A Probabilistic Modelling Approach for Rational Belief in Meta-Epistemic Contexts

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Abstract

Starting with a thorough discussion of the conceptual embedding in existing schools of thought and literature we develop a framework that aims to be empirically adequate yet scalable to epistemic states where an agent might testify to uncertainly believe a propositional formula based on the acceptance that a propositional formula is possible, called accepted truth. The familiarity of human agents with probability assignments make probabilism particularly appealing as quantitative modelling framework for defeasible reasoning that aspires empirical adequacy for gradual belief expressed as credence functions. We employ the inner measure induced by the probability measure, going back to Halmos, interpreted as estimate for uncertainty. Doing so omits generally requiring direct probability assignments testified as strength of belief and uncertainty by a human agent. We provide a logical setting of the two concepts *uncertain belief* and *accepted truth*, completely relying on the the formal frameworks of 'Reasoning about Probabilities' developed by Fagin, Halpern and Megiddo and the 'Metaepistemic logic MEL' developed by Banerjee and Dubois. The purport of Probabilistic Uncertainty is a framework allowing with a single quantitative concept (an inner measure induced by a probability measure) expressing two epistemological concepts: possibilities as belief simpliciter called accepted truth, and the agents' credence called uncertain belief for a criterion of evaluation, called rationality. The propositions accepted to be possible form the meta-epistemic context(s) in which the agent can reason and testify uncertain belief or suspend judgement.

This work is part of the larger project INTEGRITY. Integrity develops a conceptual frame integrating beliefs with individual (and consensual group) decision making and action based on belief awareness. Comments and criticisms are most welcome via email.

Contents

1 Introduction

When discussing the role of context for human reasoning, we first emphasize that the vast scope of the subject is restricted in this paper to specific perspectives on reasoning and context. We take *human reasoning* to be a process involving limited and subjectively individual intellectual, physical, mental, and emotional components. Hence, in our understanding a long-standing and well-known conflict between reasoning as logical (AI treatable) process and reasoning as a human (holistically treatable) process is present, formulated by Harman (1986, p. 107),

> There is a tendency to identify reasoning with proof or argument in accordance with rules of logic. [...] But the identification is mistaken. Reasoning is not argument or proof. It is a procedure of revising one's beliefs, for changing one's view. (Reasoning also effects one's plans, intentions, desires, hopes, and so forth).

When a holistic understanding of reasoning is adopted, as in our discussion, modelling reasoning processes comes down to modelling beliefs and belief changes.

The perspective on *context* we feel to be most appropriate given the understanding of reasoning is equally holistic and concerns physical and non-physical influences a human reasoner is confronted with. That is, the context within which a reasoner holds, evaluates, and changes beliefs is vested in both physical presence and conceptual stocks. The physical presence concerns the *in sito* reasoning conditions (e.g. hearing music, being in sunshine, smelling tasty food, cf. Liu et al. (2009)) of the human. In Anderson (2006) Anderson successfully argues that human beings have many "discriminable pathways between the world and an agent's beliefs", he calls such a pathway *openness*, and that each epistemically relevant mode of openness to the world (in particular physical intervention) operates according to its own logic and for its own purpose thereby contributing to some element of our overall set of beliefs about the world. In this discussion we focus on openness in a conceptual context and not on physical intervention. The conceptual stock is conceived of as a set of beliefs, acquired consciously or unconsciously over time, for example convictions based on (scientific) information, but also regional, cultural or religious convictions, traditions, prejudices, experience, reflection etc. The conceptual stock is later modelled as a set of propositions at the disposal of a human from which subsets *can* be employed in a given reasoning process.
We hold for this discussion the today well accepted view that reasoning and rationality considered as a *function of* context is too naive a notion, what has already been observed by Putnam (1982):

> What I am saying is that the "standards" accepted by a culture or a subculture, either explicitly or implicitly, cannot define what reason is, even in context, because they presuppose reason (reasonableness) for their interpretation. On the one hand, there is no notion of reasonableness at all without cultures, practices, procedures; on the other hand, the cultures, practices, procedures we inherit are not an algorithm to be slavishly followed.

Indeed it seems appropriate to ascertain that any account of reasoning must incorporate a notion of context in some sense. However, the dualistic view that one determines the other in terms of correctness (when context determines reason) or in terms of rationality (when normative reason determines context) seems to not sufficiently respect the complexity of human reasoning. We think that one cause for the blatant divergence of actual human reasoning (and acting) and an expected reasoning outcomes according to normative accounts of rationality, is the

integrative nature of *human* reasoners. Akin to the processing of a sensation in the brain that requires the whole brain to (passively) partake[1], a reasoning process of a human requires the whole person's internal and external capacities[2] to partake.

It seems when literature is consulted, that the complex nature of integrated reasoning (in the reading of human reasoning) retaliates strict or inflexible accounts and hence is difficult to model, both for formal modelling approaches and conceptual modelling approaches.

Formal modelling approaches that are axiomatically monotone abstract logics in the sense of Tarski (1956a)[3] are criticised - inter alia - for the idealisation of logically omniscient reasoners cf. Stalnaker (1991), potentially leading to unrealistic axiomatic implications. Apart from formalisations of reasoning processes on the highly abstract level of logical symbolic representation, *quantitative accounts* are employed as formal modelling approaches. Quantitative accounts typically stipulate a set of assumptions comparable to the axioms of a logic to ensure certain qualities of functions and relations defined on sets hold, for which the assumptions are stipulated. The idea is that based on these assured qualities mathematical representations of natural phenomena as belief can be modelled and assertions can be proven *in the scope of the model*. For example, gradual belief of an agent can be modelled as a function assigning a probability to a (set of) propositions. In that case the probability assignment has to satisfy specific conditions. If, and only if these conditions are satisfied a measure is a probability measure and assignments legitimately obtain certain properties which are then interpreted in terms of belief. As Dubois et al. observe in Dubois et al. (2004)

> There is an old controversy in the framework of Artificial Intelligence (AI) between probabilistic and other numerical representations of uncertainty on the one hand, and the symbolic setting of logical reasoning methods on the other hand. Namely, the AI tradition maintains that knowledge representation and reasoning should rely on logic Minker (2000), while the probabilistic tradition sticks to numerical representations of belief.

In contrast to logical and quantitative modelling approaches, we take *conceptual modelling approaches* to be theories of reasoning that model reasoning processes in less formal (but equally rigorous) frameworks. Basically the modelling consists of interpreting and relating belief and belief change to different schools of thought or conceptual frameworks using theories of reasoning. In that sense we take conceptual approaches to be a kind of linguistic modelling where natural language terms (words) are accorded a meaning to represent - i.e. to *model* - a concept. The capacity to represent a concept requires a notion of context in which the models are meaningful. We call these natural language contexts "schools of thought" or "conceptual approaches". Examples of conceptual modelling approaches in literature are Epistemic Normativity discussed for example by Pollock (1987) or Kornblith (1993), Epistemic Instrumentalism discussed for example by Foley (1987)), or Naturalism discussed for example by Hooker (1994) and, independently, by Laudan (1990).

Consider naturalistic conceptual modelling approaches. These are typically oriented towards, and inspired by, empirical research, reflecting observable behaviours of human reasoners to derive normative conceptual frameworks. For example Hooker (1994) and Hoffmaster and Hooker (2009) write

> Roughly, reason is a capacity, operating at both individual and collective levels, for initiating processes that replace ignorance with trustworthy information, reactivity and carelessness with systematic judgement, and prejudice and partiality with broad and insightful critical appraisal, and for applying these to rational evaluation processes themselves.

Both, formal (logical or quantitative) and conceptual accounts face various criticisms that can be separated in two generalizing categories: empirical inadequacy and conceptual controversy (or inconsistency) that are of particular relevance if accounts are normative (and not purely descriptive).

For example Hoffmaster and Hooker (2009, p.221) conclude that reasoning processes, what they take them to be, are too diverse and that "Formal tools have a useful role to play in these processes, but only as one resource among several." It seems that formal tools are *per se* considered to be empirically inadequate to model multifaceted human reasoning.

Naturalistic conceptual accounts on their part are often criticised for their indefiniteness or little demanding normative standards. The way often walked to criticise non-formal accounts is an analysis of the conceptual embedding and interpretation, pointing at improper simplifications, conceptual imbalances, or downright contradictions.

For example, Chow (2016, p.3) criticises Hooker (1994) pointing at the insufficiency of internalist reasoning concepts, albeit empirically adequate, to claim normativity. He says:

[1] Even if processing may, depending on the sensation, be predominantly located in one areal.

[2] Even if reasoning may, depending on the proposition in question, be predominantly logical, emotional, intuitive, etc.

[3] E.g. propositional logic cf. Shoenfield (1967) but also modal logic concisely presented by Emerson (1990), and also binary probabilistic logics by Fagin et al. (1990) or Nilsson (1986).

> Hoffmaster and Hooker's analysis [..] considers only explicit conscious experience and fails to consider underlying (generally nonconscious) psychological processes. This reveals a danger for naturalism of mischaracterizing the activities and/or cognitive processes of agents when evaluating their performance, which can cloud the targets of normative analysis.

In our reading, Chow criticises the appointment of individual reasoning observations that include nonconscious phenomena of that individual, to general rules of reasoning that ought to apply to many individuals, as such an appointment may not provide a *normative* setting. Epistemic norms and norms ensuring empirical adequacy are relevant if the modelled theories of reason and rationality aspire to be normative and not purely descriptive. Epistemic norms are, generally speaking, concerned with knowledge, truth, and justification. Let us regard the epistemic norm *Norm of truth* by Joyce (1998).

> An epistemically rational agent must strive to hold a system of full beliefs that strikes the best attainable overall balance between the epistemic good of fully believing truths and the epistemic evil of fully believing falsehoods (where fully believing a truth is better than having no opinion about it, and having no opinion about a falsehood is better than fully believing it).

Scientific interrogation and reasoning generally ought to adhere to the epistemic norm of truth, and from that demand operational standards for empirical investigations are derived to ensure norm compliance, for example by demanding sufficiently large sample sizes, diversity of experimentees ensuring samples are representative for the population, etc. Meeting the relevant standards is crucial if an account claims to be a epistemically trustworthy analysis of reason, as the operational correctness amounts to a *justification* of claims based on empirical analysis. Claims derived from evidence of experimental setups that are designed and operated according to scientific standards are generally assumed to be justified, or at least criticizable or falsifiable, because it is assumed that the evidence is generated *lege artis*. Empirical observation and evidential confirmation are particularly relevant to justify some claims of naturalistic accounts treating observation as a source of unbiased, reliable information. Interpreting the empirical data and supporting accounts of reason with these data then should adhere to the epistemic norm of truth. As Wilholt (2013) clarifies "To invest epistemic trust in someone is to trust her in her capacity as provider of information."

If the norm of truth is not entrenched in a theory of reasoning, in particular in naturalistic accounts, the derived characterisation of cognitive processes is endangered to be a mere generalisation instead of a normative analysis.

The preferential course of action today to meet both normative standards in theory and empirically adequacy in modelling, are accounts incorporating both human limited or insufficient compliance to rules in reasoning and solid axiomatic settings supporting some notion of rationality. Obviously, this comes at the expense of simplicity. The "rigid" frameworks of classical logics, quantitative frameworks, and conceptual *de dicto* theories are "weakened" to account for observable human caprices in actual reasoning processes.

Theories are altered both formally, i.e. by introduction of axioms in formal accounts, and interpretatively, i.e. by a *de re* interpretation of theories of reasoning in conceptual accounts. The result are complex modelling approaches both in formal form and in form of conceptual theories of reasoning.

Formal accounts adapt in particular two aspects to accommodate human reasoning naturalistically in axiomatic frameworks 1) deductive cogency concerning the required strength of rule obedience (e.g. the scope of inferences), and 2) truth conductive consequence relations, leading to so-called paraconsistent logics. Informal accounts incorporate human reasoning naturalistically in particular in 1) interpretative rigour concerning aspects of a concept being (de-)emphasized relative to a "conventional" interpretation 2) conceptual interrelation, where familiarities are analysed and interpretations are related.

Let an example of a naturalistic formal account be the proposed logic of Allo (2016) that divides hard and soft information and relates both to an axiomatic setting in a *diluted logical space*. Or Lavendhomme and Lucas, who *partializes logical omniscience* while keeping a classical logic structure.

Naturalistic quantitative notions model for example "weaker" representations of gradual belief than classical probability[4]. For example ranking theories e.g. Spohn (2012), possibility theory by Zadeh (1979), or axiomatically altered notions of probability, so-called Popper-Renyi-measures, can be regarded as less stringent quantitative modelling frameworks for belief. Considering the credence assignments of an agent conditional on the context is proposed in the following sections, what amounts to a Popper-Renyi-measure where conditional probabilities are the primitive notion of probability, e.g. van Fraassen (1976). The algebraically restrictive properties of probability

[4] As is done in Probabilistic Uncertainty to incorporate and emphasize relative relevance and context.

measures (in particular the additivity property[5] may be considered unsuitable to model human belief and belief changes empirically adequate. Possibility measures (and the concomitant necessity measures) provide a less restrictive framework, as Dubois et al. (1993) clarify: "The possibilistic representation is weaker because it explicitly handles imprecision (e.g. incomplete knowledge) and because possibility measures are based on an ordering structure rather than an additive one."

The examples for complex conceptual modelling approaches developing theories of reasoning and rational belief appealing to various schools of thought are vast. We highlight but one particularly appealing combination of schools of thought, Elqayam (2012)'s Grounded Rationality, which is based on the related principles of descriptivism and (moderate) epistemic relativism.

The in the following presented proposal too is characterised by a degree of complexity. We call the approach we put up for discussion and criticisms *Probabilistic Uncertainty* and it does exactly what it says: it probabilistically models the rational uncertainty of a human agent in her/his belief in a given context, where a context is a set of propositions considered to be possible by that agent, and the evaluation is relative to an evaluation criterion.

The key concepts we employ are *uncertain belief* and *accepted truth*. Uncertain belief is a concept which quantitatively models gradual belief in terms of probabilities. Accepted truth is a concept which models belief simpliciter in terms of possibilities. A context is a sovereignly chosen set of propositions accepted to be possible (accepted truths), forming an epistemic state of the agent modelled quantitatively in a logical framework with a semantic interpretation. Rationality is defined in the scope of a context reflecting the context propositions' capacity to represent a rationality criterion according to the agents judgement.

The rest of the paper is structured as follows. In the next sections three perspectives on Probabilistic Uncertainty are discussed, providing first a conceptual frame for the two concepts of belief employed, secondly the numerical modelling of these belief concepts, and thirdly the logical modelling of these belief concepts.

The *conceptual framework* concerning accepted truth and uncertain belief ought to facilitate an unequivocal understanding of the intended meaning of the concepts, using their embedding in literature and different schools of thought, to formulate a notion of rationality. In particular the grounding in the internalism/externalism debate is discussed. Based on these considerations the adequate *conceptual paradigm* for belief modelling with Probabilistic Uncertainty is developed.

In section 3 the *quantitative framework* is presented, which is basic probability theory. In particular, considering accepted truth as a notion of possibility boils down to a binary probability assignment for any context within which the agent reasons, called the epistemic state, assuring non-negativity and hence allowing for a classical Kolmogorovian axiomatisation of uncertain beliefs determined within a context.

In section 4 the *logical frameworks* developed by Banerjee and Dubois (2014) for accepted truth and the framework developed by Fagin, Halpern, and Megiddo (in Fagin et al. (1990) and Fagin and Halpern (1991)) are discussed as frameworks formally reflecting the concepts and accommodating the probabilistic characterisation of uncertain belief in the modal operators for accepted truth $\mathcal{T}$ and uncertain belief $B\sim$.

Finally, in section 5 we discuss some expected consequences the modelling process of Probabilistic Uncertainty might deliver for the testifying agent and the modeller.

2 The Conceptual Perspective

Probabilistic Uncertainty aims to reflect human reasoning processes in an empirically adequate manner. In this sense it can be regarded as a naturalistic approach, or at least, with aspirations towards a naturalistic philosophy of reason. However, Probabilistic Uncertainty is not solely a descriptive theory. In particular, we consider the notion of rationality to be normative. On that note the epistemic norm of truth is fundamental to our definition of rational belief. However, we interpret *truth* to be a concept immanently holding both external information and internal information of a human agent.

Naturalistic accounts concentrating on empirical adequacy generally run the risk of emphasizing single case correspondence at the expense of normative austerity, so that any approach (conceptual, logical, or quantitative) must pay particular attention to balancing human individuality and modelling idealisation. The basis for the evaluation of human reasoning processes is, generally, compliance with a given set of assumptions formulated as premises, axioms, or definitions, according to the theory of reason in question. Typically, a reasoning process and

[5] The additivity property is often blamed to demand inferences in sharp contrast to empirical observation. The problem is illustrated by the Lottery Paradox and the Elsberg Paradox.

its result is called *rational* provided it conforms to the axiomatic standards. A necessary condition for rationality evaluation altogether is hence the possibility of communication and inter-subjective understanding.

2.1 Internalsim and Externalism

If a reasoning process calls upon human facilities that cannot be communicated, the compliance with axioms cannot be analysed by another agent or modeller. The inter-subjective prerequisite restricts the assignment of the "quality predicate" *rational* and potentially discriminates parts of human reasoning that are accessible exclusively *within* a human. In literature these reasoning structures and sources of information are referred to as *internal*, and the corresponding view is called *internalism*. We take internalism to hold that a human can attain insight by an information generating process that is strictly independent of external influences, for example reflection, sensing, understanding, enlightenment, change of perspective etc.

