

## Quantum and Classical Correlations in Quantum Brownian Motion

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(Received 16 November 2001; revised manuscript received 21 June 2002; published 5 September 2002)

We investigate the entanglement properties of the joint state of a distinguished quantum system and its environment in the quantum Brownian motion model. This model is a frequent starting point for investigations of environment-induced superselection. Using recent methods from quantum information theory, we show that there exists a large class of initial states for which no entanglement will be created at all times between the system of salient interest and the environment. If the distinguished system has been initially prepared in a pure Gaussian state, then entanglement is created immediately, regardless of the temperature of the environment and the nonvanishing coupling.

DOI: 10.1103/PhysRevLett.89.137902

PACS numbers: 03.67.-a, 03.65.Yz, 05.40.Jc

No quantum system is completely isolated from its environment. This basic yet fundamental observation has been one of the key insights allowing an appropriate understanding of the dynamical emergence of classical properties in quantum systems. Not all initial states are equally fragile under the interaction of a distinguished quantum system with its environment, and a relatively robust set of so-called preferred or pointer states is selected dynamically, a process typically referred to as environment-induced superselection (einselection) or simply decoherence. This process is thought to play an important role in the transition from quantum to classical [1,2]. The most frequently employed model in investigations of einselection is the quantum Brownian motion model [3,4]. In this model one considers a distinguished quantum oscillator which is linearly coupled via the position operators to an environment consisting of many harmonic oscillators. Initially, the state of the system of interest and its environment are assumed to be uncorrelated, and the state of the environment is taken to be the canonical (Gibbs) state with respect to some temperature. The typical argument is that starting from the initial situation, the product state of the composite quantum system turns into a correlated state due to the interaction. If one considers the reduced state of the distinguished system, one finds that it undergoes dissipation and decoherence. In the context of quantum Brownian motion, it is often argued that entanglement is unavoidable.

It is the aim of this Letter to revisit the question of the creation of entanglement in quantum Brownian motion with recent powerful methods from quantum information theory [5,6]. Our analysis will be split into two parts. In the first part we will show that surprisingly, quantum Brownian motion does not necessarily create entanglement between the distinguished system and its environment. The joint state of the system and its environment may be separable at all times, that is, not entangled [7,8]: All correlations are merely classical in the sense that one could prepare the same state by mixing product states, which can in turn be prepared by implementing local quantum operations only. By definition, separable states

do not violate any Bell inequality. We explicitly construct initial states with the property that no entanglement is created: they are mixed Gaussian states which are nevertheless different from Gibbs states. In contrast to the finite-dimensional setting, where a high degree of mixing automatically implies separability [9], the existence of such initial states is not obvious [10]. The second part of our analysis is concerned with the question whether there exist initial states of the distinguished oscillator for which the joint state becomes immediately entangled. This question will be answered positively, and it will be shown that *all* pure Gaussian states have this property, regardless of the initial temperature of the environment.

From now on the distinguished quantum oscillator will be called  $S$ , and the environment will be referred to as  $E$ . In the quantum Brownian motion model [3,4], the total Hamiltonian consists of three parts  $H = H^S \otimes \mathbb{1} + \mathbb{1} \otimes H^E + H^I$ , where

$$H^S = \frac{1}{2m_1} P_1^2 + \frac{m_1 \omega_1^2}{2} X_1^2, \quad H^I = -X_1 \otimes \sum_{j=2}^{N+1} \kappa_j X_j,$$

$$H^E = \sum_{j=2}^{N+1} \left( \frac{1}{2m_j} P_j^2 + \frac{m_j \omega_j^2}{2} X_j^2 \right).$$

