Aggregation and memory of models of changing volatility

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Abstract

In this paper we study the effect of contemporaneous aggregation of an arbitrarily large number of covariance stationary processes featuring short memory dynamic conditional heteroskedasticity, when heterogeneity is allowed for across units. We look at the memory properties of the limit aggregate. General conditions for long memory heteroskedasticity are obtained. More specific results relative to certain stochastic volatility models are also developed, providing some examples of how long memory heteroskedasticity can be obtained by aggregation.

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1 Introduction

Contemporaneous aggregation (in the sense of averaging across units) of stationary heterogeneous autoregressive moving average (ARMA) processes can lead to a limit stationary process displaying long memory (see Definition 1 in the Appendix), when the number of units grows to infinity; see Robinson (1978) and Granger (1980). Relatively recent research in empirical finance indicates that the long memory paradigm represents a valid description of the dependence of volatility of financial asset returns (see Ding, Granger and Engle (1993), Granger and Ding (1996) and Andersen and Bollerslev (1997) among others). In most studies the time series of stock indexes, such as the Standard & Poor's 500, have been used to support this empirical evidence, suggesting that the aggregation mechanism could be the ultimate source of long memory in the volatility of portfolio returns. This paper presents a theoretical analysis prompted by this suggestion.

Some specific results already exist according to which aggregation of the generalized autoregressive conditional heteroskedasticity (GARCH) model of Bollerslev (1986) do not lead to long memory in the sense of a non-summable autocovariance function (acf) of the squared aggregate (hereafter long memory for brevity); see Zaffaroni (2000) and Kazakevicius, Leipus and Viano (2004). One might wonder whether this negative finding applies more generally or, instead, whether there exist volatility models for which long memory can be obtained by aggregation. Therefore we investigate the memory implications of the aggregation mechanism within a large class of volatility models which nests both GARCH and stochastic volatility (henceforth SV) models. The class of square root stochastic autoregressive volatility (SR-SARV) models, introduced by Andersen (1994) and generalized by Meddahi and Renault (1996), appears suitable for this task. A general class of SV models is also proposed by Robinson (2001) but it excludes many ARCH-type models, including GARCH.

For a finite number of units n, aggregation of GARCH has been analyzed by Nijman and Sentana (1996) and generalized by Meddahi and Renault (1996) for aggregation of SR-SARV. In contrast to them, we characterize the conditions under which the aggregate displays different features, namely long memory, from those of the micro units by letting $n \to \infty$.

We start by deriving two sets of sufficient conditions for ruling out long memory with respect to the SR-SARV class. The first condition says that the micro units can be cross-sectionally correlated to the extent that the central limit theorem (clt) works. The second condition formalizes the fact that imposing bounded fourth moment of the micro units might limit the effect of innovations to the conditional variance of the limit aggregate. Many volatility models used nowadays in empirical finance violate these conditions. We then focus on the exponential SV model of Taylor (1986) and on the nonlinear moving average model (nonlinear MA) of Robinson and Zaffaroni (1998), both belonging to the SR-SARV class. Although the necessary conditions for long memory are satisfied in both cases, we show that long memory is ruled out for the exponential SV but is permitted for the nonlinear MA.

The next section presents a set of necessary conditions for long memory with respect to the SR-SARV class. Section 3, which represents the bulk of the paper, analyzes the effect of aggregation of exponential SV and nonlinear MA. Section 4 concludes. The results are stated in propositions whose proofs are reported in the final Appendix.

2 Some general results

Summarizing Meddahi and Renault (1996, Definition 3.1) a stationary square integrable process $\{x_t\}$ is called a SR-SARV(1) process with respect to the increasing filtration J_t if x_t is J_t-adapted, $E(x_t \mid J_{t-1}) = 0$ and $\mathrm{var}(x_t \mid J_{t-1}) =: f_{t-1}$ satisfying

$$f_t = \omega + \gamma f_{t-1} + v_t, \tag{1}$$

with the sequence $\{v_t\}$ satisfying $E(v_t \mid J_{t-1}) = 0$. ω and γ are constant non-negative coefficients with $\gamma < 1$. This implies $Ex_t^2 < \infty$.

In order to study the effect of aggregation over an arbitrarily large number of units, we assume that at each point in time we observe n heterogeneous units $x_{i,t}$ $(1 \leq i \leq n)$, each parameterized as a SR-SARV(1), by assuming that the coefficients ω_i, γ_i are independent identically distributed (*i.i.d.*) random drawn from a joint distribution such that $\gamma_i < 1$ almost surely (*a.s.*).

We now establish two set of conditions, illustrated by Proposition 2.1 and 2.2 respectively, which, independently, rule out the possibility of long memory. The violation of these conditions would therefore be necessary conditions for inducing long memory. Let the aggregate process be

$$X_{n,t} := \frac{1}{n} \sum_{i=1}^{n} x_{i,t}. \tag{2}$$

Proposition 2.1 *Assume that the* $x_{i,t}$ *are i.i.d. across units. When*

$$E(x_{i,t}^2) =: \sigma^2 < \infty, \tag{3}$$

then

$$\sqrt{n}\, X_{n,t} \to_d X_t, \quad \text{as } n \to \infty,$$

where the X_t *are* $N(0, \sigma^2)$ *distributed, mutually independent, and* $\to_d$ *denotes convergence in the sense of the finite dimensional distribution.*

Under *i.i.d.* $x_{i,t}$, the limit aggregate features no memory. When (3) fails, a suitably truncated version of $X_{n,t}$ converges to a sequence of serially uncorrelated normally distributed random variables (although not identically distributed) thus featuring no memory. We skip details for sake of brevity.

As recalled in the Introduction, the limit aggregate of GARCH does not have long memory. This result depends on the fact that the covariance stationarity condition of the $x_{i,t}^2$ limits the behaviour of the impulse response of shocks to the conditional variance of the limit aggregate. The following Proposition 2.2 generalizes this outcome within the wider context of the SR-SARV class.

