# FINITE ELEMENT APPROXIMATION OF A DEGENERATE ALLEN-CAHN/CAHN-HILLIARD SYSTEM

JOHN W. BARRETT[†][§] AND JAMES F. BLOWEY[‡][¶]

**Abstract.** We consider a fully practical finite element approximation of an Allen-Cahn/Cahn-Hilliard system with a degenerate mobility and a logarithmic free energy. This system arises in the modelling of phase separation and ordering in binary alloys. In addition to showing well-posedness and stability bounds for our approximation, we prove convergence in one space dimension. Finally some numerical experiments are presented.

**Key words.** degenerate second/fourth order parabolic equations, degenerate parabolic system, Allen-Cahn/Cahn-Hilliard, order-disorder, phase separation, finite elements, convergence analysis.

**AMS subject classifications.** 35B65, 35G25, 35K50, 35K55, 35K65, 65M12, 65M60, 82C26.

**1. Introduction.** Let $\Omega$ be a bounded domain in $\mathbf{R}^d, d \leq 3$, with a Lipschitz boundary $\partial\Omega$. We consider the following Allen-Cahn/Cahn-Hilliard system with degenerate mobility and logarithmic free energy:
**(P)** Find $\{u(x,t), v(x,t), w(x,t), z(x,t)\}$ such that

$$\frac{\partial u}{\partial t} = \nabla \,.\, (b(u,v)\nabla w) \qquad \text{in } \Omega_T := \Omega \times (0,T), \tag{1.1a}$$
$$\rho\,\frac{\partial v}{\partial t} = -\,b(u,v)\,z \qquad \text{in } \Omega_T, \tag{1.1b}$$
$$w = -\gamma\,\Delta u + \theta\,[\,\phi(u+v) + \phi(u-v)\,] - \alpha\,u \qquad \text{in } \Omega_T, \tag{1.1c}$$
$$z = -\gamma\,\Delta v + \theta\,[\,\phi(u+v) - \phi(u-v)\,] - \beta\,v \qquad \text{in } \Omega_T, \tag{1.1d}$$
$$u(x,0) = u^0(x), \quad v(x,0) = v^0(x) \qquad \forall\, x \in \Omega, \tag{1.1e}$$
$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = b(u,v)\,\frac{\partial w}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (0,T); \tag{1.1f}$$

where $\nu$ is normal to $\partial\Omega$ and $\rho$, $\gamma$, $\theta$, $\alpha$ and $\beta$ are given positive constants. On introducing $\Phi \in C[0,1]$ such that

$$\Phi(s) := \Phi^+(s) + \Phi^+(1-s), \qquad \text{where } \Phi^+(s) := s\,\ln\, s; \tag{1.2a}$$

then the monotone function $\phi \in C^\infty(0,1)$ in (1.1c,d) is defined to be

$$\phi(s) := \Phi'(s) := \ln\, s - \ln\,(1-s) \equiv \phi^+(s) - \phi^+(1-s), \tag{1.2b}$$

where $\phi^+ \equiv (\Phi^+)'$. The singularities in $\phi$ at 0 and 1 mean that $w(x,t)$ and $z(x,t)$ are only well-defined for $(u \pm v)(x,t) \in (0,1)$; that is, for $\{u(x,t), v(x,t)\} \in \mathcal{Q}$, where $\mathcal{Q} := \{\,\{s_1, s_2\} \in \mathbf{R}^2 : 0 < s_1 + s_2 < 1,\ 0 < s_1 - s_2 < 1\,\}$. The mobility $b \in C(\overline{\mathcal{Q}})$ in (1.1a,b) is degenerate and is defined by

$$b(s_1, s_2) := b_1(s_1)\,b_2(s_2) \qquad \forall\, \{s_1, s_2\} \in \overline{\mathcal{Q}}, \tag{1.3a}$$
$$\text{where} \qquad b_1(s) := b_2(s - \tfrac{1}{2}), \qquad b_2(s) := \tfrac{1}{4} - s^2. \tag{1.3b}$$

[†]Department of Mathematics, Imperial College, London, SW7 2BZ, U.K. (j.barrett@ic.ac.uk), phone: +44 (0)20 7594 8529, fax: +44 (0)20 7594 8517.
[§]Partially supported by the EPSRC, U.K. through grant GR/M29689.
[‡]Department of Mathematical Sciences, University of Durham, Durham, DH1 3LE, U.K. (j.f.blowey@durham.ac.uk), phone: +44 (0)191 3744532, fax: +44 (0)191 3747388.
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On introducing the total free energy

$$\mathcal{E}(u,v) := \int_\Omega \left\{ \tfrac{1}{2}\gamma \left[ |\nabla u|^2 + |\nabla v|^2 \right] + \Psi_\theta(u,v) \right\} \mathrm{d}x, \tag{1.4a}$$

where

$$\Psi_\theta(u,v) := \theta \left[ \Phi(u+v) + \Phi(u-v) \right] + \tfrac{1}{2} \left[ \alpha\, u\,(1-u) - \beta\, v^2 \right] \tag{1.4b}$$

is the homogeneous free energy; it follows that $\frac{\delta\mathcal{E}}{\delta u} = w + \frac{1}{2}\alpha$ and $\frac{\delta\mathcal{E}}{\delta v} = z$. On noting this and that (1.1a,f) implies that $\int_\Omega \frac{\partial u}{\partial t}\,\mathrm{d}x = 0$, it follows that (1.1) can be viewed as a gradient flow with $\mathcal{E}$ non-increasing in time, $t$. Furthermore, the choice $u^0 \equiv \frac{1}{2}$ yields that $u \equiv \frac{1}{2}$ and $w \equiv -\frac{1}{2}\alpha$. Hence (1.1d) collapses to $z = -\gamma\Delta v + 2\theta\phi(\frac{1}{2}+v) - \beta v$ and the system (P) to a logarithmic Allen-Cahn equation with a degenerate mobility. Whereas, the choice $v^0 \equiv 0$ yields that $v \equiv z \equiv 0$. Hence (1.1c) collapses to $w = -\gamma\Delta u + 2\theta\phi(u) - \alpha u$ and the system (P) to a logarithmic Cahn-Hilliard equation with a degenerate mobility. Therefore for general initial data $\{u^0(x), v^0(x)\} \in \mathcal{Q}$, for all $x \in \Omega$, (P) can be considered as a system encompassing both the logarithmic Allen-Cahn and Cahn-Hilliard equations with a degenerate mobility.

The system (1.1) was derived in [6] to model the simultaneous order-disorder and phase separation in binary alloys on a BCC lattice, for example in Fe-Al alloys. Here $u$ denotes the average concentration of one of the components and, as noted above, is a conserved quantity; and $v$ is a non-conserved order parameter. The parameter $\theta$ denotes the absolute temperature. We note that if $8\,\theta < \max\{\alpha,\beta\}$, then $\Psi_\theta$ is non-convex.

Existence, uniqueness and regularity has been established for (1.1) with constant mobility and with $\Phi(s)$ in (1.2a) replaced by the quartic $s^4$ in [5]. Existence, uniqueness and regularity has been established in [2] for (1.1) with a non-degenerate mobility, $b(s_1,s_2) \geq b^{\min} > 0$ for all $\{s_1,s_2\} \in \overline{\mathcal{Q}}$. In addition, these issues are also addressed for the deep quench limit ($\theta = 0$) of (P) in [2]. Furthermore, an error bound for a fully practical piecewise linear finite element approximation of (P) and its deep quench limit are proved in [2]. The error bound being optimal in the deep quench limit.

In [7] and [12] formal asymptotics are used to describe the long time behaviour of the system (1.1) close to the deep quench limit. We note that the local minima of $\Psi_\theta(u,v)$ are transcendentally close, $O(e^{-c/\theta})$, to $\{\frac{1}{2}, \pm\frac{1}{2}\}$, $\{0,0\}$ and $\{1,0\}$; the vertices of $\mathcal{Q}$, for $\theta \approx 0$. The first pair of local minimizers are known as ordered variants and the second pair as disordered phases. Which pair are global minimizers depends on the ordering of $\alpha$ and $\beta$, since $\Psi_0(\frac{1}{2}, \pm\frac{1}{2}) = \frac{1}{8}(\alpha-\beta)$ and $\Psi_0(0,0) = \Psi_0(1,0) = 0$. Partitions between an ordered variant and a disordered phase or between the two disordered phases are known as interphase boundaries (IPBs), whereas partitions between the two ordered variants are known as antiphase boundaries (APBs); see [7] and [12] for details.

Existence of a weak solution to (P) in one space dimension ($d = 1$) was established in [9] for the degenerate mobility $b(\cdot,\cdot)$, defined in (1.3), which vanishes in the pure regions: the two ordered variants, $\{\frac{1}{2}, \pm\frac{1}{2}\}$, and the two disordered phases comprising of just one of the two components of the alloy, $\{0,0\}$ and $\{1,0\}$. This specific choice for $b$ leads to a number of mathematical difficulties since it is degenerate. The key difficulty being that there is no uniqueness proof, as is common for fourth order nonlinear degenerate parabolic problems; see e.g. [10] for the logarithmic Cahn-Hilliard equation with a degenerate mobility ((P) with $v^0 \equiv 0$). The restriction of existence to one space dimension in [9] is because they adopt the very weak solution concept introduced in [4] for fourth order nonlinear degenerate parabolic equations. It should be noted that the degenerate system (P) is far more complicated than the corresponding degenerate Cahn-Hilliard equation. This is because for (P) the mobility $b$, (1.3), vanishes only at the vertices of $\mathcal{Q}$; whereas the derivatives of the homogeneous free energy $\Psi_\theta$, (1.4b), are singular around the whole boundary of $\mathcal{Q}$. For the corresponding degenerate Cahn-Hilliard equation ($v^0 \equiv 0 \Longrightarrow v \equiv 0$); the zeroes of $b$ and the singularities in $\Psi_\theta$ are concurrent, both occurring at $\{0,0\}$ and $\{1,0\}$.

The aim of this paper is show that the fully practical finite element approximation of (P), with a non-degenerate mobility $b$, introduced and analysed in [2] is appropriate for the degenerate case, (1.3). In fact the results in this paper can be applied to a general degenerate $b \in C^1(\overline{\mathcal{Q}})$, which vanishes at the vertices of $\mathcal{Q}$ and is strictly positive otherwise. However, for ease of exposition in some aspects of the proofs, we restrict ourselves to the model case (1.3). The layout of this paper is as follows. In the next section we introduce our finite element approximation, $(\mathrm{P}^{h,\tau})$, of (P) and prove well-posedness and stability bounds for space dimensions 1, 2 and 3. We note that $(\mathrm{P}^{h,\tau})$ with $V^0 \equiv 0$ collapses to the finite element approximation of the degenerate Cahn-Hilliard equation with logarithmic free energy analysed in [3]. In §3 we prove convergence in one space dimension. Finally, in §4 we report on some numerical experiments. Therefore this paper is the natural extension of [3] for the degenerate Cahn-Hilliard equation with logarithmic free energy to (P).

**Notation and auxiliary results.** Throughout we adopt the standard notation for Sobolev spaces, denoting the norm of $W^{m,p}(G)$ ($m \in \mathbf{N}$, $p \in [1,\infty]$ and $G$ a bounded domain in $\mathbf{R}^d$ with a Lipschitz boundary) by $\|\cdot\|_{m,p,G}$ and semi-norm by $|\cdot|_{m,p,G}$. For $p = 2$, $W^{m,2}(G)$ will be denoted by $H^m(G)$ with the associated norm and semi-norm written, as respectively, $\|\cdot\|_{m,G}$ and $|\cdot|_{m,G}$. For ease of notation, in the common case when $G \equiv \Omega$ the subscript "$\Omega$" will be dropped on the above norms and semi-norms. Throughout $(\cdot,\cdot)$ denotes the standard $L^2$ inner product over $\Omega$ and $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $(H^1(\Omega))'$ and $H^1(\Omega)$. In addition we define

$$⨍ \eta := \tfrac{1}{|\Omega|}(\eta, 1) \qquad \forall\ \eta \in L^2(\Omega).$$

For later purposes, we recall also the following well-known Sobolev interpolation results, e.g. see [1]: Let $p \in [1,\infty]$, $m \geq 1$,

$$r \in \begin{cases} [p,\infty] & \text{if } m - \frac{d}{p} > 0, \\ [p,\infty) & \text{if } m - \frac{d}{p} = 0, \\ [p, -\frac{d}{m-(d/p)}] & \text{if } m - \frac{d}{p} < 0; \end{cases}$$

and $\mu := \frac{d}{m}\left(\frac{1}{p} - \frac{1}{r}\right)$. Then there is a constant $C$ depending only on $\Omega$, $p$, $r$ and $m$ such that

$$|v|_{0,r} \leq C\, |v|_{0,p}^{1-\mu}\, \|v\|_{m,p}^{\mu} \quad \forall\ v \in W^{m,p}(\Omega). \tag{1.5}$$

It is convenient to introduce the "inverse Laplacian" operator $\mathcal{G} : \mathcal{F} \to \Xi$ such that

$$(\nabla \mathcal{G} f, \nabla \eta) = \langle f, \eta\rangle \quad \forall\ \eta \in H^1(\Omega), \tag{1.6}$$

where

$$\mathcal{F} := \left\{ f \in (H^1(\Omega))' : \langle f, 1\rangle = 0 \right\}, \qquad \Xi := \{\xi \in H^1(\Omega) : (\xi, 1) = 0\}. \tag{1.7}$$

The well-posedness of $\mathcal{G}$ follows from the Lax-Milgram theorem and the Poincaré inequality

$$|\eta|_{0,p} \leq C(\,|\eta|_{1,p} + |(\eta,1)|\,) \quad \forall\ \eta \in W^{1,p}(\Omega) \quad \text{and} \quad p \in [1,\infty]. \tag{1.8}$$

One can define a norm on $\mathcal{F}$ by

$$\|f\|_{-1} := |\mathcal{G}f|_1 \equiv \langle f, \mathcal{G}f\rangle^{\frac{1}{2}} \quad \forall\ f \in \mathcal{F}. \tag{1.9}$$

Throughout $C$ denotes a generic constant independent of the three key parameters, $\varepsilon$, $h$ and $\tau$; the regularization, mesh and temporal discretization parameters, respectively.

**2. Finite element approximation.** We consider the finite element approximation of (P) under the following assumptions on the mesh:

**(A)** Let $\Omega$ be a convex polyhedron. Let $\mathcal{T}^h$ be a quasi-uniform partitioning of $\Omega$ into disjoint open simplices $\kappa$ with $h_\kappa := \text{diam}(\kappa)$ and $h := \max_{\kappa\in\mathcal{T}^h} h_\kappa$, so that $\overline{\Omega} = \cup_{\kappa\in\mathcal{T}^h}\overline{\kappa}$. In addition, it is assumed that $\mathcal{T}^h$ is a (weakly) acute partitioning; that is, for (a) $d = 2$ the sum of opposite angles relative to any side does not exceed $\pi$, (b) $d = 3$ the angle between any two faces of the same tetrahedron does not exceed $\pi/2$. Let $\{x_j\}_{j=0}^J$ be the co-ordinates of the nodes of $\mathcal{T}^h$. Associated with $\mathcal{T}^h$ is the finite element space

$$S^h := \{\chi \in C(\overline{\Omega}) : \chi\,|_\kappa \text{ is linear } \forall\ \kappa \in \mathcal{T}^h\} \subset H^1(\Omega).$$

It is convenient to introduce also for a given $m \in (0,1)$

$$S^h(m) := \{\,\chi \in S^h : {\textstyle\fint}\,\chi = m\,\} \tag{2.1a}$$

$$\text{and} \qquad S^h_I := \{\,\chi \in S^h : \chi(x_j) \in (0,1)\ j = 0 \to J\,\}. \tag{2.1b}$$

Let $\{\Theta_j\}_{j=0}^J$ be the standard basis functions for $S^h$; that is, $\Theta_j \in S^h$ and $\Theta_j(x_i) = \delta_{ij}$ for all $i$, $j = 0 \to J$. We introduce $\pi^h : C(\overline{\Omega}) \to S^h$, the interpolation operator, such that $\pi^h\eta(x_j) = \eta(x_j)$, $j = 0 \to J$. A discrete semi-inner product on $C(\overline{\Omega})$, is defined by

$$(\eta_1,\eta_2)^h := \int_\Omega \pi^h(\eta_1\,\eta_2)\,\mathrm{d}x \equiv \sum_{j=0}^J m_j\,\eta_1(x_j)\,\eta_2(x_j), \tag{2.2}$$

where $m_j := (1,\Theta_j)$. We introduce also

$$|\eta|_h := [\,(\eta,\eta)^h\,]^{\frac{1}{2}} \qquad \forall\ \eta \in C(\overline{\Omega}). \tag{2.3}$$

In addition to the interpolation operator $\pi^h$, we introduce the $L^2$ projection $Q^h$ : $L^2(\Omega) \to S^h$ and the more practical $\widehat{Q}^h : L^2(\Omega) \to S^h$ such that

$$(Q^h\eta,\chi) = (\widehat{Q}^h\eta,\chi)^h = (\eta,\chi) \quad \forall\ \chi \in S^h. \tag{2.4}$$

It follows from (2.4) that

$$(\widehat{Q}^h\eta)(x_j) \equiv \frac{(\eta,\Theta_j)}{(1,\Theta_j)} \quad j = 0 \to J \qquad \Longrightarrow$$
$$-\,\|[\eta]_-\|_{0,\infty} \leq \widehat{Q}^h\eta(x) \leq \|[\eta]_+\|_{0,\infty} \quad \forall\ x \in \overline{\Omega}, \quad \forall\ \eta \in L^\infty(\Omega); \tag{2.5}$$

where $[\,\cdot\,]_- := \min\{\cdot,0\}$ and $[\,\cdot\,]_+ := \max\{\cdot,0\}$.