The frequencies  $\omega_1, \dots, \omega_{N+1}$  and coupling constants  $\kappa_2, \dots, \kappa_N$  are taken to be positive. For convenience, we set  $\omega_1 = 1$ , all masses to be equal,  $m_j = 1$  for  $j = 1, \dots, N+1$ , and we require that the  $(N+1) \times (N+1)$ -matrix  $V$  corresponding to the potential energy is positive, where  $V_{1,1} = \omega_1^2/2$  and  $V_{j,j} = \omega_j^2/2$ ,  $V_{1,j} = V_{j,1} = -\kappa_j$  for  $j = 2, \dots, N+1$ , and all other entries of  $V$  are zero. Typically, one assumes product initial conditions [3,4],  $\rho_0 = \rho_0^S \otimes \rho_0^E$ , where the environment is initially in the Gibbs state  $\rho_0^E = \exp(-\beta H^E) / \text{tr}[\exp(-\beta H^E)]$  associated with some inverse temperature  $\beta$ . This model together with the above additional assumptions will later be referred to as the QBM model (quantum Brownian motion model) in the more specific sense. The time evolution of the reduced state with respect to  $S$  can be determined without approximations [4]: for all

spectral densities  $I(\omega) = \sum_{j=2}^{N+1} \kappa_j^2 \delta(\omega - \omega_j)/(2\omega_j)$  one can derive a differential equation that specifies the dynamical map. This completely positive map  $\mathcal{E}_t$ ,  $t \in [0, \infty)$ , maps an initial state  $\rho_0^S$  of  $S$  on the state  $\rho_t^S = \mathcal{E}_t(\rho_0^S) = \text{tr}_E[U_t(\rho_0^S \otimes \rho_0^E)U_t^\dagger]$  at a later time  $t$ , where  $U_t := \exp(-iHt)$ .

We will first clarify the notation that will be used subsequently. It will turn out to be appropriate not to investigate the state on the infinite-dimensional Hilbert space of the joint system directly, but rather its associated covariance matrix. Throughout the paper we will make repeated use of the formalism of covariance matrices and their manipulation by means of symplectic transformations [11–13]. The  $2n$  canonical self-adjoint operators corresponding to position and momentum of a system with  $n$  degrees of freedom can be collected in a vector  $\mathbf{O} = (O_1, \dots, O_{2n}) = (X_1, P_1, \dots, X_n, P_n)$ . The canonical commutation relations (CCR) can then be written in matrix form as  $[O_j, O_k] = i(\Sigma_{2n})_{j,k}$ , giving rise to the skew-symmetric block diagonal real  $2n \times 2n$ -matrix  $\Sigma_{2n}$  (or  $\Sigma$  when the size of the matrix is clear from the context). Gaussian states, which are defined through their property that the characteristic function is a Gaussian function in phase space, can be characterized in a convenient way through their moments. The first moments  $\langle O_j \rangle_\rho$ ,  $j = 1, \dots, 2n$ , are the expectation values of the canonical coordinates. The  $2n \times 2n$  covariance matrix  $\Gamma$ ,

$$\Gamma_{j,k} = 2\text{tr}[\rho(O_j - \langle O_j \rangle_\rho)(O_k - \langle O_k \rangle_\rho)] - i\Sigma_{j,k}$$

satisfying the Heisenberg uncertainty principle  $\Gamma + i\Sigma \geq 0$ , embodies the second moments of a state  $\rho$ . For later considerations we give the set of covariance matrices of a system with  $n$  degrees of freedom the name  $C_{2n} := \{\Gamma \in M_{2n} : \Gamma = \Gamma^T, \Gamma + i\Sigma \geq 0\}$ , where  $M_{2n}$  is the set of real  $2n \times 2n$  matrices. Particularly important will be covariance matrices of Gibbs states  $\exp(-\beta H)/\text{tr}[\exp(-\beta H)]$  with respect to a Hamiltonian  $H$  and the inverse temperature  $\beta$ , which are important examples of Gaussian states. For brevity, the corresponding covariance matrix will from now on be denoted as  $\Gamma(\beta H)$ . We will frequently employ linear transformations from one set of canonical coordinates to another which preserves the CCR, meaning that  $S\Sigma S^T = \Sigma$ . Such transformations form the group of (real linear) symplectic transformations  $Sp(2n, \mathbb{R})$ . A symplectic transformation  $S \in Sp(2n, \mathbb{R})$  results in a change of the covariance matrix according to  $\Gamma \mapsto S\Gamma S^T$ ; on the level of the states it is associated with a unitary operation  $\rho \mapsto U(S)\rho U(S)^\dagger$ .

