Homological perspective on edge modes in linear Yang-Mills theory

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Abstract

We provide an elegant homological construction of the extended phase space for linear Yang-Mills theory on an oriented and time-oriented Lorentzian manifold M with a time-like boundary ∂M that was proposed by Donnelly and Freidel [JHEP **1609**, 102 (2016)]. This explains and formalizes many of the rather ad hoc constructions for edge modes appearing in the theoretical physics literature.

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1 Introduction and summary

Inspired by the work of Donnelly and Freidel [DF16], there is presently a revived and growing interest in gauge and gravity theories on manifolds with boundaries, see e.g. [Gei17, BMV18, GR18, FP18, FLP19] for some follow-up papers. One of the main observations of these studies is that, in the presence of boundaries, there exist additional degrees of freedom (called edge modes) that are localized at the boundary. These edge modes are then used to determine a gauge-invariant symplectic structure on the relevant solution spaces and hence to define an appropriate concept of extended phase space for the gauge or gravity theory of interest.

The goal of this short paper is to provide an elegant and rigorous construction of the extended phase space of [DF16] for the case of linear Yang-Mills theory on an oriented and time-oriented globally hyperbolic Lorentzian manifold M with a time-like boundary ∂M. Our construction employs some basic techniques from homological algebra and the theory of groupoids, which are necessary to describe the higher categorical structures featuring in gauge theory. We refer the reader to [BS19, Section 3] for a rather non-technical introduction to these techniques and [Sch13] for an extensive overview.

The main benefit of adopting this more abstract homological perspective is that many of the *ad hoc* constructions for edge modes in [DF16] become very natural. For example, as we explain in detail in Remark 2.2, the appearance of edge modes is a direct consequence of implementing a *topological* boundary condition that identifies the bulk principal bundles with a fixed principal bundle on the boundary. Our main novel observation is that the extended phase space of [DF16] can be directly constructed from a very simple action functional (2.8), provided that one distinguishes carefully between the time-like boundary ∂M of M and the space-like boundary that is introduced by a choice of Cauchy surface $\Sigma \subset M$. This is in contrast to the construction in [DF16], which introduces by hand additional terms to the ordinary symplectic structure in order to restore gauge-invariance in the presence of a boundary.

Our construction of the extended phase space consists of two consecutive steps that allow us to distinguish between the two different kinds of boundaries (time-like boundary ∂M vs. space-like Cauchy surface Σ) that feature in [DF16]. In the first step, we construct the derived critical locus of the action functional (2.8) and its canonical $[-1]$-shifted symplectic structure, which depends on the time-like boundary ∂M but does *not* require a choice of Cauchy surface $\Sigma \subset M$. (In the physics literature, derived critical loci are called BRST/BV formalism and the shifted symplectic structure is called the antibracket.) In the second step, we obtain from this data and the *choice of a Cauchy surface* $\Sigma \subset M$ an unshifted symplectic structure by applying our simple construction from Definition 4.1. We explain in detail in Remark 4.4 that the 0-truncation of our homological construction reproduces the extended phase space of [DF16]. As a novel result, we obtain an extension of the symplectic structure on the extended phase space to ghost fields and antifields, whose explicit form is given in (4.5).

We would like to note that our approach differs from the BV-BFV formalism for gauge theories on manifolds with boundaries developed by Cattaneo, Mnev, Reshetikhin and collaborators, see e.g. the original paper [CMR14] and the more recent [MSW19]. The latter approach does not consider the two distinct ways in which two respective types of boundaries feature in the models discussed in [DF16], i.e. the boundary ∂M (on which one implements a topological boundary condition) and the Cauchy surface Σ (on which one assigns initial data). In the case where the boundary $\partial M = \emptyset$ is empty, our approach coincides with the BV-BFV formalism, see Remark 4.5. We believe that it should be possible to generalize the BV-BFV formalism [CMR14, MSW19] to encode this difference. This would provide a powerful framework for also studying the non-linear gauge and gravity theories in [DF16].

The outline of the remainder of this paper is as follows: In Section 2 we introduce our model of interest, namely linear Yang-Mills theory on an oriented and time-oriented Lorentzian manifold M with a time-like boundary ∂M, together with a topological boundary condition (leading to the edge modes, see Remark 2.2) and the novel action functional (2.8). In Section 3 we construct explicitly the (linear) derived critical locus for our model (3.6) and its canonical $[-1]$-shifted symplectic structure (3.7). In Section 4 we derive, from the choice of a Cauchy surface $\Sigma \subset M$, an unshifted symplectic structure (4.5) and show that the 0-truncation of our homological construction reproduces the extended phase space of [DF16], see Remark 4.4.

Notation and conventions for chain complexes: The main constructions and results in this paper are stated in the category $\mathbf{Ch}_{\mathbb{R}}$ of (possibly unbounded) chain complexes of vector spaces over the field of real numbers $\mathbb{R}$. We use homological degree conventions, i.e. the differentials $\mathrm{d} : V_n \to V_{n-1}$ lower the degree by 1. The tensor product $V \otimes W$ of two chain complexes is given by $(V \otimes W)_n = \bigoplus_{m\in\mathbb{Z}} V_m \otimes W_{n-m}$ together with the differential $\mathrm{d}(v\otimes w) = (\mathrm{d}v)\otimes w + (-1)^{|v|}\, v\otimes(\mathrm{d}w)$ determined by the graded Leibniz rule, where $|v| \in \mathbb{Z}$ denotes the degree of v. The tensor unit is $\mathbb{R} \in \mathbf{Ch}_{\mathbb{R}}$, regarded as a chain complex concentrated in degree 0 with trivial differentials. Given a chain complex V and an integer $p \in \mathbb{Z}$, the $[p]$-shifted chain complex $V[p]$ is defined by $V[p]_n = V_{n-p}$ and $\mathrm{d}^{V[p]} = (-1)^p\, \mathrm{d}^V$.

The homology $H_\bullet(V)$ of a chain complex V is the graded vector space defined by $H_n(V) := \mathrm{Ker}(\mathrm{d} : V_n \to V_{n-1})/\mathrm{Im}(\mathrm{d} : V_{n+1} \to V_n)$, for all $n \in \mathbb{Z}$. A chain map $f : V \to W$ is called a quasi-isomorphism if the induced map $H_\bullet(f) : H_\bullet(V) \to H_\bullet(W)$ in homology is an isomorphism. Quasi-isomorphic chain complexes are considered as 'being the same', which can be made precise by endowing $\mathbf{Ch}_\mathbb{R}$ with the usual (projective) model category structure, see e.g. [Hov99]. We refer to [BS19, Section 3] for a brief non-technical introduction to model categories in the context of classical and quantum gauge theory.

