}]$ and not as an only biadditive map on $L_A \times L_A$. The convex set $[0, \mathbb{I}] = \{a \in A : 0 \leq a \leq \mathbb{I}\}$ is the so-called effect space, playing a major role in some approaches to reconstruct quantum mechanics. With (LT), the linear extension of the map \otimes to $A \times A$ is not immediately given and will be derived later in lemma 5.1; in proposition 7.1, it will be shown that it is one-to-one.

If $A = H_k(\mathbb{C})$, then $H_{k^2}(\mathbb{C})$ is the tensor product $A \otimes A$ and, in this case, the quantum logic L_A satisfies (LT). However, such a solution is not available in the real case, since $H_k(\mathbb{R}) \otimes H_k(\mathbb{R})$ and $H_{k^2}(\mathbb{R})$ have different dimensions $[(k(k+1)/2)^2 \neq k^2(k^2+1)/2$ for $k > 1$].

Moreover, the case $H_2(\mathbb{C})$ is special; (LT) is satisfied not only with the usual map $\otimes : L_{H_2(\mathbb{C})} \times L_{H_2(\mathbb{C})} \rightarrow L_{H_4(\mathbb{C})}$, but as well with many other maps, which can be generated by the discontinuous automorphisms of this quantum logic. For every $q \in L_{H_2(\mathbb{C})}$ with $0 \neq q \neq \mathbb{I}$, an example of such a discontinuous automorphism can be defined in the following way: $\pi_q(p) := p$ for $p \in L_{H_2(\mathbb{C})}$ with $q \neq p \neq q'$ and $\pi_q(q) := q'$, $\pi_q(q') := q$. An alternative map \otimes_π is then given by $p_1 \otimes_\pi p_2 := \pi(p_1) \otimes \pi(p_2) \in L_{H_4(\mathbb{C})}$ for $p_1, p_2 \in L_{H_2(\mathbb{C})}$.

Furthermore, the quantum logics L_{A_1} and L_{A_2} of two spin factors A_1 and A_2 are isomorphic if and only if L_{A_1} and L_{A_2} have the same cardinality: The minimal propositions in L_{A_1} can be organized into the sets $\{p, p'\}$ and so can those in L_{A_2} . The only rules that an isomorphism π from L_{A_1} to L_{A_2} has to obey are that $\pi(0) = 0$, $\pi(\mathbb{I}) = \mathbb{I}$ and that the two-element subsets of L_{A_1} are mapped one-to-one to those of L_{A_2} . This only requires that L_{A_1} and L_{A_2} have the same cardinality.

The quantum logic L_A of every finite-dimensional spin factor A has the cardinality of the continuum, becomes thus isomorphic to $L_{H_2(\mathbb{C})}$ and satisfies (LT) with $A^2 = H_4(\mathbb{C})$. The origin of this unattractive feature lies in the weak algebraic structure of the quantum logic of a spin factor, since it contains only minimal propositions and 0 and \mathbb{I} . Therefore, spin factors will often play an exceptional role in the following.

Lemma 5.1. *Assume that (LT) holds for the quantum logic L_A of a finite-dimensional formally real Jordan algebra A .*

- (i) A^2 is the linear hull of $\{p \otimes q : p, q \in L_A\}$.
- (ii) Assume furthermore that the decomposition of A into simple subalgebras does not include spin factors.
 - a) Suppose $p_0 = \sum_{i=1}^k r_i p_i$ and $q_0 = \sum_{j=1}^l s_j q_j$ in A with $p_0, p_1, p_2, \dots, p_k, q_0, q_1, q_2, \dots, q_l \in L_A$ and $r_i, s_j \in \mathbb{R}$. Then

$$p_0 \otimes q_0 = \sum_{i,j=1}^{k,l} r_i s_j p_i \otimes q_j$$

in A^2 .

- b) The map $\otimes : L_A \times L_A \rightarrow A^2$ can be extended in a unique way to a bilinear map $\otimes : A \times A \rightarrow A^2$.

Proof. (i) Assume that $\{p \otimes q : p, q \in L_A\}$ does not generate A^2 . Then there is a linear functional $\mu : A^2 \rightarrow \mathbb{R}$ with $\mu \neq 0$ and $\mu(p \otimes q) = 0$ for all $p, q \in L_A$. It can be written as $\mu = t_1 \mu_1 - t_2 \mu_2$ with non-negative real numbers t_1, t_2 and two states μ_1, μ_2 on A^2 . Since $0 = \mu(\mathbb{I} \otimes \mathbb{I}) = t_1 - t_2$, we get $t_1 = t_2$. Either

$t_1 = t_2 = 0$ and $\mu = 0$, or $\mu_1(p \otimes q) = \mu_2(p \otimes q)$ for all $p, q \in L_A$. In the second case, (CS5) implies $\mu_1 = \mu_2$ and thus $\mu = 0$ in both cases, which yields the desired contradiction.

