The development of renormalization group methods for particle physics: Formal analogies between classical statistical mechanics and quantum field theory

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ABSTRACT: Analogies between classical statistical mechanics (CSM) and quantum field theory (QFT) played a pivotal role in the development of renormalization group (RG) methods for application in the two theories. This paper focuses on the analogies that informed the application of RG methods in QFT by Kenneth Wilson and collaborators in the early 1970's (Wilson and Kogut 1974). The central task that is accomplished is the identification and analysis of the analogical mappings employed. The conclusion is that the analogies in this case study are formal analogies, and not physical analogies. That is, the analogical mappings relate elements of the models that play formally analogous roles and that have substantially different physical interpretations. Unlike other cases of the use of analogies in physics, the analogical mappings do not preserve causal structure. The conclusion that the analogies in this case are purely formal carries important implications for the interpretation of QFT, and poses challenges for philosophical accounts of analogical reasoning and arguments in defence of scientific realism. Analysis of the interpretation of the cutoffs is presented as an illustrative example of how physical disanalogies block the exportation of physical interpretations from from statistical mechanics to QFT. A final implication is that the application of RG methods in QFT supports non-causal explanations, but in a different manner than in statistical mechanics.

> It is held by some that the "Renormalization Group"—or, better, renormalization groups or, let us say, *Renormalization Group Theory* (or RGT)—is "one of the underlying ideas in the theoretical structure of Quantum Field Theory." That belief suggests the potential value of a historical and conceptual account of RG theory and the ideas and sources from which it grew, as viewed from the perspective of statistical mechanics and condensed matter physics. Especially pertinent are the roots in the theory of critical phenomena.
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> The proposition just stated regarding the significance of RG theory for Quantum Field theory (or QFT, for short) is certainly debatable, even

though experts in QFT have certainly invoked RG ideas. Indeed, one may ask: How far is some concept only instrumental? How far is it crucial? ... (Fisher 1999, 91)

1 Introduction

One of the insights that inspired developments in many fields of physics in the twentieth century was that some problems in particle physics take the same form as some problems in condensed matter physics. This similarity in form was not obvious *a priori* because particle physics and condensed matter physics concern disjoint domains of phenomena. The core phenomena of particle physics are scattering experiments, in which the results of particle collisions are detected. A recently celebrated example is the experiments at the Large Hadron Collider that resulted in the discovery of a particle consistent with the Higgs boson. A central example of the phenomena of interest in condensed matter physics is phase transitions, such as the transition from liquid to gas in a steaming coffee cup and the transition from being demagnetized to spontaneously magnetized in an iron bar. The physical processes probed by particle colliders are entirely different from those in systems undergoing phase transitions; however, the theoretical treatments employ a similar mathematical formalism. More specifically, renormalization group (RG) methods are applied in both quantum field theory (QFT), the theory which underlies particle physics, and classical statistical mechanics (CSM), one of the theories underlying condensed matter physics. The theoretical problem of how to renormalize (certain) quantum field theories turns out to take the same form as the theoretical problem of how to explain the universality of the values of the critical exponents for diverse classes of condensed matter systems.

In the above quote, renowned condensed matter physicist Michael Fisher raises conceptual questions about the application of RG theory in QFT and suggests that the historical origins of RG methods in condensed matter physics could yield some insight. For philosophers, these historical developments are also of interest because they furnish a case study of two features of scientific practice: analogical reasoning and the application of mathematics. The fact that both analogical reasoning and applied mathematics work in concert is, of course, not a unique feature of this case study. A historical example is the application of equations taking a similar form in theories of heat, fluids, and electromagnetism in the nineteenth century. Two other contemporary examples are the applications of information theory in statistical mechanics and quantum theory and of thermodynamics to black holes. All of these cases of simultaneous employment of analogical reasoning and application of similar pieces of mathematics raise a set of obvious questions: Should the analogies and application of the similar mathematical formalisms be taken as an indication that there are substantive physical similarities between the domains? If not, what types of analogies are used? Why do these analogies hold and why is the same mathematical formalism applicable in both domains? These questions need to be answered on a case-by-case basis. The focus here will be on the former questions about the types of analogies employed in the case of the application of RG methods in QFT.

This paper will examine only one facet of the multi-faceted analogies which have been drawn between particle physics and condensed matter physics: the analogies between CSM and QFT which underlay the development of RG methods for application in QFT, culminating in the work of Wilson and his collaborators in the early 1970's (Wilson and Kogut 1974). This strand of analogy can be disentangled from the other strands (to a first-order approximation) by locating it in its historical context. Over the course of the twentieth century, the fields of quantum theory and statistical mechanics became more and more intertwined. The birth of quantum mechanics, at the beginning of the twentieth century, was prompted in part by the statistical mechanical analysis of black body radiation. Beginning in the 1920s and 1930s, applications of quantum theory in solid state physics were facilitated by analogies between fundamental particles and quasi-particles. Mid-century, Feynman diagrams were borrowed from QFT for applications in statistical mechanics. The quantum statistical mechanical BCS model for superconductors was a source of inspiration for the introduction of spontaneous symmetry breaking into particle physics in the 1960's (Jona-Lasinio 2010; Fraser and Koberinski 2016). In the early 1970's, Wilson drew on contemporary treatments of critical phenomena to first provide a satisfactory theoretical framework for critical phenomena within statistical mechanics[1] and then to provide a satisfactory framework for renormalization within QFT (Wilson 1971; Wilson and Kogut 1974). Analogies between statistical mechanics and QFT proved to be useful heuristics in both directions. Kadanoff glosses this history of interactions between statistical mechanics and QFT in terms of renormalization: "renormalization methods were developed first in classical field theory (i.e., classical statistical mechanics), extended to quantum field theory, brought to maturity in application to phase transitions, and then triumphantly reapplied to quantum field theory" (2013, 24).[2]

[1] The application of RG methods to statistical mechanics was the work for which Wilson won the 1982 Nobel prize in physics. The citation reads "for his theory for critical phenomena in connection with phase transitions."

[2] Jona-Lasinio portrays the final chapter of the story somewhat differently, describing RG methods from particle physics as being reformulated in probabilistic terms and then re-applied in SM to explain universality (2010). For present purposes, this difference of opinion is not important.

This paper will analyze the key development presented in Wilson and Kogut's canonical 1974 article: the application of RG methods to QFT by analogy to CSM. (N.B. The analogue is *classical* statistical mechanics, not quantum statistical mechanics.) Their central examples are a variant of the classical statistical mechanical Ising model and a scalar ϕ^4 QFT. Wilson was conscious of the role that analogical reasoning played in his discoveries. He explicitly notes the role that the analogy between statistical mechanics and QFT played in his development of RG methods for application in QFT in both Wilson and Kogut (1974) and the autobiographical account in his Nobel prize lecture (1983). Wilson started out working in QFT, but he reports that even as a new graduate student (in the mid-1950's) his attention was directed to the Ising model in statistical mechanics by Murray Gell-Mann (1983, 591).[3] Wilson eventually followed up on this suggestion in 1965, which led him to the work of Widom, Kadanoff, and Fisher (among others) on critical phenomena. He recalls that reviewing this work on block scaling laws lead him to the analogy that is the focus here:

> I now amalgamated my thinking about field theories on a lattice and critical phenomena. I learned about Euclidean (imaginary time) quantum field theory and the "transfer matrix" method for statistical-mechanical models and found there was a close analogy between the two. (1983, 592-3)

He cites Wilson and Kogut (1974) at the end of this passage.

Wilson's insight was that, when posed in a certain form, there were analogies between the theoretical problems facing, respectively, QFT and statistical mechanics circa 1970. In order to apply QFT to interacting systems (such as those subject to experiment in particle colliders), renormalization is required. In essence, renormalization techniques serve to ensure that meaningful predictions can be extracted from the theory. Without renormalization, for instance, the quantities that encode the probability amplitudes for the outcomes of scattering experiments are infinite. In the late 1940's, Feynman and others had developed perturbative renormalization techniques (including Feynman diagrams) and successfully applied them to electromagnetic interactions, formulating quantum electrodynamics (QED). While perturbative renormalization methods proved very successful for calculating scattering matrices in QED, physicists worried about the cogency and foundations of QED. Furthermore, perturbative renormalization techniques could not be straightforwardly applied to

[3]Specifically, Gell-Mann presented the problem of solving the equation for the partition function for the three-dimensional Ising model.

strong interactions because the coupling constant is large, thus not amenable to perturbative expansion. Wilson recollects that "[b]y 1963 it was clear that the only subject I wanted to pursue was quantum field theory applied to strong interactions" (1983, 590). From the point of view of particle physics, one of the chief achievements of Wilsonian RG methods was to bring conceptual clarity to renormalization procedures.

In parallel, the field of statistical mechanics was facing its own crisis (Kadanoff 2013). In the 1960's, physicists recognized that the existing theoretical treatments of critical phenomena were inadequate. A characteristic feature of critical phenomena is the presence of correlations between localized properties which extend across the entire system. For example, in a ferromagnet (e.g., iron bar) the *critical point* is the temperature at which spontaneous magnetization becomes possible. The magnetic properties of an iron bar can be modeled using the Ising model, in which the iron bar is represented by an infinitely[4] long lattice of atoms (i.e., the atoms are separated by a finite distance, the lattice spacing). Each of the atoms has a "spin" which can either point up or down. The bar becomes magnetized when the majority of spins across the bar point in the same direction. When the bar is put in a uniform external magnetic field, the magnetic field causes the spins to align. Spontaneous magnetization occurs when the spins "spontaneously" align in the absence of an external magnetic field. This is only possible below a certain temperature—the *critical temperature*. Above the critical temperature, thermal agitation tends to knock the spins out of alignment. Below the critical temperature, once a block of spins becomes aligned, it tends to force its neighbours to align in the same direction. Eventually, all of the spins in the iron bar point in the same direction; that is, the directions of the spins are correlated across the entire length of the iron bar. The symbol ξ_{CSM} represents the correlation length. At the critical point, the correlation length diverges: $\xi_{CSM} \to \infty$. This is a characteristic feature of the critical point. The behaviour of systems near criticality is characterized by critical exponents. By the late 1960's physicists had measured the values for the critical exponents of many different systems. They found that there was remarkable agreement between the values of critical exponents for broad classes of systems. This was surprising because the systems in each class have much different atomic structures. This phenomenon—which is known as *universality*—required a proper theoretical treatment and cried out for an explanation.

A substantial portion of this paper will be devoted to the conceptual bookkeeping task of identifying the analogies between CSM and QFT underlying the development

[4]The assumption that the lattice is infinite in extent is known as taking the *thermodynamic limit*. In this paper, the thermodynamic limit is accepted as a background assumption for the purposes of investigating the relationship between CSM and QFT (see Section 3.1 below).

of RG methods for application in particle physics. Since this a complex example of analogical reasoning, this is a non-trivial and worthwhile exercise. The philosophical focus of the paper will be classifying the analogies according to type. In brief, a *physical analogy* relates elements of two theories which are relevantly physically similar. In the most straightforward cases, a *formal analogy* relates elements of two theories that play the same formal role in a common mathematical formalism. For example, there are formal analogies between corresponding elements of theories for different domains that share the same mathematical structure. (In the case of purely formal analogies, the shared mathematical structure is given different physical interpretations). The types of formal and physical analogies relevant to the case study are laid out in Sec. 2. The main thesis is that the Wilsonian approach to renormalization invokes purely formal analogies. The analysis in Sec. 4-6 supports this conclusion by identifying formal analogies and substantial physical disanalogies. For philosophers of physics, the main pay off of a careful analysis of the analogies between CSM and QFT is the light shed on whether interpretative morals can be carried over from CSM to QFT (or vice versa). It is particularly enticing to transfer physical interpretations from CSM to QFT because the physical interpretation of CSM is clearer than that of QFT. However, appreciation of the physical disanalogies undermines this interpretative strategy, as the critical analysis of a textbook interpretation of the cutoffs in Sec. 6.3 illustrates. For philosophers of science, the CSM-QFT case study is primarily of interest for the light that it sheds on the role of analogical reasoning, the applicability of mathematics, and the scientific realism debate. This case challenges the assumption that purely formal analogies are not heuristically useful and also challenges a standard argumentative strategy for defending scientific realism.

