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Path-Tracking of a Tractor-Trailer Vehicle Along Rectilinear and Circular Paths: A Lyapunov-Based Approach

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***Abstract*—The problem of asymptotic stabilization for straight and circular forward/backward motions of a tractor-trailer system is addressed using Lyapunov techniques. Smooth, bounded, nonlinear control laws achieving asymptotic stability along the desired path are designed, and explicit bounds on the region of attraction are provided. The problem of asymptotic controllability with bounded control is also addressed.**

***Index Terms*—Articulated vehicles, autonomous vehicles, Lyapunov design, mobile robots, nonlinear stabilization.**

I. Introduction

This paper addresses the problem of designing a controller for a tractor-trailer autonomous vehicle which has to follow a prescribed path. The only control variable is the steering angle of the tractor front wheels. The desired path either consists of a circle of a given radius or of a straight line to be followed at a specified speed, in either a forward or backward maneuver. Observe that restricting the attention to circular and rectilinear paths is not too severe a limitation, since any path can be suitably approximated by a sequence of circular and rectilinear arcs.

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This problem is of great interest, both in applications and in purely theoretical contexts. On one side, its solution provides the potential for automatic guidance of a large class of industrial articulated vehicles, such as mining trucks, earth-removal and road-paving vehicles, buses for intercity travels, automated guided vehicles (AGVs), etc. (see, e.g., [1]–[6]). On the other side, the problem has constituted a challenging benchmark for testing the effectiveness of several advanced nonlinear control techniques (see, e.g., [7]–[20]). Most of the methods based on either Jacobian or input–output linearization [17], [20] have the drawback that convergence to the prescribed path is ensured only if the vehicle initial configuration is sufficiently close to the desired one, and the size of the region of attraction is difficult to evaluate. On the other hand, exact state-feedback linearization [11] yielding global convergence is effective only in the case of on-axle hitching.

More advanced nonlinear control techniques, such as those based on chained form [12], [13], [21], [22] or flatness [7], [8], [18] require, in general, a nonobvious selection of the guidepoint and the simultaneous use of two control variables, namely, the longitudinal velocity and the steering velocity. Moreover, they are generally not robust with respect to uncertainty in the system parameters.

This paper makes use of Lyapunov techniques as a tool for the design of a path-tracking controller for the tractor-trailer vehicle. The guidepoint is located in the middle of the tractor rear axle, and the approach of [9] is followed to decouple geometric path tracking from the velocity control. Therefore, independent design of the longitudinal velocity controller and the steering controller is possible. Herein, only the steering controller will be considered. The purpose of the paper is twofold. First, to cope in a unified framework with the general case of positive/null/negative off-axle distance. Second, to design control laws which allow for a precise characterization of the stability domain of the closed-loop system. Furthermore, all the designed controllers take possible limitations on the control action (saturation of the steering angle) explicitly into account.

The paper is organized as follows. Section II states the path-tracking control problem and introduces the model of the path-tracking offsets. The construction of stabilizing control laws is presented in Section III, while Section IV contains some simulation results. The paper ends with some concluding remarks in Section V.

II. Path-Tracking Offsets Model

The vehicle (see Fig. 1) consists of a wheeled tractor with two rear-drive wheels and a front steering wheel, towing a trailer, possibly with off-axle hitching $(c \neq 0)$. The off-axle length c has to be regarded as a variable with the sign being negative when the joint is in front of the wheel axle, and positive otherwise. The longitudinal speed v_1 and the steering angle δ of the tractor are the control variables to be (independently) manipulated so that the guidepoint P_1 follows a desired path with an assigned velocity. We will be concerned with two particular yet significant cases, a rectilinear path and a circular path of radius R_1. For simplicity, it is assumed that the path must be followed at constant speed, but an appropriate time scaling can be used to deal with the (more general) case of variable, yet sign-definite, speed.

Let $l_{\rm os}, \vartheta_{\rm os}$ denote the tractor lateral offset and its orientation offset, respectively (see Fig. 1). They are measured with reference to the projection of the point P_1 of the tractor onto the path. Moreover, let $\varphi_{\rm os} = \varphi - \varphi_p$ be the difference between the current angle φ between tractor and trailer and its steady-state value φ_p along the prescribed path. Path tracking can be viewed as the task of driving these offsets asymptotically to zero.

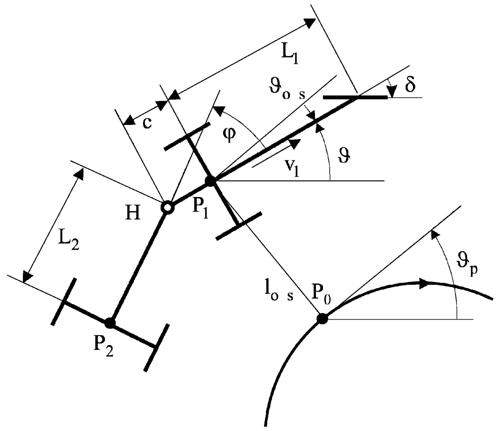

Fig. 1. Vehicle's geometry and path-tracking offsets $l_{\rm os}$ and $\vartheta_{\rm os}$.

Following [9], the dynamics of the offsets are described by

$$\dot{l}_{\rm os} = -\sigma |v_1| \sin \vartheta_{\rm os} \tag{1a}$$

$$\dot{\vartheta}_{\rm os} = v_1 \frac{u}{L_1} - \sigma |v_1| \frac{\cos \vartheta_{\rm os}}{R_1 + l_{\rm os}} \tag{1b}$$

$$\dot{\varphi}_{\rm os} = -\frac{v_1}{L_2} \sin(\varphi_{\rm os} + \varphi_p) - \frac{v_1}{L_1 L_2}(c \cos(\varphi_{\rm os} + \varphi_p) + L_2) u \tag{1c}$$

where $u = \tan \delta$ is the manipulated variable, and the parameter σ is used to distinguish between counterclockwise ($\sigma = 1$) or clockwise ($\sigma = -1$) directions. The equations corresponding to the case of a rectilinear desired path can be formally obtained from (1) by letting R_1 approach infinity with $\sigma = 1$. To rule out meaningless circular path assignments, the radius R_1 must satisfy the inequality $R_1^2 > L_2^2 - c^2$.

