

Purification via entanglement swapping and conserved entanglement

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We investigate the purification of entangled states by local actions using a variant of entanglement swapping. We show that there exists a measure of entanglement which is conserved in this type of purification procedure. [S1050-2947(99)06807-9]

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The resource of entanglement [1] has many useful applications in quantum information processing, such as secret key distribution [2], teleportation [3], and dense coding [4]. Polarization entangled photons have been used to demonstrate both dense coding [5] and teleportation [6] in the laboratory. Teleportation has also been realized using path-entangled photons [7] and entangled electromagnetic field modes [8]. Accompanying the practical applications of entanglement are some useful schemes for entanglement manipulation which may help in the distribution of entanglement between distant parties. One such scheme is entanglement swapping [9,10], which enables one to entangle two quantum systems that have never interacted directly with each other; we have discussed how this may be used in constructing a quantum telephone exchange [10]. Recently, entanglement swapping has been demonstrated experimentally [11]. There exists yet another useful manipulation of entanglement in which local actions and classical communication are used by two distant parties to distill a certain number of shared Bell states from a larger number of shared but less entangled states. When the initial shared but less entangled states are pure, this manipulation is termed as entanglement concentration [12,13], while for the more general case of the initial shared states being mixed, the process is termed as entanglement purification or distillation [14,15]. The importance of such a scheme in the distribution of entanglement is obvious as Bell pairs are essential for the implementation of quantum communication schemes with perfect fidelity. Curiously, such an important procedure remains to be realized in an experiment. In this paper, we will show that a simple variant of the entanglement swapping scheme can be viewed as a type of entanglement concentration procedure and has a physical realization with polarization entangled photons. Moreover, we show that there exists a certain measure of entanglement which remains conserved on average in this type of entanglement concentration. Note that in this paper we will often use the terms “entanglement concentration” and “entanglement purification” in an interchangeable manner, though what we demonstrate is, in the strict sense, only the concentration of pure shared entanglement.

Let pairs of photons (1,2) and (3,4) be in the following polarization entangled states:

$$|\Phi(\theta)\rangle_{12} = \cos\theta|H_1, H_2\rangle + \sin\theta|V_1, V_2\rangle, \quad (1a)$$

$$|\Phi(\theta)\rangle_{34} = \cos\theta|H_3, H_4\rangle + \sin\theta|V_3, V_4\rangle, \quad (1b)$$

where the *phase angle* θ satisfies $0 < \theta < \pi/2$. There are a number of ways to prepare photons in such polarization entangled states. For example, one may first use type-II down conversion followed by suitable birefringent crystals to prepare two photons in the Bell state $|\Phi\rangle^+ = |HH\rangle + |VV\rangle$ [16] (the other Bell states being $|\Phi\rangle^- = |HH\rangle - |VV\rangle$ and $|\Psi\rangle^\pm = |HV\rangle \pm |VH\rangle$). This may be followed by placing a dichroic element (such as the local filters described in Ref. [17]), which selectively absorbs any one of the polarization components (say H) along the path of one of the photons. In cases when this photon exits the element unabsorbed, the pair of photons is in the state $e^{-\gamma L}|HH\rangle \pm |VV\rangle$ (not normalized), where L is the length of the crystal and γ is the absorption per unit length. Thereby states of the type given by Eqs. (1a) and (1b) with $\sin\theta = \sqrt{1/(1 + e^{-2\gamma L})}$ are generated. This procedure may seem inefficient because of the possibility of the photon being absorbed by the dichroic element. However, due to the absence of two qubit logic gates for polarization entangled photons, there is no way to proceed unitarily from $|HH\rangle \pm |VV\rangle$ to the states given by Eqs. (1a) and (1b) and dissipative processes are necessary. Alternatively, one can use the recently suggested tunable ultrabright source of polarization entangled photons [18] to directly produce the states given in Eqs. (1a) and (1b). Photons 2 and 3 are brought together, while photons 1 and 4 are allowed to travel to separate distant locations as shown in Fig. 1. If one now performs a Bell state measurement on photons 2 and 3, then immediately the states of the photons 1 and 4 become entangled. This effect, called entanglement swapping, has been tested for maximally entangled photons (when $\theta = \pi/4$) [11]. We shall refer to this tested case (i.e., when

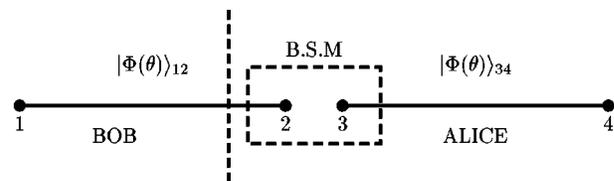


FIG. 1. The figure illustrates the procedure of purification via entanglement swapping. The solid lines connect initially entangled photon pairs. The dashed line divides the set of photons to those held by Alice and those held by Bob. B.S.M denotes the local Bell state measurement done by Alice.