2 Definition of the model

Let M be an oriented and time-oriented Lorentzian manifold with a time-like smooth boundary ∂M. Denote by $\iota : \partial M \to M$ the boundary inclusion and by $m = \dim(M) \geq 2$ the dimension of M. The orientation, time-orientation and Lorentzian metric on M induces on ∂M the structure of an oriented and time-oriented Lorentzian manifold (without boundary) of dimension $\dim(\partial M) = m - 1$. We interpret M as a physical spacetime whose boundary is another spacetime ∂M.

Let us now introduce the field content of our model of interest. As bulk fields on M we consider principal $\mathbb{R}$-bundles with connections, together with their gauge transformations. These data are described by the groupoid

$$\mathbf{B}\mathbb{R}_{\mathrm{con}}(M) \;:=\; \begin{cases} \mathrm{Obj}: & A \in \Omega^1(M) \\ \mathrm{Mor}: & A \stackrel{\epsilon}{\longrightarrow} A + \mathrm{d}\epsilon \quad \text{with } \;\epsilon \in \Omega^0(M) \end{cases} \quad , \tag{2.1}$$

whose objects are interpreted as gauge fields and morphisms as gauge transformations between gauge fields. (Recall that every principal $\mathbb{R}$-bundle is isomorphic to the trivial principal $\mathbb{R}$-bundle. Hence, up to equivalence of groupoids, one may consider only the trivial principal $\mathbb{R}$-bundle, as we have done in (2.1).) Take a principal $\mathbb{R}$-bundle on the boundary ∂M, which is described by a map of groupoids (i.e. a functor)

$$p \,:\, \{*\} \;\longrightarrow\; \mathbf{B}\mathbb{R}(\partial M) \tag{2.2}$$

from the point $\{*\}$ to the groupoid

$$\mathbf{B}\mathbb{R}(\partial M) \;:=\; \begin{cases} \mathrm{Obj}: & * \\ \mathrm{Mor}: & * \stackrel{\chi}{\longrightarrow} * \quad \text{with } \;\chi \in \Omega^0(\partial M) \end{cases} \tag{2.3}$$

of principal $\mathbb{R}$-bundles on ∂M and their gauge transformations. Observe that there is a map of groupoids

$$\mathrm{res} \,:\, \mathbf{B}\mathbb{R}_{\mathrm{con}}(M) \;\longrightarrow\; \mathbf{B}\mathbb{R}(\partial M) \tag{2.4}$$

which forgets the bulk connection and restricts the bulk principal $\mathbb{R}$-bundle to the boundary ∂M. Concretely, this functor acts on objects as $A \mapsto *$ and on morphisms as $(\epsilon : A \to A + \mathrm{d}\epsilon) \mapsto (\iota^*\epsilon : * \to *)$, where $\iota^*\epsilon \in \Omega^0(\partial M)$ denotes the pullback of $\epsilon \in \Omega^0(M)$ along the boundary inclusion $\iota : \partial M \to M$. We would like to impose a (topological) boundary condition that identifies the restriction of the bulk principal $\mathbb{R}$-bundle with the fixed principal $\mathbb{R}$-bundle on ∂M. This is formalized by considering the *homotopy pullback* (or equivalently a 2-categorical pullback)

$$\begin{array}{ccc} \mathfrak{F}(M) & \dashrightarrow & \mathbf{B}\mathbb{R}_{\mathrm{con}}(M) \\ \downarrow & h & \Big\downarrow \mathrm{res} \\ \{*\} & \xrightarrow[p]{} & \mathbf{B}\mathbb{R}(\partial M) \end{array} \tag{2.5}$$

in the model category (or 2-category) of groupoids. The resulting groupoid $\mathfrak{F}(M)$ plays the role of the groupoid of fields for our model of interest.

Proposition 2.1. *A model for the homotopy pullback in* (2.5) *is given by the groupoid*

$$\mathfrak{F}(M) \;=\; \begin{cases} \mathrm{Obj}: & (A,\varphi) \in \Omega^1(M) \times \Omega^0(\partial M) \\ \mathrm{Mor}: & (A,\varphi) \stackrel{\epsilon}{\longrightarrow} \big(A + \mathrm{d}\epsilon, \varphi + \iota^\ast\epsilon\big) \quad \text{with} \;\; \epsilon \in \Omega^0(M) \end{cases} \quad . \tag{2.6}$$

Proof. This is a direct computation using the usual model for homotopy pullbacks of groupoids, see e.g. [Hol08, Section 2]. Explicitly, an object in the homotopy pullback (2.5) is a pair of objects $(\ast, A) \in \{\ast\} \times \mathbf{B}\mathbb{R}_{\mathrm{con}}(M)$ together with a $\mathbf{B}\mathbb{R}(\partial M)$-morphism $p(\ast) = \ast \stackrel{\varphi}{\longrightarrow} \ast = \mathrm{res}(A)$. Hence, an object in $\mathfrak{F}(M)$ is given by a pair $(A,\varphi) \in \Omega^1(M) \times \Omega^0(\partial M)$. A morphism $(A,\varphi) \to (A',\varphi')$ in the homotopy pullback (2.5) is a pair of morphisms $(\mathrm{id}_\ast : \ast \to \ast, \epsilon : A \to A + \mathrm{d}\epsilon = A') \in \{\ast\} \times \mathbf{B}\mathbb{R}_{\mathrm{con}}(M)$ that is compatible with φ and φ', i.e. the diagram

$$\begin{array}{ccc} \mathrm{res}(A) = \ast & \xrightarrow{\;\iota^\ast\epsilon\;} & \ast = \mathrm{res}(A') \\ {\scriptstyle\varphi}\Big\uparrow & & \Big\uparrow{\scriptstyle\varphi'} \\ p(\ast) = \ast & \xrightarrow[\mathrm{id}_\ast]{} & \ast = p(\ast) \end{array} \tag{2.7}$$

in $\mathbf{B}\mathbb{R}(\partial M)$ commutes. Hence, a morphism in $\mathfrak{F}(M)$ is given by $(A,\varphi) \stackrel{\epsilon}{\longrightarrow} (A + \mathrm{d}\epsilon, \varphi + \iota^\ast\epsilon)$, where $\epsilon \in \Omega^0(M)$. □

Remark 2.2. Note that an object of the groupoid $\mathfrak{F}(M)$ in (2.6) is a pair $(A,\varphi) \in \Omega^1(M) \times \Omega^0(\partial M)$ consisting of a gauge field A in the bulk M and a gauge transformation φ on the boundary ∂M. Hence, the groupoid of fields $\mathfrak{F}(M)$ contains both bulk and boundary fields. It is one of the main goals of the present paper to explain that these φ are precisely the edge modes introduced in [DF16]. As a first piece of evidence for this claim, we note that the morphisms of the groupoid $\mathfrak{F}(M)$ in (2.6) are precisely the gauge transformations on bulk and boundary fields in [DF16].