Let $0 \equiv t_0 < t_1 < \cdots\, t_{N-1} < t_N \equiv T$ be a partitioning of $[0,T]$ into possibly variable time steps $\tau_n := t_n - t_{n-1}$, $n = 1 \to N$. Let $\tau := \max_{n=1\to N} \tau_n$. In this paper we consider the following fully practical finite element approximation of (P):

**(P$^{h,\tau}$)** For $n = 1 \to N$ find $\{U^n, V^n, W^n, Z^n\} \in [S^h]^4$ such that

$$\left(\tfrac{U^n-U^{n-1}}{\tau_n}, \chi\right)^h + \left(\pi^h[b(U^{n-1},V^{n-1})]\,\nabla W^n, \nabla\chi\right) = 0 \quad \forall\, \chi \in S^h, \tag{2.6a}$$

$$\rho\,\left(\tfrac{V^n-V^{n-1}}{\tau_n}, \chi\right)^h + \left(b(U^{n-1},V^{n-1})\,Z^n, \chi\right)^h = 0 \quad \forall\, \chi \in S^h, \tag{2.6b}$$

$$\gamma\,(\nabla U^n, \nabla\chi) + \theta\,(\phi(U^n+V^n) + \phi(U^n-V^n), \chi)^h = (W^n + \alpha\, U^{n-1}, \chi)^h \quad \forall\, \chi \in S^h, \tag{2.6c}$$

$$\gamma\,(\nabla V^n, \nabla\chi) + \theta\,(\phi(U^n+V^n) - \phi(U^n-V^n), \chi)^h = (Z^n + \beta\, V^{n-1}, \chi)^h \quad \forall\, \chi \in S^h; \tag{2.6d}$$

where $\{U^0, V^0\} \in [S^h]^2$, satisfying $U^0 \pm V^0 \in S^h_I$, are approximations to $u^0$, $v^0 \in H^1(\Omega)$, satisfying $(u^0 \pm v^0)(x) \in (0,1)$ for *a.e.* $x \in \Omega$; e.g. $U^0 \equiv \widehat{Q}^h u^0$, $V^0 \equiv \widehat{Q}^h v^0$ on noting (2.4) and (2.5), or if $d = 1$ $U^0 \equiv \pi^h u^0$, $V^0 \equiv \pi^h v^0$.

Remark. The discretisation (2.6) is more practical than the one studied in [2] in that there (i) $b(U^{n-1}, V^{n-1})$ is used instead of $\pi^h[b(U^{n-1}, V^{n-1})]$ in (2.6a), (ii) numerical integration is not used on the $(b(U^{n-1}, V^{n-1})\, Z^n, \chi)$ term in (2.6b). We note that the scheme in [2] requires polynomials of degree six to be integrated exactly over each simplex $\kappa$ even in the case of the model mobility (1.3). It is for this reason that we consider the more practical approximation (2.6). Whereas the stability and convergence results in this paper are easily adapted to the less practical scheme in [2], it is not clear that the error bounds established there for a non-degenerate mobility carry across to the more practical scheme (2.6). Finally we remark that it is the less practical scheme in [2] with $V^0 \equiv 0$ that collapses to the finite element approximation of the degenerate Cahn-Hilliard equation with logarithmic free energy analysed in [3], which for the model mobility (1.3) only requires quadratics to be integrated exactly over each simplex $\kappa$.

Below we recall some well-known results concerning $S^h$:

$$|(I - \widehat{Q}^h)\eta|_0 + h\,|(I - \widehat{Q}^h)\eta|_1 \le C\,h\,|\eta|_1 \quad \forall\, \eta \in H^1(\Omega). \tag{2.7}$$

$$\int_\kappa \chi^2 \,\mathrm{d}x \le \int_\kappa \pi^h[\chi^2]\,\mathrm{d}x \le (d+2) \int_\kappa \chi^2 \,\mathrm{d}x \quad \forall\, \chi \in S^h. \tag{2.8}$$

$$\left| \int_\kappa (I - \pi^h)(\eta^h\,\chi)\,\mathrm{d}x \right| \le C\,h^{1+m}\,\|\eta^h\|_{m,\kappa}\,\|\chi\|_{1,\kappa} \quad \forall\, \eta^h, \chi \in S^h, \quad m = 0 \text{ or } 1. \tag{2.9}$$

If $d = 1$

$$|(I - \pi^h)\eta|_{m,\kappa} \le C\,h^{1-m}\,|\eta|_{1,\kappa} \quad \forall\, \eta \in H^1(\kappa), \quad m = 0 \text{ or } 1; \tag{2.10}$$

$$\lim_{h\to 0} \|(I - \pi^h)\eta\|_1 = 0 \quad \forall\, \eta \in H^1(\Omega). \tag{2.11}$$

If $d = 1$, then a simple consequence of (2.9) and (2.10) is that

$$\begin{aligned}
&\left| \int_\kappa (I - \pi^h)(f\,\eta)\,\mathrm{d}x \right| \\
&\quad = \left| \int_\kappa [\,(I-\pi^h)(\,(\pi^h f)\,(\pi^h \eta)\,) + (\pi^h \eta)\,(I-\pi^h) f + f\,(I-\pi^h)\eta\,]\,\mathrm{d}x \right| \\
&\quad \leq C\,[\,|(I-\pi^h) f|_{0,\kappa} + h\,|f|_{0,\kappa}\,]\,\|\eta\|_{1,\kappa} \qquad \forall\ f \in C(\overline{\kappa}), \quad \forall\ \eta \in H^1(\kappa). \qquad (2.12)
\end{aligned}$$

In addition, we introduce the "discrete-Laplacian" operator $\Delta^h : S^h \to S^h$ such that

$$(\Delta^h \eta^h, \chi)^h = -(\nabla \eta^h, \nabla \chi) \quad \forall\ \chi \in S^h. \qquad (2.13)$$

Similarly to (1.6), we introduce the operator $\widehat{\mathcal{G}}^h : \mathcal{F}^c \to \Xi^h$ such that

$$(\nabla \widehat{\mathcal{G}}^h f, \nabla \chi) = (f, \chi)^h \quad \forall\ \chi \in S^h, \qquad (2.14)$$

where $\Xi^h := \{\xi^h \in S^h : (\xi^h, 1) = 0\} \subset \mathcal{F}^c := \{f \in C(\overline{\Omega}) : (f,1)^h = 0\}$. We have for all $\mu > 0$ that

$$(f,\chi)^h \equiv (\nabla \widehat{\mathcal{G}}^h f, \nabla \chi) \leq |\widehat{\mathcal{G}}^h f|_1\,|\chi|_1 \leq \tfrac{1}{2\mu}\,|\widehat{\mathcal{G}}^h f|_1^2 + \tfrac{\mu}{2}\,|\chi|_1^2 \quad \forall\ f \in \mathcal{F}^h,\ \forall\ \chi \in S^h. \qquad (2.15)$$

For $q_1^h,\ q_2^h \in S^h$ with $q_1^h \pm q_2^h \in S_I^h$, on noting (1.3) we have that

$$\begin{aligned}
b^{\min}_{q_1^h,q_2^h} &:= \min_{x\in\overline{\Omega}} b(q_1^h(x), q_2^h(x)) > 0 \\
\text{and} \qquad b^{\max}_{q_1^h,q_2^h} &:= \max_{x\in\overline{\Omega}} b(q_1^h(x), q_2^h(x)) \leq b(\tfrac{1}{2}, 0) = \tfrac{1}{16}. \qquad (2.16)
\end{aligned}$$

We then introduce $\widehat{\mathcal{G}}^h_{q_1^h,q_2^h} : \mathcal{F}^c \to \Xi^h$ and $\widehat{\mathcal{M}}^h_{q_1^h,q_2^h} : C(\overline{\Omega}) \to S^h$ such that

$$(\pi^h[b(q_1^h,q_2^h)]\,\nabla \widehat{\mathcal{G}}^h_{q_1^h,q_2^h} f, \nabla \chi) = (b(q_1^h,q_2^h)\,\widehat{\mathcal{M}}^h_{q_1^h,q_2^h} f, \chi)^h = (f,\chi)^h \quad \forall\ \chi \in S^h. \quad (2.17)$$

Their well-posedness follows from (2.16) and (1.8). It is easily deduced from (2.16), (2.14), (2.17), (2.3) and (2.2) that

$$\begin{aligned}
|\nabla \widehat{\mathcal{G}}^h f|_0^2 &\leq b^{\max}_{q_1^h,q_2^h}\,|[\pi^h[b(q_1^h,q_2^h)]\,]^{\frac{1}{2}} \nabla \widehat{\mathcal{G}}^h_{q_1^h,q_2^h} f|_0^2 \qquad \forall\ f \in \mathcal{F}^c, \qquad (2.18a)\\
|\chi|_h^2 &\leq b^{\max}_{q_1^h,q_2^h}\,|[b(q_1^h,q_2^h)]^{\frac{1}{2}} \widehat{\mathcal{M}}^h_{q_1^h,q_2^h} \chi|_h^2 \qquad \forall\ \chi \in S^h. \qquad (2.18b)
\end{aligned}$$

Clearly (2.6c,d) implicitly imply that $U^n \pm V^n \in S_I^h$. We will show that $(\mathrm{P}^{h,\tau})$ has the property that $U^0 \pm V^0 \in S_I^h$ implies that $U^n \pm V^n \in S_I^h$ for all $n \geq 1$. We prove well-posedness of this approximation via a regularization procedure. The logarithmic convex function $\Phi$ is replaced for $\varepsilon \in (0, \frac{1}{2})$ by the twice continuously differentiable convex function

$$\Phi_\varepsilon(s) := \Phi_\varepsilon^+(s) + \Phi_\varepsilon^+(1-s), \qquad (2.19a)$$

$$\text{where} \qquad \Phi_\varepsilon^+(s) := \begin{cases} \frac{1}{2\varepsilon}\,(s^2 - \varepsilon^2) + s\,\ln\varepsilon & \text{if } s \leq \varepsilon, \\ \Phi^+(s) & \text{if } \varepsilon \leq s. \end{cases} \qquad (2.19b)$$

We note for future reference that

$$\Phi_\varepsilon(s) \geq \begin{cases} \frac{1}{2\,\varepsilon}\,(\,[s]_-^2 + [s-1]_+^2 - \varepsilon^2\,) & \text{if } s \leq 0 \text{ or } s \geq 1, \\ \Phi(\frac{1}{2}) = -\ln 2 & \text{if } s \in [0,1]. \end{cases} \tag{2.20}$$

The monotone function $\phi_\varepsilon \in C^1(\mathbf{R})$ defined by

$$\phi_\varepsilon(s) := \Phi'_\varepsilon(s) \equiv \phi_\varepsilon^+(s) - \phi_\varepsilon^+(1-s), \qquad \text{where } \phi_\varepsilon^+(s) := (\Phi_\varepsilon^+)'(s), \tag{2.21}$$

has the following properties: For $\varepsilon \in (0, \frac{1}{4})$ and for all $r,\, s \in \mathbf{R}$

$$\begin{aligned}(\phi_\varepsilon(r) - \phi_\varepsilon(s))^2 &\leq \phi'_\varepsilon(\tfrac{1}{2} + \max\{|r-\tfrac{1}{2}|, |s-\tfrac{1}{2}|\})(\phi_\varepsilon(r) - \phi_\varepsilon(s))(r-s) \\ &\leq \tfrac{4}{3\varepsilon}(\phi_\varepsilon(r) - \phi_\varepsilon(s))(r-s).\end{aligned} \tag{2.22}$$

As the partitioning $\mathcal{T}^h$ is (weakly) acute it follows from (2.22) that for all $\varepsilon \in (0, \frac{1}{4})$

$$\begin{aligned}|\nabla \pi^h[\phi_\varepsilon(\chi)]\,|_0^2 &\leq \phi'_\varepsilon(\tfrac{1}{2} + \|\chi - \tfrac{1}{2}\|_{0,\infty})\,(\nabla \chi, \nabla \pi^h[\phi_\varepsilon(\chi)]) \\ &\leq \tfrac{4}{3}\,\varepsilon^{-1}\,(\nabla \chi, \nabla \pi^h[\phi_\varepsilon(\chi)]) \quad \forall\, \chi \in S^h,\end{aligned} \tag{2.23}$$

see [8] and [11, §2.4.2].

For later use we need to bound below $\Psi_{\theta,\varepsilon}$, the corresponding regularized version of $\Psi_\theta$ (see (1.4b)). To achieve this we first note from using a Young's inequality that for all $\{s_1, s_2\} \in \mathbf{R}^2$

$$\begin{aligned}\alpha\, s_1\,(1-s_1) - \beta\, s_2^2 &= \tfrac{1}{4}\,[2\,\alpha\,(r_1 + r_2) - (\alpha+\beta)\,(r_1^2 + r_2^2) + 2\,(\beta-\alpha)\,r_1\,r_2\,] \\ &\geq \tfrac{1}{2}\sum_{i=1}^{2}\,[\,\alpha\, r_i - \max\{\alpha,\beta\}\,r_i^2\,],\end{aligned} \tag{2.24a}$$

where $r_1 := s_1 + s_2$ and $r_2 := s_1 - s_2$. Next we note, again from using a Young's inequality, that for all $r \in \mathbf{R}$

$$\begin{aligned}2\,[\,\alpha\, r - \max\{\alpha,\beta\}\,r^2\,] \geq &- \max\{\alpha, 4\beta - 3\alpha\} \\ &- \max\{3\alpha, 4\beta-\alpha\}\,(\,[r]_-^2 + [r-1]_+^2\,).\end{aligned} \tag{2.24b}$$

Combining (2.20) and (2.24) we have for all $\varepsilon < \varepsilon_0 := \min\{\frac{1}{4}, \frac{2\theta}{\max\{3\alpha, 4\beta-\alpha\}}\}$ and for all $\{s_1, s_2\} \in \mathbf{R}^2$ that

$$\begin{aligned}\Psi_{\theta,\varepsilon}(s_1, s_2) &:= \theta\,[\,\Phi_\varepsilon(s_1 + s_2) + \Phi_\varepsilon(s_1 - s_2)\,] + \tfrac{1}{2}\,[\,\alpha\, s_1\,(1-s_1) - \beta\, s_2^2\,] \\ &\geq -\tfrac{1}{4}\,\max\{\alpha, 4\beta - 3\alpha\} - \theta\,(\,\tfrac{1}{4} + 2\,\ln\,2\,) + \tfrac{\theta}{4\varepsilon}\,(\,[s_1 + s_2]_-^2 \\ &\quad + \,[s_1 - s_2]_-^2 + [s_1 + s_2 - 1]_+^2 + [s_1 - s_2 - 1]_+^2\,).\end{aligned} \tag{2.25}$$

Finally, we introduce the discrete Lyapunov functional $\mathcal{E}^h_{(\varepsilon)} : [S^h]^2 \to \mathbf{R}$ defined by

$$\mathcal{E}^h_{(\varepsilon)}(\eta_1^h, \eta_2^h) := \tfrac{\gamma}{2}\,[\,|\eta_1^h|_1^2 + |\eta_2^h|_1^2\,] + (\Psi_{\theta(,\varepsilon)}(\eta_1^h, \eta_2^h), 1)^h \quad \forall\, \eta_1^h,\, \eta_2^h \in S^h, \tag{2.26}$$

where the notation "$\cdot_{(\varepsilon)}$" is an abbreviation for either "with" or "without" the subscript "$\varepsilon$". The following theorem is an adaption of Theorem 4.1 in [2] for a non-degenerate mobility to the degenerate case.

