Follow the Flow: sets, relations, and categories as special cases of functions with no domain

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Disclaimer

This is a first draft of an ongoing research project. This text is supposed to work as a simple reference for seminars to be delivered in Brazil.

Updated versions will be available soon.

Abstract

We introduce, develop, and apply a new approach for dealing with the intuitive notion of function, called Flow Theory. Within our framework functions have no domain at all. Sets and even relations are special cases of functions. In this sense, functions in Flow are not equivalent to functions in ZFC. Nevertheless, we prove both ZFC and Category Theory are naturally immersed within Flow. Besides, our framework provides major advantages as a language for axiomatization of standard mathematical and physical theories. Russell's paradox is avoided without any equivalent to the Separation Scheme. Hierarchies of sets are obtained without any equivalent to the Power Set Axiom. And a clear principle of duality emerges from Flow, in a way which was not anticipated neither by Category Theory nor by standard set theories.

Key words: functions, set theory, category theory.

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1 Introduction

Throughout the ages mathematicians have considered their objects, such as numbers, points, etc., as substantial things in themselves. Since these entities had always defied attempts at an adequate description, it slowly dawned on the mathematicians of the nineteenth century that the question of the meaning of these objects as substantial things does not make sense within mathematics, if at all. The only relevant assertions concerning them do not refer to substantial reality; they state only the interrelations between mathematically "undefined objects" and the rules governing operations with them. What points, lines, numbers "actually" *are* cannot and need not be discussed in mathematical science. What matters and what corresponds to "verifiable" fact is structure and relationship, that two points determine a line, that numbers combine according to certain rules to form other numbers, etc. A clear insight into the necessity of a dissubstantiation of elementary mathematical concepts has been one of the most important and fruitful results of the modern postulational development.

Richard Courant, *What is Mathematics*, 1941.

All usual mathematical approaches for well-known physical theories can be easily associated to either differential equations or systems of differential equations. Newton's second law, Schrödinger's equation, Maxwell's equations, and Einstein field equations are all differential equations which ground classical mechanics, quantum mechanics, classical electromagnetism, and general relativity, respectively. Other similar examples may be found in thermodynamics, gauge theories, the Dirac electron, etc. Solutions for those differential equations (when they exist) are either functions or classes of functions. So, the concept of function plays a major role in theoretical physics. Actually, functions are more relevant than sets, in a very precise sense [5] [6].

In pure mathematics the situation is no different. Continuous functions, linear transformations, homomorphisms, and homeomorphisms, for example, play a fundamental role in topology, linear algebra, group theory, and differential geometry, respectively. And category theory emphasizes such a role in a very clear, elegant, and comprehensive way.

Functions allow us to talk about the dynamics of the world, in the case of physical theories. Regarding mathematics, functions allow us to talk about invariant properties, whether those properties refer to either algebraic operations or order relations.

From a historical point of view, many authors have advocated the idea that functions are supposed to play a strategic role into the foundations of mathematics [18] and even mathematics teaching [12].

Intuitively speaking, a function is supposed to be a term that allows us to uniquely associate certain terms to other terms. In standard set theories, for example, a function is a special case of set, namely, a specific set of ordered pairs of sets. In contrast, in category theory morphisms have an intended interpretation which is somehow associated to functions. In both cases, functions are attached to domains and codomains which are sets in set theories and identity morphisms in category theories. In this paper we develop a new approach - Flow Theory - for dealing with the intuitive notion of function. In a precise sense, in Flow Theory functions have no domain at all. Within our approach, a set is a special case of function. Russell's paradox is avoided without any equivalent to the Separation Scheme. We provide a comprehensive discussion of Flow Theory as a new foundation for mathematics, where functions explicitly play a more fundamental role.

The name Flow is a reference to Heraclitean flux doctrine, according to which things are constantly changing. Accordingly, in Flow theory all terms are "active objects" under the action of other "active objects".

This paper is strongly motivated by [18] and [21] and related papers as well ([5] [6]). It is quite easy to show that in many natural axiomatic formulations of physical and even mathematical theories, there are many superfluous concepts usually assumed as primitive ([5] [6]). That happens mainly when those theories are formulated in the language of standard set theories, such as Zermelo-Fraenkel's. In 1925, however, John von Neumann introduced a set theory where sets are definable by means of functions [18]. And in [21] it was provided a reformulation of von Neumann's original ideas (termed $\mathcal{N}$ theory) which allowed the authors to reformulate standard physical and mathematical theories with much less primitive concepts in a very natural way. Nevertheless, in $\mathcal{N}$ theory there are two fundamental constants which are not clarified in any way. Those constants, namely, $\underline{0}$ and $\underline{1}$, allow us to define sets as particular cases of functions, in a way which is somehow analogous to the usual sense of characteristic functions in standard set theories.

In this paper Flow theory is introduced as a generalized formulation of concepts derived from $\mathcal{N}$ theory. Constants $\underline{0}$ and $\underline{1}$ are still necessary. Notwithstanding, we are able to define them from our proposed axioms and some related theorems. And that fact entails an algebra defined over functions. Such an algebra shows us that both category theory and ZFC set theory are naturally present within our framework.

Besides the presentation and discussion of Flow axioms, we introduce several applications and foundational issues by comparing Flow with ZFC set theories and category theory.

Our punch line may be summarized by something like this: (i) the concept of set (as a collection of objects) is somehow implicitly assumed through ZF axioms; (ii) nevertheless, sets play a secondary role in mathematics and applied mathematics, since the true actors are always functions, while sets work as just a stage (setting) for such actors; (iii) so, why cannot we implicitly assume the notion of function right at the start on the foundations of mathematical theories?

2 Flow theory

Flow is a first-order theory with identity, where the formula $x = y$ should be read as "x is equal to y". The formula $\neg(x = y)$ is abbreviated as $x \neq y$. Flow has one functional letter $f_1^2(f, x)$, where f and x are terms. If $y = f_1^2(f, x)$, we abbreviate this by $f(x) = y$, and we say y is the *image* of x by f. We call f_1^2 *evaluation*. All terms of Flow are called *functions*. We use lowercase Latin and Greek letters to denote functions. Uppercase letters are used to denote predicates (which are eventually defined). The axioms of Flow follow in the next subsections. But first we need to make a remark. Any explicit definition in Flow is an abbreviative one, in the sense that for a given formula F, the *definiendum* is just a metalinguistic abbreviation for the *definiens* given by F.

2.1 Functions

P1 - Weak Extensionality $\forall f \forall g(((f(g) = f \wedge g(f) = g) \vee (f(g) = g \wedge g(f) = f)) \Rightarrow f = g))$.

This first axiom is tricky. Any function f such that $f(g) = f$ is said to be *rigid with* g. And any function f such that $f(g) = g$ is said to be *flexible with* g. So, if both f and g are rigid with each other, then we are talking about the very same function ($f = g$). Another possibility to identify a function is by checking if f and g are both flexible with each other. If that is the case, then again $f = g$.

P2 - Self-Reference $\forall f(f(f) = f)$.

Our first theorem has a very intuitive meaning.

Theorem 1 $\forall f \forall g(f = g \Leftrightarrow \forall x(f(x) = g(x)))$.

Proof: By using the substitutivity of identity in the formula $f(x) = f(x)$ (which is a theorem in any first-order theory with identity), proof of the $\Rightarrow$ part is quite straightforward. After all, if $f = g$, then $f(x) = g(x)$, for any x. In particular, we have $f(f) = g(g) = f(g) = g(f) = f = g$. Concerning the $\Leftarrow$ part, suppose for any x we have $f(x) = g(x)$. In particular, for $x = f$, we have $f(f) = g(f)$. And for $x = g$, we have $f(g) = g(g)$. Nevertheless, according to **P2**, $f(f) = f$ and $g(g) = g$. So, $g(f) = f$ and $f(g) = g$. And from **P1**, that entails $f = g$.

Axiom **P2** says every function is rigid and flexible with itself. That fact deserves a more detailed discussion. Our main purpose here is to avoid any Flow-theoretic version of Russell's paradox. Consider, for example, the next statement: y is a function such that

$$\forall x(y(x) = r \Leftrightarrow x(x) \neq r).$$

In the formula above we are explicitly trying to *define* a function y. On the left side of $\Leftrightarrow$ we have the *definiendum* and on the right side we have the *definiens*. If we ignore **P2**, what about $y(y)$? If $y(y) = r$, then we are considering $y(x) = r$ where x is y. Hence, according to the formula above we entail $y(y) \neq r$. Analogously, if $y(y) \neq r$, we are considering $x(x) \neq r$ where x is again y. And according to the formula above we have $y(y) = r$. Consequently, we have $y(y) = r \Leftrightarrow y(y) \neq r$. That is Russell's paradox! To avoid such an embarrassment (which could explode Flow theory, since we are grounding our axiomatic system within classical logic) all we need to do is to introduce axiom **P2**. According to **P2**, any function y defined by the formula above guarantees that y cannot be equal to x. Since for any x we have $x(x) = x$ and the *definiens* above demands that $x(x) \neq r$, that entails $x \neq r$. But the *definiendum* states $y(x) = r$. Hence, $y(x) \neq x = x(x)$. Therefore, Theorem 1 guarantees $y \neq x$, since x and y do not share all their images. Hence, there is no paradox! After all, the paradox was entailed from the possibility that $x = y$. Axiom **P2** prohibits the definition of a function like y. Otherwise, a formula like the one proposed above would be creative, allowing us to derive contradictions. That is a much simpler solution to Russell's paradox than any equivalent to the Separation Scheme in Zermelo-Fraenkel-like set theories. Besides, as we shall see below, Flow theory allows us to talk about sets and proper classes in the usual sense of standard set theories, like ZFC with classes, NBG and their variations.

It is worth to observe that axioms **P1** and **P2** could be rewritten as one single axiom as it follows:

P1' - Alternative Weak Extensionality $\forall f \forall g (((f(g) = f \wedge g(f) = g) \vee (f(g) = g \wedge g(f) = f)) \Leftrightarrow f = g))$.

If that was the case, then **P2** would be a consequence from **P1'**. Ultimately, $f = g$ would entail that $f(g) = f$ (from **P1'**). And substitutivity of identity entails $f(f) = f$. On the other hand, we prefer to keep axioms **P1** and **P2** (instead of **P1'**) for pedagogical purposes. From **P1** and **P2**, we can analogously see that **P1'** is a theorem.

One philosophical remark concerning axiom **P2** refers to Richard Courant's quote presented in the Introduction. Functions, by themselves, are irrelevant. What matters is what they do. That point is gradually clearer thanks to the next postulates.

P3 - Identity $\exists f \forall x (f(x) = x)$.

This is the first axiom which guarantees the existence of a specific function. Any function f which satisfies **P3** is said to be an *identity function*.

Theorem 2 *The identity function is unique.*

Proof: Suppose both f and g satisfy axiom **P3**. Then, for any x we have $f(x) = x$ and $g(x) = x$. Thus, $f(g) = g$ and $g(f) = f$. Hence, according to **P1**, $f = g$.

In other words, there is one single function f which is flexible to every function. In that case we simply say f is *flexible*. That means "flexible" and "identity" are synonyms.

P4 - Rigidness $\exists f \forall x (f(x) = f)$.

In other words, there is at least one function f which is rigid with any function. Observe the symmetry between axioms **P3** and **P4**! Any function f which satisfies this last postulate is simply said to be *rigid*.

Theorem 3 *The rigid function is unique.*

Proof: Suppose both f and g satisfy axiom **P4**. Then, for any x we have $f(x) = f$ and $g(x) = g$. Thus, for any x we have $f(g(x)) = f(g) = f$ and $g(f(x)) = g(f) = g$. Thus, according to **P1**, $f = g$.

Now we are able to justify the extensionality axiom **P1**. Our purpose here is to define constants $\underline{0}$ and $\underline{1}$, in order to accommodate our view about von Neumann's ideas. So, $\underline{1}$ is the identity (flexible) function and $\underline{0}$ is the rigid function, since we proved they are both unique. In other words

$$\forall x(\underline{1}(x) = x \quad \wedge \quad \underline{0}(x) = \underline{0})$$

If we recall that $f(x) = y$ is an abbreviation for $f_1^2(f, x) = y$, we can read axioms **P3** and **P4** as statements regarding the existence of two "spurs". Axiom **P3** states there is a function f such that for any x we have $f_1^2(f, x) = x$, while **P4** says there is f such that for any x we have $f_1^2(f, x) = f$.

Theorem 4 *$\underline{0}$ is the only function which is rigid with $\underline{0}$.*

Proof: The statement above is equivalent to say that $\forall x(x \neq \underline{0} \Rightarrow x(\underline{0}) \neq x)$. In other words, $\forall x(x(\underline{0}) = x \Rightarrow x = \underline{0})$. But we already know that $\underline{0}(x) = \underline{0}$. Therefore, if we have $x(\underline{0}) = x \wedge \underline{0}(x) = \underline{0}$, according to **P1**, we have $x = \underline{0}$.

Theorem 5 *$\underline{1}$ is the only function which is flexible with $\underline{1}$.*

Proof: The statement above is equivalent to say that $\forall x(x \neq \underline{1} \Rightarrow x(\underline{1}) \neq \underline{1})$. In other words, $\forall x(x(\underline{1}) = \underline{1} \Rightarrow x = \underline{1})$. But we already know that $\underline{1}(x) = x$. Therefore, if we have $x(\underline{1}) = \underline{1} \wedge \underline{1}(x) = x$, according to **P1**, we have $x = \underline{1}$.

The last two theorems do not say what are the images $x(\underline{0})$ or $x(\underline{1})$ (when $x \neq \underline{1}$, in the last case). Nevertheless, such values prove to be rather important for future applications of Flow Theory. But before discussing that, we need another axiom.

P5$_\alpha$ **- Creation** $\forall x \exists! y(\alpha(x, y)) \Rightarrow \exists! h \forall x \forall y(x \neq h \Rightarrow (h(x) = y \Leftrightarrow \alpha(x, y)))$, where $\alpha(x, y)$ is an atomic formula such that there is at least one occurrence of x, one occurrence of y, and all those occurrences are free.

P5$_\alpha$ is an axiom scheme which allows us to define unique functions from certain formulas α in Flow. For example, if $\alpha(x, y)$ is the formula $x = y$, then we are able to show **P3** can be entailed from **P5**$_\alpha$. In other words, **P3** is not an independent axiom. The syntactic restrictions imposed over formula α are necessary to avoid certain inconsistencies. We go back to this some paragraphs below. Notwithstanding, **P5**$_\alpha$ does more than just guaranteeing the existence of $\underline{1}$. Consider, for example, the following definition.

