PHYSICAL REVIEW LETTERS

VOLUME 78 | 24 MARCH 1997 | NUMBER 12

Quantifying Entanglement

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(Received 28 October 1996)

We present conditions every measure of entanglement has to satisfy, and construct a whole class of "good" entanglement measures. The generalization of our class of entanglement measures to more than two particles is straightforward. We present a measure which has a statistical operational basis that might enable experimental determination of the quantitative degree of entanglement. [S0031-9007(97)02751-8]

PACS numbers: 89.70.+c, 89.80.+h, 03.65.Bz

We have witnessed great advances in quantum information theory in recent years. There are two distinct directions in which progress is currently being made: quantum computation and error correction on the one hand (for a short survey see [1,2]), and nonlocality, Bell's inequalities, and purification, on the other hand [3,4]. There has also been a number of papers relating the two methods (e.g., [5,6]). Our present work belongs to this second group. Recently it was realized that the CHSH (Clauser-Horne-Shimony-Holt) form of Bell's inequalities are not a sufficiently good measure of quantum correlations in the sense that there are states which do not violate the CHSH inequality, but, on the other hand, can be purified by local interactions and classical communications to yield a state that does violate the CHSH inequality [3]. Subsequently, it was shown that the only states of two two-level systems which cannot be purified are those that can be written as the sum over density operators which are direct product states of the two subsystems [7]. Therefore, although it is possible to say whether a quantum state is entangled or not, the amount of entanglement cannot easily be determined for general mixed states. Bennett *et al.* [5] have recently proposed a measure of entanglement for a general mixed state of two quantum subsystems. However, this measure has the disadvantage that it is hard to compute for a general state, even numerically. In this Letter we specify conditions which any measure of entanglement has to satisfy and construct a whole class of "good" entanglement measures. Our measures are geometrically intuitive.

Unless stated otherwise, the following considerations apply to a system composed of two quantum subsystem of arbitrary dimensions. First, we define the term *purification procedure* more precisely. There are three distinct ingredients in any protocol that aims at increasing correlations between two quantum subsystems locally.

Local general measurements (LGM).—These are performed by the two parties (A and B) separately and are described by two sets of operators satisfying the completeness relations $\sum_i A_i^\dagger A_i = \mathbf{I}$ and $\sum_j B_j^\dagger B_j = \mathbf{I}$. The joint action of the two is described by $\sum_{ij} A_i \otimes B_j$, which again describes a local general measurement.

Classical communication (CC).—This means that the actions of A and B can be classically correlated. This can be described by a complete measurement on the whole space $A + B$ which, as opposed to local general measurements, is not necessarily decomposable into a direct product of two operators as above, each acting on only one subsystem. If ρ_{AB} is the joint state of subsystems A and B then the transformation involving "LGM + CC" would look like

$$\rho_{AB} \rightarrow \sum_i A_i \otimes B_i \rho_{AB} A_i^\dagger \otimes B_i^\dagger , \tag{1}$$

i.e., the actions of A and B are "correlated." The mapping given in Eq. (1) is completely positive. To ensure that it is also trace preserving we have to require $\sum_i A_i^\dagger A_i \otimes B_i^\dagger B_i = \mathbf{I}$. Both LGM and CC are linear transformations on the set of states. Note that as the third ingredient all purification schemes use LGM and CC but also reject part of the original ensemble, making the whole transformation nonlinear [4].

We note that all entangled (inseparable) states can be purified to an ensemble of maximally entangled states [7]. This implies that any good measure of entanglement has to be zero if and only if the state is disentangled (defined by a convex sum of the form $\sum_i p_i \rho_A^i \otimes \rho_B^i$). Here we would like to quantify the degree of entanglement. In the following we briefly review some measures of entanglement between two quantum systems (for a review of correlation measures see [8]).

Entanglement of creation.—Bennett *et al.* [5] define the entanglement of creation of a state ρ by

$$E(\rho) := \min \sum_i p_i S(\rho_A^i), \tag{2}$$

where $S(\rho_A)$ is the von Neumann entropy [to be defined in Eq. (3)] and the minimum is taken over all the possible realizations of the state, $\rho_{AB} = \sum_j p_j |\psi_j\rangle\langle\psi_j|$ with $\rho_A^i = \mathrm{tr}_B(|\psi_i\rangle\langle\psi_i|)$. The entanglement of creation cannot be increased by the combined action of LGM + CC [5].

Entanglement of distillation [5].—This is the number of maximally entangled pairs that can be purified from a given state. This measure depends on the particular process of purification, and it is not yet clear how to compute it in an efficient and unique way.