Observe that system (1) is not feedback linearizable when $c \neq 0$, see, e.g., [15].

In many applications, the absolute value of the steering angle δ is bounded by a saturation value $\delta_M < \pi/2$. Such a limitation may cause difficulties in backward maneuvers, because the driver cannot recover from a severely jack-knifed initial configuration unless the motion is switched to the forward direction. As a matter of fact, it can be shown that for $L_2 > |c|$ and $\tan \delta_M \leq L_1/\sqrt{L_2^2 - c^2}$, there exist initial values of φ which cannot be driven to the prescribed value φ_p in reverse motion. The same kind of trouble may arise even if $\delta_M = \pi/2$ (no saturation), whenever the length of the trailer is smaller than the absolute value of the off-axle length ($L_2 < |c|$). In the sequel, the symbol $u_M = \tan \delta_M$ will be used to denote the maximum allowable control magnitude.

III. Design of Stabilizing Control Laws

In this section, we address the asymptotic stabilization problem for system (1). Four different situations are considered, namely, the cases of a vehicle moving forward or backward along a straight line or a circle. It will be shown that these problems exhibit substantial differences, hence, it is not possible to derive a general (unified) result. For convenience of exposition, we introduce the following definition.

Definition 1: The scalar valued function $y = {\rm sat}_\varepsilon(x)$ is said to be a unitary ε-saturation function if it is smooth and it is such that $d({\rm sat}_\varepsilon(0))/dx = \varepsilon$, $|{\rm sat}_\varepsilon(x)| \leq \varepsilon$ for all x, and ${\rm sat}_\varepsilon(x)x > 0$ for all nonzero x.

The main results of this section make use of standard Lyapunov theory, as can be found in [23], together with some recent results on stabilization of cascaded systems and of feedforward systems, see [24]–[26] and references therein for further detail.

A. Path Tracking in Forward Motion Along a Straight Line

Proposition 1: Consider the offset dynamics (1). Assume $v_1 > 0$, $R_1 = \infty$, $\sigma = 1$, and $\varphi_p = 0$. Then, for any $0 < \bar{\varphi} < \pi$ and any $u_M > 0$, there exists a feedback control law $u = u(l_{\rm os}, \vartheta_{\rm os})$ such that:

- the zero equilibrium of the closed loop system is locally exponentially stable;
- for any $l_{\rm os}$ and $\vartheta_{\rm os}$ one has $|u(l_{\rm os}, \vartheta_{\rm os})| \leq u_M$;
- any trajectory of the closed-loop system starting in the set

$$M_1 = \{(\varphi_{\rm os}, l_{\rm os}, \vartheta_{\rm os}) \,|\, (\varphi_{\rm os}, l_{\rm os}, \vartheta_{\rm os}) \in [-\bar{\varphi}, \bar{\varphi}] \times I\!R \times I\!R\}$$

remains in M_1 and converges to zero.

Finally, one such control law is

$$u = \eta_1 \tanh l_{\rm os} \frac{\sin \vartheta_{\rm os}}{\vartheta_{\rm os}} - {\rm sat}_{\eta_2}(\vartheta_{\rm os}) \tag{2}$$

with $\eta_1 > 0$, $\eta_2 > 0$, and

$$\eta_1 + \eta_2 < \sin \bar{\varphi} \frac{L_1}{|c| + L_2}. \tag{3}$$

Remark 1: The result expressed in *Proposition 1* lends itself to the following interpretation. If the angle $\varphi_{\rm os}$ does not exceed (in absolute value) π, then there exists an arbitrarily small control, which requires only a partial knowledge of the state of the system, achieving asymptotic (exponential) regulation. Note, moreover, that the limitation on the maximum allowable offset angle $\varphi_{\rm os}$ is very mild in any significant application, and the control law (2) does not depend on any system parameter except for the constraint (3).

Proof: To prove the first claim, consider the $(l_{\rm os}, \vartheta_{\rm os})$ subsystem and the (partial) Lyapunov function

$$V(l_{\rm os}, \vartheta_{\rm os}) = \frac{\eta_1}{L_1} \log(\cosh l_{\rm os}) + \frac{1}{2}\vartheta_{\rm os}^2.$$

Differentiating with respect to time along the trajectories of the closed-loop system yields

$$\dot{V} = -\frac{v_1}{L_1} \vartheta_{\rm os} {\rm sat}_{\eta_2}(\vartheta_{\rm os}) \leq 0.$$

We conclude that $\vartheta_{\rm os}$ converges to zero, and by a straightforward application of La Salle invariance principle, that also $l_{\rm os}$ converges to zero, i.e., the $(l_{\rm os}, \vartheta_{\rm os})$ subsystem in closed loop with the control (2) is globally asymptotically stable.

Note now that the $\varphi_{\rm os}$ subsystem with $u = 0$ is locally asymptotically stable, hence, the overall closed-loop system is a cascaded interconnection of two locally asymptotically stable systems. By a general results on cascaded systems, it is locally asymptotically stable. To conclude local exponential stability, note that the characteristic polynomial of the closed-loop linearized system at the origin is

$$\lambda^3 + v_1 \frac{\eta_2 L_2 + L_1}{L_1 L_2} \lambda^2 + v_1^2 \frac{\eta_1 L_2 + \eta_2}{L_1 L_2} \lambda + v_1^3 \frac{\eta_1}{L_1 L_2}$$

and this is a Hurwitz polynomial for any positive η_1 and η_2.