Proposition 2.2 *Let* $E(x_t^4 \mid J_{t-1}) = c\, f_{t-1}^2$*, some* $0 < c < \infty$*, and let* v_t *be strictly stationary with*

$$E(v_t^2 \mid J_{t-1}) = g_{t-1} + \kappa\, f_{t-1}^2, \tag{4}$$

for a real $0 \leq \kappa \leq 1 - \gamma^2$*, with* γ *from (1), and some* J_t*-measurable function* g_t *satisfying* $Eg_t < \infty$ *for any* $0 < \gamma < 1$*. Then*

$$\gamma^2 + \kappa < 1$$

represents the necessary condition for $Ex_t^4 < \infty$*.*

3 Main results

We now fully investigate the effect of aggregation for two particular elements of the SR-SARV class, namely the nonlinear MA and the exponential SV, whose definitions are recalled below. In both cases we establish the limit in mean square of $X_{n,t}$ and, as a by-product, analyze their memory properties.

3.1 Theoretical results

The nonlinear MA, introduced by Robinson and Zaffaroni (1998), is given in its simplest formulation by

$$x_t = u_t \, (\sum_{k=1}^{\infty} \alpha^k \epsilon_{t-k}), \tag{5}$$

with $|\alpha| < 1$, where $\{u_t, \epsilon_t\}$ denotes a bivariate sequence satisfying:

$$\begin{pmatrix} u_t \\ \epsilon_t \end{pmatrix} \; i.i.d. \text{ with mean } \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ and covariance matrix } \begin{pmatrix} 1 & \sigma_{u\epsilon} \\ \sigma_{u\epsilon} & \sigma^2_\epsilon \end{pmatrix}. \tag{6}$$

It is easy to verify that the nonlinear MA belongs to the SR-SARV(1) class: define the stationary AR(1)

$$h_t = \alpha h_{t-1} + \epsilon_t, \;\; |\alpha| < 1. \tag{7}$$

Then (1) holds for $f_t = h_t^2$ with $v_t = (\epsilon_t^2 - \sigma_\epsilon^2) + 2\alpha\epsilon_t h_{t-1}$ and $\omega = \sigma_\epsilon^2$, $\gamma = \alpha^2$. Consider hereafter the case when n heterogeneous stationary $x_{i,t}$ are observed, each parameterized as a nonlinear MA (5), with coefficient α_i and innovations $\{u_{i,t}, \epsilon_{i,t}\}$. Throughout this section we focus on the case where $u_{i,t} = u_t$, $\epsilon_{i,t} = \epsilon_t$, implying that Proposition 2.1 does not apply and, thus, no loss of memory occurs. This assumption can be generalized to the case of heterogeneous factor loadings yielding $u_{i,t} = a_i \, u_t$, $\epsilon_{i,t} = b_i \epsilon_t$ with *i.i.d.* $\{a_i, \, b_i\}$, without changing the memory implications. Based on the SR-SARV representation, $E(v_t^2 \mid J_{t-1}) = \alpha^4 \text{var}(\epsilon_0^2) + 4\alpha^4 \, \sigma_\epsilon^2 \, f_{t-1}$ (assume for simplicity's sake $E\epsilon_0^3 = 0$), and (4) holds with $\kappa = 0$ and g_t being an affine function of f_t. Therefore, according to Proposition 2.2, no memory restriction arises from $Ex_{i,t}^4 < \infty$.

The second model under consideration is the exponential SV(1) of Taylor (1986):

$$x_t = u_t e^{\frac{1}{2} h_{t-1}}, \tag{8}$$

with h_t defined in (7) and

$$\text{Gaussian } \epsilon_t. \tag{9}$$

The exponential SV belongs to the SR-SARV(∞) class, which generalizes (1); see Meddahi and Renault (2004, section 6) for details. Consider hereafter the case when n heterogeneous stationary $x_{i,t}$ are observed, each parameterized as an exponential SV (8), with heterogeneous coefficient α_i but common innovations $\{u_t, \epsilon_t\}$, thus ensuring that Proposition 2.1 does not apply. Second, rather than generalizing Proposition 2.2 to the case of SR-SARV(∞), we proceed by direct calculations yielding $Ex_t^2 = \exp(\frac{\sigma_\epsilon^2}{2}\frac{1}{1-\alpha^2})$ and $Ex_t^4 = Eu_0^4 \exp(\frac{2\sigma_\epsilon^2}{1-\alpha^2})$. In both cases boundedness requires $\mid \alpha \mid < 1$.

Summarizing, both the nonlinear MA and the exponential SV do satisfy the necessary conditions to induce long memory of the limit aggregate. This is not so common. For instance, Proposition 2.2 applies with a strictly non-zero κ for GARCH, the Glosten, Jaganathan and Runkle (1993) model, asymmetric GARCH, VGARCH and nonlinear asymmetric GARCH, all listed in Engle and Ng (1993), and the Heston and Nandi (2000) model. This is easily verifiable using the SR-SARV representation of these models obtained in Meddahi and Renault (2004, Proposition 4.2).

Hereafter, let c define a bounded constant, not necessarily the same, let ϑ be a finite positive constant and the symbol $\sim$ denotes asymptotic equivalence ($a(x) \sim b(x)$ as $x \to x_0$ when $a(x)/b(x) \to 1$ as $x \to x_0$).

Assumption $A(\vartheta)$**.** *The* α_i *are i.i.d. drawn with an absolutely continuous distribution with support* $[0, \vartheta)$ *and density*

$$f(\alpha) \sim c\, L(\frac{1}{\vartheta - \alpha})\, (\vartheta - \alpha)^{\delta}\, e^{-\frac{\beta}{(\vartheta - \alpha^2)}}, \quad \text{as } \alpha \to \vartheta^-, \tag{10}$$

for real $\beta \geq 0$ *and* $\delta > -1$ *and slowly varying function* $L(\cdot)$*.*

We only require the local behaviour of the cross-sectional distribution of the α_i around unity. However, Assumption $A(\vartheta)$ includes a large class of parametric specifications of $f(\alpha)$. Particular examples are the uniform distribution ($L(\cdot) = 1$ and $\beta = \delta = 0$) and the Beta distribution ($L(\cdot) = 1$ and $\beta = 0$). When $\beta = 0$ (10) becomes

$$f(\alpha) \sim c\, L(\frac{1}{\vartheta - \alpha})\, (\vartheta - \alpha)^{\delta} \quad \text{as } \alpha \to \gamma^-, \tag{11}$$

and $\delta > -1$ ensures integrability. Instead, when $\beta > 0$ then $f(\alpha)$ has a zero of exponential order at ϑ.