From our groupoid perspective, the origin of edge modes can be explained very naturally. The groupoid of fields $\mathfrak{F}(M)$ is obtained by identifying the restriction of the bulk principal $\mathbb{R}$-bundle with the fixed principal $\mathbb{R}$-bundle on ∂M, i.e. we implement a (topological) boundary condition via the homotopy pullback diagram (2.5). Boundary conditions in a gauge theory are quite subtle because gauge fields are *not* compared by equality but rather by gauge transformations, i.e. morphisms in the relevant groupoids. Hence, a boundary condition in a gauge theory is not a *property* of the gauge fields but an additional *structure* given by gauge transformations acting as witnesses of the boundary condition. The edge modes φ in (2.6) are precisely the witnesses for the statement that the restriction of the bulk principal $\mathbb{R}$-bundle is 'the same as' the fixed boundary principal $\mathbb{R}$-bundle. △

In the next step we introduce a gauge-invariant action functional in order to specify the dynamics of our model of interest. This is described by a map of groupoids $S : \mathfrak{F}(M) \to \mathbb{R}$ from our groupoid of fields (2.6) to the real numbers $\mathbb{R}$, regarded as a groupoid with only identity morphisms. We define

$$S(A,\varphi) \;:=\; \int_M \frac{1}{2}\,\mathrm{d}A \wedge \ast\mathrm{d}A + \int_{\partial M} \frac{1}{2}\,\mathrm{d}_A\varphi \wedge \ast_\partial \mathrm{d}_A\varphi \quad , \tag{2.8}$$

where $\ast_{(\partial)}$ denotes the Hodge operator on $(\partial)M$ and the affine covariant differential is given by

$$\mathrm{d}_A\varphi \;:=\; \mathrm{d}\varphi - \iota^\ast A \quad . \tag{2.9}$$

Clearly, the action (2.8) is gauge-invariant because $\mathrm{d}A$ and $\mathrm{d}_A\varphi$ are invariant under the gauge transformations in (2.6). (In the physics literature, the quantity $\mathrm{d}_A\varphi$ is also referred to as a 'dressing', see e.g. [AFLM18].)

Upon varying the action with respect to compactly supported variations $(\alpha,\psi) \in \Omega^1_{\mathrm{c}}(M) \times \Omega^0_{\mathrm{c}}(\partial M)$, a straightforward calculation using Stokes' theorem yields the expression

$$\delta_{(\alpha,\psi)} S(A,\varphi) \,=\, \int_M \alpha \wedge \mathrm{d} * \mathrm{d}A + \int_{\partial M} \iota^*\alpha \wedge \big(\iota^*(*\mathrm{d}A) - *_\partial \mathrm{d}_A\varphi\big) - \int_{\partial M} \psi \wedge \mathrm{d} *_\partial \mathrm{d}_A \varphi \quad . \tag{2.10}$$

The corresponding Euler-Lagrange equations are

$$\mathrm{d} * \mathrm{d}A \,=\, 0 \qquad \text{(linear Yang-Mills equation on } M) \quad , \tag{2.11a}$$

$$\mathrm{d} *_\partial \mathrm{d}_A\varphi \,=\, 0 \qquad \text{(inhomogeneous Klein-Gordon equation on } \partial M) \quad , \tag{2.11b}$$

$$\iota^*(*\mathrm{d}A) - *_\partial \mathrm{d}_A \varphi \,=\, 0 \qquad \text{(matching constraint on } \partial M) \quad . \tag{2.11c}$$

Remark 2.3. We would like to emphasize that the third equation in (2.11) arises from the fact that we also allow for bulk field variations $\alpha \in \Omega^1_{\mathrm{c}}(M)$ with support on the boundary ∂M, which is different from [DF16] where the usual linear Yang-Mills action $\int_M \frac{1}{2}\mathrm{d}A \wedge *\mathrm{d}A$ is varied by variations $\alpha \in \Omega^1_{\mathrm{c}}(M)$ that are assumed to vanish on the boundary, i.e. $\iota^*\alpha = 0$. The advantage of our approach is that it allows us to interpret the matching constraint in (2.11) as an Euler-Lagrange equation of the action (2.8), while in [DF16] this constraint was implemented by hand in the construction of the extended phase space. △

Remark 2.4. We would like to explain very briefly that, up to this point, our construction admits a straightforward generalization to non-Abelian Yang-Mills theory. To simplify the presentation in this remark, let us assume that $M \cong \mathbb{R}^{m-1} \times [0,\infty)$ is diffeomorphic to a half-space. Let G be a compact matrix Lie group and denote its Lie algebra by $\mathfrak{g}$. As a consequence of our assumptions, there exist no non-trivial principal G-bundles on both M and ∂M, hence the groupoid of principal G-bundles with connection on M reads as

$$\mathbf{B}G_{\mathrm{con}}(M) \,:=\, \begin{cases} \mathrm{Obj}: & A \in \Omega^1(M,\mathfrak{g}) \\ \mathrm{Mor}: & A \stackrel{g}{\longrightarrow} g^{-1}Ag + g^{-1}\mathrm{d}g \quad \text{with} \ \ g \in C^\infty(M,G) \end{cases} \tag{2.12}$$

and the groupoid of principal G-bundles on ∂M reads as

$$\mathbf{B}G(\partial M) \,:=\, \begin{cases} \mathrm{Obj}: & * \\ \mathrm{Mor}: & * \stackrel{h}{\longrightarrow} * \quad \text{with} \ \ h \in C^\infty(\partial M,G) \end{cases} \quad . \tag{2.13}$$

The two maps in the homotopy pullback diagram (2.5) exist also in the non-Abelian setting. An explicit computation as in Proposition 2.1 yields the groupoid of fields

$$\mathfrak{F}_G(M) \,=\, \begin{cases} \mathrm{Obj}: & (A,u) \in \Omega^1(M,\mathfrak{g}) \times C^\infty(\partial M,G) \\ \mathrm{Mor}: & (A,u) \stackrel{g}{\longrightarrow} \big(g^{-1}Ag + g^{-1}\mathrm{d}g, u\,\iota^*g\big) \quad \text{with} \ \ g \in C^\infty(M,G) \end{cases} \quad . \tag{2.14}$$