This paper is complementary to most of the recent philosophical work on RG methods. To orient the reader, here is a rough sketch of how this paper fits into that literature. Wilson and his collaborators had to put together many ideas in order to formulate RG methods; the analogies between CSM and QFT emphasized here were only one strand of the reasoning. For example, another important insight, published for the first time in Wilson 1970, was that RG methods permitted the QFT Hamiltonian at a high energy scale to contain arbitrarily many coupling constants while the effective, low energy Hamiltonian essentially depended on only a finite number of coupling constants (Wilson 1983; Kadanoff 2013). The idealization involved in taking the thermodynamic limit has inspired a lively debate; in the CSM models considered here, the thermodynamic limit is taken and this move will be accepted uncritically. Papers that focus on RG methods in particle physics have also honed in on emergence and reduction as a central issue, often in the context of the 'effective field theory' program for QFT (e.g., Hartmann (2001), Batterman (2011), Butterfield and Bouatta (2014)).[5] The effective field theory viewpoint has also informed a new variant of scientific realism (Fraser 2017; Williams 2017). The scope of this paper is restricted to the issues raised by the analogies from CSM to QFT that informed Wilson's application of RG methods to QFT circa 1974. The effective field theory approach was a later development first suggested by Weinberg in 1979 that was prompted by the application of RG methods to explain universality in statistical mechanics (Weinberg 2009). This thread of the analogies between statistical mechanics and QFT makes brief appearances in this paper, but neither the effective field theory program nor emergence will be discussed. However, the analogies examined in this paper are conceptually prior to the effective field theory program, and as a result offer deeper insight into some of the same interpretative issues, such as the role of causation in QFT models. The claim that effective QFTs are only defined with respect to a finite physical cutoff—often associated with the Wilsonian approach to renormalization—will be critically examined from the perspective of Wilson's actual approach to renormalization circa 1974 in Sec. 6.3.

The CSM-QFT case study is incredibly rich. This paper will examine only the analogical mappings invoked by the Wilsonian approach to renormalization. Wilson's source for Euclidean quantum field theory was a 1966 paper by Kurt Symanzik (Wilson 1983, 592-3; Symanzik 1966). At roughly the same time as Wilson and collaborators were developing RG methods, this paper by Symanzik also inspired an alternative program for renormalizing QFT within the axiomatic approach to QFT.[6] The Euclidean approach to constructing models of a set of axioms for QFT (e.g., the Wightman or Haag-Kastler axioms) for particular interactions was pursued by, among others, Guerra, Simon, Rosen, Jaffe, and Glimm. This approach is also based on analogies between CSM (including a variant of the Ising model) and QFT (including a scalar ϕ^4 interaction); furthermore, it also involves a scaling transformation similar to the RG transformation. A companion paper will compare and contrast the analogies drawn by the two approaches. Interestingly, the approaches agree on a core set of analogical mappings, but also introduce different, incompatible sets of mappings.

Finally, a note on terminology. In the physics literature, the formal analogies

[5] Even many of the discussions that focus on particle physics shift to condensed matter physics (e.g., Butterfield and Bouatta (2014, 32) and Batterman (2011)). This is understandable, since the condensed matter physics case is more tractable. However, the main question asked in this paper is whether interpretative morals can be exported from one context to the other. The substantial physical disanalogies mean that caution is needed in adopting this approach.

[6] For various reasons, Symanzik's suggestions were not taken up by the axiomatic and constructive QFT community immediately. See Wightman (1976, 4–6) and Jaffe (2008, 224–225) for discussion.

examined in this paper have led to terminology being transferred back and forth between CSM and QFT. For example, the statistical mechanical term "correlation function" is often used in QFT. Itzykson and Drouffe explain in the Preface to their textbook *Statistical Field Theory*, "[w]e often switch from one to the other interpretation, assuming that it will not be disturbing once it is realized that the exponential of the action [in QFT] plays the role of the Boltzmann-Gibbs statistical weight [in SM]" (xii-xiii). While the sharing of terminology may not be problematic for purposes of applying the theories, it does make it difficult to identify and analyze the analogies between the theories. In this paper, I will refrain from using the same terminology to label formal analogues in CSM and QFT. (For example, "correlation function" will be reserved for CSM and "vacuum expectation value" for QFT.) I will also reserve "QFT" for the relativistic quantum field theories which get applied in particle physics.

2 Types of formal and physical analogies

The CSM-QFT case study is simultaneously an example of the heuristic use of sophisticated analogical reasoning and the development of applied mathematics. The primary distinction required for the analysis of the case study is the distinction between formal and physical analogies. Intuitively, physical analogies are based on physical similarities and formal analogies are based on formal (e.g., mathematical) similarities. These categories are not mutually exclusive; an analogical mapping may be both physical and formal. Subcategories of formal and physical analogies will also be distinguished in order to characterize novel features of this case study. The philosophical framework set out in this section draws on aspects of Hesse's and Bartha's accounts of analogies, adapted to suit contemporary physics. (A more comprehensive discussion of how this framework relates to earlier accounts will be presented elsewhere.)

To characterize *strict formal analogies*, consider the example of the applications of the wave equation

$$\frac{\partial^2 f(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 f(x,t)}{\partial t^2} \tag{1}$$

to describe idealized sound and water waves. For water, the physical interpretation of $f(x,t)$ is displacement and v is the phase velocity; for sound, $f(x,t)$ is pressure and v is the speed of sound. The wave equation mediates an analogy between water (the source domain) and sound (the target domain). Applying Hesse's framework, there are horizontal relations (aka analogical mappings) between displacement and pressure as well as phase velocity and speed. There are vertical relations between displacement and phase velocity in the source domain and vertical relations between pressure and the speed of sound in the target domain. Following Hesse's analysis (but not her terminology), there is a *strict formal analogy* between these horizontally related elements because there is a "one-to-one correspondence between different interpretations of the same formal [i.e., uninterpreted] theory" (68). That is, there is "no horizontal similarity independent of the vertical relation" (68).

In contrast to formal analogies, physical analogies require some physical similarity between the source and target domains. For example, in the sound–water wave case, there are physical similarities between horizontal relata (e.g., phase velocity, speed of sound). There are also physical similarities between the corresponding vertical relations (i.e., causal relations map to causal relations). (Of course, there are also physical dissimilarities; otherwise, the correspondence would be an identity rather than an analogy.) Bartha (2010, 207–210) draws a similar contrast between formal and physical analogies. Material analogies are an important sub-category of physical analogy. Following Hesse loosely,[7] a *material analogy* obtains when the vertical relations in the source and target domains are causal relations of the same kind.

Strict formal analogies are the simplest and tidiest cases of formal analogies, but in the context of discovery the use of analogies and the development of applied mathematics is not always this tidy. Often similar mathematical formalisms are applied but there is not a single, common uninterpreted mathematical formalism that is given different physical interpretations in the two domains. Bartha captures this case in his characterization of formal similarity as "[t]wo features are formally similar if they occupy corresponding positions in formally analogous theories" (195). These *liberal formal analogies* are a species of formal analogy because they are in the same spirit as the stricter notion. In both cases, the horizontal relations are determined by the formal or mathematical vertical relations and there is no requirement that the analogical mappings be determined by physical similarities.

To foreshadow the arguments in the following sections, the Wilsonian approach to renormalization is based on liberal formal analogies between CSM and QFT. The liberal formal analogies are not underwritten by either material or physical analogies. The argumentative strategy for establishing these conclusions has a positive and a negative component. The positive strand identifies the formal roles that horizontally related elements play in CSM and QFT. The negative lines of argument identify substantial physical disanalogies between the horizontal relata and rule out the possibility that the analogical mappings preserve the causal, modal, or temporal structure of CSM. The Wilsonian derivation of a model of a renormalized, continuum, effective QFT from a CSM model is broken down into three steps. After the

[7] Hesse imposes additional requirements such as that some of the relata be observables.

explication of each step, the analogies that are invoked will be identified and both the positive and negative arguments will be laid out.

3 The Wilsonian approach to renormalization

Wilson remarks that "[p]roblems with infinitely many variables can be very difficult to solve" (Wilson 1975, 773). This remark—which was somewhat of an understatement in 1975!—captures the core mathematical similarity which Wilson perceived between the problems of treating critical phenomena in CSM and interactions in QFT. Two aspects of this mathematical similarity deserve special emphasis at the outset. First, the similarity is between *mathematical problems* generated by CSM and QFT models rather than between the models directly. Second, the focus is on an infinite number of variables because this is the case in which divergences occur, which makes the problems difficult to solve. In QFT, the divergences occur in the (unrenormalized) expressions for scattering amplitudes and, in CSM at a critical point, the correlation length diverges. Figuring out how to take appropriate limits is the key to solving both problems.

In more detail, in their "philosophical discussion of the renormalization group" in Sec. 1.1, Wilson and Kogut explain that the problems of treating critical phenomena in CSM and interactions in QFT share the characteristic feature that they involve an enormous number of degrees of freedom within a correlation length (79). Both sets of problems are also "noted for their intransigence" (79). For QFT, their assessment is that "[t]here has been sensational progress in *calculating* quantum electrodynamics, but very little progress in *understanding* it; and strong interactions are neither calculable nor understood" (79). RG methods are presented as having two objectives: (1) "the practical one of simplifying the task of solving systems with many degrees of freedom contained within a correlation length" and (2) "to explain how the qualitative features of cooperative behavior arise" (79, 81). Objective (1) is the one that pertains to QFT.[8] Attaining this objective involves establishing that RG methods afford a renormalization procedure for (some) interactions which furnishes both understanding and the ability to calculate scattering amplitudes. Sections 10-12 of their paper explain how the renormalization group transformation and the analogy with CSM can be deployed to define a renormalization procedure. In broad outline, the first step of the renormalization procedure is to artificially impose a

[8]In statistical mechanics, Objective (2) is attained by recognizing that qualitative features of cooperative behaviour are determined by the fixed point of the RG transformation, not the initial Hamiltonian for the system. For further philosophical discussion of universality in critical phenomena see Batterman (2002). (See also Sec. 6.1 below.)

lattice ($a = const$), which renders the infinite quantities finite. The second step is to carefully remove the lattice ($a \to 0$) in such a way as to keep the quantities of interest finite. Doing this carefully involves using the correspondence between CSM and QFT to control the limit.

The primary aim of Sec. 4-6 will be a conceptual bookkeeping task: keeping track of the analogical mappings between CSM and QFT which underlie the Wilsonian approach to renormalization in QFT. The presentation will closely follow that in Wilson and Kogut (1974, Sec. 12.2).

For concreteness and simplicity, the CSM model chosen for consideration is a variant of the classical Ising model for a ferromagnet. Each point on a d-dimensional spatial lattice is associated with a spin variable s_n that may take the values $+1$ or -1 (Wilson & Kogut 1974, Sec. 1.3). The Hamiltonian contains only nearest-neighbour interactions.[9] The quantum field theoretic system under consideration is represented by a scalar quantum field $\phi(x)$ on a $(d-1)$-dimensional spatial lattice and continuous time. The Hamiltonian contains a scalar ϕ^4 self-interaction term.

In the CSM model, the lattice is presumed to have a physical interpretation—the lattice represents the atomic lattice spacing. In the QFT model, the spatial lattice is imposed as a regularization technique for making the theory tractable. Without the lattice, the initially-defined propagators for the system would be infinite. From our QFT perspective, the goal is to find a way to remove the lattice in the QFT (i.e., to take the limit lattice spacing $a \to 0$) while keeping the propagators well-defined. The strategy for achieving this goal is to exploit the analogy with CSM. Wilson and Kogut characterize this procedure as a "construction" of QFT from CSM (151, 152). Their construction of continuum QFT from CSM can be broken down into three

[9] In order to set up the correspondence between the correlation functions of the CSM system and the lattice propagator of QFT outlined in Sec. 4 below, Wilson and Kogut modify the standard classical Ising model and make several changes of variables. The standard classical Ising model has partition function

$$Z = \sum_{\{s\}} \exp\left\{ K \sum_n \sum_i s_n s_{n+\hat{\imath}} \right\}$$

where $K = J/kT$ (90, Eq. 2.9). Essentially, this model is modified by smearing with a Gaussian and adding a small quartic term to the interaction (see p.95, and Sec. 3 and 4 for more details):

$$Z = \prod_m \int_{-\infty}^{\infty} ds_m \exp\left\{ -\frac{1}{2} b s_m^2 - u s_n^2 \right\} \exp\left\{ K \sum_n \sum_i s_n s_{n+\hat{\imath}} \right\}$$

where b is an arbitrary constant and u is positive. Fourier transforming to the momentum representation induces a change of variables from (b, u) to (r, s). For the purposes of tracking the analogical mappings, these details are not important. They will be consigned to footnotes.

steps:

STEP 1: Establish the following identity between lattice CSM and the Wick-rotated lattice QFT:

$$\Gamma_{n,m} = \zeta^2 D_m(-in\tau) \tag{Identity}$$

where $\Gamma_{n,m}$ is a statistical mechanical spin-spin correlation function on an m by n dimensional spatial lattice, $D_m(t)$ is a quantum field theoretic propagator on an m dimensional spatial lattice, and ζ is a scale factor. The key to obtaining this identity is that the QFT is *Wick rotated*—that is, the time coordinate $t \to -it'$. Here $t' = n\tau$, where τ is an arbitrary normalization constant.

STEP 2: Set constraint

$$\Lambda_0 = \mu_R \xi_{CSM} \tag{Constraint}$$

where Λ_0 is a (high) momentum cutoff, μ_R is the renormalized, physical mass of the QFT system that is determined by experiment, and ξ_{CSM} is the correlation length of the CSM system. This is the momentum space equivalent of setting $a = \frac{1}{\mu_R \xi_{CSM}}$.

STEP 3: Take the limit in which the momentum cutoff $\Lambda_0 \to \infty$ to obtain the continuum QFT.