Assumption B. *(6) holds with*

$$E\, u_0^4 < \infty, \;\; E\, \epsilon_0^4 < \infty.$$

Assumptions $A(\vartheta)$ and B are assumed to hold without further notice.

Proposition 3.1 (**nonlinear MA**) *Assume that the $x_{i,t}$ are described by (5)-(7).*
(i) When $\vartheta < 1$ or $\vartheta = 1$, $\beta > 0$ or $\vartheta = 1$, $\beta = 0$, $\delta > -1/2$ then $X_{n,t} \to_2 X_t$ as $n \to \infty$ where

$$X_t := u_t \left(\sum_{j=1}^{\infty} \nu_j \epsilon_{t-j-1} \right), \quad \nu_k := E\alpha_i^k. \tag{12}$$

Under the above conditions $\mid X_t \mid < \infty$ a.s. and the X_t are both strictly and weakly stationary and ergodic.
(ii) Under the above conditions the X_t^2 are covariance stationary. When $\vartheta < 1$ or $\vartheta = 1$, $\beta > 0$ then $\mathrm{cov}(X_t^2, X_{t+h}^2) = O(c^h)$ as $h \to \infty$, some $0 < c < 1$. When $\vartheta = 1$, $\beta = 0$, $\delta > -1/2$ then $\mathrm{cov}(X_t^2, X_{t+h}^2) \sim c\, h^{-4\delta-2}$ as $h \to \infty$.

The limit aggregate (12) is precisely the long memory nonlinear MA introduced by Robinson and Zaffaroni (1998) and Proposition 3.1 represents a sound 'rationale' for the model. In particular, when $\vartheta = 1$, $\beta = 0$ the acf of the squared limit aggregate decays at an hyperbolic rate and long memory is achieved when $-1/2 < \delta \leq -1/4$. In the event that $u_t = \epsilon_t$, one obtains the (one-shock) long memory nonlinear MA of Robinson and Zaffaroni (1997).

Proposition 3.2 (**exponential SV**) *Assume that the $x_{i,t}$ are described by (6)-(9).*
(i) When $\vartheta < 1$ or $\vartheta = 1$, $\beta > \sigma_\epsilon^2/4$ then $X_{n,t} \to_2 X_t$ as $n \to \infty$ where

$$X_t := u_t \sum_{r=0}^{\infty} \left(\frac{\sigma_\epsilon}{2} \right)^r N_r(t-1), \tag{13}$$

$$\text{with } N_r(t) := \sum_{\substack{i_0,..,i_r=1 \\ i_0+\ldots+i_r=r \\ 1\, i_1+\ldots+r\, i_r=r}}^{\infty} \frac{1}{0!^{i_0} \ldots r!^{i_r}} \sum_{{}^1 j_1 \neq \ldots \neq {}^1 j_{i_1} \neq \ldots \neq {}^r j_1 \neq \ldots \neq {}^r j_{i_r}}^{\infty\, *} \zeta_{({}^1 j_1+..+{}^1 j_{i_1})+2({}^2 j_1+..+{}^2 j_{i_2})+\ldots+r({}^r j_1+..+{}^r j_{i_r})}$$
$$\times \prod_{h_1=1}^{i_1} H_1(\tilde{\epsilon}_{t-{}^1 j_{h_1}}) \ldots \prod_{h_r=1}^{i_r} H_r(\tilde{\epsilon}_{t-{}^r j_{h_r}}),$$

where $H_m(\cdot)$, $m = 0, 1, ..$, define the real-valued Hermite polynomials (see Hannan (1970)), $\sum^*_{{}^a j_0} = 1\, (1 \leq a \leq r)$, $\tilde{\epsilon}_t := \epsilon_t / \sigma_\epsilon$ and $\zeta_k := E \exp(\frac{\sigma_\epsilon^2}{8} \frac{1}{(1-\alpha_i^2)}) \alpha_i^k$ for any $k \geq 0$.
Under the above conditions $|X_t| < \infty$ a.s. and the X_t are both strictly and weakly stationary and ergodic.

(ii) When $\vartheta < 1$ or $\vartheta = 1$, $\beta > \sigma_\epsilon^2/2$ the X_t^2 are covariance stationary. Under these conditions $\mathrm{cov}(X_t^2, X_{t+h}^2) = O(c^h)$ *as $h \to \infty$, some $0 < c < 1$.*

The asymptotic behaviour of $X_{n,t}$ is prominently different from that of the geometric mean aggregate $G_{n,t} := u_t \left(\prod_{i=1}^n exp(\frac{1}{2}h_{i,t-1})\right)^{\frac{1}{n}}$. In fact $|G_{n,t}|$ represents a very mild lower bound for $|X_{n,t}|$. When $\vartheta = 1$, $\beta = 0$ and (11) holds, the limit of $X_{n,t}$ is unbounded in probability whereas, as noted by Andersen and Bollerslev (1997) and using Zaffaroni (2004, Theorem 5), under the same conditions plus $\delta > -1/2$ then $G_{n,t}$ converges in probability to $u_t exp(0.5 \sum_{j=0}^{\infty} \nu_j \, \epsilon_{t-j-1}) =: G_t$ as $n \to \infty$, with $\nu_k = E\alpha_i^k \sim c\, k^{-(\delta+1)}$ as $k \to \infty$ (cf Lemma 2 in the Appendix). Long memory is obtained when $\delta < 0$. G_t, not (13), is a semiparametric generalization of the long memory SV model of Harvey (1998) and Breidt, Crato and de Lima P. (1998).