Observe now, that for any $l_{\rm os}$ and $\vartheta_{\rm os}$, one has

$$|u(l_{\rm os}, \vartheta_{\rm os})| \leq \eta_1 + \eta_2$$

which can be rendered arbitrarily small, and note that

$$\dot{\varphi}_{\rm os}|_{\varphi_{\rm os} = \bar{\varphi}} \leq -\frac{v_1}{L_2} \left(\sin \bar{\varphi} - \frac{|c| + L_2}{L_1} (\eta_1 + \eta_2) \right)$$

and this can be made negative, selecting η_1 and η_2 as in (3). Therefore, if condition (3) holds, the set M_1 is a positively invariant set for the closed-loop system, hence

$$\lim_{t\to\infty} l_{\rm os}(t) = \lim_{t\to\infty} \vartheta_{\rm os}(t) = \lim_{t\to\infty} u(l_{\rm os}(t), \vartheta_{\rm os}(t)) = 0$$

and, by asymptotic stability of the $\varphi_{\rm os}$ subsystem with $u = 0$, $\lim_{t\to\infty} \varphi_{\rm os}(t) = 0$. ■

B. Path Tracking in Forward Motion Along a Circle

Proposition 2: Consider the offset dynamics (1). Assume $v_1 > 0$, $R_1^2 \geq L_2^2 - c^2$, and $L_2 < R_1$. Then, for any $u_M > L_1)/(L_2$, there exists a feedback control law $u = u(\vartheta_{\rm os})$ such that:

- the zero equilibrium of the closed-loop system is locally exponentially stable;
- for any $\vartheta_{\rm os}$ one has

$$|u(\vartheta_{\rm os})| \leq u_M \tag{4}$$

- any trajectory of the closed-loop system starting in the set

$$M_2 = \left\{(\varphi_{\rm os}, l_{\rm os}, \vartheta_{\rm os}) \,|\, (\varphi_{\rm os}, l_{\rm os}, \vartheta_{\rm os}) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times (-R_1, +\infty) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right\}$$

remains in M_2 and converges to zero.

Finally, one such control law is

$$u = \sigma \frac{L_1}{R_1} \cos\vartheta_{\rm os} - {\rm sat}_\varepsilon(\vartheta_{\rm os}) \tag{5}$$

where

$$0 < \varepsilon \leq \frac{L_1}{L_2} - \frac{L_1}{R_1}. \tag{6}$$

Remark 2: The result in *Proposition 2* can be interpreted as follows. If the angle $\varphi_{\rm os}$ does not exceed (in absolute value) $\pi/2$, if the angle $\vartheta_{\rm os}$ is not larger (in absolute value) than $\pi/2$, and finally, if the vehicle position does not coincide with the center of the circle to be tracked, then there exists a (bounded) feedback control law, requiring only the knowledge of the angle $\vartheta_{\rm os}$, achieving local exponential stability and asymptotic convergence in the set M_2.

Proof: Define

$$z = \log\left(1 + \frac{l_{\rm os}}{R_1}\right)$$

and note that, in the coordinates $(z, \vartheta_{\rm os}, \varphi_{\rm os})$, the offset dynamics (1) can be rewritten as

$$\begin{aligned}
\dot{z} &= -\sigma \frac{v_1}{R_1} \sin\vartheta_{\rm os} e^{-z} \\
\dot{\vartheta}_{\rm os} &= \frac{v_1}{L_1} u - \sigma \frac{v_1}{R_1} \cos\vartheta_{\rm os} e^{-z} \\
\dot{\varphi}_{\rm os} &= -\frac{v_1}{L_2} \sin(\varphi_{\rm os} + \varphi_p) \\
&\quad - \frac{v_1}{L_1 L_2}(c\cos(\varphi_{\rm os} + \varphi_p) + L_2) u
\end{aligned}$$

and does not have any singularity.

Consider now the $(z, \vartheta_{\rm os})$ subsystem and the function

$$W(z, \vartheta_{\rm os}) = -\log(\cos\vartheta_{\rm os}) + e^z - z - 1$$

which is positive definite and proper in the set

$$\tilde{M}_2 = \{(z, \vartheta_{\rm os}) \,|\, (z, \vartheta_{\rm os}) \in I\!R \times (-\pi/2, \pi/2)\}$$

and note that, along the trajectories of the closed-loop system

$$\dot{W} = -v_1 \frac{1}{L_1} \tan\vartheta_{\rm os} {\rm sat}_\varepsilon(\vartheta_{\rm os}) \leq 0.$$

We conclude that the $(z, \vartheta_{\rm os})$ subsystem is rendered asymptotically stable in the region $\tilde{M}_2$ by the feedback control (5), which is bounded (in magnitude) by $L_1/R_1 + \varepsilon$.

Consider now the $\varphi_{\rm os}$ subsystem, regarded as a locally asymptotically stable system driven by external disturbances. Arguments similar to those used in the proof of *Proposition 1* allow us to conclude that if ε satisfies (6), any trajectory of the closed-loop system with $\varphi_{\rm os}(0) \in (-\pi/2, \pi/2)$ is such that $\varphi_{\rm os}(t) \in (-\pi/2, \pi/2)$ for all t and converges toward the zero equilibrium.