Each of these steps is thoroughly explained in sections 4, 5, and 6 below. Step 3, in particular, involves a number of sub-steps, including the introduction of the RG transformation. One point that will be emphasized is that the identity introduced in Step 1 applies to CSM theories and QFTs in general—it is not restricted to CSM systems near a critical point and continuum QFTs. Step 3 is the stage at which the scope of the construction is narrowed to these special cases. This means that (Identity) is widely applicable to problems in CSM and QFT. (Identity) has proven to be very fruitful because it suggests further analogical mappings between SM and QFT that have inspired solutions to problems in domains of SM and QFT outside of critical phenomena and construction of a continuum theory. For example, for constructing models of spontaneous symmetry breaking, the analogical mapping from the generating function of the SM correlation functions to the generating functional of the QFT propagators (or VEVs) plays a significant role (Jona-Lasinio 2002; Peskin and Schroeder 1995).

4 Step 1: Establish (Identity)

The first step in the construction of continuum QFT from CSM is to establish that the following identity holds between CSM on a lattice and QFT on a lattice[10]

$$\Gamma_{n,m} = \zeta^2 D_m(-in\tau) \qquad \text{(Identity)}$$

where $\Gamma_{n,m}$ is a statistical mechanical spin-spin correlation function on an m by n dimensional spatial lattice and $D_m(t)$ is a quantum field theoretic propagator on an m dimensional spatial lattice (Wilson and Kogut 1974, 150). n has dimension 1 because it is interpreted as time in the QFT model. (ζ is a scale factor and the time coordinate t has been set equal to $n\tau$, where τ is an arbitrary normalization constant.) A proof of (Identity) is offered in Wilson and Kogut (1974, 149-150). Proving or deriving (Identity), rather than merely stipulating it, amounts to setting out identifications between other quantities in CSM and QFT and then showing that (Identity) follows. The identifications invoked in the proof of (Identity) point to further analogical mappings between CSM and QFT. The derivation proceeds by equating (up to scale factor) the spin s_m with the field ϕ_m, rewriting and manipulating the expression for the spin-spin correlation function $\Gamma_{n,m}$, specifying a particular form for the QFT Hamiltonian, and then recognizing that the resulting expressions differ by a Wick rotation.[11] Before going through this proof of the identity to pick out the analogical mappings, let's take a moment to explicate the terms in the identity.

In CSM, correlation functions $\Gamma_{n,m}$ are "descriptors... [used] to describe localized fluctuations and their correlations over large distances" (Kadanoff 2013, 24). As Kadanoff explains, fluctuations are spatial variations in the physical properties of materials. For example, a characteristic difference between liquid and gaseous forms of a substance is a difference in densities. In particular, "near a liquid-gas phase transition, the density will vary in such a way that one rather large region of the material can have properties appropriate for a gas while another large region looks more like a liquid" (10). In the context of the Ising model for a ferromagnet, the fluctuations of interest are the spatial variations in the values of the spin field $s_{n,m}$ on the spatial lattice. The spin-spin correlation function $\Gamma_{n,m}$ represents the correlation between the values of the spin field at spatially-separated lattice points (n, m) and

[10] Qualification (explained below): for $n \geq 0$. More generally, an identity can be established for any n-field expectation value from QFT (Wilson and Kogut 1974, 149).

[11] An additional minor assumption that enters into the proof is that the Hermitian operator V (see Eq. (4) and (5) below) has a unique largest eigenvalue.

$(0,0)$. When expanded,

$$\Gamma_{n,m} = \frac{\langle s_{n,m} s_{0,0} \exp(-\beta\mathcal{H})\rangle}{\langle \exp(-\beta\mathcal{H})\rangle} \tag{2}$$

where $\langle\cdots\rangle$ is the standard sum over configurations, $\mathcal{H}$ is the SM Hamiltonian, β is the inverse temperature ($\beta = \frac{1}{kT}$ where k is Boltzmann's constant and T is temperature) (Wilson and Kogut 1974, 87).

On the QFT side of (Identity), the propagator $D_m(t)$ expands as

$$D_m(t) = \langle\Omega|T\phi_m(t)\phi_0(0)|\Omega\rangle \tag{3}$$

where T is the time-ordering operator and $|\Omega\rangle$ is the vacuum state. In practical terms, the role of the propagator is to encode the probability amplitudes for the outcomes of scattering experiments. The propagator is interpreted (crudely!)[12] as the amplitude for a free particle to propagate between spatial locations $x = 0$ and $x = ma$, where a is the small but non-zero lattice spacing (Peskin and Schroeder 1995, 92). (Identity) generalizes to n-point vacuum expectation values (VEVs) (Wilson and Kogut 1974, 149).[13]

Note that the spin-spin correlation function on the left-hand side of (Identity) is defined over an n by m dimensioned spatial lattice and the propagator on the right-hand side is on a spatial lattice (denoted by m). This means that the CSM model is defined on d *space* dimensions and the QFT model is defined on d *spacetime* dimensions (i.e., $d-1$ space dimensions plus one time dimension). The Wick rotated time variable in the QFT model is mapped to one of the discrete space dimensions in the CSM model. Thus, Wilson and Kogut express the content of (Identity) as follows: "the spin-spin correlation function of the statistical mechanical theory is equal to the propagator of the lattice quantum theory at discrete values ($n\tau$) of the (imaginary) time variable" (150).

Wilson and Kogut begin their proof of (Identity) with the stipulation $s_m = \varsigma\phi_m$, which is made "in order to make an explicit comparison" between the CSM correlation functions and QFT propagators (148). That is, s_m and ϕ_m are physically distinct; they are identified (up to scale factor ς) for the purpose of tracking formal similarities between the theories. The stipulation that $s_m = \varsigma\phi_m$ takes care of the analogical mapping between the subset m of spatial dimensions of the CSM theory

[12] See Fraser (2008).

[13] A *(Feynman) propagator* is a time-ordered two-point VEV (e.g., Eq. (2)). An n-point VEV is is the expression (for continuum fields) $\langle\Omega|\phi_1(\mathbf{x}_1,t_1)\phi_2(\mathbf{x}_2,t_2)\cdots\phi_n(\mathbf{x}_n,t_n)|\Omega\rangle$. (Unfortunately, n is used to denote a different quantity in this context than in the Wilson-Kogut construction.)

and the complete set m of spatial dimensions of the QFT. The work of the proof goes into relating the n spatial dimension of the CSM theory to the time dimension t of the QFT. To this end, the expression for $\Gamma_{n,m}$ is written in terms of the transfer matrix formalism. The transfer matrix V is so-called because, as Kadanoff explains, "it transfers us from one [spatial lattice] site to the next" (2000, 47). Rewriting in terms of V involves introducing the following expression for the partition function Z (where N is extent of the lattice in the n dimension):

$$Z = tr(V^N) \tag{4}$$

The advantage of casting $\Gamma_{n,m}$ in this form is that it allows a Hilbert space of wave functions to be defined and s_m to be represented as an operator acting on this Hilbert space. This is, of course, the mathematical framework for quantum theory. Furthermore, the transfer matrix formalism allows a QFT Hamiltonian to be plugged into $D_m(t)$. In order to satisfy QFT locality requirements,[14] the field theoretic Hamiltonian H is chosen to take the form (where τ is an arbitrary normalization constant)

$$H = -\frac{1}{\tau}\ln V \tag{5}$$

To obtain a strict equality between $\Gamma_{n,m}$ and $D_m(t)$, the thermodynamic limit of the CSM system must be taken (i.e., $N \to \infty$, where N is the extent of the spatial lattice in the n dimension) (150).

The final step is to recognize that the expressions for $\Gamma_{n,m}$ and $D_m(t)$ that result from these manipulations differ by a Wick rotation, $t \to -it'$, applied on the QFT to $D_m(t)$. A Wick rotation transforms Lorentz four-vector products into (negative) Euclidean four-vector products:

$$x^2 = t^2 - |\mathbf{x}|^2 \to x'^2 = -t'^2 - |\mathbf{x}|^2 \tag{6}$$

The Wick rotation also has the practical advantage that it is easier to impose an ultraviolet cutoff in Euclidean space, after the Wick rotation has been performed (Peskin and Schroeder 1995, 394).

[14] This form for the Hamiltonian is chosen in order to make it additive over distant lattice sites. Wilson and Kogut remark that

> When K [the CSM coupling]... is zero it is easy to show that $\ln V$ is actually a sum over lattice sites (in this case V is the direct product of independent operators, one for each site). This can be verified in perturbation theory in K. Otherwise proving the locality of $\ln V$ is a non-trivial problem which will not be discussed further. (145)

While the main contribution of Wilson and Kogut (1974) is the procedure for taking the continuum limit of the QFT described in Sec. 6 below, the proof of (Identity) is also original. However, (Identity) itself was already well-known. The fact that there are formal similarities between the expression for a correlation function in CSM and the expression for a VEV in QFT was not a novel insight in 1974. Indeed, Wilson and Kogut characterize the "analogy between correlation functions of spins and vacuum expectation values of fields" as "obvious." Historically, recognition of this formal analogy goes back at least as far as Bogoliubov and Shirkov (1959) (Jona-Lasinio 1964, 1792). Wilson and Kogut also describe as "obvious" the "connection that the Feynman graphs [for calculating critical exponents in SM] are similar to the (unrenormalized) Feynman graphs for a ϕ^4 field theory" (138). Wilson and Kogut also include an argument that the Wick-rotated Feynman diagrams for a ϕ^4 interaction in QFT are identical to the diagrams for the partition function in SM in their 1974 paper (Sec. 9). They characterize this argument for the identity of the diagrams as "brief" and the argument for the identity of $\Gamma_{n,m}$ and $D_m(t)$ surveyed in this subsection as "established at a more fundamental level and with greater care" (143). They elaborate that "[g]reater care means, for example, introducing a specific (and noncovariant) cutoff procedure for the ϕ^4 theory, and not making use of the Feynman expansion" (143). That is, an original and important virtue of their proof of (Identity) is that it is non-perturbative in the sense that it does not rely on the perturbative expansion on which the Feynman diagrams are based.

A few observations about this argument establishing the identity. This argument is general in scope in a number of respects. While the modified classical Ising model and scalar ϕ^4 interacting QFT model were used as concrete examples for deriving (Identity), the argument generalizes to QFT interactions of other types. The argument is also general in that there are no restrictions on the state of the CSM system (for instance, that it be at the critical point or far from the critical point). Furthermore, there are no restrictions on the dimensionality of space (respectively, spacetime) in the CSM theory (respectively, QFT). Finally, if attention is restricted to one spatial dimension of the CSM theory and the time dimension of the QFT, the identity holds between CSM and ordinary (i.e., non-relativistic) quantum theory (Kadanoff 2000, 46-48).

4.1 Formal analogies invoked in the derivation of (Identity)

Table 1 summarizes the analogical mappings between CSM and QFT which enter into Wilson and Kogut's proof of (Identity); these are the quantities which are equated in order to derive (Identity).

Table 1: Analogical mappings invoked in Wilson and Kogut's proof of (Identity)

CSM	**QFT**
s_m	ϕ_m
x_d	$(x_{d-1}, -it)$
$\ln V$	H
$\Gamma_{n,m}$	$D_m(t)$

The analogies laid out in Table 1 are clearly formal analogies. Recall from Sec. 2 that (following Hesse) in a formal analogy the horizontal relations are entirely dependent on the vertical relations. Further, as Bartha emphasizes, the elements mapped by the horizontal relations occupy the same formal mathematical roles in the theories. The quantities in Table 1 play the same abstract mathematical roles. For example, the spin field on the spatial lattice s_m and the scalar field on the spatial lattice ϕ_m are the dependent variables. d-dimensional space x_d and d-dimensional spacetime $(x_{d-1}, -it)$ are the independent variables. Exponentiating $\ln V$ and H produces expressions that represent translations. The correlation functions $\Gamma_{n,m}$ and propagators (or, more generally, VEVs) are expectation values of products of the fields (dependent variables) at different space or spacetime points (independent variables).[15]

The argument establishing the identity takes the form that one would expect from an argument that aims to draw out formal similarities between CSM and QFT: it relies entirely on the mathematical form of expressions in CSM and QFT. A sequence of manipulations is performed for the sake of obtaining expressions that take the same mathematical form. The key step is the Wick rotation of the time coordinate of the QFT. The main justification for this step is that performing a Wick rotation permits equating the correlation functions and VEVs. More generally, the Wick rotation is applied to a Minkowski covariant spacetime vector in order to obtain a Euclidean covariant space vector.[16] (The pragmatic justification for the Wick rotation is that it is easier to impose an ultraviolet cutoff in Euclidean space, after the Wick rotation

[15] Wilson (1975, 773) makes this point; this passage is discussed on p.32 below.