It seems to be difficult to calculate the degree of entanglement for a general state using these two definitions, and a closed form would be very much desired for further progress [9]. The problem is quite involved as one has to minimize over all possible decompositions of the density operator in question or over all possible purification schemes. There are other measures of entanglement which are simpler to calculate but which cannot distinguish between quantum and classical correlations. We discuss two and show how they can be generalized to give good measures of entanglement; in fact, we show how to derive a whole class of measures of entanglement.

Von Neumann entropy.—Given a pure state ρ_{AB} of two subsystems A and B we define the states $\rho_A = \mathrm{tr}_B\{\rho_{AB}\}$ and $\rho_B = \mathrm{tr}_A\{\rho_{AB}\}$, where the partial trace has been taken over one subsystem, either A or B. Then the von Neumann entropy of the reduced density operators is given by

$$S(\rho_A) := -\,\mathrm{tr}(\rho_A \ln \rho_A) = -\mathrm{tr}(\rho_B \ln \rho_B). \tag{3}$$

In the case of a disentangled pure joint state $S(\rho_A)$ is zero, and for maximally entangled states it gives $\ln 2$. However, for mixed states ρ_{AB} this measure fails to distinguish classical and quantum mechanical correlations.

Von Neumann mutual information.—This is defined by

$$I_N(\rho_A{:}\rho_B;\rho_{AB}) := S(\rho_A) + S(\rho_B) - S(\rho_{AB}), \tag{4}$$

which essentially reduces to Eq. (3) for pure states of the joint system ρ_{AB}. It is known that I_N *cannot* increase under local general measurement only [6,10], but *can* increase under LGM + CC, showing that it cannot properly distinguish between classical and quantum mechanical correlations. The von Neumann mutual information can intuitively be understood as follows: The mutual information calculates a "distance" between a given state ρ_{AB} and *one* of its disentangled counterparts $\rho_A \otimes \rho_B$. The crucial word here is "one," as there are many other disentangled states for which we could calculate I_N, which indicates the failure of this measure for general mixed states but also suggests its successful generalization.

Before we generalize the von Neumann mutual information, we present the following necessary conditions any measure of entanglement $E(\sigma)$ has to satisfy.

(i) $E(\sigma) = 0$ iff σ is separable.

(ii) Local unitary operations leave $E(\sigma)$ invariant, i.e., $E(\sigma) = E(U_A \otimes U_B \sigma U_A^\dagger \otimes U_B^\dagger)$.

(iii) The measure of entanglement $E(\sigma)$ cannot increase under LGM + CC given by Θ, i.e., $E(\Theta\sigma) \le E(\sigma)$.

The origin of condition (i) is that separable states are known to contain no entanglement, i.e., they *cannot* be purified by LGM + CC to maximally entangled states; however, any inseparable state can be purified and therefore contains some entanglement. The reason for condition (ii) is that local unitary transformations represent a local change of basis only and leave quantum correlations unchanged. The reason for condition (iii) is that any increase in correlations achieved by LGM + CC should be classical in nature, and therefore entanglement should not be increased.

In the following we construct a new class of measures that satisfy the conditions (i)–(iii). Let us consider a set $\mathcal{T}$ of all density matrices of two quantum subsystems, A and B (see Fig. 1). Let us further divide $\mathcal{T}$ into two disjunctive subsets: a set containing all disentangled states—hereafter labeled by $\mathcal{D}$—and a set of all the entangled states (all states in $\mathcal{T} - \mathcal{D}$)—hereafter labeled by $\mathcal{E}$. Note that both $\mathcal{T}$ and $\mathcal{D}$ (*but not* $\mathcal{E}$) are convex sets, i.e., $\rho_1, \rho_2 \in \mathcal{T}(\mathcal{D}) \Rightarrow \lambda\rho_1 + (1-\lambda)\rho_2 \in \mathcal{T}(\mathcal{D})$. The entanglement of a matrix $\sigma \in \mathcal{T}$ will now be defined as

$$E(\sigma) := \min_{\rho \in \mathcal{D}} D(\sigma \,\|\, \rho), \tag{5}$$

where D is any measure of *distance* between the two density matrices ρ and σ such that $E(\sigma)$ satisfies the above three conditions. To satisfy condition (i) it is sufficient to demand that $D(\sigma \,\|\, \rho) = 0$ iff $\sigma = \rho$. Because of the invariance of $\mathcal{D}$ under local unitary transformations condition (ii) is automatically satisfied. For condition (iii) to be satisfied it is sufficient to demand that $D(\sigma \,\|\, \rho)$ has the property that it is nonincreasing under every completely positive trace preserving map Θ, i.e., $D(\Theta\sigma \,\|\, \Theta\rho) \le D(\sigma \,\|\, \rho)$. This can easily be