3.2 Discussion

Aggregation of exponential SV and nonlinear MA deliver different outcomes, permitting long memory in the latter case but not in the former case. This negative outcome for the exponential SV case (8) depends on the fact that $f(\alpha)$ must decay exponentially fast toward zero as $\alpha \to 1^-$, requiring $\beta > 0$. This implies $E\alpha_i^k = O(c^k)$, $0 < c < 1$, as $k \to \infty$ (cf Lemma 1 in the Appendix). One might wonder whether long memory can be achieved if one could control somehow for the exponential function by suitably renormalizing exponential SV $x_{i,t}$. This possibility is explored in the following Proposition reported without proof for sake of brevity.

Proposition 3.3 *Assume $A(\vartheta)$, B with $\beta = 0$ and let $\tilde{x}_{i,t} := x_{i,t}\sigma_\epsilon(1-\alpha_i^2)^{-\frac{1}{2}} exp(-\frac{\sigma_\epsilon^2}{4(1-\alpha_i^2)})$ with $x_{i,t}$ described by (6)-(9). Then* $\mathrm{var}(\tilde{x}_{i,t} \mid \alpha_i) = \sigma_\epsilon^2(1-\alpha_i^2)^{-1}$ *and $n^{-1}\sum_{i=1}^n \tilde{x}_{i,t} \to_2 \tilde{X}_t$ as $n \to \infty$, with* $\mathrm{cov}(\tilde{X}_t^2, \tilde{X}_{t+h}^2) = O(c^h)$*, some $0 < c < 1$, as $h \to \infty$.*

The limit aggregate of the $\tilde{x}_{i,t}$ does not exhibit long memory. The normalization implies that the $\tilde{x}_{i,t}$ have the same conditional variance $\mathrm{var}(x_{i,t} \mid \alpha_i)$ of the nonlinear MA. However, it also implies the occurrence of a compensation effect, similar to the case $\beta > \sigma_\epsilon^2/4$ in Proposition 3.2, such that the limit aggregate displays short memory. More in general, the compensation effect occurs when aggregating $x_{i,t}\sigma_\epsilon(1-\alpha_i^2)^{-\frac{1}{2}} exp(-\frac{a}{(1-\alpha_i^2)})$, with exponential SV $x_{i,t}$, for $a > \sigma_\epsilon^2/8$, whereas the limit aggregate is not *a.s.* bounded for $0 \leq a < \sigma_\epsilon^2/8$. In contrast, under the same assumptions ($\beta = 0$) the limit aggregate of nonlinear MA can display long memory, as indicated by Proposition 3.1.

Figure 1 presents the plot of the simulated time series of the aggregate for these different models. In particular, we simulated the AR(1) processes

$$h_{i,t}^s = \alpha_i^s h_{i,t-1}^s + \epsilon_t^s, \ \ i = 1, ..., n; t = 1, ..., 2000; s = 1, ..., 10.$$

where $n = 10$ or $1,500$, the ϵ_t^s are $NID(0,1)$ for each t, s and the α_i^s are $i.i.d$ drawn from a Beta distribution with parameters $1, b$ and support $[0, 0.999]$ for each i, s. With respect to Assumption $A(\vartheta)$, the Beta distribution corresponds to the case $L(\cdot) = 1$, $\beta = 0$ and $\delta = b - 1$. `nma` refers to the cross-sectional average of the $u_t^s h_{i,t-1}^s$, with u_t^s being $NID(0,1)$ for each t, s, independent from the ϵ_t^s. `sv` refers to the cross-sectional average of $u_t^s exp(0.5 h_{i,t-1}^s)$ and `mod.sv` refers to the cross-sectional average of $u_t^s exp(0.5 h_{i,t-1}^s)(1 - \alpha_i^{s\,2})^{-\frac{1}{2}} exp(-\frac{1}{4(1-\alpha_i^{s\,2})})$. For each $s = 1, ..., 10$ and each model we calculate the finite-n aggregate $X_{n,t}^s$ and plot the average $10^{-1} \sum_{s=1}^{10} X_{n,t}^s$. In this way the results will not be the outcome of a single realization and would therefore be more representative. The first three columns of Figure 1 refer to $n = 10$ and the last three columns refer to $n = 1,500$. The first, second and third rows correspond to the case $b = 1$, $b = 0.75$ and $b = 0.5$ respectively. Case $b = 1$ corresponds to the case of α_i uniformly distributed over the unit interval, whereas cases $b = 0.75$ and 0.5 corresponds to a denser distribution of the α_i around unity, the denser the distribution, the smaller is b. By an extension of Proposition 3.1, it can be shown that the `nma` limit aggregate is nonstationary when $0 < b \leq 0.5$. It clearly emerges that for the exponential SV, denoted by `sv`, as n increases, the magnitude of the peaks increases sharply. This instability suggests $a.s.$ unboundedness of the process. In contrast, for the `nma` and `mod.sv` models, the increase of n does not lead to any visible changes in the pattern of the time series. This is to be expected, given the theoretical results. In particular, the compensation appears to be effective for the `mod.sv` model. The latter appears more noisy than the `nma` model and this difference does not appear to depend on the value of n nor on the value of b.

Table 1 reports semiparametric estimates of the long memory parameter based on the squares of $10^{-1} \sum_{s=1}^{10} X_{n,t}^s$ as the observable, obtained with the logperiodogram regression estimator of Robinson (1995). We also report the estimates of the memory parameter of the aggregate of the $h_{i,t}^s$, denoted `linear`, to provide a benchmark. We recall that the `nma` limit aggregate displays long memory squares for $b < 0.75$ and indeed, the estimates of the long memory parameter are significantly positive, at conventional significance values, when $b = 0.25$, 0.5, and increasingly so as n rises from 10 to $1,500$. For the cutting-hedge case $b = 0.75$ the estimates are not significantly different from zero but they increase sizeably as n increases. For the `mod.sv` the estimates are always numerically smaller than the corresponding estimates for the `nma`, and never significantly different from zero, at conventional significance values. Both for the `nma` and the `mod.sv`, the estimates are larger, the smaller is b, as expected. For the `sv`, instead, the estimates, never significantly different from zero, decrease with b. It is reasonable to interpret this outcome as another symptom of the instability of the `sv` model, when $\beta = 0$. For the `linear` case the estimates, highly significant when $b = 0.25$, 0.5, are larger than any other cases, in particular larger than the `nma` case. This is to be expected since the memory parameter of the latter case equals $2d - 1/2 < d$ for any $0 < d < 1/2$, indicating by d the memory parameter of the limit aggregate for the `linear` case.