(ii) a) Let $\mu : A^2 \rightarrow \mathbb{R}$ be a linear functional on A^2 ; it can again be written as $\mu = t_1\mu_1 - t_2\mu_2$ with non-negative real number t_1, t_2 and states μ_1, μ_2 on A^2 .

Define $\mu_1^{p_0}$ on L_A by $\mu_1^{p_0}(q) := \mu_1(p_0 \otimes q)$ for $q \in L_A$. By the extension of Gleason's theorem to Jordan algebras [12], $\mu_1^{p_0}$ has a unique linear extension to A . Therefore

$$\mu_1^{p_0}(q_0) = \sum_{j=1}^l s_j \mu_1^{p_0}(q_j).$$

For each j now define $\mu_1^{q_j}$ on L_A by $\mu_1^{q_j}(p) := \mu_1(p \otimes q_j)$ for $p \in L_A$. Again each $\mu_1^{q_j}$ has a unique linear extension to A and

$$\mu_1^{q_j}(p_0) = \sum_{i=1}^k r_i \mu_1^{q_j}(p_i).$$

Then

$$\begin{aligned} \mu_1(p_0 \otimes q_0) &= \mu_1^{p_0}(q_0) = \sum_{j=1}^l s_j \mu_1^{p_0}(q_j) = \sum_{j=1}^l s_j \mu_1(p_0 \otimes q_j) \\ &= \sum_{j=1}^l s_j \mu_1^{q_j}(p_0) = \sum_{j=1}^l s_j \sum_{i=1}^k r_i \mu_1^{q_j}(p_i) \\ &= \sum_{i,j=1}^{k,l} r_i s_j \mu_1(p_i \otimes q_j). \end{aligned}$$

The identity

$$\mu_2(p_0 \otimes q_0) = \sum_{i,j=1}^{k,l} r_i s_j \mu_2(p_i \otimes q_j)$$

follows for μ_2 in the same way, and we get for $\mu = t_1\mu_1 - t_2\mu_2$

$$\mu(p_0 \otimes q_0) = \sum_{i,j=1}^{k,l} r_i s_j \mu(p_i \otimes q_j) = \mu \left(\sum_{i,j=1}^{k,l} r_i s_j p_i \otimes q_j \right).$$

Since this holds for any linear functional μ on A^2 , we get the desired identity of lemma 5.1 (ii) a).

(ii) b) A is the linear hull of L_A . Pick a basis in A with elements $p_i \in L_A$, $i = 1, \dots, n_A$, and map the pair $\sum_{i=1}^{n_A} r_i p_i$ and $\sum_{j=1}^{n_A} s_j p_j$ in A to $\sum_{i,j=1}^{n_A} r_i s_j p_i \otimes p_j$ in A^2 . By (ii) a), the pairs (p_0, q_0) with $p_0, q_0 \in L_A$ are mapped to $p_0 \otimes q_0$ in A^2 . The uniqueness of the extension is then implied by (i). \square

Lemma 5.2. *Assume that (LT) holds for the quantum logic L_A of a finite-dimensional formally real Jordan algebra A . If the propositions $p_1, p_2 \in L_A$ lie in the center of A , then $p_1 \otimes p_2$ lies in the center of A^2 .*

Proof. Suppose that $p_1, p_2 \in L_A$ lie in the center of A . By lemma 4.4, p_1 and p_2 are compatible with every proposition in L_A and, by lemma 3.1, $p_1 \otimes p_2$ is compatible with every $q_1 \otimes q_2$ in L_{A^2} , $q_1, q_2 \in L_A$. By lemma 4.4, $q_1 \otimes q_2$ operator-commutes with every such $q_1 \otimes q_2$ and therefore with every element of A^2 by lemma 5.1 (i). \square

Lemma 5.3. *If (LT) holds for the quantum logic L_A of a finite-dimensional formally real Jordan algebra A and if A is the direct sum of the subalgebras A_1 and A_2 , then (LT) holds for each quantum logic L_{A_1} and L_{A_2} .*

Proof. Let \mathbb{I}_1 and \mathbb{I}_2 be the multiplicative identities of A_1 and A_2 . They lie in the center of A and $\mathbb{I}_1 + \mathbb{I}_2 = \mathbb{I}$. By lemma 5.2, $\mathbb{I}_i \otimes \mathbb{I}_j$ lies in the center of A^2 for $i, j = 1, 2$. The sum of these four propositions is $\mathbb{I} \otimes \mathbb{I}$ and therefore

$$A^2 = \oplus_{i,j=1,2} \{ \mathbb{I}_i \otimes \mathbb{I}_j, A^2, \mathbb{I}_i \otimes \mathbb{I}_j \}.$$

For $p, q \in L_{A_1}$, we have $p, q \in L_A$ and $p, q \leq \mathbb{I}_1$. By lemma 3.2, $p \otimes q \leq \mathbb{I}_1 \otimes \mathbb{I}_1$ in A^2 and therefore $p \otimes q \in \{ \mathbb{I}_1 \otimes \mathbb{I}_1, A^2, \mathbb{I}_1 \otimes \mathbb{I}_1 \}$ by lemma 4.1 (iii). The restriction of the map \otimes to $L_{A_1} \times L_{A_1}$ then satisfies (CS1-4); $L_1 = L_2 = L_{A_1}$ and L_{12} is the quantum logic of the formally real Jordan algebra $\{ \mathbb{I}_1 \otimes \mathbb{I}_1, A^2, \mathbb{I}_1 \otimes \mathbb{I}_1 \}$.

