The Kochen-Specker and Conway-Kochen Theorems

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Abstract

This essay provides an analysis of two important theorems that arise in the context of quantum mechanics: the Kochen-Specker theorem, which challenges the existence of hidden variable theories; and the Conway-Kochen (free will) theorem, which can be seen as an improvement over the Kochen-Specker theorem, but instead focuses on challenging determinism.

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Introduction

The Kochen-Specker theorem [1] was originally posed by its authors as a proof of the non-existence of hidden variable theories. Indeed, it succeeds in ruling out a major class of these hypothetical theories, namely: non-contextual hidden variable theories. The theorem is based on earlier work by Gleason [2], and provides a stronger version of von Neumann's famous impossibility proof [3]. It also paves the way to the free will theorem that was posed by Conway and Kochen [4, 5] as an argument against determinism.

In the following sections, we will analyse the two theorems in some detail, and we will discuss some of the ideas that arise naturally in light of their results. In section 1, we will introduce the problem of hidden variables with the basic assumptions about how to construct a hidden variable theory. We will then provide a proof of the Kochen-Specker theorem; pointing out the core of the theorem's result. In section 2, we will investigate the possibility of escaping the Kochen-Specker theorem (by questioning its assumptions), and how that leads us to consider different notions of contextuality. Finally, in section 3, we will review the free will theorem by considering Conway and Kochen's argument [5], as well as a more recent formulation due to Cator and Landsman [6, 7]. We will also look closely at the locality condition that is assumed in the theorem.

1 The Kochen-Specker Theorem

1.1 Description of the Problem of Hidden Variables

The theory of quantum mechanics (QM) has some peculiar features, as opposed to classical mechanics, that have led to a lot of controversy—since the time it was first conceived—in its interpretation as a theory that describes physical reality. The two main features we need to consider here are:

(I) **Intrinsic Probabilistic Nature:** Outcomes of measurement of an observable are confined to a set of real numbers (the eigenvalues) and the best we can do with QM is to predict the probability of each outcome as opposed to the case of classical mechanics where we can, subject to measurement precision, predict which outcome will pop up in a certain experiment.

(II) **The Role of Apparatus and Measurement:** In (orthodox) QM there is a split between what we call "apparatus" and the quantum system, which naturally leads to a distinction between what we call "measurement" and quantum interactions. This distinction is manifested in the fact that the "measurement" is mentioned explicitly in the postulates of QM, as opposed to the case of classical mechanics where the measurement is described (as it should be) as a physical process within the same theory. For all practical purposes, the postulated view works just fine but it is unsatisfactory if we want an ontological description of the measurement process. Treating the measurement as a special case of interactions within QM leads to the usual measurement problem (a discussion of which is out of the scope of this essay); nonetheless, the important thing is to keep in mind the participatory role that the apparatus plays in the physical process called "measurement" when searching for an ontological interpretation.

These two features have led to the idea that the formalism of QM does not give a complete description of reality and it needs to be supplemented by some additional structure that resolves both (I) and (II).

Thus the problem of hidden variables is the problem of realizing such structure by postulating the existence of extra dynamical variables that are not accessible to us (thus the word "hidden") as well as to provide a physical (ontological) interpretation for these hidden variables. This is generally what one should mean by a hidden variable theory (HVT). It is, however, the act of trying to resolve (I) alone using hidden variables (with no regard to (II)) that gets called a HVT[1]. We will work with this description of the problem for now; however, as we will discuss later, this division of the problem (which might be regarded as unfair) can lead to severe consequences.

1.2 Constructing a Hidden Variable Theory

To make the idea clear, let us now illustrate how we can formulate a HVT that resolves (I) in the way done by most no-go theorems, including the Kochen-Specker theorem under discussion in this section. We assume that the QM state $|\psi\rangle$ is an ensemble of states denoted by $|\psi\rangle_\lambda$. In each such state, all observables have sharp values and outcomes of all measurements can be predicted with absolute certainty: hence, these states are essentially dispersion-free states. In this notation, we take λ to denote a configuration of values for the hidden variables[2], they are hidden because we cannot measure (observe) them and also because we cannot prepare dispersion-free states with desired values for λ at will. Let us develop this idea in a more formal way:

- $\lambda \in \Lambda$ are the hidden variables, Λ is a probability space.
- QM states $|\psi\rangle$ are associated with a probability distribution $\rho_\psi(\lambda)$ over Λ , which extends to a probability distribution over dispersion-free states denoted by $|\psi\rangle_\lambda$.
- Each dispersion-free state $|\psi\rangle_\lambda$ naturally induces a value assignment map (over self-adjoint operators)
$$V_{|\psi\rangle_\lambda} : \Sigma \longrightarrow \mathbb{R}, \qquad \Sigma := \{\hat{A} : \mathbb{H} \longrightarrow \mathbb{H} \,|\, \hat{A} = \hat{A}^\dagger\}. \tag{1.1}$$
- Expectation value of an observable O is given by averaging over the assigned values in dispersion-free states, i.e. by
$$\langle O\rangle_\psi = \int_\Lambda V_{|\psi\rangle_\lambda}(\hat{O})\, \rho_\psi(\lambda)\, d\lambda. \tag{1.2}$$
- Probability that a measurement of an observable Q yields a value q_i is given by
$$Prob_Q^{|\psi\rangle}(q_i) = \int_\Lambda V_{|\psi\rangle_\lambda}(\hat{P}_{q_i})\, \rho_\psi(\lambda)\, d\lambda. \tag{1.3}$$

  where $\hat{P}_{q_i}$ is the spectral projector of the eigenvalue q_i.

[1]This is usually done by the mathematically inclined investigator.

[2]Instead, you can take λ_n to denote a value of the nth hidden variable and denote the state by $|\psi\rangle_{\{\lambda_n\}}$, but we will stick to the simpler notation that we defined.

Obviously the value assignment map[3] V cannot be arbitrary and should satisfy certain conditions, It is the conditions that we assume V has to satisfy a priori that lead to the so called no-go theorems against HVTs. Let us proceed by assuming some reasonable conditions on V :

(i) Value Realism: Each observable corresponds to an element of physical reality and the values assigned correspond to the set of possible outcomes (which is the set of eigenvalues for the corresponding operator), i.e. for any observable Q represented by the self adjoint operator $\hat{Q}$, we have

$$V(\hat{Q}) \in \{q_i\}, \tag{1.4}$$

where $\{q_i\}$ are the eigenvalues of operator $\hat{Q}$. In particular, in order to reproduce the QM average $< \hat{Q} >_{|q_i\rangle} = q_i$, value assignment on eigenstates (or more precisely: on their dispersion-free versions) renders the corresponding eigenvalue (i.e. $V_{|q_i\rangle_\lambda}(\hat{Q}) = q_i \quad \forall \lambda$).