Recalling the curvature $F(A) = \mathrm{d}A + A \wedge A$ and introducing $\mathrm{d}_A \log u := u^{-1}\mathrm{d}u - \iota^*A$, one easily checks that

$$S_G(A,u) \,:=\, \int_M \frac{1}{2}\,\mathrm{Tr}\big(F(A) \wedge *F(A)\big) + \int_{\partial M} \frac{1}{2}\,\mathrm{Tr}\big(\mathrm{d}_A \log u \wedge *_\partial \mathrm{d}_A \log u\big) \tag{2.15}$$

defines a gauge-invariant action. We will not develop this non-Abelian generalization of our model any further, because linearity is crucial to simplify our constructions in the remainder of this paper. △

Because our model of interest is a linear gauge theory, we can reformulate it in the language of chain complexes of vector spaces. The key ingredient for this construction is given by the Dold-Kan correspondence between simplicial vector spaces and (non-negatively graded) chain complexes of vector spaces, see e.g. [BSS15] for an application in the context of gauge theory. Explicitly, the

Dold-Kan correspondence assigns to (the nerve of) our groupoid of fields (2.6) the chain complex (denoted with abuse of notation by the same symbol)

$$\mathfrak{F}(M) \,=\, \Big(\overset{(0)}{\mathfrak{F}_0(M)} \xleftarrow{\;\;Q\;\;} \overset{(1)}{\mathfrak{F}_1(M)} \Big) \,=\, \Big(\overset{(0)}{\Omega^1(M) \times \Omega^0(\partial M)} \xleftarrow{\;\;Q\;\;} \overset{(1)}{\Omega^0(M)} \Big) \tag{2.16a}$$

concentrated in homological degrees 0 and 1, with differential given by

$$Q(C) \,=\, \big(\mathrm{d}C, \iota^* C\big) \quad , \tag{2.16b}$$

for all $C \in \Omega^0(M)$. From now on, we shall denote gauge transformations by $C \in \Omega^0(M)$. This choice of notation is explained in Remark 3.2 below, where C will be interpreted as a ghost field. Observe that elements $(A, \varphi) \in \Omega^1(M) \times \Omega^0(\partial M)$ in degree 0 are the fields of the theory, elements $C \in \Omega^0(M)$ in degree 1 are the gauge transformations and the differential Q encodes the action $(A, \varphi) \to (A, \varphi) + Q(C) = (A + \mathrm{d}C, \varphi + \iota^* C)$ of gauge transformations. The variation of the action (2.10) determines a linear differential operator

$$P \,:\, \Omega^1(M) \times \Omega^0(\partial M) \,\longrightarrow\, \Omega^{m-1}(M) \times \Omega^{m-2}(\partial M) \times \Omega^{m-1}(\partial M) \tag{2.17a}$$

given by

$$P(A, \varphi) \,=\, \Big((-1)^{m-1}\, \mathrm{d} * \mathrm{d}A, (-1)^{m-2}\, \big(\iota^*(*\mathrm{d}A) - *_\partial \mathrm{d}_A \varphi\big), -\mathrm{d} *_\partial \mathrm{d}_A \varphi \Big) \quad , \tag{2.17b}$$

for all $(A, \varphi) \in \Omega^1(M) \times \Omega^0(\partial M)$. The signs in (2.17) are due to the following choice of conventions: The codomain of P is given by the smooth Lefschetz dual

$$\mathfrak{F}_{0,\mathrm{c}}(M)^* \,:=\, \Omega^{m-1}(M) \times \Omega^{m-2}(\partial M) \times \Omega^{m-1}(\partial M) \tag{2.18a}$$

of the degree 0 component $\mathfrak{F}_{0,\mathrm{c}}(M) = \Omega^1_\mathrm{c}(M) \times \Omega^0_\mathrm{c}(\partial M)$ of the compactly supported analog of the field complex (2.16). The evaluation pairing $\langle \,\cdot\,, \,\cdot\, \rangle : \mathfrak{F}_{0,\mathrm{c}}(M)^* \times \mathfrak{F}_{0,\mathrm{c}}(M) \to \mathbb{R}$ reads as

$$\langle (A^\dagger, a^\dagger, \varphi^\dagger), (A, \varphi) \rangle = \int_M A^\dagger \wedge A + \int_{\partial M} \big(a^\dagger \wedge \iota^* A + \varphi^\dagger \wedge \varphi\big) \quad , \tag{2.18b}$$

for all $(A^\dagger, a^\dagger, \varphi^\dagger) \in \Omega^{m-1}(M) \times \Omega^{m-2}(\partial M) \times \Omega^{m-1}(\partial M)$ and $(A, \varphi) \in \Omega^1_\mathrm{c}(M) \times \Omega^0_\mathrm{c}(\partial M)$. The linear differential operator P is defined by (2.10) and the equation $\delta_{(\alpha,\psi)} S(A, \varphi) = \langle P(A, \varphi), (\alpha, \psi) \rangle$, for all $(A, \varphi) \in \Omega^1(M) \times \Omega^0(\partial M)$ and $(\alpha, \psi) \in \Omega^1_\mathrm{c}(M) \times \Omega^0_\mathrm{c}(\partial M)$. Hence, the signs in (2.17) are a consequence of graded commutativity of the $\wedge$-product.

3 Derived critical locus and shifted symplectic structure

Instead of enforcing the Euler-Lagrange equations (2.11) in a strict sense, we consider their homological enhancement given by the (linear) derived critical locus construction. Our motivation and reasons for this are twofold: 1.) Enforcing the Euler-Lagrange equations strictly as in (2.11) is in general incompatible with quasi-isomorphisms in the category $\mathbf{Ch}_\mathbb{R}$ of (possibly unbounded) chain complexes, i.e. if one takes two different quasi-isomorphic field complexes, the naive solution complexes assigned to them are in general no longer quasi-isomorphic. This is problematic because it violates the main principle of homological algebra that all sensible constructions must respect quasi-isomorphisms. 2.) Every derived critical locus carries a canonical $[-1]$-shifted symplectic structure (see e.g. [PTVV13, CPTVV17, Pri18] for the corresponding results in derived algebraic geometry) which has various physical applications. For instance, in the context of (quantum) field theory, this shifted symplectic structure is the starting point for constructing a factorization algebra [CG17] or an algebraic quantum field theory [BBS19]. Below, we give a novel application of this $[-1]$-shifted symplectic structure: It will be used to construct the extended phase space introduced in [DF16]. We note that in physics terminology, derived critical loci are called the BRST/BV formalism and the shifted symplectic structure is called the antibracket.