We are now able to state the first proposition. It is concerned with the fact that there exist initial states of the distinguished system and an initial temperature of the environment such that the joint state stays separable for all times. In the second part of the proof a lower bound for the required inverse temperature of the bath will be given.

*Proposition 1: In the QBM model, for any choice of coupling constants  $(\kappa_2, \dots, \kappa_{N+1})$ ,  $\kappa_j \geq 0$ , and frequen-*

*cies  $(\omega_2, \dots, \omega_{N+1})$ ,  $\omega_j > 0$ , there exists a Gaussian initial state of the system  $\rho_0^S$  with covariance matrix  $\Gamma_0^S$  and a temperature  $1/\beta > 0$  of the environment such that*

$$\rho_t = U_t(\rho_0^S \otimes \exp(-\beta H^E)/\text{tr}[\exp(-\beta H^E)])U_t^\dagger$$

*is not entangled for all times  $t \in [0, \infty)$ . Let  $\omega_\infty := \max\{\omega_2, \dots, \omega_{N+1}\}$ ,  $\delta := 2 \sum_{j=2}^{N+1} \kappa_j^2$ , and  $\Omega := (\omega_\infty^2 + 2\delta^{1/2})^{1/2}$ , then the above inverse temperature is bounded from below by the smallest  $\beta$  that satisfies  $\Gamma_0^S \oplus \Gamma(\beta H^E) \geq \Gamma(\gamma H)$ , where  $\gamma := \min\{2, \log(1 + 2/\Omega)/\Omega\}$ .*

*Proof:* System  $S$  and environment  $E$  form a system with  $N + 1$  canonical degrees of freedom, where now  $\mathbf{O} = (O_1, \dots, O_{2N+2}) = (X_1, P_1, X_2, P_2, \dots, P_{N+1})$ . There is a  $T \in Sp(2N + 2, \mathbb{R})$  such that the Hamiltonian  $\tilde{H}$  in the new canonical coordinates  $\tilde{\mathbf{O}}^T = T\mathbf{O}^T$  is the Hamiltonian

$$\tilde{H} = \sum_{j=1}^{N+1} \frac{\tilde{O}_{2j-1}^2}{2} + \sum_{j=1}^{N+1} \tilde{\omega}_j^2 \frac{\tilde{O}_{2j}^2}{2},$$

of  $N + 1$  uncoupled oscillators, with real numbers  $\tilde{\omega}_j^2/2$ ,  $j = 1, \dots, N + 1$ , which are the eigenvalues of the positive matrix  $V$ . As the coupling is restricted to the coordinates associated with positions,  $T$  is both orthogonal and symplectic. This symplectic transformation can be achieved by applying an appropriate orthogonal matrix  $M \in SO(N + 1)$  on the canonical variables corresponding to position,  $(\tilde{O}_1, \tilde{O}_3, \dots, \tilde{O}_{2N+1})^T = M(O_1, O_3, \dots, O_{2N+1})^T$ , such that  $H$  becomes diagonal in position, and the same matrix  $M$  on the momentum variables,  $(\tilde{O}_2, \tilde{O}_4, \dots, \tilde{O}_{2N+2})^T = M(O_2, O_4, \dots, O_{2N+2})^T$ . In these canonical coordinates the covariance matrix corresponding to the Gibbs state with respect to  $\delta$  becomes

$$\begin{aligned} \Gamma(\delta \tilde{H}) &= \text{diag}(\Gamma(\delta \tilde{H})_{1,1}, \dots, \Gamma(\delta \tilde{H})_{2N+2,2N+2}), \\ \Gamma(\delta \tilde{H})_{2j-1,2j-1} &= f(\delta \tilde{\omega}_j)/\tilde{\omega}_j, \\ \Gamma(\delta \tilde{H})_{2j,2j} &= f(\delta \tilde{\omega}_j)\tilde{\omega}_j, \end{aligned}$$

