

# Teleportation, Entanglement and Thermodynamics in the Quantum World

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## Abstract

Quantum mechanics has many counter-intuitive consequences which contradict our intuition which is based on classical physics. Here we discuss a special aspect of quantum mechanics, namely the possibility of entanglement between two or more particles. We will establish the basic properties of entanglement using quantum state teleportation. These principles will then allow us to formulate quantitative measures of entanglement. Finally we will show that the same general principles can also be used to prove seemingly difficult questions regarding entanglement dynamics very easily. This will be used to motivate the hope that we can construct a thermodynamics of entanglement.

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## I. INTRODUCTION

Quantum mechanics is a non-classical theory and therefore exhibits many effects that are counter-intuitive. This is because in our everyday life we experience a classical (macroscopic) world with respect to which we define “common sense”. One principle that lies at the heart of quantum mechanics is the superposition principle. In itself it might still be understood within classical physics, as it crops up e.g. in classical electrodynamics. However, unlike in classical theory the superposition principle in quantum mechanics also gives rise to a property called entanglement between quantum mechanical systems. This is due to the Hilbert space structure of the quantum mechanical state space. In classical mechanics particles can be correlated over long distances simply because one observer can prepare a system in a particular state and then instruct a different observer to prepare the same state. However, all the correlations generated in this way can be understood perfectly well using classical probability distributions and classical intuition. The situation changes dramatically when we consider correlated systems in quantum mechanics. In quantum mechanics we can prepare two particles in such a way that the correlations between them cannot be explained classically. Such states are called entangled states. It was the great achievement of Bell to recognize this fact and to cast it into a mathematical form that, in principle, allows the test of quantum mechanics against local realistic theories [3–5,13]. Such tests have been performed, and the quantum mechanical predictions have been confirmed [1] although it should be noted that an experiment that has no loopholes (these are insufficiencies in the experiment that allow the simple construction of a local hidden variable theory) has not yet been performed [33]. With the formulation of the Bell inequalities and the experimental demonstration of their violation, it seemed that the question of the non-locality quantum mechanics had been settled once and for all. However, in recent years it turned out this conclusion was premature. While indeed the entanglement of pure states can be viewed as well understood, the entanglement of mixed states still has many properties that are mysterious, and in fact new problems (some of which we describe here) keep appearing. The reason for

the problem with mixed states lies in the fact that the quantum content of the correlations is hidden behind classical correlations in a mixed state. One might expect that it would be impossible to recover the quantum content of the correlations but this conclusion would be wrong. Special methods have been developed that allow us to 'distill' out the quantum content of the correlations in a mixed quantum state [7,8,18,21,22]. In fact, these methods showed that a mixed state which does not violate Bell inequalities can nevertheless reveal quantum mechanical correlations, as one can distill from it pure maximally entangled states that violate Bell inequalities. Therefore, Bell inequalities are not the last word in the theory of quantum entanglement. This has opened up a lot of interesting fundamental questions about the nature of entanglement and we will discuss some of them here. We will study the problem of how to quantify entanglement [9,37,39,40], the fundamental laws that govern entanglement transformation and the connection of these laws to thermodynamics.

On the other hand, the new interest in quantum entanglement has also been triggered by the discovery that it allows us to transfer (teleport) an unknown quantum state of a two-level system from one particle to another distant particle without actually sending the particle itself [6]. As the particle itself is not sent, this represents a method of secure transfer of information from sender to receiver (commonly called Alice and Bob), and eavesdropping is impossible. The key ingredient in teleportation is that Alice and Bob share a publicly known maximally entangled state between them. To generate such a state in practise one has to employ methods of quantum state distillation as mentioned above which we review in Section 3. The protocol of quantum teleportation has been recently implemented experimentally using single photons in laboratories in Innsbruck [44] and Rome [10], which only adds to the enormous excitement that the field of quantum information is currently generating.

But perhaps the most spectacular application of entanglement is the quantum computer, which could allow, once realized, an exponential increase of computational speed for certain problems such as for example the factorization of large numbers into primes, for further explanations see the reviews [41,19,2]. Again at the heart of the idea of a quantum computer lies the principle of entanglement. This offers the possibility of massive par-

allelism in quantum systems as in quantum mechanics  $n$  quantum systems can represent  $2^n$  numbers simultaneously [19,25,41]. The disruptive influence of the environment makes the realization of quantum computing extremely difficult [29,30] and many ideas have been developed to combat the noise in a quantum computer, incidentally again using entanglement [12,20,35,36]. Many other applications of entanglement are now being developed and investigated, e.g. in frequency standards [24], distributed quantum computation [14,16], multiparticle entanglement swapping [11] and multiparticle entanglement purification [28].

In this article we wish to explain the basic ideas and problems behind quantum entanglement, address some fundamental questions and present some of its consequences, such as teleportation and its use in (quantum) communication. Our approach is somewhat unconventional. Entanglement is usually introduced through quantum states which violate the classical locality requirement (i.e. violate Bell's inequalities) as we have done above. Here we abandon this approach altogether and show that there is much more to entanglement than the issue of locality. In fact, concentrating on other aspects of entanglement helps us to view the nature of quantum mechanics from a different angle. We hope that the reader will, after studying this article, share our enthusiasm for the problems of the new and rapidly expanding field of quantum information theory, at the heart of which lies the phenomenon of quantum correlations and entanglement.

## II. QUANTUM TELEPORTATION

We first present an example that crucially depends on the existence of quantum mechanical correlations, i.e. entanglement. The procedure we will analyse is called quantum teleportation and can be understood as follows. The naive idea of teleportation involves a protocol whereby an object positioned at a place  $A$  and time  $t$  first “dematerializes” and then reappears at a distant place  $B$  at some later time  $t+T$ . Quantum teleportation implies that we wish to apply this procedure to a quantum object. However, a genuine quantum teleportation differs from this idea, because we are not teleporting the whole object but just

its state from particle  $A$  to particle  $B$ . As quantum particles are indistinguishable anyway, this amounts to ‘real’ teleportation. One way of performing teleportation (and certainly the way portrayed in various science fiction movies, e.g. *The Fly*) is first to learn all the properties of that object (thereby possibly destroying it). We then send this information as a classical string of data to  $B$  where another object with the same properties is re-created. One problem with this picture is that, if we have a single quantum system in an unknown state, we cannot determine its state completely because of the uncertainty principle. More precisely, we need an infinite ensemble of identically prepared quantum systems to be able completely to determine its quantum state. So it would seem that the laws of quantum mechanics prohibit teleportation of single quantum systems. However, the very feature of quantum mechanics that leads to the uncertainty principle (the superposition principle) also allows the existence of entangled states. These entangled states will provide a form of quantum channel to conduct a teleportation protocol. It will turn out that there is no need to learn the state of the system in order to teleport it. On the other hand, there is a need to send some classical information from  $A$  to  $B$ , but part of the information also travels down an entangled channel. This then provides a way of distinguishing quantum and classical correlations, which we said was at the heart of quantifying entanglement. After the teleportation is completed, the original state of the particle at  $A$  is destroyed (although the particle itself remains intact) and so is the entanglement in the quantum channel. These two features are direct consequences of fundamental laws that are central for understanding entanglement as we explain in a more detail in the next subsection.

### **A. A basic description of teleportation**

Let us begin by describing quantum teleportation in the form originally proposed by Bennett et al [6]. Suppose that Alice and Bob, who are distant from each other, wish to implement a teleportation procedure. Initially they need to share a maximally entangled pair of quantum mechanical two level systems. A two level system in quantum mechanics is

also called a quantum bit, or qubit [34], in direct analogy with the classical bit of information (which is just two distinguishable states of some system). Unlike the classical bit, a qubit can be in a superposition of its basis states, like  $|\Psi\rangle = a|0\rangle + b|1\rangle$ . This means that if Alice and Bob both have one qubit each then the joint state may for example be

$$|\Psi_{AB}\rangle = (|0_A\rangle|0_B\rangle + |1_A\rangle|1_B\rangle)/\sqrt{2}, \quad (1)$$

where the first ket (with subscript A) belongs to Alice and second (with subscript B) to Bob. This state is entangled meaning, that it cannot be written as a product of the individual states (like e.g.  $|00\rangle$ ). Note that this state is different from a statistical mixture ( $|00\rangle\langle 00| + |11\rangle\langle 11|$ )/2 which is the most correlated state allowed by classical physics.

Now suppose that Alice receives a qubit in a state which is unknown to her (let us label it  $|\Phi\rangle = a|0\rangle + b|1\rangle$ ) and she has to teleport it to Bob. The state has to be unknown to her because otherwise she can just phone Bob up and tell him all the details of the state, and he can then recreate it on a particle that he possesses. If Alice does not know the state, then she cannot measure it to obtain all the necessary information to specify it. Therefore she has to resort to using the state  $|\Psi_{AB}\rangle$  that she shares with Bob. To see what she has to do, we write out the total state of all three qubits

$$|\Phi_{AB}\rangle := |\Phi\rangle|\Psi_{AB}\rangle = (a|0\rangle + b|1\rangle)(|00\rangle + |11\rangle)/\sqrt{2}. \quad (2)$$

However, the above state can be written in the following convenient way (here we are only rewriting the above expression in a different basis, and there is no physical process taking place in between)

$$\begin{aligned} |\Phi_{AB}\rangle &= (a|000\rangle + a|011\rangle + b|100\rangle + b|111\rangle)/\sqrt{2} \\ &= \frac{1}{2} \left[ |\Phi^+\rangle(a|0\rangle + b|1\rangle) + |\Phi^-\rangle(a|0\rangle - b|1\rangle) + |\Psi^+\rangle(a|1\rangle + b|0\rangle) + |\Psi^-\rangle(a|1\rangle - b|0\rangle) \right], \quad (3) \end{aligned}$$

where

$$|\Phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2} \quad (4)$$

$$|\Phi^-\rangle = (|00\rangle - |11\rangle)/\sqrt{2} \quad (5)$$

$$|\Psi^+\rangle = (|01\rangle + |10\rangle)/\sqrt{2} \quad (6)$$

$$|\Psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2} \quad (7)$$

form an ortho-normal basis of Alice's two qubits (remember that the first two qubits belong to Alice and the last qubit belongs to Bob). The above basis is frequently called the Bell basis. This is a very useful way of writing the state of Alice's two qubits and Bob's single qubit because it displays a high degree of correlations between Alice's and Bob's parts: to every state of Alice's two qubits (i.e.  $|\Phi^+\rangle, |\Phi^-\rangle, |\Psi^+\rangle, |\Psi^-\rangle$ ) corresponds a state of Bob's qubit. In addition the state of Bob's qubit in all four cases looks very much like the original qubit that Alice has to teleport to Bob. It is now straightforward to see how to proceed with the teleportation protocol [6]:

1. Upon receiving the unknown qubit in state  $|\Phi\rangle$  Alice performs projective measurements on her two qubits in the Bell basis. This means that she will obtain one of the four Bell states randomly, and with equal probability.
2. Suppose Alice obtains the state  $|\Psi^+\rangle$ . Then the state of all three qubits (Alice + Bob) collapses to the following state

$$|\Psi^+\rangle(a|1\rangle + b|0\rangle) . \quad (8)$$

(the last qubit belongs to Bob as usual). Alice now has to communicate the result of her measurement to Bob (over the phone, for example). The point of this communication is to inform Bob how the state of his qubit now differs from the state of the qubit Alice was holding previously.