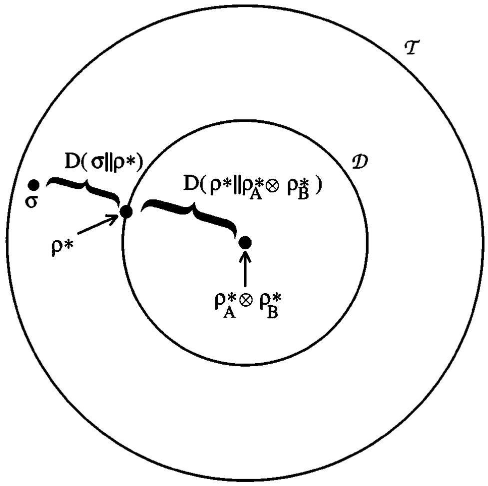

FIG. 1. The set of all density matrices, $\mathcal{T}$ is represented by the outer circle. Its subset, a set of disentangled states $\mathcal{D}$ is represented by the inner circle. A state σ belongs to the entangled states, and ρ^* is the disentangled state that minimizes the distance $D(\sigma \parallel \rho)$, thus representing the amount of quantum correlations in σ. State $\rho_A^* \otimes \rho_B^*$ is obtained by tracing ρ^* over A and B. $D(\rho^* \parallel \rho_A^* \otimes \rho_B^*)$ represent the classical part of correlations in the state σ.

seen from the following. If ρ^* is a separable density operator that realizes the minimum of Eq. (5), then, because $\Theta\mathcal{D} \subset \mathcal{D}$, we find

$$\begin{aligned} E(\sigma) &:= D(\sigma \parallel \rho^*) \geq D(\Theta\sigma \parallel \Theta\rho^*) \\ &\geq \min_{\rho\in\mathcal{D}} D(\Theta\sigma \parallel \rho) = E(\Theta\sigma)\,. \end{aligned}$$

The amount of entanglement given by Eq. (5) can be interpreted as finding a state ρ^* in $\mathcal{D}$ that is closest to σ under the measure D. Such a closest state ρ^* approximates the classical correlations of the state σ "as close as possible." Therefore $E(\sigma)$ measures the remaining quantum mechanical correlations. This suggests a division of correlations of the state σ into two distinct contributions: *quantum correlations*, $E(\sigma)$, and *classical correlations*, $D(\rho^* \parallel \rho_A^* \otimes \rho_B^*)$, where ρ^* is the disentangled state that minimizes D and ρ_A^* and ρ_B^* are its reduced parts (see Fig. 1 for a pictorial representation).

In the following we make special choices for $D(\sigma \parallel \rho)$. We use an entropic measure of distance between the two density matrices, σ and ρ, also called the von Neumann relative entropy, which is defined by analogy with the classical Kullback-Leibler distance as [6,10–12]

$$S(\sigma \parallel \rho) := \mathrm{tr}\left\{\sigma \ln \frac{\sigma}{\rho}\right\}, \tag{6}$$

where $\ln\frac{\sigma}{\rho} = \ln\sigma - \ln\rho$. Note that this quantity, although frequently referred to as a distance, does not actually satisfy the usual metric properties, e.g., $S(\sigma \parallel \rho) \neq S(\rho \parallel \sigma)$. We now define the entanglement of a state σ to be

$$E(\sigma) = \min_{\rho\in\mathcal{D}} S(\sigma \parallel \rho)\,. \tag{7}$$

Note that this is a direct generalization of the von Neumann mutual information which is obtained for $\rho = \sigma_A \otimes \sigma_B$. It is now quite easy to check that this measure in fact satisfies conditions (i)–(iii), because it is known that for the relative entropy $S(\sigma \parallel \rho) = 0$ iff $\sigma = \rho$, and that for any completely positive trace preserving map Θ we have $S(\Theta\sigma \parallel \Theta\rho) \leq S(\sigma \parallel \rho)$ [10,13].

To illustrate some properties of this measure we now restrict ourselves to two spin-$1/2$ subsystems only. First we calculate $E(\sigma)$ for a pure maximally entangled state.

Proposition 1.—Entropic entanglement reduces to the von Neumann entropy (of $\ln 2$) for pure, maximally entangled states defined by $|\Phi^\pm\rangle = (|00\rangle \pm |11\rangle)/\sqrt{2}$ and $|\Psi^\pm\rangle = (|10\rangle \pm |01\rangle)/\sqrt{2}$.