(ii) Linearity over commuting operators (Quasi-linearity):

$$V(a\,\hat{A}+b\,\hat{B}) = a\,V(\hat{A})+b\,V(\hat{B}), \qquad \text{for } [\hat{A},\hat{B}]=0, \qquad \forall\, a,b \in \mathbb{R}. \tag{1.5}$$

(iii) Non-contextuality: Value assignment to observables is non-contextual, meaning that all observables are assigned values simultaneously regardless of what else is being measured with a given observable (the measurement *context*).

Let us briefly discuss each condition.

Condition (i) assumes two things. First it assumes that every observable is a *beable*[4]: an element of reality that exists with a definite value at all times (for example, the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are beables of Maxwell's theory). That is the "realism" part. Second, it assumes that these values are revealed *faithfully* in a measurement (this gives the "value" part). Since the set of outcomes of a measurement of a certain observable is the set of eigenvalues of the corresponding operator, it immediately follows that the values assigned to observables in the HVT must belong to their respective sets of eigenvalues.

Condition (ii) is a weaker and a more physically plausible version of von Neumann's linearity condition [3] that applied to all operators not just the commuting ones. Bell (1966)[9] refuted von Neumann's condition by arguing reasonably that a linear combination of incompatible observables cannot be measured by a linear combination of their corresponding measurement devices (since the two devices are incompatible as well) and that we need a new device to measure that linear combination. While condition (ii) is physically plausible to assume, it is interesting to note that it can be deduced from condition (i)[5]. But we need to assume it explicitly because it can hold independently from condition (i). There is an important rule that can be deduced from condition (ii), it is

[3] For convenience we will use V without the subscript when we talk about any value assignment map, we will only use the subscript when we want to refer to a state specific value map.

[4] The terminology is originally due to Bell [8].

[5] by arguing that if $\hat{C} = a\hat{A} + b\hat{B}$, where $[\hat{A},\hat{B}] = 0, \quad a,b \in \mathbb{R}$, and $\hat{A}\,|\alpha_i\rangle = \alpha_i\,|\alpha_i\rangle\,, \quad \hat{B}\,|\beta_i\rangle = \beta_i\,|\beta_i\rangle$, then from QM: $\hat{C}$ is an observable satisfying $\hat{C}\,(a\,|\alpha_i\rangle \oplus b\,|\beta_i\rangle) = (a\,\alpha_i + b\,\beta_i)\,(a\,|\alpha_i\rangle \oplus b\,|\beta_i\rangle)$ and hence condition (ii) follows if we apply condition (i).

called the Functional Composition Rule (or FUNC for short), which states that if $\hat{A} = f(\hat{B})$, then $V(\hat{A}) = f(V(\hat{B}))$. FUNC tells us that the value map has to preserve the algebraic structure of the operators. In other words, the algebraic relations between values of observables mirror the algebraic relations between respective operators (see lemma 6.4 in Landsman (2017) [10] for a derivation of FUNC from the quasi-linearity condition of (ii))[6].

Condition (iii), the non-contextuality, can be seen to naturally follow from condition (i), since value realism implies that all observables possess sharp values prior to measurement (i.e. elements of reality exist with sharp values regardless of being observed or not), but here we state it explicitly to emphasize the word "observable" used in the definition. We will discuss issues related to contextuality in section 2.

As we have discussed, it can be seen that value realism is the core condition here (a fact that we will return to in section 2), and it seems reasonable enough (to the extent that it is usually only implicitly assumed!).

1.3 The Kochen-Specker Contradiction

Let us now consider what happens if we assume the innocent-looking conditions that we discussed. The bottom line is that they lead to a contradiction, and that gives us the Kochen-Specker Theorem (or KS contradiction) which is usually stated as follows:

> There are no non-contextual, quasi-linear, dispersion-free states.

Let us now proceed with the proof of the contradiction: we will follow a treatment similar to Redhead (1987)[11] and the original paper of Kochen and Specker (1967)[1].

From (1.4), it follows that

$$V(\hat{I}) \quad = \quad 1, \tag{1.6}$$

$$V(\hat{P}_k) \in \{0, 1\}, \tag{1.7}$$

where $\hat{I}$ is the identity operator and $\hat{P}_k$ is a projector on some state $|k\rangle$.

From QM we know that for a complete orthonormal set of states $|k\rangle$ we have

$$\sum_k \hat{P}_k = \hat{I}. \tag{1.8}$$

Since orthogonal projectors commute, we can apply the linearity condition eq. (1.5) to get

$$\sum_k V(\hat{P}_k) = 1, \tag{1.9}$$

but since the value of projectors is restricted to $\{0, 1\}$, therefore only one projector gets a value of 1 while the rest get values of 0. e.g.

$$V(\hat{P}_i) = 1, \qquad V(\hat{P}_j) = 0 \qquad \forall j \neq i.$$

[6]Equivalently, we could have assumed FUNC and derived the quasi-linearity condition (see Redhead (1987) [11] p. 121).

The problem is that for an n dimensional Hilbert space with $n > 2$ there is no way to assign values in this manner to all possible complete sets of projectors without reaching a contradiction.

Following Redhead (1987), this problem of value assignment can be mapped to a colouring problem where we have two colours: red and blue, corresponding to the values of 1 and 0 respectively. One dimensional projectors can be represented by rays in the Hilbert space. Since rays intersect a hypersphere in two antipodal points, the colouring of a ray maps to the colouring of the corresponding points (that the ray intersects) on the hypersphere. This defines a colouring map $C : S^2 \to \{red, blue\}$. So our colouring rule (CR hereafter) should be as follows:

Each point on S^{n-1} gets either *red* or *blue* such that for points belonging to a set of orthogonal rays, only one point gets *red*.

In $\mathbb{R}^2$, we can colour S^1 as follows:

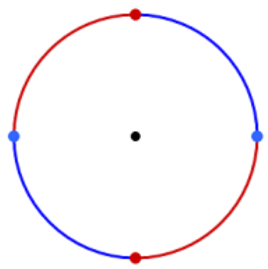

Figure 1.1: A Possible colouring of S^1 that obeys CR.

It is important to note that in $\mathbb{R}^2$ each projector (ray) is a part of only one possible duo of orthogonal projectors (rays), i.e. for a given ray X, the space of rays orthogonal to X is one dimensional. This is why it is possible to achieve this colouring scheme in $\mathbb{R}^2$. In higher dimensions (i.e. $n > 2$) this is not possible. To prove this we only need to prove that it is not possible in $\mathbb{R}^3$, and the case for higher dimensions ($n > 3$) follows by noting that their 3-dimensional subspaces will fail the colouring rule (i.e. if we cannot colour S^2, then we cannot colour $S^{n-1}\ \forall n > 3$ since $S^2 \subset S^{n-1}\ \forall n > 3$)[7]. For our proof, we need the following important lemma.