3. Now Bob knows exactly what to do in order to complete the teleportation. He has to apply a unitary transformation on his qubit which simulates a logical NOT operation:  $|0\rangle \rightarrow |1\rangle$  and  $|1\rangle \rightarrow |0\rangle$ . He thereby transforms the state of his qubit into the state  $a|0\rangle + b|1\rangle$ , which is precisely the state that Alice had to teleport to him initially. This completes the protocol. It is easy to see that if Alice obtained some other Bell state

then Bob would have to apply some other simple operation to complete teleportation. We leave it to the reader to work out the other two operations (note that if Alice obtained  $|\Phi^+\rangle$  he would not have to do anything). If  $|0\rangle$  and  $|1\rangle$  are written in their vector form then the operations that Bob has to perform can be represented by the Pauli spin matrices, as depicted in Fig. 1.

An important fact to observe in the above protocol is that all the operations (Alice's measurements and Bob's unitary transformations) are *local* in nature. This means that there is never any need to perform a (global) transformation or measurement on all three qubits simultaneously, which is what allows us to call the above protocol a genuine teleportation. It is also important that the operations that Bob performs are independent of the state that Alice tries to teleport to Bob. Note also that the classical communication from Alice to Bob in step 2 above is crucial because otherwise the protocol would be impossible to execute (there is a deeper reason for this: if we could perform teleportation without classical communication then Alice could send messages to Bob faster than the speed of light, see e.g. [38]).

Important to observe is also the fact that the initial state to be teleported is at the end destroyed, i.e it becomes maximally mixed, of the form  $(|0\rangle\langle 0| + |1\rangle\langle 1|)/2$ . This has to happen since otherwise we would end up with two qubits in the same state at the end of teleportation (one with Alice and the other one with Bob). So, effectively, we would clone an unknown quantum state, which is impossible by the laws of quantum mechanics (this is the no-cloning theorem of Wootters and Zurek [42]). We also see that at the end of the protocol the quantum entanglement of  $|\Psi_{AB}\rangle$  is completely destroyed. Does this have to be the case in general or might we save that state at the end (by perhaps performing a different teleportation protocol)? Could we for example have a situation as depicted in Fig. 2, where Alice teleports a quantum state from to Bob and afterwards the quantum channel is still preserved. This would be of great practical advantage, because we could use a single entangled state over and over again to teleport an unlimited number of quantum states from



Alice to Bob (this question was first suggested to the authors by A. Ekert). Unfortunately the answer to the above question is NO: the entanglement of the quantum channel has to be destroyed at the end of the protocol. The analytical proof of this seems to be extremely hard, because it appears that we have to check all the possible purification protocols (infinitely many). However, the rest of this article introduces new ideas and principles that will allow us to explain more easily why this needs to be so. This explanation will be presented at the end of this article. First, however, we need to understand why entanglement is necessary for teleportation in the first place.

### B. Why is entanglement necessary?

Quantum teleportation does not work if Alice and Bob share a disentangled state. If we take that  $|\Psi_{AB}\rangle = |00\rangle$  and run the same protocol as the above, then Bob's particle stays the same at the end of the protocol, i.e. there is no teleportation. In this case the total state of the three qubits would be

$$|\Phi_1\rangle = (a|0\rangle + b|1\rangle)|00\rangle . \tag{9}$$

We see that whatever we do (or, rather, whatever Alice does) on the first two qubits and however we transform them, the last qubit (Bob's qubit) will always be in the state  $|0\rangle$ ; it is thus completely uncorrelated to Alice's two qubits and no teleportation is possible.

Thus one might be tempted to say that teleportation is unsuccessful because there are no correlations between  $A$  and  $B$ , i.e.  $A$  and  $B$  are statistically independent from each other. So, let us therefore try a state of the form

$$\rho_{AB} = 1/2 (|00\rangle\langle 00| + |11\rangle\langle 11|) . \tag{10}$$

This state is a statistical mixture of the states  $|00\rangle$  and  $|11\rangle$ , both of which are disentangled. This is equivalent to Alice and Bob sharing either  $|00\rangle$  or  $|11\rangle$ , but being completely uncertain about which state they have. This state is clearly correlated, because if Alice has 0 so does

Bob, and if Alice has 1 so does Bob. However, since both the states are disentangled and neither one of them achieves teleportation then their mixture cannot do it either. The interested reader can convince himself of this fact by actually performing the necessary calculation, which is messy but straightforward. It is important to stress that Alice is in general allowed to perform any measurement on her qubits and Bob any state independent transformation on his qubit, but the teleportation would still not work with the above state [31]. In fact, it follows that if  $\{|\alpha_A^i\rangle\}$  is a set of states belonging to Alice and  $\{|\beta_B^j\rangle\}$  a set of states belonging to Bob, then the most general state that cannot achieve teleportation is of the form

$$\sigma_{AB} = \sum_{ij} p_{ij} |\alpha_A^i\rangle\langle\alpha_A^i| \otimes |\beta_B^j\rangle\langle\beta_B^j| , \quad (11)$$

where  $p_{ij}$  are a set of probabilities such that  $\sum_{ij} p_{ij} = 1$ . This is therefore the most general disentangled state of two qubits. This state might have a certain amount of classical correlations as we have seen above, but any form of quantum correlations, i.e. entanglement, is completely absent [22]. So we can now summarize: both classical and quantum correlations are *global* properties of two correlated systems, however, they can be distinguished because classical correlations alone cannot lead to teleportation. This establishes an important fact: entanglement plays a key role in the manipulation of quantum information.

### C. The non-increase of entanglement under local operations

The above discussion leads us to postulate one of the central laws of quantum information processing. We now wish to encapsulate the fact that if Alice and Bob share no entanglement

they can by no local means and classical communication achieve teleportation.

**The fundamental law of quantum information processing.**

Alice and Bob cannot, with no matter how small a probability, by local operations and communicating classically turn a disentangled state  $\sigma_{AB}$  into an entangled state.

The gist of the proof relies on reductio ad absurdum. Suppose they could turn a disentangled state  $\sigma_{AB}$  into an entangled state by local operations and classical communication. If so, then they can use the so obtained entangled state for teleportation. Thus in the end it would be possible to teleport using disentangled states which contradicts the previous subsection. Note the last part of the fundamental law which says “with no matter how small probability”. This is, of course, very important to stress as we have seen that teleportation is not possible at all with disentangled states.

In this paper we will work with a more general variant of the above law, which is more suitable for our purposes. We have seen that non-local features (i.e. entanglement) cannot be created by acting locally. This implies that if Alice and Bob share a certain amount of entanglement (the notion of the amount of entanglement will be made more precise lateron) initially, they cannot increase it by only local actions aided with the classical communication. So we can now restate the fundamental law in the following, more general, way.

**The fundamental law of quantum information processing (2. formulation).**

By local operations and classical communication alone, Alice and Bob cannot increase the total amount of entanglement which they share.

Note that, contrary to the previous formulation, the addition “with no matter how small probability” is missing. This law thus says that the total (or rather, expected) entanglement

cannot be increased. This still leaves room, that with some probability, Alice and Bob can obtain a more entangled state. Then, however, with some other probability they will obtain less entangled states so that on average the mean entanglement will not increase. The above law, it must be stressed, looks deceptively simple, but we will see that it leads to some profound implications in quantum information processing. Although it is derived from considerations of the teleportation protocol, it nevertheless has much wider consequences. For example, we have established that if Alice and Bob share disentangled states of the form in Eq. (11) then no teleportation is possible. But what about the converse: if they share a state not of the form given in Eq. (11) can they always perform teleportation? Namely, even if the state contains a small amount of entanglement, can that always be used for teleportation? This amounts to asking whether, given any entangled state (i.e. a state not of the form in Eq. (11)), Alice and Bob can, with some probability, obtain the state  $(|00\rangle + |11\rangle)/\sqrt{2}$  by acting only locally and communicating classically. Also we stated that entanglement cannot increase under local operations, but in order to check whether it has increased we need some measure of entanglement. All these questions will be discussed in the following section. At the end, we stress that the above law is a working assumption and it cannot be proved mathematically. It just so happens that by assuming the validity of the fundamental law we can derive some very useful results, as will be shown in the rest of the article.

### III. CAN WE AMPLIFY AND QUANTIFY ENTANGLEMENT?

In the previous section we have learnt that entanglement is a property that is essentially different from classical correlations. In particular entanglement allows the transmission of an *unknown quantum state* using only local operations and classical communication. Without Alice and Bob sharing one maximally entangled state this task can not be achieved perfectly. This impossibility is directly related to the fact that it is not possible to create quantum correlations, i.e. entanglement, using only local operations and classical communication.

This means that if we start with a completely uncorrelated state, e.g. a product state, then local operations and classical communication can only produce a *classically correlated state*, which is the essence of the fundamental law stated in the previous section. We will now discuss quantum state teleportation again but now not under ideal conditions but under circumstances that may occur in an experiment, in particular under circumstances where decoherence and dissipation are important. This new, realistic, situation gives rise to a new idea which is called entanglement purification.

### A. Entanglement purification

In the previous section we have learnt that starting from a product state and using only local operations and classical communication, the best we can achieve is a classically correlated state, but we will never obtain a state that contains any quantum correlations. In particular we will not be able to teleport an unknown quantum state if we only share a classically correlated quantum state.