Finally, to prove local exponential convergence, note that the characteristic polynomial of the closed-loop linearized system at the zero equilibrium is

$$\left(\lambda^2 R_1^2 L_1 + \lambda v_1 \varepsilon R_1^2 + v_1^2 L_1\right) \times \left(\lambda R_1 L_2 + v_1\sqrt{c^2 + R_1^2 - L_2^2}\right)$$

which is a Hurwitz polynomial for any $\varepsilon > 0$, hence the claim. ■

Remark 3: The control bound provided in (4) is very conservative. If $\sigma = 1$, a more precise bound is given by

$$-\varepsilon \leq u(\vartheta_{\rm os}) \leq \frac{L_1}{R_1} + \varepsilon$$

whereas, if $\sigma = -1$, one has

$$-\frac{L_1}{R_1} - \varepsilon \leq u(\vartheta_{\rm os}) \leq \varepsilon.$$

Finally, tighter bounds can be derived by selecting a particular saturation function and computing the extrema of the function (5) for $\vartheta_{\rm os} \in (-\pi/2, \pi/2)$.

C. Path Tracking in Backward Motion Along a Straight Line

Proposition 3: Consider the offset dynamics (1). Assume $v_1 < 0$, $R_1 = \infty$, $\sigma = 1$, $\varphi_p = 0$, and $L_2 > |c|$. Then, for any

$$u_M > \frac{L_1}{L_2 - |c|}$$

there exists a feedback control law $u = u(l_{\rm os}, \varphi_{\rm os}, \vartheta_{\rm os})$ such that:

- the zero equilibrium of the closed-loop system is globally asymptotically and locally exponentially stable;
- for any $l_{\rm os}$, $\varphi_{\rm os}$, and $\vartheta_{\rm os}$, one has $|u(l_{\rm os}, \varphi_{\rm os}, \vartheta_{\rm os})| \leq u_M$.

Finally, one such control law is

$$u = -\frac{L_1}{\beta(\varphi_{\rm os})} \sin\varphi_{\rm os} - {\rm sat}_{\varepsilon_1}(\varphi_{\rm os}) + w \tag{7}$$

where

$$\beta(\varphi_{\rm os}) = L_2 + c\cos\varphi_{\rm os},$$

$$w = -{\rm sat}_{\varepsilon_2}\left(\frac{\sin\varphi_{\rm os}}{{\rm sat}_{\varepsilon_1}(\varphi_{\rm os})} \frac{\gamma\eta}{\beta(\varphi_{\rm os})} + \frac{\beta(\varphi_{\rm os})}{L_1 L_2}\varphi_{\rm os}\right) + r \tag{8}$$

$$\eta = \vartheta_{\rm os} + \psi(\varphi_{\rm os}) \tag{9}$$

the function $\psi(\varphi_{\rm os})$ is the smooth solution of the differential equation

$$\frac{d\psi}{d\varphi_{\rm os}} = \frac{\sin\varphi_{\rm os}}{{\rm sat}_{\varepsilon_1}(\varphi_{\rm os})} \frac{L_1 L_2}{\beta^2(\varphi_{\rm os})} + \frac{L_2}{\beta(\varphi_{\rm os})}$$

satisfying $\psi(0) = 0$

$$r = {\rm sat}_{\varepsilon_3}(k l_{\rm os}) \tag{10}$$

with $\gamma > 0$, $k > 0$ appropriately chosen, and $\varepsilon_i > 0$, for $i = 1, 2, 3$ with ε_3 sufficiently small.

Proof: To begin with, note that the assumption $L_2 > |c|$ implies

$$\beta(\varphi_{\rm os}) = L_2 + c\cos\varphi_{\rm os} > 0$$

for all $\varphi_{\rm os}$, hence, the feedback transformation (7) is globally defined.

Consider now system (1), with $R_1 = \infty$, in closed loop with the control (7). This system is in strict feedforward form, hence, a globally stabilizing control law can be designed using the methodology of [26]. Consider the $(\vartheta_{\rm os}, \varphi_{\rm os})$ subsystem in closed loop with the control in (7)

$$\dot{\vartheta}_{\rm os} = -v_1\frac{\sin\varphi_{\rm os}}{\beta(\varphi_{\rm os})} - v_1\frac{{\rm sat}_{\varepsilon_1}(\varphi_{\rm os})}{L_1} + \frac{v_1}{L_1}w$$
$$\dot{\varphi}_{\rm os} = v_1\frac{\beta(\varphi_{\rm os})}{L_1L_2}{\rm sat}_{\varepsilon_1}(\varphi_{\rm os}) - v_1\frac{\beta(\varphi_{\rm os})}{L_1L_2}w$$

and consider the variable η defined in (9). Note that

$$\dot{\eta} = -\frac{\sin\varphi_{\rm os}}{{\rm sat}_{\varepsilon_1}(\varphi_{\rm os})}\frac{v_1}{\beta(\varphi_{\rm os})}w$$

and define

$$V(\eta, \varphi_{\rm os}) = \gamma\frac{\eta^2}{2} + \frac{\varphi_{\rm os}^2}{2}.$$

Simple computations show that

$$\dot{V} = v_1\varphi_{\rm os}{\rm sat}_{\varepsilon_1}(\varphi_{\rm os})\frac{\beta(\varphi_{\rm os})}{L_1L_2} - v_1\left(\frac{\sin\varphi_{\rm os}}{{\rm sat}_{\varepsilon_1}(\varphi_{\rm os})}\frac{\gamma\eta}{\beta(\varphi_{\rm os})} + \frac{\beta(\varphi_{\rm os})}{L_1L_2}\varphi_{\rm os}\right)w$$

hence, selecting w as in (8) yields

$$\dot{V}|_{r=0} < 0$$

for all nonzero $(\eta, \varphi_{\rm os})$.

Finally, consider the system (1) with u as in (7) and w as in (8). This system can be regarded as the cascaded interconnection of a globally stable subsystem, the $l_{\rm os}$ subsystem, with a globally asymptotically stable subsystem, the $(\eta, \varphi_{\rm os})$ subsystem, and with the control signal r.