[16] When considering the relationship between 1-dimensional CSM and 1-dimensional quantum theory (i.e., time only), Kadanoff remarks

> Note that we have used the symbol, H, normally connected with a Hamiltonian to describe space translation. Somewhere in the back of our minds sits a view in which space and time are not so different. (Think of relativity.) (2000, 48)

In the case of one dimension, relativity is merely invoked as an analogy. In more than one dimension—i.e., with space in addition to time—there is an identity, not merely an analogy: the Wick rotation transforms a Euclidean covariant vector into a Minkowski covariant (i.e., relativistic) vector.

has been performed.) No physical justification is offered for Wick rotating. Wick rotation does not represent a physical operation, for example. The only assumption in the argument for (Identity) that receives a physical justification is the form chosen for the Hamiltonian, which conforms to locality requirements of QFT. However, the rationale for introducing this assumption is to ensure that Hamiltonians that are admissible in QFT can be plugged into $D_m(t)$; the identity would be inapplicable otherwise. As noted above, the argument does not rely on the specific dynamics of the CSM or the QFT system (i.e., the form of the interaction term); consequently, physical justification for the identity cannot come from this source.

4.2 Physical disanalogies and possible material disanalogies underlying (Identity)

Do the formal analogies drawn between quantities in CSM and QFT reflect deeper physical similarities? No—in fact there are relevant physical dissimilarities between the expressions which are horizontally related in Table 1. The most obvious one is that CSM on d space dimensions is mapped to QFT on d spacetime dimensions. The work in the proof of the identity goes in to establishing that the n space dimension of CSM corresponds to the time dimension t of QFT. This correspondence is established when $\ln V$ is set equal to the QFT Hamiltonian H. In QFT, the operator associated with $\exp(-iHt)$ is the time translation operator. In lattice CSM, V represents translations from one lattice site to the next. Time translations in QFT thus correspond to spatial translations on the lattice in CSM.

Space and time are quantities with different physical interpretations which play different roles in physical explanations of phenomena. Presumably, this means that the mapping of space to time rules out many physical similarities between CSM systems and QFT systems which one might have hypothesized to underwrite (Identity). In particular, since time and causation are so intimately related, this appears to be an indication that the analogical mappings do not preserve causal structure. Recall that the core requirement that Hesse imposes on a material analogy is that the vertical relations in both the source and target domains be causal relations of the same type. She was inspired in part by the historical case study of electromagnetism, in which the fluid ether models could be considered candidate causal-mechanical models for electromagnetism. The fact that the horizontal relations map a spatial dimension in CSM to the temporal dimension in QFT appears to block our similarly regarding CSM models as candidate casual-mechanical models for QFT. While the precise relationship between causation and time is fraught, the causal sequence of states or events in a causal process typically lines up with the temporal sequence of events.

It would seem, then, that the analogical mappings do not map causal processes in the CSM model to causal processes in the QFT model. These considerations are suggestive, but not decisive. We will return to this issue in Sec. 6.3, when the causal structure of the CSM model becomes relevant to the analogies.

These physical disanalogies indicate that (Identity) and the analogies it invokes are purely formal analogies, and not physical analogies.

5 Step 2: Impose (Constraint)

The end goal of this construction of QFT from CSM is a continuum QFT. (Identity) relates CSM on a spatial lattice to QFT on a spatial lattice. In Step 3, the spatial lattice in the QFT model will be removed by taking the limit of analogue quantities in CSM. Before the lattice can be removed, another ingredient is needed: a constraint on how the parameter values in the CSM theory change as $a \to 0$. This constraint is justified by appeal to the physical interpretation of the quantities in (Identity) at long distance scales. In Step 3, it will be shown that this constraint serves the purpose for which it is needed in QFT: the constraint ensures that the propagators (or VEVs) in QFT remain well-defined as the lattice is removed.

In order to evaluate the continuum limit of the VEVs, the dependence of the CSM variables b, u_0, K (which describe the CSM interaction[17]) on a must be known. CSM does not fix this dependence. As Wilson and Kogut put it, "[i]n [CSM] one has no a priori rules for how b, u_0, K should depend on the lattice spacing a" (151). The following choice is motivated by the identity between $\Gamma_{n,m}$ and $D_m(t)$:

$$a = \frac{1}{\mu_R \xi_{CSM}} \tag{7}$$

where μ_R is the renormalized, physical mass of the particle from QFT in physical units and ξ_{CSM} is the correlation length from CSM in dimensionless units (i.e., units of the lattice spacing). ξ_{CSM} represents the longest length scale over which there are significant correlations among thermal fluctuations. ξ_{CSM} is a function $\xi_{CSM}(b, u_0, K)$, so (7) fixes the dependence of b, u_0, K on a. Bear in mind that ξ_{CSM} is a statistical mechanical quantity, calculated from the (dimensionless) CSM Hamiltonian. The factor of a enters into the equation because ξ_{CSM} is a dimensionless quantity (i.e., expressed in units of the lattice spacing a) while μ_R is not a dimensionless quantity, but a quantity given in physical units.

[17] K represents energy per kT for a spin (or, after application of the RG transformation in the block spin representation, a block of spins). For the definitions of b, u see footnote 9.

Wilson and Kogut argue that (7) is suggested by a comparison of the long-range behaviour of $\Gamma_{n,m}$ and $D_m(t)$ (150-151). This is a legitimate presupposition because the QFT that is constructed from CSM is an effective, renormalized theory which is effectively valid at relatively large distance scales. They characterize the following argument as a proof. In the limit of large n with $m = 0$

$$\Gamma_{n,0} \propto \exp\left(-\frac{n}{\xi_{CSM}}\right) \tag{8}$$

For the corresponding QFT expression $D_0(-int)$, consideration of the energy eigenstates and symmetry of $D_0(-in\tau)$ (with n large, $m = 0$, and $\tau = a$) yields the conclusion that the state that dominates is the first excited state: "According to conventional wisdom the first excited state is a single particle state" and "[t]he energy difference $E_v - E_0$ for this state is the mass μ_R of the particle" (151).[18] Assuming this bit of conventional wisdom,

$$D_0(-ina) \propto \exp\left(-\mu_R na\right) \tag{9}$$

Applying (Identity) to (8) and (9) yields (7).

While in CSM models such as the Ising model it is natural to place the system on a lattice (to reflect the atomic lattice spacing), in particle physics it is customary to instead eliminate the degrees of freedom at small distance scales by imposing a high momentum cutoff. In accordance with the practice in particle physics, the continuum limit which will be taken in the next section is formulated in terms of a momentum cutoff Λ_0.[19] The momentum space version of (7) is the following:

$$\Lambda_0 = \mu_R \xi_{CSM} \tag{Constraint}$$

Naturally, the physical motivation for (Constraint) is the same as for (7).

[18] Huggett gives a variation on this argument which appeals to the usual interpretation of the propagator and the uncertainty relations:

> ...the propagator, loosely speaking, is the quantum probability for a particle at 0 to be found at x, and so the distance ξ, or rather ξ/Λ [where Λ is the momentum cutoff] in physical distance units, is a measure of the uncertainty of the particle's location. If we take the physical–renormalized–mass, [μ_R], as a measure of the momentum uncertainty of the particle, then the uncertainty relations give $\mu_R\xi/\Lambda = 1$. (2002, 271)

The arguments are intertranslatable in the sense that the one cited by Huggett is framed using the momentum and space representations while the argument given by Wilson and Kogut uses the energy and time representations.

[19] The subscript $_0$ indicates that this is the momentum cutoff of the "bare" theory. See Sec. 6.

5.1 Formal analogies invoked by (Constraint)

For bookkeeping purposes, the analogical mapping between CSM and QFT underlying (Constraint) is straightforward. (See Table 2.) Is this analogical mapping physical or purely formal? The indication that it is at least a formal analogy is that equations (8) and (9) take the same mathematical form, and ξ_{CSM} and μ_R^{-1} play the same mathematical role in the equations. (Bear in mind that (8) and (9) were derived under the assumption of long-range behaviour.)

Table 2: Analogical mappings associated with (Constraint)

CSM	**QFT**
$\frac{\xi_{CSM}}{\Lambda_0}$	μ_R^{-1}

5.2 Physical disanalogies

(7) is derived from (Identity) by appeal to separate analyses of Γ and D. The derivation does not rely on physical similarities between the interpretations of the expressions. Indeed, the reasoning on the QFT side is based on the energy of a single particle state, which is not applicable to the CSM system. ξ_{CSM} cannot be interpreted as inverse mass.

ξ_{CSM} is dubbed the "correlation length" because (in Wilson and Kogut's words) it represents the "effective range of correlations" between thermal fluctuations.[20] Intuitively, this interpretation is apparent from the position of ξ_{CSM} in the denominator of the exponent. Wilson and Kogut informally characterize the correlation length as the minimum size to which one could reduce a system without qualitatively changing its properties (78). μ_R^{-1} is sometimes represented using the same symbol ξ, which will be distinguished from ξ_{CSM} by the notation ξ_{QFT}. μ_R^{-1} also appears in the denominator of the exponent in (9). However, when interpreting μ_R^{-1}, the Wick rotation must be taken into account. Prior to performing the Wick rotation, x represents a point of Minkowski spacetime. In analogy to ξ_{CSM}, μ_R^{-1} could be interpreted as representing the spatiotemporal range of quantum correlations. Alternatively, after the Wick rotation is performed, μ_R^{-1} could be interpreted as representing the spatial range of correlations in $d+1$-dimensional Euclidean space (Peskin and Schroeder 1995, 293-294). In either case, μ_R^{-1} is not a measure of the maximum spatial extent of correlations between fluctuations in Minkowski space. Furthermore, the fluctuations are thermal fluctuations in CSM and quantum fluctuations in QFT. As a result, the analogical mapping in Table 2 is purely formal, not physical.

[20]Wilson and Kogut redefine ξ_{CSM} in this way to suit the context of RG methods. They note that "$\xi_{[CSM]}$ is customarily defined in terms of the behavior of $\Gamma(x)$ for $|x| \to \infty$" (98). See pp.98–99 for discussion.

To underscore the point that the rationale for adopting (7) (respectively, (Constraint)) comes from QFT—more specifically, the goal of obtaining a continuum QFT—consider how (7) appears from the perspective of CSM. The choice of (7) is conventional; it is not incorrect, but in the context of CSM it is unnatural. As Wilson and Kogut point out, "[n]o such constant [a] appears in the statistical mechanics because one naturally expresses lengths in units of the lattice spacing" (152). In contrast, "for a field theorist the natural unit of length is the reciprocal of μ_R" (152). Consider CSM with units $\mu_R = 1$. The lattice spacing $a = \frac{1}{\xi(b,u_0,K)}$ "is a somewhat peculiar change of units since it depends on the parameters b, u_0, and K" (152). a does not even appear in the usual dimensionless formulation of CSM; *a fortiori*, the motivation for taking the $a \to 0$ limit does not come from CSM. The continuum limit is physically motivated from the QFT side.

6 Step 3: Take the continuum limit

(Identity) and (Constraint)—the key equations derived in the preceding two sections—relate a CSM model on a spatial lattice to a QFT model on a spatial lattice (or, the loose equivalent, with a high momentum cutoff). From the perspective of QFT, what is desired is a QFT defined on continuous space, not on a spatial lattice. One of Wilson's insights was that, armed with (Identity) and (Constraint), a continuum QFT can be constructed from a CSM theory. The key is to recognize that the continuum limit of the QFT corresponds to the critical point of the CSM. Wilson and Kogut's construction furnishes a method for renormalizing a QFT: a propagator $D_m(t)$ that figures in (Identity) at the beginning of the derivation is a function of bare parameters (μ_0, λ_0); the continuum propagator $D\,(x, t)$ that will be derived is a function of renormalized parameters (μ_R, λ_R). The product of the construction is a well-defined renormalized, continuum QFT in the sense that what is produced is a set of propagators $D(x, t)$ (or VEVs) on continuous space (i.e., $a \to 0$).[21]

The basic strategy of the Wilson-Kogut construction of QFT from CSM is to take the continuum limit $\Lambda_0 \to \infty$ of the propagators, using the relationship between the correlation functions and propagators set out in (Identity) to move back and forth between CSM and QFT, and using (Constraint) as a constraint to keep the propagators well-defined. An intuitive sense of how (Constraint)—$\mu_R^{-1} = \xi_{CSM}/\Lambda_0$—functions as a constraint can be gained by considering how the quantities in the

[21]Note that the construction does not generate other quantities which one might associate with a well-defined continuum QFT, such as a well-defined expression for the Hamiltonian. Obtaining a QFT that is well-defined in other respects too is the goal of axiomatic and constructive programs for QFT (see Summers (2012) for a review).

expression vary as $\Lambda_0 \to \infty$. The renormalized mass μ_R has a fixed, finite value which is supplied by experiment. In order for the QFT that is constructed to be physically relevant, μ_R must take this fixed, finite value. For $\mu_R^{-1} = \xi_{CSM}/\Lambda_0$ to hold and μ_R to remain finite, as $\Lambda_0 \to \infty$, it is necessary for $\xi_{CSM} \to \infty$. Wilson and Kogut discuss this in terms of the equivalent $a \to 0$ limit. They observe that "[i]f one does not make a mass renormalization, (that is, if μ_0 is held fixed as $a \to 0$) then μ_R will be proportional to the cutoff momentum a^{-1} rather than constant." They further note that it is "rather trivial" to arrange matters so that the physical (renormalized) mass μ_R remains finite; the difficult part is "to obtain definite limits for the vacuum expectation values as $a \to 0$" (152). The argument that the propagators have well-defined values in the continuum limit relies on the special properties of $\xi_{CSM} \to \infty$, which represents the approach to a critical point of the CSM system.