Definition 1 *For any functions f and g we may define an atomic formula $\alpha(x, y)$ as $y = f(g(x))$. So, according to axiom* **P5**$_\alpha$ *there is a unique function h such that for all $x \neq h$ we have $h(x) = y \Leftrightarrow y = f(g(x))$, which means $h(x) = f(g(x))$. In that case, we say h is the* composition *of f with g and denote this by $h = f \circ g$.*

The last identity in the definition above is sound, since uniqueness of composition is demanded by $\mathbf{P5}_\alpha$ itself. The intuitive idea for this last definition, granted by axiom $\mathbf{P5}_\alpha$, is quite simple. Without that postulate, an arbitrary formula $y = f(g(x))$ is not enough to guarantee y is the image of some x by a single function h. That happens due to the impossibility of guaranteeing h will satisfy the self-reference axiom. Thanks to $\mathbf{P5}_\alpha$, we can now guarantee the existence of such a function. But beware! We never calculate $(x \circ y)(x \circ y)$ as $x(y(x \circ y))$, since $(x \circ y)(x \circ y)$ is always $x \circ y$.

Theorem 6 *Composition $\circ$ is associative.*

The proof is straightforward. All we have to do is to calculate $(f \circ g) \circ h$ and $f \circ (g \circ h)$ over an arbitrary t, and then to compare both terms.

This last theorem is somehow interesting, since evaluation f_1^2 is not associative and composition is defined from evaluation. Consider, for example, $x(\underline{1}(x))$, for x different from $\underline{0}$ and different from $\underline{1}$. Thus, $x(\underline{1}(x)) = x(x) = x$. If evaluation was associative, we would have $x(\underline{1}(x)) = x(\underline{1})(x) = \underline{0}(x) = \underline{0}$. A contradiction! Of course this *rationale* works only if we prove the existence of other functions besides $\underline{0}$ and $\underline{1}$. That happens thanks to the next axiom. So, although evaluation f_1^2 is not associative, we are still able to define a binary functional letter $\circ$ from f_1^2 such that $\circ$ is associative. That happens because f and $f(x)$ are not necessarily the same thing. So, contrary to the usual slogan from category theory [14], evaluation is not a special case of composition.

Actually, it is good news that evaluation is not associative. According to **P2**, we have, for all t, $g(t) = (g(g))(t)$, since for any g we have $g(g) = g$. If evaluation was associative, we would have $g(t) = g(g(t))$ for any t, and thus, $g = g \circ g$. So, composition would be an idempotent operation. That would be an undesirable result for anyone who intends to develop, e.g., category theory within Flow.

The next formulas are all theorems: $\underline{0} \circ \underline{0} = \underline{0}$; $\underline{0} \circ \underline{1} = \underline{0}$; $\underline{1} \circ \underline{0} = \underline{0}$; $\underline{1} \circ \underline{1} = \underline{1}$. The next formulas are theorems for any function x: $\underline{0} \circ x = \underline{0}$; $x \circ \underline{1} = x$; $\underline{1} \circ x = x$. Concerning $x \circ \underline{0}$, that value is supposed to be discussed later, due to Theorem 4.

Theorem 7 *Suppose f is idempotent with respect to composition, and there are some x and y such that $f(x) = y$. Then, f is flexible with y.*

Proof: If $f(x) = y$, then $f(f(x)) = f(y)$. But $f(f(x)) = (f \circ f)(x)$ for $x \neq f \circ f$. Since f is idempotent with respect to composition, then $f \circ f = f$. Thus, $f(x) = f(y)$. Hence, $y = f(y)$.

This last theorem is not important for further developments of Flow Theory. We just proved it to show that our framework is able to mimic well known results regarding the usual way composition is defined within standard set theories. Similar results about idempotent functions can be generated.

The next axiom can be read more easily if we introduce the definition below.

Definition 2 $\sigma_f = g \neq \underline{0}$ *iff* $g \neq f \wedge f(g) = \underline{0} \wedge \forall x((x \neq g \wedge x \neq \sigma) \Rightarrow g(x) = f(x))$.

The intuitive idea of a term like σ_f is that of *successor* of a given function f. If the successor of f is a non-$\underline{0}$ term g, then f and g share the same images for any x different of g and σ, although f and g are different. Besides, $f(g) = \underline{0}$. That is why f and g are different, since $f(g) = \underline{0}$ while $g(g) = g$ (remember we are considering the case where g is a non-$\underline{0}$ term). In the case where there is no g which satisfies such demands, then σ_f is simply $\underline{0}$, and again f and g are different (if, of course, we guarantee the existence of any function like σ). Finally, it is worth to remark that σ is supposed to be a function whose existence is granted by the next axiom. Nevertheless, since σ is proven in the next paragraphs to play a strategic role in Flow Theory, we employ the special notation σ_f instead of $\sigma(f)$. So,

P6 - Expansion $\exists! \sigma$.

Axioms **P1-P5**$_\alpha$ work as "a soil prep to enhance the germination of functions". Axiom **P6**, on the other hand, states the existence of another function σ. And that fact entails the existence of infinitely many other functions. Besides $\underline{0}$ and $\underline{1}$, there is a unique function $\sigma_{\underline{0}}$ whose images are either $\underline{0}$ or $\sigma_{\underline{0}}$ itself, where $\sigma_{\underline{0}}$ is different from $\underline{0}$. In other words, **P6** says there is a $\sigma_{\underline{0}} \neq \underline{0}$ such that for any t different from $\sigma_{\underline{0}}$, both $\underline{0}$ and $\sigma_{\underline{0}}$ share the same images $\sigma_{\underline{0}}(t)$ and $\underline{0}(t)$. Such a function $\sigma_{\underline{0}}$ is denoted by ϕ_0.

Observation 1 *A word of caution is necessary here. Rigorously speaking, the label "Definition 2" does not necessarily refer to a definition. Consider, for example, there is a function ϕ'_0 such that $\phi'_0 \neq \phi_0$ and $\phi'_0 = \sigma_{\underline{0}}$. In that case, we have $\phi_0(\phi'_0) = \underline{0}$ and $\phi'_0(\phi_0) = \underline{0}$. That is a result which confirms $\phi_0 \neq \phi'_0$, according to the axiom of extensionality. On the other hand, something odd is happening here, since there seems to be* two *successors for the same function $\underline{0}$, despite the fact that $\underline{0}$ is unique. From an intuitive point of view, we cannot actually* see *or* decide *which is which. It does not matter which function is the successor of $\underline{0}$, if there is more than one successor σ which satisfies the allegedly definition 2. All that matters is how this successor does work. An analogous remark can be done about the successor of any function f which admits a non-$\underline{0}$ successor (as we intend to pursue in the next paragraphs). Nevertheless, if $\phi_0 = \sigma_{\underline{0}}$ and $\phi'_0 = \sigma'_{\underline{0}} = \sigma_{\underline{0}}$, that entails $\phi_0 = \phi'_0$, which conflicts with the assumption that $\phi_0 \neq \phi'_0$. That means, from a rigorous point of view, "Definition 2" may somehow be a creative statement. After all, if "Flow without Definition 2" is consistent, then "Flow with Definition 2" may allow us to entail a contradiction. That means our choice above for stating Definition 2 and axiom* **P6** *has a pedagogical* rationale. *That is why we used the quantifier $\exists!$ in* **P6**. *In the next postulate, we intend to talk about the successor of some other functions, in the sense that the successor of the successor of $\underline{0}$ does exist and so on. But from now on we don't have to worry with the $\exists!$ quantifier, since the uniqueness of σ guarantees the uniqueness of σ_f for any f. Our pedagogical solution to cope with Flow is based on the convenience of how to easily read our axioms.*

P7 - Infinity $\exists i((\forall t(i(t) = t \vee i(t) = \underline{0})) \wedge \sigma_i \neq \underline{0} \wedge (i(\sigma_{\underline{0}}) = \sigma_{\underline{0}} \wedge \forall x(i(x) = x \Rightarrow (i(\sigma_x) = \sigma_x \neq \underline{0}))))$.

Definition 3 *Any function i which satisfies axiom* **P7** *is said to be* inductive.

Since the existence of ϕ_0 is granted by **P6**, we can now apply σ again to get a function $\phi_1 = \sigma_{\phi_0} = \sigma_{\sigma_{\underline{0}}}$ such that $\phi_1(\phi_1) = \phi_1$, $\phi_1(\phi_0) = \phi_0$, and for the remaining functions t (functions t which are neither ϕ_0 nor ϕ_1) we have $\phi_1(t) = \underline{0}$. Concerning **P7**, this postulate states the existence of another function i. It says if a function x admits a non-$\underline{0}$ successor σ_x (in a way such that $i(x) = x$), then $i(\sigma_x) = \sigma_x$. Analogously we can get (from **P6**) functions ϕ_2, ϕ_3, and so on. Besides, according to **P7**, any inductive function i admits its own non-$\underline{0}$ successor σ_i.

Subscripts 0, 1, 2, 3, etc., are simply metalinguistic symbols based on an alphabet of ten symbols (the usual decimal numeral system) which follows the lexicographic order. The lexicographic order is denoted here by $\prec$, where $0 \prec 1 \prec 2 \prec 3 \prec 4 \prec 5 \prec 6 \prec 7 \prec 8 \prec 9$. If n is a subscript, then $n+1$ corresponds to the next subscript, in accordance to the lexicographic order. In that case, we write $n \prec n+1$. $n+m$ is an abbreviation for $(...(...((n+1)+1)+...1)...)$ with m occurrences of $+$ and m occurrences of pairs of parentheses. And again we have $n \prec n+m$. As it is well known for any finite alphabet, $\prec$ is a strict total order. That fact allows us to talk about a minimum value between two subscripts m and n. Within that context, $\min\{m, n\}$ is m iff $m \prec n$, it is n iff $n \prec m$, and it is either one of them if $m = n$. Of course, $m = n$ iff $\neg(m \prec n) \wedge \neg(n \prec m)$.

Such a vocabulary of ten symbols endowed with $\prec$ is called here (meta) language $\mathcal{L}$.

Thus, **P6** provides us some sort of "recursive definition" for functions ϕ_n, while **P7** allows us to guarantee the existence of inductive functions:

- ϕ_0 is such that $\phi_0(x)$ is ϕ_0 if $x = \phi_0$ and $\underline{0}$ otherwise.
- ϕ_{n+1} is such that $\phi_{n+1}(\phi_{n+1}) = \phi_{n+1}$, $\phi_{n+1} \neq \phi_n$, and $\phi_{n+1}(x) = \phi_n(x)$ for any x different from ϕ_{n+1}.

Observe that $\phi_{n+1}(\phi_n) = \phi_n(\phi_n) = \phi_n$, while $\phi_n(\phi_{n+1}) = \underline{0}$. More than that, $\phi_{n+2}(\phi_{n+1}) = \phi_{n+1}$, and $\phi_{n+2}(\phi_n) = \phi_{n+1}(\phi_n) = \phi_n$; while $\phi_n(\phi_{n+2}) = \underline{0}$. For a generalization of such results, see Propositions 1, 2, and 3.

Notwithstanding, **P7** says much more, since it states function i itself has its own successor σ_i.

The diagrams below (Figure 1) help us to illustrate how can we represent any function f in a quite straightforward way. Each diagram is formed by a rectangle. The left top corner of any rectangle introduces the label f of the function which is represented by the diagram. The remaining labels refer to functions x such that $f(x) \neq \underline{0}$. For each label x there is a unique corresponding arrow which indicates the image of x by f. Since for any function f we have $f(f) = f$, then the function represented at the left top corner of the rectangle does not need to be attached to any arrow. So, our first three examples below refer to functions ϕ_0, ϕ_1, and ϕ_2.

From left to right, the first diagram refers to ϕ_0. It says, for any x, $\phi_0(x)$ is $\underline{0}$, except for ϕ_0 itself. The second diagram says $\phi_1(\phi_1) = \phi_1$, and $\phi_1(\phi_0) = \phi_0$. Observe the circular arrow attached to label ϕ_0 in the second diagram is not a reference to the fact that $\phi_0(\phi_0) = \phi_0$. Circular arrows referring to axiom **P2** are simply omitted. So, the circular arrow associated to ϕ_0 in the second diagram says solely that $\phi_1(\phi_0) = \phi_0$. Finally, the third diagram says $\phi_2(\phi_2) = \phi_2$, $\phi_2(\phi_1) = \phi_1$, and $\phi_2(\phi_0) = \phi_0$. The diagram representations for $\underline{0}$ and $\underline{1}$ are, respectively, a blank rectangle and a filled in black rectangle. More sophisticated examples of functions are represented by diagrams in the next Section.

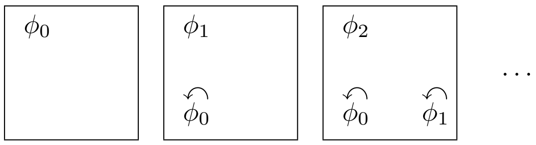

Figure 1: From left to right, diagram representations for functions ϕ_0, ϕ_1, and ϕ_2.

Observe those diagrams above may be easily identified with reflexive graphs, from Graph Theory. Since objects and morphisms of a category (in the sense of Category Theory) may be viewed as, respectively, the vertices and edges of a graph, that fact is supposed to ease our discussion in Section 4 concerning Category Theory.

Theorem 8 *If i is inductive, then for any n of language $\mathcal{L}$ we have $i(\phi_n) = \phi_n$*

The proof is straightforward.

The next propositions are provable by induction.

Proposition 1 *For any m and n of the vocabulary given above, $\phi_{m+n}(\phi_m) = \phi_m$ and $\phi_{m+n}(\phi_n) = \phi_n$.*

Proposition 2 *For any m and n of the vocabulary given above, if at least one of them is different from 0, then $\phi_m(\phi_{m+n}) = \underline{0}$ and $\phi_n(\phi_{m+n}) = \underline{0}$.*

Recall our previous argument for the non-associativity of evaluation holds, since we can now guarantee the existence of other functions besides $\underline{0}$ and $\underline{1}$.

Definition 4 $f[t]$ *iff* $t \neq f \wedge f(t) \neq \underline{0}$.

While $f(t)$ is a term for any f and t, $f[t]$ is a metalinguistic abbreviation for a formula. We read $f[t]$ as "*f acts on t*". And f acts on t iff t is not f itself and $f(t) \neq \underline{0}$. The intuitive idea of this last definition is to allow us to talk about what effectively a function f does. For example, both $\underline{0}$ and ϕ_0 do nothing at all, since there is no t on which they act. On the other hand, there is a term t on which ϕ_1 acts, namely, ϕ_0.