$j = 1, \dots, N + 1$ , where the function  $f: \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is defined as  $f(x) = 1 + 2/[\exp(x) - 1]$ . Let  $\gamma > 0$  be defined as above. We now show that

$$\Gamma(\gamma \tilde{H}) \geq \mathbb{1}_{2N+2}. \quad (1)$$

The largest eigenvalue of  $V$  is given by its operator norm  $\|V\|$  [14]. By adding and subtracting the same term one obtains  $\|V\| \leq \max\{\omega_1^2/2, \omega_2^2/2, \dots, \omega_{N+1}^2/2\} + \|V - \text{diag}(\omega_1^2/2, \omega_2^2/2, \dots, \omega_{N+1}^2/2)\|$ , giving rise to  $\|V\| \leq \omega_\infty^2/2 + \delta^{1/2}$ . The value of  $\|V\|$  is related to the largest frequency  $\tilde{\omega}_\infty := \max\{\tilde{\omega}_j, j = 1, \dots, N + 1\}$  by  $\|V\| = \tilde{\omega}_\infty^2/2$ . One finds after a few steps that indeed  $\Gamma(\gamma \tilde{H}) \geq \mathbb{1}_{2N+2}$ . Now we need to invoke the concept of partial transposition. It has been shown in Ref. [11] that in a system consisting of one oscillator in a system  $S$  and  $N$  oscillators in an environment  $E$ , a Gaussian state is separable iff its partial transpose is a quantum state. By using the matrix  $\Sigma^{TE} := \Sigma_2 \oplus (-\Sigma_{2N})$ , the criterion can also be written in the form that a state with covariance matrix  $\Gamma$

is separable if and only if  $\Gamma + i\Sigma^{T_E} \geq 0$  [11]. The next step is to see that with  $\gamma > 0$  as in Eq. (1)

$$T^T \Gamma (\gamma \tilde{H}) T + i\Sigma^{T_E} \geq \mathbb{1}_{2N+2} + i\Sigma^{T_E} \geq 0,$$

since  $\|i\Sigma^{T_E}\| = 1$  and  $T \in \text{SO}(2N+2)$ . Equipped with these tools, one can construct a class of product initial states such that the joint state of the system and its environment is separable at all times. These are of the form states  $\rho_0^S \otimes \rho_0^E$ , where  $\rho_0^E = \exp(-\beta H^E) / \text{tr}[\exp(-\beta H^E)]$  with respect to a certain inverse temperature  $\beta > 0$ .  $\rho_0^E$  is a Gaussian state, and the state of the distinguished system  $\rho_0^S$  is taken to be Gaussian as well. On the level of covariance matrices such an initial state is represented as  $\Gamma_0 = \Gamma_0^S \oplus \Gamma(\beta H^E)$ , that is,

$$\Gamma_0 = \Gamma_0^S \oplus \text{diag}((\Gamma_0)_{3,3}, \dots, (\Gamma_0)_{2N+2,2N+2}),$$

$$(\Gamma_0)_{2j-1,2j-1} = f(\beta \omega_j) / \omega_j, \quad (\Gamma_0)_{2j,2j} = f(\beta \omega_j) \omega_j,$$

for  $j = 2, \dots, N+1$ , where  $\Gamma_0^S \in C_2$ . The covariance matrix  $\Gamma_0^S \in C_2$  and  $\beta > 0$  are now chosen to be such that  $\Gamma_0 - T^T \Gamma (\gamma \tilde{H}) T \geq 0$  holds. Such covariance matrices and inverse temperatures always exist. For a given covariance matrix  $T^T \Gamma (\gamma \tilde{H}) T$  one can always choose  $\beta > 0$  and  $\Gamma_0^S \in C_2$  in such a way that  $\Gamma_0 - T^T \Gamma (\gamma \tilde{H}) T$  is diagonally dominant [14]. Since it is a symmetric matrix, diagonal dominance implies that  $\Gamma_0 - T^T \Gamma (\gamma \tilde{H}) T$  is a positive matrix. From the definition of the covariance matrix  $\Gamma(\gamma \tilde{H})$  one can infer that  $\Gamma_0 - T^T \Gamma (\gamma \tilde{H}) T \geq 0$  is equivalent to the requirement  $\Gamma_0^S \oplus \Gamma(\beta H^E) \geq \Gamma(\gamma H)$ , which is the inequality in Proposition 1 giving rise to the lower bound for  $\beta$ .