Figure 1: Construction of continuum QFT from CSM

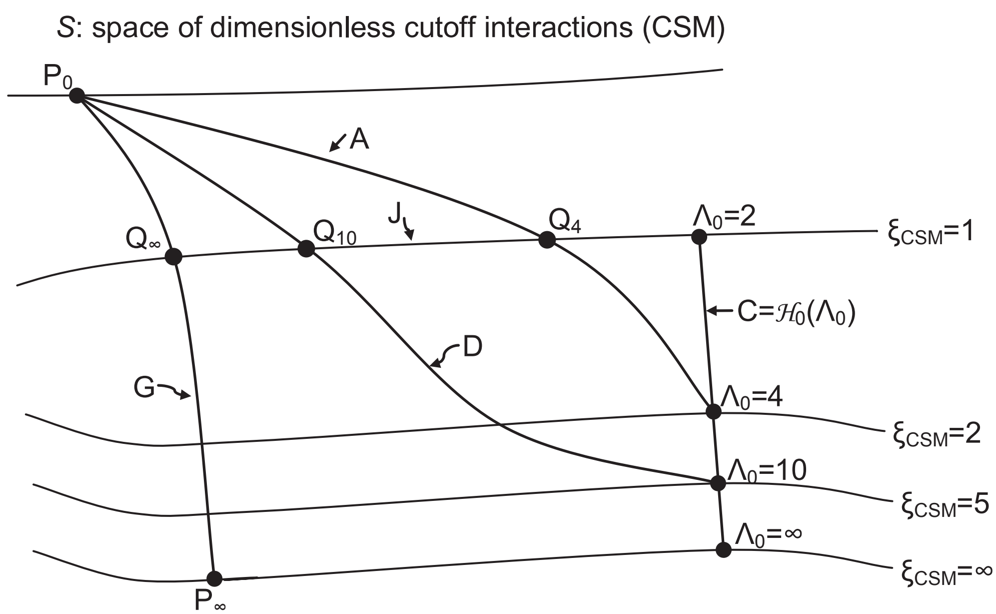

Figure 1 (adapted from Wilson and Kogut's Figure 12.7) depicts the construction of the continuum QFT from the CSM theory. Each point in the diagram represents an interacting theory with a momentum cutoff. The labels on the diagram represent elements of the CSM theories: the CSM interactions are represented by Hamiltonians

$\mathcal{H}$, the space S of interactions is parametrized by dynamical parameters r and u[22] and momentum cutoff Λ, and ξ_{CSM} is the correlation length of the interaction. These interactions are dimensionless. "Dimensionless" means that momenta are given in units of the cutoff Λ: e.g., $\mathcal{H}(q)$, where $q = \frac{k}{\Lambda}$ (Wilson 1975, 779). Furthermore, each dimensionless $\mathcal{H}$ has unit cutoff (160). This means that for each $\mathcal{H}$ there is a "scale factor" that must be applied to obtain a theory in physical units.

Wilson and Kogut speak interchangeably of S as representing interacting CSM and QFT interactions, and the ability to "go back and forth" between the CSM and the QFT interpretations is crucial for the construction. It is possible to do so because (Identity)—which identifies correlation functions in a CSM theory with propagators in a QFT—and the equations derived in the course of establishing (Identity) allow the space S of interacting theories to be interpreted either in QFT terms or CSM terms. That is, the equations allow one to move back and forth between an interacting QFT model and an interacting CSM model. In QFT terms, the interactions are represented by Hamiltonians H and S is parametrized by mass μ, charge λ, and momentum cutoff Λ. Since the renormalization of the charge parameter λ is trivial for the ϕ^4 interaction under consideration, this dimension of the space is suppressed in the diagram. (That is, the curves should actually be two-dimensional surfaces.)

The point labelled Q_∞ represents the goal of the Wilson-Kogut construction: the effective, continuum, renormalized QFT (= set of propagators) with the appropriate finite value for μ_R. μ_R is determined by experiment. Q_∞ is reached by taking the limit of the theories Q_n on the curve labelled J. Each of the Q_ns has the physical value for μ_R. Each of the Q_ns is obtained by applying the renormalization group (RG) transformation to the counterpart theory with cutoff $\Lambda_0 = n$ on curve C. For example, Q_4 is the product of applying the RG transformation to the model on curve C with $\Lambda_0 = 4$ (in dimensionless units); both Q_4 and C_4 lie on RG trajectory A. The curve C is the sequence of bare theories with different values of Λ_0 subject to (Constraint), namely $\mu_R^{-1} = \xi_{CSM}(\Lambda_0)/\Lambda_0$. As anticipated, the $\Lambda_0 \to \infty$ limit corresponds to $\xi_{CSM}(\Lambda_0) \to \infty$, the approach to the critical point in the CSM model.

In more detail, equation (Constraint) is imposed as a constraint to determine curve C in the following way. There is an element of choice in determining C in that how μ_0 varies with Λ_0 is not fixed by QFT (cf. discussion of element of choice in

[22]The bare dimensionless Hamiltonian for the CSM model with s^4 interaction in momentum space is

$$\mathcal{H}[\sigma] = -\frac{1}{2}\int_q \left(q^2 + r\right)\sigma_q\sigma_{-q} - u\int_{q_1}\int_{q_2}\int_{q_3}\sigma_{q_1}\sigma_{q_2}\sigma_{q_3}\sigma_{-q_1-q_2-q_3}$$

where the σ_qs are the spins in momentum space (i.e., the Fourier transforms of the s_ns, the spins on the spatial lattice) (96-97).

dependence of CSM parameters b, u, K on lattice spacing a in Sec. 5.2). Put another way, points on C represent different physical possibilities; each of these theories has a different upper limit to momentum (or, equivalently, lattice spacing). Different paths through this space of physically inequivalent theories could be chosen. This element of choice is exploited to choose curve C in such a way as to produce a finite renormalized theory with mass μ_R. As Wilson and Kogut explain, "[p]erforming a renormalization conventionally means giving μ_0 and λ_0 a Λ_0 dependence such that the renormalized mass μ_R and renormalized coupling constant[23] are cutoff independent" (167). The first step in determining curve C is to set the value of renormalized mass μ_R to the experimental value. Step two is to "*choose*"—where the emphasis is Wilson and Kogut's—to impose the constraint $\Lambda_0 = \mu_R \xi_{CSM}(r_0, u_0)$. While this is a choice in the context of settling on a curve C through the space S of interacting QFTs with varying momentum cutoff, as Sec. 5.2 explained, the choice is motivated by the relationship between correlation functions in CSM and propagators in QFT at relatively low energy scales. Crucially, (Constraint) entails that the physical *dimensioned* correlation lengths of interactions on C are all μ_R^{-1}. Solving $\Lambda_0 = \mu_R \xi_{CSM}(r_0, u_0)$ with fixed u_0 yields $r_0(\Lambda_0)$. Then (Identity), $s_m = \zeta \phi_m$, and the transfer matrix formalism (Eq. (5)) are applied to translate this constraint on the parameters of the CSM models into a constraint on the QFT models.[24] These manipulations yield $\mu_0^2 = r_0(\Lambda_0)\Lambda_0^2$. Since $\xi_{CSM}(r_0, u_0) \to \infty$ as $\Lambda_0 \to \infty$, $r_0 \to r_{0c}$ (the critical value) as $\Lambda_0 \to \infty$. μ_R is independent of Λ_0 by construction: μ_R is held constant and ξ_{CSM} compensates for the increase in Λ_0 as $\Lambda_0 \to \infty$.

After curve C has been determined, a theory Q_n can be obtained by applying the RG transformation to the corresponding point C_n. Consider the effect of decreasing the cutoff infinitesimally from Λ_0 to $\Lambda' = \Lambda_0 - \delta\Lambda$, which is implemented by the RG transformation U. The following RG equation governs the transformation U acting on the space of dimensionless cutoff interactions S:

$$\frac{\partial \mathcal{H}_t}{\partial t} = U[\mathcal{H}_t] \tag{10}$$

where t parametrizes U. A RG transformation acts on a bare Hamiltonian $\mathcal{H}_0$ (such as that at C_n) by re-parametrizing it in such a way as to produce "a set of interactions all of which describe the same physical system" (Wilson and Kogut 1974, 160). That is, a set of Hamiltonians $\mathcal{H}_t$ is a set of descriptions which are formally different—e.g., each $\mathcal{H}_t$ may assign different values to parameters r_t, u_t and $\mathcal{H}_t$ may contain

[23] As noted above, for the case of the ϕ^4 interaction, the coupling constant renormalization is trivial.

[24] Plus a change of variables from b to r. See footnote 9.

different interacting terms than $\mathcal{H}_0$. In the physically relevant context of QFT, the Hamiltonians H_n associated with points along an RG trajectory (such as A) can be interpreted as alternative descriptions of the same physical system in virtue of the fact that each produces the same set of propagators. Physical systems are individuated by their associated sets of propagators.

U rescales the distance—or, equivalently, the momentum—scales in the interacting theory. For the bare theory $\mathcal{H}_0(\Lambda_0)$, Λ_0 is the momentum cutoff. This means that Λ_0 is the upper limit of integration in the calculation of the propagators. For the effective theory $\mathcal{H}_t(\Lambda_0)$, the upper limit of integration is the rescaled cutoff

$$\Lambda_t = e^{-t}\Lambda_0 \tag{11}$$

That is, the re-parametrized $\mathcal{H}_t$ reflects the fact that momenta above Λ_t have been "integrated out"; the effects of these high momentum scales have been absorbed by the new parameters (r_t, u_t). Λ_t is the *effective* cutoff of the effective theory $\mathcal{H}_t$. Bear in mind, however, that $\mathcal{H}_t(\Lambda_0)$ is physically equivalent to $\mathcal{H}_0(\Lambda_0)$. This means that, while the *effective* momentum cutoff of $\mathcal{H}_t$ differs from the momentum cutoff of $\mathcal{H}_0$, the *physical* momentum cutoff remains Λ_0. This is why Wilson and Kogut characterize Λ_t as an "artificially small cutoff" (161).

Similarly, U rescales the correlation length:

$$\xi_{CSM}(\Lambda_t) = e^{-t}\xi_{CSM}(\Lambda_0) \tag{12}$$

where $\xi_{CSM}(\Lambda_t)$ is the *effective* correlation length of $\mathcal{H}_t(\Lambda_0)$ and $\xi_{CSM}(\Lambda_0)$ is the correlation length of the bare interaction $\mathcal{H}_0(\Lambda_0)$ (and the bare cutoff Λ_0 takes the same value for both expressions). Suppose $\xi_{CSM}(\Lambda_t)$ is the correlation length for theory Q_4 and $\xi_{CSM}(\Lambda_0)$ is the correlation length for C_4. $\xi_{CSM}(\Lambda_t)$ and $\xi_{CSM}(\Lambda_0)$ are both dimensionless. The physical correlation lengths (i.e., in physical units) ξ_{CSM}^{phys} of the effective theory $\mathcal{H}_t(\Lambda_0)$ and the bare theory $\mathcal{H}_0(\Lambda_0)$ are the same:

$$\xi_{CSM}^{phys} = \frac{\xi_{CSM}(\Lambda_t)}{\Lambda_t} = \frac{\xi_{CSM}(\Lambda_0)}{\Lambda_0} \tag{13}$$

Λ_t, Λ_0 are the "scale factors" that translate the dimensionless lengths into physical units. When the bare interaction is on curve C, ξ_{CSM}^{phys} is μ_R^{-1} as a consequence of the imposition of (Constraint).

To recap: Figure 1 provides a pictorial overview of the construction of the renormalized theory Q_∞. Q_∞ is arrived at by taking a limit of the sequence Q_n on curve J. Each Q_n is obtained from the theory C_n by applying the RG transformation. Along any RG trajectory (e.g., A, D) the application of the RG transformation rescales the momentum scale by shrinking it by a factor of e^{-t}. This has the effect of shrinking both the effective cutoff (i.e., $\Lambda_1 < \Lambda_0$ and in general $\Lambda_{t+1} < \Lambda_t$) and shrinking the effective correlation length (i.e., $\xi_{CSM}(\Lambda_1) < \xi_{CSM}(\Lambda_0)$, $\xi_{CSM}(\Lambda_{t+1}) < \xi_{CSM}(\Lambda_t)$). The points on an RG trajectory represent physically equivalent theories (i.e., with the same physical cutoff and the same physical correlation length). The points on curve C do not represent physically equivalent theories.