4 Conclusions

We have formally established the result of aggregation for two member of the SR-SARV class, namely the nonlinear MA of Robinson and Zaffaroni (1998) and the exponential SV of Taylor (1986). Although the models share many statistical properties, long memory of the limit aggregate is achieved in the former case but not in latter. Several generalizations are possible. For instance, one can analyze the result of aggregation for more general models, such as conditional heteroskedastic factor models, continuous-time SV, higher-order SV and models with a time-varying conditional mean. Several of the conditions for long memory and, more in general, for existence of the limit aggregate of the models here considered, provides a rich set of testable implications on which to develop empirical applications.

Appendix

We recall that c denotes an arbitrary positive constant not necessarily the same, the symbol $\sim$ denotes asymptotic equivalence and $P(A)$, 1_A, respectively, the probability and the indicator function of any event A. Finally

$\mathrm{E}_n(\cdot)$, $\mathrm{var}_n(\cdot)$ and $\mathrm{cov}_n(\cdot)$ define the expectation, the variance and the covariance operator conditional on the random coefficients.

Definition 1 (stationary long memory). *A covariance stationary processes $\{Z_t;\, t = 0, \pm 1, ...\}$ is long memory when*

$$\sum_{u=0}^{m} |\mathrm{cov}(Z_t, Z_{t+u})| \to \infty \ \ \textit{as}\ m \to \infty. \tag{14}$$

When $\{Z_t\}$ is a linear process $Z_t = c + \sum_{k=0}^{\infty} b_k \epsilon_{t-k}$, where the ϵ_t satisfy (6) and $\sum_{k=0}^{\infty} b_k^2 < \infty$, then $\{Z_t\}$ is long memory when

$$\sum_{u=0}^{m} |b_u| \to \infty \ \ \textit{as}\ m \to \infty. \tag{15}$$

We present three technical lemmas. Their proof is not reported for sake of brevity but are available upon request to the author.

Lemma 1 *Let z be a random variable (r.v.) with support $[0,1]$ and density $g(z) \sim c\ \ \exp(-\frac{\beta}{(1-z)})$ as $z \to 1^-$, for real $0 < \beta < \infty$.*
Then $E(z^k) \sim c\, k^{-\frac{1}{2}}\,(1+\beta)^{-k} exp(-k(1+\beta/2))$ as $k \to \infty$.

Lemma 2 *Let z be a r.v. with support $[0, \vartheta)$ and density*
$g(z) \sim c\ \ (\vartheta - z)^{\delta}$ as $z \to \vartheta^-$, for real $-1 < \delta < \infty$.
Then $E\, z^k \sim c\, \vartheta^k\, k^{-(\delta+1)}$ as $k \to \infty$.

Lemma 3 *Let z_i be i.i.d. drawn from a distribution with support $[0, \vartheta)$ and density $g(z) \sim c\ L(\frac{1}{\vartheta - z})(\vartheta - z)^{\delta} \exp(-\frac{\beta}{(\vartheta - z^2)})$ as $z \to \vartheta^-$, for real $0 \leq \beta < \infty$, $-1 < \delta < \infty$ and slowly varying $L(\cdot)$. For real $0 < \theta < \infty$ set $w_i := \exp(\theta(1 - z_i^2)^{-1})$, $d := \frac{\beta}{\theta}$. When $d > 0$, for a non-degenerate r.v. $\Gamma > 0$ a.s., as $n \to \infty$*

$$\begin{array}{lll} n^{-\frac{1}{d}} \sum_{i=1}^{n} w_i \to_d & \Gamma & \textit{for } 0 < d < 1, \\ n^{-1} \sum_{i=1}^{n} w_i \to_{a.s.} & E w_1 & \textit{for } d > 1. \end{array}$$

When $d = 0$, $P(\sum_{i=1}^{n} w_i / n^c < c') \to 0$ as $n \to \infty$, for any $0 < c, c' < \infty$.

Proof of Proposition 2.1. Given $i.i.d.$ and bounded variance of the $x_{i,t}$ the Lindeberg-Lévy clt applies, as $n \to \infty$. Moreover, for any n by the martingale difference property $\mathrm{cov}_n(\frac{1}{n^{\frac{1}{2}}} \sum_{i=1}^n x_{i,t}, \frac{1}{n^{\frac{1}{2}}} \sum_{i=1}^n x_{i,t+u}) = 0$ for any $u \neq 0$. □

Proof of Proposition 2.2. $Ex_t^4 < \infty$ when $Ef_{t-1}^2 < \infty$. Evaluating the expectation of $f_t^2 = \omega^2 + \gamma^2 f_{t-1}^2 + v_t^2 + 2\omega\gamma f_{t-1} + 2\omega v_t + 2\gamma f_{t-1} v_t$, given J_{t-1}, yields $E(f_t^2 \mid J_{t-1}) = \omega^2 + (\gamma^2 + \kappa) f_{t-1}^2 + 2\omega\gamma f_{t-1} + g_{t-1}$. By Stout (1974, Theorem 3.5.8) the f_t are strictly stationary yielding $Ef_t^2 = Ef_{t-1}^2$ when they are finite. Then collect terms and use the law of iterated expectations. □

Proof of Proposition 3.1. Adapting the proof of Zaffaroni (2004, Theorem 5) part (i) follows. The limit aggregate coincides with the long memory nonlinear MA of Robinson and Zaffaroni (1998) who establish the memory properties of part (ii), given the asymptotic behaviour of the ν_j characterized in Lemma 1 and 2. □