Reasoning processes that are strictly independent of external influences may or may not be enunciated by a reasoner. If they are not (or *cannot* be formulated in an inter-subjectively understandable way) a general suspicion of irrationality seems to accompany beliefs held on that basis. The prejudice of potential irrationality can be exemplified with subjective probability assignments, discussed in the following section.

Compliance with the epistemic norm "approximate the truth[6] for gradual belief" is facilitated by a notion of rationality respectful of both subjective conviction arising from strictly internal reasoning processes and subjective conviction arising from factual truth and evidence external to the reasoner. To relate the following discussion of the basic concepts *truth, awareness, and rationality* to the consequent modelling, we briefly present the core constituents of the approach.

In Probabilistic Uncertainty an agent has to choose propositions deemed possible, called the accepted truths, forming the context for probability assignments. We model accepted truth as belief simpliciter and gradual belief as probability and uncertainty assignments. We call the conscious choice of a context *intention*. An agent is expected to balance her/his uncertain belief arising from internal information, called internal subjective belief, and external information called external subjective belief. The quantitative modelling amounts to a probabilistic credence function reflecting radical subjective probability and objective probability if the agent's intention is non-extreme. The agent balances her/his uncertain belief based on an understanding of her/his individual extremes what reflects the reasoning process itself.

The two extremes are (1) an agent's intention to evaluate uncertain belief in purely internal subjective terms, then, accepted truths will be chosen accordingly, allowing for credence assessments arising from internal reasoning processes[7]. Or, (2) an agent's intention to evaluate uncertain belief in purely external subjective terms, then, accepted truths will be chosen accordingly allowing for objective probability assessments[8]. Both extremes satisfy the epistemic norm of truth approximation, the former in terms of subjective truth (conviction), the latter in form of inter-subjective truth (observable fact). A less extreme intention introduces a mixture of inter-subjective truth and subjective conviction such that the uncertain belief expressed as probability assignment by an agent is an individually balanced credence function for given accepted possibilities.

2.2 Truth

We considerer truth to be a quality of both the world *and* the agent evaluating the world. Evaluating the "outside" world, i.e. the inter-subjectively verifiable world, demands for external reasoning processes, while evaluating the 'inside" world, i.e. the strictly internal reality of the agent, demands for internal reasoning processes. Both have to enter in the modelling of reasoning processes. We employ two concepts of truth: on the one hand in a Tarsikan sense of correspondence for the formulation of the formal modelling in terms propositions. On the other hand, truth is understood as the aptitude of such propositions to represent a concept *for the agent in question*. The representation theory of truth can, in a sense, mirror an agent's individual understanding of the meta language when probabilities are assigned. The epistemic norm of truth applies in our understanding to both a correct correspondence (externalism) and a sincere representation of individual understanding (internalism).

The evaluation of the inter-subjective aspects of truth in the light of the correspondence theory of truth by Tarski (1944)[9] evolves around *Convention T* for the T-scheme cf. Lynch (2001), discussed critically for example by

[6]"An epistemically rational agent must evaluate partial beliefs on the basis of their gradational accuracy, and she must strive to hold a system of partial beliefs that, in her best judgement, is likely to have an overall level of gradational accuracy at least as high as that of any alternative system she might adopt." Joyce (1998)

[7]The "radical subjective" interpretation of probabilism in the spirit of de Finetti, as rational degrees of belief can accommodate such an understanding.

[8]The frequentist, classical, and propensity interpretations of probabilism can accommodate such an understanding.

[9]And Tarski (1956b).

Patterson (2002) or Davidson (1973). Roughly sketched, Tarski's T-scheme uses a sentence in an object language, say, p: 'snow is white', and defines truth using a *meta language* to which p corresponds, say X. The object language is "the language for which we are giving a theory of truth" defined as *material adequacy*, such that p is true iff snow is white, while the meta language is used to state the theory, such that X is true iff p. The introduction of a meta language responds to the problem of the liar paradox, as truth is not decidable on the same "level", i.e. within an object language, for sentences like "I am lying". The correspondence evaluation of p's truth is necessarily inter-subjective, such that a number of agents could decide p in the meta-language because p corresponds to something multiple agents *can* evaluate in a meta-language. This requirement is called formal correctness by Tarski.

On the other hand, truth in an object language, say English, defined as material adequacy may not always be obvious. Whether a sentence in the object language *is* materially adequate for an agent, is a highly individual, non-standardised process involving subjective understanding, interpretation and meaning. We consider the capacity of a term (or sentence) in an object language to materially adequately represent the understanding of an individual agent to be gradual. In *model theory* the term *interpretation* is generally taken to be a bit of information making a statement true or false. Therefore, we employ the term *representation* to explicitly denote the not necessarily "correspondence-truth" relevant information added or lost when a human agent testifies on a proposition, which is implicit in the agent's understanding of the proposition. We consider the statement that an agent is uncertain that a proposition is true in the object language not in the logical sense of truth but rather as a kind of *appropriateness* of the proposition depending on the agent's understanding.

The semantic representation theory of truth developed by Kamp (2013, p.190) (emphasis by Kamp), *in a sense* captures the spirit of truth qua representation we think of:

> A sentence S, or discourse D, with representation m is true in a model M if and only if M is compatible with m; and compatibility of M with m can be defined as the existence of a proper embedding of m into M, where a *proper embedding* is a map from the universe of m into that of M which, roughly speaking preserves all the properties and relations which m specifies of the elements of its domain.

In that vein, a person evaluating p would consider p to be a representation, just in case it conforms to the agents' individual understanding of the concept(s) p refers to. In our interpretation, the proper embedding of representation m in a model M is present if the map of m to M indeed preserves *all* the properties and relations an agent individually[10] holds to exist. However, a representation can hold but *some* of the properties and relations an agent individually accredits. A "degraded" representation from the agent's point of view, is a representation m in M that does not preserve *all* properties and relations the agent considers relevant to call m in M embedded properly. We model a degraded representation using a quantitative assessment, the probability assignment, what allows the agent to rate the properties and relations missing in the representation of m in M, individually according to the agent's judgement of relevance. We map m to M using a probability assignment where a probability assignment of 1 corresponds to a complete representation and a probability assignment of 0 corresponds to complete mismatch of properties and/or relations the agent considers relevant to call m in M embedded properly. In the modelling we evaluate with uncertain belief the agent's judgement to what extent a sentence we transferred to a different context via syntactic consequence, still corresponds to the *same* sentence in the object language for the agent, in a sense we ask the agent if the sentence "feels" equally materially adequate in her/his understanding[11].

Let us provide an example to illustrate what we mean by internal reasoning process, external reasoning process, correspondence, and representation. Let p denote 'I have a mother'. Now, obviously this sentence is true for *every* human, by biological necessity. The inter-subjective part of p's material adequacy is evaluated for the term "mother", defined as a person who gives birth to a child, what is inter-subjectively verifiable.

However, we do not all have the *same* mother, such that the term "mother" is multifaceted and *means* different things to different people. The extent to which an individual person accords truth to p reflects that persons' internal truth that p *represents* her/his understanding of the concept "mother" in the context of that sentence. The extent to which p is true by virtue of representation is determined by the agent's individuality as it is the basis for the concept's properties and/or relations the agent considers relevant. The degree to which (a term figuring in) a sentence, represents an agent's understanding of the concept it denotes, depends on a persons' understanding which is influenced by, for example, historical experiences, education, upbringing, desires, social environment, physical environment, and political, religious, philosophical convictions.

Maybe an individual agent was adopted and feels inclined to say "I have *two* mothers. A biological and a social mother". Maybe the mother of an agent has deceased and the agent feels inclined to say "I don't have a mother any more".

[10] We emphasize that *individual* refers to an agent's specific situation taking into account historic experience, internal reasoning structures, physical sensations, mental relations, etc.

[11] So we could say that Probabilist Uncertainty is *not* a ceteris paribus modelling approach.

The meaning of the concept "mother" by an agent is individual and is evaluated by criteria specific to a person, e.g. being present and caring, being alive. However, the individual understanding is not generally equivalent to (gradual) *disbelief* in p. If the agent is uncertain that the representation of "mother" in p captures her/his individual understanding of the concept we say that she/he holds uncertain belief in p that is, in her/his understanding p is a degraded representation, though not necessarily false. That is a very basic example to illustrate what we take representation to hold in the sense of individual meaning for the agent. In the modelling process the agent is provided an evaluation criterion, called rationality, for which p is assessed. We are interested how strong the agent believes p represents the evaluation criterion - whatever the agents' understanding of p and the evaluation criterion may be. In the modelling process we are not interested in the agents the individual context, her/his biography or why a specific understanding arose. Rather, we are interested in the modelling context: how the agent's understanding of a proposition is related to other propositions, and to the evaluation criterion. Uncertainty is defined accordingly in definition 2. The subjective probability assigned to p models a degraded representation of the evaluation criterion by $P(p) < 1$ making the uncertainty and strength of belief quantitatively explicit. Uncertain belief, expressed quantitatively as probability and uncertainty is denoted by $B \sim p$ in the logical framework. To sum up, assigning a probability to a sentence in an object language reflects the capacity of that sentence to represent truth *individually for the assigning agent*. That which is uncertainly believed can be gradually represented in the object language, reflecting how an agent understands a concept in terms of relations and properties.

2.3 The Awareness Paradigm

In our view, the purpose of beliefs is not primarily the correct reflection of an inter-subjectively observable reality. The beliefs we model are conceived as representation of an agent's individual attitude expressed vis-à-vis a concept reflecting the agent's acceptance, commitment and "identification", if you will. An agent's choice to accept a possibility and her/his assignment of her/his uncertain belief in that possibility are inseparable from internal information structures. The attitude is modelled in the two notions of truth: correspondence for accepted possibility and representation for uncertain belief.

In contrast, the gradational norm of truth seems to be predominantly understood as a kind of expectation based on factual correct probability calculus. Probabilism and countable additivity are epistemic norms for credence functions, i.e. functions describing the strength of belief. The assignment of a probabilistic credence function based on objective evidence reflects awareness of frequencies in a calibrated sense.

The Principal Principle by Lewis published in Carnap and Jeffrey (1980), says that a rational agent conforms their credences to the chances: "the chance distribution at a time and a world comes from any reasonable initial credence function by conditionalising on the complete history of the world up to the time, together with the complete theory of chance for the world." Mayo-Wilson and Wheeler (2016) call such a view "accuracy-first" epistemology. Rationality, if defined as credences conform to the chances, is a function of a world's history up to the time and the complete theory of chance for the world.

Rationality would become a function of resource availability. Clearly such an understanding facilitates all sorts of discrimination. A blind person does not dispose of visual "instruments" to assess aspects of the complete history for conditionalisation. Poor people are not likely to dispose of technical instruments to assess aspects of complete history for conditionalisation. A person who lacks schooling is not likely to have access to the complete theory of chance for the world. Understood as such, modelling *beliefs* is a decision problem: a belief based on a *more* complete history of a world is more rational and the agent who wants to believe rationally should conform her/his credence. But what is the (part of a) world's complete history to conditionalise on? What is the complete theory of chance for that world to determine one's credence?

Rational credence understood as chance gives advantage to the agent who disposes of the best instruments to test factual correct history and a complete theory of chance. The agent endowed with these prerequisites holds the "right" beliefs and believes *rational* after conditionalisation for a given theory of chance. That in turn bears on the blind or poor person's aptitude to conform to the norm of truth as it impacts the possibility to assess material adequacy. The reason why (today) a blind person is typically not considered irrational qua physical condition, is that the person can assess factual truth by 'other means' than visual stimuli-processing. The reason why *today* a poor person is typically not considered irrational qua economic condition is that the person *could* assess factual truth if only she/he were given the technical instruments. What 'other means' and what technical instruments an agent relies on, or *what sources of external information she/he trusts* to generate the chances to which the credence ought to be conform, is a *decision problem*.

We propose that for belief in our understanding the epistemic norm of truth should demand for both factual correctness and individual understanding. In other words, belief is not a decision problem.

Belief is an awareness problem.

In our modelling approach an agent *uses* beliefs to become aware of (1) her/his accepted propositions, (2) her/his uncertain belief in accepted propositions, and (3) her/his interpretation of a proposition w.r.t a "rationality". Awareness rather than correctness is desirable and normative when beliefs in our understanding are evaluated. The question shifts from 'whom should I trust to hold the "right" (rational) beliefs', to "whom do I (rationally) trust, *including myself*". When answering the latter question the agent investigates what information is trusted reflecting the agent's acceptance, commitment and "identification". Non-judgemental awareness fosters recognition of all beliefs held by the agent, not just the "correct" beliefs.

An overemphasis of inter-subjectively verifiable aspects of belief may be disadvantageous for agents complying to the epistemic norm of truth. Associating rationality with a world's history and chance theory may severely mismatch the true convictions of an agent, potentially leading to neglect and a biased self understanding what we consider to be a form of self-deception. A practical consequence we anticipate if "chance rationality" is overemphasized in the norm of truth, is that the "designed" human, the normatively correct agent, is potentially at odds with what the "true" human actually believes, considers reasonable, and feels. Therefore, we do not interpret the norm of truth solely in terms of chance rationality but in the paradigm of awareness.

2.4 Rationality in the Awareness Paradigm

A proposition p modelled in a meta-language is said to be true if it (1) corresponds to some concept in a object language, i.e. an inter-subjective language e.g. English, and (2) can bear (gradual) truth of being a representation for the human evaluating her/his belief. Let the representation p can bear be called *rationality*. In other words, the proposition p *represents* for the non-ideal agent an instance of a rationality.

In contrast to interpretations of probability as the Frequentist account, the classical account, and the propensity account, the probabilities we instrumentalize to model gradual beliefs in the paradigm of awareness need not necessarily reflect occurrence. Different rationalities, i.e. evaluation criteria, apart from *occurrence rationality* are natural to the human agent, for example *economic rationality* (e.g. how strong does p represent least cost?), *practical rationality* (e.g. how strong does p represent least effort?), *moral rationality* (e.g. how strong does p represent goodness?), *consensual rationality* (e.g. how strong does p represent expert consent?), *instrumental rationality* (e.g. how strong does p represent effectiveness?), *aesthetic rationality* (e.g. how strong does p represent beauty?), etc. Uncertain belief in a proposition p is interpreted as the non-ideal agent's evaluation of p's capacity to represent a rationality in a given context, where the context is determined by propositions accepted to be true forming the set $\mathbf{T}$. Accepting a proposition means that the agent asserts she/he considers that it is *possible* for p to represent the evaluation criterion. In order for p to gradually represent something, it must be *possible* for p to represent that something.

We entertain the view that a given set of accepted truths *necessitates* a unique strength of uncertain belief in a proposition accepted, specific to an agent and specific to a belief evaluation. We contend that an individual agent *knows* that strength of uncertain belief for an intentional set of accepted truths if the agent calls on both internal and external information resources[12]. That tacit knowledge is understood as hard knowledge in the sense of Aumann (1999), as irrevocable S5 knowledge for *a given belief evaluation situation*. Given these assumptions we say that an agent believes rationally if her/his strength of belief does not differ from her/his assigned subjective probability. Note, that subjective probability is understood as credence based on internal information and/or external information.

Definition 1. *(Rational Belief)*
It is rational to uncertainly believe, denoted $B\sim$, a proposition p exactly *with the probability $P(p)$ and uncertainty $\Psi(p)$ the proposition has for a non-ideal agent given the set of propositions considered possible, called accepted truths, for a rationality in a single belief evaluation.*

In other words, it is rational mischief to believe with a different strength or be uncertain with a different strength than testified in the probability assignment. This is applicable for both "grounds" of subjective probability, internal and external information. Moreover, believing rational is about the *most adequate context* for the probability assignment reflecting the intention of the non-ideal agent. To be clear, our terminology of *uncertain* belief does not refer to an agents insecurity, indecisiveness, or some psychological restraint. It models the agents awareness that the object language and meta-language can be understood in different ways as these languages may support different relations and properties resulting in different meanings. Conforming convictions to an inter-subjective language

[12]Calling on both internal and external information amounts to investigating the extreme intentions.

introduces uncertainty in that sense, what is captured by our notion of context. Context is modelled as accepted possibilities, serving the purpose of becoming aware of all accepted propositions and their relative strength in terms of uncertain belief.

Answering the question what a *rational* agent should believe on the *object level* in a belief modelling framework does not make sense for an account aspiring empirical adequacy. The object level is the (uncertain) belief of an agent in a proposition p modelled. In other words, rationality on the object level is concerned with distinguishing p based on qualities p may have, reflecting material adequacy. The predicate "rational" is then conferred to agents, and only those agents, who happen to decide to believe the "right" p, and with the right strength, from the view of chance theory.

Our notion of rationality is vested in the meta-level. The question we pose is what a *rational* agent *does* believe given the possibilities accepted. In general we consider the agent to *testify sincerely*. In other words, we say: whatever the propositions the agent accepts are, and whatever the strength of belief in these propositions is, based on what the agent takes them to mean, if the credence function we model (the probability assignment) reflects the strength of belief of the agent, the agent is considered to be rational.

In the paradigm that belief is an awareness problem, rather than a decision problem, evaluating one's beliefs demands for a thorough investigation of accepted possibilities and their gradual representation capacity by the human agent, potentially laying bare implicit beliefs, entrenched beliefs, and contextual beliefs that remained unconscious prior to the belief evaluation.