maximally entangled photons are used) as standard entanglement swapping. For the more general case of an arbitrary θ , the combined state of the photons 1 and 4 is projected to either of the following four states, depending on the outcome of the Bell state measurement on photons 2 and 3:

$$|\Phi'(\theta)\rangle_{14}^+ = \frac{1}{N}(\cos^2\theta|H_1, H_4\rangle + \sin^2\theta|V_1, V_4\rangle), \quad (2a)$$

$$|\Phi'(\theta)\rangle_{14}^- = \frac{1}{N}(\cos^2\theta|H_1, H_4\rangle - \sin^2\theta|V_1, V_4\rangle), \quad (2b)$$

$$|\Psi\rangle_{14}^+ = \frac{1}{\sqrt{2}}(|H_1, V_4\rangle + |V_1, H_4\rangle), \quad (2c)$$

$$|\Psi\rangle_{14}^- = \frac{1}{\sqrt{2}}(|H_1, V_4\rangle - |V_1, H_4\rangle), \quad (2d)$$

where $N = \sqrt{\cos^4\theta + \sin^4\theta}$. The probabilities to obtain the four states are

$$P(|\Phi'(\theta)\rangle_{14}^+) = P(|\Phi'(\theta)\rangle_{14}^-) = \frac{\cos^4\theta + \sin^4\theta}{2}, \quad (3a)$$

$$P(|\Psi\rangle_{14}^+) = P(|\Psi\rangle_{14}^-) = \cos^2\theta\sin^2\theta, \quad (3b)$$

where the symbol $P(|\psi\rangle)$ is used to denote the probability of the state $|\psi\rangle$. We now proceed to explain the sense in which the above variant of standard entanglement swapping is a purification procedure. Say photons 2, 3, and 4 are held by one party (Alice) and photon 1 is possessed by the other party (Bob) as shown in Fig. 1. Now, Alice can change the magnitude of the entanglement she shares with Bob by doing a Bell state measurement on photons 2 and 3 and thereby projecting photons 1 and 4 to one of the states given by Eqs. (2a)–(2d). In the cases when she projects photons 1 and 4 to either $|\Psi\rangle_{14}^+$ or $|\Psi\rangle_{14}^-$ (which are Bell states), she actually increases the magnitude of entanglement she shares with Bob. In the other cases she reduces the magnitude of entanglement she shares with Bob even further. If she initially shared a large enough ensemble of photons in the state $|\Phi(\theta)\rangle_{12}$ with Bob and applied the procedure described above on each shared pair separately, then she would be able to change the states of a certain fraction of the shared pairs to Bell states at the expense of degrading the entanglement of the other shared pairs even further. This qualifies as a type of purification procedure because local actions (by Alice) are used to concentrate the entanglement of a fraction of the shared states while the entanglement of the remaining fraction is being diluted, just as in other purification procedures [12,15].

We should pause here briefly to mention a fact relevant from the experimental viewpoint. It is known that a complete Bell state measurement is not possible with only linear elements [19]. However, a complete Bell state measurement is *not* really *necessary* to implement the above purification protocol. One only needs to be able to discriminate the states $|\Psi\rangle_{23}^+$ and $|\Psi\rangle_{23}^-$ from each other and from the rest of the Bell states, which can be done in existing experiments [5,6,9,11]. The reason for this is that the photons 1 and 4 are

projected to Bell states ($|\Psi\rangle_{14}^+$ or $|\Psi\rangle_{14}^-$) only when the outcome of the Bell state measurement on photons 2 and 3 is either $|\Psi\rangle_{23}^+$ or $|\Psi\rangle_{23}^-$. To verify whether the purification has indeed taken place, one simply has to pass the resultant Bell states of photons 1 and 4 through a Bell state analyzer [5,6,9,11].

An interesting feature of the original collective entanglement concentration procedure (called the ‘‘Schmidt projection method’’) described in Ref. [12] was that the average of the entropy of entanglement (the von Neumann entropy of the partial density matrix seen by either party) of all the shared pairs remained constant under purification (an asymptotic result). In the scheme described here, the average von Neumann entropy of entanglement of the shared pairs is, in fact, decreased. However, we shall show that there is a different measure of entanglement whose average remains conserved in this procedure of purification via entanglement swapping. This measure of entanglement is defined as the maximum probability with which two parties sharing a pure entangled state can convert it to a Bell state by classically communicating and performing local actions on their respective sides. As this is already the maximum probability, it can only remain constant or decrease under any set of local general measurements and classical communications. (This is the basic criterion to be satisfied by any measure of entanglement [14,20].) Thereby, it qualifies as a measure of entanglement which can be termed as *entanglement of single pair purification*. (This measure, however, is not an additive measure.) We shall denote this measure by E_S and refer to it henceforth simply as entanglement.