Proposition 3 *For any m and n of the vocabulary of language $\mathcal{L}$, $\phi_m \circ \phi_n = \phi_n \circ \phi_m = \phi_{\min\{m,n\}}$.*

Proof: We present here a sketch for the proof. Without loss of generality, suppose first $m \prec n$. That is equivalent to say there is some p such that $m + p = n$. So, we can use the previous propositions regarding functions ϕ_n. According to Definition 1, if $x \neq \phi_m \circ \phi_n$, then the images of $\phi_m \circ \phi_n$ are given by $(\phi_m \circ \phi_n)(x) = \phi_m(\phi_n(x))$. But according to the last two propositions, those are exactly the same images of ϕ_m. Since those functions of kind ϕ_n are generated by axiom **P6**, then $\phi_m \circ \phi_n$ is exactly ϕ_m. An analogous argument shows that $\phi_n \circ \phi_m = \phi_m$. For the case where $m = n$, the proof is straightforward.

This last proposition proves all functions ϕ_n are idempotent with respect to composition. Besides, composition is commutative among functions ϕ_n. In our discussion about the ZFC counterpart of Flow we show how to identity composition with the usual notion of intersection among sets.

Theorem 9 *For any n from language $\mathcal{L}$, $\sigma \circ \phi_n$ is a function g such that: (i) $g(g) = g$; (ii) $g(\phi_m) = \phi_{m+1}$ for any $m = n \vee m \prec n$; and $g(x) = \phi_0$ for the remaining values of x.*

Proof: If $\sigma \circ \phi_n = g$, then $g(x) = \sigma(\phi_n(x))$, for $x \neq g$. If either $m = n$ or $m \prec n$, then $g(\phi_m) = \sigma(\phi_n(\phi_m)) = \sigma(\phi_m) = \phi_{m+1}$. For the remaining values x ($x \neq g \wedge x \neq \phi_m$ for each m), we have $\phi_n(x) = \underline{0}$. Therefore, $g(x) = \sigma(\phi_n(x)) = \sigma(\underline{0}) = \phi_0$.

Theorem 10 $\sigma_{\underline{1}} = \underline{0}$.

Proof: Suppose $g = \sigma_{\underline{1}}$. According to definition 2, $\underline{1}(g) = \underline{0}$. That happens only if $g = \underline{0}$. That means the successor of $\underline{1}$ does not share all images of $\underline{1}$. That happens because the successor of $\underline{1}$ is not a non-$\underline{0}$ term.

Theorem 11 $\sigma \circ \underline{1} = \underline{1} \circ \sigma = \sigma$.

Proof: That is a corollary from the fact that for any x, we have $x \circ \underline{1} = \underline{1} \circ x = x$.

Definition 5 $\mathcal{C}_y(f)$ *iff* $\exists y \forall x (x \neq f \Rightarrow f(x) = y)$.

Formula $\mathcal{C}_y(f)$ is read as "f is a constant function with constant value y" or simply "f is a constant function", if there is no risk of confusion. That means a constant function is a term f such that, for a given y, $f(x)$ is either y (for any $x \neq f$) or f itself (which is consistent with **P2**).

The next theorem says the composition $\sigma \circ \underline{0}$ is a constant function with constant value ϕ_0.

Theorem 12 $\mathcal{C}_{\phi_0}(\sigma \circ \underline{0})$.

Proof: For any t we have $(\sigma \circ \underline{0})(t) = \sigma(\underline{0}(t)) = \sigma(\underline{0}) = \sigma_{\underline{0}} = \phi_0$. That means $\sigma \circ \underline{0}$ has images ϕ_0 for any $t \neq \sigma \circ \underline{0}$ and image $\sigma \circ \underline{0}$ for $t = \sigma \circ \underline{0}$. Thus, $\sigma \circ \underline{0}$ is a constant function with constant value ϕ_0.

Theorem 13 *$\underline{0}$ is the only constant function which assumes one single image for any x.*

Proof: According to **A5**, any constant function f has at most two images, namely, either a constant value c or f itself (see axiom **P2**). So, if f is a constant function and it has one single image for any x, then that image is supposed to be f itself, according to **P2**. Well, that is exactly the statement of axiom **P4**. And according to theorem 3, there is one single function like this, namely, $\underline{0}$.

On the other hand, **P6** guarantees the existence of at least one other constant function, namely, ϕ_0 such that $\phi_0(x) = \underline{0}(x)$ for any x different from ϕ_0. Nevertheless, ϕ_0 is not $\underline{0}$, despite the fact that both $\underline{0}$ and ϕ_0 are constant functions with the same constant value $\underline{0}$.

Theorem 14 *If f is a constant function, then its constant value is either $\underline{0}$ or ϕ_0.*

Proof: To be done...

The last theorems state there are only three constant functions in Flow, namely, $\underline{0}$, ϕ_0, and $\sigma \circ \underline{0}$.

It is worth to observe as well, both evaluation f_1^2 and composition $\circ$ are not commutative. For example, while $\sigma \circ \underline{0}$ is a constant function with constant value ϕ_0, $\underline{0} \circ \sigma$ is $\underline{0}$. Besides, $\underline{0}(\sigma) = \underline{0}$, but $\sigma(\underline{0}) = \sigma_{\underline{0}} = \phi_0$. Other examples are provided in the next paragraphs.

Now we are able to discuss about the syntactic restrictions imposed over formula $\alpha(x, y)$ in axiom $\mathbf{P5}_\alpha$. According to that postulate, $\forall x \exists ! y(\alpha(x, y)) \Rightarrow \exists ! h \forall x \forall y(x \neq h \Rightarrow (h(x) = y \Leftrightarrow \alpha(x, y)))$, where $\alpha(x, y)$ is *an atomic formula such that there is at least one occurrence of x, one occurrence of y, and all those occurrences are free*. Suppose we did not demand at least one free occurrence of x in $\alpha(x, y)$. In that case, we could have as formula $\alpha(x, y)$ the following one: $y = \phi_1$. After all, for any x there is one single y such that $y = \phi_1$. That would entail the existence of a constant function h with constant value ϕ_1. Nevertheless, that is forbidden, according to Theorem 14. On the other hand, if we allowed $\alpha(x, y)$ to be molecular (i.e., to have logical connectives), then $\alpha(x, y)$ could be this one: $y = \phi_1 \wedge x(x) = x$. Once again axiom $\mathbf{P5}_\alpha$ would allow the existence of the same constant function h with constant value ϕ_1, despite the fact that we have free occurrences of x and y in this new formula. Hence, we need those restrictions: $\alpha(x, y)$ is supposed to be an atomic formula such that there is at least one occurrence of x, one occurrence of y, and all those occurrences are free. That is why we remarked that all explicit definitions in Flow are abbreviative. No ampliative explicit definition is allowed in Flow.

Otherwise, we could hide molecular formulas into atomic formulas $\alpha(x, y)$. That could be done with predicate letters which are definable by means of molecular formulas.

A final word of caution is necessary here regarding evaluation versus composition. When we say $f(x) = \underline{1}$ for any $x \neq f$, that does not allow us to entail that x is necessarily some sort of right inverse of f (with respect to some misguided vision about composition). We cannot make confusion between formulas $f(x) = \underline{1}$ and $f \circ x = \underline{1}$, since the latter means $f(x(t)) = \underline{1}(t) = t$ for any t. So, if $f \circ x = \underline{1}$, it makes sense to talk about x as a right inverse of f. But no analogous conclusion can be entailed before $f(x) = \underline{1}$.

Next we want to guarantee the existence of *proper restrictions* of a given function. By *proper restriction* of a function f we mean a function g such that: (i) $g(f) = \underline{0}$; and (ii) for the remaining values x where $f(x) \neq \underline{0}$, we may have either $g(x) = f(x)$ or $g(x) = \underline{0}$ (except, of course, when x is g; in that case, $g(g) = g$).

So, if $F(x)$ is a formula where all occurrences of x are free and such that there is no free occurrences of g in $F(x)$, then the following is an axiom.

P8$_F$ **- Restriction** $\forall f(f \neq \underline{0} \Rightarrow \exists! g(g \neq \underline{0} \wedge (g \neq f \Rightarrow g(f) = \underline{0}) \wedge \forall x \forall y((x \neq g \wedge x \neq f) \Rightarrow (g(x) = y \Leftrightarrow ((f(x) = y \wedge F(x)) \vee (y = \underline{0} \wedge \neg F(x)))))))$.

The antecedent for the first conditional $\Rightarrow$ above guarantees the necessary condition for the existence of any *restriction* g of a given function f, namely, $f \neq \underline{0}$.

Hence, if f is different from $\underline{0}$, then there is a unique function g such that: (i) $g(f) = \underline{0}$ if $g \neq f$; (ii) $g(g) = g$; (iii) g and f share non-$\underline{0}$ images for some x as long x does satisfy formula F; and (iv) when g and f do not share non-$\underline{0}$ images for any x, then $g(x) = \underline{0}$. We call g a *restriction* of f under $F(x)$, or simply a restriction of f. In a sense, this last axiom is very similar to the Separation Scheme in ZFC. Nevertheless, Separation Scheme's role is not limited to guarantee the existence of subsets. Thanks to that postulate, ZFC avoids antinomies like Russell's paradox. In the case of Flow Theory, those antinomies are avoided by means of the self-reference axiom **P2**.

To adjust the mathematics of Flow into common practice, the next definition if quite handy.

Definition 6 *For any function f different from $\underline{0}$, its* restriction *g is either f itself, or any proper restriction g of f, as long g is not $\underline{0}$. Formally, we denote this by*

$$g \subseteq f \text{ iff } (g = f) \vee (g(f) = \underline{0} \wedge \forall x \forall y((x \neq g \wedge x \neq f) \Rightarrow (g(x) = y \Rightarrow f(x) = y)),$$

where both f and g are different of $\underline{0}$.

Proper restrictions are defined as:

Definition 7 $g \subset f$ *iff* $g \subseteq f \wedge g \neq f$.

We abbreviate $\neg(g \subseteq f)$ and $\neg(g \subset f)$ as, respectively, $g \not\subseteq f$ and $g \not\subset f$. Accordingly, for all f we have $\underline{0} \not\subseteq f$ and $f \not\subseteq \underline{0}$.

As an example, consider $f = \phi_2$.

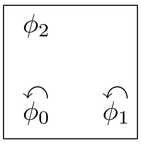

Figure 2: Diagram of function ϕ_2.

According to axiom $\mathbf{P8}_F$, there are *three* proper restrictions g to ϕ_2. If $F(x)$ is the formula "$x = \phi_0$", then $g = \phi_0$. If $F(x)$ is the formula "$x = \phi_0 \vee x = \phi_1$", then $g = \phi_1$. If $F(x)$ is "$x = x$", then again $g = \phi_1$. If $F(x)$ is "$x \neq x$", then once more $g = \phi_0$. Both ϕ_0 and ϕ_1 have their diagrams already represented some paragraphs above. The novelty here, however, happens for the formula $F(x)$ given by "$x = \phi_1$". In that case we have a proper restriction γ such that $\gamma \neq \phi_1$, $\gamma(\gamma) = \gamma$, $\gamma(\phi_1) = \phi_1$, and $\gamma(x) = \underline{0}$ for any x different from ϕ_1 and γ itself. Its diagram is as follows.

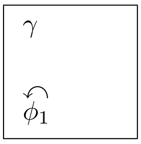

Figure 3: Diagram of function γ, a special restriction of ϕ_2.

Thus, ϕ_2 admits four restrictions: ϕ_0, ϕ_1, function γ in the diagram above, and ϕ_2.

For practical purposes, it seems useful to adopt a rule of thumb for a better understanding of the concept of restriction. Any function f which satisfies the antecedent of the first conditional in axiom $\mathbf{P8}_F$ is a function which "acts on something". That means there is a t different from f such that $f(t)$ is not $\underline{0}$. For example, ϕ_2 acts on ϕ_0 and ϕ_1. So, all restrictions of ϕ_2 correspond, intuitively speaking, to all possible combinations of ϕ_0 and ϕ_1. Those possible combinations, in that case, are: (1) *nothing at all*, since ϕ_0 does not act on anyone; (2) ϕ_0, since ϕ_1 acts only on ϕ_0; (3) ϕ_1, since γ acts only on ϕ_1; and, finally, (4) *everything*, since ϕ_2 acts both on ϕ_0 and ϕ_1.

There are infinitely many other functions (besides $\underline{1}$) which do not have any non-$\underline{0}$ successor, as stated by one of the next theorems. But before that, it is useful to adopt the next convention. The term below

$$f\big|_{F(x)}$$

denotes a restriction of f by use of axiom $\mathbf{P8}_F$ and formula $F(x)$.

Theorem 15 *There is a function ψ such that $\sigma_\psi = \underline{1}$*

Proof: Consider $\psi = \underline{1}\big|_{x \neq \underline{1}}$. That means $\psi(x) = \underline{0}$ only if either $x = \underline{0}$ or $x = \underline{1}$, which satisfies the definition of successor of $\underline{1}$.

Theorem 16 *The successor σ_g for any $g = \Phi_n =_{def} \underline{1}\big|_{x \neq \phi_n}$ is $\underline{0}$, if n belongs to language $\mathcal{L}$.*

Proof: According to $\mathbf{P8}_F$, $g(\phi_n) = \underline{0}$. That means g is different of $\underline{1}$, since $\underline{1}(\phi_n) = \phi_n$. And Theorem 1 entails that $g \neq \underline{1}$. Consequently, $g(\underline{1}) = \underline{0}$, according to $\mathbf{P8}_F$; and $g(\underline{0}) = \underline{0}$, since g is a restriction of $\underline{1}$, and $\underline{1}(\underline{0}) = \underline{0}$. Now, suppose there is $\sigma_g \neq \underline{0}$. Definition 2 demands that $g(\sigma_g) = \underline{0}$. Besides, $g \neq \sigma_g$ and g and σ_g are supposed to share the same images for any x different of σ_g. But $g(x) = \underline{0}$ iff $x = \underline{0}$ or $x = \underline{1}$ (as already established) or $x = \phi_n$, according to $\mathbf{P8}_F$. For the remaining values of x, $g(x) = x \neq \underline{0}$ (axiom $\mathbf{P8}_F$ again). That means there are only three possible values for σ_g, namely, $\underline{0}$, $\underline{1}$ or ϕ_n. And only two of them are different of $\underline{0}$. Now consider ϕ_m, where $n \prec m$. Function ϕ_m is different of either one of those three possible values. So, if anyone of them is σ_g, it is supposed to share the same images of g, for $x = \phi_m$. Notwithstanding, $\phi_n(\phi_m) = \underline{0}$ (according to the recursive definition of functions ϕ_n), while $g(\phi_m) = \phi_m$. That means σ_g cannot be ϕ_n. Finally, $\underline{1}(\phi_n) = \phi_n$, while $g(\phi_n) = \underline{0}$. That means σ_g cannot be $\underline{1}$ either. Thus, the only possible value for σ_g is $\underline{0}$, despite the fact that g and $\underline{0}$ do not share the same images for any x.