Since the Hamiltonian  $H$  is a quadratic polynomial in the canonical coordinates, time evolution  $\rho_0 \mapsto \rho_t = U_t \rho_0 U_t^\dagger$  is affected by a symplectic transformation on the level of covariance matrices. There exists a continuous map  $t \in [0, \infty) \mapsto S_t \in \text{Sp}(2N+2, \mathbb{R})$ , such that given a covariance matrix  $\Gamma_0 \in C_{2N+2}$  at time  $t = 0$ , the covariance matrix of  $\rho_t$  becomes  $\Gamma_t := S_t \Gamma_0 S_t^T$ . The aim is to show that this matrix corresponds to a separable state, i.e.,  $\Gamma_t + i\Sigma^{T_E} \geq 0$  for all times. As  $\Gamma(\gamma H)$  is the covariance matrix of a Gibbs state with respect to  $H$ ,  $S_t \Gamma(\gamma H) S_t^T = \Gamma(\gamma H)$  for all  $t \in [0, \infty)$ , and hence,  $S_t \Gamma(\gamma H) S_t^T + i\Sigma^{T_E} \geq 0$ . Moreover,  $S_t[\Gamma_0 - \Gamma(\gamma H)] S_t^T \geq 0$ , because  $\Gamma_0 - \Gamma(\gamma H) \geq 0$ . It follows that

$$\Gamma_t + i\Sigma^{T_E} = S_t[\Gamma_0 - \Gamma(\gamma H)] S_t^T + \Gamma(\gamma H) + i\Sigma^{T_E} \geq 0$$

for all  $t \in [0, \infty)$ . As the state was initially a Gaussian state, it remains Gaussian under time evolution. Having a positive partial transpose is equivalent to being separable for systems where one of the parts consists of only one oscillator, which means that we can conclude that  $\Gamma_t$  corresponds to a separable state for all times. ■

So we have shown that for these initial conditions, no entanglement will be created at all times. At this point, a remark might be appropriate concerning the inverse temperature  $\beta$  in Proposition 1. The question of the behavior of  $\beta$  is particularly relevant when one performs a con-

tinuum limit as is typically done when deriving quantum master equations. In this context, it is of interest to see that a lower bound for  $\beta$  can be found that is independent of the number of oscillators. We consider a sequence of Hamiltonians  $\{H_N\}_{N=1}^\infty$  of a joint system with an environment consisting of  $N$  oscillators, each equipped with a coupling constant  $\kappa_j^N$  and a frequency  $\omega_j^N$ ,  $j = 2, \dots, N+1$ . Take for each  $N$  an equidistant distribution of frequencies, such that  $\omega_j^N = (j-1)\omega_\infty/N$ , where  $\omega_\infty$  is the largest (cutoff) frequency. Concerning the spectral densities we make only the assumptions that  $\kappa_j^N = \alpha_N (\omega_j^N)^p$  with  $p > 0$  and  $\alpha_N > 0$ , which covers the Ohmic ( $p = 1$ ), the sub-Ohmic ( $p < 1$ ), and the supra-Ohmic ( $p > 1$ ) cases. The refinement must in all instances be made such that  $\sum_{j=2}^{N+1} (\kappa_j^N)^2 = 2 \int d\omega I(\omega) \omega$  remains constant. Then one can show that there exists a strictly positive lower bound for  $\beta$  as in Proposition 1 which is independent of the number of oscillators  $N$ , and the limit  $N \rightarrow \infty$  may be performed. For more details see Ref. [15].

In turn, having this observation in mind, one may ask whether there are initial states  $\rho_0^S$  for which one can be sure that entanglement will be created immediately, no matter how weak the interaction is between the system and its environment, and given any possibly very high initial temperature of the environment. We shall see that there exist such states: all pure Gaussian states have this property.