Lemma 1. *Let p, $q \in S^2$ and $\boldsymbol{u}_p$, $\boldsymbol{u}_q$ are their respective unit vectors. If $C(p) \neq C(q)$, then $\exists$ a finite angle (namely $\sin^{-1}(\frac{1}{3})$) between $\boldsymbol{u}_p$ and $\boldsymbol{u}_q$.*

It is interesting to note that Bell (1966) arrived at this result as a corollary of Gleason's theorem [2] and deduced (although not rigorously) the same result of the Kochen-Speker theorem, i.e. the non-existance of non-contextual quasi-linear dispersion-free states.

We now proceed with the proof of the lemma. We can represent points on the sphere and the orthogonality relations between their rays by using a *Kochen-Specker diagram* (KS diagram hereafter) where:

[7]This extends to complex spaces as well, since $\mathbb{C}^n \simeq \mathbb{R}^{2n}$.

- Vertices represent points on the Sphere.
- Lines between vertices represent orthogonality between corresponding rays.

For example, consider the points a, b and c with Cartesian coordinates $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ respectively. The three points have corresponding orthogonal rays and thus we get the following KS diagram:

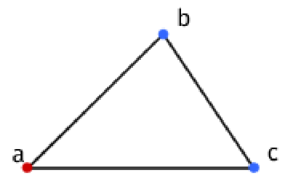

Figure 1.2: Kochen-Specker diagram for points a, b and c with a possible colouring according to CR.

Now let us consider the following diagram for our proof:

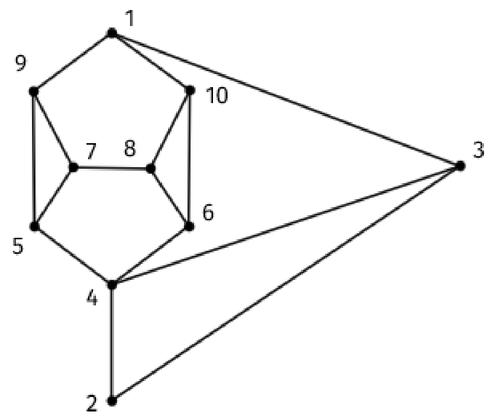

Figure 1.3: Kochen-Specker diagram that can only be constructed if $0 \leq \sin\theta_{12} \leq \frac{1}{3}$.

We will denote the unit vector of each point i by $\boldsymbol{u}_i$.

We can show that the KS diagram in figure 1.3 cannot be constructed unless $0 \leq \sin\theta_{12} \leq \frac{1}{3}$, where θ_{12} is the angle between $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$[8]. The proof is rather elementary (see Readhead (1987) p. 126), what is more important is using that fact to prove our lemma. Consider that we now have this diagram constructed already, so now we must have $0 \leq \theta_{12} \leq \sin^{-1}(\frac{1}{3})$, so θ_{12} can be arbitrarily small. Therefore to prove our lemma, we need to show that this diagram cannot be coloured by the CR such that points 1 and 2 get opposite colours. To show

[8]Since in principle we care about rays (i.e. we do not care about directions of unit vectors), we will consider the case where the angle θ_{12} is acute

this, let $C(1) = red$ and $C(2) = blue$. Using the orthogonality relations of the diagram, we can deduce the colouring of the rest of the points as follows:

$$\because C(1) = red \;\Rightarrow \therefore C(3) = C(9) = C(10) = blue,$$

$$\because C(2) = blue = C(3) \;\Rightarrow \therefore C(4) = red \;\Rightarrow \therefore C(5) = C(6) = blue.$$

Now focusing on the triangle of points $9, 7, 5$ we have

$$C(9) = C(5) = blue \;\Rightarrow \therefore C(7) = red,$$

and similarly for the triangle of points $10, 6, 8$ we get

$$C(10) = C(6) = blue \;\Rightarrow \therefore C(8) = red,$$

but 7 is connected to 8 (i.e. $\boldsymbol{u}_7 \perp \boldsymbol{u}_8$), therefore they should get different colours, and thus we have a contradiction. Therefore for any two points 1 and 2 with different colours we must have $\theta_{12} > \sin^{-1}(\frac{1}{3})$, otherwise we can construct the diagram in fig. 1.3 and achieve a colouring contradiction, and that proves our lemma.

Now we start employing our lemma to prove our main theorem. Consider six successive points $1, 2, 3, 4, 5,$ and 6 on the equator of S^2 such that the angle between each two successive points is $\theta_{ij} = 18°,\ j = i+1$ (i.e. $\theta_{12} = \theta_{23} = \theta_{34} = \theta_{45} = \theta_{56} = 18°$). Let us consider colouring $C(1) = red$. Since $\theta_{12} = 18° < \sin^{-1}(\frac{1}{3})$, therefore (by using the lemma) we must have $C(2) = C(1) = red$. By similar reasoning, we will have all the points coloured by *red*. So now we have $C(6) = C(1) = red$, but $\theta_{16} = 5 \times 18° = 90°$, and hence CR is violated (since now we have two *red* points in an orthogonal triplet, while CR allows only one *red* member of an orthogonal triplet).

Kochen and Specker illustrated the contradiction in a physically realisable scenario. They considered the simultaneous measurement of the squared spin components of a spin one system in three orthogonal directions. The squared spin components S_x^2, S_y^2 and S_z^2 (in three orthogonal directions x,y and z) obey the relation (in units of $\hbar = 1$)

$$S_x^2 + S_y^2 + S_z^2 = 2. \tag{1.10}$$

And since each squared component has eigenvalues $\in \{0, 1\}$, therefore the value assignment to each squared component must obey

$$(V(S_x^2), V(S_y^2), V(S_z^2)) \in \{(1,1,0),\ (0,1,1),\ (1,0,1)\} \tag{1.11}$$

in order to satisfy eqn. (1.10). This is equivalent to our colouring problem in $\mathbb{R}^3$ if we map 0 to *red* and 1 to *blue*. For the simultaneous measurement of S_x^2, S_y^2 and S_z^2, Kochen and Specker showed that we can do this by considering an orthohelium[9] atom in a weak electric field with rhombic symmetry. By measuring the energy shift of the lowest orbital state, we can infer (by calculation) the values of S_x^2, S_y^2 and S_z^2 by using their algebraic relation to the perturbation Hamiltonian. The perturbation Hamiltonian H_s can be shown to have the following form

$$H_s = aS_x^2 + bS_y^2 + cS_z^2, \qquad a, b, c \in \mathbb{R}, \quad a \neq b \neq c. \tag{1.12}$$

[9] An orthohelium atom is a helium atom in which the two electrons are in the triplet total spin state, and thus this is a spin 1 system.