THEOREM 2.1. *Let $\Omega$ and $\mathcal{T}^h$ be such that the assumption (A) holds. Let $\{U^0, V^0\} \in S^h(m^h) \times S^h$, with $m^h \in (0,1)$, be such that $U^0 \pm V^0 \in S^h_I$. Then for all time partitions $\{\tau_n\}_{n=1}^N$ of $[0,T]$ there exists a unique solution $\{U^n, V^n, W^n, Z^n\}_{n=1}^N$ to $(P^{h,\tau})$ such that $U^n \in S^h(m^h)$ and $U^n \pm V^n \in S^h_I$, $n = 1 \to N$, and*

$$\begin{aligned}
&\max_{n=0\to N} [\, \|U^n\|_1^2 + \|V^n\|_1^2 \,] + \sum_{n=1}^N [\, \|U^n - U^{n-1}\|_1^2 + |V^n - V^{n-1}|_1^2 \,] \\
&\quad + \sum_{n=1}^N \tau_n \, [\, b^{\max}_{n-1} ]^{-1} [\, |\tfrac{V^n - V^{n-1}}{\tau_n}|_0^2 + \|\widehat{\mathcal{G}}^h \left(\tfrac{U^n - U^{n-1}}{\tau_n}\right) \|_1^2 \,] \\
&\quad + \sum_{n=1}^N \tau_n \, [\, |\, [\pi^h [b(U^{n-1}, V^{n-1})]\,]^{\frac{1}{2}} \nabla W^n |_0^2 + |\, [b(U^{n-1}, V^{n-1})]^{\frac{1}{2}} \, Z^n |_h^2 \,] \\
&\quad + \min\{(m^h)^2, \, (1-m^h)^2\} \sum_{n=1}^N \tau_n \, b^{\min}_{n-1} \, [\, \theta^2 \, |\, \pi^h [\phi(U^n \pm V^n)] \,|_0^2 + |\Lambda^n|^2 \,] \\
&\qquad\qquad \le C \, [\, 1 + |U^0|_1^2 + |V^0|_1^2 \,],
\end{aligned} \tag{2.27}$$

*where $b^{\min}_{n-1} := b^{\min}_{U^{n-1}, V^{n-1}} > 0$ and $b^{\max}_{n-1} := b^{\max}_{U^{n-1}, V^{n-1}} \le b(\frac{1}{2}, 0) = \frac{1}{16}$.*

*Proof.* For $n = 1 \to N$, given $U^{n-1} \in S^h(m^h)$ and $U^{n-1} \pm V^{n-1} \in S^h_I$ firstly we prove existence of $\{U^n, V^n, W^n, Z^n\}$ satisfying (2.6) and show that $U^n \in S^h(m^h)$ and $U^n \pm V^n \in S^h_I$. Choosing $\chi \equiv 1$ in (2.6a) yields that $⨍ \, U^n = ⨍ \, U^{n-1} \Longrightarrow U^n \in S^h(m^h)$. As $U^{n-1} \pm V^{n-1} \in S^h_I$, it follows from (2.6a,b), (2.16), (2.17), (1.8) and (2.6c) with $\chi \equiv 1$ that

$$W^n \equiv -\widehat{\mathcal{G}}^h_{n-1} \left(\tfrac{U^n - U^{n-1}}{\tau_n}\right) + \Lambda^n, \quad Z^n \equiv -\rho \, \widehat{\mathcal{M}}^h_{n-1} \left(\tfrac{V^n - V^{n-1}}{\tau_n}\right) \tag{2.28a}$$

$$\text{and} \quad \Lambda^n := ⨍ \, \pi^h [\, \theta \, \phi(U^n + V^n) + \theta \, \phi(U^n - V^n) - \alpha \, U^{n-1} \,]; \tag{2.28b}$$

where for notational convenience we set $\widehat{\mathcal{G}}^h_{n-1} \equiv \widehat{\mathcal{G}}^h_{U^{n-1}, V^{n-1}}$ and $\widehat{\mathcal{M}}^h_{n-1} \equiv \widehat{\mathcal{M}}^h_{U^{n-1}, V^{n-1}}$. Therefore on substituting (2.28a) into (2.6c,d) and on adding and subtracting the resulting equations, (2.6) can be rewritten as:
Find $\{U^n, V^n\} \in S^h(m^h) \times S^h$ such that

$$\begin{aligned}
&\gamma \, (\nabla (U^n \pm V^n), \nabla \chi) + (\widehat{\mathcal{G}}^h_{n-1} \left(\tfrac{U^n - U^{n-1}}{\tau_n}\right) - \Lambda^n \pm \rho \, \widehat{\mathcal{M}}^h_{n-1} \left(\tfrac{V^n - V^{n-1}}{\tau_n}\right), \chi)^h \\
&\quad + (2 \, \theta \, \phi(U^n \pm V^n) - (\alpha \, U^{n-1} \pm \beta \, V^{n-1}), \chi)^h = 0 \quad \forall \, \chi \in S^h,
\end{aligned} \tag{2.29}$$

where $\Lambda^n$ is defined by (2.28b) and $\{W^n, Z^n\}$ can be obtained from (2.28a). To prove existence of a solution to (2.29), and hence $(\mathrm{P}^{h,\tau})$, we consider the regularized version obtained by replacing the singular $\phi$ by $\phi_\varepsilon$ for any chosen $\varepsilon \in (0, \varepsilon_0)$:
Find $\{U^n_\varepsilon, V^n_\varepsilon\} \in S^h(m^h) \times S^h$ such that

$$\begin{aligned}
&\gamma \, (\nabla (U^n_\varepsilon \pm V^n_\varepsilon), \nabla \chi) + (\widehat{\mathcal{G}}^h_{n-1} \left(\tfrac{U^n_\varepsilon - U^{n-1}}{\tau_n}\right) \pm \rho \, \widehat{\mathcal{M}}^h_{n-1} \left(\tfrac{V^n_\varepsilon - V^{n-1}}{\tau_n}\right), \chi)^h \\
&\quad + (2 \, \theta \, \phi_\varepsilon(U^n_\varepsilon \pm V^n_\varepsilon) - \Lambda^n_\varepsilon - (\alpha \, U^{n-1} \pm \beta \, V^{n-1}), \chi)^h = 0 \quad \forall \, \chi \in S^h,
\end{aligned} \tag{2.30}$$

$$\text{where} \quad \Lambda^n_\varepsilon := ⨍ \, \pi^h [\, \theta \, \phi_\varepsilon(U^n_\varepsilon + V^n_\varepsilon) + \theta \, \phi_\varepsilon(U^n_\varepsilon - V^n_\varepsilon) - \alpha \, U^{n-1} \,]. \tag{2.31}$$

Existence and uniqueness of $\{U_\varepsilon^n, V_\varepsilon^n\}$ follows by noting that (2.30) is the Euler-Lagrange equation of the strictly convex minimization problem

$$\min_{\eta_1^h \in S^h(m^h),\, \eta_2^h \in S^h} \Big\{ \tfrac{\gamma}{2} [\, |\eta_1^h|_1^2 + |\eta_2^h|_1^2 \,] + \theta\, (\, \Phi_\varepsilon(\eta_1^h + \eta_2^h) + \Phi_\varepsilon(\eta_1^h - \eta_2^h), 1)^h$$
$$+ \tfrac{1}{2\tau_n} \,|\, [\pi^h[b_{n-1}]\,]^{\frac{1}{2}} \nabla \widehat{\mathcal{G}}_{n-1}^h(\eta_1^h - U^{n-1}) \,|_0^2 - \alpha\, (U^{n-1}, \eta_1^h)^h$$
$$+ \tfrac{\rho}{2\tau_n} \,|\, b_{n-1}^{\frac{1}{2}} \widehat{\mathcal{M}}_{n-1}^h(\eta_2^h - V^{n-1}) \,|_h^2 - \beta\, (V^{n-1}, \eta_2^h)^h \Big\}, \tag{2.32}$$

where for notational convenience we set $b_{n-1} \equiv b(U^{n-1}, V^{n-1})$. Choosing $\chi \equiv (U_\varepsilon^n - U^n) \pm (V_\varepsilon^n - V^n) \in S^h$ in (2.30), then adding together these $\pm$ versions yields that

$$\gamma\, [\, (\nabla U_\varepsilon^n, \nabla (U_\varepsilon^n - U^{n-1})) + (\nabla V_\varepsilon^n, \nabla (V_\varepsilon^n - V^{n-1}))\,]$$
$$+ \theta\, (\phi_\varepsilon(U_\varepsilon^n + V_\varepsilon^n), (U_\varepsilon^n - U^{n-1}) + (V_\varepsilon^n - V^{n-1}))^h$$
$$+ \theta\, (\phi_\varepsilon(U_\varepsilon^n - V_\varepsilon^n), (U_\varepsilon^n - U^{n-1}) - (V_\varepsilon^n - V^{n-1}))^h$$
$$+ \tau_n \,|\, [\pi^h[b_{n-1}]\,]^{\frac{1}{2}} \nabla \widehat{\mathcal{G}}_{n-1}^h \left(\tfrac{U_\varepsilon^n - U^{n-1}}{\tau_n}\right) |_0^2 + \tau_n\, \rho \,|\, b_{n-1}^{\frac{1}{2}} \widehat{\mathcal{M}}_{n-1}^h \left(\tfrac{V_\varepsilon^n - V^{n-1}}{\tau_n}\right) |_h^2$$
$$= \alpha\, (U^{n-1}, U_\varepsilon^n - U^{n-1})^h + \beta\, (V^{n-1}, V_\varepsilon^n - V^{n-1})^h. \tag{2.33}$$

Rearranging (2.33) on noting the convexity of $\Phi_\varepsilon$ and the identity

$$2(s-r)s = s^2 - r^2 + (s-r)^2 \quad \forall\, r,\, s \in \mathbf{R}, \tag{2.34}$$

yields that

$$\tfrac{\gamma}{2} [\, |U_\varepsilon^n|_1^2 + |V_\varepsilon^n|_1^2 - |U^{n-1}|_1^2 - |V^{n-1}|_1^2 + |U_\varepsilon^n - U^{n-1}|_1^2 + |V_\varepsilon^n - V^{n-1}|_1^2 \,]$$
$$+ \theta\, (\Phi_\varepsilon(U_\varepsilon^n + V_\varepsilon^n) + \Phi_\varepsilon(U_\varepsilon^n - V_\varepsilon^n), 1)^h$$
$$- \theta\, (\Phi_\varepsilon(U^{n-1} + V^{n-1}) + \Phi_\varepsilon(U^{n-1} - V^{n-1}), 1)^h$$
$$+ \tau_n \,|\, [\pi^h[b_{n-1}]\,]^{\frac{1}{2}} \nabla \widehat{\mathcal{G}}_{n-1}^h \left(\tfrac{U_\varepsilon^n - U^{n-1}}{\tau_n}\right) |_0^2 + \tau_n\, \rho \,|\, b_{n-1}^{\frac{1}{2}} \widehat{\mathcal{M}}_{n-1}^h \left(\tfrac{V_\varepsilon^n - V^{n-1}}{\tau_n}\right) |_h^2$$
$$\leq \alpha\, (U^{n-1}, U_\varepsilon^n - U^{n-1})^h + \beta\, (V^{n-1}, V_\varepsilon^n - V^{n-1})^h$$
$$\leq \tfrac{\alpha}{2} [\, |U_\varepsilon^n|_h^2 - |U^{n-1}|_h^2 \,] + \tfrac{\beta}{2} [\, |V_\varepsilon^n|_h^2 - |V^{n-1}|_h^2 \,]. \tag{2.35}$$

On noting (2.26), (2.25), (2.19), $U_\varepsilon^n - U^{n-1} \in \Xi^h$ and our assumptions on $U^{n-1}$, $V^{n-1}$; it follows from (2.35) that

$$\mathcal{E}_\varepsilon^h(U_\varepsilon^n, V_\varepsilon^n) + \tfrac{\gamma}{2} [\, |U_\varepsilon^n - U^{n-1}|_1^2 + |V_\varepsilon^n - V^{n-1}|_1^2 \,]$$
$$+ \tau_n \,|\, [\pi^h[b_{n-1}]\,]^{\frac{1}{2}} \nabla \widehat{\mathcal{G}}_{n-1}^h \left(\tfrac{U_\varepsilon^n - U^{n-1}}{\tau_n}\right) |_0^2 + \tau_n\, \rho \,|\, b_{n-1}^{\frac{1}{2}} \widehat{\mathcal{M}}_{n-1}^h \left(\tfrac{V_\varepsilon^n - V^{n-1}}{\tau_n}\right) |_h^2$$
$$\leq \mathcal{E}_\varepsilon^h(U^{n-1}, V^{n-1}) \leq C\, [\, 1 + |U^{n-1}|_1^2 + |V^{n-1}|_1^2 \,]. \tag{2.36}$$

Hence on noting (2.36), (2.26) and (2.25) there exist positive constants $C_i$, independent of $\varepsilon$, $h$ and $\tau_n$, such that

$$C_1\, [\, \|U_\varepsilon^n\|_1^2 + \|V_\varepsilon^n\|_1^2 \,] - C_2 \leq \mathcal{E}_\varepsilon^h(U_\varepsilon^n, V_\varepsilon^n) \leq C\, [\, 1 + |U^{n-1}|_1^2 + |V^{n-1}|_1^2 \,]. \tag{2.37}$$

On choosing $\chi \equiv \tau_n\, (2\theta\, \pi^h[\phi_\varepsilon(U_\varepsilon^n \pm V_\varepsilon^n)] - \Lambda_\varepsilon^n)$ in (2.30), we have on noting (2.23), a Young's inequality, (2.3), (2.2), (2.8), (1.8), our assumptions on $U^{n-1}$ and $V^{n-1}$, (2.16) and (2.36) that

$$\begin{aligned}
\tau_n\,[\,\tfrac{3}{2}\gamma\theta\,\varepsilon\,|\,\pi^h[\phi_\varepsilon(U_\varepsilon^n \pm V_\varepsilon^n)]\,|_1^2 + \tfrac{1}{2}\,|2\,\theta\,\phi_\varepsilon(U_\varepsilon^n \pm V_\varepsilon^n) - \Lambda_\varepsilon^n|_h^2\,] \\
\leq \tfrac{1}{2}\tau_n\,|\widehat{\mathcal{G}}_{n-1}^h\left(\tfrac{U_\varepsilon^n - U^{n-1}}{\tau_n}\right) \pm \rho\,\widehat{\mathcal{M}}_{n-1}^h\left(\tfrac{V_\varepsilon^n - V^{n-1}}{\tau_n}\right) - (\alpha\,U^{n-1} \pm \beta\,V^{n-1})|_h^2 \\
\leq \tau_n\,C\,[\,b_{n-1}^{\min}\,]^{-1}\,[\,1 + |\,[\pi^h[b_{n-1}]\,]^{\frac{1}{2}}\nabla\widehat{\mathcal{G}}_{n-1}^h\left(\tfrac{U_\varepsilon^n - U^{n-1}}{\tau_n}\right)|_0^2 \\
+ |\,b_{n-1}^{\frac{1}{2}}\widehat{\mathcal{M}}_{n-1}^h\left(\tfrac{V_\varepsilon^n - V^{n-1}}{\tau_n}\right)|_h^2\,] \\
\leq C\,[\,b_{n-1}^{\min}\,]^{-1}\,[\,1 + |U^{n-1}|_1^2 + |V^{n-1}|_1^2\,].
\end{aligned} \tag{2.38}$$

Adding together the $\pm$ versions of (2.30) with $\chi \equiv (U_\varepsilon^n \pm V_\varepsilon^n) - \mu$, respectively, for any $\mu \in \mathbf{R}$ yields that

$$\begin{aligned}
&(\Lambda_\varepsilon^n, \mu - U_\varepsilon^n) \\
&= -\,\gamma\,[\,|U_\varepsilon^n|_1^2 + |V_\varepsilon^n|_1^2\,] + \alpha\,(U^{n-1}, U_\varepsilon^n - \mu)^h + \beta\,(V^{n-1}, V_\varepsilon^n)^h \\
&\quad -\;(\widehat{\mathcal{G}}_{n-1}^h\left(\tfrac{U_\varepsilon^n - U^{n-1}}{\tau_n}\right), U_\varepsilon^n)^h - \rho\,(\widehat{\mathcal{M}}_{n-1}^h\left(\tfrac{V_\varepsilon^n - V^{n-1}}{\tau_n}\right), V_\varepsilon^n)^h \\
&\quad +\;\theta\,(\phi_\varepsilon(U_\varepsilon^n + V_\varepsilon^n), \mu - (U_\varepsilon^n + V_\varepsilon^n))^h +\;\theta\,(\phi_\varepsilon(U_\varepsilon^n - V_\varepsilon^n), \mu - (U_\varepsilon^n - V_\varepsilon^n))^h \\
&\leq C\,(1 + |\mu| + |U_\varepsilon^n|_0 + |V_\varepsilon^n|_0\,) + C\,[\,b_{n-1}^{\min}\,]^{-\frac{1}{2}}\,|\,[\pi^h[b_{n-1}]\,]^{\frac{1}{2}}\nabla\widehat{\mathcal{G}}_{n-1}^h\left(\tfrac{U_\varepsilon^n - U^{n-1}}{\tau_n}\right)|_0\,|U_\varepsilon^n|_0 \\
&\quad +\;C\,[\,b_{n-1}^{\min}\,]^{-\frac{1}{2}}\,|\,b_{n-1}^{\frac{1}{2}}\widehat{\mathcal{M}}_{n-1}^h\left(\tfrac{V_\varepsilon^n - V^{n-1}}{\tau_n}\right)|_h\,|V_\varepsilon^n|_0 \\
&\quad +\;\theta\,(2\,\Phi_\varepsilon(\mu) - \Phi_\varepsilon(U_\varepsilon^n + V_\varepsilon^n) - \Phi_\varepsilon(U_\varepsilon^n - V_\varepsilon^n), 1)^h,
\end{aligned} \tag{2.39}$$

where we have noted (2.2), (2.3), (2.8), our assumptions on $U^{n-1}$ and $V^{n-1}$, (1.8) and the convexity of $\Phi_\varepsilon$. It follows from (2.39) on choosing $\mu = 0$ and 1, and noting (2.37), (2.19), (2.20), $U_\varepsilon^n \in S^h(m^h)$, $m^h \in (0,1)$, and (2.36) that

$$\begin{aligned}
\min\{(m^h)^2,\,(1-m^h)^2\}\;\tau_n\;|\Lambda_\varepsilon^n|^2 \\
\leq \tau_n\,C\,[\,b_{n-1}^{\min}\,]^{-1}\,[\,1 + |U^{n-1}|_1^2 + |V^{n-1}|_1^2 \\
+ |\,[\pi^h[b_{n-1}]\,]^{\frac{1}{2}}\nabla\widehat{\mathcal{G}}_{n-1}^h\left(\tfrac{U_\varepsilon^n - U^{n-1}}{\tau_n}\right)|_0^2 + |\,b_{n-1}^{\frac{1}{2}}\widehat{\mathcal{M}}_{n-1}^h\left(\tfrac{V_\varepsilon^n - V^{n-1}}{\tau_n}\right)|_h^2\,] \\
\leq C\,[\,b_{n-1}^{\min}\,]^{-1}\,[\,1 + |U^{n-1}|_1^2 + |V^{n-1}|_1^2\,].
\end{aligned} \tag{2.40}$$