Now let μ_1 and μ_2 be states on this quantum logic with $\mu_1(p_1 \otimes q_1) = \mu_2(p_1 \otimes q_1)$ for all $p_1, q_1 \in L_{A_1}$. Define states ν_1 and ν_2 on L_{A^2} by

$$\nu_k(x) := \mu_k(\{ \mathbb{I}_1 \otimes \mathbb{I}_1, x, \mathbb{I}_1 \otimes \mathbb{I}_1 \})$$

for $x \in L_{A^2}$, $k = 1, 2$. Note that $\{ \mathbb{I}_1 \otimes \mathbb{I}_1, x, \mathbb{I}_1 \otimes \mathbb{I}_1 \}$ is idempotent for $x \in L_{A^2}$ because $\mathbb{I}_1 \otimes \mathbb{I}_1$ lies in the center of A^2 .

Suppose $p, q \in L_A$. Then $p = p_1 + p_2$ and $q = q_1 + q_2$ with $p_1, q_1 \in L_{A_1}$, $p_2, q_2 \in L_{A_2}$ and hence $p \otimes q = \sum_{i,j} p_i \otimes q_j$. By (C3) and lemma 4.1 (iii), $\{ \mathbb{I}_1 \otimes \mathbb{I}_1, p_i \otimes q_j, \mathbb{I}_1 \otimes \mathbb{I}_1 \} = 0$ for $i \neq 1$ or $j \neq 1$. Thus $\{ \mathbb{I}_1 \otimes \mathbb{I}_1, p \otimes q, \mathbb{I}_1 \otimes \mathbb{I}_1 \} = \{ \mathbb{I}_1 \otimes \mathbb{I}_1, p_1 \otimes q_1, \mathbb{I}_1 \otimes \mathbb{I}_1 \} = p_1 \otimes q_1$ by lemma 3.2. Therefore

$$\nu_1(p \otimes q) = \mu_1(p_1 \otimes q_1) = \mu_2(p_1 \otimes q_1) = \nu_2(p \otimes q).$$

Since this holds for all $p, q \in L_A$ and L_A satisfies (LT), we get $\nu_1 = \nu_2$ on L_{A^2} . By lemma 4.1 (iii), the restriction of ν_k to the quantum logic of $\{ \mathbb{I}_1 \otimes \mathbb{I}_1, A^2, \mathbb{I}_1 \otimes \mathbb{I}_1 \}$ coincides with μ_k , $k = 1, 2$, and we get $\mu_1 = \mu_2$. That L_{A_2} satisfies (LT) follows in the same way. \square

6. Auxiliaries

In this section, it shall always be assumed that A is a finite-dimensional formally real Jordan algebra, that its decomposition into simple subalgebras does not include spin factors, and that its quantum logic L_A satisfies (LT).

Lemma 6.1. *Suppose $\{p_1, q_1, p_1\} = r_1 p_1$ and $\{p_2, q_2, p_2\} = r_2 p_2$ with $p_1, p_2, q_1, q_2 \in L_A$ and $r_1, r_2 \in \mathbb{R}$. Then*

$$\{p_1 \otimes p_2, q_1 \otimes q_2, p_1 \otimes p_2\} = r_1 r_2 p_1 \otimes p_2$$

in A^2 .

Proof. If $r_1 = 0$, then q_1 and p_1 are orthogonal by lemma 4.1 (iii), therefore $p_1 \otimes p_2$ and $q_1 \otimes q_2$ are orthogonal by (C3) and both sides of the identity above become 0.

Now assume $r_1 > 0$, and let μ be a state on A^2 with $\mu(p_1 \otimes p_2) = 1$. Then $1 \geq \mu(\mathbb{I} \otimes p_2) = \mu(p_1 \otimes p_2) + \mu(p_1' \otimes p_2) \geq 1$ and thus $\mu(\mathbb{I} \otimes p_2) = 1$. Therefore

$$\mu_1(a) := \mu(a \otimes p_2), \quad a \in A,$$

defines a state on A with $\mu_1(p_1) = 1$; note that $a \otimes p_2$ is defined by lemma 5.1 (ii) b). By lemma 4.1 (vi),