It is easy to check explicitly in terms of the matrix representation of spin 1 operators

$$S_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix} \quad S_z = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{1.13}$$

that the squared components can be written in terms of H_s as

$$\begin{aligned} S_x^2 &= (a-b)^{-1}(c-a)^{-1}(H_s-(b+c))(H_s-2a), \\ S_y^2 &= (b-c)^{-1}(a-b)^{-1}(H_s-(c+a))(H_s-2b), \\ S_z^2 &= (c-a)^{-1}(b-c)^{-1}(H_s-(a+b))(H_s-2c). \end{aligned} \tag{1.14}$$

1.4 The Core of The Contradiction

It is important to note that the contradiction arises only for what is called *nonmaximal* observables (or *nonmaximal* operators). A nonmaximal operator is an operator that has degenerate (repeated) eigenvalues, while a maximal operator is an operator that has non-degenerate eigenvalues. We can show[10] that if $\hat{A}$ is nonmaximal, then we can write

$$\hat{A} = f(\hat{B}), \qquad \hat{A} = g(\hat{C}), \qquad \text{with } [\hat{B}, \hat{C}] \neq 0, \tag{1.15}$$

where $\hat{B}$ and $\hat{C}$ are some maximal operators, while f and g are some Borel functions. By applying FUNC (and equating both sides), we get the condition that

$$V(\hat{A}) = f(V(\hat{B})) = g(V(\hat{C})). \tag{1.16}$$

which is the core of the contradiction.

To see that, recall that the value assignment problem was concerned with one dimensional projectors which are highly nonmaximal[11]. The problem was that we demanded that the projectors get assigned the same value no matter which set of orthogonal projectors it was considered to be a part of. This is equivalent to considering a projector as a function of different (non-commuting) maximal operators.

To illustrate this, consider the example of the squared spin components. Let us write S_z^2 in eqn. (1.14) as

$$S_z^2 = f(H_s). \tag{1.17}$$

Now consider the following operator in terms of $S_{x'}^2$, $S_{y'}^2$ and S_z^2 (corresponding to orthogonal directions x', y' and z, which can be achieved from x, y and z by a rotation about z axis)

$$H_s^{'} = aS_{x'}^2 + bS_{y'}^2 + cS_z^2 \tag{1.18}$$

We can write S_z^2 also as

$$S_z^2 = f(H_s^{'}). \tag{1.19}$$

[10]See Redhead (1987) p. 20.

[11]In a Hilbert space of dimension n, the degeneracy of a one dimensional projector is $n-1$ (i.e. a projector has $n-1$ repeated eigenvalues, which are equal to 0).

Clearly $[H_s, H_s^{'}] \neq 0$, and both H_s and $H_s^{'}$ are maximal (since their eigenvalues are $a+b$, $b+c$ and $a+c$, which are distinct because $a \neq b \neq c$). Getting the value of the nonmaximal operator S_z^2 using (1.17) is equivalent to considering it with S_y^2 and S_z^2, while using (1.19) is equivalent to considering it with $S_{x'}^2$ and $S_{y'}^2$, and the contradiction arises by demanding that the two values are the same.

2 Escaping the Kochen-Specker Theorem

Now we turn to consider an interesting question that turns up, having discussed the KS theorem; is the KS theorem an impossibility proof or can we avoid its contradiction somehow? The short answer is that we can escape it by introducing some contextuality. But how would this contextuality arise? (and how is it justified?). To see how different notions of contextuality arise we need to go back and inspect the assumptions that have led to the contradiction in the first place. We would expect that denying different assumptions leads to different notions of contextualities. As we discussed before, value realism was the core assumption, leading naturally to both FUNC and non-contextuality. We will see, in fact, that the different notions of contextuality considered here come from some form of a weakening of value realism (if not denying it altogether!).

2.1 Ontological Contextuality

Recalling that the contradiction manifests in that a non-maximal operator obeying FUNC will be assigned the same value regardless of which maximal operator it is considered to be a function of (eqn. (1.16)), we see that one way of avoiding the contradiction while keeping value realism in some sense is to demand that , in the context of eqns (1.15) and (1.16),

$$f(V(\hat{B})) \neq g(V(\hat{C})). \tag{2.1}$$

This way, which was first posed by van Fraassen (1973)[12], obviously solves the problem, but what does it mean? This, in fact, amounts to denying a hidden assumption (that we have been taking for granted all along) about the one to one correspondence between observables and self-adjoint operators.

To illustrate this, let $\mathcal{O}$ be the set of all observables, and let Σ be the set of all self-adjoint operators. Our hidden assumption was that $\phi : \mathcal{O} \to \Sigma$ is injective. Now let us consider if that was not the case. So let us return to eqn. (1.15), and let $X, Y \in \mathcal{O}$ be two (somehow) distinct observables such that

$$\phi(X) = f(\hat{B}), \quad \phi(Y) = g(\hat{C}). \tag{2.2}$$

So from eqn (1.15),

$$\phi(X) = \phi(Y) = \hat{A}, \tag{2.3}$$

i.e. the two distinct observables are mapped to the same self-adjoint operator $\hat{A}$ but they are distinctly defined by their functional relation to operators $\hat{B}$ and $\hat{C}$ respectively.

But what is the nature of this distinction other than being algebraic, since if we consider them as related to $\hat{A}$ we can just assume that ϕ is injective (as we usually do) and say that they are observationally the same? The answer is that if they are really two different observables, then the difference must be ontological. Therefore the value assigned to each one need not be the same, i.e.[12]

$$V(\phi(X)) \neq V(\phi(Y)) \Rightarrow V(f(\hat{B})) \neq V(g(\hat{C})) \Rightarrow f(V(\hat{B})) \neq g(V(\hat{C})), \quad (2.4)$$

and thus this rescues us from the contradiction.

The above result is what Redhead called *ontological contextuality.* But how does the measurement context come into play here? To answer this question, let us elaborate more on what the ontological difference between observables really means. So far, we have been using the notions of observables and beables interchangeably, but now we need to note the distinction between the two. Observables are the things that we, well.., can observe using the physical process that we call measurement, while beables are the elements of physical reality attaining definite values at all times. The usual assumption is that a certain observable tells us direct information about a unique beable. This is one to one observable realism (i.e. each observable corresponds to a unique element of physical reality) which we drop[13] here to escape the KS contradiction. The ontological difference of $\hat{A} = f(\hat{B})$ and $\hat{A} = g(\hat{C})$ then means that the observable A actually corresponds to two different (and also here observationally incompatible) beables behaving observationally similar in different measurement contexts (i.e. on an observational level, it seems that we are getting information about the same element of reality that can be measured in two different ways, but in essence, each way reveals information about different elements of reality).

To illustrate, recall that we can simultaneously measure A with B or A with C (but not B with C). The two measurement schemes are incompatible and hence need different incompatible measurement instruments, let us call them $\mathcal{I}_{AB}$ and $\mathcal{I}_{AC}$ respectively. We can think about the measurement using $\mathcal{I}_{AB}$ for example as an instrument that measures B (and displays its outcome) then applies the function f through a physical process[14]. The two measurements need not yield the same outcome[15]; they are physically distinct processes using different (and incompatible) instruments each of which is designed to reveal a certain aspect of physical reality (or equivalently, designed to get information about a different beable), and this leads to the contextual dependence in this case.