Our construction of the (linear) derived critical locus and its shifted symplectic structure is a relatively straightforward generalization of the case of linear Yang-Mills theory on spacetimes without boundaries presented in [BBS19, BS19]. To make the present paper self-contained, we shall briefly explain this construction. By analogy with (2.18), we define the smooth Lefschetz dual

$$\mathfrak{F}_{1,\mathrm{c}}(M)^{*} \,:=\, \Omega^{m}(M)\times\Omega^{m-1}(\partial M) \tag{3.1a}$$

of the degree 1 component $\mathfrak{F}_{1,\mathrm{c}}(M)=\Omega^0_{\mathrm{c}}(M)$ of the compactly supported analog of the field complex (2.16). The evaluation pairing $\langle\,\cdot\,,\,\cdot\,\rangle : \mathfrak{F}_{1,\mathrm{c}}(M)^{*}\times\mathfrak{F}_{1,\mathrm{c}}(M)\to\mathbb{R}$ reads as

$$\langle (C^{\dagger},c^{\dagger}),C\rangle = \int_M C^{\dagger}\wedge C + \int_{\partial M} c^{\dagger}\wedge \iota^{*}C \quad , \tag{3.1b}$$

for all $(C^{\dagger},c^{\dagger})\in\Omega^{m}(M)\times\Omega^{m-1}(\partial M)$ and $C\in\Omega^0_{\mathrm{c}}(M)$. We denote by

$$Q^{*} \,:\, \Omega^{m-1}(M)\times\Omega^{m-2}(\partial M)\times\Omega^{m-1}(\partial M) \;\longrightarrow\; \Omega^{m}(M)\times\Omega^{m-1}(\partial M) \tag{3.2a}$$

the formal adjoint of the linear differential operator Q in (2.16), which is defined implicitly by $\langle Q^{*}(A^{\dagger},a^{\dagger},\varphi^{\dagger}),C\rangle = \langle (A^{\dagger},a^{\dagger},\varphi^{\dagger}),Q(C)\rangle$, for all $(A^{\dagger},a^{\dagger},\varphi^{\dagger})\in\Omega^{m-1}(M)\times\Omega^{m-2}(\partial M)\times\Omega^{m-1}(\partial M)$ and $C\in\Omega^0_{\mathrm{c}}(M)$. A straightforward calculation using Stokes' theorem then provides the explicit expression

$$Q^{*}(A^{\dagger},a^{\dagger},\varphi^{\dagger}) \,=\, \Big((-1)^{m}\,\mathrm{d}A^{\dagger}, (-1)^{m-1}\,\big(\mathrm{d}a^{\dagger}+\iota^{*}A^{\dagger}\big)+\varphi^{\dagger}\Big) \quad , \tag{3.2b}$$

for all $(A^{\dagger},a^{\dagger},\varphi^{\dagger})\in\Omega^{m-1}(M)\times\Omega^{m-2}(\partial M)\times\Omega^{m-1}(\partial M)$. The smooth Lefschetz dual of the compactly supported analog of the field complex (2.16) is thus given by

$$\begin{aligned}\mathfrak{F}_{\mathrm{c}}(M)^{*} \;&=\; \Big(\overset{(-1)}{\mathfrak{F}_{1,\mathrm{c}}(M)^{*}} \xleftarrow{\;-Q^{*}\;} \overset{(0)}{\mathfrak{F}_{0,\mathrm{c}}(M)^{*}}\Big)\\ &=\; \Big(\overset{(-1)}{\Omega^{m}(M)\times\Omega^{m-1}(\partial M)} \xleftarrow{\;-Q^{*}\;} \overset{(0)}{\Omega^{m-1}(M)\times\Omega^{m-2}(\partial M)\times\Omega^{m-1}(\partial M)}\Big) \quad .\end{aligned} \tag{3.2c}$$

This chain complex is used to define the total space $T^{*}\mathfrak{F}(M) := \mathfrak{F}(M)\times\mathfrak{F}_{\mathrm{c}}(M)^{*}\in\mathbf{Ch}_{\mathbb{R}}$ of the cotangent bundle over the field complex (2.16) as a Cartesian product of chain complexes. The variation of the action (2.10), or equivalently, the associated differential operator P in (2.17), defines a section

$$\begin{array}{c}\mathfrak{F}(M)\\ {\scriptstyle \delta S}\big\downarrow\\ T^{*}\mathfrak{F}(M)\end{array} \;=\; \left(\begin{array}{ccccc} 0 & \xleftarrow{\quad 0 \quad} & \mathfrak{F}_0(M) & \xleftarrow{\quad Q \quad} & \mathfrak{F}_1(M)\\ {\scriptstyle 0}\big\downarrow & & {\scriptstyle (\mathrm{id},P)}\big\downarrow & & {\scriptstyle \mathrm{id}}\big\downarrow\\ \mathfrak{F}_{1,\mathrm{c}}(M)^{*} & \xleftarrow[-Q^{*}\pi_2]{} & \mathfrak{F}_0(M)\times\mathfrak{F}_{0,\mathrm{c}}(M)^{*} & \xleftarrow[\iota_1 Q]{} & \mathfrak{F}_1(M)\end{array}\right) \tag{3.3}$$

of the cotangent bundle. The zero-section of the cotangent bundle is given by

$$\begin{array}{c}\mathfrak{F}(M)\\ {\scriptstyle 0}\big\downarrow\\ T^{*}\mathfrak{F}(M)\end{array} \;=\; \left(\begin{array}{ccccc} 0 & \xleftarrow{\quad 0 \quad} & \mathfrak{F}_0(M) & \xleftarrow{\quad Q \quad} & \mathfrak{F}_1(M)\\ {\scriptstyle 0}\big\downarrow & & {\scriptstyle (\mathrm{id},0)}\big\downarrow & & {\scriptstyle \mathrm{id}}\big\downarrow\\ \mathfrak{F}_{1,\mathrm{c}}(M)^{*} & \xleftarrow[-Q^{*}\pi_2]{} & \mathfrak{F}_0(M)\times\mathfrak{F}_{0,\mathrm{c}}(M)^{*} & \xleftarrow[\iota_1 Q]{} & \mathfrak{F}_1(M)\end{array}\right) \quad . \tag{3.4}$$

In order to enforce the dynamics encoded by the action functional (2.8), we intersect δS with the zero-section 0 (in the derived sense) by forming the homotopy pullback

$$\begin{array}{ccc} \mathfrak{S}(M) & \dashrightarrow & \mathfrak{F}(M) \\ {\scriptstyle h}\ \downarrow & & \downarrow {\scriptstyle \delta S} \\ \mathfrak{F}(M) & \xrightarrow[0]{} & T^*\mathfrak{F}(M) \end{array} \tag{3.5}$$

in the model category $\mathbf{Ch}_{\mathbb{R}}$.