This last theorem does not consider all possible cases of functions f with no successor $\sigma_f \neq \underline{0}$. Similar results may be obtained, e.g., for $\underline{1}\big|_{x \neq \phi_n \vee x \neq \phi_m}$ (where $m \neq n$), $\underline{1}\big|_{x \neq \phi_n \vee x \neq \phi_m \vee x \neq \phi_p}$ (for $m \neq n$, $n \neq p$ and $m \neq p$) etc. Even if we consider $F(x)$ as any finite disjunction of the form $x \neq \phi_{n_1} \vee x \neq \phi_{n_2} \vee \cdots \vee x \neq \phi_{n_m}$ for any m, we still cannot guarantee that all possible cases of functions with no successor different of $\underline{0}$ are ran out. But this last theorem is proven to be rather important for our discussion regarding the translation of ZFC axioms into Flow, as we see in the next Section.

The last theorems teach us the following:

1. If a function f does have a successor $\sigma_f \neq \underline{0}$, that does not necessarily entail that any restriction of f has a non-$\underline{0}$ successor. For example, any function Φ_n of Theorem 16 is a proper restriction of ψ (Theorem 15). Nevertheless, although there is a successor of ψ, which is different of $\underline{0}$, no Φ_n has a successor different of $\underline{0}$.

2. If a function f has $\underline{0}$ as its successor, that does not entail that every restriction of f has its successor equal to $\underline{0}$. For example, every Φ_n has successor $\underline{0}$. Nevertheless, ϕ_3 is a proper restriction of any Φ_n, for $n \neq 3$. And despite the fact that such a Φ_n has successor $\underline{0}$, ϕ_3 has its successor different of $\underline{0}$.

3. If a function f has successor $\sigma_f \neq \underline{0}$, that does not entail that σ_f has successor different of $\underline{0}$. For example, the successor of ψ is $\underline{1}$. But the successor of $\underline{1}$ is $\underline{0}$. So, there is $\sigma_\psi \neq \underline{0}$, but there is no $\sigma_{\sigma_\psi} \neq \underline{0}$.

Hence, Flow teaches us that restrictions of a function f are not informative enough about the behavior of any f. We need something else. And that something else is provided by axiom **P11**, which is displayed some pages below.

So far, most functions f in Flow behave like "children" of $\underline{1}$, in the sense that for any x we have either $f(x) = x$ or $f(x) = \underline{0}$. One exception for this rule is σ. Notwithstanding, if we want to ground standard mathematics, we need much more than that. So, in order to discuss about that, we need something which resembles the usual notion of ordered pair.

Definition 8 *f is an* ordered pair (a, b) *iff there are α and β such that $\alpha \neq f$, $\beta \neq f$, $\alpha \neq a$, $\beta \neq b$ and*

$$f(x) = \begin{cases} \alpha & \textit{if } x = \alpha \\ \beta & \textit{if } x = \beta \\ \underline{0} & \textit{if } x \neq f \wedge x \neq \alpha \wedge x \neq \beta \end{cases}$$

where $\alpha(a) = a$, $\alpha(x) = \underline{0}$ if x is neither a nor α, $\beta(a) = a$, $\beta(b) = b$, $\beta(x) = \underline{0}$ if x is neither a nor b or β.

Observe we did not demand $\alpha \neq b$. That means we may have two kinds of ordered pairs, namely, those where $\alpha \neq b$ (first kind) and those where $\alpha = b$ (second kind). The diagram for an ordered pair $f = (a, b)$, where $\alpha \neq b$, may be written as follows:

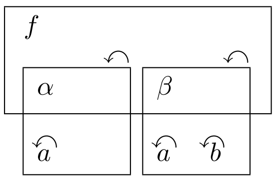

Figure 4: Diagram of an ordered pair $f = (a, b)$ of the first kind.

The diagram above says f acts only on α and β, while α acts only on a, and β acts only on a and b. In the particular case where $a = b$, we have $\alpha = \beta$, and the ordered pair f is denoted by (a, a). Observe that in the diagram above $f(a) = f(b) = \underline{0}$, which means that f never acts neither on a nor on b, if f is the ordered pair (a, b) of the first kind. In other words, f is an ordered pair (a, b) iff f acts only on functions α and β which act, respectively, only on a and only on a and b. To get (b, a), all we have to do is to exchange α by a function α' which acts only on b. Hence, our definition for ordered pair is obviously inspired on the standard notion by Kuratowski. In standard set theory an ordered pair (a, b) is a set $\{\{a\}, \{a, b\}\}$ such that neither a nor b belong to (a, b). In Flow, on the other hand, an ordered pair (a, b) of the first kind is a function which does not act neither on a nor on b.

Nevertheless, the second kind of ordered pair shows our approach is not equivalent to Kuratowki's. In the case where $\alpha = b$, we have the following diagram.

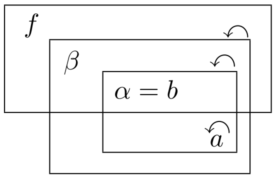

Figure 5: Diagram of an ordered pair $f = (a, b)$ of the second kind.

In this non-Kuratowskian kind of ordered pair $f = (a, b)$, f acts on b, although it does not act on a. In the general case, no ordered pair $f = (a, b)$ ever acts on a.

Since any ordered pair (a, b) is a function, we abbreviate $x((a, b)))$ as $x(a, b)$ for a given function x.

We intend to use the notion of ordered pair to guarantee the existence of other functions, besides our previous "children" of $\underline{1}$ (which are "children" of $\underline{0}$ as well, since usually most of their images are $\underline{0}$). So, our idea is as follows. Consider, for example, function ϕ_2, whose restrictions are ϕ_0, ϕ_1, ϕ_2 and γ, as previously discussed. If we guarantee the existence of a function f which acts only on ϕ_1 and ϕ_2 (in a way such that $f(\phi_1) = \phi_1$ and $f(\phi_2) = \phi_2$), then we can easily prove f is the ordered pair (ϕ_0, ϕ_1). After all, ϕ_1 acts only on ϕ_0; and ϕ_2 acts only on ϕ_0 and ϕ_1. On the other hand, if we can guarantee the existence of function g such that g acts only on γ and ϕ_2 (in a way such that $g(\gamma) = \gamma$ and $g(\phi_2) = \phi_2$), we can easily prove that g is the ordered pair (ϕ_1, ϕ_0). Ultimately, γ acts only on ϕ_1; and ϕ_2 acts only on ϕ_1 and ϕ_0. Observe that (ϕ_0, ϕ_1) is a non-Kuratowskian ordered pair (second kind), while (ϕ_1, ϕ_0) is a Kuratowskian ordered pair (first kind).

Once Flow is endowed with ordered pairs (ϕ_0, ϕ_1) and (ϕ_1, ϕ_0), all we have to do is to guarantee the existence, e.g., of functions l and m such that $l(\phi_0) = \phi_1$ and $m(\phi_1) = \phi_0$. In that case we are no longer restricted to functions f such that $f(x)$ is either x itself or $\underline{0}$.

Fortunately, the next theorem guarantees we can always define ordered pairs (a, b) for any functions a and b as long we state that none of them is $\underline{1}$. Such a restriction comes from the fact that we use restrictions applied to $\underline{1}$ in order to prove the next result. And any proper restriction of $\underline{1}$ is a function f such that $f(\underline{1}) = \underline{0}$.

Theorem 17 *If a and b are both different of $\underline{1}$, then there is a function f such that $f = (a, b)$.*

Proof: First, we use axiom $\mathbf{P8}_F$ to define the proper restriction of $\underline{1}$ for "$x = a \vee x = b$" as formula $F(x)$. Such a proper restriction can be denoted as β. So, β is a function such that $\beta(a) = a$, $\beta(b) = b$, $\beta(\beta) = \beta$, and $\beta(x) = \underline{0}$ for all remaining values of x. Analogously, the proper restriction of $\underline{1}$ for "$x = a$" as formula $F(x)$ in $\mathbf{P8}_F$ gives us the function α such that $\alpha(a) = a$, $\alpha(\alpha) = \alpha$, and $\alpha(x) = \underline{0}$ for the remaining values of x. Finally, the proper restriction of $\underline{1}$ for "$x = \alpha \vee x = \beta$ as formula $F(x)$ in $\mathbf{P8}_F$ provides us a function f such that $f(\alpha) = \alpha$, $f(\beta) = \beta$, $f(f) = f$, and $f(x) = \underline{0}$ for all the remaining values of x. But function f is exactly that one in definition 8. Hence, $f = (a, b)$.

This last theorem says we do not need ϕ_2 to produce ordered pairs (ϕ_0, ϕ_1) and (ϕ_1, ϕ_0), as we did above. Since axiom **P6** guarantees the existence of functions ϕ_n, we can use $\mathbf{P8}_F$ to obtain any ordered pair (ϕ_m, ϕ_n).

Observation 2 *Now, what is the first valuable lesson taught by Flow? From the first five axioms we learn the existence of two functions, namely,* $\underline{0}$ *and* $\underline{1}$*. Besides, we learn how to distinct one from the other, thanks to* **P1***. That fact,* per se*, suggests some notion of duality which is reinforced by the concept of successor: the successor of* $\underline{1}$ *is* $\underline{0}$*, and the successor of* $\underline{0}$ *is* ϕ_0*. Thus, more than a principle of duality, we have a principle of complementarity, where the function successor establishes some sort of cycle which connects those two extremes,* $\underline{0}$ *and* $\underline{1}$*. In its turn, axiom* $\mathbf{P5}_\alpha$ *teaches us how to compose functions. But what is the advantage of composing functions if all we have is a couple of privileged functions? Compositions between* $\underline{0}$ *and* $\underline{1}$ *do not produce any new functions at all. So, axioms* **P6** *and* **P7** *allow us to build infinitely many functions from* $\underline{0}$*. Those are functions* ϕ_n *(*$n = 0, 1, 2, \cdots$*). On the other hand, axiom* $\mathbf{P8}_F$ *allows us to "deconstruct"* $\underline{1}$ *to achieve another vast myriad of functions, including ordered pairs. Without* **P7***,* $\mathbf{P8}_F$ *is useless, for the latter demands the existence of a function* f *which acts on some* t*. And no function can do that in a universe where all we know is the existence of* $\underline{0}$ *and* $\underline{1}$*. And without* $\mathbf{P8}_F$*,* **P7** *is very poor. Hence,* $\underline{0}$*, under the influence of* **P6** *and* **P7***, can be seen as a* creation *function. Analogously,* $\underline{1}$*, under the influence of* $\mathbf{P8}_F$*, can be seen as an* annihilation *function. Both, creation and annihilation, allow us to shape a whole universe of functions. First we create, then we destroy. That is the main difference between our approach and the usual notions of standard set theories. Standard set theories like ZFC build whole universes of sets from one single source, the empty set. That means the standard approach for deriving sets is by means of a single process of creation. In Flow, however, we build new terms from two fronts: good and evil, light and darkness, creation and annihilation, expansion (***P6***) and restriction (*$\mathbf{P8}_F$*). That is how Flow Theory flows.*

Theorem 18 $\forall f (f \neq \underline{0} \Rightarrow (\phi_0 \subseteq f))$.

Proof: If $f = \phi_0$, the proof is trivial, since according to Definition 6 every function different of $\underline{0}$ is a restriction of itself. If $f \neq \phi_0$, all we have to do is to use "$x \neq x$" as formula $F(x)$ in axiom $\mathbf{P8}_F$. In that case ϕ_0 is a proper restriction of f.

Definition 9 *z is the* power function of *f (or simply the* power of *f) iff* $z \neq f \wedge \forall x (x \neq z \Rightarrow ((z(x) = x \Leftrightarrow x \subseteq f) \wedge (z(x) = \underline{0} \Leftrightarrow x \not\subseteq f)))$*. We denote z as* $\mathcal{P}(f)$*.*

Theorem 19 *For any function h different from $\underline{0}$ there is a unique $\mathcal{P}(h)$.*

Proof: All we have to do is to apply axiom $\mathbf{P8}_F$ over function $f = \underline{1}$ and assume "$x \subseteq h$" as formula $F(x)$. Function g guaranteed by $\mathbf{P8}_F$ is precisely $\mathcal{P}(h)$.

So, for example, $\mathcal{P}(\phi_0) = \phi_1$, $\mathcal{P}(\phi_1) = \phi_2$, and $\mathcal{P}(\phi_2)$ is a function f such that $f \neq \phi_2$, $f(\phi_0) = \phi_0$, $f(\phi_1) = \phi_1$, $f(\phi_2) = \phi_2$, $f(\gamma) = \gamma$, $f(f) = f$, and $f(x) = \underline{0}$ for all the remaining values of x.

One interesting side effect of the concept of power function is that $\mathcal{P}(\underline{1})$ is somehow a "smaller" function than $\underline{1}$. What do we mean by that? It means that $\underline{1}$ acts on every single function, with the only exceptions of $\underline{0}$ and $\underline{1}$, since $\underline{1}(\underline{0}) = \underline{0}$ and $\underline{1}(\underline{1}) = \underline{1}$. But $\mathcal{P}(\underline{1})$ is a function z which is different of $\underline{1}$ and such that z acts only on those functions t such that for any x we have either $t(x) = x$ or $t(x) = \underline{0}$. So, Flow is apparently free of any paradox regarding the notion of power.

The next theorem is a first step to prove the existence of relations in Flow. So, contrary to usual set-theoretic notions, relations are special cases of functions.

Theorem 20 *Let l and m be functions such that they are both different from $\underline{0}$ and ϕ_0. Then there is a function g such that for any $t \neq g$ we have $g(t) \neq \underline{0}$ iff $t = (a, b)$, where a and b are such that $a \neq l$, $b \neq m$, $l(a) \neq \underline{0}$, and $m(b) \neq \underline{0}$.*

Proof: All we have to do is to apply Axiom $\mathbf{P8}_F$ over function $f = \underline{1}$, by assuming as formula $F(x)$ the following one, for a given l and a given m: $a \neq l \wedge b \neq m \wedge l(a) \neq \underline{0} \wedge m(b) \neq \underline{0} \wedge x = (a, b)$.

This unique function g is called the *product* between l and m, and it is denoted by $l \otimes m$.