$$\mu_1(q_1) = \mu_1(\{p_1, q_1, p_1\}) = r_1 \mu_1(p_1) = r_1.$$

Now define, again using lemma 5.1 (ii) b),

$$\mu_2(b) := \frac{1}{r_1} \mu(q_1 \otimes b), \quad b \in A.$$

Then $\mu_2(p_2) = \mu_1(q_1)/r_1 = 1$. Furthermore, $p_1 \otimes p_2$ and $q_1 \otimes p_2'$ are orthogonal by (C3) and $1 = \mu(\mathbb{I} \otimes \mathbb{I}) \geq \mu(p_1 \otimes p_2 + q_1 \otimes p_2') = \mu(p_1 \otimes p_2) + \mu(q_1 \otimes p_2') = 1 + \mu(q_1 \otimes p_2') \geq 1$ implies $\mu(q_1 \otimes p_2') = 0$, hence $\mu_2(p_2') = 0$ and therefore $\mu_2(\mathbb{I}) = \mu_2(p_2) + \mu_2(p_2') = 1$. This means that μ_2 becomes a state on A with $\mu_2(p_2) = 1$. Once more applying lemma 4.1 (vi) yields

$$\mu_2(q_2) = \mu_2(\{p_2, q_2, p_2\}) = r_2 \mu_2(p_2) = r_2.$$

Therefore

$$\mu(q_1 \otimes q_2) = r_1 \mu_2(q_2) = r_1 r_2.$$

Now let ν be any state on A^2 with $\nu(p_1 \otimes p_2) > 0$. Then

$$\mu(x) := \frac{1}{\nu(p_1 \otimes p_2)} \nu(\{p_1 \otimes p_2, x, p_1 \otimes p_2\}), \quad x \in A^2,$$

becomes a state with $\mu(p_1 \otimes p_2) = 1$ and, as shown before,

$$\mu(q_1 \otimes q_2) = r_1 r_2.$$

This means

$$\nu(\{p_1 \otimes p_2, q_1 \otimes q_2, p_1 \otimes p_2\}) = r_1 r_2 \nu(p_1 \otimes p_2).$$

This identity holds for all states ν including those with $\nu(p_1 \otimes p_2) = 0$, since then $0 \leq \nu(\{p_1 \otimes p_2, q_1 \otimes q_2, p_1 \otimes p_2\}) \leq \nu(p_1 \otimes p_2) = 0$ by lemma 4.1 (i). Therefore

$$\{p_1 \otimes p_2, q_1 \otimes q_2, p_1 \otimes p_2\} = r_1 r_2 p_1 \otimes p_2.$$

□

Lemma 6.2. *Let p_1, p_2 be minimal propositions in L_A .*

- (i) $p_1 \otimes p_2$ is a minimal proposition in L_{A^2} .
- (ii) $\mathbb{P}_{p_1 \otimes p_2}(q_1 \otimes q_2) = \mathbb{P}_{p_1}(q_1)\mathbb{P}_{p_2}(q_2)$ for all $q_1, q_2 \in L_A$.
- (iii) If q_1, q_2 are propositions in L_A such that $p_1 \otimes p_2$ and $q_1 \otimes q_2$ are orthogonal in L_{A^2} , then either p_1 and q_1 or p_2 and q_2 or both pairs are orthogonal in L_A .

Proof. First note that $p_1 \otimes p_2 \neq 0$ by (C2) in section 3.

(i) Let $q_1, q_2 \in L_A$. By lemma 4.1 ii), $\{p_1, q_1, p_1\} = r_1 p_1$ and $\{p_2, q_2, p_2\} = r_2 p_2$ with reals r_1, r_2 . By lemma 6.1, $\{p_1 \otimes p_2, q_1 \otimes q_2, p_1 \otimes p_2\} = r_1 r_2 p_1 \otimes p_2$.

By lemma 5.1 (i), A^2 is the linear hull of $\{q_1 \otimes q_2 : q_1, q_2 \in L_A\}$, and we get $\{p_1 \otimes p_2, A^2, p_1 \otimes p_2\} = \mathbb{R} p_1 \otimes p_2$. By lemma 4.1 (ii), $p_1 \otimes p_2$ becomes minimal in L_{A^2} .

(ii) This follows from Definition 4.2, lemma 6.1 and lemma 6.2 (i).

(iii) Suppose that $p_1 \otimes p_2$ and $q_1 \otimes q_2$ are orthogonal in A^2 . By lemma 6.2 (ii) then $\mathbb{P}_{p_1}(q_1)\mathbb{P}_{p_2}(q_2) = \mathbb{P}_{p_1 \otimes p_2}(q_1 \otimes q_2) = 0$. Therefore $\mathbb{P}_{p_1}(q_1) = 0$ and p_1 and q_1 are orthogonal, or $\mathbb{P}_{p_2}(q_2) = 0$ and p_2 and q_2 are orthogonal. □

Lemma 6.3. *Suppose that A is simple, and let $p_1, p_2, q_1, q_2 \in L_A$ be minimal propositions. Then $p_1 \otimes p_2$ and $q_1 \otimes q_2$ belong to the same simple direct summand of A^2 .*