The impossibility of creating entanglement locally poses an important practical problem to Alice and Bob when they want to do teleportation in a realistic experimental situation. Imagine Alice wants to teleport a quantum state to Bob. Furthermore assume that Alice and Bob are really far apart from each other and can exchange quantum states only for example through an optical fibre. The fibre, which we will occasionally call a quantum channel, is really long and it is inevitable that it contains faults such as impurities which will disturb the state of a photon that we send through the fibre. For teleportation Alice and Bob, need to share a maximally entangled state, e.g. a singlet state. However, whenever Alice prepares a singlet state on her side and then sends one half of it to Bob the impurities in the fibre will disturb the singlet state. Therefore, after the transmission Alice and Bob will **not** share a singlet state but some mixed state that is no longer maximally entangled. If Alice attempts teleportation with this perturbed state, Bob will not receive the quantum state Alice tried to send but some perturbed (and usually mixed) state. Facing this situation, Alice and Bob

become quite desperate, because they have learnt that it is not possible to create quantum entanglement by local operations and classical communication alone. Because Alice and Bob are so far apart from each other, these are the only operations available to them. Therefore Alice and Bob conclude that it will be impossible to 'repair' the state they are sharing in order to obtain a perfect singlet between them. Luckily Alice and Bob have some friends who are physicists (called say Charles, Gilles, Sandu, Benjamin, John and William) and they tell them of their predicament and ask for advice. In fact Charles, Gilles, Sandu, Benjamin, John and William confirm that it is impossible to create entanglement from nothing (i.e. local operations and classical communication starting with a product state). However, they inform Alice and Bob that while it is impossible to create quantum entanglement locally when you have no initial entanglement, you can in some sense amplify or, better, concentrate entanglement from a source of weakly entangled states to obtain some maximally entangled states [7,8,21,20,22] (this was the more general formulation of the fundamental law). The purpose of this section is to explain briefly two particular implementations (there are too many to discuss all of them) of these entanglement purification methods in order to convince Alice, Bob and the reader that these methods really work.

One main difference between the existing purification schemes is their generality, i.e. whether they can purify an arbitrary quantum state or just certain subclasses such as pure states. In fact the first purification schemes [21,7] were not able to purify any arbitrary state. One scheme could purify arbitrary pure states [7] (to be described in the following subsection) while the other could purify certain special classes of mixed state [21]. Here we will present a scheme that can purify arbitrary (pure or mixed) bipartite states, if these states satisfy one general condition. This condition is expressed via the fidelity  $F(\rho)$  of the state  $\rho$ , which is defined as

$$F(\rho) = \max_{\{\text{all max. ent. } |\psi\rangle\}} \langle \psi | \rho | \psi \rangle . \quad (12)$$

In this expression the maximization is taken over all maximally entangled states, i.e. over all states that one can obtain from a singlet state by local unitary operations. The scheme

we are presenting here requires that the fidelity of the quantum state is larger than 0.5 in order for it to be purifiable.

Although one can perform entanglement purification acting on a single pair of particles only [7,21,38], it can be shown that there are states that cannot be purified in this way [27]. Therefore we present a scheme that acts on two pairs simultaneously. This means that Alice and Bob need to create initially two non-maximally entangled pairs of states which they then store. This and the following operations are shown in Fig. 3. Now that Alice and Bob are holding the two pairs, both of them perform two operations. First Alice performs a rotation on the two particles she is holding. This rotation has the effect that

$$|0\rangle \rightarrow \frac{|0\rangle - i|1\rangle}{\sqrt{2}} \quad (13)$$

$$|1\rangle \rightarrow \frac{|1\rangle - i|0\rangle}{\sqrt{2}} . \quad (14)$$

Bob performs the inverse of this operation on his particles. Subsequently both Alice and Bob, perform a controlled NOT (CNOT) gate between the two particles they are holding. The particle of the first pair serves as the control bit, while the particle of the second pair serves as the target [2]. The effect of a CNOT gate is that the second bit gets inverted (NOT) when the first bit is in the state 1 while it remains unaffected when the first bit is in the state 0, i.e.

$$|0\rangle|0\rangle \rightarrow |0\rangle|0\rangle \quad (15)$$

$$|0\rangle|1\rangle \rightarrow |0\rangle|1\rangle \quad (16)$$

$$|1\rangle|0\rangle \rightarrow |1\rangle|1\rangle \quad (17)$$

$$|1\rangle|1\rangle \rightarrow |1\rangle|0\rangle . \quad (18)$$

The last step in the purification procedure consists of a measurement that both Alice and Bob perform on their particle of the second pair. They inform each other about the measurement result and keep the first pair if their results coincide. Otherwise they discard both pairs. In each step they therefore discard at least half of the pairs. From now on we are only

interested in those pairs that are not discarded. In the Bell basis of Eqs. (4)-(7) we define the coefficients

$$A = \langle \Phi^+ | \rho | \Phi^+ \rangle \quad (19)$$

$$B = \langle \Psi^- | \rho | \Psi^- \rangle \quad (20)$$

$$C = \langle \Psi^+ | \rho | \Psi^+ \rangle \quad (21)$$

$$D = \langle \Phi^- | \rho | \Phi^- \rangle \quad . \quad (22)$$

For the state of those pairs that we keep we find that

$$\tilde{A} = \frac{A^2 + B^2}{N} \quad (23)$$

$$\tilde{B} = \frac{2CD}{N} \quad (24)$$

$$\tilde{C} = \frac{C^2 + D^2}{N} \quad (25)$$

$$\tilde{D} = \frac{2AB}{N} \quad . \quad (26)$$

Here  $N = (A + B)^2 + (C + D)^2$  is the probability that Alice and Bob obtain the same results in their respective measurements of the second pair, i.e. the probability that they keep the first pair of particles. One can quite easily check that  $\{A, B, C, D\} = \{1, 0, 0, 0\}$  is a fixed point of the mapping given in Eqs. (23 - 26) and that for  $A > 0.5$  one also has  $\tilde{A} > 0.5$ . The ambitious reader might want to convince himself numerically that indeed the fixed point  $\{A, B, C, D\} = \{1, 0, 0, 0\}$  is an attractor for all  $A > 0.5$ , because the analytical proof of this is quite tricky and not of much interest here. The reader should also note that the map Eqs. (23) - (26) actually has two fixed points, namely  $\{A, B, C, D\} = \{1, 0, 0, 0\}$  and  $\{A, B, C, D\} = \{0, 0, 1, 0\}$ . This means that if we want to know towards which maximally entangled state the procedure will converge, we need to have some more information about the initial state than just the fidelity according to Eq. (12). We will not go into further technical details of this purification procedure and instead we refer the reader to the literature [8,9,18]

Now let us return to the problem that Alice and Bob wanted to solve, i.e. to achieve teleportation over a noisy quantum channel. We summarize in Fig. 6 what Alice and Bob



have to do to achieve their goal. Initially they are given a quantum channel (for example an optical fibre) over which they can transmit quantum states. As this quantum channel is not perfect, Alice and Bob will end up with a partially entangled state after a single use of the fibre. Therefore they repeat the transmission many times which gives them many partially entangled pairs of particles. Now they apply a purification procedure such as the one described in this section which will give them a smaller number of now maximally entangled pairs of particles. With these maximally entangled particles Alice and Bob can now teleport an unknown quantum state, e.g.  $|\psi\rangle$  from Alice to Bob. Therefore Alice and Bob can achieve perfect transmission of an unknown quantum state over a noisy quantum channel.

The main idea of the first two sections of this article are the following. Entanglement cannot be increased if we are allowed to performed only local operations, classical communication and subselection as shown in Fig. 4. Under all these operations the expected entanglement is non-increasing. This implies in particular that, starting from an ensemble in a disentangled state, it is impossible to obtain entangled states by local operations and classical communication. However, it does not rule out the possibility that using only local operations we are able to select from a ensemble described by a partially entangled state a subensemble of systems that have higher average entanglement. This is the essence of entanglement purification procedures, for which the one outlined here is a particular example. Now we review another important purification protocol.

## **B. Purification of pure states**

The above title is not the most fortunate choice of wording, because it might wrongly imply purifying something that is already pure. The reader should remember, however, that the purification means entanglement concentration and pure states need not be maximally entangled. For example a state of the form  $a|00\rangle + b|11\rangle$  is not maximally entangled unless  $|a| = |b| = 1/\sqrt{2}$ . In this subsection we consider the following problem first analysed by

Bennett and coworkers in [7]: Alice and Bob share  $n$  entangled qubit pairs, where each pair is prepared in the state

$$|\Psi_{AB}\rangle = a|00\rangle + b|11\rangle, \quad (27)$$

where we take  $a, b \in \mathbf{R}$ , and  $a^2 + b^2 = 1$ . How many maximally entangled states can they purify? It turns out, that the answer is governed by the von Neumann reduced entropy  $S_{vN}(\rho_A) \equiv \text{tr} \rho_A \ln \rho_A$  and is asymptotically given by  $n \times S_{vN}(\rho_A) = n \times (-a^2 \ln a^2 - b^2 \ln b^2)$ . To see why this is so, consider the total state of  $n$  pairs given by

$$\begin{aligned} |\Psi_{AB}^{\otimes n}\rangle &= (a|00\rangle + b|11\rangle) \otimes (a|00\rangle + b|11\rangle) \otimes \dots \otimes (a|00\rangle + b|11\rangle) \\ &= a^{2n}|0000\dots 00\rangle + a^{2(n-1)}b^2(|0000\dots 11\rangle + \dots + |1100\dots 00\rangle) \\ &\quad + \dots b^{2n}|1111\dots 11\rangle. \end{aligned} \quad (28)$$

(The convention in the second and the third line is that the states at odd positions in the large joint ket states belong to Alice and the even states belong to Bob). Alice can now perform projections (locally, of course) onto the subspaces which have no states  $|1\rangle$ , 2 states  $|1\rangle$ , 4 states  $|1\rangle$ , and so on, and communicates her results to Bob. The probability of having a successful projection onto a particular subspace with  $2k$  states  $|1\rangle$  can easily be seen for the above equation to be

$$p_{2k} = a^{2(n-k)}b^{2k} \binom{n}{k}, \quad (29)$$

which follows directly from Eq. (28). It can be shown that this state can be converted into approximately  $\ln \binom{n}{k}$  singlets [7]. If we assume that the unit of entanglement is given by the entanglement of the singlet state then the total expected entanglement is seen to be

$$E = \sum_{k=0}^n a^{2(n-k)}b^{2k} \binom{n}{k} \ln \binom{n}{k}. \quad (30)$$

We wish to see how this sum behaves asymptotically as  $n \rightarrow \infty$ . It can be seen easily that the term with the highest weight is

$$E \sim (a^2)^{na^2} (b^2)^{nb^2} \binom{n}{b^2 n} \ln \binom{n}{b^2 n}, \quad (31)$$

which can, in turn, be simplified using Stirling's approximation to obtain

$$\begin{aligned}
E &\sim e^{-nS_{vN}(\rho_A)} e^{n \ln n - a^2 n \ln a^2 n - b^2 n \ln b^2 n} (n \ln n - a^2 n \ln a^2 n - b^2 n \ln b^2 n) \\
&= e^{-nS_{vN}(\rho_A)} e^{nS_{vN}(\rho_A)} \times nS_{vN}(\rho_A) = nS_{vN}(\rho_A).
\end{aligned}
\tag{32}$$

This now shows that for pure states the singlet yield of a purification procedure is determined by the von Neumann reduced entropy. It is also important to stress that the above procedure is *reversible*, i.e. starting from  $m$  singlets Alice and Bob can locally produce a given state  $a|0,0\rangle + b|1,1\rangle$  with an asymptotic efficiency of  $m \ln 2 = nS_{vN}(\rho_A)$ . This will be the basis of one of the measures of entanglement introduced by Bennett et al. [7]. Of course, Alice and Bob cannot do better than this limit, since both of them see the initial string of qubits as a classical 0,1 string with the corresponding probabilities  $a^2$  and  $b^2$ . This cannot be compressed to more than its Shannon entropy  $S_{Sh}$  ( $S_{Sh} = -a^2 \ln a^2 - b^2 \ln b^2$  which in this case coincides with the von Neumann entropy) [17]. However, another, less technical reason, and more in the spirit of this article, will be given section 5.