For example, if $l = \phi_3$ and $m = \phi_2$ (both do satisfy the conditions demanded by the theorem above), then $g = \phi_3 \otimes \phi_2$ is the following function.

$$\boxed{\begin{array}{lll} g & & \\ (\phi_0, \phi_0) & (\phi_0, \phi_1) & (\phi_1, \phi_0) \\ (\phi_1, \phi_1) & (\phi_2, \phi_0) & (\phi_2, \phi_1) \end{array}}$$

Figure 6: The product between ϕ_3 and ϕ_2.

The arrows in Figure 6 say that $g(\phi_0, \phi_0) = (\phi_0, \phi_0)$, $g(\phi_0, \phi_1) = (\phi_0, \phi_1)$, $g(\phi_1, \phi_0) = (\phi_1, \phi_0)$, $g(\phi_1, \phi_1) = (\phi_1, \phi_1)$, $g(\phi_2, \phi_0) = (\phi_2, \phi_0)$, and $g(\phi_2, \phi_1) = (\phi_2, \phi_1)$. Besides, $g(g) = g$ and $g(x) = \underline{0}$ for the remaining values of x.

As expected, this operation $\otimes$ is not commutative since, e.g., $\phi_3 \otimes \phi_2$ is different from $\phi_2 \otimes \phi_3$. That means we can define relations as it follows:

Definition 10 *If g is the product between l and m, then any f such that $f \subseteq g$ is called a* relation with domain l and co-domain m.

If we want to define a relation f with domain l and co-domain m, we just need to apply $\mathbf{P8}_F$ over $l \otimes m$ for a given formula $F(x)$. As an example, consider the following definition:

Definition 11 *Let l and m be functions such that they are both different from $\underline{0}$ and ϕ_0. Function g is a* trivially arbitrary function with domain l and co-domain m *iff $f \subseteq l \otimes m$ and for all a such that $l[a]$ there is a unique b such that $m[b]$ and $g(a, b) = (a, b)$. We denote this by $\mathcal{T}_{l \to m}(g)$.*

That means trivially arbitrary functions are special cases of relations.

Theorem 21 *For any functions l and m which are both different of $\underline{0}$ and ϕ_0, there is at least one g such that $\mathcal{T}_{l \to m}(g)$.*

Proof: All we have to do is to apply Axiom $\mathbf{P8}_F$ over function $f = l \otimes m$, by assuming as formula $F(x)$ the following one: $\forall a((a \neq l \wedge l(a) \neq \underline{0}) \Rightarrow \exists! b(b \neq m \wedge m(b) \neq \underline{0} \wedge x = (a, b)))$.

If we use the particular case illustrated in Figure 6, one example of trivially arbitrary function f with domain ϕ_3 and co-domain ϕ_2 is the following:

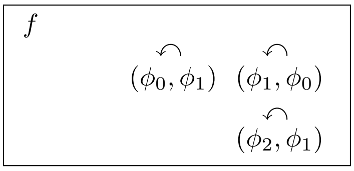

Figure 7: Example of a trivially arbitrary function f with domain ϕ_3 and co-domain ϕ_2.

Notwithstanding, despite all those results above, all functions in Flow work as some some sort of restriction of $\underline{1}$, in the sense that all our functions f (until now) are such that for any x we have $f(x)$ is either x or $\underline{0}$. To accommodate arbitrary functions, we need the next axiom.

P9 - Freedom $\forall l \forall m \forall f(\mathcal{T}_{l \to m}(f) \Rightarrow \exists! g(\forall a \forall b(f(a, b) \neq \underline{0} \Rightarrow g(a) = b)))$.

The intuitive idea of this last axiom is quite simple. If we have a trivially arbitrary function f with domain l and co-domain m which acts on ordered pairs (a, b) in a way such that $f(a, b)$ is always (a, b), then there is a function g such that $g(a) = b$. That means we have now new functions g where $g(a)$ is not necessarily a.

If we apply axiom **P9**, e.g., over function f illustrated in Figure 7, we can get now the following:

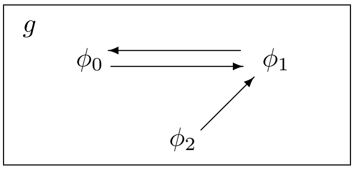

Figure 8: Example of a function g obtained from f (of Figure 7) by use of Axiom **P9**.

The example above refers to a function g such that $g(\phi_0) = \phi_1$, $g(\phi_1) = \phi_0$, $g(\phi_2) = \phi_1$, $g(g) = g$, and $g(r) = \underline{0}$ for the remaining values of r.

By using the same ideas, we can define as well, from $i \otimes i$ (where i is an inductive function), a function λ such that $\lambda(\phi_n) = \phi_{n+1}$, $\lambda(\lambda) = \lambda$ and $\lambda(r) = \underline{0}$ for the remaining values of r. That function λ is particularly useful in later discussions.

The next definition is quite useful for dealing with unions, as we intend to do in the next axiom:

Definition 12 *Let g, h and t be functions. Then,*

1. $\mathcal{X}_{\Box}(g, h \leftarrow t)$ *iff* $g[t] \wedge h[t] \wedge g(t) = h(t)$,
2. $\mathcal{X}_{\Diamond}(g, h \leftarrow t)$ *iff* $g[t] \wedge h[t] \wedge g(t) \neq h(t)$,
3. $\mathcal{X}_{\triangle}(g, h \leftarrow t)$ *iff* $t \neq g \wedge t \neq h \wedge (g(t) = \underline{0} \vee h(t) = \underline{0}) \wedge \neg(g(t) = \underline{0} \wedge h(t) = \underline{0})$,
4. $\mathcal{X}_{\bigcirc}(g, h \leftarrow t)$ *iff* $t \neq g \wedge t \neq h \wedge g(t) = \underline{0} \wedge h(t) = \underline{0}$.

$\mathcal{X}_{\Box}(g, h \leftarrow t)$ says both g and h act on a given t and share the same value for $g(t)$ and $h(t)$. $\mathcal{X}_{\Diamond}(g, h \leftarrow t)$ says both g and h act on a given t; but in that case they do not share the same image for t. $\mathcal{X}_{\triangle}(g, h \leftarrow t)$ says either g or h acts on a given t, while the other one gives us an image equal to $\underline{0}$. Finally, $\mathcal{X}_{\bigcirc}(g, h \leftarrow t)$ simply says both g and h share the same image for a given t, and that image is $\underline{0}$.

P10 - Union $\forall f((f \neq \underline{0} \wedge \forall x(f[x] \Rightarrow \sigma_x \neq \underline{0})) \Rightarrow$
$\exists! u(\sigma_u \neq \underline{0} \wedge (\forall g \forall h((f[g] \wedge f[h]) \Rightarrow \forall t(t \neq u \Rightarrow$
$(\mathcal{X}_{\Box}(g, h \leftarrow t) \Rightarrow u(t) = g(t)) \wedge$
$((\mathcal{X}_{\Diamond}(g, h \leftarrow t) \vee \mathcal{X}_{\bigcirc}(g, h \leftarrow t)) \Rightarrow u(t) = \underline{0})) \wedge$
$(\mathcal{X}_{\triangle}(g, h \leftarrow t) \Rightarrow (u(t) \neq \underline{0} \wedge (u(t) = g(t) \vee u(t) = h(t))))))).$

We hope the reader does not feel intimidated by the apparent complexity of this last formula. Actually, this postulate is quite intuitive.

Suppose f acts on many functions, like g and h. So, we have four possibilities for an arbitrary t (as long neither g nor h is t): (i) both g and h act on t and share the same value ($g(t) = h(t)$); (ii) both g and h act on t, but do not share the same value ($g(t) \neq h(t)$); (iii) either g or h does not act on t, but one of them does act on t; (iv) both $g(t)$ and $h(t)$ have value $\underline{0}$. In the first case, $u(t)$ has the value shared by both g and h on t. In the second case, $u(t)$ is $\underline{0}$. And the same happens for the fourth case. Finally, in the third case, $u(t)$ has the same value of either $g(t)$ or $h(t)$, as long we are talking about the only one which acts on t.

This last axiom allows us to obtain arbitrary unions of functions, even if they do not share the same images. And the resultant unique union u is a function. So, in a precise sense, this last axiom generalizes the standard notion of union in theories like ZFC, NBG, and others. We denote function u as

$$u = \bigcup_{f[g]} g,$$

where f acts on g.

Consider the following example.

Let f be such that $f(g) = g$, $f(h) = h$, $f(f) = f$, and $f(r) = \underline{0}$ for the remaining values of r, where g and h are represented below:

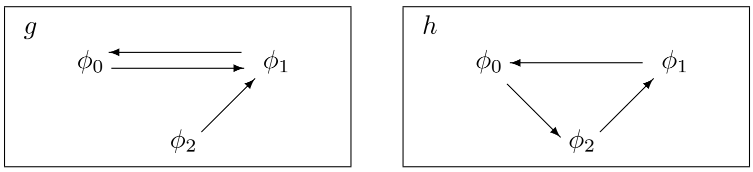

Figure 9: Example of functions to be unified.

That means $g(\phi_0) = \phi_1$, $g(\phi_1) = \phi_0$, $g(\phi_2) = \phi_1$, $g(g) = g$, and $g(r) = \underline{0}$ for the remaining values of r. Besides, $h(\phi_0) = \phi_2$, $h(\phi_1) = \phi_0$, $h(\phi_2) = \phi_1$, $h(h) = h$, and $h(r) = \underline{0}$ for the remaining values of r. Observe that for both cases we have $g(x) \neq \underline{0} \Rightarrow \sigma_x \neq \underline{0}$ and $h(x) \neq \underline{0} \Rightarrow \sigma_x \neq \underline{0}$. That fact entails that both g and h have their respective non-$\underline{0}$ successors. In other words, there is a union u which is associated to f. So, we have $\mathcal{X}_\square(g, h \leftarrow \phi_1)$, $\mathcal{X}_\square(g, h \leftarrow \phi_2)$, $\mathcal{X}_\Diamond(g, h \leftarrow \phi_0)$, and $\mathcal{X}_\bigcirc(g, h \leftarrow r)$ for the remaining values of r. By applying axiom **P10**, we have that $u = \bigcup_{f[i]} i$ (where f acts on i) is simply

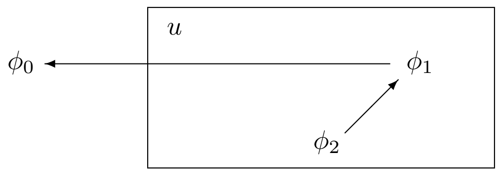

Figure 10: Union between functions g and h of Figure 9.

where the arrow which escapes the diagram says that $u(\phi_0) = \underline{0}$, despite the fact that $u(\phi_1) = \phi_0$.

As a second example, consider a function f' such that $f'(g) = h$, $f'(h) = g$, $f'(f') = f'$, and $f'(r) = \underline{0}$ for the remaining values of r. Clearly, $f' \neq f$. Nevertheless, we have

$$u = \bigcup_{f'[j]} j = \bigcup_{f[i]} i,$$

where both f' and f act, respectively, on j and i. That means different functions may generate the same union u, a result which is analogous to what happens, e.g., in ZFC.

If we want the particular case of standard union, all we need to do is to consider the definition given below.

Definition 13 *Any function f is* strictly unifiable *iff f is not $\underline{0}$ and* $\forall g \forall h((f(g) \neq \underline{0} \wedge f(h) \neq \underline{0}) \Rightarrow \forall t((g(t) \neq \underline{0} \wedge h(t) \neq \underline{0}) \Rightarrow g(t) = h(t)))$. *We denote this by* $\mathcal{U}(f)$.

So, if f is strictly unifiable, its arbitrary union corresponds, intuitively speaking, to the standard notion of union. That is proved in the next section.

Theorem 22 *Let x and y be functions such that both σ_x and σ_y are different of $\underline{0}$. If $x \circ y \neq \underline{0}$ and y does not act on σ_x, then $\sigma_x \circ \sigma_y \subseteq \sigma_{x \circ y}$.*

Proof:

$$(\sigma_x \circ \sigma_y)(t) = \sigma_x(\sigma_y(t)) = \begin{cases} \sigma_x \circ \sigma_y & \text{if } t = \sigma_x \circ \sigma_y \\ \sigma_x(y(t)) & \text{if } t \neq \sigma_x \circ \sigma_y \wedge t \neq \sigma_y \\ \sigma_x(\sigma_y) & \text{if } t = \sigma_y \end{cases} =$$

$$\begin{cases} \sigma_x \circ \sigma_y & \text{if } t = \sigma_x \circ \sigma_y \\ x(y(t)) & \text{if } t \neq \sigma_x \circ \sigma_y \wedge t \neq \sigma_y \wedge y(t) \neq \sigma_x \\ \sigma_x(\sigma_y(t)) & \text{if } t = \sigma_x \\ \sigma_x(\sigma_y) & \text{if } t = \sigma_y \end{cases}$$

So, according to the second line of the last brace, unless y acts on σ_x, we have that $\sigma_x \circ \sigma_y \subseteq \sigma_{x \circ y}$.

P11 - Coherence $\forall f((\sigma_f \neq \underline{0} \wedge \forall x(f[x] \Rightarrow \sigma_x \neq \underline{0})) \Rightarrow$
$(\forall g(g \subseteq f \Rightarrow \sigma_g \neq \underline{0}) \wedge \forall g \forall h((h[g] \Rightarrow g \subseteq f) \Rightarrow \sigma_h \neq \underline{0})))$.

This last postulate allows us to establish a frontier between standard objects of Flow Theory and those who are non-standard. For now, standard objects are those directly associated to the concept of a non-$\underline{0}$ successor. If f has a non-$\underline{0}$ successor and it acts only on terms who have non-$\underline{0}$ successor, then any restriction of f has a non-$\underline{0}$ successor and any h which acts on those restrictions has a non-$\underline{0}$ successor. Later on we identify those standard objects to those terms who can be found in ZFC. Its intuitive appeal is quite clear.

Definition 14 $f \sim g$ *iff* $\forall t((t \neq f \wedge t \neq g) \Rightarrow f(t) = g(t))$

This last definition has an important role to be discussed at the end of this paper. For now, all the reader needs to know is that its main purpose is to be used in the next postulate.

P12 - Choice $\forall f(\forall x \forall y((f(x) \neq \underline{0} \wedge f(y) \neq \underline{0} \wedge x \neq y) \Rightarrow (x \neq \phi_0 \wedge \neg \exists s(x(s) = y(s) \wedge x(s) \neq \underline{0}))) \Rightarrow$
$\exists c \forall r(f(r) \neq \underline{0} \Rightarrow \exists! w(c(w) = r(w) \wedge r(w) \neq \underline{0})) \wedge \forall d(d \sim c \Rightarrow c = d))$.