The epistemic value of increased awareness becomes manifest whenever an agent learns something about herself/himself - gets to know herself/himself better - due to a belief evaluation. In order to realize the epistemic value of a belief evaluation, the agent ought to aspire holding rational beliefs, as rational beliefs are likely to foster awareness in an internal and inter-subjective sense. If an agent does not seek to hold rational beliefs according to definition 1, three forms of rational misconduct can be identified.

2.5 The good, the bad, and the ugly

Rational misconduct of an agent can occur in three forms: wishful believing (the good), doubtful believing (the bad), and denial (the ugly). In these three cases the agent biases her/his belief. Intentional and unintentional forms of misconduct are possible.

Intentional misconduct occurs when the agent introduces accepted truths to *purposefully* change, that is to bias, her/his probability and uncertainty of a proposition. Unintentional misconduct occurs when the agent believes with a different strength or uncertainty than her/his probability assignment expresses due to a disparity in the perception of belief and probability, for example an over- or underestimation.

Wishful believing is belief biased towards a "positive" probability and uncertainty, from the agent's point of view. The agent, for some reason, desires a proposition to be more/less probable than she actually considers it to be. The agent *wants* to belief a proposition stronger (less strong) than she/he actually can, for a given set of accepted truths.

In the case of doubtful believing, for some reason, the agent wants to be more uncertain of, a proposition than she/he can, a given set of accepted truths.

Denial, (not facing the ugly truth) can also occur intentionally and unintentionally. An agent commits unintentional denial if beliefs influence an agents' belief evaluation that remain *unconscious*, for example in form of implicit beliefs, prejudices, etc. Intentional denial is the purposeful exclusion of propositions whose relevance the agent is aware of from the accepted truths of a belief evaluation.

We do not care too much about intentional rational mischief. The intentional introduction, denial or neglect of relevant propositions is known to the agent and biases are designed purposefully. We do not judge intentional rational mischief and respect an agents choice to do so in the modelling. Many are the reasons we can think of for such "irrational" behaviour. For example agents who have undergone trauma may know to be unable to emotionally cope without biases. Or agents who know to be jeopardized by depression without positive biases. Intentional rational mischief is not considered problematic because the agent *knows* that biases are at play, and we assume that these biases are created for a reason. The most obvious reason we assume is that these biases allow the agent to function in a practical way, that is in a social environment, and hence are actually not irrational. Their rationality is simply not covered by our definition of rationality. It is covered by social behaviourism. We assume that an agent will stop performing intentional mischief when the time has come.

Studies in cognitive psychology and psychological behaviourism underpin the existence of structural biases we call rational mischief. The *availability heuristic* is, roughly, the phenomenon that familiarity with an interpretation or an outcome in a decision situation increases the subjective probability an agent assigns. Tversky and Kahneman (1973, p.230) write:

> Continued preoccupation with an outcome may increase its availability, and hence its perceived likelihood. People are preoccupied with highly desirable outcomes, such as winning, or highly undesirable outcomes, such as an air plane crash. Consequently availability provides a mechanism by which occurrences of extreme utility (or disutility) may appear more likely than they actually are.

In the case of subjective belief generated by purely internal reasoning processes, it is not possible to "proof" or criticise someone else performs intentional or unintentional misconduct. If the subjective probability assignments are based on both internal and external reasoning processes, it may - to a limited extent - be possible to point at another person's rational mischief.

From a conceptual perspective judgement and/or denunciation of rational misconduct is not our primary concern. The epistemic norm of truth is expected to guide a human agent to the awareness that she/he commits intentional and/or unintentional rational misconduct in repeated belief evaluation situations. Here we do not necessarily think of repeated belief evaluations for the same set of accepted truths, say T_a, giving rise to rational mischief. It may be easier or necessary for an agent to become aware of implicit assumptions or structural biases relevant for context T_a in a different reasoning context T_b. However, eventually, if the agent complies to the epistemic norm of truth, subjective belief based on both purely internal reasoning process and reasoning process involving external verifiable information will guide the agent to rational, i.e. unbiased, beliefs.

The framework provides a setting that *allows and facilitates* the evaluation of rational belief *by the agent*. If the awareness paradigm is accepted the agent will seek to correct unintentional rational mischief. Indeed, as will become clear in later sections, formally, the agent's beliefs are modelled *as testified* reflecting intentional and unintentional mischief in Probabilistic Uncertainty.

Defining rationality as the sincere truthful assignment of individual credences reflects a non-discriminative, respectful attitude in a belief evaluation. Frameworks which distinguish between rational and irrational *beliefs* potentially neglect in their *modelling* beliefs disobedient with criteria. While this may be appropriate for strictly normative modelling accounts, for Probabilistic Uncertainty it is inappropriate as we aspire empirical adequacy.

In particular, an agent may be unaware of rational mischief. Misconduct need not necessarily be purposeful by an agent. The agent may simply not know that relevant propositions are not included in the set of accepted propositions, or an included proposition can be more/less probable by the light of evidence. Generally, we do not call any belief irrational in the awareness paradigm, as any probability assignment reflects the agent's individual internal and external information.

The interplay between subjective beliefs based on internal information and subjective beliefs based on objective evidence is not considered to be a matter of one dominating the other. Rather, the agent allows one to relativize the other, according to the agent's (more or less extreme) intention introducing contextual possibilities. That is, an observation classified as evidence does not *per se* have the power to change the aspects of subjective belief that originate in internal reasoning structures of an agent - safe upon choice.

Equivalently, we do not consider internal subjective belief to have the power to change evidence. In contrast to many rational belief theories we assert that *rational belief* is insufficiently characterised by confirmation, in the sense of objective evidence. In particular, we do not consider "correcting" internal subjective beliefs to match empirical evidence as rational action. In the awareness paradigm our aspiration is the presentation and modelling of testified beliefs by a rational agent, not the discrimination, denunciation, or disdain of what the agent beliefs on the object level.

The above mentioned prejudice (or suspicion) that subjective beliefs originating in internal reasoning processes of an gent are potentially irrational is a major criticism of subjective probability. The attitude towards probabilities that are assigned based predominantly on internal subjective judgement, called radical subjectivism, is formulated in literature. For example Hájek (2007, p. 577) writes

> The radical subjectivist's epistemology is so spectacularly permissive that it sanctions opinions that we would normally call ridiculous. For example, you may with no insult to rationality assign probability 0.999 to George Bush turning into a prairie dog, provided that you assign 0.001 to this not being the case (and that your other assignments also obey the probability calculus). And you are no more or less worthy of praise if you assign it 0,12 or 0.17485, or 1/e or whatever you like. Your probability assignments can be completely at odds with the way the world is, and thus are 'guides' in name only.

Radical subjectivism indeed conflicts with rationality *if* we want probability assignments to reflect expectancy of what might happen, "a guide to life" as Hájek's formulates it. The criticism is sometimes adopted in probabilistic belief modelling. Basically the argument is that if a belief is rational it ought not be at odds with the "real world". But beliefs are intrinsically subjective. Associating gradual belief with probability (and uncertainty) necessitates a

radically subjective dimension if we want to express *beliefs* and not mere observational facts. An agent modelled in Probabilistic Uncertainty is not considered to be a "device", learning and adjusting beliefs to chance with the aim to ever better conform belief to observable reality. *Expressing* beliefs as probabilities is fundamentally distinct from expressing objective reality as probability distribution. The former employs probability theory to give form to beliefs (in a quantitative setting), the latter employs observation to give form to a probability distribution what is a quantitative representation of the observations *per se*.

If beliefs are "designed" - both in modelling and understanding - such that they represent a least-error duplicate of observable reality we do not exploit the full potential of beliefs and fail using them for their very purpose. The purpose of beliefs, in our view, is to be the bridging (or combining) concept between observable facts and their interpretation by an agent.

In Probabilistic Uncertainty we do not focus on the conviction-observability conflict because the quantitative probability setting is *employed* to model *beliefs* in contrast to using probability as 'guide' to life denoting occurrence expectancy. The focus is reflecting internal reasoning processes and external reasoning processes in an (individually) balanced manner for a human agent. The underlying normative doctrine is that - even if an individual balances her/his internal/external reasoning to one extreme - the subjective probability assigned is adequate to meet the norm of truth in the spirit of belief awareness.

It is our opinion that a non-judgemental attitude in the modelling of an agent's beliefs fosters rational belief understood as sincere truthful credence assignment, because in that understanding rationality demands the *recognition* of all involved beliefs. Only if context beliefs are laid bare and scrutinized for internal and external truthfulness, completeness, and (un-)intentional biases by the agent, their effect on uncertain belief can be acknowledged by an agent. We think that awareness of the agent regarding the specific, individual context beliefs held - which may be unconscious, neglected, discriminated by the agent - empowers the agent to *consciously* accept or refuse them. This in turn allows the agent to make sovereign, independent context choices reducing unintentional and intentional rational mischief.

3 The Quantitative Perspective

We emphasize that we take subjective probabilities to be the result of internal reasoning processes *and/or* external reasoning processes *and/or* both. Indeed we are convinced that a probability assignment, called a *belief evaluation*, is a reasoning process that reflects a unique, instantaneous, adequate mixture of internal/external aspects of the human reasoner. Every belief evaluation *demands* for a specific share of internal/external reasoning aspects for a specific person in a specific context reflecting the agent's intention. The probability and uncertainty we formally define now is to be understood as subjective probabilities resulting from a human agents' (individually adequately balanced) internal and external reasoning processes.

Definition 2. *(Uncertainty and Probability Measure)*
Let $\mathcal{V}$ be a measurable space, let $\mathcal{F}$ be a σ-algebra[13] *of $\mathcal{V}$. Let an event*[14] *$w \in \mathcal{F}$. A probability measure $P(\cdot)$ is a finitely additive measure on $\mathcal{F}$, such that P: $\mathcal{F} \to [0,1]$ assigns a real number $P(w)$ to every set w_i in $\mathcal{F}$, satisfying the properties (1) $P(w) \geq 0$ for all $w \in \mathcal{F}$, (2) $P(\mathcal{V}) = 1$, (3) $P(\bigcup_{i=1}^{n} w_i) = \sum_{i=1}^{n} P(w_i)$ for disjoint sets $w_i \in \mathcal{F}$. From these axioms one can deduce $P(\emptyset) = 0$ by taking $w = \mathcal{V}$ and monotonicity, that is, if $w_1 \subseteq w_2$ then $P(w_1) \leq P(w_2)$. Let the uncertainty Ψ of an event w, denoted by $\Psi(w)$, be assigned concomitantly as the probability of the complement event(s) of w. In other words, the uncertainty of any (collection of) events in a finite sample space can be computed by subtracting the events' probability from the sample space probability, which is by definition 1, that is $\Psi(w) = 1 - P(w) = P(w^C)$.*

With this definition we formally capture that uncertainty is *not* a notion for insecurity of a non-ideal agent for accepted truths and uncertain belief. In other words, the agent is considered being capable of assigning probabilities. For accepted truth the binary probability assignment 1 and uncertainty assignment 0 are the formal expression of accepting the proposition as true possibility. Uncertain belief is expressed as quantitative notion on $[0, 1]$ for the gradual strength of the agent's belief. Such a credence function must respect that an agent may not want to testify a probability assignment to *every* proposition. We follow in for the modelling of propositions the agent suspends judgement on the framework developed by Fagin and Halpern (1991) [henceforth FHM] where a proposition p (and

[13] A collection $\mathcal{F}$ of subsets of $\mathcal{V}$ is said to be a σ-algebra $\mathcal{F}$ of subsets of $\mathcal{V}$ provided that (1) $\emptyset \in \mathcal{F}$, (2) $\mathcal{F}$ is closed under finite intersections $\bigcap_{i=1}^{n} w_i \in \mathcal{F}$ for countable collections of w_i, and (3) if $w \in \mathcal{F}$ then its complement $w^C \in \mathcal{F}$, i.e. $\mathcal{F}$ is closed under complementation and countable unions.

[14] As will become clear we are concerned with propositional valuations w, but for simplicity we denote the "objects" of the probability assignment here as events.

a propositional formula) that is not assigned a probability and uncertainty, is modelled as a *nonmeasurable set*. To provide lower and upper bounds on the degree of belief in a nonmeasurable set, the subsets in which the proposition figures are consulted where a probability *is* assigned, i.e. where the agent testified uncertain belief. To do so, the probability measure P is extended to the power set of $\mathcal{V}$, the set of all subsets, and the *inner and the outer measure induced by the probability measure* is defined, $P_\bigstar(\cdot)$ and $P^\bigstar(\cdot)$ for all events w. The original formulation of inner and outer measures was brought forward by Halmos in 1950, the following definition is adapted from FHM and Haenni et al. (2011).

Definition 3. *Let $\mathcal{V}$ be a measurable space, let $\mathcal{F}$ be a σ-algebra and $P(\cdot)$ be a probability measure as defined in definition 2. For the probability space ($\mathcal{V}$, $\mathcal{F}$, P) the probability measure $P(\cdot)$ can be extended to $2^{\mathcal{V}}$ by defining the functions $P_\bigstar(\cdot)$ and $P^\bigstar(\cdot)$, called the inner and outer measure induced by P for an arbitrary subset $w \subseteq \mathcal{V}$ as follows*

$$P_\bigstar(w) = sup\{P(p)|p \subseteq w \text{ and } p \in \mathcal{F}\}$$

$$P^\bigstar(w) = inf\{P(p)|p \supseteq w \text{ and } p \in \mathcal{F}\}$$

where sup denotes the least upper bound and inf denotes the greatest lower bound.

A nonmeasurable set can, by definition 3, be assigned an interval-valued probability by the inner and outer measure. The inner measure is the smallest measurable subset of $2^{\mathcal{V}}$ that contains the nonmeasurable event, and the outer measure can be seen as the largest measurable subset of $2^{\mathcal{V}}$ containing the nonmeasurable event. If w is a *measurable set* then $P_\bigstar(w) = P^\bigstar(w) = P(w)$. If there are no nonempty measurable sets contained in w, i.e. no p exist such that $p \subseteq w$, then $P_\bigstar(w) = 0$. Equivalently, if there are no nonempty sets containing w, i.e. no p exist such that $p \supseteq w$, then the whole sample space still contains w such that $P^\bigstar(w) = 1$.

Criticisms have been formulated emphasizing that the assignment of point probabilities, also called precise probabilities, are in a sense problematic both concerning their meaning when used as credence function and concerning their legitimacy regarding uncertain (background) assumptions, cf. Weichselberger and Pöhlmann (1990, p.2). Using interval probabilities to represent credences of agents has become popular in belief modelling based on the pioneering work of - inter alia - Strassen and Huber (1981), Walley (1991), and Dempster. The in our view appropriate axiomatic frame to model imprecise probabilities in our approach has been developed by Weichselberger and Pöhlmann (1990); Weichselberger (1996, 2000); Weichselberger and Augustin (2003) and Augustin and Hable (2010); Augustin and Coolen (2004). We consider the framework as particularly suitable because the scope of the theory fits our intentions. In particular the theory accommodates the concept of ambiguity, belief functions and related concepts, an interpretation of interval-estimates in classical theory, and is not bound to a certain interpretation of probability cf. Weichselberger (2000, p.194). Especially the last quality is of relevance if we seek to model beliefs of an agent arising from both internal reasoning processes and external reasoning processes. However, the objections raised by Mayo-Wilson and Wheeler (2016) should be born in mind when using the imprecise framework for our idea of credence assignments capable of reflecting both external information translated to accuracy in the norm of truth and internal information translated to probability (interval) assignments reflecting the credence of rational agents.

The implementation of an imprecise probability framework is published by Weichselberger, so we only provide a sketch of the axiomatic frame here. The basic idea of Weichselbergers account is based on Komogorovian probability extended by two additional axioms, generalizing classical probability to two types of interval probability: R-probability ("reasonable probability") and F-probability ("feasible probability"). "The first one is characterized by employing interval-limits which are not self-contradictory, while the interval-limits of the second (F-probability) fit exactly to the set of classical probabilities in accordance to these interval-limits (structure)" Weichselberger (1996, p.391).

Given by definition 2.1 in Weichselberger (2000), with adapted notation, we define interval probabilities similarly.

Definition 4. *Let a sample space $\mathcal{V}$ be a measurable space, let $\mathcal{F}$ be a σ-field and a set function $\mathcal{P}(\cdot)$ be an interval assignment, denoted by $[a] := [a; a]$ on $\mathcal{F}$, then $\mathcal{P}(\cdot)$ is called an R-probability if it obeys the following two axioms. Note, that a probability assignment P denotes a classical (Kolmogorovian) probability, while $\mathcal{P}(\cdot)$ denotes an interval probability.*

(IV) $\mathcal{P}(w) = [L(w); U(w)]$ *for every* $w \in \mathcal{F}$
with $0 \leq L(w) \leq U(w) \leq 1$ *for every* $w \in \mathcal{F}$.

(V) *The set of classical probability functions $\mathscr{M}$ satisfying def 2 on $\mathcal{F}$with*
$L(w) \leq P(w) \leq U(w)$ *is not empty for every* $w \in \mathcal{F}$.