The latter argument—about the unnecessity of equality of the two discussed measurements—was Bell's (1966) solution to the KS contradiction. Redhead regarded ontological contextuality and Bell's argument as seperate solutions to the KS contradiction[16]. We here acknowledge the relation as we have discussed.

[12]Recall that V is defined over operators not observables. If we wanted to define a value map for observables, we can simply define it as $\tilde{V} := V \circ \phi$.

[13]Here is where the original value realism is weakened in a sense, since the one to one correspondence was implicitly assumed in it.

[14]We usually consider that measuring B and then applying the function f to the result ourselves suffices as a simultaneous measurement to A and B, but if you insist on a more integrated way, you can think of an instrument that measures B and transforms the result into binary and then inputs the result in a digital circuit that simulates the function f (which is a genuine physical process!).

[15]We can never know, since we cannot measure B and C simultaneously.

[16]Perhaps because Bell's argument, which was motivated by Bohr's view as well as Bohm's pilot wave theory, had the merit of potentiality contextuality that we are going to discuss later.

We should note that value assignment to observables is still, strictly speaking, non-contextual (although the value map that we defined for operators is now contextual). It is just that a nonmaximal observable correspond to ontologically distinct observables that are assigned different values, and the measurement context manifests in our ignorance about the ontological distinction.

2.2 Relational (Environmental) Contextuality

Another way to escape the contradiction is to say that the true dispersion-free state (that we assign values in) is actually a state of the system as well as its environment. In other words, we can say that even if the QM state $|\psi_{sys.}\rangle$ of the system can be described without dependence on the environment (i.e. $|\psi_{total}\rangle = |\psi_{sys.}\rangle \otimes |\psi_{env.}\rangle$), the complete (dispersion-free) state $|\psi_{sys.}\rangle_\lambda$ is described by hidden variables of both the system and its environment. So how can this save us from the value assignment problem, since it seems that all we have just done is to expand our list of hidden variables to include the environment?

The answer is to remember that a measurement is a physical process, so the measurement instrument will indeed change the environment (and particularly the hidden variables of the environment) and hence it will change the dispersion free state before revealing the outcome, which will be the value of the observable in the new state. For example, consider that we want to measure an observable A, and let the initial state of the system be $|\psi\rangle_{\lambda_1}$ with $V_{|\psi\rangle_{\lambda_1}}(\hat{A}) = a_1$. When the measurement instrument $\mathcal{I}_A$ is introduced, the hidden variables of the environment change to λ_2[17]. Therefore the state changes to $|\psi\rangle_{\lambda_2}$ with $V_{|\psi\rangle_{\lambda_2}}(\hat{A}) = a_2$. Now the measurement happens[18] and the outcome revealed is a_2. In essence a_2 can be the same as a_1[19] but this need not be the case. The idea that the outcome revealed is not the same as the value possessed by the observable before the act of measurement is not new. After all, the measurement is not instantaneous and one could also argue that the true act of (faithful) measurement was the final part of the process, where the final value (here a_2) was faithfully revealed.

More importantly, this still does not save us! It seems that we have introduced some contextual dependence, by including the environment in the game[20], so the outcome now is contextual in some sense, but the value map still assigns values to all observables simultaneously (non-contextually), i.e. the final state $|\psi\rangle_{\lambda_2}$ (just before faithful measurement)[21] pre-assigns values to all observables,

[17]Here we can assume that the measurement instrument has hidden variables that can interact with the hidden variables of the environment and thus change their values. The description of the interaction depends on the HVT at hand and thus should not be confused with the usual QM interaction.

[18]This is the usual QM measurement (with the collapse of the wavefunction etc.), and here we are still retaining faithful measurement.

[19]This should happen, for example, if the initial state was an eigenstate of $\hat{A}$, since even if the the hidden variables change, the (dispersion-free) state will still be an eigenstate of $\hat{A}$ with the same eigenvalue. The HVT should satisfy this in order to reproduce QM results (about measurement of observables in eigenstates) and also to ensure that successive measurements lead to the same revealed outcome regardless of the change of the values of hidden variables.

[20]You might want to argue that this solves the problem by arguing that observables have different versions depending on the relational dispersion-free state, but that would just be ontological contextuality that we discussed before, though in a relational sense.

[21]Here you might want to argue that we can escape the contradiction by presuming that

which leads to the contradiction that we know. Kochen and Specker (1967) commented on this[22] saying:

> "This is nevertheless no argument against the above proof. For in a classical interpretation of quantum mechanics observables such as spin will still be functions on the phase space of the combined apparatus and system and as such should be simultaneously predictable."

So what have we gained by introducing this whole argument? What we have gained is that thinking in this direction relaxes the restriction that the set of possible outcomes is the same as the set of possible assigned values (i.e. we argue that the condition of eqn.(1.4) does not necessarily have to hold). This prompts the idea that the set of possible outcomes might only be a subset of the set of possible values that can be assigned to observables (or more correctly: assigned to the beables corresponding to observables). With this idea in mind, we can now have more values (colours) to solve the (colouring) problem. These extra non-eigenvalues are simply not observable in any measurment context. In other words, the beable underlying an observable has a range of values that exceeds the range we can observe it to have when being measured. In this sense, all possible measurements of an observable only provides us with partial information about the underlying beable[23].

This may seem to threaten us with scepticism, but let us remember that not every aspect of physical reality is directly accessible by measurement, and that does not undermine its reality; neither does it imply a conspiracy by nature. For example, we cannot observe the transition of an atom that absorbs a photon without ruining the whole transition, but the underlying theory (QM in this case) tells us that this process is happening. In this sense we should replace the word "measurement" by "observation" which is a physical phenomenon that tells us information about a certain beable in a certain context. Beables are only observed in certain situations (measurement contexts), and what we observe is their behaviour (values) in these circumstances . It is up to a coherent theory to tell us about the underlying picture of reality when we are not observing. For this we quote Heisenberg [14]:

> "What we observe is not nature itself, but nature exposed to our method of questioning."

The notion of contextuality introduced here can be related to Heywood and Redhead's (1983)[15] environmental contextuality, where they proposed the existence of some non-quantum interaction between the system and the environment that takes place before the measurement and alters the values of observables. In our case, the notion of non-quantum interaction was realized in the form of hidden variable interactions. Shimony (1984)[16] also discussed the idea of environmental contextuality, though in a more general way. We note that

dispersion-free states appearing in a certain context (i.e. for a certain value of λ) are not dispersion-free in all observables, but that is a different story. Here we are still states that assigns sharp values to all observables.

[22] They were actually commenting on Bohm's interpretation of spin in his hidden variable theory[13], but their argument is relevant in our case here as well.

[23] One can deduce here that the beable in question is a special kind of a (semi-)hidden variable!

both their analyses were not as explicit as discussed in this section, in particular, they did not explicitly show how to escape the KS problem in light of the comment of Kochen and Specker that was mentioned earlier.