*Proposition 2: In the QBM model, for any initial pure Gaussian state  $\rho_0^S$  of  $S$ , any coupling constants  $(\kappa_2, \dots, \kappa_{N+1})$ ,  $\kappa_j \geq 0$ , any frequencies  $(\omega_1, \dots, \omega_{N+1})$ ,  $\omega_j > 0$ , and any  $\beta > 0$ , the state  $\rho_t = U_t(\rho_0^S \otimes \exp(-\beta H^E) / \text{tr}[\exp(-\beta H^E)]) U_t^\dagger$  is entangled for all times  $t \in (0, \varepsilon]$  for an appropriate  $\varepsilon > 0$ .*

*Proof:* A Gaussian state  $\rho_0^S$  with a covariance matrix  $\Gamma_0^S \in C_2$  is pure iff  $(\Sigma_2 \Gamma)^2 = -\mathbb{1}_2$ . The task is to find an  $\varepsilon > 0$  such that  $\mathcal{E}_t(\rho_0^S \otimes \exp(-\beta H^E) / \text{tr}[\exp(-\beta H^E)])$  is entangled for all  $t \in (0, \varepsilon]$ . Assume without loss of generality that  $\langle O_j \rangle_{\rho_0} = 0$  for  $j = 1, 2$ . Since  $(d/dt)|_{t=0} \langle O_j \rangle_{\rho_t} = 0$ , at  $t = 0$  the covariance matrix  $\Gamma_t$  of  $\rho_t = U_t \rho_0 U_t^\dagger$  satisfies

$$\left. \frac{d}{dt} \right|_{t=0} \Gamma_t = W^T \Gamma_0 W$$

with some  $W \in M_{2N+2}$ , which can be evaluated by making use of  $i \text{tr}[O_j O_k (d\rho_t/dt)|_{t=0}] = \text{tr}[O_j O_k [H, \rho_0]]$ ,  $j, k = 1, \dots, 2N+2$ . The state  $\rho_t$  is entangled at a time  $t > 0$  if the reduced state with respect to the system  $S$  and one of the oscillators of the environment  $E$  is an entangled state. The covariance matrix of  $S$  and the oscillators of  $E$  with label 2 is given by a real  $4 \times 4$  principal submatrix of  $\Gamma_t$ . This matrix will from now on be called  $\gamma_t \in C_4$ ; it can be written in block form as

$$\gamma_t = \begin{pmatrix} A_t & C_t \\ C_t^T & B_t \end{pmatrix}, \quad (2)$$

where  $A_t, B_t, C_t \in M_2$ . The statement that the reduced state of  $S$  and the oscillator with label 2 is entangled is

equivalent to the statement that  $\gamma_t + i\Sigma^{T_E}$  is not positive, where now  $\Sigma^{T_E} = \Sigma_2 \oplus (-\Sigma_2)$ . This in turn is equivalent with the smallest eigenvalue of  $(i\Sigma^{T_E}\gamma_t)^2$  being smaller than 1 (i.e., one of the symplectic eigenvalues of the partial transpose of  $\gamma_t$  is smaller than 1 [16]). Denote the smallest eigenvalue of  $(i\Sigma_4^{T_E}\gamma_t)^2$  as  $\lambda_t$ , and let for brevity  $d_t := \det(A_t) + \det(B_t) - 2\det(C_t)$ . The smallest eigenvalue  $\lambda_t$  can then be expressed as  $\lambda_t = d_t/2 - [d_t^2/4 - \det(\gamma_t)]^{1/2}$  [16]. At  $t = 0$ , the covariance matrix  $\gamma_0$  is of the form of Eq. (2) with  $A_0 \in C_2$ ,  $C_0 = 0$ , and  $B_0 = \text{diag}[f(\beta\omega_2)/\omega_2, f(\beta\omega_2)\omega_2]$ , the latter matrix satisfying  $\det(B_0) > 1$  by definition. As  $S$  is initially in a pure state,  $\det(A_0) = 1$ , so that  $\lambda_0 = 1$ . The first derivatives of  $A_t$ ,  $B_t$ , and  $C_t$  at  $t = 0$  are

$$\begin{aligned} \frac{d}{dt} \Big|_{t=0} A_t &= \begin{pmatrix} 0 & (A_0)_{2,2} - (A_0)_{1,1} \\ (A_0)_{2,2} - (A_0)_{1,1} & -2(A_0)_{1,2} \end{pmatrix}, \\ \frac{d}{dt} \Big|_{t=0} C_t &= \begin{pmatrix} 0 & -\kappa_2(A_0)_{1,1} \\ 0 & \kappa_2(A_0)_{1,2} \end{pmatrix}, \\ \frac{d}{dt} \Big|_{t=0} B_t &= \begin{pmatrix} 0 & (B_0)_{2,2} - (B_0)_{1,1} \\ (B_0)_{2,2} - (B_0)_{1,1} & 0 \end{pmatrix}. \end{aligned}$$