Let a R-probability field $\mathscr{R}$ on $(\mathcal{V}, \mathcal{F})$ be denoted by $\mathscr{R} = (\mathcal{V}, \mathcal{F}, L(\cdot), U(\cdot))$. Then a structure $\mathscr{M}$ of $\mathscr{R}$ is the non-empty set of Kolmogorovian probability functions $\mathscr{M} = \{P(\cdot)|L(w) \leq P(w) \leq U(w)\}$ for every $w \in \mathcal{F}$. The existence of a non-empty structure is a sufficient condition for any R-field. Necessary conditions for R-probability are $L(\emptyset) = 0$ and $U(\mathcal{V}) = 1$. If an R-probability obeys axiom VI it is named an F-probability.

$$(VI) \quad \left.\begin{array}{l} \inf\limits_{P \in \mathscr{M}} P(w) = L(w) \\ \sup\limits_{P \in \mathscr{M}} P(w) = U(w) \end{array}\right\} \text{for every } w \in \mathcal{F}$$

Weichselberger interprets R-probability as "not contradictory, but not necessarily perfect" because a structure is permissible, but the limits may not be narrow enough with respect to this structure, what gives rise to "redundant R-probability fields". F-probability is interpreted as perfect generalization of classical probability to an interval-valued one. The structure and the interval limits imply each other and hence information about limits is sufficient to construct a F-probability field $\mathscr{M}$. The quantitative *uncertainty* associated with an interval probability corresponds to the R-probability of an F-probability.

The three concepts briefly presented, Kolmogorovian probability, inner and outer measures, and F-probability and R-probability are obviously related. Inner and outer measures are related to interval probability as they give rise to coherent probabilities which can be used to construct F-probability fields.

In literature aspects of both precise and imprecise probabilities have been extensively researched. Though mathematically the relations are obvious, different schools of thought and the developed theories are to be regarded independently. For example, for lower previsions introduced by Walley (2000) the natural extension of previsions avoiding sure loss corresponds to the procedure of constructing the derivable F-probability field formulated by Weichselberger (2000, p.153). On the conceptual level however, Weichselberger (2000, p.150) explicitly stresses that "the domain of application is neither limited to purely formal aspects nor is bound by a certain interpretation of probability". A thorough analysis in terms of conceptual implications must be left to future work at this point, but will be comparable to the discussion of the conceptual setting of Walley's account by Miranda (2008), who focus on the behavioural interpretation of imprecise probabilities. For a mathematical discussion of generalised imprecise probabilities see Troffaes and Destercke (2011).

3.1 Two kinds of belief

The two notions of belief we employ correspond to distinct concepts. A proposition is considered possible by an agent as belief simpliciter. We call relevant propositions p_i that are part of an epistemic state E_i considered possible by an agent *accepted truths*, with a strictly bivalent probability assignment denoting *the truth assignment* of the agent, that is, belief and disbelief. The *intended meaning* we accord to an agent's bivalent probability assignment to a proposition p is 'the agent confers complete subjective confidence in the *possibility* that p is true.

The second kind of belief modelled is uncertain belief. Uncertain belief denotes gradual belief expressed as probability and uncertainty on [0,1]. Uncertain belief *does not bear on* the truth assignment. Uncertain belief is assigned to propositions or propositional formulae, given a valuation w, that is, a truth assignment by acceptance. The agent is modelled to be uncertain that the meaning a proposition in the object language has, represents completely what the agent takes that proposition to mean in terms of relations and properties for a given meta-epistemic context.

Using the conceptual framework of Probabilism for uncertain belief allows emphasizing the informative potency of the (super)additive notion in contrast to (pre-)orderings or rankings. We want uncertain belief to reflect the complete certainty in a given epistemic state the agent is capable of, and willing to testify. The familiarity of human agents with probability, and the intuitive representation of a credence function as probability assignment, is expected to contribute to naturalistic qualities of the modelling.

Several propositions or propositional formulae can be accepted to be true forming a meta-epistemic context E for a number of valuations $w \in E$. The meta-epistemic context, i.e. the epistemic states E_i, are subsets of $\mathcal{V}$ containing all valuations of propositions. The set of valuations $\mathcal{V}$ is finite and contains propositions considered relevant for the agent (for a belief evaluation). The powerset $2^{\mathcal{V}}$ is the set of all subsets of $\mathcal{V}$, including the empty set and $\mathcal{V}$ itself.

Assigning the strongest quantitative notion of subjective belief (probability 1) means to accept a propositions' truth is possible. This may in an actual reasoning process of an agent demand for additional propositions to simultaneously be possible, for example as prerequisite or as consequence. These propositions are the *context* in which possibility is epistemically created by an agent. Modelling reasoning processes with Probabilistic Uncertainty facilitates acknowledgement of implicit assumptions an agent holds by the notion of accepted truth. In a belief

modelling experiment we would typically offer an agent a number of propositions to choose the relevant propositions from, to limit the scope of related and relevant propositions to a few of particular interest in the experiment. But the mechanism by which the agent accepts propositions to be true, forming the meta-epistemic context. In essence a met-epistemic context concerns the relevance of a context proposition for the meaning of a proposition under evaluation.

Increasing the transparency of implicit assumptions through explicit context acceptance is of independent epistemic relevance because it allows the re-assessment of accepted propositions and the reduction of default truth assignments based on habit, previous experiences, peer example, etc. We take this feature to respond to the criticism by Chow, cited above: the approach [of Hoffmaster and Hooker] encourages an agent to complement explicitly conscious experience with adjacent beliefs held, and fosters considering underlying (generally nonconscious) psychological processes in form of mutual bearing on the strength of belief. The conscious choice of propositions accepted to be true by an agent for a given belief evaluation may alleviate biases in reasoning as for example prejudice, cultural biases, or underlying psychological processes by rendering them transparent as explicit acceptance.

A truth assignment in our understanding reflects a (verifiable) state of the world external to the agent *and* a reasoning result of an strictly internal highly individual reasoning process. The assignment of an agent's "certainty that something is possible" calls on an evaluation in terms of both internal and external reasoning what is emphasized by the term *accepted truth* and modelled as belief simpliciter. The evaluation of uncertain belief is the consequent reasoning process reflecting the accepted propositions' capacity to represent an evaluation criterion. The following two sections discuss in more detail how we interpret the two kinds of belief in the quantitative framework of Probabilism.

3.1.1 Interpretation of Accepted Truth and Uncertain Belief in Quantitative Modelling

A belief modelling framework for non-ideal agency, in particular human agency, inevitably faces the challenge of representing human psychological effects, as for example selective attention or context-dependency. We take selective attention to be a natural feature of a human mind that, according to some preferential or repellent mode, restricts the possibility of perception by targeted attention. When modelling beliefs with Probabilistic Uncertainty this effect is relevant because it impacts the choice of propositions potentially considered possible. This in turn affects the evaluation of uncertain belief, as the probability and uncertainty assigned to a proposition are evaluated based on context propositions considered possible.

A subset of propositions, named an epistemic state or a possible world, of all propositions the agent disposes of, named the sample space, is called a reference class. A probability assignment in a reference class is a belief evaluation of an epistemic state of an agent. The phenomenon that a proposition can be assigned *different* probabilities within the axiomatic framework of probability theory, given a different subset of the sample space, i.e. a different reference class, is called the reference class problem Reichenbach (1971), Hájek (2007). In the context of Probabilistic Uncertainty this means that uncertain belief in a proposition p (or propositional formula α, expressed as probability and uncertainty, may be held with a different strength, given a different epistemic state (possible world).

3.1.2 Uncertain Belief as Probability Assignment

The reference class "problem", i.e. the relativity of any probability w.r.t. the reference class (i.e. the accepted truths), becomes a *tool* to enhance awareness if one refrains from viewing belief as a decision problem. Recognising that the strength of (uncertain) belief, understood as the attitude of an agent w.r.t. a proposition (deficiently) representing the actual world, is variable given different contexts or evaluation criteria - by the agent's own lights - is the aim of a belief evaluation in the awareness paradigm. The epistemic awareness that the (conscious) *choice* of accepted possibilities provides the agent's reasoning grounds is expected to foster a kind of "doxastic humility" what potentially has far reaching practical consequences in social interaction, decision practices, and self-understanding.

The translation of the meta language to the object language is, in our view, possible because the probability assignment understood as representation quality - relative to a reference class - assures that the mapping of the model to the object language respects the agent's individual understanding and relevance of *relations and properties* in the epistemic context. That translation corresponds to a *bridge principle* as understood by Allo. He writes:

> Bridge-principles do not simply relate claims about logical necessity and possibility (or validity) to claims about what one should or should not believe, but in fact relate formal claims about validity to informal claims about deontic statements about beliefs. Bridge-principles are, therefore, not just connections between alethic modalities and some combination of deontic and doxastic modalities, but also between formal and natural-language modalities.

In that sense the (below in more detail discussed) modal operator $B\sim$ figures as a bridge between the modelling of a human reasoner's self-understanding and concept understanding, expressed in natural language. The formal claims about validity are vindicated by the axioms of probability. The deontic statements about beliefs embedded in $B\sim$ are vested in the informal claims resulting from the definition of rationality which employs the epistemic norm of truth on both strictly internal and external reasoning processes. The legitimacy of translation becomes particularly important when epistemic contexts are constructed or inference rules are applied to propositions accepted by the agent in the logical framework discussed below.

The notion of uncertain belief as gradual belief in terms of probability and uncertainty is interpreted as a *complete* description of credences, because uncertainty defined as the Kolmogorovian probability of all alternative events the agent considers possible is based on the theorem of total probability, cf. Suppes (2000). That is, for a proposition p the uncertainty is defined from the probability, as $\Psi(p) = 1 - P(p)$.

Complete descriptions in form of two-dimensional approaches are discussed in literature for example Dubois and Prade (2012), and advocated to be beneficiary for example by van Horn (2003) who cites Shafer[15] Shafer (1976, p.42):

> One's beliefs about a proposition A are not fully described by one's degree of belief Bel(A), for Bel(A) does not reveal to what extent one doubts A – i.e., to what extent one believes its negation A. A fuller description consists of the degree of belief Bel(A) together with the degree of doubt Dou(A)=Bel(¬A)

Probabilistic uncertainty provides such a double perspective. The psychological effect of assigning both probability and uncertainty to a proposition p is a kind of "accuracy evaluation" based on consistency. That is, given the total probability of 1, the agent can reconsider her/his credence assignment to mutually exclusive propositions by "changing the perspective" to the alternatives which are implicitly assigned strength of *belief* from the definition of uncertainty. If the uncertainty assigned to one proposition accurately captures the strength of belief the agent rationally holds for the alternatives, we say a balanced strength of belief and uncertainty is assigned.

For example, in a single proposition case the alternative to a proposition is its negation. In a multi-proposition case a probability and uncertainty assignment implies that gradual *belief* in all alternative propositions must not exceed the uncertainty assigned to the proposition in question.

In our view the criticism of Hájek underestimates the power compliance to the axioms of probability theory together with a definition of rationality as sincere and truthful probability assignment has, in terms of rationality. The ("ridiculous") subjective probability assignment of 0.999 to p_8: George Bush turns into a prairie dog, is licensed *provided that you assign 0.001 to this not being the case (and that your other assignments also obey the probability calculus).* If only p_8 as single proposition is evaluated, the uncertainty of the agent is equivalent to her/his *strength of uncertain belief* in $\neg p_8$, that is, her strength of belief that p_8 is *not* the case, by $P(\neg p_8) = 0.001 = \Psi(p_8) = 0.001$. If the agent rationally introspects her/his uncertainty about p_8, what is equivalent to her/his belief in $\neg p_8$, she/he might re-adjust her probability assignment to p_8 until $P(p_8)$ *and* $\Psi(p_8)$ reflect her/his sincere and truthful uncertain belief in p_8 *and* $\neg p_8$. From the perspective of belief in $\neg p_8$ it might prove for the agent that relevant context is missing. This may lead to an adjustment of uncertain belief in p_8 or, if accepted possibilities are introduced as context, the agent's "ridiculous" beliefs might become understandable inter-subjectively.

In the single proposition case uncertain belief is a consistent notion for belief w.r.t. the axioms of probability. In comparison to Dempster-Shafer belief functions, uncertain belief is a belief function solely for *probabilistic events* in the spirit of Shafer (1976).

3.1.3 Accepted Truth as binary Probability Assignment

Accepted truths, i.e. propositions believed to be possible with a subjective probability of 1, are interpreted as subjective probability assignments. That is, accepted propositions are *self-sufficient and selected by choice.* They emerge from a sovereign, unrestricted, unprovable, subjective choice. Accepted truths[16] are the expression of the agent which propositions she/he is willing to include in her/his assessment, forming the context that conforms most to the agent's intention in a belief evaluation. Accepted truths are belief simpliciter.

An accepted truth is conceived of as sovereign because it need not be justified, explained or referential. It can, but it need not. Accepted truth is considered to be unrestricted, that is, accepted truth can be viewed in relation to

[15] However, Shafer and Vovk (2005) detail in response to question 1 "Continuity" that it is unnecessary to study both $P_\star(A)$ (interpreted as measure of the degree to which we expect A to happen) and $P^\star$ (interpreted as a measure of plausibility of A happening), because they can be defined in terms of the other.

[16] for convenience the correct notion *a proposition p or propositional formula α accepted to be true, denoted by a binary probability assignment of 1, with the intended meaning that 'p is considered possible in context E by the non-ideal agent'* is shortened hereafter to "an accepted truth".

other propositions (for example two agents can compare their accepted truths) but the acceptance is not conditional. Accepted truth is a subjective probability assignment based on internal and/or external reasoning processes of a human agent. Propositions accepted to be true are *the basis* for the evaluation of gradual commitment in form of uncertain belief for an evaluation criterion, a rationality. To use Hájeks example again, the agent may choose to accept George Bush turning into a prairie dog to be possible, provided the agent can *with the same credence disbelieve* that this is impossible. Accepted truth is considered to be unprovable in general because it encodes possibility not probability. It can be provable possible that is, the proposition occurred to be true in the past w.r.t. the rationality of evaluation. If it is, uncertain belief can be evaluated in terms of relative frequency.

Establishing a firm commitment to a propositions' *possibility*, related to the evaluation of the propositions' moral, economic, or occurrence rationality (to name just a few), aims at ensuring the relevance of propositions accepted to be true, and makes uncertain belief meaningful to start with. The evaluation of a proposition's rationality requires in our view acceptance that the proposition is possible for the agent, as an expression of subjective engagement necessary to avoid rational mischief. The commitment captured in acceptance is "all-or-nothing" for a state of affairs in the actual world to be possible (given a rationality), modelled as a proposition representing that state of affairs to be true. For example, suppose the rationality we are interested in is a moral rationality, and the state of affairs in the actual world we are interested in is anthropogenic climate change, we might model p_4: 'The earth's climate is changed by human action'. An agent accepting p_4 accredits the proposition a moral dimension in her understanding which she/he can testify. In other words, the agent accepts that it is possible that p_4 is morally good or bad. From that we know that the evaluation of uncertain belief in p_4 is relevant for the agent.

The evaluation of uncertain belief is *based on* the accepted propositions the agent chooses. This resonates with Putnam's observation cited above that "there is no notion of reasonableness at all without context".

And lastly, accepted truth is considered to be created *by choice*. It is a true statement of self-expression based on an individuals' self-understanding in a given moment. The sovereign, unrestricted, unprovable choice of accepted propositions allows the non-ideal agent to become aware what it *means* to hold a belief in a proposition, in terms of accepted truth and uncertain belief in a given evaluation. That is, the agent defeasibly reasons about what she/he considers possible and how strong she/he holds a belief, exploring what context propositions are relevant to her/him and how the context propositions impact her/his gradual belief in these propositions, given an evaluation criterion.

4 The Logical Perspective

Non-ideal agency is of particular interest if theoretical frameworks are developed aiming at practical relevance. A non-ideal agent is taken to be an agent with limited resources as for example memory capacity and computational power, hence not a logically omnicsient agent. Finding a balance between empirical accuracy for sound single-case modelling of an agent in a given belief evaluation and normative rigour to allow scalability seems to be a particularly demanding task[17].

Probabilistic Uncertainty is a framework that aspires to be empirically adequate for non-ideal agents and concomitantly not being short of logical coherence. The fundamental question one has to address to meet that aspiration is how a human agent can be both logically consistent and limited.

We do not consider the ideas presented so far to be revolutionary. Indeed the grounding in literature and ongoing scientific discussion shows that many of the intuitions we assemble in Probabilistic Uncertainty figure independently in one context or another. And this is true for the logical modelling too.

In the following we adapt the original notions of the authors, however, we emphasize that all of the following is developed and presented by the cited authors. None of this is our genuine work, in particular *full recognition for the conceptual development of the logics and the formal proofs is on the part of the authors cited.*

We apply their work as a formal model of belief conceptualised as accepted truth and uncertain belief. We do so encouraged by Banerjee and Dubois (2014, p. 652) [henceforth BD] statement that "One of the merits of MEL is to potentially offer a logical grounding to uncertainty theories of incomplete information.", and the statement of Fagin and Halpern (1991, p.172) "We hope that this characterization [the logic for reasoning about probabilities] leads to better tools for reasoning about uncertainty".

[17] Allo (2016), for example, addresses that tension by allowing for exceptions (a view called "sohpisticated revisionist") and the consequentialist criticism that a defeasible principle must relate formal systems, philosophical notions, and sets of norms (a view called "critical sceptic").

4.1 Expressing Accepted Truth in a Logic

In this section we discuss how our understanding of accepted truth can be modelled logically. As discussed above, we consider accepting a proposition to be a sovereign, unrestricted choice. Modelled in a logic, sovereignty amounts to a setting where the logic models epistemic states of an agent, rather than truth in congruence with observable reality. Propositions figuring as accepted truth are unrestricted, what means that the agent is not dependent on a previous reasoning process to accept a proposition. However, it does not mean that within a belief evaluation the agent may assign truth values independent of any axiomatic setting - syntactic consequences are governed by logical axioms.