Proof of Proposition 3.2. (i) Any instantaneous transformation of a normally distributed r.v. $g(Z)$, where Z is Gaussian, can be expanded in terms of Hermite polynomials when $E\, g(Z)^2 < \infty$ (see Hannan (1970)). Hence, given $\prod_{k=0}^{\infty} E_n\, exp(\alpha_i^k \epsilon_{t-k}) = exp(\sigma_\epsilon^2 (2(1-\alpha_i^2))^{-1}) < \infty$ $a.s.$, expanding the terms $exp(0.5\alpha_i^k \epsilon_{t-k})$ yields

$$X_{n,t} = u_t \sum_{\substack{m_j=0 \\ j=0,1,..}}^{\infty} \left(\frac{\sigma_\epsilon}{2}\right)^{\sum_{j=0}^{\infty} m_j} \frac{1}{\prod_{j=0}^{\infty} m_j!} \hat{\zeta}_{\sum_{j=0}^{\infty} j m_j} \prod_{j=0}^{\infty} H_{m_j}(\tilde{\epsilon}_{t-j-1}), \tag{16}$$

setting $\hat{\zeta}_k := \frac{1}{n} \sum_{i=1}^n \exp(\frac{\sigma_\epsilon^2}{8} \frac{1}{(1-\alpha_i^2)}) \alpha_i^k$ for any real $k \geq 0$. When $\vartheta = 1$, $\beta > \sigma_\epsilon^2/4$ the ζ_k are finite and the law of iterated logarithm for $i.i.d.$ variates (see Stout (1974, Corollary 5.2.1)) applies yielding $\mid \hat{\zeta}_k - \zeta_k \mid = O\left(\frac{\theta_k^{\frac{1}{2}}}{n^{\frac{1}{2}}} (\ln\ln(n\,\theta_k))^{\frac{1}{2}}\right)$ $a.s.$ for $k \to \infty$, where $\theta_k := E\, \exp(\frac{\sigma_\epsilon^2}{4} \frac{1}{(1-\alpha_i^2)}) \alpha_i^{2k}$, bounded by assumption for any $k \geq 0$. By the independence of the ϵ_t and the orthogonality of the Hermite polynomials, for some $0 < c = c(b, \sigma_\epsilon^2) < 1$ using Lemma 1, $E_n (X_{n,t} - X_t)^2 = \sum_{\substack{m_j, n_j = 0 \\ j=0,1,...}}^{\infty} (\frac{\sigma_\epsilon}{2})^{\sum_{j=0}^{\infty} (m_j + n_j)} \frac{1}{\prod_{j=0}^{\infty} m_j! n_j!} \times$
$(\hat{\zeta}_{\sum_{j=0}^{\infty} j m_j} - \zeta_{\sum_{j=0}^{\infty} j m_j})(\hat{\zeta}_{\sum_{j=0}^{\infty} j n_j} - \zeta_{\sum_{j=0}^{\infty} j n_j}) \mathrm{cov}\left(\prod_{j=0}^{\infty} H_{m_j}(\tilde{\epsilon}_{t-j-1}), \prod_{j=0}^{\infty} H_{n_j}(\tilde{\epsilon}_{t-j-1})\right) = O\left(\frac{\ln\ln n}{n} \exp(\frac{\sigma_\epsilon^2}{4(1-c)})\right)$.
The nonlinear moving average representation (13) of X_t follows by replacing $\hat{\zeta}_k$ by ζ_k in (16) and re-arranging terms. We show covariance stationarity of the X_t. By Schwarz inequality $E_n\, X_{n,t}^2 \leq n^{-2} \sum_{i=1}^n E_n\, exp(h_{i,t-1}) +$

$\left(n^{-1}\sum_{i=1}^{n}(E_n\, exp(h_{i,t-1}))^{\frac{1}{2}}\right)^2$. By Lemma 3 $E_nX_{n,t}^2$ diverges to infinity in probability at rate $n^{2(\frac{\sigma_\epsilon^2}{4\beta}-2)}$ when $\beta < \sigma_\epsilon^2/4$ and $E_n\, X_{n,t}^2$ converges $a.s.$ to $EX_{n,t}^2 < \infty$ when $\beta > \sigma_\epsilon^2/4$. Finally, by Lemma 3, $E_n\, X_{n,t}^2 \to_d \Gamma$ as $n \to \infty$, when $\beta = \sigma_\epsilon^2/4$ for a non-degenerate r.v. Γ. Strict stationarity and ergodicity follows by using Stout (1974, Theorem 3.5.8) and Royden (1980, Proposition 5 and Theorem 3) to (13).
(ii) Focusing for simplicity's sake on case $\vartheta = 1$, by the same arguments used in part (i) then $E_nX_{n,t}^4$ converges to a bounded constant when $\beta > \sigma_\epsilon^2/2$, diverges to infinity in probability at rate $n^{2(\sigma_\epsilon^2/\beta-2)}$ when $\beta < \sigma_\epsilon^2/2$ and converges to a non-degenerate r.v. when $\beta = \sigma_\epsilon^2/2$.
Let us deal with the acf. By the cumulants' theorem (see Leonov and Shiryaev (1959))

$$\begin{aligned}\mathrm{cov}_n(X_{n,t}^2, X_{n,t+h}^2) &= \frac{1}{n^4}\sum_{a,b,c,d=1}^{n} e^{\frac{\sigma_\epsilon^2}{8}\sum_{k=0}^{h-1}(\alpha_c^k+\alpha_d^k)^2}\sum_{k=0}^{\infty} e^{\frac{\sigma_\epsilon^2}{8}\sum_{j=0}^{k-1}(\alpha_a^j+\alpha_b^j+\alpha_c^{j+h}+\alpha_d^{j+h})^2} \\ &\times\left[e^{\frac{\sigma_\epsilon^2}{8}(\alpha_a^k+\alpha_b^k+\alpha_c^{k+h}+\alpha_d^{k+h})^2} - e^{\frac{\sigma_\epsilon^2}{8}(\alpha_a^k+\alpha_b^k)^2}e^{\frac{\sigma_\epsilon^2}{8}(\alpha_c^{k+h}+\alpha_d^{k+h})^2}\right] \\ &\times e^{\frac{\sigma_\epsilon^2}{8}\sum_{j=k+1}^{\infty}(\alpha_a^j+\alpha_b^j)^2}e^{\frac{\sigma_\epsilon^2}{8}\sum_{j=k+1}^{\infty}(\alpha_c^{j+h}+\alpha_d^{j+h})^2}, \quad h > 0.\end{aligned}$$