Let r be as in (10), then global asymptotic stability of the $(\eta, \varphi_{\rm os})$ subsystem implies a local input-to-state stability with restriction property with respect to the input r. Hence, if ε_3 is sufficiently small, the set

$$\Omega_\alpha = \left\{(l_{\rm os}, \eta, \varphi_{\rm os})\ \middle|\ \gamma\eta^2 + \varphi_{\rm os}^2 \le \alpha^2\right\}$$

where α is a positive constant depending on the parameters of the system, is attractive and positively invariant for the closed-loop system (1), (7), (8), (10). Moreover, the set Ω_α can be arbitrarily shrunk, reducing the parameter ε_3, and if α is sufficiently small, the dynamics of the closed-loop system (1), (7), (8), (10) inside the set Ω_α can be approximated by a linear system, namely, the system that is obtained discarding all high-order terms in $\vartheta_{\rm os}$ and $\varphi_{\rm os}$. The characteristic polynomial of such a system is

$$\lambda^3 + \left(\frac{\varepsilon_2}{\varepsilon_1^2(c+L_2)^2} + \frac{(c+L_2)^2\varepsilon_2}{L_1^2L_2^2} + \frac{(c+L_2)\varepsilon_1}{L_1L_2}\right)|v_1|\lambda^2 + \left(\frac{v_1^2\varepsilon_2\gamma}{\varepsilon_1L_1L_2(c+L_2)} + \frac{k\varepsilon_3v_1|v_1|}{L_1}\right)\lambda + \frac{v_1^2|v_1|k\varepsilon_3}{L_1L_2}$$

and this can be rendered a Hurwitz polynomial for any $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, $\varepsilon_3 > 0$, and $\gamma > 0$, by a proper selection of $k > 0$. As a result, all trajectories in the set Ω_α converge exponentially to the origin. Finally, the control signal is bounded for any $(l_{\rm os}, \varphi_{\rm os}, \vartheta_{\rm os})$ by

$$\frac{L_1}{L_2 - |c|} + \varepsilon_1 + \varepsilon_2 + \varepsilon_3.$$

Hence the claims. ■

Remark 4: Strictly speaking, *Proposition 3* does not provide an explicit formula for a stabilizing control law, because of the presence of the function $\psi(\varphi_{\rm os})$, which is a solution of a differential equation. However, from a practical point of view, it is possible to construct simple approximations of the function $\psi(\varphi_{\rm os})$, using numerical integration procedures. Such approximations turn out to be adequate, i.e., the properties of the closed-loop system are not modified if an approximation, rather than the real function $\psi(\varphi_{\rm os})$, is used.

Remark 5: It is interesting to compare the result in *Proposition 3* with the result in *Proposition 1*. The former is a global result, i.e., no constraint on the initial condition of the system is imposed, whereas the latter is a local property, although an estimation of the region of convergence is computable, and this is, for any practical purpose, sufficiently *large*. However, the former requires full state feedback, whereas the latter needs simply measurement of $l_{\rm os}$ and $\vartheta_{\rm os}$.

Remark 6: There is a substantial difference between a forward motion and a backward motion. In the former, the equation describing the dynamics of $\varphi_{\rm os}$ can be regarded as a locally (exponentially) stable system driven by an exogenous disturbance, whereas in the latter, it can be regarded as a driven exponentially unstable system. This explains the need to feed back the variable $\varphi_{\rm os}$ in the backward maneuver. However, the instability is due to a bounded function, hence, stabilization can be achieved with small, yet not arbitrarily small, control.

D. Path Tracking in Backward Motion Along a Circle

Proposition 4: Consider the offset dynamics (1). Assume $v_1 < 0$ and $L_2 > |c|$. Then for any $u_M > L_1/(L_2 - |c|)$, there exists a feedback control law $u = u(l_{\rm os}, \varphi_{\rm os}, \vartheta_{\rm os})$ such that:

- the zero equilibrium of the closed-loop system is locally asymptotically stable;
- any trajectory of the closed-loop system starting in the set

  $$M_4 = \{(\varphi_{\rm os}, l_{\rm os}, \vartheta_{\rm os})\,|\,(l_{\rm os}, \vartheta_{\rm os}) \in (-R_1, +\infty) \times (-\pi, \pi]\}$$

  remains in M_4 and converges to zero;
- for any $l_{\rm os}$, $\varphi_{\rm os}$, and $\vartheta_{\rm os}$, one has

$$|u(l_{\rm os}, \varphi_{\rm os}, \vartheta_{\rm os})| \le u_M. \tag{11}$$

Moreover, one such control law is

$$u = L_1\mu(\varphi_{\rm os})\left(-\frac{\sin(\varphi_{\rm os}+\varphi_p)}{L_2} - {\rm sat}_{\varepsilon_1}\left(\frac{k_1}{\varepsilon_1}\varphi_{\rm os}\right) + w\right) \tag{12}$$

where

$$w = -{\rm sat}_{\varepsilon_2}\left(\frac{k_2}{\varepsilon_2}\psi\right)$$
$$\psi = \varphi_{\rm os} + (\chi + C(\varphi_{\rm os}, \chi, l_{\rm os})) \times \left(\gamma(\vartheta_{\rm os}, l_{\rm os})\mu(\varphi_{\rm os})\left(1 + \frac{\partial C}{\partial\chi}\right) + \frac{\partial C}{\partial\varphi_{\rm os}}\right)$$
$$\chi = \log(1 + k_3\eta)$$
$$\eta = \frac{1}{2}\frac{l_{\rm os}^2}{R_1} + (l_{\rm os} + R_1)(1 - \cos\vartheta_{\rm os})$$
$$\mu(\varphi_{\rm os}) = \frac{L_2}{c\cos(\varphi_{\rm os}+\varphi_p) + L_2}$$
$$\gamma(\vartheta_{\rm os}, l_{\rm os}) = -\frac{k_3(l_{\rm os}+R_1)\sin\vartheta_{\rm os}}{1 + k_3\eta}$$