Hence, it follows that  $(d\lambda_t/dt)|_{t=0} < 0$ . This means that there exists an  $\varepsilon > 0$  such that  $\lambda_t < 1$  for all  $t \in (0, \varepsilon]$ , which implies that the partial transpose of the state corresponding to  $\gamma_t$  is not a state. We can conclude that the state  $\rho_t$  associated with the covariance matrix  $\Gamma_t$  is an entangled state [17]. ■

To summarize, we have investigated the entanglement properties of the joint state of a distinguished system and its environment in quantum Brownian motion. Surprisingly, we found that there exists a large set of initial states of the system for which no entanglement is created at all times. Also, we have shown that for pure initial Gaussian states of the distinguished system entanglement will be immediately created. The tools we used were mostly taken from the field of quantum information theory. In fact, we hope that this Letter can contribute to the line of thought of applying methods from quantum information theory to the issue of emerging classicality in quantum physics.

We thank P. Hänggi, P. Talkner, C. Simon, H. Wiseman, K. J. Boström, J. Kempe, V. Scarani, J. R. Anglin, and W. H. Zurek for inspiring discussions. This work has been supported by the EU (EQUIP, IST-1999-11053), the A.v.-Humboldt Foundation, and the EPSRC.

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[15] The goal is to show that there exists a  $\beta > 0$  and a  $\Gamma_0^S \in C_2$  such that  $\Gamma_0^S \oplus \Gamma(\beta H_N^E) \geq \Gamma(\gamma H_N)$  for all  $N \in \mathbb{N}$ . This means that the continuum limit may be performed, and the temperature of the environment that has to be chosen in order to satisfy the assumptions of Proposition 1 does not diverge in the limit  $N \rightarrow \infty$ : (i) The first step is to observe that it is sufficient to prove that  $\Gamma_0^S \oplus \Gamma(\beta H_N^E) \geq 2(A_N \oplus B_N)$ , where  $A_N$  is the  $2 \times 2$  upper leading principle submatrix of  $\Gamma(\gamma H_N)$ , and  $B_N$  is the  $2N \times 2N$  lower principle submatrix of  $\Gamma(\gamma H_N)$ . This is sufficient, as if the latter is true, then  $\Gamma_0^S \oplus \Gamma(\beta H_N^E) - \Gamma(\gamma H_N) = \Gamma_0^S \oplus \Gamma(\beta H_N^E) - 2(A_N \oplus B_N) + P\Gamma(\gamma H_N)P \geq 0$ , where  $P := \text{diag}(-1, -1, 1, \dots, 1)$ . (ii) There exists a number  $q > 0$  such that  $2\Gamma(\gamma H_N^E) + q\mathbb{1} \geq B_N$  for all  $N \in \mathbb{N}$ . (iii) There always exists a  $\beta > 0$  such that  $f(\omega\beta)\omega \leq 2f(\omega\gamma)\omega + q$  and  $f(\omega\beta)/\omega \leq 2f(\omega\gamma)/\omega + q$  for all  $N \in \mathbb{N}$  and all  $\omega \in [0, \omega_\infty]$ , and therefore, also  $\Gamma(\beta H_N^E) \geq 2\Gamma(\gamma H_N^E) + q\mathbb{1}$  holds for all  $N \in \mathbb{N}$ . (iv) The last step is to see that there exists a  $\Gamma_0^S \in C_2$  such that  $\Gamma_0^S \geq A_N$  for all  $N \in \mathbb{N}$ . This is, however, obvious from the requirement that  $\sum_{j=2}^{N+1} (\kappa_j^N)^2$  is kept constant.

[16] G. Vidal and R. F. Werner, *Phys. Rev. A* **65**, 032314 (2002).

[17] Compare S. Bose *et al.*, *Phys. Rev. Lett.* **87**, 050401 (2001).