Proposition 3.1. *A model for the homotopy pullback in* (3.5) *is given by the chain complex*

$$\mathfrak{S}(M) \;=\; \Big(\overset{(-2)}{\mathfrak{F}_{1,\mathrm{c}}(M)^*} \xleftarrow{\;Q^*\;} \overset{(-1)}{\mathfrak{F}_{0,\mathrm{c}}(M)^*} \xleftarrow{\;P\;} \overset{(0)}{\mathfrak{F}_0(M)} \xleftarrow{\;Q\;} \overset{(1)}{\mathfrak{F}_1(M)} \Big) \quad , \tag{3.6}$$

with differentials defined in (2.16)*,* (2.17) *and* (3.2)*.*

Proof. The proof is completely analogous to the one of [BS19, Proposition 3.21] and will not be repeated here. □

Remark 3.2. The chain complex (3.6) admits an interpretation in terms of the BRST/BV formalism. Elements $C \in \mathfrak{S}_1(M) = \Omega^0(M)$ in degree 1 are the ghost fields and elements $(A,\varphi) \in \mathfrak{S}_0(M) = \Omega^1(M) \times \Omega^0(\partial M)$ in degree 0 are the fields of the theory. Furthermore, elements $(A^\dagger, a^\dagger, \varphi^\dagger) \in \mathfrak{S}_{-1}(M) = \Omega^{m-1}(M) \times \Omega^{m-2}(\partial M) \times \Omega^{m-1}(\partial M)$ in degree -1 are the antifields and elements $(C^\dagger, c^\dagger) \in \Omega^m(M) \times \Omega^{m-1}(\partial M)$ in degree -2 are the antifields for ghosts. The differential operator Q encodes the gauge symmetries and P encodes the equation of motion of our model. In particular, the 0-th homology $H_0(\mathfrak{S}(M))$ of (3.6) is the ordinary vector space of gauge equivalence classes of solutions of the Euler-Lagrange equations (2.11). Note that, in contrast to the usual BRST/BV formalism on manifolds without a boundary, our model of interest (3.6) also contains boundary fields φ and boundary antifields $a^\dagger$, $\varphi^\dagger$ and $c^\dagger$. It is important to emphasize that this field content is not arbitrary, but it is dictated (up to quasi-isomorphism) by our homological approach, i.e. by the homotopy pullbacks in (2.5) and (3.5). △

To conclude this section, we explicitly write out the canonical $[-1]$-shifted symplectic structure that exists on the (linear) derived critical locus (3.5). Denoting by $\mathfrak{S}_{\mathrm{c}}(M)$ the compactly supported analog of the solution complex $\mathfrak{S}(M)$ in (3.6), the $[-1]$-shifted symplectic structure is the chain map $\omega_{-1} : \mathfrak{S}_{\mathrm{c}}(M) \otimes \mathfrak{S}_{\mathrm{c}}(M) \to \mathbb{R}[-1]$ defined in terms of the integration pairings (2.18) and (3.1) by

$$\omega_{-1}\big((C^\dagger, c^\dagger), C\big) \;=\; \int_M C^\dagger \wedge C + \int_{\partial M} c^\dagger \wedge \iota^* C \quad , \tag{3.7a}$$

$$\omega_{-1}\big(C, (C^\dagger, c^\dagger)\big) \;=\; -\omega_{-1}\big((C^\dagger, c^\dagger), C\big) \quad , \tag{3.7b}$$

$$\omega_{-1}\big((A^\dagger, a^\dagger, \varphi^\dagger), (A,\varphi)\big) \;=\; \int_M A^\dagger \wedge A + \int_{\partial M} \big(a^\dagger \wedge \iota^* A + \varphi^\dagger \wedge \varphi\big) \quad , \tag{3.7c}$$

$$\omega_{-1}\big((A,\varphi), (A^\dagger, a^\dagger, \varphi^\dagger)\big) \;=\; -\omega_{-1}\big((A^\dagger, a^\dagger, \varphi^\dagger), (A,\varphi)\big) \quad , \tag{3.7d}$$

for all $(C^\dagger, c^\dagger) \in \Omega^m_{\mathrm{c}}(M) \times \Omega^{m-1}_{\mathrm{c}}(\partial M)$, $C \in \Omega^0_{\mathrm{c}}(M)$, $(A^\dagger, a^\dagger, \varphi^\dagger) \in \Omega^{m-1}_{\mathrm{c}}(M) \times \Omega^{m-2}_{\mathrm{c}}(\partial M) \times \Omega^{m-1}_{\mathrm{c}}(\partial M)$ and $(A,\varphi) \in \Omega^1_{\mathrm{c}}(M) \times \Omega^0_{\mathrm{c}}(\partial M)$.

4 Construction of the unshifted symplectic structure

From now on, we assume that M is globally hyperbolic in the sense of Lorentzian manifolds with a time-like boundary, see e.g. [Sol06] and also [BDS18] for a review. Let us choose any Cauchy surface $\Sigma \subset M$ and note that Σ is a manifold with boundary $\partial\Sigma \subset \partial M$. The aim of this section is to construct from the datum of a Cauchy surface $\Sigma \subset M$ and the $[-1]$-shifted symplectic structure ω_{-1} in (3.7) an unshifted symplectic structure ω_0^Σ. We will then show that the extended phase space proposed by Donnelly and Freidel in [DF16] is given by the 0-truncation of this homological construction.

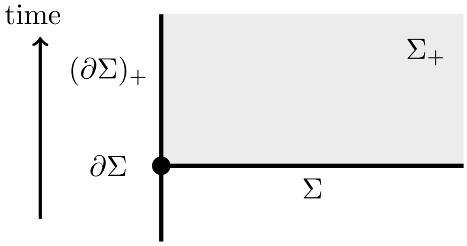
Before we can state our definition of the unshifted symplectic structure ω_0^Σ, we will need to introduce some simple concepts from Lorentzian geometry. Let us denote by

$$\Sigma_+ \,:=\, J_M^+(\Sigma) \,\subseteq\, M \tag{4.1a}$$

the causal future of the Cauchy surface $\Sigma \subset M$, which is the set of all points $p \in M$ that can be reached from $\Sigma \subset M$ via future-pointing causal curves, including all points $p \in \Sigma$ in the Cauchy surface. Note that, by definition, $\Sigma \subset \Sigma_+$ is a subset. We denote by

$$(\partial\Sigma)_+ \,:=\, \Sigma_+ \cap \partial M \,\subseteq \partial M \tag{4.1b}$$

the intersection of Σ_+ with the boundary of M. The following picture visualizes our geometric setup



(4.2)

We observe that Σ_+ has two different kinds of boundary components, given by the time-like boundary $(\partial\Sigma)_+$ and the (space-like) Cauchy surface Σ, as well as a codimension 2 corner $\partial\Sigma$.