It follows from (2.37) that there exist $U^n \in S^h(m^h)$ and $V^n \in S^h$ and subsequences $\{U_{\varepsilon'}^n, V_{\varepsilon'}^n\}$ such that $U_{\varepsilon'}^n \to U^n$ and $V_{\varepsilon'}^n \to V^n$ as $\varepsilon' \to 0$. It follows from (2.38) and (2.40) that there exist $\phi_\pm^n \in S^h$ such that $\pi^h[\phi_{\varepsilon'}(U_{\varepsilon'}^n \pm V_{\varepsilon'}^n)] \to \phi_\pm^n$ as $\varepsilon' \to 0$. Noting that $\phi_{(\varepsilon)}^{-1} \in C^1(\mathbf{R})$ and $[\phi_\varepsilon]^{-1}(s) \to \phi^{-1}(s)$ as $\varepsilon \to 0$, for all $s \in \mathbf{R}$, we have that $\phi_\pm^n \equiv \pi^h[\phi(U^n \pm V^n)]$. Therefore we may pass to the limit $\varepsilon' \to 0$ in (2.30) to prove existence of a solution to (2.29) at time level $t_n$. Uniqueness of $\{U^n, V^n\}$ follows, as for $\{U_\varepsilon^n, V_\varepsilon^n\}$, from the monotonicity of $\phi$. Hence noting (2.28), we have existence and uniqueness of a solution $\{U^n, V^n, W^n, Z^n\}$ to (2.6) at time level $t_n$. In addition non-regularized versions of the bounds (2.36)–(2.38) and (2.40) hold; that is, with the $\varepsilon$ subscript removed and $\varepsilon = 0$ on the left hand side of (2.38). The non-regularized versions of (2.38) and (2.40) immediately yield that $U^n \pm V^n \in S_I^h$. Hence we have that $U^n \in S^h(m^h)$ and $U^n \pm V^n \in S_I^h$ for $n = 1 \to N$ by the above induction process. Finally, summing the non-regularized versions of (2.36), (2.38) and (2.40), and noting the non-regularized version of (2.37), (1.8), (2.28a), (2.18), (2.3), (2.2) and (2.8) yield the bounds in (2.27). ◻

REMARK. If $u^0$, $v^0 \in H^1(\Omega)$ are such that $(u^0 \pm v^0)(x) \in (0,1)$ for $a.e.\ x \in \Omega$; it follows on noting (2.4), (2.5) and (2.7) that $U^0 \equiv \widehat{Q}^h u^0$, $V^0 \equiv \widehat{Q}^h v^0$ satisfy for all $h > 0$ the assumptions of Theorem 2.1 with $m^h \equiv \fint u^0 \in (0,1)$ and $|U^0|_1 + |V^0|_1 \leq C$. Similarly, in the case $d = 1$ it follows from (2.10) and (2.11) that $U^0 \equiv \pi^h u^0$, $V^0 \equiv \pi^h v^0$ satisfy for all $h > 0$ the assumptions of Theorem 2.1 with $m^h \in [\delta, 1-\delta]$ for some $\delta \in (0, \frac{1}{2})$ and $|U^0|_1 + |V^0|_1 \leq C$.

**3. Convergence in one space dimension.** Throughout this section we assume that $d = 1$, $\Omega \equiv (a,b)$ and, without loss of generality, that the nodes are ordered

$$a = x_0 < x_1 \cdots < x_{J-1} < x_J = b \qquad \text{with} \qquad h_j := x_j - x_{j-1}, \quad j = 1 \to J. \tag{3.1}$$

Noting this, (2.3) becomes

$$|\eta|_h^2 \equiv \tfrac{1}{2} \sum_{j=1}^{J} h_j \left[\, (\eta(x_j))^2 + (\eta(x_{j-1}))^2 \,\right] \qquad \forall\, \eta \in C(\overline{\Omega}). \tag{3.2}$$

Given $\{\chi^n\}_{n=0}^N$, $\chi^n \in S^h$, we introduce

$$\chi(\cdot,t) := \tfrac{t-t_{n-1}}{\tau_n}\, \chi^n(\cdot) + \tfrac{t_n - t}{\tau_n}\, \chi^{n-1}(\cdot) \qquad t \in [t_{n-1}, t_n] \quad n \geq 1 \tag{3.3a}$$

and

$$\chi^+(\cdot,t) := \chi^n(\cdot), \qquad \chi^-(\cdot,t) := \chi^{n-1}(\cdot) \qquad t \in (t_{n-1}, t_n] \quad n \geq 1. \tag{3.3b}$$

We note for future reference that

$$\chi - \chi^{\pm} = (t - t_n^{\pm})\, \tfrac{\partial \chi}{\partial t} \qquad t \in (t_{n-1}, t_n) \quad n \geq 1, \tag{3.4}$$

where $t_n^+ := t_n$ and $t_n^- := t_{n-1}$. We introduce also

$$\overline{\tau}(t) := \tau_n \qquad t \in (t_{n-1}, t_n) \quad n \geq 1. \tag{3.5}$$

Using the above notation, (2.6) can be restated as:
Find $\{U, V, W, Z\} \in H^1(0,T; S^h(m^h)) \times H^1(0,T;S^h) \times [L^2(0,T;S^h)]^2$ such that

$$\int_0^T \left[\, \left(\tfrac{\partial U}{\partial t}, \chi\right)^h + \left(\pi^h[b(U^-, V^-)]\, \nabla W^+, \nabla \chi\right) \right] \mathrm{d}t = 0 \quad \forall\, \chi \in L^2(0,T;S^h), \tag{3.6a}$$

$$\int_0^T \left[\, \rho\, \left(\tfrac{\partial V}{\partial t}, \chi\right)^h + \left(b(U^-, V^-)\, Z^+, \chi\right)^h \right] \mathrm{d}t = 0 \quad \forall\, \chi \in L^2(0,T;S^h), \tag{3.6b}$$

$$\int_0^T \left[\, \gamma\, (\nabla U^+, \nabla \chi) + \theta\, (\phi(U^+ + V^+) + \phi(U^+ - V^+), \chi)^h \,\right] \mathrm{d}t = \int_0^T (W^+ + \alpha\, U^-, \chi)^h \, \mathrm{d}t \qquad \forall\, \chi \in L^2(0,T;S^h), \tag{3.6c}$$

$$\int_0^T \left[\, \gamma\, (\nabla V^+, \nabla \chi) + \theta\, (\phi(U^+ + V^+) - \phi(U^+ - V^+), \chi)^h \,\right] \mathrm{d}t = \int_0^T (Z^+ + \beta\, V^-, \chi)^h \, \mathrm{d}t \qquad \forall\, \chi \in L^2(0,T;S^h). \tag{3.6d}$$

For later purposes we prove the following result, which is based on an argument in the proof of Proposition 3.5 in [9].

LEMMA 3.1. *For all $\delta \in (0, \frac{1}{2})$ and for all $\{s_1, s_2\} \in \mathcal{Q}$ with $s_1 \in (\delta, 1-\delta)$, it follows that*

$$|\phi(s_1 \pm s_2)| < |\phi(s_1 + s_2) - \phi(s_1 - s_2)| - \ln \delta - \ln (1-\delta). \tag{3.7}$$

*Proof.* Firstly, we note that

$$\delta \le s_1 \pm s_2 < 1 \Longrightarrow \ln \delta \le \ln (s_1 \pm s_2) < 0. \tag{3.8}$$

Secondly, we note from (1.2b) that

$$\begin{aligned} & 0 < s_1 \pm s_2 \le \delta, \quad 0 < s_1 \mp s_2 < 1 \text{ and } \delta < s_1 \\ & \quad \Longrightarrow 1 > s_1 \mp s_2 = 2s_1 - (s_1 \pm s_2) > \delta \\ & \quad \Longrightarrow \ln \delta \ge \ln (s_1 \pm s_2) \ge \ln (s_1 \pm s_2) + \ln (1 - (s_1 \mp s_2)) \\ & \qquad\qquad = [\phi(s_1 \pm s_2) - \phi(s_1 \mp s_2)] + \ln (s_1 \mp s_2) + \ln (1 - (s_1 \pm s_2)) \\ & \qquad\qquad > [\phi(s_1 \pm s_2) - \phi(s_1 \mp s_2)] + \ln \delta + \ln (1-\delta). \end{aligned} \tag{3.9}$$

Therefore on combining (3.8) and (3.9), we have that

$$\begin{aligned} & \{s_1, s_2\} \in \mathcal{Q} \text{ with } \delta < s_1 \Longrightarrow \\ & \quad 0 > \ln (s_1 \pm s_2) > -|\phi(s_1 \pm s_2) - \phi(s_1 \mp s_2)| + \ln \delta + \ln (1-\delta). \end{aligned} \tag{3.10}$$

Similarly to (3.10), we have that

$$\begin{aligned} & \{s_1, s_2\} \in \mathcal{Q} \text{ with } s_1 < 1-\delta \Longrightarrow \\ & \quad 0 < -\ln (1 - (s_1 \pm s_2)) < |\phi(s_1 \pm s_2) - \phi(s_1 \mp s_2)| - \ln \delta - \ln (1-\delta). \end{aligned} \tag{3.11}$$

Noting (1.2b) and combining (3.10) and (3.11) yields the desired result (3.7). □

LEMMA 3.2. *Let $d = 1$ and $u^0,\ v^0 \in H^1(\Omega)$ with $⨍\ u^0 = m \in (0,1)$ and $(u^0 \pm v^0)(x) \in (0,1)$ for all $x \in \Omega$. Let $\{\mathcal{T}^h, U^0, V^0, \tau\}_{h>0}$ be such that*

*(a) $\{U^0, V^0\} \in [S^h]^2$ with $U^0 \pm V^0 \in S^h_I$ and $U^0 \to u^0$, $V^0 \to v^0$ in $H^1(\Omega)$ as $h \to 0$,*

*(b) $\{\mathcal{T}^h\}_{h>0}$ fulfill assumption (A),*

*(c) $\tau \to 0$ as $h \to 0$.*

*Then there exists a subsequence of $\{U, V\}_h$ and functions*

$$u \in L^\infty(0,T; H^1(\Omega)) \cap C^{\frac{1}{2},\frac{1}{8}}_{x,t}(\overline{\Omega}_T), \quad v \in L^\infty(0,T; H^1(\Omega)) \cap C^{\frac{1}{2},\frac{1}{4}}_{x,t}(\overline{\Omega}_T) \tag{3.12}$$

*with $⨍\ u(\cdot,t) = m$ and $\{u(\cdot,t), v(\cdot,t)\} \in \overline{\mathcal{Q}}$ for all $t \in [0,T]$, such that as $h \to 0$*

$$U,\ U^\pm \to u, \quad V,\ V^\pm \to v \qquad \textit{uniformly on } \overline{\Omega}_T, \tag{3.13a}$$

$$U,\ U^\pm \to u, \quad V,\ V^\pm \to v \qquad \textit{weakly in } L^2(0,T; H^1(\Omega)). \tag{3.13b}$$

*Proof.* The assumptions (a) and (b) yield that the right hand side of (2.27) is uniformly bounded. Hence the definitions (3.3), (3.5) and the first six bounds in (2.27) imply that

$$\begin{aligned} & \|U\|^2_{L^\infty(0,T;H^1(\Omega))} + \|V\|^2_{L^\infty(0,T;H^1(\Omega))} + \|\tau^{\frac{1}{2}} \tfrac{\partial U}{\partial t}\|^2_{L^2(0,T;H^1(\Omega))} \\ & \quad + \|\tau^{\frac{1}{2}} \tfrac{\partial V}{\partial t}\|^2_{L^2(0,T;H^1(\Omega))} + \|\tfrac{\partial V}{\partial t}\|^2_{L^2(\Omega_T)} + \|\widehat{\mathcal{G}}^h \tfrac{\partial U}{\partial t}\|^2_{L^2(0,T;H^1(\Omega))} \le C. \end{aligned} \tag{3.14}$$

We deduce from (3.4), (3.5) and (3.14) that

$$\|U - U^{\pm}\|^2_{L^2(0,T;H^1(\Omega))} \le \|\tau \, \tfrac{\partial U}{\partial t}\|^2_{L^2(0,T;H^1(\Omega))} \le C\,\tau, \tag{3.15a}$$

$$\|V - V^{\pm}\|^2_{L^2(0,T;H^1(\Omega))} \le \|\tau \, \tfrac{\partial V}{\partial t}\|^2_{L^2(0,T;H^1(\Omega))} \le C\,\tau. \tag{3.15b}$$

In the next step we show that the discrete solutions $U$ and $V$ are uniformly Hölder continuous. The first two bounds in (3.14) give via a standard imbedding result that

$$|U(y_2,t) - U(y_1,t)| + |V(y_2,t) - V(y_1,t)| \le C\,|y_2 - y_1|^{\frac{1}{2}} \quad \forall\; y_1,\, y_2 \in \overline{\Omega}, \quad \forall\; t \ge 0. \tag{3.16}$$

In addition it follows from (1.5), (2.8), (2.2), (2.15) and (3.14) that

$$\begin{aligned}
&\|U(\cdot,t_b) - U(\cdot,t_a)\|_{0,\infty} \\
&\quad \le C\,\|U(\cdot,t_b) - U(\cdot,t_a)\|_0^{\frac{1}{2}}\;\|U(\cdot,t_b) - U(\cdot,t_a)\|_1^{\frac{1}{2}} \\
&\quad \le C\,|\widehat{\mathcal{G}}^h\,(U(\cdot,t_b) - U(\cdot,t_a))\,|_1^{\frac{1}{4}}\;\|U(\cdot,t_b) - U(\cdot,t_a)\|_1^{\frac{3}{4}} \\
&\quad \le C\left|\widehat{\mathcal{G}}^h\left[\int_{t_a}^{t_b} \tfrac{\partial U}{\partial t}(\cdot,t)\,\mathrm{d}t\right]\right|_1^{\frac{1}{4}}\;\left(2\,\|U\|_{L^\infty(0,T;H^1(\Omega))}\right)^{\frac{3}{4}} \\
&\quad \le C\left|\int_{t_a}^{t_b} \widehat{\mathcal{G}}^h\,\tfrac{\partial U}{\partial t}(\cdot,t)\,\mathrm{d}t\right|_1^{\frac{1}{4}} \;\le\; C\,(t_b - t_a)^{\frac{1}{8}}\;\left(\int_{t_a}^{t_b}\left|\widehat{\mathcal{G}}^h\,\tfrac{\partial U}{\partial t}\right|_1^2\mathrm{d}t\right)^{\frac{1}{8}} \\
&\quad \le C\,(t_b - t_a)^{\frac{1}{8}} \qquad \forall\; t_b \ge t_a \ge 0.
\end{aligned} \tag{3.17}$$

Similarly it follows that

$$\begin{aligned}
\|V(\cdot,t_b) - V(\cdot,t_a)\|_{0,\infty} &\le C\left|\int_{t_a}^{t_b} \tfrac{\partial V}{\partial t}(\cdot,t)\,\mathrm{d}t\right|_0^{\frac{1}{2}}\;\left(2\,\|V\|_{L^\infty(0,T;H^1(\Omega))}\right)^{\frac{1}{2}} \\
&\le C\,(t_b - t_a)^{\frac{1}{4}} \qquad \forall\; t_b \ge t_a \ge 0.
\end{aligned} \tag{3.18}$$

An immediate consequence of (3.17) and (3.18) is that

$$\|U - U^{\pm}\|^2_{L^\infty(\Omega_T)} + \|V - V^{\pm}\|_{L^\infty(\Omega_T)} \le C\,\tau^{\frac{1}{4}}. \tag{3.19}$$

Now (3.14), (3.16), (3.17) and (3.18) imply that the $C^{\frac{1}{2},\frac{1}{8}}_{x,t}(\overline{\Omega}_T)$ norm of $U$ and the $C^{\frac{1}{2},\frac{1}{4}}_{x,t}(\overline{\Omega}_T)$ norm of $V$ are bounded independently of $h$, $\tau$ and $T$. Hence, under the stated assumption (c) on $\tau$, every sequence $\{U,V\}_h$ is uniformly bounded and equicontinuous on $\overline{\Omega}_T$, for any $T > 0$. Therefore by the Arzelà-Ascoli theorem there exists a subsequence such that

$$U \to u \in C^{\frac{1}{2},\frac{1}{8}}_{x,t}(\overline{\Omega}_T), \quad V \to v \in C^{\frac{1}{2},\frac{1}{4}}_{x,t}(\overline{\Omega}_T) \qquad \text{uniformly on } \overline{\Omega}_T \text{ as } h \to 0. \tag{3.20}$$

As $U(\cdot,t) \in S^h(\textstyle\fint U^0)$ and $(U \pm V)(\cdot,t) \in S^h_I$ for all $t \in [0,T]$, it follows that $\fint u(\cdot,t) = m$ and $\{u(\cdot,t), v(\cdot,t)\} \in \overline{\mathcal{Q}}$ for all $t \in [0,T]$. Moreover (3.14) implies that this same subsequence is such that

$$U \to u, \quad V \to v \qquad \text{weakly in } L^2(0,T;H^1(\Omega)) \text{ as } h \to 0. \tag{3.21}$$

Therefore the desired results (3.12) and (3.13) follow on noting the assumption (c), and combining (3.20), (3.19), (3.21) and (3.15). □

REMARK. We note from (2.11) that $U^0 \equiv \pi^h u^0$ and $V^0 \equiv \pi^h v^0$ satisfy the assumptions (a) of Lemma 3.2.