Proof.—We prove proposition 1 for the Bell state $\sigma \equiv |\Phi^+\rangle\langle\Phi^+|$. All other maximally entangled states can be generated from this one by local unitary transformations which do not change $E(\sigma)$. As σ is a pure state we have

$$E(\sigma) = \min_{\rho\in\mathcal{D}} \mathrm{tr}\left\{\sigma \ln \frac{\sigma}{\rho}\right\} = \min_{\rho\in\mathcal{D}} -\mathrm{tr}\{\sigma \ln \rho\}\,. \tag{8}$$

Now we use the fact that the function $f(x) = -\ln x$ is convex, which results in

$$f(\langle\phi|A|\phi\rangle) \leq \langle\phi|f(A)|\phi\rangle \tag{9}$$

for any operator A and any normalized state $|\phi\rangle$. This leads to

$$E(\sigma) = \min_{\rho\in\mathcal{D}} -\langle\Phi^+|\ln\rho|\Phi^+\rangle \geq \min_{\rho\in\mathcal{D}} -\ln\langle\Phi^+|\rho|\Phi^+\rangle\,. \tag{10}$$

It is known [14] that $\rho \in \mathcal{D} \Rightarrow \langle\Phi^+|\rho|\Phi^+\rangle \leq \frac{1}{2}$, and therefore $E(\sigma) \geq \ln 2$. This lower limit can be reached, for example, by the state $\rho = \frac{1}{2}\{|00\rangle\langle 00| + |11\rangle\langle 11|\}$. Therefore we have $E(\sigma) = \ln 2$.

For any pure, entangled state with coefficients α and β (e.g., $\alpha|00\rangle + \beta|11\rangle$) we conjecture that this measure reduces to the usual von Neumann reduced entropy $-|\alpha|^2\ln|\alpha|^2 - |\beta|^2\ln|\beta|^2$, but the rigorous proof has not been found.

Now we also calculate the entanglement of Bell-diagonal states [7]. We define the density operators $\sigma_{1/2} = |e_{1/2}\rangle\langle e_{1/2}| = |\Psi^\pm\rangle\langle\Psi^\pm|$ and $\sigma_{3/4} = |e_{3/4}\rangle\langle e_{3/4}| = |\Phi^\pm\rangle\langle\Phi^\pm|$, where $|\Psi^\pm\rangle, |\Phi^\pm\rangle$ is the usual Bell basis. Then a Bell-diagonal state has the $W = \sum_i \lambda_i\sigma_i$. We now prove the following.

Proposition 2.—For a Bell-diagonal state $\sigma = \sum_i \lambda_i\sigma_i$, where all $\lambda_i \in [0, \frac{1}{2}]$, we find

$$E(\sigma) = 0\,, \tag{11}$$

while for $\lambda_1 \geq \frac{1}{2}$ we obtain

$$E(\sigma) = \lambda_1 \ln \lambda_1 + (1 - \lambda_1) \ln(1 - \lambda_1) + \ln 2 \quad (12)$$

and analogously for $\lambda_i \geq \frac{1}{2}$.

Proof.—The first case is simple once we remember that a Bell-diagonal state ρ is separable, i.e., $\rho \in \mathcal{D}$, iff its spectrum lies in $[0, \frac{1}{2}]$ [14]. Therefore $E(\sigma) = 0$.

To prove the theorem for $\lambda_1 \geq \frac{1}{2}$ we again utilize the fact that $f(x) = -\ln x$ is convex. We obtain

$$\begin{aligned} E(\sigma) &= \sum_i \lambda_i \ln \lambda_i + \min_{\rho \in \mathcal{D}} -\mathrm{tr}\{\sigma \ln \rho\} \\ &\geq \sum_i \lambda_i \ln \lambda_i + \min_{\rho \in \mathcal{D}} -\sum_i \lambda_i \ln \langle e_i | \rho | e_i \rangle . \end{aligned} \quad (13)$$

We know that $\rho \in \mathcal{D}$ implies that all $\rho_{ii} \leq \frac{1}{2}$ (or otherwise the state can be purified [4,7]). Therefore we can determine the minimum not over the states from $\mathcal{D}$ but over the space $\mathcal{B}$ of all Bell-diagonal states with spectrum in $[0, \frac{1}{2}]$. This gives a lower bound to Eq. (13) because

$$\min_{\rho \in \mathcal{D}} -\sum_i \lambda_i \ln \langle e_i | \rho | e_i \rangle = \min_{\rho \in \mathcal{B}} -\sum_i \lambda_i \ln \langle e_i | \rho | e_i \rangle .$$