Accepted truth in the logical modelling need not be provable from a semantic point of view. That is, whether the accepted propositions conform to actual states of the world is irrelevant. In this way we do not require the agent to conform her/his credences to the chances (the principal principle). We will see how this is captured in the meta-epistemic logic MEL in detail, by *encapsulating* propositional formulae referring to the actual world (and merely assuming that all tautologies are believed).

A more subtle aspect of non-provability is that the perspective of the modeller is always imperfect. The perspective adopted for the logical modelling of accepted truth and uncertain belief is external to the agent. However, there is no judgement, weighing, or dismissive conduct of the modeller w.r.t. the testimony of a modelled non-ideal agent - a human. The agent's information is, in line with BD, considered to be *incomplete* if the agent does not know whether the accepted proposition is true or not. The modeller's information is imperfect as accepted truth is regarded unprovable. In other words, the modeller takes the testimony of the agent to be true and takes the risk of both misinterpreting the agent and an agent's deceit, though generally sincere testimony is assumed.

Accepted truth ought to represent possibility. However, contrary to possibilistic logic, cf. Dubois and Prade (2004), accepted truth is designed as a binary notion of belief: either the agent accepts a proposition to be possible or not, given a rationality. The reason for possibility as belief simpliciter is that acceptance reflects the agent's commitment as subjective *choice*.

Possibilistic logic in the (generally used) gradual modelling establishes a lower bound of necessity degrees that forms, in contrast to probability assignments, the threshold above which the *validity is maintained* across inference steps. On these grounds possibilistic logic reflects the idea that the concept possibility can be used for deductive reasoning, as inconsistencies do not arise above the threshold. It is a measurement of a conclusion's validity by the *least possible* proposition involved.

This is not what accepted truth ought to express. Possibility, conceived as unrestricted and sovereign should not reflect partial inconsistency. Therefore accepted truth is modelled, in terms of possibilistic logic, by only allowing a threshold of necessity degree 1. In a sense our understanding is comparable in that we assume that validity is maintained, but not in a gradual sense. However, we do not feel that formulating our ideas purely in possibilistic logic would reflect the purpose of possibilistic logic frameworks, which are in essence a way for inconsistency modelling through weighs. In contrast, accepted *truth* ought to express a non-restricted commitment of the agent to a possibility modelled as proposition p, not a gradual truth implying inconsistency. However, *uncertain belief* is gradual. We prefer using conditional probabilities to model gradual belief to exploit the super-additive characteristics of both accepted truth as *binary* truth bearing inner measure and uncertain belief as the concomitant expected degree of belief and uncertainty testified as probability assignment or as inner measure induced by a probability measure in the non-measurable case. Put plainly, we model that an agent can err in her/his judgement that a proposition is true (possible), hence *accepted* truth, but given the agent considers a proposition is true, she/he *cannot* err in her gradual belief in that proposition.

We follow closely the structure of the original presentation of MEL by BD, both to maintain traceability, comparability, and to illustrate that their framework need not be contorted in order to express the ideas of accepted truth and uncertain belief.

In particular, we employ the MEL logic developed by BD to logically capture the *truth assignments* to propositions by the agent. We emphasize that the unrestricted commitment of the agent, i.e. the binary probability assignment, is only concerned with *possibility*.

MEL is a meta-epistemic logic where the agent is modelled from an external point of view. The accepted truth modality $\mathcal{T}$ corresponds in MEL to the modality $\Box$, it depicts what the non-ideal agent reports to have accepted. The "items" the agent can accept are propositional formulae α, β, based on propositional variables $p_1, \ldots, p_n$ with the connectives $\neg$, $\wedge$ of a classical propositional logic PL forming $\mathcal{L}$. If accepted, the unary modality $\mathcal{T}$ is added in front of the propositional formulae which, with the connectives $\neg$, $\wedge$, form a MEL-formula ϕ.

To reflect the subjective, sovereign, unrestricted nature of the accepted truth modality MEL is particularly suitable because the logic "encapsulates" propositional formulae through association of that unary connective $\mathcal{T}$, and derived from it $B\sim$ in front of all sentences in $\mathcal{L}$ forming the *disjoint* language $\mathcal{L}_{\mathcal{T}}$. The agent possesses a set

of accepted true propositional formulae, distinguished from propositions and propositional formulae referring to the real world formally by adding the modal operator $\mathcal{T}$, to be read as "the agent asserts acceptance of this proposition to be true". Note however, that we assume the agent accepts α that are valid by axiom N, but no elements of $\mathcal{L}$ can be assorted with accepted truths.

The modality $B\sim$, uncertain belief, corresponds to the $\Diamond$ modality in MEL, and reads 'a PL-formula can be believed uncertainly, $B\sim$, given the agent does not accept truth of the negation of the PL-formula'. $B\sim$ is defined as follows.

Definition 5. *$B\sim\alpha := \neg\mathcal{T}\neg\alpha$, where $\alpha \in \mathcal{L}_{\mathcal{T}}$.*

Self-sufficiency of accepted truth is translated for the modality $\mathcal{T}$ in MEL by the prohibition of modal operator iteration. That is, the agent does not formally introspect her/his accepted truths and uncertain beliefs. The formal modelling is *as reported by the agent.*

The axioms and inference rule in MEL by BD are, for any MEL-formula $\phi, \psi, \mu, \in \mathcal{L}_{\mathcal{T}}$ and any PL-formula $\alpha,\ \beta$ *in* $\mathcal{L}$

(PL)	(i)	$\phi \to (\psi \to \phi)$
	(ii)	$(\phi \to (\psi \to \mu)) \to ((\phi \to \psi) \to (\phi \to \mu))$
	(iii)	$(\neg\phi \to \neg\psi) \to (\psi \to \phi)$
(K)		$\mathcal{T}(\alpha \to \beta) \to (\mathcal{T}\alpha \to \mathcal{T}\beta)$
(N)		$\mathcal{T}\alpha,\ whenever\ \vdash_{PL} \alpha$
(D)		$\mathcal{T}\alpha \to B\sim\alpha$
RULE	(MP)	$If\ \phi,\ \phi \to \psi\ \ then\ \psi.$

(PL) are axioms of propositional logic. Axiom K expresses that if an agent accepts that a proposition is entailed by another, it follows that if the former is accepted, then the latter is accepted. The N-axiom denotes that the agent is expected to accept all tautologies.

We read this axiom as the relation of the radically subjective realm of belief and the inter-subjectively observable (inter-subjective) reality. BD emphasize, that this relation is an axiom in MEL, not an inference rule because α is not a member of the language $\mathcal{L}_{\mathcal{T}}$. We could say the agent's accepted truths are "oriented" towards the actual world as we expect α to be accepted if α is entailed by (PL).

Axiom D is the connection between accepted truth and uncertain belief. BD themselves clarify that it is "the counterpart of the numerical inequality between belief and plausibility functions, necessity and possibility measures". We interpret the axiom as counterpart for accepted truth and uncertain belief as numerical inequalities in terms of probability and uncertainty. Axiom D is taken by BD in a hierarchical way in terms of strength, i.e. asserting acceptance of α is stronger than asserting uncertain belief in α. That reflects our intention of firm commitment for acceptance, modelled as binary notion, that is, accepted truth is assigned a subjective probability of 1 compared to uncertain belief that is assigned a probability of ≤ 1. However, we would like to broaden the interpretation of D in the sense that accepting is a prerequisite for uncertainly believing. That interpretation is modelled in the definition of uncertain belief, definition 5, which is *based on* a notion of acceptance.

In related work by Dubois, Fragier, and Prade in Dubois et al. (2004, p.24) point at important features of accepted beliefs. They write

> Usually, accepted beliefs are understood as propositions whose degrees of belief are high enough to be considered as true, and they form a deductively closed set. [...] The specific requirement for p to be an accepted belief is that the agent is ready to infer from p in a classical sense, as if p were true.

We model the agent's commitment to "infer from p as if it were true" by the modality $\mathcal{T}$ denoting the strongest subjective probability assignment, i.e. $\mathcal{T}\alpha$ iff $P(\alpha) = 1$, or $P(\neg\alpha) = 1$.

The probability assignment is formally modelled by the Boolean function τ discussed in section 4.4. Accepted truth is a *binary* probability assignment so that accepting a propositional formula implies not accepting the negation of the propositional formula in question (what is equivalent to the definition of uncertain belief). However, for later reference we note here, that accepting a proposition (or the negation of a proposition) in one context does *not* mean that proposition is accepted (or its negation) in a different context. Indeed this would not reflect that we are interested in the changed meaning of a proposition in a different context.

In comparison, Dubois et al. (2004, p.24) define accepted belief according to a confidence relation, a relation comparing propositions in terms of their relative credibility for an agent : "a minimal requirement for a proposition p to be a belief is that the agent believes p more than its negation. This is the weakest definition one may think of".

We consider demanding a strong commitment for acceptance appropriate, given the agent is considered to be 'ready to infer from p in a classical sense, as if p were true'. As we conceptualise acceptance as unrestricted sovereign choice (even if it may be partially unconscious) we are not primarily interested in the relative credibility compared to other propositions. In a sense *our* minimal requirement is that the agent can accept, for a number of propositions, that the affairs of the actual world the propositions *represent for the agent* are materially adequate (considered possible) w.r.t. a rationality.

However, we are aware that agents may be reluctant to testify strong commitments in practice, given the agent might know to (partially) derive her/his beliefs from semitransparent internal reasoning structures, incomplete observation of actual phenomena, unreliable information sources, insincere testimony of others, biased perception and attention, social norms, etc. To alleviate the agent's reticence, the strongest commitment (probability 1) is used for the weakest concept (possibility) in terms of consequence, so that correctness judgements (right/wrong) are avoided. The strong commitment to a possibility allows us to construct epistemic states with deductive closure that *hold uncertainty*. To express the agent's uncertainty we need 1) a gradual notion distinct from possibility and 2) an evaluation criterion for the agents uncertainty. The former is the probability and uncertainty assignment as credence function whereas the latter is the rationality we are interested in.

We believe that to be in the spirit of BD, as they write:

> A set-function is typically used as a representation of uncertainty. In other words, a propositional valuation of the modal language can be viewed as an uncertainty measure. In the case of MEL, we can define exactly which kind of uncertainty measure is induced by valuations.

BD model uncertainty using a Boolean *necessity measure* of possibility theory developed by Zadeh (1979). We employ a Boolean probability measure to model the binary probability assignment of accepted truth. As Dubois and Prade (2015, p.31) clarify, possibility theory is an uncertainty theory concerned with the modelling of incomplete information. "To a large extent, it is comparable to probability theory because it is based on set-functions. It differs from the latter by the use of a pair of dual set functions (possibility and necessity measures) instead of only one. Besides, it is not additive and makes sense on ordinal structures." In the case of uncertainty understood as in definition 2 above, the set function is also dual. By property (3) of the σ-algebra, noted in the definition's footnote, it is assured that we can *always* find the uncertainty corresponding to a probability assignment.

For accepted truth a binary probability is assigned, what corresponds to the necessity measure BD use. They specify the necessity measure *for the Boolean finite case*:

A set-function $\mathcal{N}$ with range on the unit interval is a necessity measure in possibility theory provided that $\mathcal{N}(\emptyset) = 0,\ \mathcal{N}(\mathcal{V}) = 1,\ \mathcal{N}(A \cap B) = min(\mathcal{N}(A), \mathcal{N}(B))$. "It is a special case of belief function and lower probability measure." In our modelling, it is the corresponding Boolean inner measure induced by a probability measure, discussed in section 4.4.

4.2 Propositional and Epistemic Semantics

The modelling "separates" two levels. On the one hand the language $\mathcal{L}$ refers to the actual world, on the other hand, the language $\mathcal{L}_{\mathcal{T}}$ encapsulates members of $\mathcal{L}$ and refers to the agents independent choice of what is accepted to be true, related by axiom N to what the agent considers materially adequate in the actual world.

The atomic propositions $p_1,\ p_2, \ldots$ and the propositional formulae constructed from the atomic propositions $\alpha,\ \beta, \ldots$ of $\mathcal{L}$, are the representation of states of affairs in the actual world. We call a PL-formula α a 'minimal context'. Valuations of atomic propositions reflect material adequacy in the agent's understanding.

Let a propositional valuation be a map $w : PV \to \{0, 1\},\ where\ PV := p_1, ..., p_n$. The set of all propositional valuations (interpretations) is denoted by the finite set $\mathcal{V}$. For a PL-formula α, $w \vDash \alpha$ indicates that w satisfies α or w is a model of α, i.e. $w(\alpha) = 1$ $(true)$, what reads 'the agents considers α to be possible', that is, α is a representation of a state of affairs that *can* represent a rationality for the agent. The set of all valuations that model α is denoted by $[\alpha]$.

The second level epistemic semantics is based on standard propositional semantics for the propositional language $\mathcal{L}_{\mathcal{T}}$. A minimal context α that is accepted to be true by an agent, is assigned the unary modality $\mathcal{T}$, forming the MEL-atoms $\mathcal{T}\alpha$ and MEL-formulae $\phi,\ \psi$ in $\mathcal{L}_{\mathcal{T}}$. A propositional model of a MEL-formula is an interpretation v of $\mathcal{L}_{\mathcal{T}}$. A MEL-formula is satisfied if an interpretation, i.e. mapping form MEL-atoms to $\{0, 1\}$, makes a MEL-formula true. If a MEL-formula $\phi \in \mathcal{L}_{\mathcal{T}}$ is satisfied in all epistemic states, it is called a valid MEL-formula by BD.

An epistemic state E is represented by a subset of mutually exclusive propositional valuations. Each valuation in $\mathcal{V}$ models a possible world. $\mathcal{T}\alpha$ is said to be true if "in all possible worlds compatible with what the agent believes, it is the case that α", denoted by $[\alpha]$ as the set of valuations modelling[18] α in correspondence with BD p. 640.

[18] That is $\{w : w \vDash \alpha\}$

The satisfaction of a MEL-formula is defined recursively. For any $\alpha \in \mathcal{L}$, and $\phi, \psi \in \mathcal{L}_{\mathcal{T}}$, for a non-empty epistemic state $E \subseteq \mathcal{V}$, a MEL formula is satisfied given.

$$\begin{aligned} E \vDash \mathcal{T}\alpha &\text{ iff } E \subseteq [\alpha] \\ E \vDash \neg\phi &\text{ iff } E \nvDash \phi \\ E \vDash \phi \wedge \psi &\text{ iff } E \vDash \phi \textit{ and } E \vDash \psi \end{aligned}$$

E is an epistemic state of subsets of valuations $\mathcal{V}$, such that α is accepted by the agent if α is modelled in E. If $E \vDash \phi$, E is called an epistemic model of ϕ, and if $E \vDash \phi$, $\phi \in \mathcal{L}_{\mathcal{T}}$, for all E, ϕ is said to be a valid MEL-formula.

Definition 6. *An agent is said to be inconsistent if $E = \emptyset$. The assumption is that agents are consistent, so generally it is assumed that $E \neq \emptyset$. This, together with definition 5 for uncertain belief gives rise to*

$$E \vDash \ B \sim \alpha \textit{ if and only if } E \bigcap [\alpha] \neq \emptyset.$$

Epistemic contexts can be constructed from an agent's testimony by the epistemic semantics for consequence and equivalence. Semantic equivalence, denoted by $\equiv$, and semantic consequence, denoted by $\vDash_{MEL}$ are notions to relate formulae. Two formulae ϕ and ψ are semantically equivalent, if for any epistemic state E, E satisfies ϕ if and only if E satisfies ψ, i.e. $E \vDash \phi$ iff $E \vDash \psi$.

Semantic entailment for a set Γ of MEL formulae ϕ is based on the satisfaction of *all* members of Γ in an epistemic state E, i.e. $E \vDash \Gamma$ means $E \vDash \phi$ for each $\phi \in \Gamma$. The epistemic models of Γ, denoted by $[\Gamma]$, are the set of epistemic states $\{E : E \vDash \Gamma\}$. Then, for any set $\Gamma \cup \{\phi\}$ of MEL-formulae, ϕ is a *semantic consequence* of Γ, $\Gamma \vDash_{MEL} \phi$ provided that for every epistemic state E, ϕ is satisfied if Γ is satisfied, i.e. $E \models \Gamma$ implies $E \models \phi$.

In other words, ϕ is a semantic consequence if Γ is satisfied in an epistemic state. Basing semantic consequence on all epistemic models of Γ asseverates monotonicity and captures Davies and Brachman's observation of *metalevel* monotonicity of probability theory.

By definition 6 it is assured that there is at least one possible world where α is accepted by the agent to be possible. "As a consequence, the epistemic state E of the agent is known to be consistent with $[\alpha]$" p. 643 BD. Furthermore we thereby formally capture the requirement that accepted truths be *relevant*. If the agent does not assign uncertain belief in any epistemic state, that is, the agent suspends judgement in every epistemic state (called full ignorance by BD) then $E \bigcap [\alpha] \neq \emptyset$ and the complement set of valuations is also consistent $E \bigcap [\alpha]^C \neq \emptyset$.

The quantitative modelling of uncertain belief allows nonetheless to evaluate $B \sim$ quantitatively in terms of the inner and outer measure across possible worlds and epistemic states. Indeed we think that screening different contexts is a common reasoning strategy of human agents to get one's bearings, if the strength of belief and uncertainty in a proposition is "difficult" to find, for example because it is an unprecedented context, or an experience (or awareness) has fundamentally changed the agents' attitude. A comparative reasoning strategy is calling upon uncertain belief in the proposition in different contexts as "benchmarks" or orientation, and from that knowledge the agent is extrapolating to the context under evaluation. That extrapolation is possible by adjustment of the strength of belief and uncertainty for the context of evaluation taking into account the (mutual) influences of additional (missing) accepted truths as premises. Modelling such reasoning in terms of the inner and the outer measure can assist or even encourage the agent to testify uncertain belief explicitly, using a *bound of uncertainty* derived from different contexts as suggestion. The lower bound of uncertainty is the *highest assigned probability* in any subset of $\mathcal{V} \neq \emptyset$ (all other assignments have a lower probability and hence a higher uncertainty).