#### IV. ENTANGLEMENT MEASURES

In the first two sections we have seen that it is possible to concentrate entanglement using local operations and classical communication. A natural question that arises in this context is that of the efficiency with which one can perform this concentration. Given  $N$  partially entangled pairs of particles each in the state  $\sigma$ , how many maximally entangled pairs can one obtain? This question is basically one about the amount of entanglement in a given quantum state. The more entanglement we have initially, the more singlet states we will be able to obtain from our supply of non-maximally entangled states. Of course one could also ask a different question, such as for example: How much entanglement do we need to create a given quantum state by local operations and classical communication alone? This question is somehow the inverse of the question of how many singlets we can obtain from a supply of non-maximally entangled states.

All these questions have been worrying physicists in the last two-three years and a complete answer is still unknown. The answer to these questions lies in entanglement measures and in this section we will discuss these entanglement measures a little bit more. First we will explain conditions every 'decent' measure of entanglement should satisfy. After that we will then present some entanglement measures that are known today. Finally we will compare these different entanglement measures. This comparison will tell us something about the way in which the amount of entanglement changes under local quantum operations.

### A. Basic properties of entanglement measures

To determine the basic properties every 'decent' entanglement measure should satisfy we have to recall what we have learnt in the first two sections of this article. The first property we realized is that any state of the form Eq. (11), which we call separable, does not have any quantum correlations and should therefore be called disentangled. This gives rise to our first condition:

- 1) For any separable state  $\sigma$  the measure of entanglement should be zero, i.e.

$$E(\sigma) = 0 . \tag{33}$$

The next condition concerns the behaviour of the entanglement under simple local transformations, i.e. local unitary transformations. A local unitary transformation simply represents a change of the basis in which we consider the given entangled state. But a change of basis should not change the amount of entanglement that is accessible to us, because at any time we could just reverse the basis change. Therefore in both bases the entanglement should be the same.

- 2) For any state  $\sigma$  and any local unitary transformation, i.e. a unitary transformation of the form  $U_A \otimes U_B$ , the entanglement remains unchanged. Therefore

$$E(\sigma) = E(U_A \otimes U_B \sigma U_A^\dagger \otimes U_B^\dagger) . \tag{34}$$

The third condition is the one that really restricts the class of possible entanglement measures. Unfortunately it is usually also the property that is the most difficult to prove for potential measures of entanglement. We have seen in section 1 that Alice and Bob cannot create entanglement from nothing, i.e. using only local operations and classical communication. In section 2 we have seen that given some initial entanglement we are able to select a subensemble of states that have higher entanglement. This can be done using only local operations and classical communication. However, what we cannot do is to increase the total amount of entanglement. We can calculate the total amount of entanglement by summing up the entanglement of all systems after we have applied our local operations, classical communications and subselection. That means that in Fig. 4 we take the probability  $p_i$  that a system will be in particular subensemble  $\mathcal{E}_i$  and multiply it by the average entanglement of that subensemble. This result we then sum up over all possible subensembles. The number we obtain should be smaller than the entanglement of the original ensemble.

**3)** Local operations, classical communication and subselection cannot increase the expected entanglement, i.e. if we start with an ensemble in state  $\sigma$  and end up with probability  $p_i$  in subensembles in state  $\sigma_i$  then we will have

$$E(\sigma) \geq \sum_i p_i E(\sigma_i) . \quad (35)$$

This last condition has an important implication as it tells us something about the efficiency of the most general entanglement purification method. To see this we need to find out what the most efficient purification procedure will look like. Certainly it will select one subensemble, which is described by a maximally entangled state. As we want to make sure that we have as many pairs as possible in this subensemble, we assume that the entanglement in all the other subensembles vanishes. Then the probability that we obtain a maximally entangled state from our optimal quantum state purification procedure is bounded by

$$p_{singlet} \leq \frac{E(\sigma)}{E_{singlet\ state}} . \quad (36)$$

The considerations leading to Eq. 36 show that every entanglement measure that satisfies the three conditions presented in this section can be used to bound the efficiency of entanglement purification procedures from above. Before the reader accepts this statement (s)he should, however, carefully reconsider the above argument. In fact, we have made a hidden assumption in this argument which is not quite trivial. We have assumed that the entanglement measures have the property that the entanglement of two pairs of particles is just the sum of the entanglements of the individual pairs. This sounds like a reasonable assumption but we should note that the entanglement measures that we construct are initially purely mathematical objects and that we need to prove that they behave reasonably. Therefore we demand this additivity property as a fourth condition

4) Given two pairs of entangled particles in the total state  $\sigma = \sigma_1 \otimes \sigma_2$  then we have

$$E(\sigma) = E(\sigma_1) + E(\sigma_2) . \quad (37)$$

Now we have specified reasonable conditions that any 'decent' measure of entanglement should satisfy and in the next section we will briefly explain some possible measures of entanglement.

### **B. Three measures of entanglement**

In this subsection we will present three measures of entanglement. One of them, the entropy of entanglement, will be defined only for pure states. Nevertheless it is of great importance because there are good reasons to accept it as the unique measure entanglement for pure states. Then we will present the entanglement of formation which was the first measure of entanglement for mixed states and whose definition is based on the entropy of entanglement. Finally we introduce the relative entropy of entanglement which was developed from a completely different viewpoint. Finally we will compare the relative entropy of entanglement with the entanglement of formation.

The first measure we are going to discuss here is the entropy of entanglement. It is defined in the following way. Assume that Alice and Bob share an entangled pair of particles in a state  $\sigma$ . Then if Bob considers his particle alone he holds a particle whose state is described by the reduced density operator  $\sigma_B = \text{tr}_A\{\sigma\}$ . The entropy of entanglement is then defined as the von Neumann entropy of the reduced density operator  $\sigma_B$ , i.e.

$$E_{vN} = S_{vN}(\sigma_B) = -\text{tr}\{\sigma_B \ln \sigma_B\} \quad (38)$$

One could think that the definition of the entropy of entanglement depends on whether Alice or Bob calculate the entropy of their reduced density operator. However, it can be shown that for a pure state  $\sigma$  this is not the case, i.e. both will find the same result. It can be shown that this measure of entanglement, when applied to pure states, satisfies all the conditions that we have formulated in the previous section. This certainly makes it a good measure of entanglement. In fact many people believe that it is the only measure of entanglement for pure states. Why is that so? In the previous section we have learnt that an entanglement measure provides an upper bound to the efficiency of any purification procedure. For pure states it has been shown that there is a purification procedure that achieves the limit given by the entropy of entanglement [7]. We reviewed this procedure in the previous section. In addition the inverse property has also been shown. Assume that we want to create  $N$  copies of a quantum state  $\sigma$  of two particles purely by local operations and classical communication. As local operations cannot create entanglement, it will usually be necessary for Alice and Bob to share some singlets before they can create the state  $\sigma$ . How many singlet states do they have to share beforehand? The answer, again, is given by the entropy of entanglement, i.e. to create  $N$  copies of a state  $\sigma$  of two-particles one needs to share  $N E(\sigma)$  singlet states beforehand. Therefore we have a very interesting result. The entanglement of pure states can be concentrated and subsequently be diluted again in a reversible fashion. One should note, however, that this result holds only when we have many (actually infinitely many) copies of entangled pairs at once at our disposal. For finite  $N$  it is not possible to achieve the theoretical limit exactly [26]. This observation suggests a close

relationship between entanglement transformations of pure states and thermodynamics. We will see in the following to what extent this relationship extends to mixed entangled states.

We will now generalize the entropy of entanglement to mixed states. It will turn out that for mixed states there is not one unique measure of entanglement but that there are several different measures of entanglement.

How can we define a measure of entanglement for mixed states? As we now have agreed that the entropy of entanglement is a good measure of entanglement for pure states, it is natural to reduce the definition of mixed state entanglement to that of pure state entanglement. One way of doing that is to consider the amount of entanglement that we have to invest to create a given quantum state  $\sigma$  of a pair of particles. By creating the state we mean that we represent the state  $\sigma$  by a statistical mixture of pure states. It is important in this representation that we do not restrict ourselves to pure states that are orthonormal. If we want to attribute an amount of entanglement to the state  $\sigma$  in this way then this should be the smallest amount of entanglement that is required to produce the state  $\sigma$  by mixing pure states together. If we measure the entanglement of pure states by the entropy of entanglement, then we can define the entanglement of formation by

$$E_F(\sigma) = \min_{\sigma = \sum_i p_i |\psi_i\rangle\langle\psi_i|} \sum_i p_i E_{vN}(|\psi_i\rangle\langle\psi_i|) . \quad (39)$$

The minimization in Eq. (39) is taken over all possible decompositions of the density operator  $\sigma$  into pure states  $|\psi\rangle$ . In general, this minimization is extremely difficult to perform. Luckily for pairs of two-level systems one can solve the minimization analytically and write down a closed expression for the entanglement of formation which can be written entirely in terms of the density operator  $\sigma$  and does not need any reference to the states of the optimal decomposition. In addition the optimal decomposition of  $\sigma$  can be constructed for pairs of two-level systems. To ensure that Eq. (39) really defines a measure of entanglement, one has to show that it satisfies the four conditions we have stated in the previous section. The first three conditions can actually be proven analytically (we do not present the proof here) while the fourth condition (the additivity of the entanglement) has so far only been con-



firmed numerically. Nevertheless the entanglement of formation is a very important measure of entanglement especially because there exists a closed analytical form for it [43].