Expanding the exponential terms and using the inequality $(a+b+c+d)^2 < 4(a^2+b^2+c^2+d^2)$, true for any real a, b, c, d except for case $a=b=c=d$, when $\beta > \sigma_\epsilon^2/2$, we can find a $0 < c < \infty$ such that $E\,\mathrm{cov}_n(X_{n,t}^2, X_{n,t+h}^2)$

$$= O\left(\int_0^1\int_0^1\int_0^1\int_0^1(\alpha_a^k+\alpha_b^k)^r(\alpha_c^{k+h}+\alpha_d^{k+h})^r e^{-c(\frac{1}{1-\alpha_a}+\frac{1}{1-\alpha_b}+\frac{1}{1-\alpha_c}+\frac{1}{1-\alpha_d})}d\alpha_a d\alpha_b d\alpha_c d\alpha_d\right).$$

Expanding the two binomial terms and using Lemma 1 repeatedly yields, for some $0 < \bar{c} < 1$,

$$E\,\mathrm{cov}_n(X_{n,t}^2, X_{n,t+h}^2) = O(\sum_{r=1}^{\infty}\frac{\sigma_\epsilon^{2r}}{r!}\frac{\bar{c}^{hr}}{1-\bar{c}^{2r}}) = O(e^{\sigma_\epsilon^2\bar{c}^h}-1) = O(\bar{c}^h) \quad \text{as } h \to \infty. \ \square$$

References

Andersen, T., 1994. Stochastic autoregressive volatility: a framework for volatility modelling. Mathematical Finance 4, 75–102.

Andersen, T., Bollerslev, T., 1997. Heterogeneous information arrivals and return volatility dynamics: uncovering the long-run in high frequency returns. Journal of Finance 52, 975–1005.

Bollerslev, T., 1986. Generalized autoregressive conditional heteroskedasticity. Journal of Econometrics 31, 302–327.

Breidt, F., Crato, N., de Lima P., 1998. Modelling long-memory stochastic volatility. Journal of Econometrics 73, 325–334.

Ding, Z., Granger, C. W. J., Engle, R. F., 1993. A long memory property of stock market and a new model. Journal of Empirical Finance 1, 83–106.

Engle, R., Ng, V., 1993. Measuring and testing the impact of news on volatility. Journal of Finance 48, 1749–1778.

Glosten, L., Jaganathan, R., Runkle, D., 1993. On the relation between the expected value and the volatility of the nominal excess returns on stocks. Journal of Finance 48, 1779–1801.

Granger, C., 1980. Long memory relationships and the aggregation of dynamic models. Journal of Econometrics 14, 227–238.

Granger, C. W. J., Ding, Z., 1996. Varieties of long memory models. Journal of Econometrics 73, 61–77.

Hannan, E., 1970. Multiple time series. Wiley, New York.

Harvey, A., 1998. Long memory in stochastic volatility. In: J. Knight, S. Satchell (Ed.), Forecasting volatility in financial markets. Butterworth-Heineman, London.

Heston, S., Nandi, S., 2000. A closed-form GARCH option pricing model. Review of Financial Studies 13, 586–625.

Kazakevicius, V., Leipus, R., Viano, M., 2004. Stability of random coefficient ARCH models and aggregation schemes. Journal of Econometrics 120, 139–158.

Leonov, V., Shiryaev, A., 1959. On a method of calculation of semi-invariants. Theory of Probability Applications 4, 319–329.

Meddahi, N., Renault, E., 1996. Aggregation and marginalization of GARCH and Stochastic Volatility models. University of Montreal.

Meddahi, N., Renault, E., 2004. Temporal aggregation of volatility models. Journal of Econometrics 119, 355–379.

Nijman, T., Sentana, E., 1996. Marginalization and contemporaneous aggregation in multivariate GARCH processes. Journal of Econometrics 71, 71–87.

Robinson, P. M., 1978. Statistical inference for a random coefficient autoregressive model. Scandinavian Journal of Statistics 5, 163–168.

Robinson, P. M., 1995. Log-periodogram regression of time series with long-range dependence. Annals of Statistics 23, 1048–1072.

Robinson, P. M., 2001. The memory of stochastic volatility models. Journal of Econometrics 101, 195–228.

Robinson, P. M., Zaffaroni, P., 1997. Modelling nonlinearity and long memory in time series. In: C. Cutler, D. Kaplan (Ed.), Nonlinear Dynamics and Time Series. American Mathematical Society, Providence (RI).

Robinson, P. M., Zaffaroni, P., 1998. Nonlinear time series with long memory: a model for stochastic volatility. Journal of Statistical Planning and Inference 68, 359–371.

Royden, H., 1980. Real analysis. Macmillan, London.

Stout, W., 1974. Almost sure convergence. Academic Press, London.

Taylor, S., 1986. Modelling Financial Time Series. Wiley, Chichester (UK).

Zaffaroni, P., 2000. Contemporaneous aggregation of GARCH processes. Econometrics Discussion Paper 378. LSE.

Zaffaroni, P., 2004. Contemporaneous aggregation of linear dynamic models in large economies. Journal of Econometrics 120, 75–102.