Proof. By lemma 4.5, there are minimal propositions e_1 and e_2 in L_A such that e_1 is neither orthogonal to p_1 nor to q_1 and e_2 is neither orthogonal to p_2 nor to q_2 . By lemma 6.2 (iii), $e_1 \otimes e_2$ is neither orthogonal to $p_1 \otimes p_2$ nor to $q_1 \otimes q_2$ in A^2 . Therefore, both $p_1 \otimes p_2$ and $q_1 \otimes q_2$ belong to the same direct summand as $e_1 \otimes e_2$, because minimal propositions from different direct summands are orthogonal. □

Lemma 6.4. *If A is simple, then A^2 is simple too.*

Proof. Let p_i , $i = 1, 2, \dots, k_A$, be pairwise orthogonal minimal propositions in L_A with $\sum p_i = \mathbb{I}$. By lemma 6.3, the $p_i \otimes p_j$, $i, j = 1, 2, \dots, k_A$, all belong to the same simple direct summand of A^2 . This direct summand then includes

$$\sum_{i,j=1}^{k_A} p_i \otimes p_j = \left(\sum_{i=1}^{k_A} p_i \right) \otimes \left(\sum_{j=1}^{k_A} p_j \right) = \mathbb{I} \otimes \mathbb{I}$$

and thus becomes the complete algebra A^2 . Therefore, A^2 is simple. □

Findings, analogous to the last two lemmas above, are contained in [32], but another framework and other assumptions are used; local tomography is not postulated and the real version of quantum mechanics remains included.

7. Results

Proposition 7.1. *Let A be a finite-dimensional formally real Jordan algebra such that its decomposition into simple subalgebras does not include spin factors and its quantum logic L_A satisfies (LT). The two characteristic numbers, dimension and rank, then factorize in the following way: $n_{A^2} = n_A^2$ and $k_{A^2} = k_A^2$.*

Proof. If $\mathbb{I} = \sum_{i=1}^{k_A} p_i$ in A with pairwise orthogonal minimal propositions p_i in L_A , then $\mathbb{I} \otimes \mathbb{I} = \sum_{i,j=1}^{k_A} p_i \otimes p_j$ in A^2 . By lemma 6.2 (i), the $p_i \otimes p_j$ are minimal propositions in L_{A^2} and therefore $k_{A^2} = k_A^2$.

Now let $q_i, i = 1, 2, \dots, n_A$, be a basis of A such that each q_i is a proposition in L_A . It shall be shown that the $q_i \otimes q_j, i, j = 1, 2, \dots, n_A$, become a basis in A^2 .

Assume $0 = \sum_{ij} r_{ij} (q_i \otimes q_j)$ with $r_{ij} \in \mathbb{R}$. For any two minimal propositions $p_1, p_2 \in L_A$ then by lemma 6.2 (i) and (ii)

$$\begin{aligned} 0 &= \mathbb{P}_{p_1 \otimes p_2} \left(\sum_{ij} r_{ij} (q_i \otimes q_j) \right) = \sum_{ij} r_{ij} \mathbb{P}_{p_1 \otimes p_2} (q_i \otimes q_j) \\ &= \sum_{ij} r_{ij} \mathbb{P}_{p_1} (q_i) \mathbb{P}_{p_2} (q_j) = \mathbb{P}_{p_2} \left(\sum_{ij} r_{ij} \mathbb{P}_{p_1} (q_i) q_j \right). \end{aligned}$$

Since this holds for all minimal propositions $p_2 \in L_A$, we get by lemma 4.3

$$0 = \sum_{ij} r_{ij} \mathbb{P}_{p_1} (q_i) q_j.$$

The linear independence of the q_j implies

$$0 = \sum_i r_{ij} \mathbb{P}_{p_1} (q_i) = \mathbb{P}_{p_1} \left(\sum_i r_{ij} q_i \right)$$

for each j . Since this still holds for all minimal propositions $p_1 \in L_A$, we get by lemma 4.3 again

$$0 = \sum_i r_{ij} q_i$$

for each j , and the linear independence of the q_i implies $r_{ij} = 0$ for all $i, j = 1, 2, \dots, n_A$.

Thus we have the linear independence of the $q_i \otimes q_j, i, j = 1, 2, \dots, n_A$, in A^2 . By lemma 5.1, these elements become a basis of A^2 and therefore $n_{A^2} = n_A^2$. \square

Proposition 7.2. *The quantum logics of $H_k(\mathbb{R})$, $H_k(\mathbb{H})$ with $k \geq 3$ and $H_3(\mathbb{O})$ do not satisfy (LT).*

Proof. Note that the dimensions of $H_k(\mathbb{R})$, $H_k(\mathbb{C})$, $H_k(\mathbb{H})$ and $H_3(\mathbb{O})$ are $k(k+1)/2$, k^2 , $k(2k-1)$ and 27, respectively. Let A be one of the algebras $H_k(\mathbb{R})$, $H_k(\mathbb{H})$ with $k \geq 3$ or $H_3(\mathbb{O})$.