We now define a map $\omega_{-1}^\Sigma : \mathfrak{S}_{\mathrm{c}}(M) \otimes \mathfrak{S}_{\mathrm{c}}(M) \to \mathbb{R}[-1]$ of graded vector spaces by recalling the definition of the $[-1]$-shifted symplectic structure in (3.7) and restricting the integrations therein from M to Σ_+ and from ∂M to $(\partial\Sigma)_+$. Explicitly, this gives

$$\omega_{-1}^\Sigma\big((C^\dagger, c^\dagger), C\big) \,=\, \int_{\Sigma_+} C^\dagger \wedge C + \int_{(\partial\Sigma)_+} c^\dagger \wedge \iota^* C \quad , \tag{4.3a}$$

$$\omega_{-1}^\Sigma\big((A^\dagger, a^\dagger, \varphi^\dagger), (A, \varphi)\big) \,=\, \int_{\Sigma_+} A^\dagger \wedge A + \int_{(\partial\Sigma)_+} \big(a^\dagger \wedge \iota^* A + \varphi^\dagger \wedge \varphi\big) \quad , \tag{4.3b}$$

for all $(C^\dagger, c^\dagger) \in \Omega_{\mathrm{c}}^m(M) \times \Omega_{\mathrm{c}}^{m-1}(\partial M)$, $C \in \Omega_{\mathrm{c}}^0(M)$, $(A^\dagger, a^\dagger, \varphi^\dagger) \in \Omega_{\mathrm{c}}^{m-1}(M) \times \Omega_{\mathrm{c}}^{m-2}(\partial M) \times \Omega_{\mathrm{c}}^{m-1}(\partial M)$ and $(A, \varphi) \in \Omega_{\mathrm{c}}^1(M) \times \Omega_{\mathrm{c}}^0(\partial M)$. It is important to emphasize that, in contrast to the $[-1]$-shifted symplectic structure in (3.7), the restricted integrations in (4.3) *do not* define a chain map, i.e. the pre-composition $\omega_{-1}^\Sigma \circ \mathrm{d}^\otimes \neq 0$ with the differential $\mathrm{d}^\otimes$ of the tensor product chain complex $\mathfrak{S}_{\mathrm{c}}(M) \otimes \mathfrak{S}_{\mathrm{c}}(M)$ is non-zero. However, we obtain a chain map $\omega_{-1}^\Sigma \circ \mathrm{d}^\otimes : \mathfrak{S}_{\mathrm{c}}(M) \otimes \mathfrak{S}_{\mathrm{c}}(M) \to \mathbb{R}$ to the unshifted real numbers, because the differential $\mathrm{d}^\otimes$ has degree -1 and the chain map property $\omega_{-1}^\Sigma \circ \mathrm{d}^\otimes \circ \mathrm{d}^\otimes = 0$ is a consequence of nilpotency ${\mathrm{d}^\otimes}^2 = 0$ of the differential.

We are now in a position to define the *unshifted* symplectic structure associated with a Cauchy surface Σ.

Definition 4.1. The unshifted symplectic structure is the chain map

$$\omega_0^\Sigma \,:=\, \omega_{-1}^\Sigma \circ \mathrm{d}^\otimes \,:\, \mathfrak{S}_{\mathrm{c}}(M) \otimes \mathfrak{S}_{\mathrm{c}}(M) \,\longrightarrow\, \mathbb{R} \quad . \tag{4.4}$$

Proposition 4.2. *The unshifted symplectic structure is explicitly given by*

$$\omega_0^\Sigma\big((A, \varphi), (A', \varphi')\big) \,=\, \int_\Sigma \big(A \wedge *\mathrm{d}A' - A' \wedge *\mathrm{d}A\big) - \int_{\partial\Sigma} \big(\varphi \wedge *_\partial \mathrm{d}_{A'}\varphi' - \varphi' \wedge *_\partial \mathrm{d}_A \varphi\big) \quad , \tag{4.5a}$$

$$\omega_0^\Sigma\big((A^\dagger, a^\dagger, \varphi^\dagger), C\big) \,=\, (-1)^m \int_\Sigma A^\dagger \wedge C - (-1)^{m-1} \int_{\partial\Sigma} a^\dagger \wedge \iota^* C \quad , \tag{4.5b}$$

for all $(A,\varphi),(A',\varphi')\in\Omega^1_{\mathrm{c}}(M)\times\Omega^0_{\mathrm{c}}(\partial M)$, $(A^\dagger,a^\dagger,\varphi^\dagger)\in\Omega^{m-1}_{\mathrm{c}}(M)\times\Omega^{m-2}_{\mathrm{c}}(\partial M)\times\Omega^{m-1}_{\mathrm{c}}(\partial M)$ *and* $C\in\Omega^0_{\mathrm{c}}(M)$.

Proof. The proof is a straightforward calculation using Stokes' theorem for manifolds with boundaries and corners, see e.g. [BMPR18]. Thus, we will not write out the details of this calculation. However, for the benefit of the reader, we note that there are two different instances of Stokes' theorem that enter this calculation (consider the picture in (4.2) for a helpful visualization). First, for any $\zeta\in\Omega^{m-1}_{\mathrm{c}}(\Sigma_+)$ in the bulk Σ_+, Stokes' theorem with corners yields

$$\int_{\Sigma_+}\mathrm{d}\zeta\,=\,\int_\Sigma\zeta+\int_{(\partial\Sigma)_+}\zeta\quad, \tag{4.6}$$

because $\partial(\Sigma_+)=(\partial\Sigma)_+\cup\Sigma$. Second, for any $\eta\in\Omega^{m-2}_{\mathrm{c}}((\partial\Sigma)_+)$ on the time-like boundary component $(\partial\Sigma)_+$, ordinary Stokes' theorem yields

$$\int_{(\partial\Sigma)_+}\mathrm{d}\eta\,=\,-\int_{\partial\Sigma}\eta\quad, \tag{4.7}$$

because $\partial((\partial\Sigma)_+)=-\partial\Sigma$ is the boundary of Σ with the opposite orientation. □