As the entanglement of formation is a measure of entanglement it represents an upper bound on the efficiency of purification procedures. However, in addition it also gives the amount of entanglement that has to be used to create a given quantum state. This definition of the entanglement of formation alone guarantees already that it will be an upper bound on the efficiency of entanglement purification. This can be seen easily, because if there would be a purification procedure that produces, from  $N$  pairs in state  $\sigma$ , more entanglement than  $N E_F(\sigma)$  then we would be able to use this entanglement to create more than  $N$  pairs in the state  $\sigma$ . Then we could repeat the purification procedure and we would get even more entanglement out. This would imply that we would be able to generate arbitrarily large amounts of entanglement by purely local operations and classical communication. This is impossible and therefore the entanglement of formation is an upper bound on the efficiency of entanglement purification. What is much more difficult to see is whether this upper bound can actually be achieved by any entanglement purification procedure. On the one hand we have seen that for pure states it is possible to achieve the efficiency bound given by the entropy of entanglement. On the other hand for mixed states the situation is much more complicated because we have the additional statistical uncertainty in the mixed state. We would expect that we have to make local measurements in order to remove this statistical uncertainty and these measurements would then destroy some of the entanglement. On the other hand we have seen that in the pure state case we could recover all the entanglement despite the application of measurements. This question was unresolved for some time and it was possible to solve it when yet another measure of entanglement, the relative entropy of entanglement, was discovered.

The relative entropy of entanglement has been introduced in a different way than the two entanglement measures presented above [37,40]. The basic ideas in the relative entropy of entanglement are based on distinguishability and geometrical distance. The idea is to compare a given quantum state  $\sigma$  of a pair of particles with disentangled states. A canonical

disentangled state that one can form from  $\sigma$  is the state  $\sigma_A \otimes \sigma_B$  where  $\sigma_A$  ( $\sigma_B$ ) is the reduced density operator that Alice (Bob) are observing. Now one could try to define the entanglement of  $\sigma$  by any distance between  $\sigma$  and  $\sigma_A \otimes \sigma_B$ . The larger the distance the larger is the entanglement of  $\sigma$ . Unfortunately it is not quite so easy to make an entanglement measure. The problem is that we have picked a particular (although natural) disentangled state. Under a purification procedure this product state  $\sigma_A \otimes \sigma_B$  can be turned into a sum of product states, i.e. a classically correlated state. But what we know for sure is that under any purification procedure a separable state of the form Eq. (11) will be turned into a separable state. Therefore it would be much more natural to compare a given state  $\sigma$  to all separable states and then find that separable state that is closest to  $\sigma$ . This idea is presented in Fig. 5 and can be written in a formal way as

$$E_{RE}(\sigma) = \min_{\rho \in \mathcal{D}} D(\sigma || \rho) . \quad (40)$$

Here the  $\mathcal{D}$  denotes the set of all separable states and  $D$  can be any function that describes a measure of separation between two density operators. Of course, not all distance measures will generate a 'decent' measure of entanglement that satisfies all the conditions that we demand from an entanglement measure. Fortunately, it is possible to find some distances  $D$  that generate 'decent' measures of entanglement and a particularly nice one is the relative entropy which is defined as

$$S(\sigma || \rho) = tr\{\sigma \ln \sigma - \sigma \ln \rho\} . \quad (41)$$

The relative entropy is a slightly peculiar function and is in fact not really a distance in the mathematical sense because it is not even symmetric. Nevertheless it can be proven that Eq. (40) together with the relative entropy of Eq. (41) generates a measure of entanglement that satisfies all the conditions we were asking for in the previous section. It should be said here that the additivity of the relative entropy of entanglement has only been confirmed numerically as for the entanglement of formation. All other properties can be proven analytically and it should also be noted that for pure states the relative entropy of entanglement reduces to the entropy of entanglement which is of course a very satisfying property.

But why does the relative entropy of entanglement answer the question whether the upper bound on the efficiency of entanglement purification procedures that we found from the entanglement of formation can actually be achieved or not? The answer comes from a direct comparison of the two measures of entanglement for a particular kind of state. These, called Werner states, are defined as

$$\rho_F = F|\psi^-\rangle\langle\psi^-| + \frac{1-F}{3}(|\psi^+\rangle\langle\psi^+| + |\phi^-\rangle\langle\phi^-| + |\phi^+\rangle\langle\phi^+|) \quad (42)$$

where we have used the Bell basis defined in Eq. (4-7). The parameter  $F$  is the fidelity of the Werner state and lies in the interval  $[\frac{1}{4}, 1]$ . For Werner states it is possible to calculate both the entanglement of formation and the relative entropy of entanglement analytically. In Fig. 7 the entanglement of the Werner states with fidelity  $F$  is plotted for both entanglement measures. One can clearly see that the relative entropy of entanglement is smaller than the entanglement of formation. But we know that the relative entropy of entanglement, because it is an entanglement measure, is an upper bound on the efficiency of any entanglement purification procedure too. Therefore we reach the following very interesting conclusion. Assume we are given a certain amount of entanglement that we invest in the most optimal way to create by local means some mixed quantum states  $\sigma$  of pairs of two-level systems. How many pairs in the state  $\sigma$  we can produce is determined by the entanglement of formation. Now we try to recover this entanglement by an entanglement purification method whose efficiency is certainly bounded from above by the relative entropy of entanglement. The conclusion is that the amount of entanglement that we can recover is always smaller than the amount of entanglement that we originally invested. Therefore we arrive at an irreversible process, in stark contrast to the pure state case where we were able to recover all the invested entanglement by a purification procedure. This result again sheds some light on the connection between entanglement manipulations and thermodynamics and in the next section we will elaborate on this connection further.

## V. THERMODYNAMICS OF ENTANGLEMENT

Here we would like to elucidate further the fundamental law of quantum information processing by comparing it to the Second Law of Thermodynamics. The reader should not be surprised that there are connections between the two. First of all, both laws can be expressed mathematically by using an entropic quantity. The second law says that thermodynamical entropy cannot decrease in an isolated system. The fundamental law of quantum information processing, on the other hand, states that entanglement cannot be increased by local operations. Thus both of the laws serve to prohibit certain types of processes which are impossible in nature (this analogy was first emphasised by Popescu and Rohrlich in [32], but also see [23,40]). The rest of the section shows the two principles in action by solving two simple, but important problems.

### A. Reversible and irreversible processes

We begin by stating more formally a form of the Second Law of thermodynamics. This form is due to Clausius, but it is completely analogous to the no increase of entropy statement we gave above. In particular it will be more useful for what we are about to investigate.

**The Second Law of thermodynamics (Clausius).** There exists no thermodynamic process the *sole* effect of which is to extract a quantity of heat from the colder of two reservoirs and deliver it to the hotter of the two reservoirs.

Suppose now that we have a thermodynamical system. We want to invest some heat into it so that at the end our system does as much work as possible with this heat input. The efficiency is therefore defined as

$$\eta = \frac{W_{\text{out}}}{Q_{\text{in}}} . \quad (43)$$

Now it is a well known fact that the above efficiency is maximized if we have a reversible process (simply because an irreversible process wastes useful work on friction or some other lossy mechanism). In fact, we know the efficiency of one such process, called the Carnot cycle. With the Second Law on our mind, we can now prove that no other process can perform better than the Carnot cycle. This boils down to the fact that we only need to prove that no other reversible process performs better than the Carnot cycle. The argument for this can be found in any undergraduate book on Thermodynamics and briefly runs as follows (again *reductio ad absurdum*). The Carnot engine takes some heat input from a hotter reservoir, does some work and delivers an amount of heat to the colder reservoir. Suppose that there is a better engine,  $E$ , that is operating between the same two reservoirs (we have to be fair when comparing the efficiency). Suppose also that we run this better machine backwards (as a refrigerator): we would do some work on it, and it would take a quantity of heat from the cold reservoir and bring some heat to the hot reservoir. For simplicity we assume that the work done by a Carnot engine is the same as the work that  $E$  needs to run in reverse (this can always be arranged and we lose nothing in generality). Then we look at the two machines together, which is just another thermodynamical process: they extract a quantity of heat from the colder reservoir and deliver it to the hot reservoir with all other things being equal. But this contradicts the Second Law, and therefore no machine is more efficient than the Carnot engine.

In the previous section we have learnt about the purification scheme of Bennett et al [7] for pure states. Efficiency of any scheme was defined as the number of maximally entangled states we can obtain from a given  $N$  pairs in some initial state, divided by  $N$ . This scheme is in addition reversible, and we would suppose, guided by the above thermodynamic argument, that no other reversible purification scheme could do better than the Bennett et al. Suppose that there is a more efficient (reversible) process. Now Alice and Bob start from a certain number  $N$  of maximally entangled pairs. They apply a reverse of the scheme of Bennett et al [7] to get a certain number of less entangled states. But then they can run the more efficient purification to get  $M$  maximally entangled states out. However, since the second purification

is more efficient than the first one, then we have that  $M > N$ . So, locally Alice and Bob can increase entanglement, which contradicts the fundamental law of quantum information processing. We have to stress that as far as the mixed states are concerned there are no results regarding the best purification scheme, and it is not completely understood whether the same strategy as above could be applied (for more discussion see [40]).

In any case, the above reasoning shows that the conceptual ideas behind the Second law and the fundamental law are similar in nature. Next we show another attractive application of the fundamental law. We return to the question at the beginning of the article that started the whole discussion: can Alice teleport to Bob as many qubits as she likes using only one entangled pair shared between them?

## **B. What can we learn from the non-increase of entanglement under local operations?**

If the scheme that we are proposing could be utilized then it would be of great technological advantage, because to create and maintain entangled qubits is at present very hard. If a single maximally entangled pair could transfer a large amount of information (i.e. teleport a number of qubits), then this would be very useful. However, there is no free lunch. In the same way that we cannot have an unlimited amount of useful work and no heat dissipation, we cannot have arbitrarily many teleportations with a single maximally entangled pair. In fact, we can prove a much stronger statement: in order to teleport  $N$  qubits Alice and Bob need to share  $N$  maximally entangled pairs!