What remains to be shown is that the model Q_∞ satisfies the desideratum that motivated the construction in the first place: that it have a set of well-defined propagators.[25] Here is the argument from Wilson and Kogut (1974, 167) that the renormalized theory Q_∞ has a set of well-defined propagators.[26]

1. "For each Λ_0 there is a value of t for which $\mathcal{H}_t(\Lambda_0)$ intersects the surface of interactions with $\xi_{CSM} = 1$. The set of such interactions defines the curve J shown in [Figure 1]." [i.e., each point on curve J has the dimensionless correlation length $\xi_{CSM} = 1$]

2. Consider interactions Q_4 and Q_{10} on curve J. "The interaction Q_4 describes the same physics as the interaction $\mathcal{H}_0(4)$. Likewise the interaction Q_{10} defines the same physics as the interaction $\mathcal{H}_0(10)$. The only change is a scale transformation [i.e., Λ_t]. It is easily seen that the scale transformation is the same for both Q_4 and Q_{10}. The reason is that $\mathcal{H}_0(\Lambda_0)$ was defined to have a constant correlation length in physical units (namely μ_R^{-1}) independent of Λ_0." i.e., from Equations (13) and (Constraint)

$$\text{For } \mathcal{H}_t \text{ at } Q_4\text{:} \quad \Lambda_t = \frac{(\Lambda_0 = 4)}{\xi_{CSM}(\Lambda_0 = 4)} \cdot \xi_{CSM}(\Lambda_t) = \frac{1}{\mu_R^{-1}} \cdot 1 = \mu_R \tag{14}$$
$$\text{For } \mathcal{H}_t \text{ at } Q_{10}\text{:} \quad \Lambda_t = \frac{(\Lambda_0 = 10)}{\xi_{CSM}(\Lambda_0 = 10)} \cdot \xi_{CSM}(\Lambda_t) = \frac{1}{\mu_R^{-1}} \cdot 1 = \mu_R$$

3. "Now let $\Lambda_0 \to \infty$. In this limit Q_4 and Q_{10} are replaced by Q_∞, the intersection of J with the curve G. Q_∞ is a well-defined, cutoff interaction so it describes a well-defined physics including a complete set of VEVs." Q_∞ is "a well-defined, cutoff interaction" in the sense that the *effective* cutoff Λ_t is finite (cf. Equations (14)):

For $\mathcal{H}_t$ approaching Q_∞ in the limit:

[25] This is the relatively modest goal of Wilson and Kogut (1974). RG methods were later the basis for a more ambitious explanation of why the interactions in models in particle physics turn out to be perturbatively renormalizable. See Butterfield and Bouatta (2014) for a thorough philosophical discussion and Polchinski (1984) for a mathematical treatment.

[26] I have changed the notation in the following quotes to agree with that used in this section.

$$\Lambda_t = \frac{(\Lambda_0 \to \infty)}{\xi_{CSM}(\Lambda_0 \to \infty)} \cdot \xi_{CSM}(\Lambda_t) = \frac{1}{\mu_R^{-1}} \cdot 1 = \mu_R \tag{15}$$

> This is relevant because the effective cutoff Λ_t is used to calculate the propagators from the effective QFT Hamiltonian (i.e., is the upper limit of integration). But, of course, Q_∞ is on the same RG trajectory as $\mathcal{H}_0(\Lambda_0 \to \infty)$, so the *physical* cutoff $\Lambda_0 \to \infty$. Wilson and Kogut conclude "[h]ence, the limit $\Lambda_0 \to \infty$ exists: there is a finite (renormalized) theory in the limit $\Lambda_0 \to \infty$."

This method for obtaining the renormalized model Q_∞ only works if certain conditions are satisfied. In particular, it hinges on the existence of a non-trivial fixed point of the RG transformation (Wilson and Kogut 1974, 152).[27] A fixed point is a point such that the interacting Hamiltonian $\mathcal{H}^*$ remains unchanged by application of the RG transformation:[28]

$$U[\mathcal{H}^*] = 0 \tag{16}$$

The fixed point P_∞ is the important one for the construction. The subscript $_\infty$ indicates that P_∞ lies on the curve $\xi_{CSM} = \infty$. (The fixed point P_0 lies on the curve $\xi_{CSM} = 0$.) In order to take the limit $\Lambda_0 \to \infty$, there must exist a critical point $\xi_{CSM} = \infty$ of the CSM theory. Theory Q_∞ is the intersection of the trajectory G emanating from P_∞ and the curve $\xi_{CSM} = 1$. It is not possible to obtain Q_∞ by applying the RG transformation to $\mathcal{H}_0(\Lambda_0 = \infty)$ directly due to the mathematical properties of fixed points. Repeated applications of the RG transformation to any point on the critical surface $\xi_{CSM} = \infty$—such as $\mathcal{H}_0(\Lambda_0 = \infty)$—will eventually produce the theory at the fixed point P_∞. Subsequent applications of the RG transformation at a fixed point such as P_∞ always produce the same theory $\mathcal{H}^*$, by definition (see Equation (16)). Thus, Q_∞ cannot be reached by further applications of the RG transformation to P_∞. However, repeated applications of the RG transformation starting with a bare interaction $\mathcal{H}_0(\Lambda_0)$ infinitesimally close to (but not on) the critical surface will produce trajectories that veer away from the critical surface near P_∞ and intersect $\xi_{CSM} = 1$ near Q_∞. As Wilson and Kogut explain, this is due to the fact that the direction of $\partial\mathcal{H}_t/\partial t$ is not defined at P_∞ (166). This is why Q_∞ must be obtained by the limiting process in the construction: that is,

[27] See Wilson (1975, 779-780) for discussion of further conditions of applicability.

[28] This is more intuitive in terms of the discrete version of the RG transformation rather than the infinitesimal version. The discrete version is T: $\mathcal{H}_{l+1} = T[\mathcal{H}_l]$, where, for example, $l \to l+1$ corresponds to a change in the cutoff from $\Lambda \to \Lambda/2$ when $\mathcal{H}_l$ has cutoff $\Lambda = 2^{-l}$ (Wilson and Kogut 1974, 126). A fixed point is defined by $H^* = T[\mathcal{H}^*]$. This equation transparently shows that a fixed point of T is a Hamiltonian that remains unchanged under the application of T.

by taking the limits of the intersection points $Q_4, Q_{10,}...$ of the curves A, D, ... and their intersections with $\xi_{CSM} = 1$ as $\Lambda_0 \to \infty$.[29]

To sum up this section, what are the analogies to CSM contributing to the renormalization method for QFT laid out by Wilson and Kogut? The effective, renormalized, continuum theory Q_∞ is found by choosing curve C to intersect with the critical surface $\xi_{CSM} = \infty$. This is essential for making the $\Lambda_0 \to \infty$ limit well-defined because the fixed point P_∞ lies on the critical surface. The correspondence with CSM—(Identity), (Constraint), and related equations—is used to fix the functions $\lambda_0(\Lambda_0)$ and $\mu_0(\Lambda_0)$ so that the effective theory remains well-defined. The construction trades on the fact that the CSM Hamiltonians $\mathcal{H}$ are in dimensionless form. This both allows the physical correlation length to be assigned the physical QFT value μ_R^{-1} (while the dimensionless correlation length $\xi_{CSM}(\Lambda_0) \to \infty$) *and* allows for the RG transformation to be applied to produce an effective Hamiltonian with "artificially low"—and so low as to be finite!—momentum cutoff Λ_t.

6.1 Analogical mappings invoked by the Wilsonian approach to renormalization

What are the analogical mappings between QFT and CSM which get invoked in the course of the Wilson-Kogut construction of continuum QFT from a CSM theory at a critical point described in Sec. 6? Since the construction relies on Steps 1 and 2, it presupposes the analogical mappings identified in Tables 1 and 2. In particular, the constraint that maps $\frac{\xi_{CSM}}{\Lambda_0}$ and μ_R^{-1} (Sec. 5) plays the crucial role of keeping the continuum limit well-defined. The Wilsonian approach to renormalization characterizes the problem as finding a way to formulate a continuum QFT with well-defined VEVs. The identifications with CSM are introduced to solve this theoretical problem. From this perspective, it is natural to take the correspondences between CSM and QFT to be indicators of the relevant similarities between elements of CSM and QFT. In the Wilson-Kogut construction, the $\Lambda_0 \to \infty$ continuum limit in QFT is obtained by taking the $\xi_{CSM} \to \infty$ limit. That is, the value of ξ_{CSM} at each point on curve Q is fixed by the value of Λ_0 using the equation $\Lambda_0 = \mu_R \xi_{CSM}(r_0, u_0)$, and $\xi_{CSM}(r_0, u_0)$ in turn determines the functional dependence of μ_0 on Λ_0.[30] The $\xi_{CSM} \to \infty$ and

[29] Wilson and Kogut also note that it is essential to their method that the continuum theory be obtained as a limit of cutoff theories because "the renormalization group transformation U is defined to integrate out the momenta just below the cutoff and that is a meaningless operation if the cutoff is infinite" (160).

[30] Focusing on applications of RG methods in condensed matter physics (rather than particle physics), Batterman (2010a) also argues that limit operations (as opposed to mathematical entities

$\Lambda_0 \to \infty$ limits are analogues.

The mapping between the $\xi_{CSM} \to \infty$ and $\Lambda_0 \to \infty$ limits suggests a further mapping between the parameters of the theories. The CSM system is taken to its critical point $\xi_{CSM}(r_0, u_0) = \infty$ by taking the limit $r_0 \to r_{0c}$, where r_{0c} is defined as the critical value of parameter r_0 at $\xi_{CSM} = \infty$. The $\Lambda_0 \to \infty$ limit of QFT Hamiltonian $H_0(\Lambda_0)$ (i.e., curve C) is taken by evaluating the equation $\mu_0^2 = r_0(\Lambda_0)\Lambda_0^2$. Thus, in taking the $\Lambda_0 \to \infty$ limit, $r_0 \to r_{0c}$ fixes $\mu_0 \to \mu_{0c}$ (where $\mu_{0c} = \infty$). (Recall that r_0 is a parameter in the CSM interaction Hamiltonian $\mathcal{H}_0$ that resulted from a change of variables (see footnote 9).) In experimental applications of the Ising model, the physical variable that is typically varied to bring a system to a critical point is temperature T. T is not a parameter in the CSM Hamiltonian. T enters the model via the assumption that, at the atomic scale, the CSM system possesses a Hamiltonian $\mathcal{H}_0(m_0, g_0)$ and that the bare parameters m_0, g_0 are functions of temperature T and the other basic thermodynamic parameters (Itzykson and Drouffe 1989, 234). Further assumptions about analyticity suggest that there is a linear dependence on T.[31] Thus, in practical physical terms, the limit $\xi_{CSM}(r_0, u_0) \to \infty$ is taken by taking $T \to T_c$. At the level of parameters, the analogy between $\xi_{CSM} \to \infty$ and $\Lambda_0 \to \infty$ induces an analogy between temperature T and bare mass μ_0.

Table 3: Analogies invoked by the Wilsonian approach to renormalization

CSM	**QFT**
$\frac{\xi_{CSM}}{\Lambda_0}$	μ_R^{-1}
$\xi_{CSM} \to \infty$	$\Lambda_0 \to \infty$
T	μ_0

Table 3 summarizes the analogical mappings implicated in the Wilson-Kogut construction of continuum QFT from CSM at a critical point. Note that the mappings of $\frac{\xi_{CSM}}{\Lambda_0}$ to μ_R^{-1} and $\xi_{CSM} \to \infty$ to $\Lambda_0 \to \infty$ are compatible in virtue of the fact that ξ_{CSM} is a dimensionless quantity and $\frac{\xi_{CSM}}{\Lambda_0}$ is a dimensioned quantity. Both mappings are derived from the constraint equation $\frac{\xi_{CSM}}{\Lambda_0} = \mu_R^{-1}$ (where μ_R^{-1} is a fixed constant). Different mappings result from stressing different aspects of the constraint equation.

I've motivated the mappings in Table 3 by considering the theoretical problem of constructing continuum renormalized QFT. These mappings receive stronger motivation by considering the analogy between QFT and CSM from the perspective of CSM. A continuum, cutoff independent QFT corresponds to a CSM system at

and their properties) play a crucial role in explaining the applicability of RG methods.