2.3 Potentiality Contextuality

The only possibility left to consider is to deny value realism by denying the realism part so that now we can disregard the idea that all observables correspond to beables; this introduces the concept of potentiality.

A potentiality is a non-intrinsic property of the system that manifests (or is actualized) during a certain measurement situation (or equivalently, a certain physical phenomenon). This tells us that before the measurement is performed, a potentiality would only provide us with a range of possible outcomes (corresponding to different measurement contexts) and thus does not possess a sharp value prior to a measurement scenario. In other words, if an observable A is a potentiality, then the mere question of asking about the value that A possesses prior to a measurement is really meaningless! The value map simply does not exist for potentialities (by definition). If we needed to define any map at all, it should be called an outcome map, describing what the outcome will be in every possible measurement setting. The outcome map is clearly contextual (it literally depends on each and every possible detail of the measurement context); in particular, the KS problem is solved by noting that observables that are not measured in a certain context do not get assigned any values at all.

Another way of looking at the concept of potentiality is to regard it as a relational attribute of both the system and the measurement instrument. This can be treated as a case of the relational contextuality that we discussed, where now we can consider the true dispersion-free state as the state of the system and apparatus combined. This sheds new light on the notion of the word "measurement": which can by now be regarded as a misnomer and should be really replaced by the word "experiment". The word "measurement" suggests that the underlying process simply reveals a pre-existing value of a property of the system. But we can see now that this need not be the case, the measurement can be regarded as a physical process that is inseparable from the notion of what is being measured. This agrees with Bohr's view that the quantum phenomenon is an unanalysable whole.

But are all observables potentialities (or relational attributes)? For the sake of only escaping the KS conundrum, all we need is that nonmaximal observables satisfy this criterion, but the question at hand is a deep philosophical question, and the mere thought of addressing it here would be an exaggeration. For now let us not abandon realism altogether, and consider that we have primary observables that correspond to beables of the theory and secondary observables that are potentialities. Measurement of secondary observables, as we discussed, does not reveal any information about pre-existing properties that correspond to those secondary observables; instead, it should reveal some information about beables of the system[24], which correspond to primary observables. In other words, the observation of potentialities is in terms of primary observables.

[24]It may reveal information about beables of the measurement instrument as well, but if we know the instrument information before hand (because we prepare the instrument settings for example), we can infer the information about beables of the system.

To illustrate this, consider the measurement of a spin component by a Stern-Gerlach magnet, do we really observe spin directly? What we really observe is the deflection of the particle, and then we attribute the value of the spin component accordingly. So what is essentially observed is the position of the particle. This is not totally new, one could argue that there are directly and indirectly-observable beables, and the job of the measurement is to link what we cannot directly observe to what we can. But in light of our potentiality discussion, we can think of spin as a potentiality. In fact, this is the case in pilot wave theory [13], where the only beables of the theory are the wavefunction itself (which here has an ontological interpretation as a quantum field that "guides" the particle's motion) and the position of the particle (and of course its time derivatives).

We can see that the three notions of contextuality discussed in this section involved (to some extent) treating the measurement as a physical process. We conclude by quoting Bell, as he commented on forgetting about the role of the complete physical set up, saying [17]:

> "When it is forgotten, it is more easy to expect that the results of the observations should satisfy some simple algebraic relations and to feel that these relations should be preserved even by the hypothetical dispersion-free states of which quantum-mechanical states may be composed."

3 The Conway-Kochen Theorem

We now turn to discuss the second main theorem of this essay which is due to Conway and Kochen. Conway and Kochen argued that, subject to a seemingly plausible locality condition: if the experimenters have free will, then so do fundamental particles. They initially proposed this as "The Free Will Theorem" (2006)[4] and then strengthened their argument in (2009)[5] restating their theorem as "The Strong Free Will Theorem". We will consider the latter version, as well as Cator and Landsman's version of the theorem [6, 7, 10].

3.1 The Strong Free Will Theorem

Consider a two-wing, EPR-like [18, 19], experiment with an entangled pair of (massive) spin 1 particles a and b, where the two wings are spacelike separated. On one wing, we have Alice performing simultaneous measurements of squared spin components S_x^2, S_y^2 and S_z^2 of particle a , corresponding to three orthogonal directions[25] x, y and z. On the other wing, we have Bob performing measurements of the squared spin component S_w^2 of particle b, corresponding to some direction w.

Let us denote the result of one squared component of Alice's measurements in x direction as $R_A(x)$, so that we can denote the result of Alice's measurement in three orthogonal directions x, y and z as

$$\tilde{R}_A(x, y, z) = (R_A(x), R_A(y), R_A(z)). \tag{3.1}$$

Similarly, we can denote Bob's result in direction w as $R_B(w)$.

Conway and Kochen used three axioms for their free will theorem; they called them: SPIN, TWIN and MIN. The axioms can be stated as follows:

[25] To be more precise, by direction here we mean a ray in $\mathbb{R}^3$.

- SPIN:

$$R_A(x),\, R_B(w) \in \{1, 0\},$$
$$\tilde{R}_A(x, y, z) \in \{(1, 1, 0),\, (0, 1, 1),\, (1, 0, 1)\}.$$

- TWIN: If the direction w chosen by Bob happens to be the same as one of Alice's choices of x, y and z, then both Alice and Bob get the same results of that particular direction; i.e.

$$\text{If } \ w \in \{x, y, z\}, \text{ then } \ R_B(w) = R_A(w). \tag{3.2}$$

- MIN: IF the two wings of the experiment are spacelike separated, then Alice is free to choose any orthogonal triplet of directions without affecting Bob's outcome, and Bob is also free to choose any direction without affecting Alice's outcome.

The SPIN and TWIN axioms follow directly from QM, but Conway and Kochen wanted to emphasize this particular result without involving the whole formalism of QM. Although this entangled experiment has not been done yet, the extreme success of QM gives us a lot of faith about these axioms. The MIN axiom can be split into two assumptions:

(i) Alice and Bob are free to choose their measurement settings: this is the free will or the freedom assumption. We can see that this ambiguous notion of freedom is just an assumption about the indeterminism of the parameters of measurements. Therefore we will call it *Parameter Indeterminism.*

(ii) The outcome of one wing is independent of the choice of settings (parameters) in the opposite wing. This assumption is a locality condition[26] that is usually known in the literature as Parameter Independence, but we can see that the assumption is about the dependence of an outcome on the local context of the experiment not the global: in other words, the context dependence of the outcome is local; hence we will call this condition *Context Locality*[27].