If (LT) were satisfied, A^2 would be a simple formally real Jordan algebra and thus a matrix algebra with rank $k_{A^2} = k^2$ and, for $A = H_3(\mathbb{O})$, $k_{A^2} = 9$. This follows from lemma 6.4 and proposition 7.1. For the dimension n_{A^2} of A^2 , proposition 7.1 implies:

$$n_{A^2} = k^2(k+1)^2/4 \text{ if } A = H_k(\mathbb{R}),$$

$$n_{A^2} = k^2(2k-1)^2 \text{ if } A = H_k(\mathbb{H}), \text{ and}$$

$$n_{A^2} = 27^2 = 729 \text{ if } A = H_3(\mathbb{O}).$$

The dimension n_{A^2} must equal either $k^2(k^2+1)/2$ if A^2 were a real matrix algebra, or k^4 if A^2 were a complex matrix algebra, or $k^2(2k^2-1)$ if A^2 were a matrix algebra over the quaternions. Simple calculations show that each case is impossible for $k \geq 3$. With $A = H_3(\mathbb{O})$, $n_{A^2} = 729$ would have to be identical to one of the dimensions of $H_9(\mathbb{R})$, $H_9(\mathbb{C})$, $H_9(\mathbb{H})$, but these are 45, 81 and 153. Therefore, (LT) cannot hold. \square

Theorem 7.3. *Let A be a finite-dimensional formally real Jordan algebra such that its decomposition into simple subalgebras does not include spin factors. Its quantum logic L_A satisfies (LT) if and only if A is the self-adjoint part of a C^* -algebra.*

Proof. A is the direct sum of algebras of the following types: $H_k(K)$ with $K = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and $k \geq 3$, $H_3(\mathbb{O})$ and \mathbb{R} . If (LT) holds, lemma 5.3 and proposition 7.2 rule out all the cases with $K = \mathbb{R}, \mathbb{H}$ and $H_3(\mathbb{O})$, and the direct sum can include only the cases $H_k(\mathbb{C})$ and \mathbb{R} . This means that A is the self-adjoint part of a C^* -algebra.

Viceversa, if A is the self-adjoint part of a C^* -algebra, let A^2 be the self-adjoint part of the tensor product of two copies of this C^* -algebra; (LT) is then fulfilled with the usual embedding $L_A \times L_A \ni (p_1, p_2) \rightarrow p_1 \otimes p_2$. \square

Theorem 7.4. *The quantum logic L_A of a finite-dimensional formally real Jordan algebra A satisfies (LT) if and only if A is the direct sum of the self-adjoint part of a C^* -algebra and spin factors.*

Proof. Suppose that (LT) holds. By lemma 5.3 and proposition 7.2, only the complex matrix algebras, spin factors and \mathbb{R} can occur in the decomposition of A into simple subalgebras and A becomes the direct sum of the self-adjoint part of a C^* -algebra and spin factors.

Now suppose that A is the direct sum of the self-adjoint part of a C^* -algebra and spin factors. The quantum logic of each spin factor is isomorphic to the quantum logic of $H_2(\mathbb{C})$. This is the self-adjoint part of the C^* -algebra consisting of the 2×2 -matrices. Using these isomorphisms, A^2 can again be constructed as the self-adjoint part of the tensor product of C^* -algebras. \square

In [20, 33, 41], the identity $n_{A^2} = n_A^2$ in proposition 7.1 for the dimensions of A^2 and A is not derived, but simply introduced as one of the postulates. Results analogous to those of this section and, particularly, a proof close to the one of proposition 7.2 are included in [33]. However, the theory of the sequential product spaces and the postulates used in [33] differ a lot from the quantum logical approach. This is why the role of the spin factors becomes different in the two approaches and all except the complex 2×2 -matrices can be ruled out in [33].

8. Conclusion

Theorem 7.3 gets particularly interesting when it is combined with the results of [29]. Four postulates for a quantum logic L were there presented that are satisfied if and only if L is the quantum logic of a finite-dimensional formally real Jordan algebra the decomposition of which into simple subalgebras does not include spin factors. Local tomography becomes the perfect add-on to these four postulates in order to finally bring us to common quantum mechanics, when (LT) is replaced by the following version:

- (LT') *For the quantum logic L , there shall be another quantum logic L^2 and a map $\otimes : L \times L \rightarrow L^2$ such that the four conditions (C1-5) in section 3 hold with $L_1 = L_2 = L$ and $L_{12} = L^2$, and both L and L^2 shall satisfy the four postulates of [29].*

We then get the following result:

A quantum logic L satisfies the four postulates of [29] and (LT')

if and only if

L is the quantum logic formed by the self-adjoint projections in a finite-dimensional C^ -algebra that does not include the complex 2×2 -matrix algebra as a direct summand.*

The local tomography postulate (LT) can also be applied to complete other reconstructions of quantum mechanics that first derive the formally real Jordan algebras [3, 6, 7, 8, 23, 29, 32, 33, 36, 37, 38]. Some of these reconstructions already use local tomography, but a stronger version (see section 5) which could be replaced by the less restrictive (LT). Moreover, their methods, frameworks and assumptions differ from those of the present paper, and they often do not include the non-simple (reducible) algebras.