and $C(\varphi_{os}, \chi, l_{os})$ is the solution of the partial differential equation

$$0 = \gamma(\vartheta_{os}, l_{os})M(\varphi_{os}) - \frac{\partial C}{\partial \varphi_{os}}\mathrm{sat}_{\varepsilon_1}\left(\frac{k_1}{\varepsilon_1}\varphi_{os}\right) - \frac{\partial C}{\partial l_{os}}\sigma \sin\vartheta_{os} + \frac{\partial C}{\partial \chi}\gamma(\vartheta_{os}, l_{os})M(\varphi_{os}) \quad (13)$$

with[1]

$$\vartheta_{os} = \arccos\left(1 + \frac{\frac{1}{2}\frac{l_{os}^2}{R_1} - \frac{e^{\chi}-1}{k_3}}{l_{os} + R_1}\right) \quad (15)$$

$$M(\varphi_{os}) = \frac{\sigma}{R_1} - \mu(\varphi_{os})\left(\frac{\sin(\varphi_{os}+\varphi_p)}{L_2} + \mathrm{sat}_{\varepsilon_1}\left(\frac{k_1}{\epsilon_1}\varphi_{os}\right)\right)$$

such that $C(0, \chi, l_{os}) = 0$ for all (χ, l_{os}), and k_1, k_2, k_3, ε_1, and ε_2 are positive constants.

Proof: We break up the proof in four steps. First, we rewrite the equations of the system in the new coordinates $(\varphi_{os}, \chi, l_{os})$. Then we show that there is a solution of the partial differential equation (13), and we derive a few properties of the function $C(\varphi_{os}, \chi, l_{os})$. Then we show that the proposed control law is such that the first two claims of *Proposition 4* hold. Finally, we show that condition (11) can be enforced by a proper selection of the design parameters.

Step 1) *Change of coordinates*. A simple analysis shows that, for any pair (l_{os}, ϑ_{os}) in the set M_4, there is a unique pair (l_{os}, χ), with χ in the set specified in (14), and vice-versa. Moreover, the point $(l_{os}, \vartheta_{os}) = (0, 0)$ is mapped into the point $(l_{os}, \chi) = (0, 0)$. In the coordinates $(l_{os}, \chi, \varphi_{os})$, the offset dynamics (1) are described by

$$\begin{aligned}\dot{l}_{os} &= -\sigma|v_1|\sin\vartheta_{os}\\ \dot{\chi} &= |v_1|\gamma(\vartheta_{os}, l_{os})\left(\frac{\sigma}{R_1} + \frac{u}{L_1}\right)\\ \dot{\varphi}_{os} &= -\frac{v_1}{L_2}\sin(\varphi_{os}+\varphi_p)\\ &\quad - \frac{v_1}{L_1 L_2}(c\cos(\varphi_{os}+\varphi_p) + L_2)u \end{aligned} \quad (16)$$

where ϑ_{os} is as in (15).

Step 2) *Existence of a solution of (13)*. Consider system (16) with u as in (12)

$$\begin{aligned}\dot{l}_{os} &= -\sigma|v_1|\sin\vartheta_{os}\\ \dot{\chi} &= |v_1|\gamma(l_{os}, \vartheta_{os})(M(\varphi_{os}) + \mu(\varphi_{os})w)\\ \dot{\varphi}_{os} &= |v_1|\left(-\mathrm{sat}_{\varepsilon_1}\left(\frac{k_1}{\epsilon_1}\varphi_{os}\right) + w\right)\end{aligned}$$

where ϑ_{os} is as in (15). This system with $w = 0$ can be regarded as the cascaded interconnection of two systems, the former with state φ_{os} (globally) asymptotically stable, and the latter with state (l_{os}, χ) globally stable. This system has the same structure and the same properties as the systems studied in [24]. Hence, by the general results established in [24], there exists a function $C(\varphi_{os}, \chi, l_{os})$ solving the partial differential equation (13). Moreover, the function $C(\varphi_{os}, \chi, l_{os})$ is differentiable, in its set of definition, and it is such that the function

$$\Omega(\varphi_{os}, \chi, l_{os}) = \frac{\varphi_{os}^2}{2} + \frac{(\chi + C(\varphi_{os}, \chi, l_{os}))^2}{2}$$

[1]By definition of χ, one has

$$\frac{1}{2}\frac{l_{os}^2}{R_1} \leq \frac{e^{\chi}-1}{k_3} \leq \frac{1}{2}\frac{l_{os}^2}{R_1} + 2(l_{os}+R_1) \quad (14)$$

hence, ϑ_{os} in (15) is well defined.

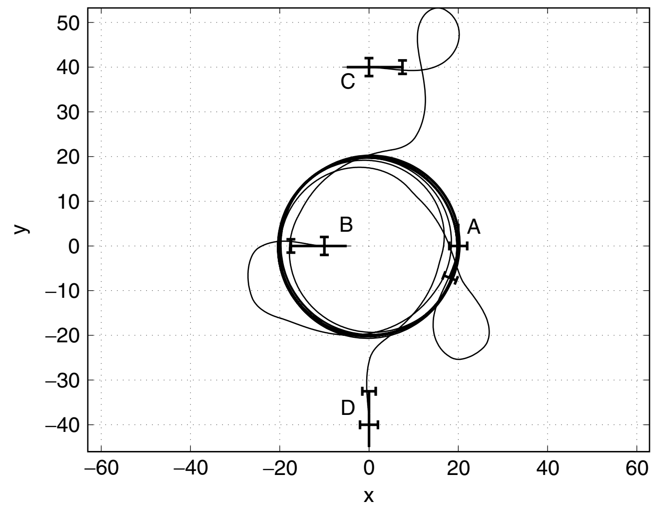

Fig. 2. Tracking a circle: trajectories of the guidepoint starting from different initial configurations.