The bottom line of Conway and Kochen's free will theorem is that if SPIN, TWIN and MIN hold, then we have to give up (outcome) determinism. This would mean that the outcome of the measurement is not fully determined by past events in the universe. Equivalently, in terms of our new definitions of the axioms, we can say

$$\begin{aligned} &\text{QM} + \text{Parameter Indeterminism} + \\ &\qquad\qquad \text{Context Locality} + \text{Outcome Determinism} \longrightarrow \text{Contradiction}, \end{aligned}$$

so that when we decide to give up outcome determinism we get

$$\text{QM+Parameter Indeterminism+Context Locality} \longrightarrow \text{Outcome Indeterminism}. \tag{3.3}$$

[26]Or more precisely: a probabilistic version of it.

[27]We should note that Landsman[7] used the name "context locality" (which he states that he learned from M. Seevinck) to distinguish this locality condition from other conditions like: Einstein locality[20] (of the commutation of spacelike separated operators), and Bell's notion of local causality[8]. But as we discussed, we have used that terminology here to emphasize the locality of the measurement context; we could have otherwise called it "the principle of local contextuality" for all that matters.

When the argument is cast in this way, it looks like we assume some form of indeterminism (namely: the agents' free choice of measurement settings) and conclude another (namely: the outcomes of the measurements).

Let us now proceed with the Conway and Kochen version of the free will theorem, but using a reformulation of the assumptions similar to Cator and Landsman's treatment, which will make it easier to show their version of the theorem later on. Let us start by the following definitions:

- Define Γ_Z as the state space of the universe (excluding free agents), where $Z \in \Gamma_Z$ represents the state of the universe.
- Denote Alice's choice of settings by A and Bob's choice by B.
- Define Γ_A as the range of A (space of possible Alice's settings) which here will be the space of all possible orthogonal triplets of directions in $\mathbb{R}^3$.
- Define Γ_B as the range of B (space of possible Bob's settings) which here will be the space of all possible rays in $\mathbb{R}^3$.
- Denote Alice's outcome function by F and Bob's by G.
- Define
$$\Gamma_F := \{(1,1,0),\ (0,1,1),\ (1,0,1)\}, \quad \Gamma_G := \{1,0\}.$$

From QM (SPIN part) we have

$$F \in \Gamma_F, \quad G \in \Gamma_G.$$

In general we would have

$$F = F(A,B,Z), \quad G = G(A,B,Z),$$

but from context locality, we get

$$F = F(A,Z), \quad G = G(B,Z).$$

From parameter indeterminism, A and B are free variables that are independent of Z, therefore F and G are not fully determined for a given choice of Z. Outcome determinism implies that if we consider a certain state Z_0, then the outcome functions are defined for all parameter choices by Alice and Bob, i.e.

$$F_0(A) \equiv F(A, Z_0) : \Gamma_A \longrightarrow \Gamma_{F,} \quad (3.4)$$
$$G_0(A) \equiv G(B, Z_0) : \Gamma_B \longrightarrow \Gamma_G. \quad (3.5)$$

We should recall that A is a choice of orthogonal triplet (x,y,z), while B is a choice of ray w in $\mathbb{R}^3$, so we can write F_0 and G_0 in a more convenient way as functions of choices of directions (rays) as

$$F_0(x,y,z) \in \Gamma_F, \quad G_0(w) \in \Gamma_G.$$

Notice that here each wing's outcome is contextual (locally), so Alice and Bob independently would have no problem with the Kochen-Specker theorem. However, due to the entanglement (or twinning) between the two wings' particles, Alice and Bob's outcomes are not independent anymore and have to obey the TWIN part of QM result. From TWIN eq. (3.2) and Alice's result eq. (3.1), we get

$$\tilde{R}_A(x,y,z) = (R_B(x), R_B(y), R_B(z)). \quad (3.6)$$

By demanding that the outcome maps must coincide with the results, i.e. $F_0 \equiv \tilde{R}_A$ and $G_0 \equiv R_B$, we get

$$F_0(x, y, z) = (G_0(x), G_0(y), G_0(z)). \tag{3.7}$$

Now the problem is that

$$F_0(x, y, z) = (G_0(x), G_0(y), G_0(z)) \in \{(1, 1, 0),\ (0, 1, 1),\ (1, 0, 1)\}; \tag{3.8}$$

therefore G_0 assigns values to a triplet (x, y, z) obeying a version of the colouring rule (CR) that we discussed in section 1. And since x, y and z are free variables (i.e. Alice can choose any possible triplet), G_0 must follow the colouring rule for all possible triplets—and this is not permitted by the Kochen-Specker theorem as was shown in section 1.

Conway and Kochen conclude that the outcome must then be indeterministic (eq. (3.3)), and that if we regard the outcome as the particle's response to our questions (measurements), then the particles have their own share of the property that we call free will[28]. Furthermore, one might argue that the particle's free will is prior to ours, since after all, we are made out of particles.

The argument of the conclusion seems almost circular, since the proof seems to imply "indeterminism in, indeterminism out" as we have pointed out earlier in (3.3). This led to some objections (e.g. [21, 22]), since we are not really in a position to judge determinism if we initially assume an ambiguous source of indeterminism. Nonetheless, we can conclude that their proof shows that the following is consistent:

$$\text{QM} \ + \ \text{Context Locality} \ + \ (\text{Parameter} \ + \ \text{Outcome}) \text{ Indeterminism.}$$

Though this only shows that indeterminism is just a viable option of escaping the (global) KS contradiction here, and is not to be seen as a necessary conclusion.

3.2 Landsman's Free Will Theorem

Cator and Landsman [6] resolved the criticism to Conway and Kochen's proof by explicitly defining a notion for the freedom of choice in a deterministic world. According to Landsman [7], the thrust of the free will theorem is "Determinism in, Constraints on determinism out". In philosophy, the view of incorporating freedom in a deterministic world is called Compatibilism. Landsman [7, 10] adopts a compatibilist notion of free will, based on a detailed philosophical discussion by David Lewis (1981)[23]: which is called "local miracle compatibilism". We will not discuss this philosophical notion here in detail, but we will show its mathematical formulation, following Landsman, and try to briefly elaborate on this notion of freedom.

We use the same definitions as before, but now we have a super-deterministic view of the world (that still allows some sort of free agents). So additionally we have the following definitions:

[28] They have to be (massive) spin 1 particles though; so in this sense, among the fundamental particles in the standard model, only the weak-interaction gauge bosons enjoy that luxury! But in principle, similar proofs can be done for more complicated quantum systems with Hilbert spaces of dimension > 2. Heywood and Redhead (1983) constructed a general abstract proof, although they were not concerned with abandoning determinism.