Erroneously, in [24], a postulate that is not even fulfilled by quantum mechanics with the complex Hilbert space was considered for the model of a composite system [26]. It requires local tomography and something more. The present paper demonstrates that pure local tomography and an additional requirement for the map \otimes are the right replacement for it in order to derive the need of the complex numbers in quantum mechanics. The new requirement for the map \otimes that was not used in [24] is the only-if part of (C2) in section 3 (the logical independence).

In this paper, only finite-dimensional formally real Jordan algebras have been considered. Their infinite-dimensional analog are the so-called JBW-algebras [2, 19]. The idempotent elements of a JBW-algebra form a quantum logic and the local tomography postulate (LT) can easily be transferred. An interesting question now becomes whether (LT) distinguishes the self-adjoint parts of the von Neumann algebras among the JBW algebras in the same way as in the finite-dimensional case (theorems 7.3 and 7.4). This problem cannot be tackled with the mathematical tools used here, since the structure theory and classification of the JBW algebras with infinite dimension are a lot richer than the finite-dimensional case.

In order to derive the quantum-mechanical need of the C*-algebras or von Neumann algebras, an alternative to local tomography, that does not involve bipartite or multipartite systems, is the notion of *dynamical correspondence*. It concerns the mathematical model of the dynamical evolution of a single system and is motivated by the quantum-mechanical feature that the dynamical group is generated by the Hamilton operator [2, 8, 28]. Further mathematical alternatives to distinguish the self-adjoint parts of the C*-algebras or von Neumann algebras among the JB-algebras or JBW-algebras (*3-ball property* and *orientability* [2, 1]) lack a physical or probabilistic motivation.

Theorem 7.4 includes the quantum logics of spin factors, but involves a high degree of ambiguity in this case, to which the two-dimensional Hilbert space and the single qubit belong. The ambiguity cannot be circumvented in the quantum logical setting because of the discontinuous automorphisms and isomorphisms of the quantum logics of spin factors (see section 5). This may be regarded as a drawback of the quantum logical approach. Its benefits lie in its fundamentality. Instead of presuming linear or convex structures from the very beginning, it starts with the more basic structure of the quantum logic which is a generalization of the classical Boolean lattice. Nonetheless, in the case of the quantum logic of a formally really Jordan algebra without spin factors as simple subalgebras, this structure is rich enough to recover from it the complete algebra with the linear and multiplicative structure [29].

The spin factors are unproblematic in the case of other approaches like the gran work by Alfsen and Shultz, the contemporary operational probabilistic theories or the sequential product spaces, since they put convex structures (the state space or the effect space) and their characteristics into the focus.

References

- [1] E. M. Alfsen and F. W. Shultz. *State spaces of operator algebras: basic theory, orientations, and C^* -products*. Basel, Switzerland: Birkhäuser, 2001.
- [2] E. M. Alfsen and F. W. Shultz. *Geometry of state spaces of operator algebras*. Basel, Switzerland: Birkhäuser, 2003.
- [3] H. Barnum, C. P. Gaebler, and A. Wilce. Ensemble steering, weak self-duality, and the structure of probabilistic theories. *Foundations of Physics*, 43(12):1411–1427, 2013.
- [4] H. Barnum, M. Graydon, and A. Wilce. Some nearly quantum theories. *Electronic Proceedings in Theoretical Computer Science*, 195:59–70, 2015.
- [5] H. Barnum, M. Graydon, and A. Wilce. Composites and categories of Euclidean Jordan algebras. *arXiv eprint math-ph:1606.09331*, 2016.
- [6] H. Barnum and J. Hilgert. Strongly symmetric spectral convex bodies are Jordan algebra state spaces. *arXiv eprint math-ph:1904.03753*, 2019.
- [7] H. Barnum, C. M. Lee, C. M. Scandolo, and J. H. Selby. Ruling out higher-order interference from purity principles. *Entropy*, 19(6), 2017.
- [8] H. Barnum, M. P. Müller, and C. Ududec. Higher-order interference and single-system postulates characterizing quantum theory. *New Journal of Physics*, 16(12):123029, 2014.
- [9] H. Barnum and A. Wilce. Local tomography and the Jordan structure of quantum theory. *Foundations of Physics*, 44(2):192–212, 2014.
- [10] H. Barnum and A. Wilce. Post-classical probability theory. In G. Chiribella and R. W. Spekkens, editors, *Quantum Theory: Informational Foundations and Foils*, pages 367–420. Dordrecht, The Netherlands: Springer, 2016.
- [11] J. Barrett. Information processing in generalized probabilistic theories. *Physical Review A*, 75(3):032304, 2007.
- [12] L. Bunce and J. M. Wright. Quantum measures and states on Jordan algebras. *Communications in Mathematical Physics*, 98(2):187–202, 1985.
- [13] G. Chiribella, G. M. D’Ariano, and P. Perinotti. Informational derivation of quantum theory. *Physical Review A*, 84(1):012311, 2011.
- [14] G. de la Torre, L. Masanes, A. J. Short, and M. P. Müller. Deriving quantum theory from its local structure and reversibility. *Physical Review Letters*, 109(9):090403, 2012.
- [15] J. Faraut and A. Korányi. *Analysis on symmetric cones*. Oxford, UK: Oxford University Press, 1994.