Defining $p_i = \langle e_i | \rho | e_i \rangle$ we have to minimize the function $f(p_1, p_2, p_3, p_4) = -\sum_i \lambda_i \ln p_i$ under the constraints $\sum_{i=1}^4 p_i = 1$ and $p_i \in [0, \frac{1}{2}]$. This minimization yields

$$p_1 = 1/2, \qquad p_i = \lambda_i / 2(1 - \lambda_1). \quad (14)$$

The state $\rho = \sum_i p_i \sigma_i$ with the values from Eq. (14) lies in $\mathcal{D}$ [14] and therefore the lower limit can be reached, which proves Eq. (12).

Note that the expression for the entanglement Eq. (12) given in proposition 2 is different from the entanglement of creation [5]. For a Werner state with $F = 0.625$ we obtain $\approx 0.04 \ln 2$, whereas the entanglement of creation is $\approx 0.117 \ln 2$. It is not clear yet what these numbers actually mean, and whether they give a bound to the maximum possible efficiency of purification schemes. For consistency, it is only important that if σ_1 is more entangled then σ_2 for one measure than it also must be for all other measures. Comparing Bennett *et al.*'s entanglement of creation with our entanglement measure for Bell-diagonal states shows that this is in fact the case.

So far we have discussed only the von Neumann relative entropy. However, there are many other possible distances that we can choose for $D(\sigma \parallel \rho)$ in Eq. (5) to quantify entanglement of two arbitrarily dimensional subsystems. An example of interest is the Bures metric $D_B(\sigma \parallel \rho) = 2 - 2\sqrt{F(\sigma, \rho)}$, where $F(\sigma, \rho) := [\mathrm{tr}\{\sqrt{\rho}\, \sigma \sqrt{\rho}\}^{1/2}]^2$ is the so-called fidelity (or Uhlmann's transition probability) [15]. It can be shown that if we use this distance in Eq. (5) we obtain a measure of entanglement that satisfies the conditions (i)–(iii) (see [16] for the proof that fidelity does not decrease under LGM + CC). Other possible measures can be found and will be discussed elsewhere. The Bures metric has a very nice statistical, operational basis for the measure of entanglement in terms of general measurements [17]. It derives from the nature of fidelity as a "measure" of distinguishability between two probability distributions $p_{1i} = \mathrm{tr}(\sigma A_i^\dagger A_i)$ and $p_{1i} = \mathrm{tr}(\rho A_i^\dagger A_i)$, where $\sum_i A_i^\dagger A_i = \mathbf{I}$. More precisely,

$$F(\sigma, \rho) = \min_{A_i^\dagger A_i} \sum_i \sqrt{\mathrm{tr}(\sigma A_i^\dagger A_i)} \sqrt{\mathrm{tr}(\rho A_i^\dagger A_i)}, \quad (15)$$

where the minimum is taken over all possible general measurements. This possibly enables us, in principle, to determine Eq. (5) and therefore also the degree of entanglement experimentally.

So far we have only defined entanglement between two subsystems of arbitrary dimensions. It is, however, straightforward to generalize this notion to more than two subsystems. Let us for simplicity assume that we have three systems, A, B, and C. Then the entanglement would be a minimum distance of Eq. (5) over all disentangled states, which, in this case, would be of the form

$$\rho_{ABC} = \sum_i p_i \rho_{AB} \rho_C + q_i \rho_{AC} \rho_B + r_i \rho_A \rho_{BC} . \quad (16)$$

Again, we can see that this class of measures has to satisfy the three imposed conditions. In the same fashion the above approach to quantifying the entanglement could be generalized to any number of quantum subsystems. However, the complexity involved in minimizing the distance increases with increasing the number of subsystems under consideration.

In this Letter we have presented conditions every measure of entanglement has to satisfy, and shown that there is a *whole class* of distance measures suitable for entanglement measures. The central idea of our construction is that we calculate the distance between a given state and all possible disentangled states, taking the minimum as the actual amount of entanglement. This construction approximates classical correlations as closely as possible and therefore measures the quantum correlations only. The generalization to entanglement measures for more than two particles is straightforward. Our work suggests further investigation is worthwhile into the relationship between purification procedures and the various measures of entanglement suggested above, as well as finding a closed form for the expression for entanglement.

We thank the Oxford Quantum Information Group for useful discussions. This work was supported by the European Community, the UK Engineering and Physical Sciences Research Council, the Alexander von Humboldt Foundation, and the Knight Trust.

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