is radially unbounded. Finally, the function $C(\varphi_{os}, \chi, l_{os})$ is such that $C(0, \chi, l_{os}) = 0$, i.e., there exists a differentiable function $C_1(\varphi_{os}, \chi, l_{os})$ such that

$$C(\varphi_{os}, \chi, l_{os}) = C_1(\varphi_{os}, \chi, l_{os})\varphi_{os}.$$

This equation implies that

$$\left.\frac{\partial C}{\partial l_{os}}\right|_{\varphi_{os}=0} = \left.\frac{\partial C}{\partial \chi}\right|_{\varphi_{os}=0} = 0$$

$$\left.\frac{\partial^2 C}{\partial l_{os}\partial\varphi_{os}}\right|_{\varphi_{os}=0} = \left.\frac{\partial C_1}{\partial l_{os}}\right|_{\varphi_{os}=0}.$$

Step 3) *Asymptotic stability*. Consider the function $\Omega(\varphi_{os}, \chi, l_{os})$ and note that, by construction

$$\dot{\Omega} = -|v_1|\varphi_{os}\mathrm{sat}_{\varepsilon_1}\left(\frac{k_1}{\varepsilon_1}\varphi_{os}\right) - |v_1|\psi\,\mathrm{sat}_{\varepsilon_2}\left(\frac{k_2}{\varepsilon_2}\psi\right) \leq 0.$$

As a result, the zero equilibrium is stable in the sense of Lyapunov. To conclude asymptotic stability, it is sufficient to invoke La Salle invariance principle, noting that $\dot{\Omega} = 0$ implies $\varphi_{os} = 0$ and $\psi = 0$. These, in turn, imply that along any trajectory of the closed-loop system such that $\dot{\Omega} = 0$, one has

$$u = -\frac{\sigma L_1}{R_1}, \quad w = 0, \quad \chi = \bar{\chi}, \quad \eta = \bar{\eta} \quad (17)$$

for some constants $\bar{\chi}$ and $\bar{\eta}$. Moreover, along such trajectories

$$0 = \psi = \bar{\chi}\left(\gamma(\vartheta_{os}, l_{os})\mu(0) + \left.\frac{\partial C}{\partial \varphi_{os}}\right|_{\varphi_{os}=0}\right)$$

$$0 = \dot{\psi} = \bar{\chi}\left(\dot{\gamma}(\vartheta_{os}, l_{os})\mu(0) + \left.\frac{\partial C_1}{\partial l_{os}}\right|_{\varphi_{os}=0}\dot{l}_{os}\right). \quad (18)$$

Consider now the trajectory of the system (1) with the constraints (17) and (18)

$$\begin{aligned}\dot{l}_{os} &= -\sigma|v_1|\sin\vartheta_{os}\\ \dot{\vartheta}_{os} &= |v_1|\sigma\left(\frac{1}{R_1} - \frac{\cos\vartheta_{os}}{R_1 + l_{os}}\right)\\ \dot{\varphi}_{os} &= 0.\end{aligned} \quad (19)$$

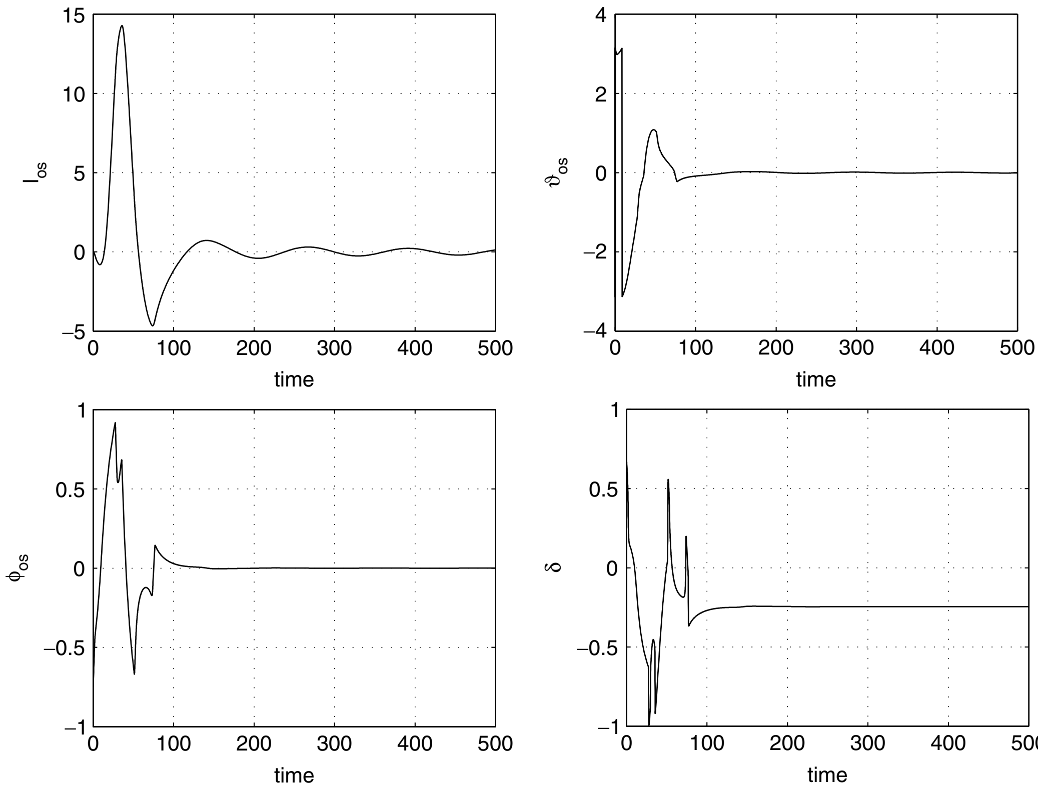

Fig. 3. Tracking a circle: time histories of the offsets and the steering angle in experiment A.