In order to prove this we need to understand another simple concept from quantum mechanics. Namely, if we can teleport a pure unknown quantum state then we can teleport an unknown mixed quantum state (this is obvious since a mixed state is just a combination of pure states). But now comes a crucial result: every mixed state of a single qubit can be thought of as a part of a pure state of two *entangled* qubits (this result is more general, and applies to any quantum state of any quantum system, but we do not need the generalization here). Suppose that we have a single qubit in a state

$$\rho = a^2|0\rangle\langle 0| + b^2|1\rangle\langle 1|. \quad (44)$$

This single qubit can then be viewed as a part of a pair of qubits in state

$$|\psi\rangle = a|00\rangle + b|11\rangle. \quad (45)$$

One obtains Eq. (44) from Eq. (45) simply by taking the partial trace over the second particle. Bearing this in mind we now envisage the following teleportation protocol. Alice and Bob share a maximally entangled pair, and in addition Bob has a qubit prepared in some state, say  $|0\rangle$ . Alice then receives a qubit to teleport in a general (to her unknown) state  $\rho$ . After the teleportation we want Bob's extra qubit to be in the state  $\rho$  and the maximally entangled pair to stay intact (or at least not to be completely destroyed). This is shown in Fig. 2.

Now we wish to prove this protocol impossible—entanglement simply has to be completely destroyed at the end. Suppose it is not, i.e. suppose that the above teleportation is possible. Then Alice can teleport any unknown (mixed) state to Bob using this protocol. But this mixed state can arise from an entangled state where the second qubit (the one to be traced out) is on Alice's side. So initially Alice and Bob share one entangled pair, but after the teleportation they have increased their entanglement as in Fig. 8. Since the initial state can be maximally mixed state ( $a = b = 1/\sqrt{2}$ ) the final entanglement can grow to be twice the maximally entangled state. But, as this would violate the fundamental law of quantum information processing it is impossible and the initial maximally entangled pair has to be destroyed. In fact, this argument shows that it has to be destroyed completely. Thus we see that a simple application of the fundamental law can be used to rule out a whole class of impossible teleportation protocols. Otherwise every teleportation protocol would have to be checked separately and this would be a very hard problem.

## VI. CONCLUSIONS

Let us briefly recapitulate what we have learnt. Quantum teleportation is a procedure whereby an unknown state of a quantum system is transferred from a particle at a place A to a particle at a place B. The whole protocol uses only local operations and classical communication between A and B. In addition, A and B have to share a maximally entangled state. Entanglement is central for the whole teleportation: if that state is not maximally entangled then teleportation is less efficient and if the state is disentangled (and only classically correlated) then teleportation is impossible. We have then derived a fundamental law of quantum information processing which stipulated that entanglement cannot be increased by local operations and classical communication only. This law was then investigated in the light of purification procedures: local protocols for increasing entanglement of a subensemble of particles. We discussed bounds on the efficiency of such protocols and emphasised the links between this kind of physics and the theory of thermodynamics. This led us to formulate various measures of entanglement for general mixed states of two quantum bits. At the end we returned to the problem of teleportation, asking how many entangled pairs we need in order to teleport  $N$  qubits. Using the fundamental law of quantum information processing we offered an elegant argument for needing  $N$  maximally entangled pairs for teleporting  $N$  qubits, a pair per qubit.

The analogy between thermodynamics and quantum information theory might be deeper, but this at present remains unknown. Quantum information theory is still at a very early stage of development and, although there are already some extraordinary results, a number of areas is still untouched. In particular the status of what we called the fundamental law is unclear. First and foremost, it is not known how it relates to other results in the field, such as, for example, the no cloning theorem [42] which states that an unknown quantum state cannot be duplicated by a physical process. We hope that research in this area will prove fruitful in establishing a deeper symbiotic relationship between information theory, quantum physics and thermodynamics. Quantum theory has had a huge input into information theory



and thermodynamics over the past few decades. Perhaps by turning this around we can learn much more about quantum theory by using information-theoretic and thermodynamic concepts. Ultimately, this approach might solve some long standing and difficult problems in modern physics, such as the measurement problem and the arrow of time problem. This is exactly what was envisaged more than 60 years ago in a statement attributed to Einstein: 'The solution of the problems of quantum mechanics will be thermodynamical in nature' [45].

## VII. ACKNOWLEDGEMENTS

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## Biographies

Martin Plenio studied in Göttingen (Germany) where he obtained both his Diploma (1992) and his PhD (1994) in Theoretical Physics. His main research area at that time was Quantum Optics and in particular the properties of single quantum systems such as single trapped ions irradiated by laser light. After his PhD he joined the Theoretical Quantum Optics group at Imperial College as a postdoc. It was here that he started to become interested in quantum computing, quantum communication and quantum information theory. Since January 1998 he is now a lecturer in the Optics Section of Imperial College.

Vlatko Vedral obtained both his first degree (1995) and PhD (1998) in Theoretical Physics from Imperial College. He is now an Elsag-Bailey Postdoctoral Research Fellow at the Center for Quantum Computing in Oxford. From October 1998 he will take up a Junior Research Fellowship at Merton College in Oxford. His main research interests are in connections between information theory and quantum mechanics, including quantum computing, error correction and quantum theory of communication.

FIGURES

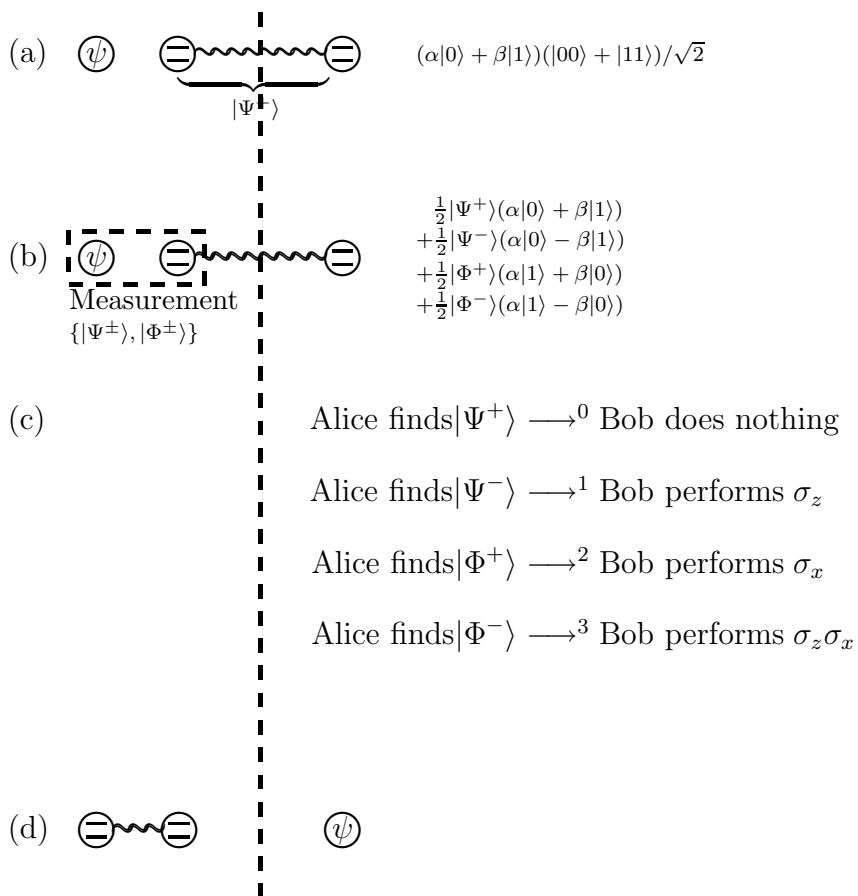


FIG. 1. The basic steps of quantum state teleportation. Alice and Bob are spatially separated, Alice on the left of the dashed line, Bob on the right. (a) Alice and Bob share a maximally entangled pair of particles in the state  $(|00\rangle + |11\rangle)/\sqrt{2}$ . Alice wants to teleport the unknown state  $|\psi\rangle$  to Bob. (b) The total state of the three particles that Alice and Bob are holding is rewritten in the Bell basis Eqs. (4-7) for the two particles Alice is holding. Alice performs a measurement that projects the state of her two particles onto one of the four Bell states. (c) She transmits the result encoded in the numbers 0, 1, 2, 3 to Bob, who performs a unitary transformation  $\mathbf{1}, \sigma_z, \sigma_x, \sigma_z\sigma_x$  that depends only on the measurement result that Alice obtained but **not** on the state  $|\psi\rangle$ ! (d) After Bob has applied the appropriate unitary operation on his particle he can be sure that he is now holding the state that Alice was holding in (a).

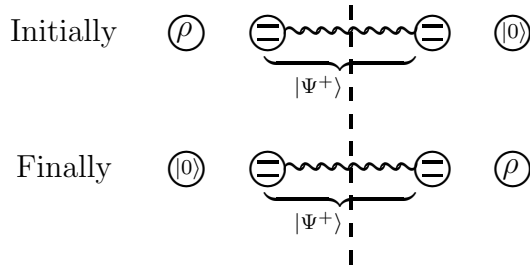


FIG. 2. Again Alice is on the left of the dashed line and Bob on the right side. Assume that initially Alice and Bob are sharing two particles in a maximally entangled state  $|\psi\rangle$ . Alice also holds a particle in an unknown state  $\rho$  while Bob holds a particle in the known state  $|0\rangle$ . The aim is that finally Alice and Bob have exchanged the states of their particles **and** that they are still sharing a pair of particles in the maximally entangled state  $|\psi\rangle$ . The question whether this protocol is possible will be answered in Section V.