Corollary 4.3. *Using the same formulas as in* (4.5)*, the unshifted symplectic structure from Definition 4.1 and Proposition 4.2 admits an extension to a chain map*

$$\omega_0^\Sigma\,:\,\mathfrak{S}_{\mathrm{sc}}(M)\otimes\mathfrak{S}_{\mathrm{sc}}(M)\,\longrightarrow\,\mathbb{R}\quad, \tag{4.8}$$

where $\mathfrak{S}_{\mathrm{sc}}(M)$ *is the space-like compactly supported analog of the solution complex* (3.6)*. (Recall that a differential form* $\zeta\in\Omega^p(M)$ *has space-like compact support if* $\mathrm{supp}(\zeta)\subseteq J^+_M(K)\cup J^-_M(K)$*, for some compact subset* $K\subseteq M$*.)*

Remark 4.4. At first sight, it seems that our unshifted symplectic structure (4.5) is different from the one proposed in [DF16]. However, upon closer inspection, one finds that this is *not* the case and that the 0-truncation of our approach reproduces the results of [DF16]. Let us recall that [DF16] are not working in a homological approach, which means that they are implementing the Euler-Lagrange equations (2.11) in the strict sense. From our perspective, this means that they are considering 0-cycles in the space-like compactly supported solution complex $\mathfrak{S}_{\mathrm{sc}}(M)$. For every two 0-cycles $(A,\varphi),(A',\varphi')\in\Omega^1_{\mathrm{sc}}(M)\times\Omega^0_{\mathrm{sc}}(\partial M)$, i.e. $P(A,\varphi)=0=P(A',\varphi')$ with P given in (2.17), one can write the unshifted symplectic structure (4.5a) equivalently as

$$\begin{aligned}\omega_0^\Sigma\big((A,\varphi),(A',\varphi')\big)\,&=\,\int_\Sigma\big(A\wedge*\mathrm{d}A'-A'\wedge*\mathrm{d}A\big)-\int_{\partial\Sigma}\big(\varphi\wedge*_\partial\mathrm{d}_{A'}\varphi'-\varphi'\wedge*_\partial\mathrm{d}_A\varphi\big)\\&=\,\int_\Sigma\big(A\wedge*\mathrm{d}A'-A'\wedge*\mathrm{d}A\big)-\int_{\partial\Sigma}\big(\varphi\wedge\iota^*(*\mathrm{d}A')-\varphi'\wedge\iota^*(*\mathrm{d}A)\big)\quad,\end{aligned} \tag{4.9}$$

where we used explicitly the matching constraint from the Euler-Lagrange equations (2.11). This equivalent form of the unshifted symplectic structure on 0-cycles coincides with the proposal in [DF16]. We note that the antifield-ghost component (4.5b) of our unshifted symplectic structure is a novel feature of our homological approach that has no corresponding analog in the 0-truncation studied in [DF16]. △

Remark 4.5. To compare our results to the BV-BFV formalism [CMR14], we specialize to the case in which the *time-like* boundary is empty, i.e. $\partial M=\emptyset$. The solution complex (3.6) then simplifies to

$$\mathfrak{S}(M)\,=\,\Big(\overset{(-2)}{\Omega^m(M)}\xleftarrow{\;(-1)^m\,\mathrm{d}\;}\overset{(-1)}{\Omega^{m-1}(M)}\xleftarrow{\;(-1)^{m-1}\,\mathrm{d}*\mathrm{d}\;}\overset{(0)}{\Omega^1(M)}\xleftarrow{\;\mathrm{d}\;}\overset{(1)}{\Omega^0(M)}\Big) \tag{4.10}$$

and the $[-1]$-shifted symplectic structure (3.7) simplifies to

$$\omega_{-1}\big(C^\dagger, C\big) \,=\, \int_M C^\dagger\wedge C \quad , \qquad \omega_{-1}\big(A^\dagger, A\big) \,=\, \int_M A^\dagger \wedge A \quad . \tag{4.11}$$

Furthermore, the unshifted symplectic structure (4.5) simplifies to

$$\omega_0^\Sigma\big(A, A'\big) \,=\, \int_\Sigma \big(A\wedge *\mathrm{d}A' - A'\wedge *\mathrm{d}A\big) \quad , \qquad \omega_0^\Sigma\big(A^\dagger, C\big) \,=\, (-1)^m \int_\Sigma A^\dagger \wedge C \quad . \tag{4.12}$$

We observe that both the $[-1]$-shifted and unshifted symplectic structure agree with the ones obtained from the BV-BFV formalism applied to electromagnetism, see in particular [CMR14, Section 5.1].

We further obtain as in [CMR14, Section 5.1.6] a $[+1]$-shifted symplectic structure in codimension 2 by iterating our construction in Definition 4.1. Concretely, let us choose any codimension 1 submanifold $S\subset\Sigma$ of the Cauchy surface (i.e. $S\subset M$ is codimension 2) and cut Σ along S. This defines two submanifolds $S_+, S_-\subset \Sigma$ with boundary $\partial(S_\pm) = \pm S$ which determine Σ by pasting $\Sigma = S_+\sqcup_S S_-$. Analogously to (4.3), we define

$$\omega_0^S\big(A, A'\big) \,=\, \int_{S_+} \big(A\wedge *\mathrm{d}A' - A'\wedge *\mathrm{d}A\big) \quad , \qquad \omega_0^S\big(A^\dagger, C\big) \,=\, (-1)^m \int_{S_+} A^\dagger \wedge C \tag{4.13}$$

by restricting the integrations from Σ to $S_+\subset\Sigma$. The $[+1]$-shifted symplectic structure can then be defined analogously to Definition 4.1 as $\omega_1^S := \omega_0^S\circ \mathrm{d}^\otimes : \mathfrak{S}_\mathrm{c}(M)\otimes\mathfrak{S}_\mathrm{c}(M)\to\mathbb{R}[1]$. By a straightforward calculation using Stokes' theorem, we obtain

$$\omega_1^S\big(A, C\big) \,=\, -\int_S *\mathrm{d}A\wedge C \,=\, -\omega_1^S\big(C, A\big) \quad . \tag{4.14}$$

Note that this matches the codimension 2 $[+1]$-symplectic structure in [CMR14, Section 5.1]. Finally, by a further iteration of our construction in Definition 4.1, one easily shows that the $[+2]$-shifted symplectic structure in codimension 3 is zero. $\triangle$

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