¿From (2.27) we see that we can only control $\nabla W^+$ and $Z^+$ on sets where $b(U^-, V^-)$
$> 0$. Therefore in order to construct the appropriate limits as $h \to 0$, we introduce the following subsets of $\Omega$ and $\Omega_T$. For any $\delta \in (0, \frac{1}{2})$, we set

$$D_{u,\delta}(t) := \{ x \in \overline{\Omega} : \delta < u(x,t) < 1-\delta \} \qquad \forall\, t \in [0,T], \tag{3.22a}$$

$$D_{v,\delta}(t) := \{ x \in \overline{\Omega} : |v(x,t)| < \tfrac{1}{2} - \delta \} \qquad \forall\, t \in [0,T], \tag{3.22b}$$

$$D_{u,\delta} := \{ (x,t) \in \overline{\Omega}_T : \delta < u(x,t) < 1-\delta \}, \tag{3.22c}$$

$$D_{v,\delta} := \{ (x,t) \in \overline{\Omega}_T : |v(x,t)| < \tfrac{1}{2} - \delta \} \tag{3.22d}$$

$$\text{and} \qquad D_\delta(t) := D_{u,\delta}(t) \cap D_{v,\delta}(t), \qquad D_\delta := D_{u,\delta} \cap D_{v,\delta}. \tag{3.22e}$$

As $u \in C^{\frac{1}{2},\frac{1}{8}}_{x,t}(\overline{\Omega}_T)$ and $⨍\, u(\cdot,t) = m \in (0,1)$ for all $t \in [0,T]$, it follows that there exists a $\delta_0^u > 0$ such that $D_{u,\delta_0^u}(t) \not\equiv \emptyset$ for all $t \in [0,T]$. Throughout, without loss of generality, we shall assume that there exists a $\delta_0 > 0$ such that $D_{\delta_0}(t) \not\equiv \emptyset$ for all $t \in [0,T]$. If this were not the case, then there would exist a $t_\star \in [0,T]$ such that $D_\delta(t_\star) \equiv \emptyset$ for all $\delta \in (0, \delta_0^u)$. This would imply that $v(\cdot, t_\star) \equiv \frac{1}{2}$ or $-\frac{1}{2}$ on $D_{u,\delta_0^u}(t_\star)$ as $v \in C^{\frac{1}{2},\frac{1}{4}}_{x,t}(\overline{\Omega}_T)$. In which case $u(\cdot,t_\star) \equiv \frac{1}{2}$ on $D_{u,\delta_0^u}(t_\star)$, as $\{u(\cdot,t_\star), v(\cdot,t_\star)\} \in \overline{\mathcal{Q}}$, and hence on $\overline{\Omega}$. Therefore this singular steady state, $u(\cdot,t_\star) \equiv \frac{1}{2}$ and $v(\cdot,t_\star) \equiv \pm\frac{1}{2}$, could only possibly occur if $m = \frac{1}{2}$. Such a situation can be avoided in this particular case by choosing $T < t_\star$. However under the assumptions of Lemma 3.2 on the data of (P), one could derive an entropy estimate for $v$; see (1.2) in [9]. It immediately follows from this, that there exists an $\varepsilon > 0$ such that $|v(x,t)| \le \frac{1}{2} - \varepsilon$ for all $(x,t) \in \overline{\Omega}_T$, see Corollary 3.7 in [9], and hence such a singular state is impossible. We should remark that our resulting weak formulation of (P), to which $(\mathrm{P}^{h,\tau})$ converges as $h \to 0$, is slightly different to that in [9]. Nevertheless, it can be deduced that this entropy estimate still holds.

¿From (3.20), we have that there exist positive constants $C_x^u$, $C_x^v$, $C_t^u$ and $C_t^v$ such that

$$|u(y_2,t) - u(y_1,t)| \le C_x^u\, |y_2 - y_1|^{\frac{1}{2}}, \quad |v(y_2,t) - v(y_1,t)| \le C_x^v\, |y_2 - y_1|^{\frac{1}{2}} \qquad \forall\, y_1,\, y_2 \in \overline{\Omega}, \quad \forall\, t \in [0,T]; \tag{3.23a}$$

$$|u(x,t_b) - u(x,t_a)| \le C_t^u\, |t_b - t_a|^{\frac{1}{8}}, \quad |v(x,t_b) - v(x,t_a)| \le C_t^v\, |t_b - t_a|^{\frac{1}{4}} \qquad \forall\, t_a,\, t_b \in [0,T], \quad \forall\, x \in \overline{\Omega}. \tag{3.23b}$$

It immediately follows from (3.22a,b) and (3.23) for any $t_a$, $t_b \in [0,T]$ and for any $\delta_1$, $\delta_2 \in (0,\delta_0)$ with $\delta_1 > \delta_2$ that

$$y_1 \in D_{\delta_1}(t_a) \text{ and } y_2 \in \partial D_{\delta_2}(t_b) \text{ with } y_2 \notin \partial\Omega \quad \Longrightarrow$$
$$C_x\, |y_2 - y_1|^{\frac{1}{2}} + \max\{ C_t^u\, |t_b - t_a|^{\frac{1}{8}},\, C_t^v\, |t_b - t_a|^{\frac{1}{4}} \} > (\delta_1 - \delta_2), \tag{3.24}$$

where $\partial D_\delta(t)$ is the boundary of $D_\delta(t)$ and $C_x := \max\{ C_x^u,\, C_x^v \}$. Therefore (3.24) implies that for any $\delta \in (0,\delta_0)$, there exists an $h_0(\delta)$ such that for all $h \le h_0(\delta)$

$$D_\delta(t) \subset D_\delta^h(t) := \cup_{\kappa \in \mathcal{T}_\delta^h(t) \subset \mathcal{T}^h} \overline{\kappa} \subset D_{\frac{\delta}{2}}(t) \qquad \forall\, t \in [0,T]. \tag{3.25}$$

Similarly it follows from (3.24) that for any $\delta \in (0, \delta_0)$, there exists a $\tau_0(\delta)$ such that for all $\tau \leq \tau_0(\delta)$

$$D_\delta(t) \subset D_{\frac{\delta}{2}}(t_n) \subset D_{\frac{\delta}{4}}(t) \qquad \forall\, t \in (t_{n-1}, t_n], \quad n = 1 \to N. \tag{3.26}$$

Clearly, we have from (3.25) and (3.26) that

$$\delta_2 < \delta_1 < \delta_0 \qquad \Longrightarrow \qquad h_0(\delta_2) \leq h_0(\delta_1) \quad \text{and} \quad \tau_0(\delta_2) \leq \tau_0(\delta_1). \tag{3.27}$$

For any $\delta \in (0, \delta_0)$ we introduce cut-off functions $\xi^n_\delta \in C^\infty(\overline{\Omega})$, $n = 1 \to N$, such that

$$\begin{aligned} &\xi^n_\delta \equiv 1 \quad \text{on } D_\delta(t_n), \qquad 0 \leq \xi^n_\delta \leq 1 \quad \text{on } D_{\frac{\delta}{2}}(t_n) \setminus D_\delta(t_n), \\ &\xi^n_\delta \equiv 0 \quad \text{on } \overline{\Omega} \setminus D_{\frac{\delta}{2}}(t_n) \quad \text{and} \quad |\nabla \xi^n_\delta| \leq C\,\delta^{-2}. \end{aligned} \tag{3.28}$$

It follows from (3.24) that this last property can be achieved. Then for any $\delta \in (0, \frac{1}{4}\delta_0)$, we have from (3.28), (2.8), (3.25), (3.26) and (3.3b) that for all $h \leq h_0(2\delta)$ and for all $\tau \leq \tau_0(4\delta)$

$$\begin{aligned} \sum_{n=1}^{N} \tau_n\, |\xi^n_\delta\, \chi^n|^2_h &\geq \sum_{n=1}^{N} \tau_n \int_{D^h_{2\delta}(t_n)} \pi^h[(\xi^n_\delta\, \chi^n)^2]\,\mathrm{d}x \\ &\equiv \sum_{n=1}^{N} \tau_n \int_{D^h_{2\delta}(t_n)} \pi^h[(\chi^n)^2]\,\mathrm{d}x \geq \sum_{n=1}^{N} \tau_n \int_{D^h_{2\delta}(t_n)} |\chi^n|^2\,\mathrm{d}x \\ &\geq \sum_{n=1}^{N} \tau_n \int_{D_{2\delta}(t_n)} |\chi^n|^2\,\mathrm{d}x \geq \int_{D_{4\delta}} |\chi^+|^2\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{3.29}$$

Finally for any $\delta \in (0, \delta_0)$, it follows from (3.22c,d), (3.20) and (3.19) that there exist an $\widehat{h}_0(\delta) \leq h_0(\delta)$ and a $\widehat{\tau}_0(\delta) \leq \tau_0(\delta)$ such that for all $h \leq \widehat{h}_0(\delta)$ and $\tau \leq \widehat{\tau}_0(\delta)$

$$\begin{aligned} &(i) \quad \text{either} \quad 0 \leq U^\pm(x,t) \leq 2\,\delta \\ &\qquad \text{or} \quad 1 - 2\,\delta \leq U^\pm(x,t) \leq 1 \qquad \forall\, (x,t) \notin D_{u,\delta}; \qquad (3.30\text{a}) \\ &(ii) \quad \text{either} \quad -\tfrac{1}{2} \leq V^\pm(x,t) \leq -\tfrac{1}{2} + 2\,\delta \\ &\qquad \text{or} \quad \tfrac{1}{2} - 2\,\delta \leq V^\pm(x,t) \leq \tfrac{1}{2} \qquad \forall\, (x,t) \notin D_{v,\delta}; \qquad (3.30\text{b}) \\ &(iii) \quad \tfrac{1}{2}\,\delta \leq U^\pm(x,t) \leq 1 - \tfrac{1}{2}\delta \qquad \forall\, (x,t) \in D_{u,\delta}, \qquad (3.31\text{a}) \\ &\qquad |V^\pm(x,t)| \leq \tfrac{1}{2} - \tfrac{1}{2}\delta \qquad \forall\, (x,t) \in D_{v,\delta}. \qquad (3.31\text{b}) \end{aligned}$$

The following result is essentially a discrete analogue of Lemma 3.8 in [9].

Lemma 3.3. *Let the assumptions of Lemma 3.2 hold. Then we have for any $\delta \in (0, 2\delta_0)$ that for all $h \leq \widehat{h}_0(\frac{\delta}{32})$ and for all $\tau \leq \widehat{\tau}_0(\frac{\delta}{32})$*

$$\int_{D_{\frac{\delta}{8}}} [\,|\nabla W^+|^2 + |Z^+|^2\,]\,\mathrm{d}x\,\mathrm{d}t \leq C\,\delta^{-2}, \tag{3.32a}$$

$$\int_{D_{\frac{\delta}{2}}} [\,|W^+|^2 + |\Delta^h U^+|^2 + |\Delta^h V^+|^2\,]\,\mathrm{d}x\,\mathrm{d}t \leq C\,\delta^{-2}, \tag{3.32b}$$

$$\int_{D_{\frac{\delta}{2}}} [\,|\pi^h[\phi(U^+ \pm V^+)]|^2 + |\phi(U^+ \pm V^+)|^2\,]\,\mathrm{d}x\,\mathrm{d}t \leq C\,\delta^{-2}. \tag{3.32c}$$

*Proof.* Firstly, we recall that the assumptions (a) and (b) of Lemma 3.2 yield that the right hand side of (2.27) is uniformly bounded. Hence similarly to (3.29), it follows from (2.27), (2.3), (2.2), (3.25), (3.31a,b), (1.3a), (3.22e), (3.26), (2.8) and (3.3b) that for all $h \leq \widehat{h}_0(\frac{\delta}{32})$ and for all $\tau \leq \widehat{\tau}_0(\frac{\delta}{32})$

$$\begin{aligned} C &\geq \sum_{n=1}^{N} \tau_n \, |[b(U^{n-1}, V^{n-1})]^{\frac{1}{2}} Z^n|_h^2 \\ &\geq \sum_{n=1}^{N} \tau_n \int_{D^h_{\frac{\delta}{16}}(t_n)} \pi^h[b(U^{n-1}, V^{n-1})\,(Z^n)^2]\,\mathrm{d}x \\ &\geq C_1\,\delta^2 \sum_{n=1}^{N} \tau_n \int_{D^h_{\frac{\delta}{16}}(t_n)} \pi^h[(Z^n)^2]\,\mathrm{d}x \geq C_1\,\delta^2 \int_{D_{\frac{\delta}{8}}} |Z^+|^2\,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{3.33a}$$

and

$$\begin{aligned} C &\geq \sum_{n=1}^{N} \tau_n \int_{\Omega} \pi^h[b(U^{n-1}, V^{n-1})]\,|\nabla W^n|^2\,\mathrm{d}x \\ &\geq C_1\,\delta^2 \sum_{n=1}^{N} \tau_n \int_{D^h_{\frac{\delta}{16}}(t_n)} |\nabla W^n|^2\,\mathrm{d}x \geq C_1\,\delta^2 \int_{D_{\frac{\delta}{8}}} |\nabla W^+|^2\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{3.33b}$$

Hence the desired result (3.32a).