[31] More specifically, it is assumed that the coefficient of the relevant operator is analytic in T, which suggests that the coefficient is linear in T (Wilson 1975, 785).

a critical point. Turning this around and conceiving of QFT as the source domain and CSM as the target domain for the analogy, QFT can be used to solve theoretical problems in CSM. In particular, QFT can be applied to calculate the values of the critical exponents of the CSM system. Peskin and Schroeder offer the following assessment:

> The ability of quantum field theory to predict the critical exponents gives a concrete application both of the formal connection between quantum field theory and statistical mechanics and of the flows of coupling constants predicted by the renormalization group. However, there is another experimental aspect of critical behavior that is even more remarkable, and more persuasive. Critical behavior can be studied not only in magnets but also in fluids, binary alloys, superfluid helium, and a host of other systems. It has long been known that, for systems with this disparity of microscopic dynamics, the scaling exponents at the critical point depend only on the dimension N of the fluctuating variables and not on any other detail of the atomic structure. Fluids, binary alloys, and uniaxial magnets, for example, have the same critical exponents. To the untutored eye, this seems to be a miracle. But for a quantum field theorist, this conclusion is the natural outcome of the renormalization group idea, in which most details of the field theoretic interaction are described by operators that become irrelevant as the field theory finds its proper, simple, large-distance behavior. (Peskin & Schroeder 1995, 437)

Peskin and Schroeder describe the phenomenon of *global universality*, which Wilson characterizes as "independence of critical behavior to large changes in the interaction (such as the change from a ferromagnet to a binary alloy or a liquid-gas transition)" (1975, 787). Global universality obtains when the same fixed point governs a class of interactions for different types of systems. Derivation of the critical exponents for a particular type of interaction within a global universality class (e.g., ferromagnetic) reveals a second type of universality. *Local universality* is "independence of critical behavior under infinitesimal changes in the interaction" (787). An example of local universality for the Ising model is that Hamiltonians with different second- and third-neighbour couplings all produce the same critical behaviour (786). As Peskin and Schroeder indicate, the explanation[32] of universality is based on the irrelevant operators in both QFT and CSM. Without entering into the details, adding second-

[32]Peskin and Schroeder explicitly use the term "explanation" elsewhere: "in general, the evidence is compelling that quantum field theory provides the basic explanation for the thermodynamic critical behavior of a broad range of physical systems" (451).

and third-neighbour couplings involves adding irrelevant operators to the Hamiltonian, which does not affect the values of the critical exponents because irrelevant operators tend to zero under repeated applications of the RG transformation. Critical exponents are measures of the behaviour of a system close to the critical point. As a corollary, the values of the coefficients of the relevant operators in the Hamiltonian must be finely tuned in order to obtain a fixed point interaction under repeated applications of the RG transformation. Physically, the system is taken to the critical point by taking $T \to T_c$, so the coefficient of the relevant operator is proportional to T. In QFT, the relevant operator is the mass operator with coefficient μ_0. As we saw in Sec. 6, μ_0 must be "finely tuned"—i.e., renormalized—in order to reach the critical surface. The fact that the analogy between T and μ_0 plays this role in the prediction of critical exponents and the explanation of universality provides further justification for the analogical mapping of T to μ_0.

Peskin and Schroeder's use of terms such as "prediction" and "explanation" to describe the relationship between CSM and QFT raises our recurring question of whether or not the "formal connection" between CSM and QFT is indicative of underlying physical similarities between the domains. While a mere formal analogy may be sufficient to underwrite the application of QFT to make predictions for CSM, most accounts of scientific explanation would require a more robust physical analogy in order for QFT to furnish genuine explanations for CSM. To be explanatorily relevant, the explanans contributed by QFT would have to state physical facts about the CSM system, not merely capture formal features of the mathematical frameworks common to QFT and CSM. I will return to the analogue question for QFT in Sec. 6.3 below.

6.2 Formal analogies

A liberal formal analogy is, in Bartha's words, an analogical mapping that maps elements that "occupy corresponding positions in formally analogous theories" (195). Wilson himself offers a nice analysis of how the analogical mappings in Table 3 satisfy this definition. The key is to recognize that the physical problems arising in CSM and QFT can be set up in such a way that they take similar mathematical forms. Wilson coins the term "statistical continuum limits" to denote problems that take this mathematical form; problems of this type may be solved by applying RG methods. $\Lambda_0 \to \infty$ is a statistical continuum limit in QFT and $\xi_{CSM} \to \infty$—the limit in which there are correlations at distances large compared to lattice spacing a—is a statistical continuum limit in CSM. In the following quote, Wilson offers a succinct comparison of these statistical continuum limits:

> There are two ways in which a statistical continuum limit can arise. The obvious way is when the independent field variables are defined on a continuous space; the case of statistical or quantum fluctuations of the electromagnetic field is an example. If one were to replace the continuum by a discrete lattice of points, the field averages would consist of integrals over the value of the field E at each lattice site n. Thus for the discrete lattice case one has a multiple integration, $\prod_n \int dE_n$, the variables of integration being the fields E_n. In the continuum limit, one has infinitely many integration variables E_n. ...
>
> The second source of statistical continuum limits is the situation where one has a lattice with a fixed lattice spacing, usually an atomic lattice. The number of independent variables (i.e., independent degrees of freedom) at each lattice site is fixed and finite. The continuum limit arises when one considers large size regions containing very many lattice sites. When the lattice is viewed on a macroscopic scale one normally expects the lattice structure to be invisible. That is, large scale effects should be describable by a continuum picture making no reference to the lattice spacing. (1975, 773)

Wilson draws attention to the infinite number of independent variables that arise in the statistical continuum limit. Both problems of physical interest take the similar mathematical form of integrals over products of these variables. In QFT, the integrals are the expressions for the propagators or VEVs. In CSM, the integrals are the expressions for the correlation functions. As we saw, the identification of propagators and correlation functions was the starting point for the analogy between QFT and CSM. The statistical continuum limits get introduced in Step 3.

The statistical continuum limits $\Lambda_0 \to \infty$ and $\xi_{CSM} \to \infty$ occupy corresponding positions in formally analogous theoretical problems, and thus satisfy the definition of liberal formal analogy, but not strict formal analogy. Recall that the strict sense of formal analogy requires a single uninterpreted formalism that is given two different physical interpretations. That is, the horizontal relations are formal relations between elements of physical theories that are the physical interpretations of the same element of the uninterpreted formalism. In the CSM-QFT case, there is no single underlying (or overarching) mathematical framework that can be given one interpretation in which the statistical continuum limit is $\Lambda_0 \to \infty$ and another interpretation in which it is $\xi_{CSM} \to \infty$. This is apparent from the method for solving for the $\Lambda_0 \to \infty$ statistical continuum limit in QFT. The renormalized continuum theory Q_∞ is obtained by taking the $\Lambda_0 \to \infty$ limit in lock step with the $\xi_{CSM} \to \infty$ limit.

The space of interactions in Fig. 1 can be physically interpreted in either quantum field theoretic or classical statistical mechanical terms; however, the method for arriving at the effective, renormalized theory Q_∞ can only be carried out under the QFT interpretation. First, determining the parameters of each theory C_n along curve C involves going back and forth between the CSM and QFT interpretations; this is not done by referring to a single abstract formalism that could be given either a quantum field theoretic or statistical mechanical interpretation. Second, the RG transformation is applied to the QFT interpretation of the C_ns and the scaling preserves *physical* equivalence under the QFT interpretation. Wilson and Kogut's presentation of RG methods for QFT does not introduce an abstract formalism that could be applied in CSM (or other contexts), but their RG methods for QFT are still formally analogous to RG methods for CSM (and other applications). As conceived by Wilson, RG methods are a set of mathematical techniques for solving statistical continuum limits that share family resemblances. The central common feature is the employment of scaling transformations such as the RG transformation U in QFT.

Wilson's formulation of RG methods in the mid-1970s did not introduce a single overarching formalism that could be applied in different cases, but that of course does not imply that it is impossible to devise an abstract formalism for RG methods. In fact, the situation is even more complicated, because even within statistical mechanics alone RG techniques take different forms. Jona-Lasinio (2001) documents confusions in the 1970s (and beyond) about the relationship between formally different approaches to RG transformations in statistical mechanics. He reports that (as of 2000) the general theory of RG methods is "still missing" (442).

6.3 Material and physical disanalogies

The formal analogy between QFT and CSM relies on pragmatic considerations to identify the relevantly similar elements of QFT and CSM. Wilson's goal in renormalizing QFT is to obtain a well-defined set of VEVs. The corresponding problem in CSM is obtaining an adequate theoretical description of the system at critical points, especially the calculation of critical exponents. The physical problems in both theories take the mathematical form of deriving statistical continuum limits. RG methods are a mathematical technique for solving mathematical problems of this type. These similarities in pragmatics are accompanied by important differences. As Wilson points out in the passage quoted in the previous section, the physical rationales for taking the statistical continuum limits differ: in QFT, the discrete lattice is artificially imposed on a continuum system and then removed in the course of taking the statistical continuum limit; in CSM, the atomic spacing a has a physical interpretation[33] and the lattice is not removed in the course of taking the statistical continuum limit. Instead, in CSM the infinite statistical continuum limit is taken by introducing the idealization of an infinitely large system.

This physical disanalogy between the pragmatic justifications for introducing the statistical continuum limits entails many substantial differences in physical interpretation between CSM and QFT. An immediate consequence of the pragmatic differences is epistemic differences. When the space S in Fig. 1 is viewed as representing CSM interactions, the space may be full of interactions which are empirically accessible. The critical surface $\xi_{CSM} = \infty$ contains the interactions for systems at their critical points, but other regions of S can contain interactions describing systems away from the critical point. These systems may be subject to experiment. In contrast, if space S is interpreted as a space of QFT interactions, much less of the space is empirically accessible. Point Q_∞ (with an appropriate experimental value inserted for μ_R) is empirically accessible; the predicted values for scattering amplitudes yielded by the propagators could in principle be empirically tested. However, according to the model, the other points Q_n do not represent empirically accessible interacting systems. This is because points Q_n each retain an artificially imposed lattice (or momentum cutoff); these points are part of the mathematical derivation of Q_∞, but according to the QFT model do not themselves perform a representational function. The analogous epistemic situation in CSM would be to have empirical access to systems at the critical point, but not to systems away from criticality. Clearly, this dissimilarity between epistemic situations holds significant implications for theorizing in QFT and CSM. In particular, the rich empirical data on a diverse range of CSM systems near and far from critical points generates the demand for an explanation of the universality of the values of critical exponents for diverse systems. Since there is—in principle—no analogous body of empirical evidence in QFT, there is no analogous explanatory demand.[34] This is an in principle difference—and not just, for example, a consequence of the fact that practical limitations currently prevent us

[33] A physical manifestation of the atomic spacing is that, at distance scale a, the fluctuations scale differently than they do at distance scales between a and the correlation length ξ (Wilson 1975, 775).

[34] The closest analogue of universality in QFT is the "effective field theory" interpretation of QFT, which stresses the implication of RG techniques that a wide range of bare Hamiltonians with different parameter values is compatible with our current, relatively low-energy experimental data. A discussion of this point of view is unfortunately beyond the scope of this paper, but note the general theme that there is one actual Hamiltonian about which we are ignorant, which is in contrast to the situation in CSM (i.e., many actual systems described by different Hamiltonians, which display approximately the same empirical values for their critical exponents).

from performing experiments on relatively high energy QFT systems[35]—because it stems from the fact that the $\Lambda_0 \to \infty$ limit plays a purely mathematical role in the set up of the QFT problem and the $\xi_{CSM} \to \infty$ limit plays a physical role in the set up of the CSM problem.

These epistemic differences arising from the different roles of the $\xi_{CSM} \to \infty$ and $\Lambda_0 \to \infty$ limits are also reflected at the level of the parameters. T can be measured by experiment; μ_0 is not a measurable parameter. In fact, in some cases (e.g., QED) $\mu_0 \to \infty$ as $\Lambda_0 \to \infty$. (See Butterfield and Bouatta (2014, 14–15, 38) for further discussion.) In QFT, μ_R takes the experimentally measured value for mass. Furthermore, unlike T in CSM, μ_0 is not a parameter that is within experimental control. $T \to T_c$ represents a physical operation that may be carried out on a single physical system; $\mu_0 \to \mu_{0c}$ does not represent a physical operation.

There are also deep metaphysical differences underlying the analogical mappings. First, there is a modal difference. Recall from Sec. 4.2 that space in CSM gets mapped to spacetime in QFT. This allows different points in the space S of CSM interactions to be interpreted as representing the same system at different times. For example, taking the $T \to T_c$ limit represents the physical process of a single system undergoing a change of temperature over some duration of time. In QFT, varying μ_0 cannot be interpreted as a physical process which takes place in time because QFT is a spacetime theory. Each point in the space S of QFT interactions represents a spacetime description of an interacting system. The best interpretation of varying μ_0 as a function of Λ_0 (i.e., curve C in Fig. 1) is as merely a mathematical technique for obtaining the renormalized continuum QFT. If, however, one wants to push for a metaphysical interpretation of varying μ_0, it would be something along the lines of different values of μ_0 representing different values of (bare) mass in different possible worlds. If one accepts this interpretation, there is a crucial modal difference in how parameters T and μ_0 represent changes. Varying T (may) represent a single system occupying a sequence of different states at different times in the actual world. Varying μ_0 represents different values of the mass which a system[36] possesses in different possible worlds.

Second, there are differences between the role that causation plays in the two models. In Sec. 4.2 it was suggested that mapping space in CSM to spacetime in

[35] A separate empirical limitation in particle physics is that (at present) we are only capable of experimentally probing systems at relatively low energy scales. This restricts the portion of the RG trajectory for $H(\Lambda_0 \to \infty)$ that is empirically testable (i.e., limits the range of experimental values for μ_R).

[36] The same system or different systems, depending on how one cashes out trans-world identity conditions.