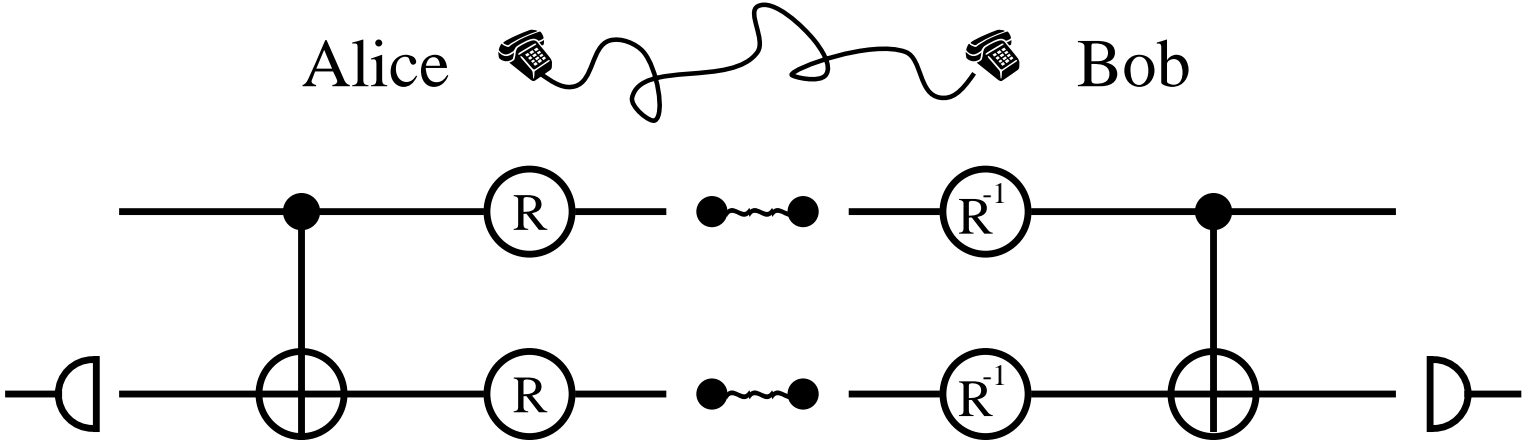
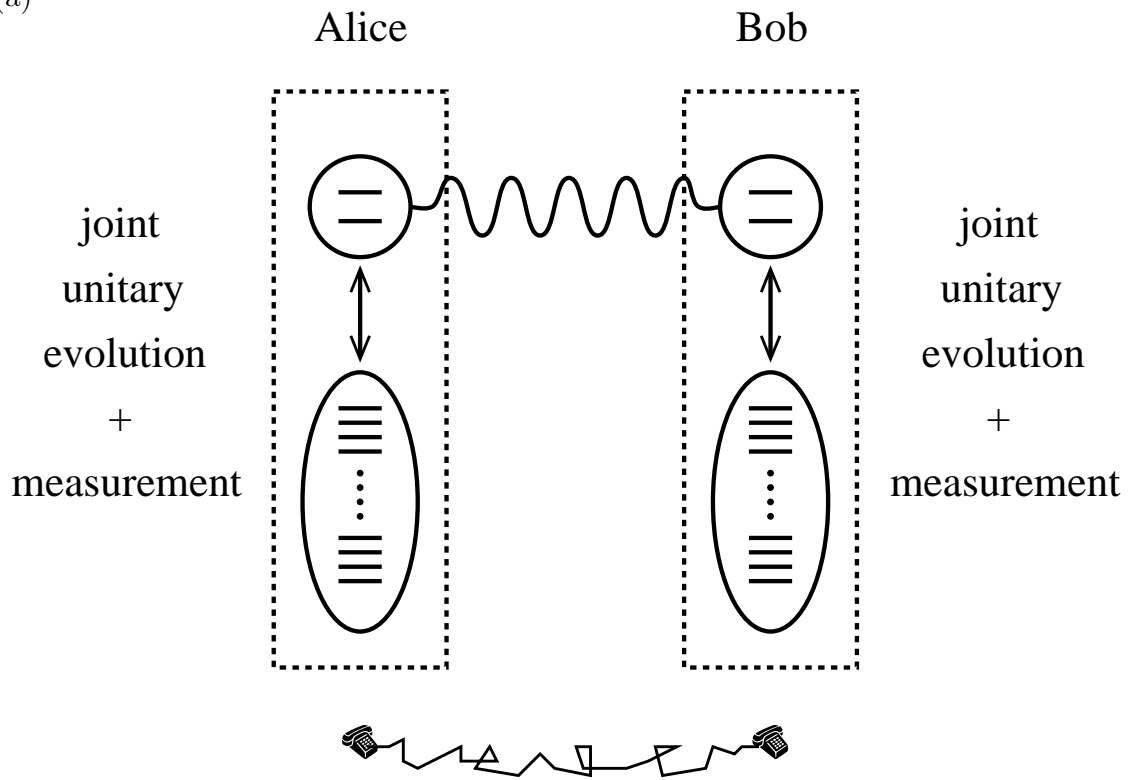


FIG. 3. The quantum network that implements quantum privacy amplification. Alice and Bob share two pairs of entangled particles. First Alice performs a one bit rotation  $\mathcal{R}$  (given by the  $R$  in a circle) which takes  $|0\rangle \rightarrow (|0\rangle - i|1\rangle)/\sqrt{2}$  and  $|1\rangle \rightarrow (|1\rangle - i|0\rangle)/\sqrt{2}$  on her particles, while Bob performs the inverse rotation on his side. Then both parties perform a CNOT gate on their particles where the first pair provides the control bits (signified by the full circle) while the second pair provides the target bits (signified by the encircled cross). Finally Alice and Bob measure the second pair in the  $\{0,1\}$  basis. They communicate their results to each other by classical communication (telephones). If their results coincide they keep the first pair, otherwise they discard it.



(a)



(b)

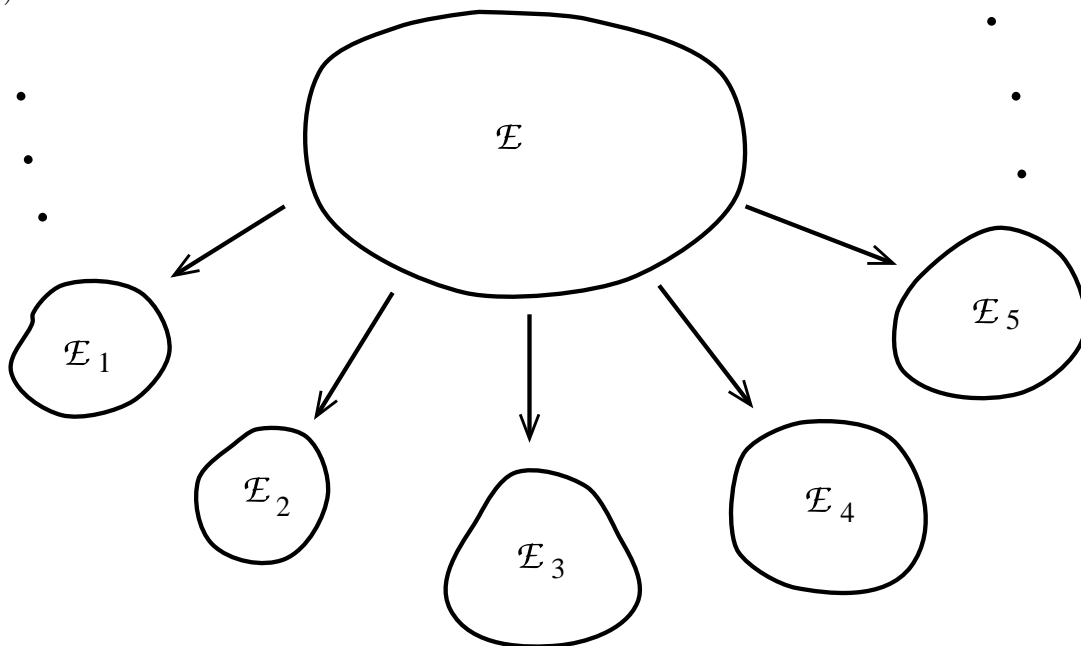


FIG. 4. In quantum state purification procedures three different kinds of operations are allowed. In part (a) of this figure the first two are depicted. Alice and Bob are allowed to perform any local operation they like. The most general form is one where Alice adds additional multi-level systems to her particle and then performs a unitary transformation on the joint system followed by a measurement of the additional multi-level system. She can communicate classically with Bob about the outcome of her measurement (indicated by the telephones). The third allowed operation is given in part (b) of the figure. Using classical communication Alice and Bob can select, based on their measurement outcomes, subensembles  $\mathcal{E}_1, \dots, \mathcal{E}_n$  from the original ensemble  $\mathcal{E}$ . The aim is to obtain at least one subensemble that is in a state having more entanglement than the original ensemble.

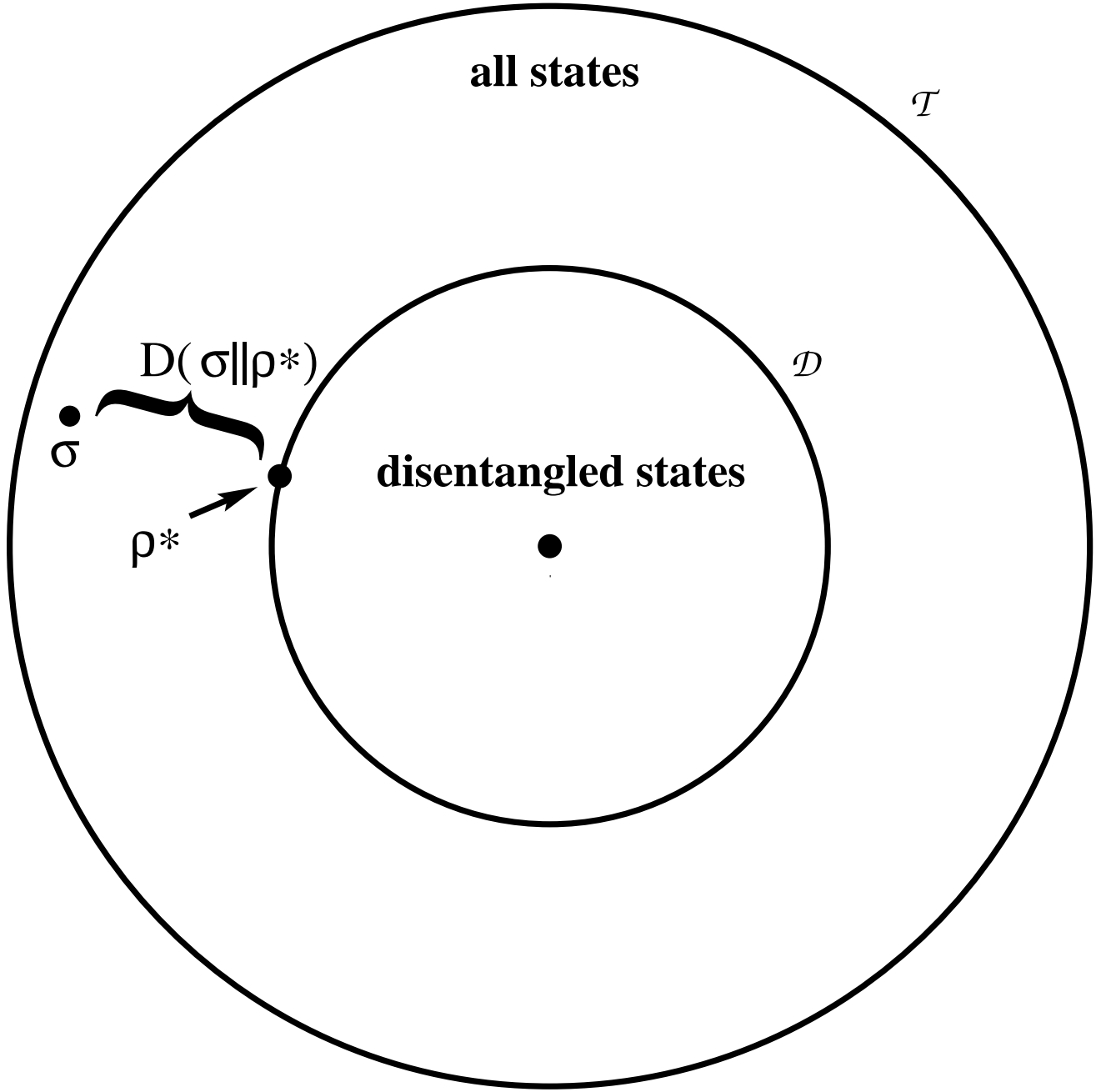


FIG. 5. A geometric way to quantify entanglement. The set of all density matrices  $\mathcal{T}$  is represented by the outer circle. Its subset of disentangled (separable) states  $\mathcal{D}$ , is represented by the inner circle. A state  $\sigma$  belongs to the entangled states, and  $\rho^*$  is the disentangled state that minimizes the distance  $D(\sigma || \rho)$ . This minimal distance can be defined as the amount of entanglement in  $\sigma$ .