We note from (2.13) and (2.6c,d) that

$$W^n = -\gamma\,\Delta^h U^n + \theta\,\pi^h[\,\phi(U^n + V^n) + \phi(U^n - V^n)\,] - \alpha\,U^{n-1} \tag{3.34a}$$
$$Z^n = -\gamma\,\Delta^h V^n + \theta\,\pi^h[\,\phi(U^n + V^n) - \phi(U^n - V^n)\,] - \beta\,V^{n-1}. \tag{3.34b}$$

It follows from (3.34b), (3.28) and a Young's inequality that

$$\begin{aligned} \gamma\,|\xi^n_{\frac{\delta}{8}}\,\Delta^h V^n|_h^2 - \theta\,(\,(\xi^n_{\frac{\delta}{8}})^2\,[\,\phi(U^n + V^n) - \phi(U^n - V^n)\,], \Delta^h V^n)^h \\ = -((\xi^n_{\frac{\delta}{8}})^2\,(Z^n + \beta\,V^{n-1}), \Delta^h V^n)^h \\ \leq \tfrac{1}{2\gamma}\,|\xi^n_{\frac{\delta}{8}}\,(Z^n + \beta\,V^{n-1})\,|_h^2 + \tfrac{\gamma}{2}\,|\xi^n_{\frac{\delta}{8}}\,\Delta^h V^n|_h^2. \end{aligned} \tag{3.35}$$

Similarly, we have from (3.34a) that

$$\begin{aligned} \gamma\,|\xi^n_{\frac{\delta}{8}}\,\Delta^h U^n|_h^2 - \theta\,(\,(\xi^n_{\frac{\delta}{8}})^2\,[\,\phi(U^n + V^n) + \phi(U^n - V^n)\,], \Delta^h U^n)^h \\ = -((\xi^n_{\frac{\delta}{8}})^2\,(W^n + \alpha\,U^{n-1}), \Delta^h U^n)^h. \end{aligned} \tag{3.36}$$

Adding together (3.35) and (3.36), and noting (2.13) yields that

$$\begin{aligned} \tfrac{\gamma}{2}\,[\,2\,|\xi^n_{\frac{\delta}{8}}\,\Delta^h U^n|_h^2 + |\xi^n_{\frac{\delta}{8}}\,\Delta^h V^n|_h^2\,] + \theta\,(\nabla\pi^h[(\xi^n_{\frac{\delta}{8}})^2\,\phi(U^n + V^n)], \nabla(U^n + V^n)) \\ + \theta\,(\nabla\pi^h[(\xi^n_{\frac{\delta}{8}})^2\,\phi(U^n - V^n)], \nabla(U^n - V^n)) \\ \leq \tfrac{1}{2\gamma}\,|\xi^n_{\frac{\delta}{8}}\,(Z^n + \beta\,V^{n-1})\,|_h^2 + (\nabla\pi^h[(\xi^n_{\frac{\delta}{8}})^2\,(W^n + \alpha\,U^{n-1})], \nabla U^n). \end{aligned} \tag{3.37}$$

We now bound the terms in (3.37). ¿From the monotonicity of $\phi$ and (3.28), it follows that

$$(\nabla\pi^h[(\xi^n_{\frac{\delta}{8}})^2\,\phi(\chi)],\nabla\chi) \equiv \sum_{j=1}^{J}\Big[(\xi^n_{\frac{\delta}{8}}(x_j))^2\,\phi(\chi_j) - (\xi^n_{\frac{\delta}{8}}(x_{j-1}))^2\,\phi(\chi_{j-1})\Big]\Big[\tfrac{\chi_j-\chi_{j-1}}{h_j}\Big]$$

$$= \sum_{j=1}^{J}\Big[(\xi^n_{\frac{\delta}{8}}(x_j) - \xi^n_{\frac{\delta}{8}}(x_{j-1}))\,(\xi^n_{\frac{\delta}{8}}(x_j)\,\phi(\chi_j) + \xi^n_{\frac{\delta}{8}}(x_{j-1})\,\phi(\chi_{j-1}))$$

$$+\,\xi^n_{\frac{\delta}{8}}(x_j)\,\xi^n_{\frac{\delta}{8}}(x_{j-1})\,(\phi(\chi_j)-\phi(\chi_{j-1}))\Big]\Big[\tfrac{\chi_j-\chi_{j-1}}{h_j}\Big]$$

$$\geq \sum_{j=1}^{J}(\xi^n_{\frac{\delta}{8}}(x_j) - \xi^n_{\frac{\delta}{8}}(x_{j-1}))\,(\xi^n_{\frac{\delta}{8}}(x_j)\,\phi(\chi_j) + \xi^n_{\frac{\delta}{8}}(x_{j-1})\,\phi(\chi_{j-1}))\Big[\tfrac{\chi_j-\chi_{j-1}}{h_j}\Big]$$

$$\forall\;\chi\in S^h; \tag{3.38}$$

where, throughout, $\chi_j \equiv \chi(x_j)$. Similarly to (3.38), we have that

$$(\nabla\pi^h[(\xi^n_{\frac{\delta}{8}})^2\,\eta],\nabla\chi) = \sum_{j=1}^{J}\Big[(\xi^n_{\frac{\delta}{8}}(x_j) - \xi^n_{\frac{\delta}{8}}(x_{j-1}))\,(\xi^n_{\frac{\delta}{8}}(x_j)\,\eta(x_j) + \xi^n_{\frac{\delta}{8}}(x_{j-1})\,\eta(x_{j-1}))$$

$$+\,\xi^n_{\frac{\delta}{8}}(x_j)\,\xi^n_{\frac{\delta}{8}}(x_{j-1})\,(\eta(x_j)-\eta(x_{j-1}))\Big]\Big[\tfrac{\chi_j-\chi_{j-1}}{h_j}\Big]$$

$$\forall\;\eta\in C(\overline{\Omega}),\quad \forall\;\chi\in S^h. \tag{3.39}$$

Next we note from a Young's inequality and (3.2) that for all $\mu > 0$

$$\left|\sum_{j=1}^{J}(\xi^n_{\frac{\delta}{8}}(x_j) - \xi^n_{\frac{\delta}{8}}(x_{j-1}))\,(\xi^n_{\frac{\delta}{8}}(x_j)\,\eta(x_j) + \xi^n_{\frac{\delta}{8}}(x_{j-1})\,\eta(x_{j-1}))\Big[\tfrac{\chi_j-\chi_{j-1}}{h_j}\Big]\right|$$

$$\leq \tfrac{\mu}{2}\sum_{j=1}^{J} h_j\,(\xi^n_{\frac{\delta}{8}}(x_j)\,\eta(x_j) + \xi^n_{\frac{\delta}{8}}(x_{j-1})\,\eta(x_{j-1}))^2$$

$$+\,\tfrac{1}{2\mu}\sum_{j=1}^{J} h_j\Big[\tfrac{\xi^n_{\frac{\delta}{8}}(x_j)-\xi^n_{\frac{\delta}{8}}(x_{j-1})}{h_j}\Big]^2\Big[\tfrac{\chi_j-\chi_{j-1}}{h_j}\Big]^2$$

$$\leq 2\mu\,|\xi^n_{\frac{\delta}{8}}\,\eta|_h^2 + \tfrac{1}{2\mu}\,\|\nabla\xi^n_{\frac{\delta}{8}}\|^2_{0,\infty}\,|\chi|_1^2 \quad \forall\;\eta\in C(\overline{\Omega}),\quad \forall\;\chi\in S^h. \tag{3.40}$$

Similarly to (3.40), we have for all $\mu > 0$ that

$$\left|\sum_{j=1}^{J}\xi^n_{\frac{\delta}{8}}(x_j)\,\xi^n_{\frac{\delta}{8}}(x_{j-1})\,(\eta(x_j)-\eta(x_{j-1}))\Big[\tfrac{\chi_j-\chi_{j-1}}{h_j}\Big]\right|$$

$$\leq \tfrac{\mu}{2}\sum_{j=1}^{J} h_j\;\Big[[\xi^n_{\frac{\delta}{8}}(x_j)\,\xi^n_{\frac{\delta}{8}}(x_{j-1})]^2\Big[\tfrac{\eta(x_j)-\eta(x_{j-1})}{h_j}\Big]^2\Big] + \tfrac{1}{2\mu}\,|\chi|_1^2$$

$$\forall\;\eta\in C(\overline{\Omega}),\quad \forall\;\chi\in S^h. \tag{3.41}$$

Therefore multiplying (3.37) by $\tau_n$, summing from $n = 1 \to N$ and noting (3.38)–

(3.41) yields for all $\mu > 0$ that

$$
\begin{aligned}
&\tfrac{\gamma}{2} \sum_{n=1}^{N} \tau_n \,[\, 2\, |\xi^n_{\frac{\delta}{8}}\, \Delta^h U^n|^2_h + |\xi^n_{\frac{\delta}{8}}\, \Delta^h V^n|^2_h \,] \\
&\quad \leq 2\mu\theta^2 \sum_{n=1}^{N} \tau_n \left[ |\xi^n_{\frac{\delta}{8}}\, \phi(U^n+V^n)|^2_h + |\xi^n_{\frac{\delta}{8}}\, \phi(U^n-V^n)|^2_h \right] \\
&\qquad + \tfrac{1}{\mu} \sum_{n=1}^{N} \tau_n \,\|\nabla \xi^n_{\frac{\delta}{8}}\|^2_{0,\infty} \,[\, |U^n|^2_1 + |V^n|^2_1 \,] + \tfrac{1}{2\mu} \sum_{n=1}^{N} \tau_n \,[\, 1 + \|\nabla \xi^n_{\frac{\delta}{8}}\|^2_{0,\infty} \,] |U^n|^2_1 \\
&\qquad + \sum_{n=1}^{N} \tau_n [\, 2\mu\, |\xi^n_{\frac{\delta}{8}}\, (W^n + \alpha\, U^{n-1})\,|^2_h + \tfrac{1}{2\gamma}\, |\xi^n_{\frac{\delta}{8}}\, (Z^n + \beta\, V^{n-1})\,|^2_h \,] \\
&\qquad + \tfrac{\mu}{2} \sum_{n=1}^{N} \tau_n \left( \sum_{j=1}^{J} h_j\, [\xi^n_{\frac{\delta}{8}}(x_j)\, \xi^n_{\frac{\delta}{8}}(x_{j-1})]^2 \left[ \tfrac{(W^n_j + \alpha U^{n-1}_j) - (W^n_{j-1} + \alpha U^{n-1}_{j-1})}{h_j} \right]^2 \right). \qquad (3.42)
\end{aligned}
$$

We now bound the terms on the right hand side of (3.42). ¿From (3.34a), (3.2), (3.31a) and (3.7) we have that

$$
\begin{aligned}
&|\xi^n_{\frac{\delta}{8}}\, (W^n + \alpha\, U^{n-1})\,|^2_h \\
&\quad \leq 3\, [\, \gamma^2\, |\xi^n_{\frac{\delta}{8}}\, \Delta^h U^n|^2_h + \theta^2\, |\xi^n_{\frac{\delta}{8}}\, \phi(U^n+V^n)|^2_h + \theta^2\, |\xi^n_{\frac{\delta}{8}}\, \phi(U^n-V^n)|^2_h \,] \\
&\quad \leq 3\gamma^2\, |\xi^n_{\frac{\delta}{8}}\, \Delta^h U^n|^2_h + 12\theta^2\, |\xi^n_{\frac{\delta}{8}}\, [\phi(U^n+V^n) - \phi(U^n-V^n)]\,|^2_h + C\,\delta^{-2}. \qquad (3.43)
\end{aligned}
$$

It follows from (3.34b) that

$$
\begin{aligned}
&\theta^2\, |\xi^n_{\frac{\delta}{8}}\, [\phi(U^n+V^n) - \phi(U^n-V^n)]\,|^2_h \\
&\qquad\qquad \leq 2\, [\, \gamma^2\, |\xi^n_{\frac{\delta}{8}}\, \Delta^h V^n|^2_h + |\xi^n_{\frac{\delta}{8}}\, (Z^n + \beta\, V^{n-1})\,|^2_h \,]. \qquad (3.44)
\end{aligned}
$$

It follows from (2.3), (2.2), (3.28), (3.25), (2.8), (3.3a), (3.14) and (3.33a) that

$$
\begin{aligned}
\sum_{n=1}^{N} \tau_n\, |\xi^n_{\frac{\delta}{8}}\, (Z^n + \beta\, V^{n-1})\,|^2_h &\leq 2 \sum_{n=1}^{N} \tau_n \left[ \beta^2\, |V^{n-1}|^2_h + \int_{D^h_{\frac{\delta}{16}}(t_n)} \pi^h[(Z^n)^2]\, \mathrm{d}x \right] \\
&\leq C\,\delta^{-2}. \qquad (3.45)
\end{aligned}
$$

Similarly to (3.45), it follows from (3.28), (3.25), (3.3a), (3.14) and (3.33b) that

$$
\begin{aligned}
&\sum_{n=1}^{N} \tau_n \left( \sum_{j=1}^{J} h_j\, [\xi^n_{\frac{\delta}{8}}(x_j)\, \xi^n_{\frac{\delta}{8}}(x_{j-1})]^2 \left[ \tfrac{(W^n_j + \alpha U^{n-1}_j) - (W^n_{j-1} + \alpha U^{n-1}_{j-1})}{h_j} \right]^2 \right) \\
&\qquad \leq 2 \sum_{n=1}^{N} \tau_n \left[ \alpha^2\, |U^{n-1}|^2_1 + \int_{D^h_{\frac{\delta}{16}}(t_n)} |\nabla W^n|^2\, \mathrm{d}x \right] \leq C\,\delta^{-2}. \qquad (3.46)
\end{aligned}
$$

Combining (3.42), (3.43), (3.46), (3.45) and (3.44), and noting (3.3a), (3.14) and

(3.28) yields for all $\mu > 0$ that

$$\frac{\gamma}{2} \sum_{n=1}^{N} \tau_n \left[ 2\, |\xi^n_{\frac{\delta}{8}} \Delta^h U^n|_h^2 + |\xi^n_{\frac{\delta}{8}} \Delta^h V^n|_h^2 \right]$$
$$\leq 32\mu\theta^2 \sum_{n=1}^{N} \tau_n\, |\xi^n_{\frac{\delta}{8}} [\phi(U^n + V^n) - \phi(U^n - V^n)]\, |_h^2$$
$$+ 6\mu\gamma^2 \sum_{n=1}^{N} \tau_n |\xi^n_{\frac{\delta}{8}} \Delta^h U^n|_h^2 + C\,(1+\mu^{-1})\,\delta^{-2}$$
$$\leq 2\mu\gamma^2 \sum_{n=1}^{N} \tau_n \left[ 32\, |\xi^n_{\frac{\delta}{8}} \Delta^h V^n|_h^2 + 3\, |\xi^n_{\frac{\delta}{8}} \Delta^h U^n|_h^2 \right] + C\,(1+\mu^{-1})\,\delta^{-2}. \tag{3.47}$$

On choosing $\mu = 1/(256\,\gamma)$ in (3.47) and noting (3.43), (3.44), (3.45), (2.8), (3.3a) and (3.14) we deduce that

$$\sum_{n=1}^{N} \tau_n \left[ |\xi^n_{\frac{\delta}{8}} W^n|_h^2 + |\xi^n_{\frac{\delta}{8}} \Delta^h U^n|_h^2 + |\xi^n_{\frac{\delta}{8}} \Delta^h V^n|_h^2 \right] + \theta^2 \sum_{n=1}^{N} \tau_n\, |\xi^n_{\frac{\delta}{8}} \phi(U^n \pm V^n)|_h^2 \leq C\,\delta^{-2}. \tag{3.48}$$

The desired result (3.32b) and the first bound in (3.32c) then follow immediately from (3.48) and (3.29).

To prove the final bound in (3.32c), we note that the monotonicity of $\phi$ implies for all $\chi \in S^h_I$

$$\int_{x_{j-1}}^{x_j} |\phi(\chi)|^2 \,\mathrm{d}x \leq h_j \max\{|\phi(\chi_{j-1})|^2,\, |\phi(\chi_j)|^2\} \leq 2 \int_{x_{j-1}}^{x_j} \pi^h[(\phi(\chi))^2] \,\mathrm{d}x. \tag{3.49}$$

Therefore similarly to (3.29), it follows from (3.3b), (3.26), (3.25), (3.49) and (3.28) that

$$\int_{D_{\frac{\delta}{2}}} |\phi(U^+ \pm V^+)|^2 \,\mathrm{d}x\,\mathrm{d}t \leq \sum_{n=1}^{N} \tau_n \int_{D^h_{\frac{\delta}{4}}(t_n)} |\phi(U^+ \pm V^+)|^2 \,\mathrm{d}x\,\mathrm{d}t$$
$$\leq 2 \sum_{n=1}^{N} \tau_n \int_{D^h_{\frac{\delta}{4}}(t_n)} \pi^h[(\phi(U^+ \pm V^+))^2] \,\mathrm{d}x\,\mathrm{d}t \leq 2 \sum_{n=1}^{N} \tau_n\, |\xi^n_{\frac{\delta}{8}} \phi(U^n \pm V^n)|_h^2. \tag{3.50}$$

Hence combining (3.50) and (3.48) yields the desired final bound in (3.32c). ☐

THEOREM 3.4. *Let the assumptions of Lemma 3.2 hold. Then there exists a subsequence of $\{U, V, W, Z\}_h$ and functions $\{u, v, w, z\}$ satisfying (3.12) and*

$$u \in H^1(0,T;(H^1(\Omega))'), \quad v \in H^1(0,T;L^2(\Omega)), \tag{3.51a}$$
$$\phi(u \pm v),\ w,\ z \in L^2_{loc}(D) \ \textit{with}\ \nabla w \in L^2_{loc}(D) \tag{3.51b}$$
$$\textit{and} \quad b(u,v)\nabla w,\ b(u,v)\,z \in L^2(D), \tag{3.51c}$$

*where $D := \{(x,t) \in \Omega_T \,:\, 0 < u(x,t) < 1$ and $|v(x,t)| < \frac{1}{2}\}$, such that as $h \to 0$ (3.13) hold and*

$$\pi^h[\phi(U^+ \pm V^+)],\ \phi(U^+ \pm V^+) \to \phi(u \pm v) \qquad \textit{in } L^2_{loc}(D), \tag{3.52a}$$
$$W^+ \to w, \quad \nabla W^+ \to \nabla w, \quad Z^+ \to z \qquad \textit{weakly in } L^2_{loc}(D). \tag{3.52b}$$