For $\mathcal{T}\alpha \vee \mathcal{T}\neg\alpha$ the intended meaning by BD is "the agent knows whether α is true or not, but her belief about α remains unknown." We interpret that statement precisely in the sense of BD and consider it as an expression of the subjective nature of belief expressed as $\mathcal{T}$ and $B \sim$. The authors draw attention to the fact that the language allows expressing an agent's omission of a truth assignment on the part of the agent, but it cannot express "that we ignore if the agent knows anything about α."

We interpret that feature as non-judgemental, sincere, and uninvolved reporting of a an agent's testimony on the part of the modeller. This may be in general be an assumption in agency and logical modelling, however, MEL *cannot* express judgement of the modeller, just as it does not allow modelling introspective reasoning of the agent. In other words, the models of a given evaluation reflect belief as it is, according to the agent's testimony. Neither does the modeller modify the agents' assertions in additional external modalities expressing the modeller's interpretation of what the agent reports, nor can the agent can express "second thought" on assertions what would be modelled by iterating the modalities $\mathcal{T}$ and $B \sim$. The testimony is represented *ipso facto* what is considered to be an expression of respect, sovereignty, sobriety, and trust.

In the following section we take a closer look at the modelling of uncertain belief in MEL and the Logic for Reasoning *about* Probabilities by FHM to fully exploit the quantitative properties of the gradual credence assignments denoting *uncertain belief*.

4.3 Expressing Uncertain Belief in a Logic

The probability assignments are not considered to be truth conductive in a semantic interpretation of the logic. What uncertain belief provides is an assessment of the agent's gradual commitment that a proposition accepted to be possible *represents* the agent's understanding of the actual world and the rationality to be evaluated. $B \sim$ models gradual representation quality of a proposition *salva veritate*. In other words, the truth value is preserved for the two co-referring modalities $\mathcal{T}$ and $B \sim$.

The statement $B \sim \alpha \ \in \Gamma$ interpreted by BD as "possible truth", in the sense that the agent "has no argument to the falsity of α". The agent either believes α or ignores whether α is true or not, cf. p.641. We, in close correspondence interpret that statement in terms of uncertain belief as *gradual belief* expressed by the probability and uncertainty assignments, based on the accepted truth of α. The agent is considered to have no argument that it is impossible that α represents the rationality. The agent either considers α to be a (gradual) representation for her/him or has not (yet) evaluated her internal and external subjective belief in α.

For $B \sim \alpha \wedge B \sim \neg\alpha \in \Gamma$ the intended meaning by BD is "the agent explicitly ignores whether α is true or not". We interpret that statement as *suspended judgement*.

As Davis and Brachman (2014, p.101) clarify, classical probabilities can be considered as a monotonic theory, however, this is only true on the metalevel. They say on p. 101 "the statement "Given a set of sentences Γ, ϕ has probability x" does not change its truth. [...] At the object level, probability theory is nonmonotonic. The statement ϕ may become more or less likely as increasing evidence is accumulated".

This corresponds to our modelling: the metalevel is modelled in a monotonic setting of MEL, while the assessment of propositions reasoned *about* in an epistemic state is governed by probability assignments that may be more or less uncertain in different minimal contexts and epistemic states, and are, if nonmeasurable sets are involved, in general non-additive.

Nonmeasurable sets are formulae involving propositions the agent does not assign a probability to. In that case the uncertain belief cannot be modelled by adding the probabilities assigned to atomic propositions. Instead, the inner measure $P_\bigstar(\alpha)$ induced by the probability measure is used to model a *bound*: the interval probability derived from the powerset presented in definition 3. Inner measures correspond "in a precise sense" (FHM) to Dempster-Sharfer belief functions. Both, belief functions and inner measures are generally non-additive, that is, the sum of belief functions need not add up to 1. In literature non-additive measures are characterized by a condition called *total monotonicity*, for example by Gilboa and Schmeidler (1994, p.44).

Total monotonicity as defined by Gilboa and Schmeidler refers a property[19] of a function. The inner and the outer measure induced by the probability measure are such functions defined from a finite algebra on $\mathcal{V}$ from the power set $2^{\mathcal{V}}$ on $\mathbb{R}$, $P2^{\mathcal{V}} \to \mathbb{R}$ is called a non-additive measure, if it is nonnegative and for every $p_1, \ldots, p_n \in 2^{\mathcal{V}}$ the generalized Bonferroni inequality holds $P(\bigcup_{i=1}^{n} p_i) \geq \sum_{\{I|\phi \neq I \subseteq \{I,\ldots,n\}\}} (-1)^{|I|+1} P(\bigcap_{i \in I} p_i)$.

Using this inequality we can compute the uncertain belief for every propositional formula accepted to be possible from both "directions", using the assigned probability or the concomitant uncertainty. That may be of particular interest if an epistemic state is modelled involving propositions in formulae the agent suspends judgement on, but chose to testify in other minimal contexts. These other contexts can be consulted to "bound" the uncertain belief. BD interpret uncertain belief $B \sim$ generally as "partial ignorance". Through the combination with the logical framework of FHM we can quantify that partial ignorance in terms of probability and uncertainty across contexts.

The lower bound of uncertainty reads 'For the agent, the uncertainty of believing α is x *or higher*, that is, a lower bound of uncertainty x is an interval probability on $[0, 1]$ with $0 \leq x \leq 1$. The highest assigned probability is the strongest uncertain belief $B \sim$, testified in α in a context $E \subseteq \mathcal{V}$ that satisfies α maximally, $E(\neq \emptyset) \in \mathcal{V} \models maxB \sim \alpha$, given by $P(\alpha) = x$ with $0 \leq x \leq 1$. Equivalently, the lowest assigned uncertainty in α can provide orientation for an explicit assignment of uncertain belief, i.e. $maxB \sim \alpha$, given by $\Psi(\alpha) = y$. Regarding the *strength of belief* in $B \sim$ can be replaced by regarding the *uncertainty* defined as $\Psi(\alpha) = 1 - P(\alpha) = P(\alpha^C)$. To shift the focus, the relation of the inner measure $P_\bigstar$ and the outer measure $P^\bigstar$ is used, given by $P^\bigstar(\alpha) = 1 - P_\bigstar(\alpha^C)$. In section 4.8 the perspective on uncertain belief for the two modelling levels is discussed, that is, the level of minimal contexts contained in $\mathcal{L}$ referring to the actual world, and the epistemic level of an agent contained in $\mathcal{L}_\mathcal{T}$ where truth of minimal contexts is accepted.

To quantitatively model $B \sim$ we use the axiomatic framework developed by FHM. Again we follow closely the presentation of FHM in Fagin and Halpern (1991) and Fagin et al. (1990), however, with adapted notation.

Let a *probability structure* be a tuple $(\mathcal{V}, \mathcal{F}, P, \pi)$, where $(\mathcal{V}, \mathcal{E}, P)$ is a probability space, and π is Boolean set function associating with each $w \in \mathcal{V}$ a truth assignment $\pi(w) : \mathcal{L} \to \{true, false\}$. It is said that α *is true at* w if $\pi(w)(\alpha) = true$, otherwise it is said that α *is false at* w. FHM think of $\mathcal{V}$ as consisting of possible states of

[19] "Observe that additive measures are totally monotone, totally monotone measures are convex and convex ones are superadditive", Gilboa and Schmeidler (1994, p. 47)

the world, where each state w *in* $\mathcal{V}$ can be associated a unique atom describing the truth values of the primitive propositions in α. The different *states of a possible world* come from the fact that different combinations of propositional valuations, i.e. multiple w, can satisfy propositional formulae characterising a possible world. FHM provide the following example. If $\mathcal{L} = \{p_1, p_2\}$, and if $\pi(w)(p_1) = true$ and $\pi(w)(p_2) = false$, then the atom $p_1 \wedge \neg p_2$ is associated with w. The same atom can have different states associated or some atoms may not be associated with any state.

Any formula α can be assigned a truth value using the rules of propositional logic for a probability structure M, consisting of all states in M where α is true, i.e. $\{w \in \mathcal{V} | \pi(w)(\alpha) = true\}$. If every primitive proposition p is measurable (i.e. is assigned a probability) then every formula α is also measurable, and such an M is a *measurable probability structure*. In the following section a bottom-up modelling approach is exemplified where this feature is used to model an expectation of uncertain belief arising from probability and uncertainty assignments to all primitive propositions, provided the minimal contexts are accepted to be true.

For measurable probability structures the axioms and inference rules for propositional reasoning to *Reason about Probabilities* are given by FHM p. 168. For every α and β that are propositional formulae of $\mathcal{L}$.

TAUT	(all instances of)	*propositional tautologies*
RULE	(MP)	$If\ \alpha,\ \alpha \rightarrow \beta\ \ then\ \beta$
(W1)	(nonnegativity)	$P(\alpha) \geq 0$
(W2)	(probability of event *true* is 1)	$P(true) = 1$
(W3)	(additivity)	$P(\alpha \wedge \beta) + P(\alpha \wedge \neg\beta) = P(\alpha)$
(W4)	(distributivity)	$P(\alpha) = P(\beta)\ if\ \alpha \Leftrightarrow \beta$
		is a propositional tautology.

A measurable probability structure requires the agent to assign a probability and uncertainty to *every* proposition what may not be desired or possible. Moreover, modelling the expected uncertain belief in propositional formulae as the additive combination of assigned probabilities is a monotonic setting.

For "arbitrary probability structures", where the *inner measure induced by the probability measure* is used to quantify uncertain belief, axiom $W3$ no longer holds, since inner measures are not finitely additive. For inner measures "P" is replaced by $P_\bigstar$ in the axioms $W1, W2, W4$, and $W3$ is *replaced by*

(W5)	(probability of event *false* is 0)	$P_\bigstar(false) = 0$
(W6)	(total monotonicity)	$P_\bigstar(\alpha_1 \vee \cdots \vee \alpha_k) \geq \sum_{I \subseteq \{1,\ldots,k\}, I \neq \emptyset} (-1)^{\lvert I \rvert + 1} P_\bigstar(\wedge_{i \in I}\ \alpha_i)$.

Using the inner measure allows us to model non-monotonic reasoning. Note, that for explicit uncertainty assignments $P_\bigstar(\alpha) = P^\bigstar(\alpha) = P(\alpha)$. The axioms (I1 - I6) an are not in detail repeated here. (I1-I6) are used to compute the inner measure of Boolean combinations of propositional weight formulae $a_1 w(\alpha_k)$, where a are integers with $k \geq 1$, w denotes the inner (outer) measure induced by a probability measure, and $\alpha_1, \ldots, \alpha_k$ are propositional formulae . They are the standard assumptions for the handling of linear inequalities, namely (I1) adding and deleting 0 terms, (I2) permutation, (I3) addition of coefficients, (I4) multiplication and division by nonzero coefficients, (I5) dichotomy, and (I6) monotonicity, shown to be the sound and complete axiomatization for Boolean combinations of linear inequalities, where $w(\alpha_i)$ is treated like a variable x_i.

We draw now attention to tautologies in the frameworks of BD and FHM, in particular in their respective primitive propositions. In the MEL framework $\mathcal{T}\alpha \vee \mathcal{T}\neg\alpha$, and generally $\mathcal{T}\alpha \vee \mathcal{T}\beta$, is not tautological because epistemic models of a disjunction form the set $\{E : E \subseteq [\alpha]\} \cup \{E : E \subseteq [\beta]\}$. In contrast, $\mathcal{T}(\alpha \vee \beta)$ would allow for epistemic states where none of α or β can be asserted.

In particular, $E \vDash B \sim \alpha \wedge B \sim \neg\alpha$ is interpreted by BD as full ignorance, and we, as mentioned above, interpret it as suspended judgement. This feature of possibility reasoning is well known in literature, and it does not impact our requirement of acceptance to be non-contradictory. This has already been acknowledged in literature, for example Dubois and Prade (2001, p.38) clarify "the proposition *possible p* is not the same as p, and possible ¬p is not the negation of possible p. Hence the fact that the proposition "possible p and possible ¬p" may be true does not question the law of non-contradiction". We model the difference between possible p and p with the unary modality $\mathcal{T}$ (and $B \sim$). In that reading p corresponds to the modelling level referring to the actual world contained in $\mathcal{L}$ whereas *possible p* corresponds to the epistemic modelling level contained in $\mathcal{L}_\mathcal{T}$ where the modal operators apply.

In the FHM framework, if p is a primitive proposition, $p \vee \neg p$ it is not an instance of a tautology, since it is not a *weight formula*, cf. Fagin et al. (1990, 85).

It is interesting to remark that FHM, by Theorem 5.5, show that the complexity of deciding whether a weight formula is satisfiable with respect to probability structures is NP-complete and "in a precise sense exactly as difficult as propositional reasoning". We consider this to be relevant not only for us as modellers, but also for reasoners. An agent who is assumed capable of performing propositional reasoning, can be assumed (and can consider herself/himself) capable of gradual belief reasoning for propositional formulae.

4.4 Relating the Logical Frameworks of BD and FHM

Of course, BD are aware of the enormous potential of MEL. They present MEL as a basis for reasoning about uncertainty in the light of various formal approaches, *including* Nilsson's probabilistic logic (which is concerned with measurable probability spaces) and FHM (which is more general, accounting in addition for nonmeasurable events), when they say "such logics can be studied as graded extensions of (fragments of) MEL".

The connecting constituent of BD's framework, FHM's framework and Probabilistic Uncertainty is the Boolean set function. BD employ a Boolean necessity measure $\mathcal{N}$, FHM use the Boolean set function π, and Probabilistic Uncertainty employs a Boolean inner measure τ induced by a probability measure.

We first note that all functions are defined on $2^{\mathcal{V}}$ with the Boolean set $\{0,1\}$ as their codomain. Our aim is to see that the Boolean function $\tau : 2^{\mathcal{V}} \to \{0,1\}$ is equivalent to $\mathcal{N}$.

Let $\tau : 2^{\mathcal{V}} \to \{0,1\}$ be an inner measure, as presented in definition 3, that is, $\tau(\alpha) = max\{P(w)|w \subseteq \alpha \text{ and } w \in \mathcal{V}\}$. For propositional valuations v of $\mathcal{L}_{\mathcal{T}}$ (a mapping of MEL-atoms to $\{0,1\}$) it is assured that the MEL axioms hold for epistemic states $A, B, E \subseteq \mathcal{V}$, if τ is a necessity measure by proposition 1 of BD. We then use τ to replace π in the framework of FHM, so that the truth assignments are respectful of the MEL axioms. The quantitative probability and uncertainty assignments in any epistemic state A, B, E are then interpreted as the quantitative counterpart to $B \sim$ embedded in the weaker (possibilistic) binary truth assignments of τ interpreted as accepted possibilities.

In the Boolean finite case for each such necessity measure $\mathcal{N}$ there exists a unique non-empty subset $E \subseteq \mathcal{V}$,

$$\mathcal{N}(A) = \begin{cases} 1 & \text{if } E \subseteq A \\ 0 & \text{otherwise.} \end{cases}$$

$\mathcal{N}(A) = 1$ corresponds to the epistemic semantics of assertion $\mathcal{T}\alpha$, where $[\alpha] = A$. The epistemic state E can be defined from propositional valuations of $\mathcal{L}$ (in contrast to v), $w \in E$ if and only if $\mathcal{N}(\mathcal{V}\backslash\{w\})$. By proposition 2 of BD, a Boolean necessity measure $\mathcal{N}$ on $2^{\mathcal{V}}$, with a valuation v defined by $v(\mathcal{T}\alpha) := \mathcal{N}([\alpha])$ for all $\alpha \in \mathcal{L}$, satisfies all instances of axioms K, N, D. By remark 2 p. 644 we are *explicitly* assured that this is also true for the Boolean function τ when BD write

> Proposition 2 holds for Boolean versions of more general set-function than necessity measures, namely, super-additive ones, i.e., such that $\tau(A \cup B) \geq \tau(A) + \tau(B)$ whenever $A \cap B = \emptyset$, since if the range of τ is the set $\{0,1\}$, then the super-additivity axioms is equivalent to $\tau(A \cap B) = min(\tau(A), \tau(B))$ for all $A, B \subseteq \mathcal{V}$.

We draw attention to the multivalued case of necessity measures, that (sort of) corresponds to the quantitative notion of uncertain belief, where axiom K is stronger than the bivalent axiom of necessity measures we employ for accepted truth. Our intention that accepting a proposition as true possibility is a prerequisite for believing that proposition uncertainly in the range of $[0,1]$, is here captured in the formal modelling, as the numerical inequality $\tau(A^C \cup B) \leq max(1 - \tau(A), \tau(B))$ "enforces τ to be a Boolean set-function[20]" p. 644 BD.

For uncertain belief, let τ range on $[0,1]$ expressing axiom K using N as the numerical inequality $min(\tau(A^C \cup B), \tau(A)) \leqslant \tau(A \cap B)$ "which is equivalent to the axiom of necessity measures in the gradual case". However, we only exploit the quantitative representation and *do not value* the probability and uncertainty assignments as *truth assignments*, so that Probabilistic Uncertainty is not a multi-valued framework. The point is to reason *about* probabilities, in correspondence with Fagin et al. (1990) p. 79:

> We expect our logic to be used for reasoning about probabilities. All formulas are either true or false. They do not have probabilistic truth values.