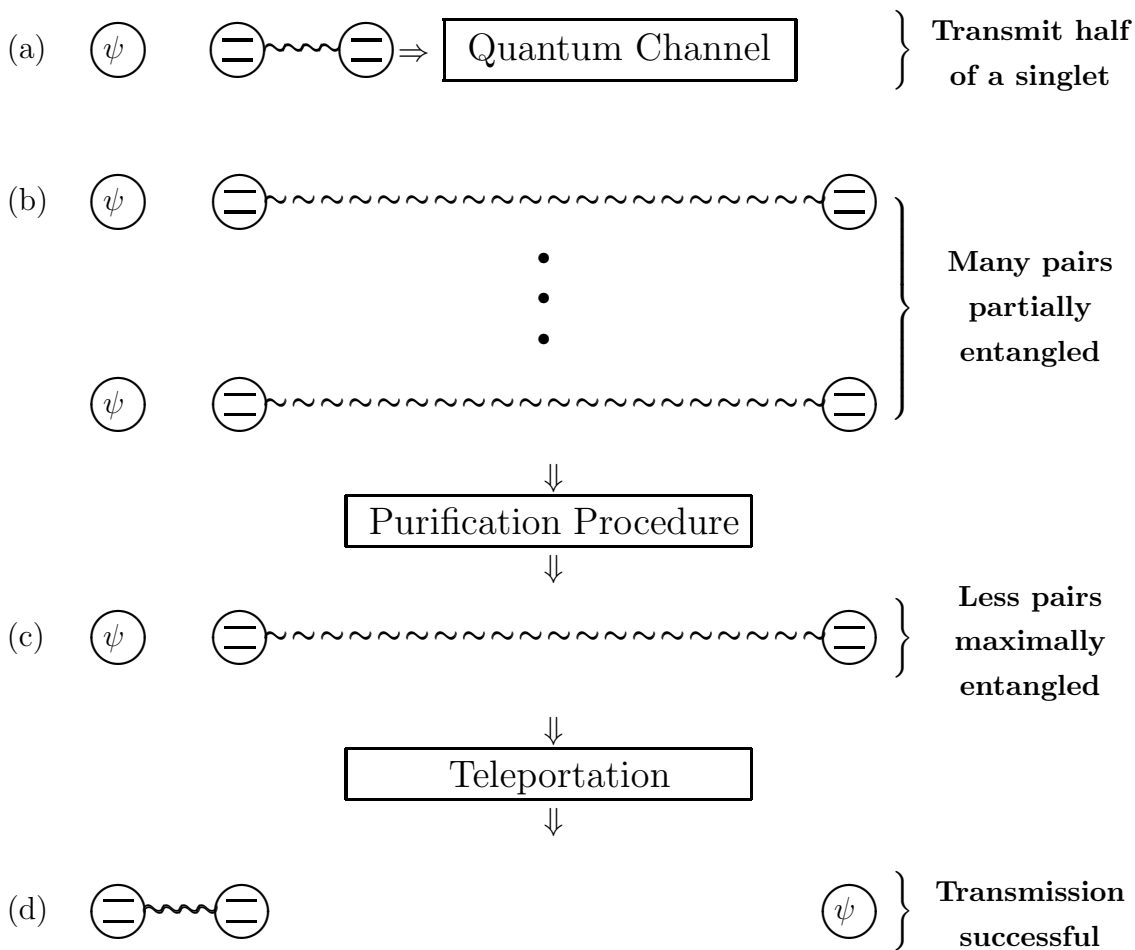


FIG. 6. Summary of the teleportation protocol between Alice and Bob in the presence of decoherence. (a) Alice (on the left side) holds an unknown quantum state  $|\psi\rangle$  which she wants to transmit to Bob. Alice creates singlet states and sends one half down a noisy channel. (b) She repeats this procedure until Alice and Bob share many partially entangled states. (c) Then Alice and Bob apply a local entanglement purification procedure to distill a subensemble of pure singlet states. (d) This maximally entangled state can then be used to teleport the unknown state  $|\psi\rangle$  to Bob.

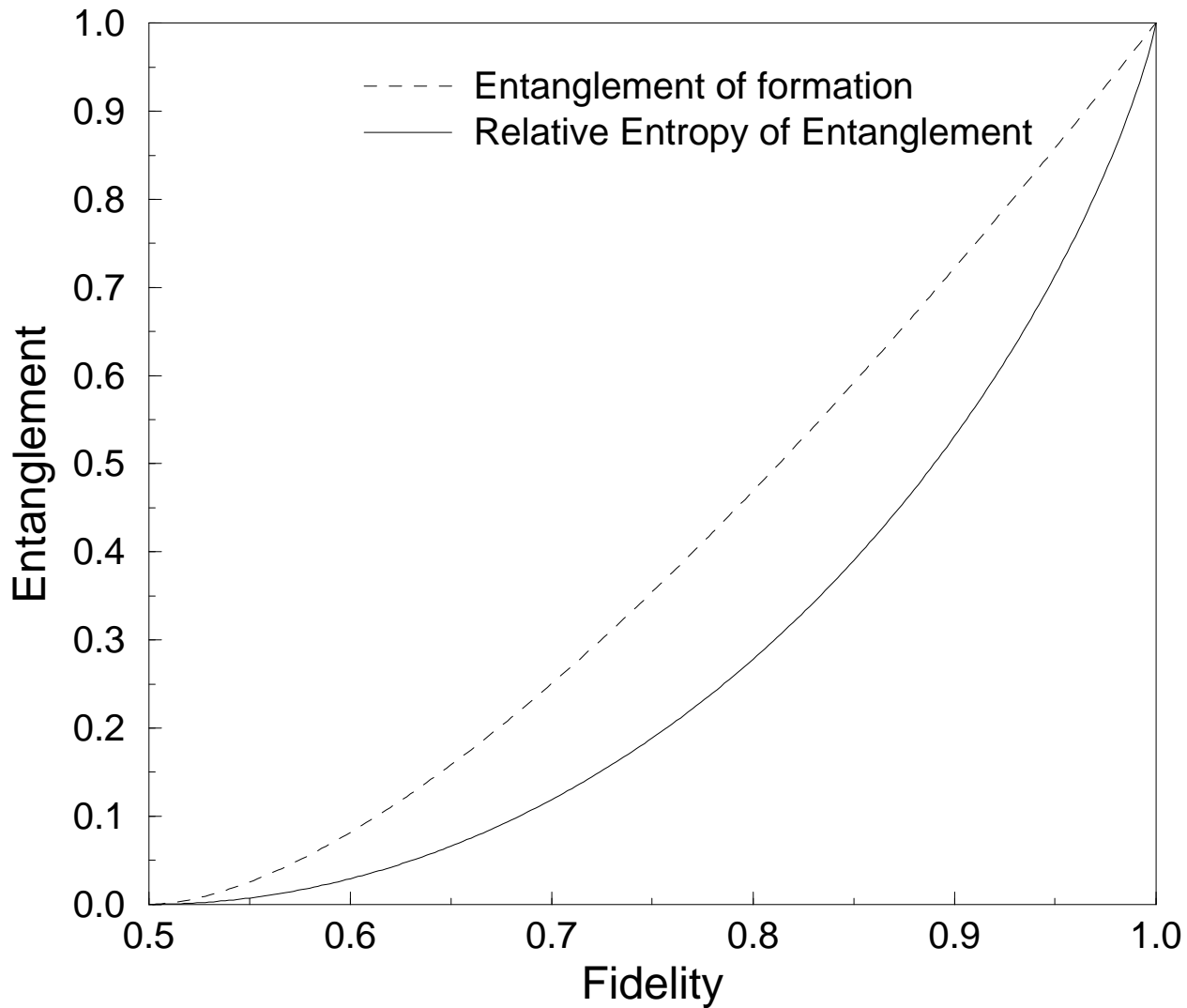


FIG. 7. Comparison of the entanglement of formation with the relative entropy of entanglement for Werner states with fidelity  $F$ . The relative entropy of entanglement is always smaller than the entanglement of formation. This proves that in general entanglement is destroyed by local operations.

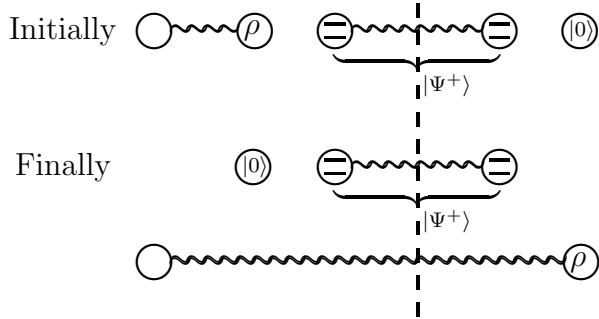


FIG. 8. A diagrammatical proof that the teleportation protocol in Fig. (2) is impossible. Alice is on the left of the dashed line, Bob on the right. Initially Alice is holding a mixed state  $\rho$  and Bob a particle in state  $|0\rangle$ . In addition Alice and Bob share a pair of maximally entangled particles in state  $|\psi^+\rangle$ . The particle in the mixed state  $\rho$  that Alice is holding can be part of a pair of entangled particles. The aim is that finally, after the teleportation Bob holds the state  $\rho$  and Alice and Bob still have their two particles in a maximally entangled state  $|\psi^+\rangle$ . However, not only the state  $\rho$  will be transferred to Bob but also its entanglement with other particles. Therefore after the envisaged teleportation Alice and Bob would be sharing more entanglement than initially. This contradicts the fundamental law of quantum information processing that entanglement cannot be increased.