*Furthermore,* $\{u, v, w, z\}$ *fulfill* $u(\cdot, 0) = u^0(\cdot)$, $v(\cdot, 0) = v^0(\cdot)$ *and*

$$\int_0^T \langle \tfrac{\partial u}{\partial t}, \eta \rangle \,\mathrm{d}t + \int_D b(u,v)\,\nabla w\,\nabla \eta \,\mathrm{d}x\,\mathrm{d}t = 0 \quad \forall\ \eta \in L^2(0,T;H^1(\Omega)), \tag{3.53a}$$

$$\int_0^T \rho\,(\tfrac{\partial v}{\partial t}, \eta)\,\mathrm{d}t + \int_D b(u,v)\,z\,\eta\,\mathrm{d}x\,\mathrm{d}t = 0 \quad \forall\ \eta \in L^2(\Omega_T), \tag{3.53b}$$

$$\int_D [\,\gamma \nabla u\,\nabla \eta + (\,\theta\,[\,\phi(u+v) + \phi(u-v)\,] - \alpha\,u - w\,)\,\eta\,]\,\mathrm{d}x\,\mathrm{d}t = 0$$
$$\forall\ \eta \in L^2(0,T;H^1(\Omega)) \textit{ with } supp(\eta) \subset D, \tag{3.53c}$$

$$\int_D [\,\gamma \nabla v\,\nabla \eta + (\,\theta\,[\,\phi(u+v) - \phi(u-v)\,] - \beta\,v - z\,)\,\eta\,]\,\mathrm{d}x\,\mathrm{d}t = 0$$
$$\forall\ \eta \in L^2(0,T;H^1(\Omega)) \textit{ with } supp(\eta) \subset D. \tag{3.53d}$$

*Proof.* For any $\eta \in H^1(0,T;H^1(\Omega))$ we choose $\chi \equiv \pi^h \eta$ in (3.6a,b) and now analyse the subsequent terms. Firstly on the left hand sides, we have that

$$\int_0^T (\tfrac{\partial U}{\partial t}, \eta)^h\,\mathrm{d}t = -\int_0^T (U, \tfrac{\partial \eta}{\partial t})^h\,\mathrm{d}t + (U(\cdot,T), \eta(\cdot,T))^h - (U(\cdot,0), \eta(\cdot,0))^h, \tag{3.54a}$$

$$\int_0^T (\tfrac{\partial V}{\partial t}, \eta)^h\,\mathrm{d}t = -\int_0^T (V, \tfrac{\partial \eta}{\partial t})^h\,\mathrm{d}t + (V(\cdot,T), \eta(\cdot,T))^h - (V(\cdot,0), \eta(\cdot,0))^h. \tag{3.54b}$$

Next we conclude using the regularity of $\eta$, (2.2), (2.8), (2.11), (2.12) and (3.13a) that for all $\eta \in H^1(0,T;H^1(\Omega))$

$$\int_0^T (U, \tfrac{\partial \eta}{\partial t})^h\,\mathrm{d}t \to \int_0^T (u, \tfrac{\partial \eta}{\partial t})\,\mathrm{d}t, \qquad \int_0^T (V, \tfrac{\partial \eta}{\partial t})^h\,\mathrm{d}t \to \int_0^T (v, \tfrac{\partial \eta}{\partial t})\,\mathrm{d}t \qquad \text{as } h \to 0. \tag{3.55}$$

We now analyse the terms involving $b$ in (3.6a,b). In view of (3.33), (3.3b), (2.12) and (2.8) we deduce for all $\eta \in L^2(0,T;H^1(\Omega))$ that

$$\begin{aligned}
&\left| \int_{\Omega_T} \pi^h[b(U^-,V^-)]\,\nabla W^+\,\nabla (I-\pi^h)\eta\,\mathrm{d}x\,\mathrm{d}t \right| \\
&\qquad \le [\,b^{\max}\,]^{\frac{1}{2}}\,\|[\pi^h[b(U^-,V^-)]\,]^{\frac{1}{2}}\nabla W^+\|_{L^2(\Omega_T)}\,\|(I-\pi^h)\eta\|_{L^2(0,T;H^1(\Omega))} \\
&\qquad \le C\,\|(I-\pi^h)\eta\|_{L^2(0,T;H^1(\Omega))}
\end{aligned} \tag{3.56a}$$

and

$$\begin{aligned}
&\left| \int_{\Omega_T} (\,\pi^h[b(U^-,V^-)\,Z^+\,\eta] - \pi^h[b(U^-,V^-)\,Z^+]\,\eta\,)\,\mathrm{d}x\,\mathrm{d}t \right| \\
&\qquad \le C\,h\,\|\pi^h[b(U^-,V^-)\,Z^+]\|_{L^2(\Omega_T)}\,\|\eta\|_{L^2(0,T;H^1(\Omega))} \\
&\qquad \le C\,h\,\|\,[\pi^h[b(U^-,V^-)\,(Z^+)^2]\,]^{\frac{1}{2}}\|_{L^2(\Omega_T)}\,\|\eta\|_{L^2(0,T;H^1(\Omega))} \\
&\qquad \le C\,h\,\|\eta\|_{L^2(0,T;H^1(\Omega))},
\end{aligned} \tag{3.56b}$$

where $b^{\max} := \max_{\{s_1,s_2\}\in \overline{\mathcal{Q}}} b(s_1,s_2) = \frac{1}{16}$.

We now consider a fixed $\delta \in (0, \frac{1}{2}\delta_0)$. On noting (3.33), (3.3b), (3.25), (3.22e), (3.30a,b), (1.3) and (2.8) we have for all $h \leq \widehat{h}_0(2\delta)$ and for all $\tau \leq \widehat{\tau}_0(2\delta)$ that

$$
\begin{aligned}
\left| \int_{\Omega_T \setminus D_\delta} \pi^h[b(U^-, V^-)]\, \nabla W^+\, \nabla \eta \,\mathrm{d}x\,\mathrm{d}t \right| &\leq \|\pi^h[b(U^-, V^-)]\,\|^{\frac{1}{2}}_{L^\infty(\Omega_T \setminus D^h_{2\delta})} \times \\
&\quad \times \|[\pi^h[b(U^-, V^-)]\,]^{\frac{1}{2}} \nabla W^+\|_{L^2(\Omega_T)}\, \|\eta\|_{L^2(0,T;H^1(\Omega))} \\
&\leq C\, \|b(U^-, V^-)\|^{\frac{1}{2}}_{L^\infty(\Omega_T \setminus D_{2\delta})}\, \|\eta\|_{L^2(0,T;H^1(\Omega))} \\
&\leq C\, \delta^{\frac{1}{2}}\, \|\eta\|_{L^2(0,T;H^1(\Omega))} \qquad \forall\ \eta \in L^2(0,T;H^1(\Omega))
\end{aligned}
\tag{3.57a}
$$

and

$$
\begin{aligned}
\left| \int_{\Omega_T \setminus D_\delta} \pi^h[b(U^-, V^-)\, Z^+]\, \eta \,\mathrm{d}x\,\mathrm{d}t \right| &\leq \|\pi^h[b(U^-, V^-)]\,\|^{\frac{1}{2}}_{L^\infty(\Omega_T \setminus D^h_{2\delta})} \times \\
&\quad \times \|\,[\pi^h[b(U^-, V^-)\,(Z^+)^2]\,]^{\frac{1}{2}}\|_{L^2(\Omega_T)}\, \|\eta\|_{L^2(\Omega_T)} \\
&\leq C\, \delta^{\frac{1}{2}}\, \|\eta\|_{L^2(\Omega_T)} \qquad \forall\ \eta \in L^2(\Omega_T),
\end{aligned}
\tag{3.57b}
$$

where $D^h_\delta := \{\,(x,t) \in \overline{\Omega}_T : x \in D^h_\delta(t)\,\}$. (3.32a,b) yield that there exists a subsequence of $\{U, V, W, Z\}_h$, and functions $w \in L^2(0,T;H^1(D_{\frac{\delta}{2}}(t)))$ and $z \in L^2(D_{\frac{\delta}{2}})$ such that $\{U, V\}_h$ satisfy (3.13) and

$$
W^+ \to w, \quad \nabla W^+ \to \nabla w, \quad Z^+ \to z \qquad \text{weakly in } L^2(D_{\frac{\delta}{2}}) \text{ as } h \to 0.
\tag{3.58}
$$

Next noting (3.32a), (1.3), (1.5), (3.13a), (2.11) and (3.12) we conclude that for all $\eta \in L^2(0,T;H^1(\Omega))$

$$
\begin{aligned}
&\left| \int_{D_\delta} [\pi^h[b(U^-, V^-)] - b(u,v)]\, \nabla W^+\, \nabla \eta \,\mathrm{d}x\,\mathrm{d}t \right| \\
&\qquad \leq \|b(u,v) - \pi^h[b(U^-, V^-)]\,\|_{L^\infty(\Omega_T))}\, \|\nabla W^+\|_{L^2(D_\delta)}\, \|\eta\|_{L^2(0,T;H^1(\Omega))} \\
&\qquad \leq C\, \delta^{-1}\, \|\eta\|_{L^2(0,T;H^1(\Omega))}\, [\,\|b(u,v) - b(U^-, V^-)\|_{L^\infty(\Omega_T)} \\
&\qquad\qquad + \|(I - \pi^h) b(u,v)\|_{L^\infty(0,T;H^1(\Omega))}\,]
\end{aligned}
\tag{3.59a}
$$

will converge to 0 as $h \to 0$. Similarly on noting (3.25), (2.12), (2.8), (1.3), (3.12), (1.5), (3.32a), and (3.13a) we conclude for all $\eta \in L^2(0,T;H^1(\Omega))$

$$
\begin{aligned}
&\left| \int_{D_\delta} (\,\pi^h[b(U^-, V^-)\, Z^+] - b(u,v)\ Z^+\,)\ \eta\ \mathrm{d}x\,\mathrm{d}t \right| \\
&\qquad \leq \int_{D^h_\delta} |\,\pi^h[\,(\,b(U^-, V^-) - b(u,v)\,)\, Z^+]\, \eta\,|\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad\qquad + \int_{D^h_\delta} |(I - \pi^h)(b(u,v)\, Z^+)\, \eta\,|\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad \leq \|b(u,v) - b(U^-, V^-)\|_{L^\infty(\Omega_T)}\, \|\,[\pi^h[(Z^+)^2]\,]^{\frac{1}{2}}\|_{L^2(D^h_\delta)}\, \|\eta\|_{L^2(\Omega_T)} \\
&\qquad\qquad + C\, h\, \|Z^+\|_{L^2(D^h_\delta)}\, \|b(u,v)\|_{L^\infty(0,T;H^1(\Omega))}\, \|\eta\|_{L^2(0,T;L^\infty(\Omega))} \\
&\qquad \leq C\, \|Z^+\|_{L^2(D_{\frac{\delta}{2}})}\, \|\eta\|_{L^2(0,T;H^1(\Omega))} \times \\
&\qquad\qquad \times [\,\|b(u,v) - b(U^-, V^-)\|_{L^\infty(\Omega_T)} + h\, \|b(u,v)\|_{L^\infty(0,T;H^1(\Omega))}\,] \\
&\qquad \leq C\, \delta^{-1}\, \|\eta\|_{L^2(0,T;H^1(\Omega))}\, [\,h + \|b(u,v) - b(U^-, V^-)\|_{L^\infty(\Omega_T)}\,]
\end{aligned}
\tag{3.59b}
$$

will converge to 0 as $h \to 0$. Combining (3.59) and (3.58), and noting (2.11), (1.3), (3.12) and (3.13a) yields for all $\eta \in L^2(0,T;H^1(\Omega))$ that

$$\int_{D_\delta} \pi^h[b(U^-,V^-)]\,\nabla W^+\,\nabla\eta\,\mathrm{d}x\,\mathrm{d}t \to \int_{D_\delta} b(u,v)\,\nabla w\,\nabla\eta\,\mathrm{d}x\,\mathrm{d}t, \tag{3.60a}$$

$$\int_{D_\delta} \pi^h[b(U^-,V^-)\,Z^+]\,\eta\,\mathrm{d}x\,\mathrm{d}t \to \int_{D_\delta} b(u,v)\,z\,\eta\,\mathrm{d}x\,\mathrm{d}t \qquad \text{as } h \to 0. \tag{3.60b}$$

We now turn our attention to the other two equations in $(\mathrm{P}^{h,\tau})$. Once again we consider a fixed $\delta \in (0,\delta_0)$. For any $\eta \in L^2(0,T;H^1(\Omega))$ with $\mathrm{supp}(\eta) \subset D_\delta$, we choose $\chi \equiv \pi^h\eta$ in (3.6c,d). It follows immediately from (3.25) that for all $\eta \in L^2(0,T;H^1(\Omega))$ and for all $h \le h_0(\delta)$

$$\mathrm{supp}(\eta) \subset D_\delta \Longrightarrow \mathrm{supp}(\pi^h\eta) \subset D_\delta^h \subset D_{\frac{\delta}{2}}. \tag{3.61}$$

We now analyse the subsequent terms in (3.6c,d). Similarly to (3.60a), it follows from (2.11) and (3.13b) that for all $\eta \in L^2(0,T;H^1(\Omega))$

$$\int_0^T (\nabla U^+,\,\nabla(\pi^h\eta))\,\mathrm{d}t \to \int_0^T (\nabla u,\,\nabla\eta)\,\mathrm{d}t, \tag{3.62a}$$

$$\int_0^T (\nabla V^+,\,\nabla(\pi^h\eta))\,\mathrm{d}t \to \int_0^T (\nabla v,\,\nabla\eta)\,\mathrm{d}t \qquad \text{as } h \to 0. \tag{3.62b}$$

We deduce for all $\eta \in L^2(0,T;H^1(\Omega))$ with $\mathrm{supp}(\eta) \subset D_\delta$ on noting (2.2), (2.12), (3.61) and (3.32) that

$$\begin{aligned}\left|\int_0^T \left[\,(W^+,\eta)^h - (W^+,\eta)\,\right]\,\mathrm{d}t\right| &\le C\,h\,\|W^+\|_{L^2(D_{\frac{\delta}{2}})}\,\|\eta\|_{L^2(0,T;H^1(\Omega))}\\ &\le C\,\delta^{-1}\,h\,\|\eta\|_{L^2(0,T;H^1(\Omega))}\end{aligned} \tag{3.63a}$$

and similarly

$$\begin{aligned}&\left|\int_0^T \left[\,(\phi(U^+\pm V^+),\eta)^h - (\pi^h[\phi(U^+\pm V^+)],\eta)\,\right]\,\mathrm{d}t\right|\\ &\qquad + \left|\int_0^T \left[\,(Z^+,\eta)^h - (Z^+,\eta)\,\right]\,\mathrm{d}t\right| \le C\,\delta^{-1}\,h\,\|\eta\|_{L^2(0,T;H^1(\Omega))}.\end{aligned} \tag{3.63b}$$

The bounds (3.32c) yield that there exists a subsequence of $\{U,V,W,Z\}_h$, of the subsequence $\{U,V,W,Z\}_h$ satisfying (3.13) and (3.58), and functions $g_1^\pm,\,g_2^\pm \in L^2(D_{\frac{\delta}{2}})$ such that

$$\phi(U^+\pm V^+) \to g_1^\pm, \quad \pi^h[\phi(U^+\pm V^+)] \to g_2^\pm, \qquad \text{weakly in } L^2(D_{\frac{\delta}{2}}) \text{ as } h \to 0. \tag{3.64}$$

The monotonicity of $\phi$, (3.32c) and the fact that $\phi^{-1}:\mathbf{R} \to (0,1)$ imply that

$$\int_{D_{\frac{\delta}{2}}} [\,(U^+\pm V^+) - \phi^{-1}(\eta)\,]\,[\,\phi(U^+\pm V^+) - \eta\,]\,\mathrm{d}x\,\mathrm{d}t \ge 0 \qquad \forall\,\eta \in L^2(D_{\frac{\delta}{2}}). \tag{3.65}$$

Taking the $h \to 0$ limit in (3.65), and noting (3.13a) and (3.64) yields that

$$\int_{D_{\frac{\delta}{2}}} [\,(u \pm v) - \phi^{-1}(\eta)\,]\,[\,g_1^{\pm} - \eta\,]\,\mathrm{d}x\,\mathrm{d}t \geq 0 \qquad \forall\, \eta \in L^2(D_{\frac{\delta}{2}}). \tag{3.66}$$

The inequality (3.66) implies that $(u \pm v) \equiv \phi^{-1}(g_1^{\pm})$ and hence that $g_1^{\pm} \equiv \phi(u \pm v)$. As $\phi(u \pm v) \in L^2(D_{\frac{\delta}{2}})$, it follows that $(u \pm v)(x,t) \in (0,1)$ for *a.e.* $(x,t) \in D_{\frac{\delta}{2}}$. Therefore (3.13a) and (3.16) yield that $\pi^h[\phi(U^+ \pm V^+)]$, $\phi(U^+ \pm V^+) \to \phi(u \pm v)$ *a.e.* in $D_{\frac{\delta}{2}}$ as $h \to 0$. Hence combining (3.64) and the above, we have that

$$\pi^h[\phi(U^+ \pm V^+)],\ \phi(U^+ \pm V^+) \to \phi(u \pm v), \qquad \text{in } L^2(D_{\frac{\delta}{2}}) \text{ as } h \to 0. \tag{3.67}$$

It follows from (3.62), (3.63), and similar results on replacing $W^+$ in (3.63a) with $U^-$ or $V^-$, (3.13a), (3.58) and (3.67) that taking the $h \to 0$ limit of (3.6c,d) with $\chi \equiv \pi^h \eta$ yields for all $\eta \in L^2(0,T;H^1(\Omega))$ satisfying $\mathrm{supp}(\eta) \subset D_\delta$

$$\int_{D_\delta} [\,\gamma \nabla u\, \nabla \eta + (\,\theta\,[\,\phi(u+v) + \phi(u-v)\,] - \alpha\, u - w\,)\,\eta\,]\,\mathrm{d}x\,\mathrm{d}t = 0, \tag{3.68a}$$

$$\int_{D_\delta} [\,\gamma \nabla v\, \nabla \eta + (\,\theta\,[\,\phi(u+v) - \phi(u-v)\,] - \beta\, v - z\,)\,\eta\,]\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{3.68b}$$

The equations (3.68) uniquely define $w$ and $z$ in terms of $u$ and $v$ on the set $D_\delta$.