QFT has the likely consequence that analogical mappings between CSM and QFT do not preserve causal structure. With the addition of the mappings in Table 3 between CSM systems at critical points and continuum QFTs, we can confirm that this is indeed the case. In the CSM model, temperature T is a contributing cause to the value of the correlation length ξ_{CSM}. The interactions between spins on the lattice tend to make them align in the same direction. However, for $T > T_c$, thermal agitation disrupts this tendency, and spins do not become correlated over long distances. As $T \to T_c$, thermal agitation decreases and the spins become aligned over longer distances (i.e., $\xi_{CSM} \to \infty$). The vertical relations between these elements of the CSM model are causal relations. The mapping of T to μ_0 and $\xi_{CSM} \to \infty$ and $\Lambda_0 \to \infty$ does not indicate that there is, analogously, a causal relationship between μ_0 and $\Lambda_0 \to \infty$. Taking the $\mu_0 \to \mu_{0c}$ limit does not cause $\Lambda_0 \to \infty$. When the cutoff is implemented by a spatial lattice, varying the bare mass does not cause the lattice spacing to decrease. Rather, along curve C in Fig. 1, the values of μ_0 and Λ_0 co-vary without standing in a causal relationship. Consequently, the analogies between CSM and QFT in Table 3 are not material analogies because the corresponding vertical relations are not causal relations.

Recently a number of authors have argued (citing a variety of reasons) that RG methods are invoked in non-causal explanations of universality in condensed matter physics (e.g., Batterman (2010b), Reutlinger (2014)).[37] This conclusion is different from the conclusion just defended, and the differences supply a nice illustration of the morals of this paper. In the condensed matter case, the claim is that the explanation of a particular explanandum is non-causal, while the present argument concludes that RG methods applied in QFT support non-causal explanations in general. The particular explanandum in the condensed matter case concerns universality, which involves features of a class of systems. There is disagreement about what makes the explanation of universality non-causal, but there is agreement that, when attention is restricted to a single system, the statistical mechanical models do afford causal descriptions of a system. For example, Batterman emphasizes that the explanation of universality ignores the detailed causal descriptions available for each individual system; Reutlinger instead argues that the relevant fact is that the RG trajectories do not represent causal relations, but agrees that "it is presupposed that Hamiltonians describe the causal interactions among the components of a system" (1166). The particle physics case is different in kind because RG methods are used by Wilson and Kogut to model a *single* system, not a class of systems (i.e., systems described by different Hamiltonians). For a single particle physics system, RG methods already

[37] In the same spirit, Morrison (2015) argues that RG methods in condensed matter physics furnish mathematical explanations.

support non-causal explanations. For example, the construction in Sec. 6 supplies an answer to question “Why is the given interaction represented by Hamiltonian H renormalizable (in Wilson’s sense of possessing an effective, continuum model)?” These differences in the ways non-causal explanations are supported are a direct result of the differences in the ways in which RG methods are applied that Wilson flags and the differing roles of space and time.

The main moral is that there are pragmatic differences between the set ups of the problems in QFT and CSM, which give rise to epistemic differences, which in turn give rise to metaphysical differences. These differences all reflect substantial physical dissimilarities between the interpretations of the expressions mapped in Table 3. The dimensioned quantity $\frac{\xi_{CSM}}{\Lambda_0}$ is not physically interpreted as inverse physical mass in CSM. The dimensionless quantity ξ_{CSM} (in $\xi_{CSM} \to \infty$) is physically interpreted as the correlation length in CSM, whereas Λ_0 (in $\Lambda_0 \to \infty$) represents the momentum cutoff. T, μ_0 have different physical interpretations and play different roles with respect to time, causation, and counterfactuals in CSM and QFT.

The substantial physical disanalogies that underlie the heuristic use of analogies are compatible with the physical intuition that set Wilson on this path. Wilson’s inspiration for drawing on critical phenomena in CSM to renormalize QFT was that at the critical point of a condensed matter system there are large-scale effects that can be described by a continuum theory. (See the passage about statistical continuum limits quoted on p.32.) When one “zooms out” to a large-scale picture of the system the lattice structure becomes invisible. This was suggestive because the goal in the QFT case was to get rid of the lattice. In both a CSM model at a critical point and a renormalized, effective, continuum QFT, the large-scale descriptions are insensitive to the small scale cutoff. This is a similarity between how the two representations function at large scales. This representational similarity is compatible with the substantial physical disanalogies highlighted in this section. For example, the lattice structure *appears* to be invisible in the “zoomed out” CSM model, whereas the momentum cutoff is actually removed in the effective, renormalized QFT.

The substantive physical disanalogies between the CSM and QFT models entail that caution needs to be exercised in transferring physical interpretations and morals from statistical mechanics to QFT. Whether exportation is permissible will need to be sorted out on a case-by-case basis. To illustrate how the analysis here can be used to address these questions, consider as a relatively simple example the interpretation of high momentum or small distance spatial cutoffs in QFT. Peskin and Schroeder’s influential QFT textbook contains the following passage:

> Wilson’s analysis takes just the opposite point of view, that any quantum field theory is defined fundamentally with a cutoff Λ that has some phys-

> ical significance. In statistical mechanical applications, this momentum scale is the inverse atomic spacing. In QED and other quantum field theories appropriate to elementary particle physics, the cutoff would have to be associated with some fundamental graininess of spacetime, perhaps a result of quantum fluctuations in gravity. (402)

The physical interpretation of the cutoffs in QFT is inferred by analogy from the physical interpretation of the cutoffs in CSM. This argument from analogy is unsound because the horizontal relations are not correctly identified. In the QFT model, the bare cutoff Λ_0 is the physical cutoff. However, Λ_0 is *not* the analogue of the physical cutoff (inverse atomic spacing) in CSM. The analogical mapping is between the limits $\Lambda_0 \to \infty$ in QFT and $\xi_{CSM} \to \infty$ in CSM. The cutoffs are on different footings in the two theories. Recall that the lynchpin of the construction is the identity between *dimensionless* ξ_{CSM} (i.e., in units of the cutoff) and *dimensioned* μ_R^{-1}. As a result, in the CSM model $\xi_{CSM} \to \infty$ while a remains constant, but $\Lambda_0 \to \infty$ in the QFT model. Thus, the analogical mappings underlying the Wilsonian approach to renormalization do not support the inference that the cutoffs in QFT should be physically interpreted as representing the "fundamental graininess of space".[38] Of course, it is possible that support for Peskin and Schroeder's interpretation could be found within QFT or drawn from quantum gravity, but this example illustrates the pitfalls of using CSM as a guide to interpreting QFT.

7 Conclusion

As we have seen, the CSM-QFT case study is a complex example of the use of analogies in theory development. It is worthwhile to carefully track the analogical mappings through the twists and turns in the reasoning and to subject them to analysis because this is a historically and philosophically important case study. It is both historically and philosophically important because the introduction of RG methods has had a profound and wide-ranging impact in physics and beyond. Historically, this sophisticated case study extends the methodology of using analogies, displaying features not present in earlier cases such as the development of electromagnetism. Philosophically, the identification and analysis of the analogies in this

[38] Wilson and Kogut do state that the interactions in space S are defined with respect to the cutoff because "the renormalization group transformation U is defined to integrate out the momenta just below the cutoff and that is a meaningless operation if the cutoff is infinite" (160). However, the point that they are making in this passage is that it is essential to their method that the continuum limit theory be obtained as a limit of cutoff theories. Moreover, they also explain that one can also define an exact RG transformation that does not require a strict cutoff for its definition.

case study informs the physical interpretation of QFT and presents *prima facie* challenges to general philosophical accounts of analogies and core principles animating the scientific realism debate.

The preliminary move in the conceptual bookkeeping exercise is to divide the analogies between CSM and QFT into two categories. The analogical mappings laid out in Tables 1 and 2 fall into the first category. These analogical mappings follow from the identification of the correlation functions $\Gamma_{n,m}$ of the CSM model with the Wick rotated propagators $D_m(t)$ (or VEVs) for the QFT model on a lattice set out in (Identity). (Identity) and the associated analogical mappings apply to the CSM model when the system is in any state and to the QFT with arbitrary finite lattice spacing (or momentum cutoff). Consequently, these analogical mappings can be used in contexts other than RG methods (e.g., spontaneous symmetry breaking). The analogical mappings set out in Table 3 fall into the second category. In contrast to the sets of analogical mappings in the first category, these sets of analogical mappings have a more limited scope. These mappings apply only when the system to which the CSM model applies is at or approaching a critical point ($\xi_{CSM} \to \infty$) and when the corresponding QFT model is a continuum model (or approaching a continuum model, $\Lambda_0 \to \infty$). These narrower sets of analogical mappings between a CSM model at a critical point and a continuum QFT model are used to obtain renormalized, continuum QFT models.

The central philosophical question raised by the application of analogical reasoning in the CSM-QFT case study is the following: Are the analogies physical or purely formal? The conclusion reached here is that the Wilsonian approach invokes purely formal analogies. The analogies count as formal in the liberal sense that the analogical mappings pick out elements that "occupy corresponding positions in formally analogous theories" (Bartha 2010, p.195). For example, the limit $\Lambda_0 \to \infty$ in QFT is the analogue of the limit $\xi_{CSM} \to \infty$ in CSM. The compelling evidence that these sets of analogical mappings are not physical is the substantial differences in the physical interpretations of the analogues. Spatial quantities in CSM are mapped to spatiotemporal quantities in QFT. This difference gives rise to a range of pragmatic, epistemic, and modal differences. In terms of the framework of Sec. 2, both the horizontal relata and the corresponding vertical relations have substantially different physical interpretations. These substantial physical disanalogies mean that caution has to be exercised in attempts to export physical interpretations from one context to the other. For example, the physical cutoff in the QFT model is not the analogue of physical cutoff in the CSM model, so RG methods do not supply a rationale for regarding the cutoffs as having similar physical interpretations.

That the analogical mappings do not respect the causal structure is a noteworthy feature of this case study. Hesse (1966) contends that arguments from analogy must use material analogies, which require that corresponding vertical relations in the source and target domains be causal relations. She bases her account on earlier historical examples, such as the development of electromagnetism, in which the analogies do respect the causal structure of the source and target domains. The use of analogies that do not respect causal structure is an innovative feature of the CSM-QFT case study. Another implication is that the application of RG methods in QFT supports non-causal explanations in a different manner than the application of RG methods in statistical mechanics. A number of philosophers have argued that non-causal explanation can be offered for the universal behaviour of a class of condensed matter systems near a critical point that individually possess causal models, while non-causal explanations can be offered for a single QFT system.

The "bottom up" approach of closely examining the operative analogies in the CSM-QFT case study provides strong support for the conclusion that the analogies are purely formal. However, from the "top-down" perspective of general philosophy of science and philosophical accounts of analogies, this conclusion may seem problematic. The heuristic strategy of using analogies between CSM and QFT to develop QFT has unquestionably proven very successful. (As has the use of analogies to QFT to develop statistical mechanics.) The scientific realists' intuition is that success in science is underwritten by getting something essentially right about the world. However, the judgment that the analogies are formal and not physical entails that scientific realists cannot appeal to the discovery of relevant physical similarities between CSM and QFT systems to explain the success of QFT because there are no such physical similarities. In their philosophical accounts of analogies, Hesse and Bartha both defend the position that arguments from analogy in science should not be based on purely formal analogies. For instance, Bartha criticizes Steiner (1998)'s examples of purely formal analogies in physics and concludes that plausible analogical arguments must include relevant similarities that "have known physical significance" (221). Does the successful employment of formal analogies between CSM and QFT then defy explanation?

A full explanation of the applicability of purely formal analogies in this case will be developed elsewhere, but the outline of an answer is suggested by Wilson's own characterization of RG methods. As he emphasizes in the passage quoted on p.32, the commonality between CSM and QFT is that they pose theoretical problems that can be expressed in terms of statistical continuum limits. This is possible because both theories describe phenomena using fluctuating fields and field fluctuations on a wide range of scales contribute to producing the phenomena. Furthermore—as this paper has stressed—the fluctuations, fields, and scales can all be given different physical interpretations in different contexts. The minimal and multiply realizable representational conditions on the possibility of formulating statistical continuum limits are satisfied in a wide range of theories, which is why RG methods have found applications in fields ranging from economics to biology to physics. Wilson aptly compares statistical continuum limits and RG methods to the calculus: RG methods are "the tool that one uses to study the statistical continuum limit in the same way that the derivative is the basic procedure for studying the ordinary continuum limit" (1975, 774). Similarly, it is not surprising that the minimal representational conditions for applying statistical continuum limits are satisfied in theories within economics, biology, and physics.

Finally, one might wonder if care needs to be taken to refrain from investing formal analogies with unwarranted physical significance, how do formal analogies perform the heuristic function of guiding the development of theories? It is perhaps revealing that Maxwell and contemporary quantum field theorists express similar ideas about this. In the course of presenting his early model of the force as lines of incompressible fluid, Maxwell offers the following suggestion:

> The substance here treated of...is not even a hypothetical fluid which is introduced to explain actual phenomena. It is merely a collection of imaginary properties which may be employed for establishing certain theorems in pure mathematics in a way more intelligible to many minds and more applicable to physical problems than that in which algebraic symbols alone are used. (Maxwell [1856] 1890, 160)

This attitude is echoed in the following reflection by Peskin and Schroeder on the role of analogies in the SM-QFT case: "[i]n essence, [the correspondence between QFT and SM] adds to our reserves of knowledge a completely new source of intuition about how field theory expectation values should behave" (294). That is, the common suggestion is that analogies furnish intuitive pictures to accompany abstract mathematics—and, as the electromagnetism case demonstrates, the pictures need not represent reality in order to serve this function.

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