Note that the origin is an equilibrium for this system and that any trajectory $(l_{\rm os}(t), \vartheta_{\rm os}(t))$ of the $(l_{\rm os}, \vartheta_{\rm os})$ subsystem with nonzero initial conditions is such that, for some time $t = \bar{t}$, $\vartheta_{\rm os}(\bar{t}) = 0$. Consider now the constraint (18) at time $\bar{t}$, i.e.,

$$0 = -\bar{\chi}|v_1|\sigma l_{\rm os}(\bar{t})\frac{k_3\mu(0)}{(1+k_3\bar{\eta})R_1}.$$

As a result, either $\bar{\chi} = 0$ or $l_{\rm os}(\bar{t}) = 0$. This implies that $(l_{\rm os}(\bar{t}), \vartheta_{\rm os}(\bar{t})) = (0, 0)$. However, this contradicts the assumption that the trajectory has nonzero initial conditions. We conclude that the only trajectory of system (19) compatible with the constraint (18) is the trivial trajectory.

Step 4) *Bound on the control variable*. This point is straightforward, once noted that the magnitude of the last two terms in the control law can be arbitrarily reduced by a proper selection of ϵ_1 and ϵ_2, and that the term

$$\frac{L_1 L_2}{c\cos(\varphi_{\rm os}+\varphi_p)+L_2}\left(-\frac{1}{L_2}\sin(\varphi_{\rm os}+\varphi_p)\right)$$

is upper bounded in magnitude by

$$\frac{L_1}{L_2-|c|}$$

hence the claim. ■

Remark 7: As in the case of a straight backward motion, the result summarized in *Proposition 4* does not provide an explicit expression for the control law yielding global asymptotic stability. Nevertheless, it is possible to construct approximations of such a control law, i.e., approximations of the solution of the partial differential equation (13).

Remark 8: It is worth noting that the control law (12) does not yield local exponential stability of the origin. In fact, a simple analysis shows that the characteristic polynomial of the matrix associated with the linear approximation of the closed-loop system (1), (12) is

$$\lambda^2(\lambda + |v_1|(k_1+k_2))$$

hence the claim.

IV. Simulation Results

The control strategies developed in this paper have been tested in a number of simulation experiments. The attention has been focused on backward maneuvers along a circular path, since it is known that they represent the most challenging tasks. The simulated vehicle consists of a tractor of length $L_1 = 5$ m towing a trailer of length $L_2 = 5$ m. The off-axle length is $c = 2.5$ m, and the longitudinal velocity is $v_1 = -1$ m/s. The maximum allowable control magnitude is taken as $u_M = 3$.

The path to be tracked coincides with a circle of radius $R_1 = 20$ m, centered at the origin, to be followed counterclockwise. Several experiments have been carried out with different initial vehicle's configurations.

The controller of Section III-D has been suitably tuned, obtaining the following values of the relevant parameters: $\varepsilon_1 = 0.1$; $\varepsilon_2 = 0.1$; $k_1 = 0.1$; $k_2 = 1$; and $k_3 = 1$.

The results are collected in Fig. 2, where the trajectories of the guide-point P_1 are shown. Making reference to experiment A, where the vehicle has to perform a U-turn maneuver, Fig. 3 reports the time histories of the offsets and the steering angle. These simulations confirm convergence of the vehicle to the prescribed path for any initial configuration.

V. Concluding Remarks

The paper has presented an application of Lyapunov techniques to the design of stabilizing control laws for the problem of path tracking a two-body articulated vehicle. In each of the four different situations considered (forward/backward motion, rectilinear/circular desired path), a precise definition of the guaranteed domain of attraction in the space of the offset variables has been given. Possible limitations of the steering angle, frequently encountered in practice, have been effectively dealt with. In some cases, the guaranteed stability region is a finite proper subset of the entire state space. Thus, the development of suitable strategies ensuring the convergence from any arbitrary initial configuration is still a matter of investigation. Many of the ideas in the paper are likely to be extended to a multitrailer vehicle, thanks to the special structure of the relevant offsets equations.

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Task-Space Tracking Control of Robot Manipulators via Quaternion Feedback

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***Abstract*—In this paper, we consider the problem of task-space tracking control of robot manipulators. Based on a quaternion representation of the end-effector orientation, we design a class of task-space controllers that ensure asymptotic end-effector position and orientation tracking. To facilitate the control design, we first develop model-based and adaptive full-state feedback controllers. We then present a model-based output feedback controller that eliminates link velocity measurements via a model-based observer. The application of the proposed control strategy to redundant robots is also discussed. Simulation results based on a six-link manipulator system are presented for the output feedback controller.**

***Index Terms*—Robot manipulator, output feedback control, task-space control, quaternion, Lyapunov.**

I. INTRODUCTION

In robotic applications, the desired task is typically defined in terms of the end-effector motion. As a result, the desired robot trajectory is described by the desired position and orientation of a Cartesian coordinate frame attached to the robot end-effector with respect to the base frame (i.e., the so-called *task-space* variables). Control of the robot motion is then performed using feedback of either the link variables (position and velocity of each robot link) or the task-space variables. Unfortunately, link-based control has the undesirable feature of requiring the solution of the inverse kinematics to convert the desired task-space trajectory into the desired link trajectory. In contrast, task-space control does not require the inverse kinematics; however, the precise tracking control of the end-effector orientation is not straightforward. Several parameterizations exist to describe the orientation angles, including three-parameter representations (e.g., Euler angles, Rodrigues parameters, etc.) and the four-parameter representation given by the unit quaternion. Whereas the three-parameter representations always exhibit singular orientations (i.e., the orientation Jacobian matrix in the kinematic equation is singular for some orientations), the

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