We adopt FHM's semantics presented in detail in Fagin et al. (1990). Let the probability structure M be the tuple $(\mathcal{V}, \mathcal{F}, P, \pi)$, so that π associates a truth assignment (a valuation) w in $\mathcal{V}$ to the primitive propositions p in $\mathcal{L}$. Define

[20] Illustrated by assuming τ on the range of $[0,1]$ and assuming $A = B$, then $max(1 - \tau(A), \tau(A)) \geq 1$

$p^M = \{w \in \mathcal{V} | \pi(w)(p) = true\}$. If the set p^M is measurable for each primitive proposition p the probability structure M is called measurable. The set p^M is interpreted as the minimal contexts. The truth assignment is extended to all propositional formulas such that with each propositional formula α the set $\alpha^M = \{w \in \mathcal{V} | \pi(w)(\alpha) = true\}$. In the non-measurable case the inner measure induced by the probability measure replaces the probability measure P. If the propositional formulae are accepted they are associated the unary modality $\mathcal{T}$ and become MEL-atoms in $\mathcal{L}_{\mathcal{T}}$. The semantics for $B \sim$ are then defined as follows.

Definition 7. $M \vDash a_1 B \sim (\alpha_1) + \cdots + a_k B \sim (\alpha_k) \geq c$ *iff* $a_1 P_\bigstar(\alpha_1^M) + \cdots + a_k P_\bigstar(\alpha_k^M) \geq c$

$B \sim \alpha$ corresponds to a *primitive weight term* in FHM's notation, forming a *weight term* together with the integers a_k, where $k \geq 1$. We will not make use of that terminology, however, it is important to unambiguously relate our notation to the original authors' work. For simplicity we assume here $a = 1$. M satisfies $B \sim \alpha \geq c$ iff there is a measurable set contained in α^M whose probability is at least c. If M is a measurable probability structure, then $P_\bigstar(\alpha) = P^\bigstar(\alpha) = P(\alpha)$ for every formula α.

Using the integers a_k for reasoning with inequalities allows modelling "relative" uncertain beliefs. FHM provide the example $B \sim (\alpha_1) \leq 2B \sim (\alpha_2)$ what would read "the uncertain belief in α_1 is twice the uncertain belief in α_2. As such we can formally capture the qualitative dimensions - the rationalities - and their relative relevance for an agent. Modelling the qualitative dimension of uncertain belief in the formal framework in detail must be left to future work, so that we merely sketch the idea here. The basic idea is that the valuations are indexed to a rationality and accordingly form subsets of valuations. If an agent agent can *prioritize* different rationalities (e.g. moral, economic, etc.) according to their *relative relevance* for the agent, these weighs can be introduced via the integers of weight terms a_k with $k \geq 1$[21].

4.5 Modelling Variants

Generally, there are two "levels" of modelling. On the one hand, propositions referring to the actual world, p and formulae α contained in $\mathcal{L}$, on the other hand the meta-level is modelled with accepted propositional formulae $\mathcal{T}\alpha$, called MEL-atoms and MEL-formulae ϕ contained in $\mathcal{L}_{\mathcal{T}}$. Uncertain belief is modelled on the former level by $P(p)$, $P(\alpha)$, or, in case of interval probability assignments $\mathcal{P}(p)$, $\mathcal{P}(\alpha)$ with associated uncertainty $\Psi(p)$, $\Psi(\alpha)$, respectively $\psi(p)$, $\psi(\alpha)$. On the meta level uncertain belief is denoted by $B \sim$ for all assignments of gradual belief associated with formulae in an epistemic state. Uncertainty is equivalent to uncertain belief in a different epistemic state, as epistemic states are subsets of mutually exclusive propositional interpretations. For interval probabilities it is calculated using the conjugate $U(w) = 1 - L(\neg w)$ respectively $L(w) = 1 - U(\neg w)$ as given in section 3.

4.6 Bottom-Up Modelling

When modelling beliefs bottom-up in Probabilistic Uncertainty we start modelling from the propositions referring to the real world and then construct epistemic states, the meta-level. The agent is asked to assign atomic propositions p a probability and uncertainty in the single proposition case for a given rationality. On this level the axiomatic setting of FHM is applied to construct propositional formulae in $\mathcal{L}$. Given the agent testifies to accept a propositional formula in $\mathcal{L}$, that propositional formula is assigned the modality $\mathcal{T}$ and forms an MEL-atom in $\mathcal{L}_{\mathcal{T}}$. On the meta-level epistemic states can be construed. If all truth values of atomic propositions p forming propositional formulae are known so are the truth values of accepted MEL-atoms and the truth-values of the other MEL-atoms are determined by axioms K, D, N. For interpretations of MEL-formulae (i.e. valuations v) satisfying axiom N (i.e. $\tau\mathcal{V} = 1$), axiom D (for all $A \subseteq \mathcal{V}, \tau(A) \leq 1 - \tau(A^C)$), and both K and N (for all $A, B \subseteq \mathcal{V}, \tau(A \cap B) = min(\tau(A), \tau(B))$, epistemic states can be constructed. Such an epistemic state is defined as $E_v := \{w \in \mathcal{V} : \tau(\mathcal{V} \backslash \{w\}) = 0\}$.

Using FHM's framework *measurable* propositional formulae can be evaluated, e.g. $\alpha = p_1 \wedge p_2$, by asking the agent to assign a probability and uncertainty to all atomic propositions p, for example, p_1: 'Mary calls on Saturdays', p_2: 'Mary is on vacation'. The propositions referring to the 'real world' are captured in $\mathcal{L}$, distinct from $\mathcal{L}_{\mathcal{T}}$, the set containing accepted propositions and propositional formulae according to the testified understanding of the actual world. From that we can model epistemic states of the agent where an expectation of the agent's beliefs can be formulated based on the testimony. However, we emphasize that assembling single proposition evaluations

[21] From the qualitative dimensions, even if $\mathcal{L}$ contains only one single proposition, a probability distribution can be created over different rationalities in which the agent testifies. This in turn raises awareness for the context dependency of uncertain belief.

to propositional formulae *creates* minimal contexts for a single proposition. The agent does not necessarily accept the compound, though we assume she/he does given the testimony, what is captured in axiom N[22].

In order to evaluate the empirical adequacy of the approach for *that specific agent*, the agent needs to accept the *compound* propositional formula by testimony. If the agent explicitly accepts $\alpha = p_1 \wedge p_2$ as PL-formula, we are justified to assume that the modelling can be empirically adequate in this case. This is important because the minimal context created by assembling atomic propositions to propositional formulae might impede acceptance. For example, the agent may testify, for the evaluation criterion *occurrence rationality*, to rationally believe p_1 and p_2 independently, but not accept that 'Mary is on vacation and calls on Saturdays', for instance because the agent has implicit knowledge (maybe there is an agreement to send emails instead of calling when on vacation).

Bottom-up modelling should be tested for empirical adequacy of formulae to ensure the set $\mathcal{L}_\mathcal{T}$ holds accepted truths of the particular agent in an actual belief evaluation situation. In a sense we "abstract away" from the agent's assertions by forming propositional formulae, so that we can evaluate the empirical adequacy of acceptance *for that agent* if we "concretise back" to actual statements of the agent (for the constructed contexts) by interrogation.

The construction of minimal contexts (i.e. propositional formulae) *by the agent* may differ from an axiomatic construction according to PL axioms. Also, the construction of MEL-formulae ϕ of accepted propositional formulae, and the inference rules to construct synthetic consequences can lead to logically legitimate but, in a sense, unintuitive contexts from the viewpoint of the agent.

As Chow states Chow (2016, p.7) "it is empirically established that various processes influence our thought and action, but escape introspection." A discrepancy between axiomatically rigorous construction and empirical evidence by testimony is sometimes considered as inconsistent reasoning. It is the very reason why naturalistic accounts to model reasoning emerged. However, we consider deviations from axiomatic compliance not *per se* as irrational human behaviour but as direct proof of implicit beliefs. Also, intentional or unintentional rational mischief could be involved caused, for example, by relations to entrenched beliefs the agent is unaware of. With Probabilistic Uncertainty implicit assumptions, prejudices, unconscious biases, (un-)intentional rational mischief can be investigated in the bottom-up modelling.

To that end, propositions p in an epistemic state are re-evaluated, reasoned about in a context by the agent, and two questions are raised. 1) what propositions need to be accepted additionally or need to be removed in order to *maintain* acceptance and the strength of belief and uncertainty the agent assigned in the single proposition case. This question serves to identify interdependencies. 2) what is the uncertain belief in a proposition p given different minimal contexts in which p figures? This question serves to evaluate the quantitative *variability* of belief in p *for the agent*. Probabilistic Uncertainty can be used to model both interrelations and quantitative uncertain belief magnitudes. In a sense, we evaluate the relevance of the context by changing the context. The extreme contexts are obviously the single proposition case and the universal conjunct of accepted propositions of a belief evaluation.

It is not obvious but important that the *perspectives on* strength of belief and uncertainty are fundamentally different on the level of primitive propositions in $\mathcal{L}$ and on the level of $\mathcal{L}_\mathcal{T}$. In the former case the perspective is on the strength of belief, while in the latter the perspective is on the uncertainty. This is so, because the inner measure of atomic propositions p without testimony is 0, while the "default" inner measure of MEL-atoms is 1. In an epistemic state E a valuation v_E of an accepted propositional formula α is given by

$$v_E(\mathcal{T}\alpha) := \begin{cases} 1 & \text{if } E \subseteq [\alpha] \\ 0 & \text{otherwise.} \end{cases}$$

This, together with $E \vDash \ B \sim \alpha$ if and only if $E \bigcap [\alpha] \neq \emptyset$ allows us to see that. In other words, once a propositional formula is accepted to be true (is a member of $\mathcal{L}_\mathcal{T}$), uncertain belief emphasizes the uncertainty that the propositional formula represents the agent's belief, implying that α is believed less uncertainly too. Uncertain belief is defined as 'not accept not α' and the uncertainty amounts to 'accept not α'. Consequently, uncertainty allows us to model an expectation of strength of belief in a different epistemic state, where the negation of α is accepted.

The modal operator $B \sim$ does not exist[23] for members of $\mathcal{L}_\mathcal{T}$. The strength of belief and associated uncertainty for elements of $\mathcal{L}$ are expressed as $P(\cdot)$ and $\Psi(\cdot)$. A probability assignment x to atomic propositions p emphasizes the strength of belief implying that weaker beliefs in the same proposition are held too. A suspended judgement collapses to complete uncertainty for propositions in $\mathcal{L}$, whereas suspended judgement for propositions in $\mathcal{L}_\mathcal{T}$ collapses to acceptance of the proposition with the corresponding uncertainty of the least upper bound (which must exist unless the agent is inconsistent).

[22] (N) $\mathcal{T}\alpha$, *whenever* $\vdash_{PL} \alpha$.

[23] "In MEL, modalities $\Box$ and $\diamond$ only apply to PL-formulae, contrary to usual modal logics." BD p. 641.

That reflects the difference between reasoning about accepted propositions and reasoning about propositions in general. We consider the single proposition assignments as tentative, indicating which propositional formulae could be accepted by the agent and how uncertainly the agent might believe them. Based on the single proposition assignments we have an expectation for the agent's uncertain belief in α, given by definition 3 $P_\bigstar(\alpha) = sup\{P(p)|p \subseteq \alpha \text{ and } p \in \mathcal{L}\}$.

If the agent does testify uncertainty, the agent holds uncertain belief in the epistemic state $E \vDash B \sim \alpha \wedge B \sim \neg\alpha$ then $\mathcal{T}\alpha \vee \mathcal{T}\neg\alpha$. Generally, epistemic models of a disjunction form the set $\{E : E \subseteq [\alpha]\} \cup \{E : E \subseteq [\beta]\}$ reading that at least one is asserted by the agent. On the meta-epistemic level the agent testifies to *accept truth* - and the gradual belief models the uncertainty that the true proposition in a given context represents the agent's understanding. Epistemic states restricted to *singletons* would "mimic classical semantics" and allow for epistemic states where none of α or β can be asserted. This makes sense because for singletons the epistemic context is equivalent to the modelling of propositions referring to the actual world, where suspending judgement leads to an inner measure of 0.

4.7 Top-Down Modelling

Suppose we are interested in an agents' uncertain belief concerning the social aspects of a state of affairs in the actual world, that is, we are interested how strong a proposition p represents a social rationality for an agent. It is known that agents tend to testify "socially accepted" beliefs in the single proposition case, in particular for social, moral, and ethical evaluation criteria. A well known example is the not-in-my-backyard flip of acceptance and uncertain belief if contexts change, or the "well-behaved" assertion that gender equality is desirable but "for female employees in our company payment isn't that important" (otherwise, they would negotiate better...). For social, moral, and ethical dimensions context is extremely important.

In a top-down setting the agent is asked whether she/he accepts simpliciter propositional formulae α providing a minimal context, and how strong the minimal contexts represent a rationality. Note that this allows for quite complex belief evaluations.

For accepted minimal contexts that share individual constituents, an expectation for uncertain belief can be evaluated across minimal contexts and for constructed epistemic contexts. We can assign an expected lower (upper) bound of uncertainty in epistemic states where the agent accepts propositional formulae, disbelieves α, or suspends judgement on α. Indeed BD provide in section 4.1 the proposition 5, clarifying that belief (plausibility) functions are numerical generalizations of MEL formulae that can be evaluated top-down. They write:

> There is a similarity between the problem of reconstructing a mass assignment m from the knowledge of a belief function and the problem of representing an epistemic state E in the language of MEL given by
>
> $$m(E) = P([\delta_E]) = P\left(\left[\mathcal{T}\alpha_E \wedge \neg \bigvee_{w \in E} \mathcal{T}\neg\alpha_w\right]\right)$$
>
> [where, in our notation $\delta_E := \mathcal{T}\alpha_E \wedge \bigwedge_{w \in E} B \sim \alpha_w$] It is literally the probability $P(\{E\})$ of "only knowing" E. [...] At the syntactic level, one could handle graded modal propositions $\Box_r\alpha$ where α is a proposition in PL, and $r \in [0,1]$ stands for a lower bound for the degree of belief of α. At the semantic level, the satisfaction relation should be of the form $m \vDash \Box_r\alpha$ whenever $\sum_{E \subseteq [\alpha]} m(E) \geq r$. This is clearly one of the perspective opened by our framework.

The uncertain belief operator $B \sim$ corresponds to $\Box_r\alpha$. Adopting the semantic definition of FHM, definition 7, then yields $P_\bigstar \vDash B \sim \alpha$ whenever $\sum_{E \subseteq [\alpha]} P_\bigstar(E) \geq c$.

Suppose $\mathcal{L}_\mathcal{T}$ contains the accepted PL-formulae $\mathcal{T}\alpha$, $\mathcal{T}\beta$ with a probability and uncertainty assignment for a rationality, and $\mathcal{T}\gamma$. Upon interrogation the agent ignores to testify on her/his uncertainty in γ. We can still reason about the agent's uncertain belief in γ which is explicitly ignored if atomic propositions in γ figure in α or β. In particular we can quantitatively assess an expected lower bound of uncertainty.

The inference rules in MEL (proposition 4 BD) apply to encapsulated PL-formulae. Testing logical implications for empirical adequacy by interrogation might lay bare an agent's rational mischief and/or the empirical inadequateness of our modelling. The MEL inference rules, for propositional formulae α, β, γ, are (1) a weakened from of PL modus ponens which "preserves consistency, not certainty (hence not truth) of inner formulae". (2) a kind of resolution rule of possibilistic logic which preserves the "weakest degree of certainty". And (3) the resolution counterpart of (1).

$$
\begin{array}{ll}
(1) & \{\mathcal{T}\alpha, B \sim (\alpha \to \beta)\} \vdash_{MEL} B \sim \beta \\
(2) & \{\mathcal{T}(\neg\alpha \vee \beta), \mathcal{T}(\alpha \vee \gamma)\} \vdash_{MEL} \mathcal{T}(\beta \vee \gamma) \\
(3) & \{\mathcal{T}(\neg\alpha \vee \beta), B \sim (\alpha \vee \gamma)\} \vdash_{MEL} B \sim (\beta \vee \gamma)
\end{array}
$$

Again, our expectation for construed contexts is compared to an agent's testimony for that context to evaluate the empirical adequacy. Unless the agent confirms our expectations, we can read the modal operator $B \sim$ "*uncertain* belief" also as our uncertainty, as modellers, that we correctly capture the agent's strength of belief and uncertainty, based on the contextual information the agent testified.

4.8 Hybrid Modelling

The hybrid modelling approach combines top-down and bottom-up to prudently construe contexts and identify significant propositions that impact the evaluation of compounds (propositional formulae and MEL formulae). We emphasize that formally members of $\mathcal{L}$ cannot be assorted with members of $\mathcal{L}_{\mathcal{T}}$. The hybrid modelling approach can be used to investigate atomic propositions in $\mathcal{L}$ which might be ignored in minimal contexts and accepted truths, for example if (un-)intentional rational mischief occurs. To investigate acceptance, (changed) meaning of a proposition in context, and (changed) meaning of a context w.r.t. a proposition, the single-case and the compound testimony is compared with our expectation as given by the MEL and FHM axioms.