Repeating (3.60) for all $\delta > 0$, and noting (3.56), (3.57) and (2.11) yields for all $\eta \in L^2(0,T;H^1(\Omega))$ that

$$\int_{\Omega_T} \pi^h[b(U^-,V^-)]\,\nabla W^+\,\nabla(\pi^h \eta)\,\mathrm{d}x\,\mathrm{d}t \to \int_D b(u,v)\,\nabla w\,\nabla \eta\,\mathrm{d}x\,\mathrm{d}t, \tag{3.69a}$$

$$\int_{\Omega_T} \pi^h[b(U^-,V^-)\,Z^+\,\eta]\,\mathrm{d}x\,\mathrm{d}t \to \int_D b(u,v)\,z\,\eta\,\mathrm{d}x\,\mathrm{d}t \qquad \text{as } h \to 0. \tag{3.69b}$$

The uniform $L^2(\Omega_T)$ bounds on $\pi^h[b(U^-,V^-)]\,\nabla W^+$ and $\pi^h[b(U^-,V^-)\,Z^+]$, see (3.56), and the limits (3.60) imply that (3.51c) holds. Taking the $h \to 0$ limit of (3.6a,b) with $\chi \equiv \pi^h\eta$, and noting (2.2), (3.54), (3.55), (3.69) and (3.13a), and arguing similarly as in (3.63), by using (2.12), we conclude that for all $\eta \in H^1(0,T;H^1(\Omega))$

$$(u(\cdot,T),\eta(\cdot,T)) - (u^0(\cdot),\eta(\cdot,0)) - \int_0^T (u, \tfrac{\partial \eta}{\partial t})\,\mathrm{d}t + \int_D b(u,v)\,\nabla w\,\nabla\eta\,\mathrm{d}x\,\mathrm{d}t = 0, \tag{3.70a}$$

$$\rho\,[\,(v(\cdot,T),\eta(\cdot,T)) - (v^0(\cdot),\eta(\cdot,0)) - \int_0^T (v, \tfrac{\partial \eta}{\partial t})\,\mathrm{d}t\,] + \int_D b(u,v)\,z\,\eta\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{3.70b}$$

As $b(u,v)\,\nabla w$, $b(u,v)\,z \in L^2(D)$, we deduce from (3.70) that $u \in H^1(0,T;(H^1(\Omega))')$ and $v \in H^1(0,T;L^2(\Omega))$. Therefore combining the above results and repeating (3.68) for all $\delta > 0$ , we obtain the desired results (3.51a,b), (3.52) and (3.53). □

Remark. The weak formulation (3.53) is slightly different to the one studied in [9] for (P). One of the main differences is that they work with variables $\{u, v, f\}$, where $f \equiv [b(u,v)]^{\frac{1}{2}}\nabla w$. Another is that the constraint on $v$ on the set $D$ is removed. This is because they derive an a priori entropy estimate for $v$; see (1.2) in [9]. It immediately follows from this, that there exists an $\varepsilon > 0$ such that $|v(x,t)| \leq \frac{1}{2} - \varepsilon$

for all $(x,t) \in \overline{\Omega}_T$, see Corollary 3.7 in [9]. It can be deduced that this entropy estimate also holds for our formulation (3.53), but it is not necessary in order to prove convergence of $(\mathrm{P}^{h,\tau})$. Choosing $v^0 \equiv \frac{1}{2}$, (3.53) collapses to the weak formulation obtained in [3] for the logarithmic Cahn-Hilliard equation with a degenerate mobility; whereas that in [9] does not. Finally we note that both of these weak formulations are based on the solution concept introduced in [4] for fourth (and higher) order nonlinear degenerate parabolic equations. Hence they are both restricted to one space dimension.

**4. Numerical experiments.** We consider four numerical experiments, highlighting (i) the differences between the mobility $b$ being degenerate, (1.3), and $b$ being constant, $b \equiv b_{\max} \equiv 2^{-4}$ (the maximum value of the above degenerate $b$); (ii) the differences between $\alpha < \beta$ and $\alpha > \beta$.

In each experiment, we considered $(\mathrm{P}^{h,\tau})$ with the following fixed data: $d = 1$, $\Omega = (0,1)$, $\gamma = 5 \times 10^{-3}$ and $\theta = 0.01$. We took $h = 1/100$, $\tau \equiv \tau_n = 1/2048$ and chose initial data $\{U^0, V^0\} \equiv \{0.55 + \delta_u^h, \delta_v^h\}$; where $\delta_u^h$, $\delta_v^h \in S^h$ are random with $\max\{\|\delta_u^h\|_{0,\infty}, \|\delta_v^h\|_{0,\infty}\} < 0.05$, but such that $\fint U^0 = 0.55$ and $\fint V^0 = 0$. For solving the nonlinear algebraic system arising at each time level from $(\mathrm{P}^{h,\tau})$, we used the iterative solver described in [2, §5] with the parameter $b^{n-1} \equiv b_{\max}$ and the relaxation parameter $\mu = 0.75$. For a stopping criterion we chose that the maximum difference in successive iterates should be smaller than $10^{-7}$.

In the first experiment we took $\alpha = 4$, $\beta = 2$, $\rho = 0.001$ with degenerate mobility. We note that the numerical steady state (i.e. the iterative solver for the nonlinear algebraic system converges in one iteration) at time $t = 35$, see Figure 4.1, has two antiphase boundaries, which are "prewetted". An antiphase boundary between two ordered variants is said to be "wetted" if it contains a region of a pure phase; and is said to be "prewetted" if it contains a region of partial phase separation $(0 < |u - \frac{1}{2}| < \frac{1}{2})$.

The data for the second experiment was the same as the first except with constant mobility, see Figure 4.2. The numerical solution at $t = 0.25$ is similar to the steady state in the first experiment; the interior region having already disappeared. However, in contrast to the case of the degenerate mobility there is no pinning and phase separation ensues. This is to be expected for $\alpha > \beta$ as the global minimizers of $\Psi_{0.01}$ are very close to the pure phases.

In the third experiment we took $\alpha = 2$, $\beta = 4$, $\rho = 0.08$ with degenerate mobility, see Figure 4.3. For this value of $\rho$, phase separation occurs on a faster time scale when compared to the previous experiments (for the role of different time scales in degenerate Allen-Cahn/Cahn-Hilliard systems see [7] and [12]). Thickening of some of the interphase boundaries then occurs. This eventually leads to a numerical steady state, where the antiphase boundaries are "prewetted" and in addition there is an interphase boundary.

The fourth and final experiment is the same as the third except with constant mobility, see Figure 4.4. The evolution is similar to that in the third experiment, except that it is on a faster time scale. However, there is no interphase boundary in the numerical steady state at $t = 4.25$; which is similar to that at $t = 35$ in the first experiment. Once again this is to be expected, since for $\alpha < \beta$ the global minimizers of $\Psi_{0.01}$ are very close to the ordered variants.

All of these experiments were repeated on a finer space-time mesh ($h = 1/200$ and $\tau \equiv \tau_n = 1/4096$) and the results were found to be graphically similar. Finally we recall that as $U^n \pm V^n \in S_I^h$, see Theorem 2.1, it follows that $0 < U < 1$ and $|V| < \frac{1}{2}$ in all of these experiments.

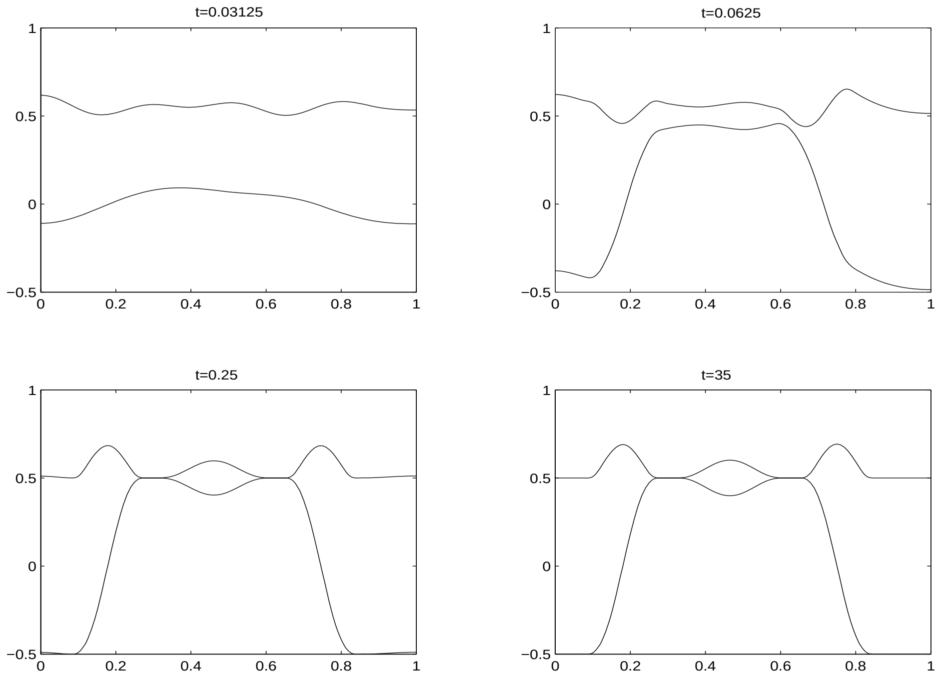


FIG. 4.1. $U(\cdot,t)$, $V(\cdot,t)$ *with degenerate mobility,* $\alpha = 4$, $\beta = 2$ *and* $\rho = 0.001$.

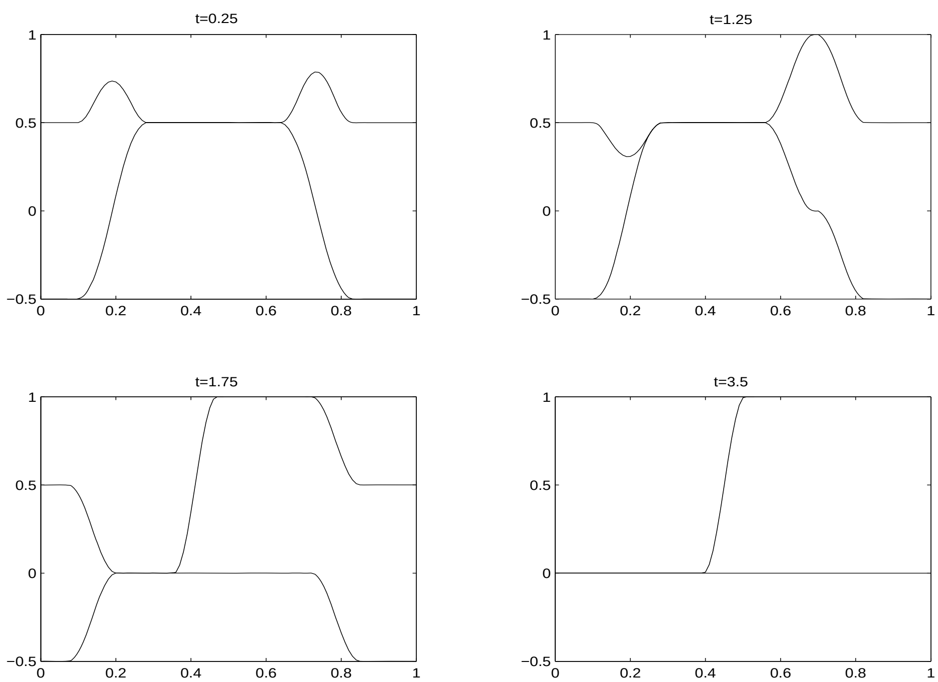


FIG. 4.2. $U(\cdot,t)$, $V(\cdot,t)$ *with constant mobility,* $\alpha = 4$, $\beta = 2$ *and* $\rho = 0.001$.

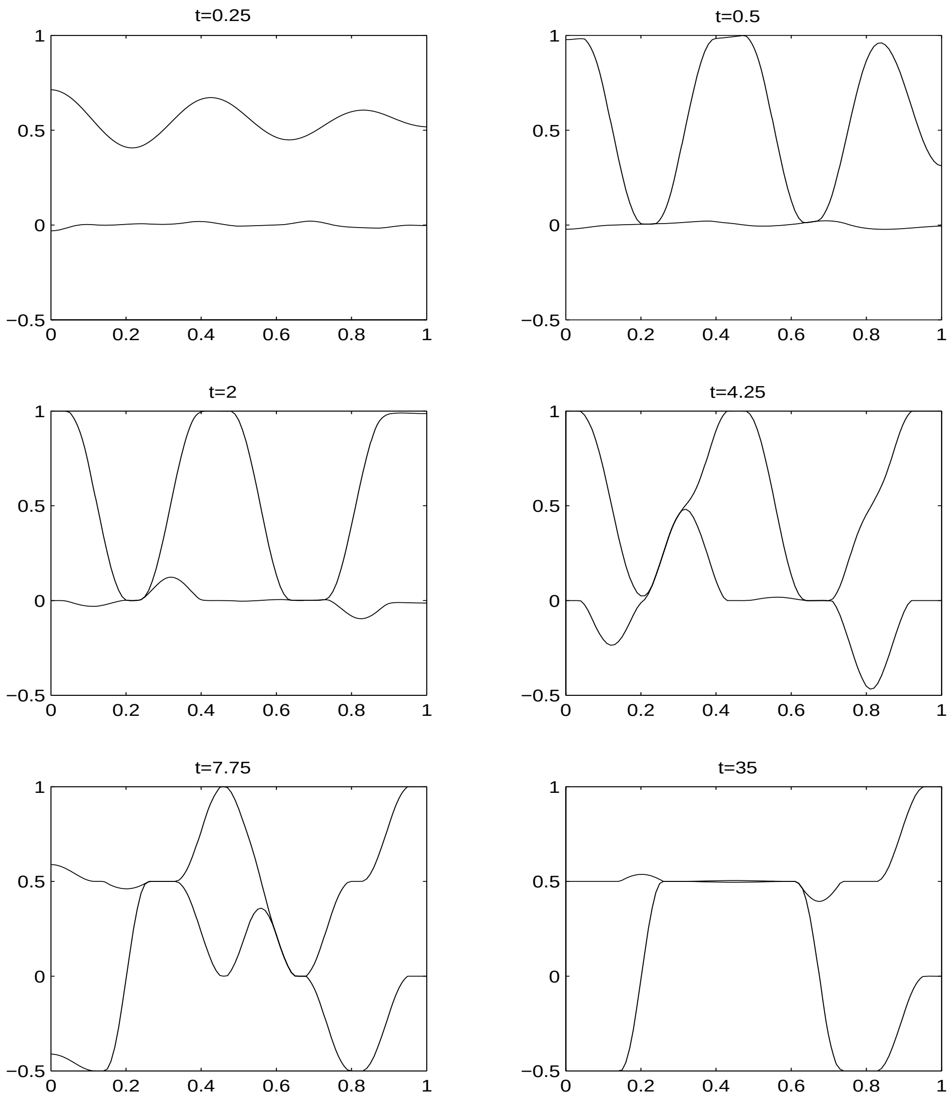


FIG. 4.3. $U(\cdot,t)$, $V(\cdot,t)$ *with degenerate mobility,* $\alpha = 2$, $\beta = 4$ *and* $\rho = 0.08$.

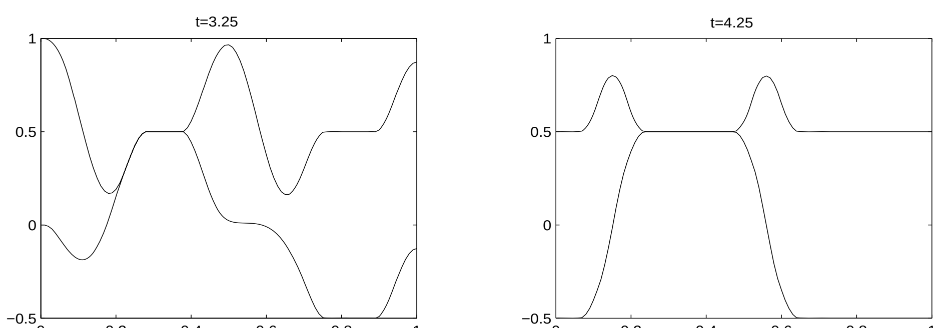


FIG. 4.4. $U(\cdot,t)$, $V(\cdot,t)$ *with constant mobility,* $\alpha = 2$, $\beta = 4$ *and* $\rho = 0.08$.

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