We now proceed to show that E_S is conserved in the purification process described above. From the results of Lo and Popescu [13], it follows that the maximum probability with which a Bell state can be obtained by purifying a single entangled pair (in a pure state) is twice the modulus square of the Schmidt coefficient of smaller magnitude. In the case of the states given by Eqs. (1a) and (1b), this is simply $2\cos^2\theta$ if $\cos\theta$ is the smaller of the two Schmidt coefficients. Therefore, before the entanglement swapping, the average value of the entanglement shared between Alice (A) and Bob (B) is

$$\langle E_S \rangle_{AB} = 2\cos^2\theta. \quad (4)$$

After the entanglement swapping, when the shared states are those given by Eqs. (2a)–(2d) with probabilities given by Eqs. (3a) and (3b),

$$\begin{aligned} \langle E_S \rangle_{AB} &= P(|\Phi'(\theta)\rangle_{14}^+)E_S(|\Phi'(\theta)\rangle_{14}^+) \\ &\quad + P(|\Phi'(\theta)\rangle_{14}^-)E_S(|\Phi'(\theta)\rangle_{14}^-) \\ &\quad + P(|\Psi\rangle^+)E_S(|\Psi\rangle^+) + P(|\Psi\rangle^-)E_S(|\Psi\rangle^-) \\ &= 2\cos^2\theta, \end{aligned} \quad (5)$$

where $E_S(|\psi\rangle)$ denotes the entanglement of the state $|\psi\rangle$. Therefore, the ensemble average of the entanglement of single pair purification is a conserved quantity in the process of purification by entanglement swapping. This result also indicates that the purification via entanglement swapping is an optimal protocol for single pair entanglement purification, as the average entanglement of the purified pairs is equal to

the original entanglement. Here, by optimality we mean the best combination of entangled states that can be finally obtained by the purification. A positive implication of this optimality is that the purification process can be separately repeated on the final subensembles, which turn out to be less entangled, without destroying any entanglement on average (though for that, one needs to be able to do complete Bell state measurements, for which schemes have been suggested [21]). If one continues this process indefinitely, in the limit of an infinite sequence, the final ensemble generated will be comprised of a certain fraction of Bell pairs and a certain fraction of completely disentangled pairs. This fraction of Bell pairs should be equal to $2\cos^2\theta$ as the average of the entanglement has been conserved in each step. Thus in the limit of an infinite sequence, the process of purification via entanglement swapping allows us to convert all the entanglement that can possibly be extracted by single pair purifications from the ensemble into Bell pairs.

Here, it is worthwhile to mention that a procedure for single pair purification called the ‘‘Procrustean method’’ had also been suggested in the original entanglement concentration paper [12]. The Procrustean method also conserves E_S and its fractional yield of Bell states is optimal (i.e., equal to E_S). From the point of view of efficiency, our method lies somewhere between the Schmidt projection method (when it is implemented with two entangled pairs, but one pair totally belonging to Alice) and the procrustean method. Our method has a fractional yield of Bell states equal to the Schmidt projection method (with two entangled pairs), but the non-Bell state outcomes are also entangled. It is because of the extra entanglement of the non-Bell state outcomes that E_S is conserved on average in our method of purification. Hence the improvement over the Schmidt projection method, brought in by entanglement swapping is, *to make the non-Bell state outcomes entangled as well*.

Next, we proceed to consider the case when photon pairs (1,2) and (3,4) are not in the same type of entangled state. Suppose photons 1 and 2 start in the entangled state $|\Phi(\theta_1)\rangle_{12}$ and photons 3 and 4 start in the entangled state $|\Phi(\theta_2)\rangle_{34}$ [defined as in Eqs. (1a) and (1b)]. Let the entanglement of the first photon pair be $E_S = \alpha$ and the entanglement of the second photon pair be $E_S = \beta$. Then a simple calculation shows that projecting photons 2 and 3 onto a Bell state basis projects the photons 1 and 4 to states $(\cos\theta_1\cos\theta_2|H_1, H_4\rangle \pm \sin\theta_1\sin\theta_2|V_1, V_4\rangle)$ and $(\cos\theta_1\sin\theta_2|H_1, V_4\rangle \pm \sin\theta_1\cos\theta_2|V_1, H_4\rangle)$ (not normalized) with specific probabilities. The average of the entanglement between the photons 1 and 4 after the projection turns out to be $\langle E_S \rangle = \min\{\alpha, \beta\}$. The manipulation of entanglement described above can be visualized as a step towards the complete generalization of entanglement swapping. The original scheme involved the use of two Bell states and Bell state measurements [9,11]. It has been generalized to cases when many-particle Greenberger-Horne-Zeilinger (GHZ) states are used and many-particle GHZ measurements are conducted [10]. The procedure presented here is a generalization of the original scheme to the case when two-particle pure states which are less entangled than Bell states are used, but measurements conducted are still Bell state measurements. This result is illustrated in Fig. 2. One can get an intuitive feel of this result from the golden rule that entanglement, on aver-