The term c above is called the *choice function* associated to f.

2.2 Sets and Proper Classes

In this subsection we introduce concepts which are intuitively associated to some notion of collection. Such collections are organized as classes, proper classes, sets, and even Universes.

Definition 15 $Col(f)$ *iff* $f \neq \underline{0} \wedge \forall x(f(x) \neq \underline{0} \Rightarrow f(f(x)) \neq \underline{0})$.

In the definition above we read $Col(f)$ as "f is a collection" or "f is a class".

Definition 16 $x \in f$ *iff* $x \neq f \wedge f(x) \neq \underline{0}$.

The negation of the formula $x \in f$ is abbreviated as $x \notin f$. It is immediate to see that $x \in f$ iff $f[x]$. Observe as well that we do not demand f to be a class. That will allow us, hopefully, to talk about fuzzy sets in the case where f is not a class.

Theorem 23 *The next formulas are all theorems: (i)* $Col(\underline{1})$*; (ii)* $\forall x((x \neq \underline{1} \wedge x \neq \underline{0}) \Rightarrow x \in \underline{1})$*; (iii)* $\forall x(x \notin x)$.

Their proof are straightforward.

Definition 17 *A* structure-free class *is a class f such that for any x we have* $f(x) \neq \underline{0} \Rightarrow f(x) = x$.

It is easy to check that every function ϕ_n is a structure-free class.

Definition 18 *Any class which is not a structure-free class is said to be a* structured class.

Definition 19 f *is a* set *iff f is a class and for any x,* $f[x] \Rightarrow \sigma_x \neq \underline{0}$.

If f is a set, we denote this by $Set(f)$. Examples of sets are each and every ϕ_n.

Inspired on **P11**, next we define ZF-sets.

Definition 20 $Z(f)$ *iff* $\forall x((f[x] \Rightarrow (f(x) = x \wedge \sigma_x \neq \underline{0})) \wedge \forall g(g \subseteq f \Rightarrow \sigma_g \neq \underline{0}))$.

We read $Z(f)$ as "f is a ZF-set". Any ZF-set f is a structure-free class, and if f acts on any x, then x has a non-$\underline{0}$ successor. Besides, every restriction of a ZF-set has its own non-$\underline{0}$ successor.

Theorem 24 *$\neg Z(\sigma)$.*

Proof: Since $\sigma(\phi_0) = \phi_1 \neq \phi_0$, that is enough to prove σ is not a ZF-set.

Theorem 25 *If f is a ZF-set, then $f = \underline{1}\big|_{f[x]}$.*

The proof is straightforward.

Theorem 26 *If f is a ZF-set, then $\sigma_f \neq \underline{0}$.*

Proof: If f is a ZF-set, then for any x, $f[x]$ entails $\sigma_x \neq \underline{0}$. But according to axiom **P11**, any restriction of f (under such assumption) has a non-$\underline{0}$ successor. Since f is a restriction of f (for any f different of $\underline{0}$), then $\sigma_f \neq \underline{0}$.

Theorem 27 *Every ϕ_n is a ZF-set.*

Proof: $Z(\phi_0)$ is vacuously valid. Now, let $n > 0$. Then any ϕ_n acts only on ϕ_m and $\phi_n(\phi_m) = \phi_m$, where $0 < m < n$. And each ϕ_m has a non-$\underline{0}$ successor, from the definition itself for ϕ_m. And according to **P11**, that entails that any restriction g of ϕ_n has a non-$\underline{0}$ successor. So, $Z(\phi_n)$ for any n from language $\mathcal{L}$.

This last theorem helps us to see how to start building ZF-sets from the axioms of Flow. Next theorem shows us how to build standard hierarchies of ZF-sets.

Theorem 28 *If f is a ZF-set, then $\mathcal{P}(f)$ is a ZF-set.*

Proof: According to Theorem 19, $\mathcal{P}(f) = \underline{1}\big|_{t\subseteq f}$. Let us denote $\mathcal{P}(f)$ by p, for the sake of abbreviation. Since f is a ZF-set, that means any restriction t of f has a non-$\underline{0}$ successor σ_t. In other words, $p[t]$ entails that $p(t) = t$ (since p is a restriction of $\underline{1}$) and $t \subseteq f$. Thus, $\sigma_t \neq \underline{0}$ (since $Z(f)$). But according to **P11**, if f is a ZF-set, then any h such that $h[t] \Rightarrow t \subseteq f$ entails $\sigma_h \neq \underline{0}$. Well, p is exactly like that, since $p[t] \Rightarrow t \subseteq f$. So, there is a non-$\underline{0}$ σ_p. Consequently, according to **P11**, every restriction g of p has its own non-$\underline{0}$ successor σ_g. That, finally, corresponds to say that p is a ZF-set. In other words, $\forall t((p[t] \Rightarrow (p(t) = t \wedge \sigma_t \neq \underline{0})) \wedge \forall g(g \subseteq p \Rightarrow \sigma_g \neq \underline{0}))$.

So, we have here a vast universe of ZF-sets who are built from ϕ_0 and the notion of successor, in a way which allows us to build hierarchies defined through the power function and corresponding restrictions. All of them are ZF-sets.

Theorem 29 *Any inductive function i is a ZF-set.*

Proof: Straightforward from the definitions of inductive function and ZF-set.

Theorem 30 *The union of ZF-sets is a ZF-set.*

Proof: ...

All previous results motivate us to define the concept of a proper class.

Definition 21 *f is a* proper class *iff f is a class and anyone of the next conditions is satisfied: either (i) $\sigma_f = \underline{0}$ or (ii) there is some x such that f acts on x but x has $\sigma_x = \underline{0}$.*

That means no proper class is a ZF-set. If a proper class f is a free-structure class, then we say f is a *free-structure proper class*. Otherwise, we say f is a *structured proper class*.

Examples of proper classes are $\underline{1}$ and $\underline{1}\big|_{x \neq \phi_n}$ (see Theorem 16). That happens because neither $\underline{1}$ nor any $\underline{1}\big|_{x \neq \phi_n}$ has a non-$\underline{0}$ successor. Another example of proper class is function ψ from Theorem 15. Although ψ has a non-$\underline{0}$ successor, ψ acts on any $\underline{1}\big|_{x \neq \phi_n}$. So, ψ acts on certain terms t such that there is no non-$\underline{0}$ σ_t.

Theorem 31 *There is one single ZF-set f such that for any x, we have $x \notin f$.*

Proof: $f = \phi_0$. And, according to **P6**, ϕ_0 is unique. In other words, ϕ_0 is the empty set, which can be denoted by $\emptyset$.

3 ZFC is immersed in Flow

There are two reasons for referring to ZFC at this point. First, presenting the theory provides a framework that will allow us to compare our proposal to a standard and well-known formulation of set theory. Second, having ZFC in place will be useful for our proof that we can still use standard mathematical results when we adopt Flow-theoretic principles. After all, as we'll show shortly, there's a translation from the language of ZFC into a variation of Flow theory such that every translated axiom of ZFC is a theorem in our proposed formal system. However, as we will see, to adopt Flow has the significant advantage of providing a whole new universe to work with.

3.1 ZFC Axioms

ZFC is a first-order theory with identity and with one predicate letter f_1^2, such that the formula $f_1^2(x, y)$ is abbreviated as $x \in y$, if x and y are terms, and is read as "x belongs to y" or "x is an element of y". The negation $\neg(x \in y)$ is abbreviated as $x \notin y$.

The axioms of ZFC are the following:

ZF1 - Extensionality $\forall x \forall y (\forall z (z \in x \Leftrightarrow z \in y) \Rightarrow x = y)$

ZF2 - Empty set $\exists x \forall y(\neg(y \in x))$

ZF3 - Pair $\forall x \forall y \exists z \forall t(t \in z \Leftrightarrow t = x \vee t = y)$

The pair z is denoted by $\{x, y\}$ if $x \neq y$. Otherwise, $z = \{x\} = \{y\}$.

Definition 22 $x \subseteq y =_{def} \forall z(z \in x \Rightarrow z \in y)$

ZF4 - Power set $\forall x \exists y \forall z(z \in y \Leftrightarrow z \subseteq x)$

If $F(x)$ is a formula in ZFC, such that there are no free occurrences of the variable y, then the next formula is an axiom of ZFC:

ZF5$_F$ - Separation Scheme $\forall z \exists y \forall x(x \in y \Leftrightarrow x \in z \wedge F(x))$

The set y is denoted by $\{x \in z/F(x)\}$.

If $\alpha(x, y)$ is a formula where all occurrences of x and y are free, then the following is an axiom scheme of ZFC:

ZF6$_\alpha$ - Replacement Scheme

$$\forall x \exists! y \alpha(x, y) \Rightarrow \forall z \exists w \forall t(t \in w \Leftrightarrow \exists s(s \in z \wedge \alpha(s, t)))$$

ZF7 - Union set $\forall x \exists y \forall z(z \in y \Leftrightarrow \exists t(z \in t \wedge t \in x))$

The set y from **ZF7** is abbreviated as

$$y = \bigcup_{t \in x} t$$

The intersection among sets is defined by using the Separation Scheme as follows:

$$\bigcap_{t \in x} t =_{def} \left\{ z \in \bigcup_{t \in x} t / \forall t(t \in x \Rightarrow z \in t) \right\}$$

ZF8 - Infinite $\exists x(\emptyset \in x \wedge \forall y(y \in x \Rightarrow y \cup \{y\} \in x))$

ZF9 - Choice $\forall x(\forall y \forall z((y \in x \wedge z \in x \wedge y \neq z) \Rightarrow (y \neq \emptyset \wedge y \cap z = \emptyset)) \Rightarrow \exists y \forall z(z \in x \Rightarrow \exists w(y \cap z = \{w\})))$

As is well known, most if not all classical mathematics can be reformulated in ZFC. As a result, ZFC provides a rich framework for the formulation of physical theories—although perhaps not the most economical. As an alternative, we will now consider a different version of set theory, and explore its use in the foundations of physics.

3.2 ZFC translation

For the sake of abbreviation, we call Flow Theory $\boldsymbol{\mathcal{F}}$.

Having presented the main features of Flow, we can now prove that standard mathematics, as formulated in Zermelo-Fraenkel set theory with the Axiom of Choice (ZFC), is preserved in a Flow-like axiom system, namely, $\boldsymbol{\mathcal{F}}$. After that, we discuss the meaning of such a result. But first, it is helpful to define the concept of arbitrary intersection.

Definition 23 *For a given $x \neq \underline{0}$, let $F(r)$ be the formula "$r = z \Leftrightarrow \forall t(x[t] \Rightarrow t[z])$". Then*

$$\bigcap_{x[t]} t =_{def} \left(\bigcup_{x[t]} t \right) \Bigg|_{F(r)}.$$

In the particular case where x acts only on two values p and q, such an arbitrary intersection may be rewritten simply as $p \cap q$.

Now, our main result from this section.

Proposition 4 *There is a translation from the language of ZFC into the language of $\mathcal{F}$ such that every translated axiom of ZFC is a theorem in $\boldsymbol{\mathcal{F}}$.*

To prove this proposition, we need to exhibit a translation from ZFC into $\boldsymbol{\mathcal{F}}$. This translation is given by the table below:

TRANSLATING ZFC INTO $\boldsymbol{\mathcal{F}}$	
ZFC	$\boldsymbol{\mathcal{F}}$
$\forall$	$\forall_Z$
$\exists$	$\exists_Z$
$x \in y$	$y[x]$
$x \subseteq y$	$x \subseteq y$

where Z is the predicate "to be a ZF-set" from Definition 20.

The proof of Proposition 4 is made through the following lemmas. The first lemma is quite sensitive. A discussion about its proof is delivered afterwards.

Lemma 1 *The translation of the Axiom of Extensionality in ZFC into Flow is a theorem. That means $\vdash_{\boldsymbol{\mathcal{F}}}$ "Translated* **ZF1**″.

Proof: The translated **ZF1** is the formula $\forall_Z x \forall_Z y(\forall_Z z(x[z] \Leftrightarrow y[z]) \Rightarrow x = y)$. If x and y are ZF-sets and $x[z]$ and $y[z]$, that means $x(z) = z$ and $y(z) = z$ (Definition 20). If $\neg x[z]$ or $\neg y[z]$, that means either $x(z) = \underline{0}$ or $y(z) = \underline{0}$; or $z = x$ or $z = y$. So, the translated **ZF1** considers the case where both x and y share the same images, except perhaps for $z = x$ or $z = y$. In other words, $x[z] \Leftrightarrow y[z]$ is equivalent to say that for any z we have $x(z) = y(z)$, except perhaps for $z = x$ or $z = y$. Now, suppose $x \neq y$, despite the fact that both x and y share the same images for any $z \neq x$ and any $z \neq y$.

After all, in principle we may have the following situation: $x(y) = \underline{0}$ while $y(y) = y$ (this last identity is due to **P2**). Analogously, we may have $x(x) = x$ while $y(x) = \underline{0}$. In both particular cases ($z = x \vee z = y$), we have $\neg x[y] \wedge \neg y[y]$, a situation which satisfies the antecedent of $\Rightarrow$ in the translated **ZF1**. Nevertheless, all functions in Flow are built from $\underline{0}$ and $\underline{1}$ through operations like composition, successor, restriction, union, freedom, and choice. And those functions built from $\underline{0}$ and $\underline{1}$ are defined by means of terms where they act. According to Theorem 2, $\underline{1}$ is unique; and according to Theorem 3, $\underline{0}$ is also unique. Besides, any successor σ_f for any f is unique. Uniqueness of composition is guaranteed in $\mathbf{P5}_\alpha$. Uniqueness of restriction is guaranteed in **P8**. Uniqueness of union is guaranteed in **P10**. Uniqueness of arbitrary functions (freedom) is guaranteed in **P9**. And the uniqueness of any given choice function c is guaranteed in **P12**, in the sense that once c is obtained and $d \sim c$, for any d, then $d = c$. That means there can be no two functions x and y which act on the same terms z in a way such that $x(z) = y(z)$.

After the proof of this first lemma, one natural question seems to be unavoidable. Why didn't we introduce a stronger version for extensionality instead of Axiom **P1**? If we had done something like this, all those strange maneuvers used for proving Lemma 1 could be easily avoided. That is true. Notwithstanding, we intend to suggest here another way of doing mathematics. If we had adopted a stronger version of extensionality, we would have a kind of mathematics which is quite similar to the standard way. So, at the end of this paper we perform a detailed philosophical discussion about this issue. Our main purpose here is to let an open door which can lead us to what we call a *Heraclitean Mathematics*. And such a Heraclitean Mathematics has no room for ZFC.