- [16] A. M. Gleason. Measures on the closed subspaces of a Hilbert space. *Journal of Mathematics and Mechanics*, 6:885–893, 1957.
- [17] R. Haag and D. Kastler. An algebraic approach to quantum field theory. *Journal of Mathematical Physics*, 5(7):848–861, 1964.
- [18] J. Hamhalter. Statistical independence of operator algebras. *Annales de l’IHP Physique Théorique*, 67(4):447–462, 1997.
- [19] H. Hanche-Olsen and E. Størmer. *Jordan operator algebras*. London, UK: Pitman, 1984.
- [20] L. Hardy. Quantum theory from five reasonable axioms. *arXiv eprint quant-ph/0101012*, 2001.
- [21] L. Hardy. Foliable operational structures for general probabilistic theories. In H. Halvorson, editor, *Deep Beauty: Understanding the Quantum World through Mathematical Innovation*, pages 409–442. Cambridge, UK: Cambridge University Press, 2011.
- [22] L. Masanes and M. P. Müller. A derivation of quantum theory from physical requirements. *New Journal of Physics*, 13(6):063001, 2011.
- [23] K. Nakahira. Derivation of quantum theory with superselection rules. *arXiv eprint quant-ph/1910.02649*, 2019.
- [24] G. Niestegge. Composite systems and the role of the complex numbers in quantum mechanics. *Journal of Mathematical Physics*, 45(12):4714–4725, 2004.
- [25] G. Niestegge. Why do the quantum observables form a Jordan operator algebra? *International Journal of Theoretical Physics*, 43(1):35–46, 2004.
- [26] G. Niestegge. Erratum: ”Composite systems and the role of the complex numbers in quantum mechanics” [J. Math. Phys. 45, 4714 (2004)]. *Journal of Mathematical Physics*, 51, 2010.
- [27] G. Niestegge. Conditional probability, three-slit experiments, and the Jordan algebra structure of quantum mechanics. *Advances in Mathematical Physics*, 2012, 2012.
- [28] G. Niestegge. Dynamical correspondence in a generalized quantum theory. *Foundations of Physics*, 45(5):525–534, 2015.
- [29] G. Niestegge. A simple and quantum-mechanically motivated characterization of the formally real Jordan algebras. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 476(2233):20190604, 2020.
- [30] C. Piron. Axiomatique quantique. *Helvetica Physica Acta*, 37(4-5):439–468, 1964.

- [31] M. Rédei. Logical independence in quantum logic. *Foundations of Physics*, 25(3):411–422, 1995.
- [32] J. van de Wetering. An effect-theoretic reconstruction of quantum theory. *arXiv eprint quant-ph/1801.05798*, 2018.
- [33] J. van de Wetering. Sequential product spaces are Jordan algebras. *Journal of Mathematical Physics*, 60(6):062201, 2019.
- [34] V. S. Varadarajan. *Geometry of Quantum Theory, vols. 1 and 2*. New York, NY: Van Nostrand Reinhold, 1968 and 1970.
- [35] J. von Neumann, P. Jordan, and E. Wigner. On an algebraic generalization of the quantum mechanical formalism. *Annals of Mathematics*, 35:29–64, 1934.
- [36] A. Wilce. Symmetry and self-duality in categories of probabilistic models. *Electronic Proceedings in Theoretical Computer Science*, 95:275–279, 2012.
- [37] A. Wilce. A royal road to quantum theory (or thereabouts). *Entropy*, 20(4):227, 2018.
- [38] A. Wilce. Conjugates, filters and quantum mechanics. *Quantum*, 3:158, 2019.
- [39] W. K. Wootters. Quantum mechanics without probability amplitudes. *Foundations of physics*, 16(4):391–405, 1986.
- [40] W. K. Wootters. Local accessibility of quantum states. In W. H. Zurek, editor, *Complexity, entropy and the physics of information*, pages 39–46. Boston, MA: Addison-Wesley, 1990.
- [41] M. Zaopo. Information theoretic axioms for quantum theory. *arXiv eprint quant-ph/1205.2306*, 2012.