To give an example, suppose the agent, in the course of a belief evaluation of an accepted propositional formula $\mathcal{T}\alpha = p_1 \wedge p_2$ testifies for an economic rationality to accept α and hold uncertainty $\Psi(\alpha) = 0.3$. That is, using the R-probability field as specified in definition 3 where Ψ corresponds to a structure $\mathscr{M}$ the non-empty set of Kolmogorovian probability functions $\Psi = \{P(\cdot)|L(w) \leq P(w) \leq U(w)\}$ *for every* $w \in \mathcal{F}$ hence $\Psi(\alpha) = [0, 0.3]$ for every meta-epistemic context where α is accepted *and* testified.

We might be interested in the variability of uncertain belief in α in the light of a different minimal context, say including p_3, for a given rationality. Suppose the agent testifies to believe p_3 is economical with a strength of belief $P(p_3) = 0.6$. Note, that we construct the belief/uncertainty probability interval from the "other side": the strength of belief without testified acceptance that p_3 *is* economical. The agent herself/himself is uncommitted to the possibility that p_3 is economical, i.e. $P(p_3) = [0, 0.6]$.

We can then evaluate the contextual beliefs of both the propositional formula and the atomic proposition in combination. Let $\beta = p_1 \wedge p_2 \wedge p_3$, then the expectation for the strength of belief and uncertainty for β is given by the the R-field, limited by the highest uncertainty (that is $\Psi(p_3) = [0.4, 1]$ as $L(0.4), U(1)$. We remark that if the agent does *not testify* to accept β, the propositional formula is contained in $\mathcal{L}$ and is in case of suspended judgement assumed to be not accepted. In other words, if an agent's acceptance is a formal consequence (by regarding $\mathcal{T}\beta$ as consequence of individual acceptance of α and β where p_3 figures), it is pure speculation.

The interval captures that if a belief of strength x with $x = [0, 1]$ is testified *for an accepted proposition*, the agent is considered to belief the proposition simpliciter. Uncertain belief denotes the agents conviction that her judgement of a proposition being - for example economical - as accepted is uncertain. We could say it is the error probability agents assign to their own judgement.

If the agent testifies to accept β, then $\mathcal{T}\beta$, and the modal operator reflects the expected interval of uncertainty. The strength of belief corresponds to the F-Probability, the uncertainty to the resulting R-Probability. For the example, given $\mathcal{T}\beta$, we can model the agent's uncertain belief that β represents an economical state of affairs given accepted truth of β by $B \sim \beta = [0.6, 1]$ (acceptance, i.e. strongest belief, "down to" weakest belief), and an uncertainty $\Psi(\beta) = [0, 0.4]$ (acceptance, i.e. no uncertainty, "up to" highest uncertainty) with $\Psi(p_1) = 0.3, \Psi(p_2) = 0.3, \Psi(p_3) = 0.4$. In non-technical terms the agent could say I accept that β is economic by a strength of *at least* 0.6, i.e. 0.6 or higher. Or, put yet differently, in at least 6 out of 10 situations where all statements about the real world β represents for the agent are true, the situation turns out to be economic for the agent.

5 Some considerations on Belief Change

A human being is subject to perpetual change. With every second passing the human being has consciously or subconsciously processed thoughts and beliefs, has undergone experiences and interacted with the environment as for example studied in phenomenology. A human reasoner may, after even a short period of time, evaluate the beliefs held in a proposition *differently* given the *same* set of accepted truths, i.e. $B \sim$ is *unstable* over time. Also, the propositions a human reasoner considers relevant for the evaluation of beliefs for a given rationality may change without obvious reason from an inter-subjective perspective, i.e. $\mathcal{L}_{\mathcal{T}}$ is *unstable* over time. That reasoning of human agents is influenced by many internal and external stimuli is sometimes perceived as psychological bias, as flaw of a

human reasoner in contrast to, say, AI. Chow points at well known phenomena, explicitly mentioning the *framing effect*. The phenomenon of judging information in resonance with the *way information is presented*, unconsciously or consciously biasing the judgement, is called the framing effect, cf. Levin et al. (1998).

In Probabilistic Uncertainty both accepted truths and uncertain beliefs are not considered as (non-physical) objects the human reasoner *possesses*, rather these notions refer to concepts the human reasoner *creates* ad hoc, for a given intention serving the purpose of gaining awareness. The reasoner is not bound to the result of a previous reasoning process.

Viewed from this perspective the reasoner does not *change* a belief, rather the human reasoner *creates* beliefs as an active, immediate response to both, herself/himself and her/his environment. This is a fundamentally different perspective than many belief modelling frameworks adopt, for example the AGM theory with deductively closed belief sets, or other approaches employing belief bases that are not necessarily deductively closed. In the same line Bayesian belief change models typically depart from an (arbitrarily chosen) set of propositions and condition the belief function according to evidence. Adopting the view that belief ought to be conditioned on evidence is sometimes called the *evidentialist* view, such that the disposition to believe is in accordance with, or a function of, evidence. In contrast one can view evidence as a virtue arising from an agent's disposition to believe, a view called *virtue epistemology*, holding that agents are at the origin of epistemic values and hence the "nature" of cognition is under scrutiny in that perspective Greco (1993).

That "nature" of cognition is typically discussed from the perspective of a moral, ethical, or value theory. An overview of arguments in the contemporary debate by the most acknowledged researchers can be found in Fernández Vargas (2016), Fairweather and Zagzebski (2001), Axtell et al. (2000).

Though it is our fundamental interest to relate our ideas to these accounts we postpone the taxonomy of Probabilistic Uncertainty as we seek in this discussion to establish a thorough understanding of *what* our ideas actually express given the modelling we propose. The foregone chapters aimed at providing a concrete form of the modelling in terms of conceptual, quantitative and logical structure. Based on an understanding of "what it is" we discuss "what it means" in future work to do justice to the complexity of the existing arguments in virtue epistemology. It might well be that in the course of this process our modelling approach will change. In our understanding that would be a natural process because the modelling *implicitly* reflects arguments of different schools of thought that may not be obvious prima facie. Our beliefs about what the modelling expresses are but one perspective, and once the "facts" are inter-subjectively communicable, i.e. are given a linguistic, quantitative, and logical form, the ideas can be regarded from different perspectives and may prove to support or contradict various arguments. However, we regard these (potential) changes not as a process of *improvement*, rather, we consider our potentially arising change in views and modelling as structurally independent. In the same sense, we consider the beliefs an agent testifies in a belief evaluation as independent from new information and no learning in the conventional meaning is modelled. A "new belief evaluation" is strictly independent of former testimony. We take a *repeated* belief evaluation to be a timely distinct evaluation of beliefs expressed as linguistically similar propositions.

Generally, beliefs in our understanding are not held "stronger" by virtue of being repeatedly part of a reasoning process, i.e. the relative frequency of a belief is considered to be a property of the reasoner, rather than a property of the belief. In other words, the human reasoner may choose to repeatedly create a belief, or not. Equivalently, we consider the "connection", i.e. the relation, connotation, or intuitive plausibility of the relation of two propositions not as "stronger" based on the re-creation of that connection in a new reasoning process.

Beliefs in our understanding are not held "more true" by virtue of being repeatedly part of a reasoning process. The relative strength of belief is an instantaneous, direct assignment independent of former results of reasoning processes (belief evaluations).

Beliefs in our understanding are not held "exclusive" in repeated reasoning processes. A reasoning process, as conceived in this discussion, is composed of a reasoner who chooses to accept propositions to be true and evaluates the strength of uncertain belief in (some of) the propositions for a given rationality. A proposition can, in the same set of accepted propositions simultaneously be assigned different strengths of uncertain belief for different rationalities. Each assignment is considered to be unique, reflecting the agent's understanding of the propositions in question in the context, the rationality, and the in sito situation of the agent during the belief evaluation. Conceptualizing beliefs as subjective, instantaneous expressions *created* by a human agent in response to (a naturally changed) Self and environment embraces the paradigm that belief is an awareness problem, not a decision problem. The human agent becomes by re-creation aware of what is believed in that moment.

Obviously, conceptualising beliefs as representations of an unique understanding based on ad hoc reasoning processes expressed as structure (epistemic states) and rational strength of belief (the agent is assumed to genuinely *know*), defies generalisations of a single belief evaluation to a reasoners beliefs held. However, modelling beliefs bottom-up, hybrid, and top-down creates contexts and expectations that can only be warranted if we *assume* gen-

erality of the agent's testimony. We assume generality in order to justify *our* expectations based on the agent's testimony and have the agent legitimate our expectations by repeated inquiry to evidentially support *our* understanding of her/him. We emphasize that an agent who is rational according to definition 1 may *not* confirm our expectation. Our understanding of the agent as modellers is limited to what has been testified and the assumption of generality can be regarded as auxiliary assumption reflecting the modellers imperfect knowledge. The more the agent testifies, the more empirical adequacy can be reflected in the modelling - we could say we get to know the agent better. But more importantly, as the modelling allows us expressing an expectation for the structure and strength of beliefs of an agent, we might create logically consistent contexts the agent may not have been aware of - we could say the agent gets to know herself/himself better when evaluating the empirical adequacy of our modelling.

What we want to investigate in future work is how our intuition that acceptance gives rise to belief unfolds in MEL. Once a proposition is accepted by an agent, that is, once the agent testifies that a proposition *can* represent a rationality for her/him, we can by supposing that the proposition is false "open" a new epistemic state for which we can formulate expectations using logical equivalence and the quantitative properties of strength of belief and uncertainty by $\Psi = P(A^C)$.

Although the modelling can be read in that way, the non-ideal agent modelled is not considered to *possess* a belief set or belief base to which a belief revision process could apply. The propositions in the sets $\mathcal{L}$ and $\mathcal{L}_{\mathcal{T}}$ are considered as inter-subjective objects that are not attached to the agent. In a sense one could view the propositions representing the actual world and forming epistemic states as place-holders, "adopted" for a period of time by the agent to exercise - to express - a reasoning process. This adoption does not inflict any commitment to the agent to accept and (uncertainly) believe a proposition in future belief evaluations or logically constructed contexts.

The reasoning process is finished when the purpose of a belief evaluation is met. The purpose is *gaining awareness*. In contrast to, for example, the the AGM framework Alchourrón et al. (1985) operations as expansion, contraction, and revision are meaningless, because the ad hoc belief base (the accepted truths) of a reasoning process at a time t_0 is not necessarily a representation of an ad hoc set of accepted truths at a time t_1. That is, there is no assumption of a "stability property" whatsoever adherent to the *structure* of reasoning in terms epistemic states and possible worlds arising from accepted truth beyond the modelled reasoning process in question. Equivalently no stability is assumed for the *result* of the reasoning process in terms of strength of belief and uncertainty. Quite on the contrary, instead of belief reinforcement to create stability through evidential support, repeated belief evaluations of the same beliefs are likely to increase the understanding of the variability of strength of belief and uncertainty.

By this conception the reasoning process is most compliant with the purpose, as *more* awareness is gained, if the non-ideal agent *re-creates*, instead of *adapts* her/his believes. In particular, an agent who acquires information that a belief held is false is modelled in Bayesian frameworks as conditionalisation on events of probability 0. Obviously this is impossible in Kolmogorovian probability where conditional probabilities are defined as ratio. "Preempting the problem by requiring that probability 0 be only assigned to impossible events amounts to stipulating that agents never have any wrong beliefs", cf. Baltag and Smets (2008, p.182), and "collapses belief into knowledge". In our modelling the concept of *accepted* truth reflects that wrong beliefs are basically different epistemic states an agent may well reason about. This, in a sense hypothetical reasoning, may bear important information for beliefs accepted to be true. The connection between these epistemic states is modelled as uncertainty defined as the probability of alternative events. In a sense the "new" information of having a false belief is contained in the modelling of a belief evaluation for a given set of propositions.

Regarding belief evaluations as independent serves the purpose in the awareness paradigm. Again we emphasize that belief conceived a such is not about believing "right", but about recognizing (1) that one believes at all, and (2) recognizing what is believed and the respective strength of belief and uncertainty in a given reasoning process. In this sense, the human agent cannot reason unsuccessfully because success is not determined by inclusion or withdrawal of sentences from a belief set. Instead success is determined by the fact that the agent *acknowledges* her/his reasoning process in terms of structure, i.e. the epistemic states and states of possible worlds of a belief evaluation, and result, i.e. the rationally assigned strength of belief and uncertainty. If this awareness is gained the reasoning process is finished what implies that the agent can use the "additional" awareness available when she/he *creates a new reasoning process.*

6 Conclusion Section and Further Research

In contrast to other belief modelling frameworks, as for example Leitgeb's Stability Theory of Belief Leitgeb (2014), we do *not* consider the proposition that is assigned a stable probability in different contexts to be *the* rational (strength of) belief. The nomination of resilient beliefs as rational does not make sense in the awareness paradigm, as awareness of uncertain belief in propositions is increased with flexibility and courage to introspect interrelations

and gradual (quantitative) credences. The modelling of epistemic states allows formulating an expectation for uncertain belief that is based on testimony in different contexts (both minimal contexts and epistemic contexts). But requiring the agent to firmly hold a strength of belief within the expectation to call her/him rational is *not* our modelling approach. Rather, we consider the agent to hold rational beliefs and attribute the deviation of expectation and testimony to our imperfect knowledge - and hence modelling - of the agent's epistemic state. That stance allows us to investigate implicit beliefs to improve our modelling.

There exist approaches to model partial inconsistency and defeasible reasoning defining *a priori* criteria for the "limit" of rational reasoning. As such the agent's *rationality is a function of compliance to a priori axioms.* However, a naturalistic account that aspires empirically adequacy ought in our view consider an agent as source of legitimation for axiomatic formulations. As such *consistency is a function of axioms complying to the agent's a priori rationality.* In Probabilistic Uncertainty we can measure the expected strength and uncertainty of belief based on axioms of FHM and BD and handle the deviation of our expectation and empirical fact (i.e. the testimony) as contextual uncertainty.

Instead of requiring the agent to conform to axioms to call the agents' beliefs rational, we solicit the agents' evaluation whether the contexts constructed *are* rational, according to the agent. We do so by interrogating acceptance as belief simpliciter and gradual strength of belief and uncertainty for the construed context, and compare our expectation with the agent's reality. Adopting this interpretation we consider the agent's testimony to be not only an expression of her/his beliefs but also as *quality evaluation of Probabilistic Uncertainty* in meeting the aspiration of being empirically adequate.

It may have appeared in the outlines that if an agent is particularly "ambitious" in her/his reasoning under the awareness paradigm, she/he might well arrive at Socrates ultimate wisdom: I know that I know nothing. The insight that characterises Probabilistic Uncertainty could best be described by "I am certain that all my knowledge is uncertain." We draw several conclusions and sense some potential consequences assuming that the framework indeed can foster an agent's ultimate conclusion that certainly all knowledge is uncertain as it is a function of context.

An expected epistemological consequence is the awareness that the *strongest notion* of belief, i.e. belief with probability 1, demands either for a "weak" (that is consequentially limited) concept or for a "weak" generality (that is a limited context). The former is captured in the assignment of binary truth to *possibility* (accepted truth), while the latter is captured in the assignment of gradual belief in *epistemic states* (uncertain belief as conditional probability). The variability of strength of belief and uncertainty becomes under the awareness paradigm a tool enhancing the subjective understanding of mutual impacts and relational significance of accepted truths under different evaluation criteria and different epistemic states.

An expected ethical consequence is the realization that accepting a proposition to be true is a responsible choice. By choice of accepted truths the agent determines the limits of what is considered possible and relevant. All reasoning contexts are derived from that choice and the agent, sooner or later, has to rise to the challenge of accepting propositions intentionally as effective way to reduce systematic biases with the ambition to avoid (un-)intentional rational mischief.

An expected psychological consequence is the experience that the agent *has* a sense of certainty in a context. One of the negative effects of the worlds' limited predictability is, for some humans, a profound feeling of *insecurity*, leading to a fatalist distortion of Socrates' wisdom from 'I know that I know nothing' to 'I know that I cannot know anything'. This is a disempowering perspective paving the way for all sorts of self-destructing feelings and conclusions. The emphasis of Probabilistic Uncertainty lies in the agent's awareness that 'I am *certain* that all my knowledge is uncertain'. The respectful treatment of individual truth and the demand to testify uncertain belief *as it is* in order to be rational nurtures an agent's "connection" to the agent's internalist truth and knowledge. Psychologically, self-truthfulness is a form of self-esteem eventually empowering the reasoner who is daring enough to confront (personal) truth.

An expected practical consequence of the awareness that beliefs are relative to acceptance is an understanding that the strength and uncertainty of beliefs is *always* subjective for the person accepting and believing and as such defies judgemental statements about another person's beliefs. We belief this to be *the* feature of Probabilistic Uncertainty enhancing sovereignty and respect in a practical, inter-subjective sense. Being aware from introspection that one's own belief *is* uncertain and its strength is derived from *possibilities chosen to be relevant* fosters tolerance for possibilities other agents choose and emerging uncertain beliefs in their reasoning contexts.

We consider Probabilistic Uncertainty as a modelling framework to illustrate and foster reasoning in the awareness paradigm because we think it is a particular responsibility *researchers* have when accounts are put forward that potentially fuel conflict, separation and discrimination by their modelling and definition of rationality, beliefs, and knowledge. Rather than calling an agent's belief wrong or irrational we encourage the agent to acknowledge that

by the agent's own light "right" is a relative notion and that being rational is eventually incompatible with self deceit.

7 Remarks

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7.2 Biography

M.Culka holds a Ph.D. in philosophy of science, a M.Sc. in Environmental Technologies and Economics, and a B.A. in Energy Economics. Recent works have focussed on methodological choices in scenario construction and quantitative vs. qualitative modelling techniques.

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