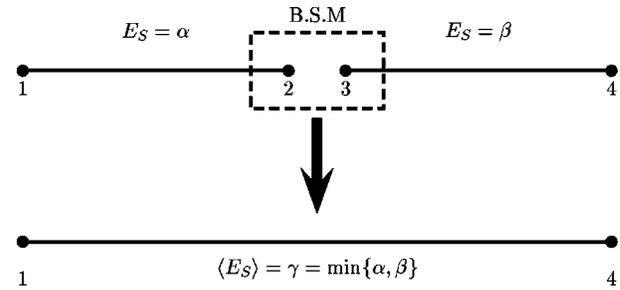


FIG. 2. The figure shows the generalization of entanglement swapping to pure states with arbitrary entanglement.

age, cannot be increased under local actions and classical communications [20]. In the situation depicted in Fig. 2, we have the freedom to choose which photons belong to Alice and which to Bob. For $\alpha < \beta$, we choose to allot the photon 1 to Bob and the rest to Alice, while for $\alpha > \beta$ we allot the photon 4 to Bob and the rest to Alice. Then a Bell state measurement on Alice’s side cannot increase the entanglement she shares with Bob on average. As this initial entanglement is the smaller of the numbers α and β , the final average entanglement has to be smaller than or equal to $\min\{\alpha, \beta\}$. However, the fact that it is actually equal is a peculiar feature of the entanglement swapping process.

One may envisage a situation in which Alice tries to purify the state $|\Phi(\theta_1)\rangle_{12}$ shared with Bob with the help of the state $|\Phi(\theta_2)\rangle_{34}$ (which we can call the purifier) in her possession. As long as $\beta < \alpha$, she degrades the entanglement shared with Bob on average, while when $\beta \geq \alpha$, she conserves the entanglement shared with Bob on average. Thus in order not to lose any entanglement, the purifier state should have at least as much entanglement as the state to be purified. Moreover, the degree to which the entanglement is concentrated (i.e., inhomogeneously redistributed among the four measurement outcomes) gets better as β approaches α . Bell pairs are produced *only* when β exactly equals α . Thus, as Alice increases the entanglement of her purifier, the degree of entanglement concentration increases yielding Bell pairs when β reaches α (a criterion which can be called *entanglement matching*). On increasing β further, the degree of concentration starts going down. When β reaches unity (maximal entanglement), there is no concentration of entanglement (all the four outcomes have an entanglement equal to that of the original state: a situation equivalent to the perfect fidelity teleportation of an entangled state).

In this paper we have shown that entanglement swapping can be used to purify single pairs of polarization entangled photons. Such a scheme may be achieved by an extension of an existing experiment [11]. In contrast, physical implementation of the entanglement purification schemes involving collective measurements (like the Schmidt projection method of Ref. [12]) will be difficult, as they would involve measurements on many photons at once. The absence of two qubit logic gates for polarization entangled photons also makes purification procedures described in Ref. [15] difficult to realize. From this point of view, the method described here should be a positive first step in the direction of implementation of purification procedures. In the paper we have also introduced a measure of entanglement for single pure pairs and demonstrated that this quantity is conserved in the

above process. This makes purification via entanglement swapping interesting from a purely theoretical angle. The nonadditive nature of the introduced measure implies that additivity is not an essential requirement for an entanglement measure to have a physical interpretation. As a natural generalization of our measure, one may use the maximum possible fractional yield of Bell states from various *finite* collections of shared pairs as physically relevant measures of entanglement. The physical relevance stems from the fact that in a real implementation of entanglement concentration one would always have access to only a finite number of

systems. All the results presented in this paper hold only for pure states. An extension to mixed states will not be trivial, as single pairs in such states cannot, in general, be purified [22]. However, it would still be interesting to investigate whether one can generalize the measure of entanglement presented here to some quantity which is conserved in entanglement swapping with mixed states.

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