- Define Γ as a super-state space of the universe as well as free agents.
- Determinism is defined by demanding that there are maps

  $$A : \Gamma \longrightarrow \Gamma_A, \quad B : \Gamma \longrightarrow \Gamma_B, \quad Z : \Gamma \longrightarrow \Gamma_Z,$$

  i.e.

  $$A = A(\gamma), \quad B = B(\gamma), \quad Z = Z(\gamma), \qquad \forall \gamma \in \Gamma.$$

- Freedom is defined here by demanding that A, B and Z are surjective: i.e.

  $$\forall (a, b, z) \in \Gamma_A \times \Gamma_B \times \Gamma_Z, \quad \exists \gamma \in \Gamma, \quad \text{with } A(\gamma) = a,\ B(\gamma) = b,\ Z(\gamma) = z.$$

  In other words, if we denote the Cartesian product by

  $$\Gamma_{ABZ} := \Gamma_A \times \Gamma_B \times \Gamma_Z,$$

  then we demand that the map

  $$\mathcal{G} : \Gamma \longrightarrow \Gamma_{ABZ}$$

  is surjective.

  The above condition ensures that all measurement settings (every possible combination) can be, in principle, chosen by Alice and Bob. So for each measurement setting on Alice's wing, there is a state[29] in which Alice chooses that setting, and thus Alice is allowed to choose any possible measurement choice, otherwise, it would not be much of a freedom.

  The choices that do happen in the actual world, depending on the actual super-state that happens, are a matter of a different story. Alice could have done otherwise only if the super-state had been different, but Alice could not have changed the super-state. So the freedom here is that there is always a state that allows Alice to choose a certain choice, although the events of choices that happen are determined by the super-state evolution. Therefore, without any unjustified assumption about some a priori super-selection rules between the states[30], Alice (and Bob!) can choose any possible measurement setting.

With these definitions, the same argument follows exactly as before, where we would get (as in eq. (3.8))

$$F_0(x, y, z) = (G_0(x), G_0(y), G_0(z)) \in \{(1, 1, 0),\ (0, 1, 1),\ (1, 0, 1)\},$$

and our notion of freedom tells us that (x, y, z) can be any triplet, and thus G_0 has to follow the colouring rule for all triplets, and that leads us to the usual contradiction. But the conclusion here is different, in this case we have

$$\text{QM} + \text{Context Locality} + \text{Freedom} + \text{Determinism} \longrightarrow \text{Contradiction},$$

which would lead to[31]

$$\text{QM} + \text{Context Locality} \longrightarrow \sim \text{(Local Miracle) Compatibilism}.$$

So here we can see that a philosophically popular notion of free will is at odds with two strong pillars of modern physics; namely, Quantum Mechanics, and a locality condition that is supported by Relativity.

[29] It does not necessarily have to be unique.

[30] That is, we imagine the situation as if there is an initial state that evolves deterministically, there is no reason to assume that some initial conditions are better than others.

[31] The symbol "$\sim$" means denying.

3.3 The Context Locality Condition

Let us now have a closer look at this condition of context locality. As we have seen, this condition is what brings the KS contradiction in the free will theorem: it basically tells us that global non-contextuality must hold. In this sense, the free will theorem is an improvement over the Kochen-Specker theorem: even if we can escape local non-contextuality (as we discussed in section 2), we cannot escape global non-contextuality without violating a very reasonable locality condition.

But what if we were to violate context locality, does it matter which notion of contextuality we use? At first glance, context locality seems insensitive to the type of contextuality that one can introduce to violate it. But, as a matter of fact, we can show that context locality is a combination of conceptually different locality conditions; each of which corresponds to a certain type of contextuality. Let us consider applying our ontological contextuality and environmental contextuality to the global context.

Applying ontological contextuality would amount to a violation of what Heywood and Redhead [15, 11] called *ontological locality* (OLOC). OLOC tells us that local observables that are maximal on one wing (which are obviously non-maximal on the opposite wing) are not ontologically split by ontological contextuality when considered as functions of different (non-commuting) maximal observables of the joint system of the two wings. In other words, OLOC tells us that (local) observables correspond to local beables[32].

Similarly, applying environmental contextuality would lead to a violation of *environmental locality* (ELOC) [15, 11]. ELOC tells us that a change in the environment of one wing (due to a change in the measurement settings for example) does not alter the value of observables at the opposite wing. Aside from mentioning the environment, this seems like a tautology of our original definition of context locality! This is true, but there is a subtlety. Violating ELOC becomes an issue only if OLOC was valid in the first place. In other words, if (local) observables correspond to non-local beables (i.e. if OLOC is violated), then altering their values by changing the environment in the opposite wing is not a locality issue anymore: they were non-local in the first place. Therefore, context locality as we defined it is the same as ELOC only if OLOC was tacitly assumed (as it is usually is).

Heywood and Redhead's result in [15] shows that a deterministic supplementation to QM (i.e. a deterministic hidden variable theory) must violate either ELOC or OLOC[33]. In either case, this would violate context locality (or parameter independence), albeit with conceptually different pictures: violating OLOC means a theory of observable non-local beables, while violating ELOC means a theory of non-local interactions between local beables.

[32] Note that OLOC does not deny the existence of non-local beables. In principle, we can have non-local beables that are not observable at all (they could be hidden variables for example). But if we deny this assumption, then OLOC becomes a condition about the locality of beables.

[33] This is the connection to the free will theorem; although Heywood and Redhead were more concerned with locality rather than determinism.

Conclusion

In this essay, we have dealt with two interesting theorems: the Kochen-Specker theorem that excludes the possibility of non-contextual hidden variable theories, and the free will theorem which excludes local (contextual) deterministic hidden variable theories. These theorems are examples of the kind of impossibility proofs that tell us about what cannot be done rather than what should be done. Despite of this, we have seen that analysing such results might leave us with interesting contemplations and useful intuition.

As we have seen, forgetting about the role of the apparatus led to problems like the KS contradiction. But what about our divided view of the world? we seem to treat the quantum realm with no regard to the existence of gravity! May be we run into problems because we forget the role of gravity, like we ran into problems when we forgot the role of the apparatus in a measurement. For example, Penrose [24] proposed that gravity plays a role in the reduction of the QM state (collapse of the wavefunction). Furthermore, some approaches to quantum gravity (e.g. [25, 26]) suggest that non-locality is fundamental; which saves determinism from the constraints of the free will theorem for example. Some even suggest (e.g. see [27, 28]) that spacetime itself is emergent from more fundamental degrees of freedom which can have non-local relations between them.

We can draw an interesting speculation by contemplating the following question: can we measure the curvature of spacetime directly? This reminds us of the same question (near the end of section 2) about the measurement of spin. Recalling the answer, in light of our discussion of potentialities, fundamental spacetime can be thought of as a potentiality[34] of the observables $\mathcal{O}_{\mathcal{M}}$ we use to measure it (the trajectory of a test particle for example); this is just the relational view of spacetime! Adding this to the picture of spacetime as emerging from more fundamental beables $\mathcal{B}_{\mathcal{F}}$ leads us to contemplate the connection between those beables and the observables $\mathcal{O}_{\mathcal{M}}$. It would be interesting to entertain the idea that all observables are essentially potentialities of $\mathcal{B}_{\mathcal{F}}$. In this sense, quantum theory itself could be fundamentally a theory of these spacetime beables.

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