Lemma 2 $\vdash_{\mathcal{F}}$ *"Translated* **ZF2**"

Proof: The translated **ZF2** is the formula $\exists_Z x \forall_Z y(\neg(x[y]))$. That result is a straightforward corollary from Theorem 31. Function x is simply ϕ_0, which is a ZF-set (and, by the way, unique).

Lemma 3 $\vdash_{\mathcal{F}}$ *"Translated* **ZF3**"

Proof: The translated **ZF3** is the formula $\forall_Z x \forall_Z y \exists_Z z \forall_Z t(z[t] \Leftrightarrow (t = x \vee t = y))$. All we have to do is to define $z = \underline{1}|_{F(t)}$ for formula $F(t)$ given by "$(t = x \vee t = y)$", where x and y are any two ZF-sets. That can be done thanks to $\mathbf{P8}_F$. Since both x and y are ZF-sets, according to **P11** there are non-$\underline{0}$ σ_x and σ_y. Hence, z is a ZF-set, since $z[x]$ and $z[y]$ entail $z(x) = x$, $z(y) = y$ (remember z is a restriction of $\underline{1}$), $\sigma_x \neq \underline{0}$, and $\sigma_y \neq \underline{0}$.

Lemma 4 $\vdash_{\mathcal{F}}$ *"Translated* **ZF4**"

Proof: The translated **ZF4** is the formula $\forall_Z x \exists_Z y \forall_Z z(y[z] \Leftrightarrow z \subseteq x)$. That corresponds exactly to Theorem 28.

Lemma 5 $\vdash_{\mathcal{F}}$ *"Translated* **ZF5**$''$

Proof: The translated **ZF5** is the formula $\forall_Z f \exists_Z g \forall_Z x(g[x] \Leftrightarrow f[x] \wedge F(x))$ (we changed the names of variables in order to facilitate the reading of our proof). According to Axiom $\mathbf{P8}_F$, $\forall f(f \neq \underline{0} \Rightarrow \exists g(g \neq \underline{0} \wedge (g \neq f \Rightarrow g(f) = \underline{0}) \wedge \forall x \forall y((x \neq g \wedge x \neq f) \Rightarrow (g(x) = y \Leftrightarrow ((f(x) = y \wedge F(x)) \vee (y = \underline{0} \wedge \neg F(x))))))$. In other words, $\mathbf{P8}_F$ says that for a given f different of $\underline{0}$ there is a g which shares the same images of f for a given x, as long $F(x)$ (where F has the same syntactical restrictions of formula F from translated **ZF5**); otherwise, g has images $\underline{0}$. That entails $g \subseteq f$. And in the case where $g \subset f$, then $g(f) = \underline{0}$. But from Definition 20, it is easy to see that any restriction g of a ZF-set f is also a ZF-set, even in the case where f acts on ZF-sets x. So, the translated **ZF5** is simply a straightforward consequence from $\mathbf{P8}_F$.

Lemma 6 $\vdash_{\mathcal{F}}$ *"Translated* **ZF6**$''$

Proof: The translated **ZF6** is the formula $\forall_Z x \exists!_Z y \alpha(x, y) \Rightarrow \forall_Z z \exists_Z w \forall_Z t(w[t] \Leftrightarrow \exists_Z s(z[s] \wedge \alpha(s, t)))$. That means we are talking about a specific formula α such that for any ZF-set x there is a unique ZF-set y where $\alpha(x, y)$. By applying Axiom $\mathbf{P8}_Z$ over $\underline{1}$ with formula $F(t)$ as "$z[s] \wedge \alpha(s, r) \Leftrightarrow r = t$", for a given ZF-set z and a given formula α like the one demanded by the translated **ZF6**, we get a function w. In other words, $w = \underline{1}\big|_{z[s] \wedge \alpha(s,r) \Leftrightarrow r=t}$. So, if w acts on any t, then $w(t) = t$ and there is a successor σ_t (due to the way formula α is defined and thanks to Theorem 26). Hence, w is a ZF-set.

Lemma 7 $\vdash_{\mathcal{F}}$ *"Translated* **ZF7**$''$

Proof: The translated **ZF7** is the formula $\forall_Z f \exists_Z u \forall_Z t(u[t] \Leftrightarrow \exists_Z r(r[t] \wedge f[r]))$. Once again we changed the names of the original variables in order to facilitate its reading. According to **P10**,
$\forall f((f \neq \underline{0} \wedge \forall x(f[x] \Rightarrow \sigma_x \neq \underline{0})) \Rightarrow$
$\exists! u(\sigma_u \neq \underline{0} \wedge (\forall g \forall h((f[g] \wedge f[h]) \Rightarrow \forall t(t \neq u \Rightarrow$
$(\mathcal{X}_{\Box}(g, h \leftarrow t) \Rightarrow u(t) = g(t)) \wedge$
$((\mathcal{X}_{\Diamond}(g, h \leftarrow t) \vee \mathcal{X}_{\bigcirc}(g, h \leftarrow t)) \Rightarrow u(t) = \underline{0})) \wedge$
$(\mathcal{X}_{\triangle}(g, h \leftarrow t) \Rightarrow (u(t) \neq \underline{0} \wedge (u(t) = g(t) \vee u(t) = h(t)))))))$.
But since we are talking about ZF-sets, the possibility of $\mathcal{X}_{\Diamond}(g, h \leftarrow t)$ is simply discarded. After all, if f acts on both g and h, and both g and h act on t, then it is impossible that $g(t) \neq h(t)$, since $g(t) = t$ and $h(t) = t$. Now observe that terms g and h from **P10** have the same role of r in translated **ZF7**. Thus, $u(t)$ has the same non-$\underline{0}$ value t of either $g(t)$ or $h(t)$ only in the case where either $g(t) = t$ or $h(t) = t$ (which corresponds to the cases $\mathcal{X}_{\Box}(g, h \leftarrow t)$ and $\mathcal{X}_{\triangle}(g, h \leftarrow t)$). That is equivalent to say that $u[t] \Leftrightarrow \exists_Z r(r[t] \wedge f[r])$. But since f is a ZF-set, then it acts on ZF-sets r. Since each r is a ZF-set, then each r acts on a ZF-set t. That means u acts only on ZF-sets, which makes itself a ZF-set.

Lemma 8 $\vdash_{\mathcal{F}}$ *"Translated* **ZF8**$''$

Proof: The translated **ZF8** is the formula $\exists_Z x(x[\emptyset] \wedge \forall_Z y(x[y] \Rightarrow x[y \cup \{y\}]))$. Axiom **P7** states that $\exists i((\forall t(i(t) = t \vee i(t) = \underline{0})) \wedge \sigma_i \neq \underline{0} \wedge (i(\sigma_{\underline{0}}) = \sigma_{\underline{0}} \wedge \forall x(i(x) = x \Rightarrow (i(\sigma_x) = \sigma_x \neq \underline{0}))))$. Well, $\emptyset$ is exactly ϕ_0. So, i acts on ϕ_0 and $i(\phi_0) = \phi_0$. Besides, ϕ_0 is a ZF-set. Besides, if y is a ZF-set, then there is σ_y. And $y \cup \{y\}$ is exactly such σ_y, where $\{y\} = \underline{1}\big|_{t=y}$. And the union of ZF-sets is a ZF-set, as already proved in the previous lemma. So, if i acts on a ZF-set t, then i acts on the ZF-set σ_t, which makes i itself a ZF-set.

Lemma 9 $\vdash_{\mathcal{F}}$ *"Translated* **ZF9**$''$

Proof: The translated **ZF9** is the formula $\forall_Z x(\forall_Z y \forall_Z z((x[y] \wedge x[z] \wedge y \neq z) \Rightarrow (y \neq \phi_0 \wedge y \cap z = \phi_0)) \Rightarrow \exists_Z y \forall_Z z(x[z] \Rightarrow \exists_Z w(y \cap z = \{w\})))$. On the other hand, **P12** says $\forall f(\forall x \forall y((f(x) \neq \underline{0} \wedge f(y) \neq \underline{0} \wedge x \neq y) \Rightarrow (x \neq \phi_0 \wedge \neg \exists s(x(s) = y(s) \wedge x(s) \neq \underline{0}))) \Rightarrow$
$\exists c \forall r(f(r) \neq \underline{0} \Rightarrow \exists! w(c(w) = r(w) \wedge r(w) \neq \underline{0})))$. Thus, the translated **ZF9** is just a particular case for a ZF-set x. Since c acts on ZF-sets, then c is a ZF-set itself.

4 Category theory is immersed in Flow

The intuitive notion of a category is quite simple. A category refers to some sort of universe where we can find two kinds of terms, namely, *objects* and *morphisms*. Within a set-theoretic interpretation, objects can be associated to either sets or proper classes, while morphisms can be associated to some sort of general notion of function. Besides, there is a binary operation called *composition*, which is applicable over some pairs of morphisms. Composition, when defined, is associative and it allows the existence of (left and right) neutral elements. Usually Category Theory is referred to as a general theory of functions. Nevertheless, we prove in this section that Category Theory corresponds to a minor fragment of Flow Theory. After all, while composition in Category Theory is not always feasible, within Flow there always exist a composition between any two functions. Those facts lead us to one more important lesson from Flow Theory.

Observation 3 *We proved in Section 3 that ZFC is immersed within Flow. Nevertheless, we did that by assuming as ZF-sets only special cases of free-structure classes. In this Section we prove Category Theory is immersed within Flow as well. And once again we do that by assuming morphisms (including their domains and co-domains) as special cases of free-structure classes. More than that, we prove next that all standard categories may be dealt with through the exclusive use of free-structure classes. From a philosophical point of view, our results point to an interesting perspective. Despite all the propaganda regarding Category Theory as a general theory of functions, the truth is that all standard* *categorical results may be reduced to a world of restrictions of* $\underline{1}$. *So, Category Theory may be reduced to a particular study of functions* f *whose images for any* x *are either* x *itself or* $\underline{0}$. *The main advantage of Category Theory lurks in its power to establish a connection between different domains, like topology and analysis, algebra and number theory. But that could be achieved within any set theory endowed with proper classes and universes. And once again we are still committed to the standard view that a function is nothing more than a collection of ordered pairs, let it be a morphism, a functor or a natural transformation. One of the epistemological barriers of Category Theory lies in the usual* set-theoretic assumption *that every morphism is somehow associated to some sort of domain (and a co-domain). And that fact yields to a quite prejudiced perspective about the dynamic nature functions are supposed to have. From a Flow-theoretic point of view, functions have no domain. And from this same perspective, a function* f *can genuinely act on a given* a *in a way such that* $f(a)$ *is not necessarily identical to* a. *So, after all this discussion about standard mathematics, we explore in the next sections the first steps towards what Flow Theory can really offer to us.*

4.1 Category axioms

We follow here a first order language recipe for defining categories as presented by William S. Hatcher in his classical book [9]. Category Theory $\mathcal{K}$ is a first order theory with identity and one ternary predicate letter K of degree three and two monadic function letters D and C. The intended interpretation of its terms is that of *morphism*. All terms are represented by lower case Latin letters. Intuitively speaking, we read $K(x, y, z)$ as z is the composition of x with y; $D(x)$ as "the domain of x"; and $C(x)$ as "the codomain of x". The proper axioms of $\mathcal{K}$ are the following.

The domain of the codomain of any morphism a is the codomain of a. And the codomain of the domain of a is the domain of a:

K-1 $\forall a(D(C(a)) = C(a) \wedge C(D(a)) = D(a))$.

Composition is unique:

K-2 $\forall a\forall b\forall c\forall d((K(a, b, c) \wedge K(a, b, d)) \Rightarrow c = d)$.

The composition of a with b is defined if and only if the codomain of a is the domain of b:

K-3 $\forall a\forall b(\exists c(K(a, b, c) \Leftrightarrow C(a) = D(b)))$.

If c is the composition of a with b, then the domain of c is the domain of a and the the codomain of c is the codomain of b:

K-4 $\forall a\forall b\forall c(K(a, b, c) \Rightarrow (D(c) = D(a) \wedge C(c) = C(b)))$.

For any a, the domain of a is a left identity for a under composition, and the codomain of a is a right identity:

K-5 $\forall a(K(D(a), a, a) \wedge K(a, C(a), a))$.

Composition is associative when it is defined:

K-6 $\forall a \forall b \forall c \forall d \forall e \forall f \forall g((K(a, b, c) \wedge K(b, d, e) \wedge K(a, e, f) \wedge K(c, d, g)) \Rightarrow f = g)$.

4.2 Defining categories in Flow

First we need the concept of *surjective trivially arbitrary function*.

Definition 24 *Let r, s, and g be functions such that $\mathcal{T}_{r \to s}(g)$. In other words, g is a trivially arbitrary function with domain r and codomain s. We say that g is* surjective *iff for any b such that $s[b]$, there is a such that $r[a]$ and $g(a, b) = (a, b)$.*

Next we define a *static morphism*.

Definition 25 *Let g, r, and s be functions. Then, $\mathcal{M}_\dagger(g, r, s)$ iff*

1. $\forall t(r[t] \Rightarrow r(t) = t) \wedge \forall t(s[t] \Rightarrow s(t) = t)$,
2. $\mathcal{T}_{r \to s}(g)$,
3. *g is surjective.*

We read the ternary predicate above as "g is a *static morphism* with *domain* r and *codomain* s". The first condition says r and s are structure-free classes. The second one says g is a trivially arbitrary function. In other words, g is a particular case of a structure-free class as well. The third condition guarantees the codomain of a trivially arbitrary function is coincident with it range.

We intend to prove that surjective trivially arbitrary functions work just fine for describing usual categories from standard mathematics. Then we use this opportunity to suggest a more general (and simpler) definition for categories.

Definition 26 *Let r, s, and g be functions such that $\mathcal{M}_\dagger(g, r, s)$. Then,*

1. $d_g^\dagger = h$ *iff* $\mathcal{M}_\dagger(h, r, r) \wedge \forall a(r[a] \Leftrightarrow h(a, a) = (a, a))$.
2. $c_g^\dagger = h$ *iff* $\mathcal{M}_\dagger(h, s, s) \wedge \forall b(s[b] \Leftrightarrow h(b, b) = (b, b))$.

Besides, both $d_g^\dagger$ and $c_g^\dagger$ have images $\underline{0}$ iff r does not act on a or s does not act on b, respectively.