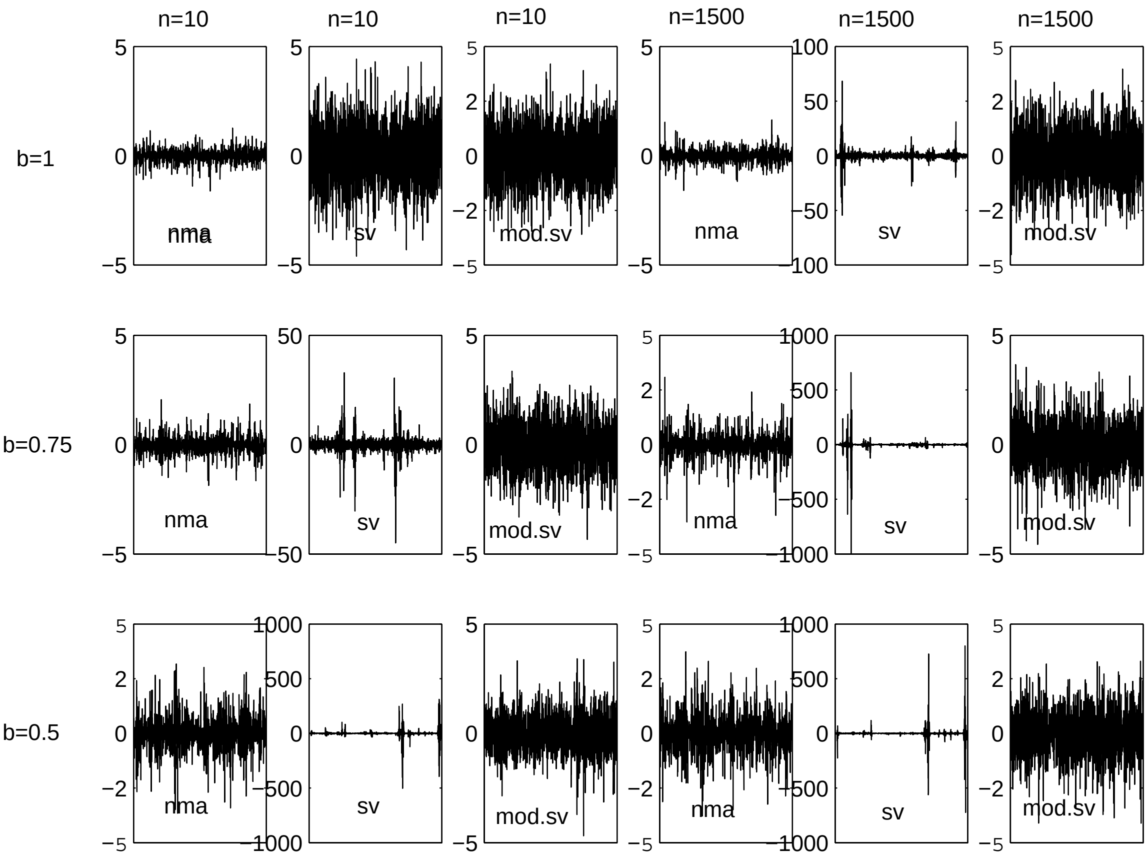
Table 1

Semiparametric estimates of the memory parameter (simulated data)

cross-sectional size $n =$	10				1500			
	nma	sv	mod.sv	linear	nma	sv	mod.sv	linear
$b = .25$	0.319	0.093	0.145	0.392	0.329	0.116	0.172	0.399
	(3.306)	(0.959)	(1.496)	(4.119)	(3.401)	(1.202)	(1.776)	(4.129)
$b = .50$	0.172	0.072	0.115	0.219	0.191	0.115	0.159	0.210
	(1.784)	(0.750)	(1.190)	(2.273)	(1.969)	(1.187)	(1.648)	(2.178)
$b = .75$	0.071	0.105	0.074	0.096	0.107	0.154	0.071	0.180
	(0.737)	(1.087)	(0.762)	(0.999)	(1.106)	(1.594)	(0.736)	(1.865)
$b = 1$	0.051	0.108	0.109	0.093	0.118	0.168	0.098	0.154
	(0.521)	(1.121)	(1.128)	(0.964)	(1.226)	(1.737)	(1.015)	(1.590)

Note: We report the semiparametric estimates of the memory parameter (asymptotic standard errors in parentheses), obtained with the logperiodogram regression method of Robinson (1995) with bandwidth $m = [2000^{\frac{1}{2}}] = 44$. The observable is $\left(10^{-1}\sum_{s=1}^{10} X_{n,t}^s\right)^2$ for `nma`, `sv` and `mod.sv` and $10^{-1}\sum_{s=1}^{10} X_{n,t}^s$ for `linear`, where in the latter case $X_{n,t}^s = n^{-1}\sum_{i=1}^{n} h_{i,t}^s$. See the Note to Figure 1 for the description of the models.

FIGURE 1



Note: We simulated the AR(1) process

$$h_{i,t}^s = \alpha_i^s h_{i,t-1}^s + \epsilon_t^s, \quad i = 1, ..., n; t = 1, ..., 2000; s = 1, ..., 10.$$

where the ϵ^s are $NID(0,1)$ for each t, s and the α_i^s for each i, s are $i.i.d$ drawn from a Beta distribution with parameters $1, b$. Let u_t^s be $NID(0,1)$ for each t, s, independent from the ϵ_t^s. Each panel reports the time series of $10^{-1}\sum_{s=1}^{10} X_{n,t}^s$, where

$$X_{n,t}^s = \begin{cases} \frac{u_t^s}{n}\sum_{i=1}^n h_{i,t-1}^s & \text{for nma}, \\ \frac{u_t^s}{n}\sum_{i=1}^n e^{0.5h_{i,t-1}^s} & \text{for sv}, \\ \frac{u_t^s}{n}\sum_{i=1}^n e^{0.5h_{i,t-1}^s}\frac{1}{(1-\alpha_i^{s\,2})^{\frac{1}{2}}}e^{-\frac{1}{4(1-\alpha_i^{s\,2})}} & \text{for mod.sv}. \end{cases}$$

The first three columns refer to $n = 10$ and the last three columns refer to $n = 1,500$. The first, second and third rows correspond to the case $b = 1$, $b = 0.75$ and $b = 0.5$ respectively. Note that the plots of the second and fifth columns, corresponding to the exponential SV model (sv), refer to a different scale.