We read $d_g^\dagger = h$ as "h is *the static domain* of g". And $c_g^\dagger = h$ says "h is *the static codomain* of g". That means $d_g^\dagger$ is a function which acts on ordered pairs (a, a), as long g acts on (a, b). Analogously, $c_g^\dagger$ acts on ordered pairs (b, b) as long g acts on (a, b). Thus, while $g[(a, b)]$ entails $g(a, b) = (a, b)$, $d_g^\dagger[(a, a)]$ entails $d_g^\dagger(a, a) = (a, a)$, and $c_g^\dagger[(b, b)]$ entails $c_g^\dagger(b, b) = (b, b)$.

Definition 27 *Let g, r, s, h, and t be functions such that $\mathcal{T}_{r\to s}(g)$ and $\mathcal{T}_{s\to t}(h)$. Then $g \circ_\dagger h$ is a function such that,*

1. $\mathcal{T}_{r\to t}(g \circ_\dagger h)$,
2. $\forall a \forall b \forall c((g[(a,b)] \wedge h[(b,c)]) \Rightarrow (g \circ_\dagger h)[(a,c)])$.
3. $\forall a \forall c((g \circ_\dagger h)[(a,c)] \Rightarrow \exists b(g[(a,b)] \wedge h[(b,c)]))$.

We read $g \circ_\dagger h$ as "*the static composition of g with h*". The notation $(g \circ_\dagger h)[(a,c)]$ says the composition $g \circ_\dagger h$ acts on (a,c).

Definition 28 *Let f be a function. Then $\mathcal{C}_\dagger(f)$ iff*

1. $f \neq \underline{0}$,
2. $\forall g(f[g] \Rightarrow f(g) = g)$,
3. $\forall g(f[g] \Rightarrow (\exists r \exists s(\mathcal{M}_\dagger(g,r,s) \wedge \forall h(\mathcal{M}_\dagger(h,r,s) \Rightarrow f[h] \wedge f[d_h^\dagger] \wedge f[c_h\dagger]))))$,
4. $\forall g \forall h((f[g] \wedge f[h] \wedge \exists i(i = g \circ_\dagger h)) \Rightarrow f[i])$.

We read the monadic predicate above as "f is a *static category*". The first two conditions above say any static category is a free-structure class. The third condition says if a static category f acts on any g, then g is a static morphism from r to s, and f acts on g's static domain and on g's static codomain. Besides, the same happens with every h which is a morphism from r to s. Finally, last condition says if f acts on g and h, then it acts on the static composition of g with h. But that happens obviously if such a static composition exists. In other words, static composition is a quite limited perception about composition, in the sense that static composition in a static category does not necessarily exist, while compositions within Flow always do exist.

Definition 29 *Let g be a function. Then $\dagger_f(g)$ iff g is a function such that a specific static category f acts on g. If there is no risk of confusion, we may rewrite $\dagger_f(g)$ simply as $\dagger(g)$.*

Before we prove static categories do satisfy all axioms of $\mathcal{K}$ (if a proper translation is provided), it might be useful to introduce here a rather simple example (although non-trivial) of a static category. Let f be given as it follows:

f
g h i $\ldots$

where g, h, and i are given as:

g		h		i
(ϕ_0,ϕ_0) (ϕ_1,ϕ_1)		(ϕ_0,ϕ_1) (ϕ_1,ϕ_2)		(ϕ_1,ϕ_1) (ϕ_2,ϕ_2)

In that case, f is a static category. Besides, $d^\dagger_g = c^\dagger_g = d^\dagger_h = g$, $d^\dagger_i = c^\dagger_i = c^\dagger_h = i$, and $g \circ_\dagger h = h \circ_\dagger i = h$, while neither $h \circ_\dagger g$ nor $i \circ_\dagger h$ do exist. The ellipsis above just indicates there are other functions with static domain (static codomain) g and static codomain (static domain) h.

The translation provided in the next subsection allows us to prove that f given above is a category in the sense given by William Hatcher [9].

4.3 Categories translation

Here we prove the main result of this Section.

Proposition 5 *There are translations from the language of Category Theory $\mathcal{K}$ into the language of $\boldsymbol{\mathcal{F}}$ such that every translated axiom of $\mathcal{K}$ is a theorem in $\boldsymbol{\mathcal{F}}$ in each translation.*

To prove this proposition scheme we need to exhibit a translation from $\mathcal{K}$ into $\boldsymbol{\mathcal{F}}$, for every possible static category f. Such a translation is given by the table below:

Translating $\mathcal{K}$ into $\boldsymbol{\mathcal{F}}$	
$\mathcal{K}$	$\boldsymbol{\mathcal{F}}$
$\forall$	$\forall_\dagger$
$\exists$	$\exists_\dagger$
$D(g)$	$d^\dagger_g$
$C(g)$	$c^\dagger_g$
$K(g,h,i)$	$i = g \circ_\dagger h$

where predicate $\dagger$ refers to the specific static category f. In other words, $\forall_\dagger x(P)$ means $\forall x(f[x] \Rightarrow P)$, and $\exists_\dagger x(P)$ means $\exists x(f[x] \wedge P)$, where P is a formula from Flow.

The proof of last proposition scheme is made through the following lemmas. We keep the same labels used for terms in $\mathcal{K}$ axioms when it is convenient for us. Otherwise, we change them.

Lemma 10 $\vdash_{\boldsymbol{\mathcal{F}}}$ *"Translated* **K-1***".*

Proof: The translated **K-1**, for the static category f, is $\forall_\dagger g(d^\dagger_{c^\dagger_g} = c^\dagger_g \wedge c^\dagger_{d^\dagger_g} = d^\dagger_g)$. Notwithstanding, $c^\dagger_g$ is a function such that $c^\dagger_g[(b,b)]$ iff $c^\dagger_g(b,b) = (b,b)$ and $s[b]$ for a given s; and $c^\dagger_g(t) = \underline{0}$ iff t is different from $c^\dagger_g$ or different from any b where that given s acts, according to Definition 26. But that is precisely the static domain of $c^\dagger_g$, according again to Definition 26 and Theorem 1 An analogous argument can be used for proving that $c^\dagger_{d^\dagger_g} = d^\dagger_g$.

Lemma 11 $\vdash_{\boldsymbol{\mathcal{F}}}$ *"Translated* **K-2***".*

Proof: The translated **K-2**, for the static category f, is $\forall_\dagger a \forall_\dagger b \forall_\dagger c \forall_\dagger d((c = a \circ_\dagger b \wedge d = a \circ_\dagger b) \Rightarrow c = d)$. According to Definition 27, both c and d share the same images, for a given a and a given b. So, from Theorem 1, $c = d$.

Lemma 12 $\vdash_{\mathcal{F}}$ *"Translated* **K-3***".*

Proof: The translated **K-3**, for the static category f, is $\forall_\dagger g \forall_\dagger h(\exists_\dagger i(i = g \circ_\dagger h \Leftrightarrow c^\dagger_g = d^\dagger_h))$. According to Definition 27, $i[(a, c)]$ iff $g[(a, b)]$ and $h[(b, c)]$. But according to Definitions 25 and 26, $g[(a, b)]$ entails $c^\dagger_g[(b, b)]$, and $h[(b, c)]$ entails $d^\dagger_h[(b, b)]$. Thus, $i[(a, c)]$ iff $c^\dagger_g[(b, b)]$ and $d^\dagger_h[(b, b)]$, which is equivalent to say that $c^\dagger_g = d^\dagger_h$, according to Theorem 1.

Lemma 13 $\vdash_{\mathcal{F}}$ *"Translated* **K-4***".*

Proof: The translated **K-4**, for the static category f, is $\forall_\dagger g \forall_\dagger h \forall_\dagger i(i = g \circ_\dagger h \Rightarrow (d^\dagger_i = d^\dagger_g \wedge c^\dagger_i = c^\dagger_h))$. According to Definition 27, $i[(a, c)]$ iff $g[(a, b)]$ and $h[(b, c)]$. But according to Definitions 25 and 26, $g[(a, b)]$ entails $d^\dagger_g[(a, a)]$, and $h[(b, c)]$ entails $c^\dagger_h[(c, c)]$. Thus, once again Definition 27 shows that $i[(a, c)]$ entails $d^\dagger_i[(a, a)]$ and $c^\dagger_i[(c, c)]$, which is equivalent to say that $d^\dagger_i = d^\dagger_g$ and $c^\dagger_i = c^\dagger_h$, according to Theorem 1.

Lemma 14 $\vdash_{\mathcal{F}}$ *"Translated* **K-5***".*

Proof: The translated **K-5**, for the static category f, is $\forall_\dagger g(d^\dagger_g \circ_\dagger g = g \wedge g \circ_\dagger c^\dagger_g = g)$. According to Definitions 26 and 25, $g[(a, b)]$ iff $d^\dagger_g[(a, a)]$ and $c^\dagger_g[(b, b)]$. And according to Definition 27, $d^\dagger_g \circ_\dagger g$ acts on (a, a), while $g \circ_\dagger c^\dagger_g$ acts on (b, b). That is equivalent to say that $d^\dagger_g \circ_\dagger g = g$ and $g \circ_\dagger c^\dagger_g = g$, according to Theorem 1.

Lemma 15 $\vdash_{\mathcal{F}}$ *"Translated* **K-6***".*

Proof: The translated **K-6**, for the static category f, is $\forall_\dagger a \forall_\dagger b \forall_\dagger c \forall_\dagger d \forall_\dagger e \forall_\dagger f \forall_\dagger g((a \circ_\dagger b = c \wedge b \circ_\dagger d = e \wedge a \circ_\dagger e = f \wedge c \circ_\dagger d = g) \Rightarrow f = g)$. According to Definition 27 and according to the antecedent of conditional $\Rightarrow$ above, we have the following: for certain values α, β, γ, and δ, $a[(\alpha, \beta)]$, $b[(\beta, \gamma)]$, and $d[(\gamma, \delta)]$. Besides, $c[(\alpha, \gamma)]$ and $e[(\beta, \delta)]$. Thus, f acts on (α, δ) iff g acts on (α, δ). That is equivalent to say $f = g$, according to Theorem 1.

Hence, as promised, any category in the general sense provided by Hatcher is reducible to a structure-free class f which acts only on structure-free classes. That is somehow identifiable with the current view that any small category is isomorphic to a subcategory of **Set** (Category of sets, in standard mathematics). Nevertheless, our result shows that no category (either small or not) demands any notion which goes beyond the intuitive concept of a (structure-free) class. That is one of the main reasons why we try to explore this new approach called Flow Theory. We did not check if Flow is reducible to Category Theory. But that is a task we intend to undertake.

4.4 Functors and natural transformations

To be written...

4.5 Set and other standard categories

Before we start to show examples of standard categories within Flow, it seems useful to show some theorems. Our first one says the class of ZF-sets is a proper class.

Theorem 32 $\underline{1}\big|_{Z(x)}$ *has no successor.*

Proof: Let us denote $\underline{1}\big|_{Z(x)}$ by c. In other words, $c[x] \Leftrightarrow Z(x)$. That means any x where c acts has a successor. Now suppose c has a successor. That would entail, from **P11**, that c is a ZF-set. So, c acts on c. But no function acts on itself. So, there is no σ_c.

According to the next theorems, there are other restrictions of $\underline{1}\big|_{Z(x)}$, besides itself, which do not admit any successor.

4.6 The Cantor-Schröder-Bernstein theorem

Despite the fact that Category Theory emphasizes the role of functions (called morphisms) in mathematics, that theory does not allow us to prove the Cantor-Schröder-Bernstein Theorem. That happens because Category Theory algebra is related to composition. In Flow theory, however, our algebra of functions is primarily based on functions valuations.

Theorem 33 (Cantor-Schröder-Bernstein) *Let $\mathcal{M}(f, r, s)$ and $\mathcal{M}(g, s, r)$ such that both f and g are injective. Then, there is a function h such that $\mathcal{M}(h, r, s)$ and h is bijective.*

Proof: All we have to do is to follow Kolmogorov-Fomin style in their book *Introductory Real Analysis* [13].

5 Axiomatization as a flow-theoretic predicate

In this section we briefly propose and discuss a Flow-theoretic version for the axiomatization program proposed by Patrick Suppes [23] [3] [4]. Roughly speaking, Suppes Program is associated to his famous slogan "to axiomatize a theory is to define a set-theoretic predicate". Our proposed slogan can be read like this: "Any theory is a function".

From now on, we are not limited to structure-free classes. Our proposal is to use the full potential of Flow Theory in order to illustrate how we can use it in everyday mathematics. We start with Group Theory.

5.1 Group theory

Definition 30 *A* binary operation *is a function f such that* $\forall t(f[t] \Rightarrow \exists a \exists b(t = (a,b)))$.

So, if $*$ is a binary operation, we may denote $*(a,b)$ simply as $a * b$.

Definition 31 *A binary operation f is* closed *iff* $\forall a \forall b \forall c \forall d((f[(a,b)] \wedge f[(c,d)]) \Rightarrow f[(f(a,b), f(c,d))]$.

Definition 32 *A binary operation f has* neutral element *iff* $\exists e \forall a \forall b(f(f(a,e),b) = f(f(e,a),b) = f(a,b))$. *Term e is the neutral element of f.*

Definition 33 *A binary operation f is* universally invertible *iff f has neutral element e and* $\forall a \exists a^{-1} \forall b(f(f(a,a^{-1}),b) = f(f(a^{-1},a),b) = f(e,b)$. *Term a^{-1} is called the inverse of a in f.*

Definition 34 *A binary operation f is associative iff* $\forall a \forall b \forall c(g(a, g(b,c)) = g(g(a,b),c)))$.

So, one way to define a group is like this:

Definition 35 *A* group *is a function g such that:*

1. $\forall t(g[t] \Rightarrow \exists a \exists b(t = (a,b)))$.
2. $\forall a \forall b \forall c \forall d((g[(a,b)] \wedge g[(c,d)]) \Rightarrow g[(g(a,b), g(c,d))]$.
3. $\forall a \forall b \forall c(g(a, g(b,c)) = g(g(a,b),c)))$.
4. $\exists e \forall a \forall b(g(g(a,e),b) = g(g(e,a),b) = g(a,b))$.
5. $\forall a \exists a^{-1} \forall b(g(g(a,a^{-1}),b) = g(g(a^{-1},a),b) = g(e,b)$.

Another way is like this:

Definition 36 *A* group *is a binary operation $*$ which is closed, associative, and universally invertible.*

Next we prove that any ZFC-theoretic group is associated to some Flow-theoretic group, but the converse is not valid.

5.2 Classical mechanics

To be written...

6 Variations of Flow

To be written...

7 Final remarks

To be written...

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